Tutorial-2, Basics on Probability & Statistics

Q.1 Let, $\Omega = \{1,2,\ldots,6\}$

$F = \{\phi, \Omega, \{1,2,3\}, \{4,5,6\}\}$

Define, $X(\omega) = \sqrt{\omega}$

$Y(\omega) = \mathbb{1}_{(\omega \in \{1,2,3\})}$

* A function $X: \Omega \to \mathbb{R}$ is a random variable w.r.t a field F if for every real number a, the set $\{x \in \Omega : X(x) \leq a\} \in F$.

(i) For, $X(\omega) = \sqrt{\omega}$,

possible values are $\{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}\}$.

consider a event $\{X \leq 1.5\} = \{\omega : \sqrt{\omega} \leq 1.5\}$

$= \{1,2\} \notin F$

So, X is not a random variable w.r.t F.

(ii) For, $Y(\omega) = \mathbb{1}_{(\omega \in \{1,2,3\})}$

Possible values are: $Y = 1$ if $\omega \in \{1,2,3\}$

$Y = 0$ if $\omega \in \{4,5,6\}$

So, $\{Y=1\} = \{1,2,3\} \in F$

$\{Y=0\} = \{4,5,6\} \in F$

$\therefore$ For any real 'a', the sets $\{Y \leq a\}$ are always in F.

Hence, Y is a random variable w.r.t. F.

Q.2

$\Omega = \{H, T\}^2$ represents the outcome of tossing two coins.

$\Rightarrow \Omega = \{HH, HT, TH, TT\}$

X takes values $\{1, 0, -1\}$ with pre-images,

$X = 1 \rightarrow \{HH\}$

$X = 0 \rightarrow \{HT, TH\}$

$X = -1 \rightarrow \{TT\}$

$\therefore$ The smallest sigma-field is

$$\sigma(X) = \{\phi, \{HH\}, \{TT\}, \{HT, TH\}, \{HH, TT\}, \{HH, HT, TH\}, \{TT, HT, TH\}, \Omega\}$$

Q.4

* A distribution function must be (a) non-decreasing, (b) right continuous. (c) limits 0 at $-\infty$ and 1 at $+\infty$.

(i) $$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1/2 \\ 1 & \text{if } x > 1/2 \end{cases}$$

Here, on $[0, \frac{1}{2}]$, $F(x) = x$ is non-decreasing and right continuous.

$F(\frac{1}{2}) = \frac{1}{2}$ but, $F > \frac{1}{2}$ for $x > \frac{1}{2}$ is 1. (i.e, $F(x)$ is not right continuous.)

As, $x \rightarrow +\infty$, $F \rightarrow 1$

$x \rightarrow -\infty$, $F \rightarrow 0$

So the given function is not a distribution function.

Q.4

(ii) $$F(x) = \begin{cases} 1-e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

if $x_1, x_2 < 0$ $\quad F(x_1) = F(x_2) = 0$ inequality holds.

if $x_1 < 0 \leq x_2$ $\quad F(x_1) = 0$ and $F(x_2) = 1-e^{-x_2} \geq 0$

$\Rightarrow F(x_1) \leq F(x_2)$

if $0 \leq x_1 < x_2$ $\quad F(x) = (1-e^{-x})$ on $[0,\infty)$

$\therefore$ for $x \geq 0$, $\frac{d}{dx}(1-e^{-x}) = e^{-x} \geq 0$

So, $F(x)$ is non decreasing on $\mathbb{R}$.

- F is right continuous as the function $(1-e^{-x})$ is continuous on $[0,\infty)$ and for $x<0$, $\lim_{t \to x} F(t) = F(x) = 0$.

- $\lim_{x \to -\infty} F(x) = 0$ as $F(x) = 0$ $\forall x < 0$.

$$\lim_{x \to +\infty} F(x) = \lim_{x \to +\infty} (1-e^{-x})$$
$$= 1-0$$
$$= 1$$

So, $F(x)$ is a distribution function.

(iii) $F(x) = \frac{1}{\pi} \tan^{-1} x \qquad -\infty < x < \infty$

$$\lim_{x \to +\infty} F(x) = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}$$

$$\lim_{x \to -\infty} F(x) = \frac{1}{\pi}\left(-\frac{\pi}{2}\right) = -\frac{1}{2}$$

These are not 1 and 0. Thus the function does not satisfy the necessary boundary limits.

So, $F(x)$ is not a distribution function.

Q.5.

A c.d.f of a discrete variable has jumps at the mass points.

The given cdf is a step function with positive jumps only at a finite set of points $(-2, 0, 5, 6)$.

So, X is a discrete random variable.

The pmf is given by the jump at each mass points:

$$P(X=-2) = \frac{1}{3}$$

$$P(X=0) = \left(\frac{1}{2}-\frac{1}{3}\right) = 1/6$$

$$P(X=5) = \left(\frac{3}{4}-\frac{1}{2}\right) = 1/4$$

$$P(X=6) = \left(1-3/4\right) = 1/4$$

i.e, $P(X=-2)+P(X=0)+P(X=5)+P(X=6) = \left(\frac{1}{3}+\frac{1}{6}+\frac{1}{4}+\frac{1}{4}\right) = 1$

So, the pmf is:

$$P_X(x) = \begin{cases} 1/3 & x=-2 \\ 1/6 & x=0 \\ 1/4 & x=5 \\ 1/4 & x=6 \\ 0 & \text{otherwise} \end{cases}$$

Q.6

(i) For a purely discrete random variables the c.d.f is a step function (constant between jumps).

Here between 0 and 1, the cdf is $F(x) = x/3$, which is a straight line, not flat.

So, it can not be purely discrete.

For a purely ~~random~~ continuous random variable, the cdf is continuous everywhere (no jumps).

But, here at $x=1$ and $x=2$, F jumps from $\frac{1}{3} \to 2/3$ and from $2/3 \to 1$.

So it can not be purely continuous.

(ii)

- $P(X=1) = F(1) - F(1^-)$
$= (2/3 - 1/3)$
$= 1/3$

- $P(X=2) = F(2) - F(2^-)$
$= (1 - 2/3)$
$= 1/3$

- $P(X=1.5)$
Since $1.5 \in (1,2)$ and F is constant on $[1,2)$, there is no point at 1.5.
So, $P(X=1.5) = 0$.

- $P(1<X<2)$
$= F(2^-) - F(1)$
$= (2/3 - 2/3) = 0$.

Q.7

(i) A distribution is continuous when its cdf has no jumps.

At $x=1$,
$$\lim_{x \uparrow 1} F(x) = \frac{1}{2} = \frac{1}{2} \text{ and } F(1) = \frac{1}{2}$$

At $x=2$, $$\lim_{x \uparrow 2} F(x) = \frac{2}{2} = 1 \text{ and } F(2) = 1$$

So, $F(x)$ is continuous everywhere, hence X is continuous random variable.

(ii) For a ~~random~~ continuous random variable,

$P(X=a) = 0$

i.e, $P(X=1) = 0$ ——— (a)

$P(1 < X \leq 2) = F(2) - F(1) = (1 - \frac{1}{2}) = \frac{1}{2}$ ——— (b)

$P(X \geq 1) = 1 - F(1^-) = 1 - \frac{1}{2} = \frac{1}{2}$ ——— (c)

(iii) For $0 < x < 1$:

$F(x) = \frac{x^2}{2} \Rightarrow f(x) = F'(x) = x$

For $1 < x < 2$

$F(x) = x/2 \Rightarrow f(x) = F'(x) = \frac{1}{2}$

Elsewhere $f(x) = 0$

So, the pdf is,

$$f(x) = \begin{cases} x & 0 < x < 1 \\ \frac{1}{2} & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(iv)
We need Q_1 (lower quartile), median, Q_3 (upper quartile).

Now, $F(q) = 0.25$.

Since $0.25 < 0.5$, solution lies in $0 \le x < 1$.

$\therefore \frac{q^2}{2} = 0.25 \Rightarrow q^2 = 0.5 \Rightarrow q = \frac{1}{\sqrt{2}} \approx 0.707$ — (Q_1)

$F(q) = 0.5$.

At $x = 1$, $F(1) = 0.5$

So, median $= 1$ — (Ans)

$F(q) = 0.75$

Since, $0.75 > 0.5$, solution lies in $1 \le x < 2$

$\therefore q/2 = 0.75 \Rightarrow q = 1.5$ — (Q_3)

(8)

i) $$f(x) = \begin{cases} cp^x & ; x = 0, 1, 2 \\ 0 & ; o.w \end{cases}$$ with $0 < p < 1$

Condition for a pmf —

$$\sum_{x=0}^{\infty} p_x(x) = 1$$

$$\sum_{x=0}^{\infty} cp^x = 1.$$

$$c\left(p^0 + p^1 + p^2 + \cdots\right) = 1.$$

$$c\left(1 + p + p^2 + \cdots\right) = 1.$$

$\because$ $1 + p + p^2 + \cdots$ is a G.P and its sum for infinite terms will be $\frac{1}{1-p}$.

So

$$c\left(\frac{1}{1-p}\right) = 1.$$

$$\boxed{c = 1-p}$$

ii) $$f(x) = \begin{cases} cxe^{-x} & ; \text{for } x > 0 \\ 0 & ; o.w \end{cases}$$

Condition for a pdf —

$$\int_x f_X(x)\, dx = 1.$$

$$\int_0^{\infty} c x e^{-x} dx = 1.$$

$$c \int_0^{\infty} x e^{-x} dx = 1$$

$$c \left[\left(-x e^{-x}\right)_0^{\infty} - \int 1 \left(-e^{-x}\right) dx \right] = 1.$$

$$c \left[0 + \int e^{-x} dx \right] = 1.$$

$$c \left(-e^{-x}\right)_0^{\infty} = 1$$

$$c \left(-0 + 1\right) = 1$$

$$\boxed{c = 1}$$

(9)
$$F(x) = \begin{cases} 0 & ; x < a \\ \dfrac{x-a}{b-a} & ; a \le x \le b \\ 1 & ; x > b \end{cases}$$

first, find the pdf from given CDF.

$$f(x) = \begin{cases} \dfrac{1}{b-a} & ; a \le x \le b \\ 0 & ; \cancel{x > b} \end{cases}$$

- $E(X) = \int_x x f(x) \, dx$

$$= \int_a^b \frac{x}{(b-a)} dx$$

$$= \frac{1}{(b-a)} \left(\frac{x^2}{2}\right)_a^b$$

$$= \frac{1}{(b-a)} \frac{(b^2-a^2)}{2}$$

$$= \frac{\cancel{(b-a)}(a+b)}{2\cancel{(b-a)}}$$

$$\boxed{E(X) = \frac{a+b}{2}}$$

- Now $V(X) = E(X^2) - [E(X)]^2$

So, we need $E(X^2)$.

$$E(X^2) = \int_a^b x^2 f(x)\, dx$$

$$= \int_a^b x^2 \frac{1}{(b-a)}\, dx$$

$$= \frac{1}{(b-a)} \left(\frac{x^3}{3}\right)_a^b$$

$$= \frac{1}{(b-a)} \frac{(b^3-a^3)}{3}$$

$$= \frac{\cancel{(b-a)}(a^2+b^2+ab)}{\cancel{(b-a)}\, 3}.$$

$$E(X^2) = \frac{(a^2+b^2+ab)}{3}.$$

then

$$V(X) = \frac{a^2+b^2+ab}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4(a^2+b^2+ab) - 3(a^2+b^2+2ab)}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$\boxed{V(X) = \frac{(b-a)^2}{12}}$$

(10) $$f(x) = \begin{cases} \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}} & \text{if } 0<x<1. \\ 0 & ; \text{o.w} \end{cases}$$

- Symmetry -

$$f(1-x) = \frac{1}{\pi} \frac{1}{\sqrt{(1-x)x}}$$

$$f(1-x) = f(x)$$

So, density is invariant under $x \to -x+1$. i.e;

$$2x = 1.$$

$$\boxed{x_0 = \frac{1}{2}}$$

- Since it is symmetric about $\frac{1}{2}$. then its expected value is exactly at the point of symmetry. i.e:

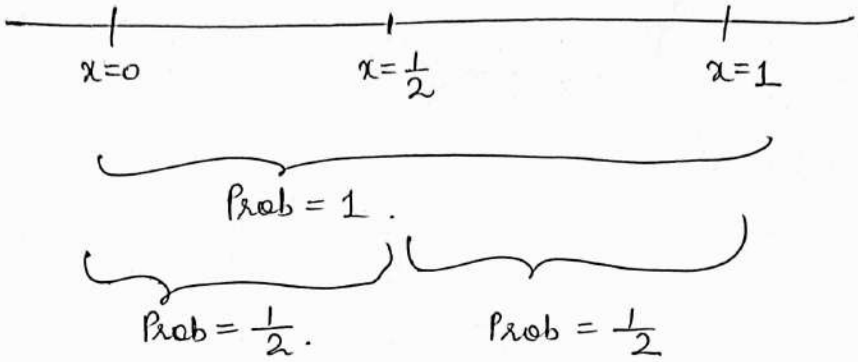
$$\boxed{E(X) = \frac{1}{2}}$$

- $\cancel{P(X>x_0)}$

$$\cancel{\Rightarrow P(X>\tfrac{1}{2})}$$

$$\cancel{= \int_0^{\frac{1}{2}}}$$

- $P(X > x_0) = P\left(X > \frac{1}{2}\right)$



i.e; $P\left(X \leq \frac{1}{2}\right) = \frac{1}{2}$ i.e; $P\left(X > \frac{1}{2}\right) = \frac{1}{2}$.

Hence $\boxed{P(X > x_0) = \frac{1}{2}}$

(11) $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \; ; -\infty < x < \infty$

- $f(-x) = \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} = f(x).$

So, the distribution of X and −X coincide i.e; $X \overset{d}{=} -X$.

- $E(X^3) = \int_{-\infty}^{\infty} x^3 f(x)\, dx.$

Here $f(x)$ is even function and x^3 is odd function.
So, the integrand is odd. Thus –

$\boxed{E(X^3) = 0}$

- Since it is symmetric about 0, then

$\boxed{P(X > 0) = \frac{1}{2}}$

(12) $P(H) = \frac{3}{4}$

1) first toss and he gets tail so got 0\$.

$$P(T) = \frac{1}{4}$$

and total \$ = 0.

2) In first toss, he gets head and won 5\$. So, he gets second toss. But in 2^{nd} toss, he gets tail and won 0\$.

$$P(H)\,P(T \mid H \text{ comes in } 1^{st} \text{ toss}) = \frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$$

and total \$ = 0 + 5 = 5.

3) In first toss, he gets head and won 5\$. so, he gets second toss and in 2^{nd} toss, he gets head and won 10\$.

$$P(H)\,P(H \mid H \text{ comes in } 1^{st} \text{ toss}) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$$

and total \$ = 5 + 10 = 15.

Expected winning amount $= 0 \times \frac{1}{4} + 5 \times \frac{3}{16} + 15 \times \frac{9}{16}$

$$= \frac{15}{16} + \frac{135}{16}$$

$$= \frac{150}{16}$$

$$= 9.375.$$

$$M_X(t) = \left(\frac{1+2e^t}{3}\right)\left(\frac{1+3e^t}{4}\right)$$

- $P(X=1)$ - ?

$$M_X(t) = \frac{1}{12}(1+2e^t)(1+3e^t)$$

$$M_X(t) = \frac{1}{12}\left(1+5e^t+6e^{2t}\right)$$

$$= \frac{1}{12} + \frac{5}{12}e^t + \frac{6}{12}e^{2t}$$

$$P(X=x) = \begin{cases} \frac{1}{12} & ; x=0 \\ \frac{5}{12} & ; x=1 \\ \frac{6}{12} & ; x=2 \end{cases} \quad \text{——— } (1)$$

and

$$= P(X=0)+P(X=1)+P(X=2)$$

$$= \frac{1}{12}+\frac{5}{12}+\frac{6}{12} = 1.$$

So, it is a pmf.

for discrete r.v, X

$$M_X(t) = \sum_x P(X=x)\, e^{tx}$$

So, the coefficient of e^t is $P(X=1)$.

from (1)

$$\boxed{P(X=1) = \frac{5}{12}}$$

(14) $M_X(t) = (1-2t)^{-\frac{n}{2}}$ for $t < \frac{1}{2}$.

- $E(X) = M_X'(0)$

$$M_X'(t) = \frac{d}{dt}(1-2t)^{-\frac{n}{2}}$$

$$= \left(-\frac{n}{2}\right)(1-2t)^{-\frac{n}{2}-1}(-2).$$

$$= n(1-2t)^{-\frac{n}{2}-1}$$

$t=0$

$$M_X'(0) = n(1-0)^{-\frac{n}{2}-1} = n.$$

$$\boxed{E(X) = n}$$

and for $V(X) = E(X^2) - (E(X))^2$

so, $E(X^2) = M_X''(0)$

$$M_X''(t) = \frac{d^2}{dt^2}(1-2t)^{-\frac{n}{2}}$$

$$= \frac{d}{dt} n(1-2t)^{-\frac{n}{2}-1}$$

$$= n\left(\frac{-n}{2}-1\right)(1-2t)^{-\frac{n}{2}-1-1}(-2)$$

$$M_X''(t) = n(n+2)(1-2t)^{-\frac{n}{2}-2}$$

at $t=0$.

$$M_X''(0) = n(n+2)(1-0)^{-\frac{n}{2}-2}.$$

$$M_X''(0) = n(n+2)$$

$$\{E(X^2) = n(n+2)\}$$

so $V(X) = n(n+2) - n^2$

$$\boxed{V(X) = 2n}$$

(15) $M_X(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t}$

$$p_X(x) = \begin{cases} \frac{1}{6} ; & x=-2 \\ \frac{1}{3} ; & x=-1 \\ \frac{1}{4} ; & x=1 \\ \frac{1}{4} ; & x=2 \end{cases}$$

and

$= P(X=-2)+P(X=-1)+P(X=1)$
$+P(X=2)$

$= \frac{1}{6}+\frac{1}{3}+\frac{1}{4}+\frac{1}{4}$

$= 1.$

So, it is a pmf.

$\cdot \; P(|X| \leq 1)$

$= P(-1 \leq X \leq 1)$

$= P(X=1) + P(X=-1)$

$= \frac{1}{3} + \frac{1}{4}$

$= \frac{7}{12}$

$$\boxed{P(|X| \leq 1) = 0.5833}$$

16. Given that X is the no. scored in a throw of a fair die.

$$\mu = \frac{1+2+\dots+6}{6} = \frac{n+1}{6} = 3.5$$

$$Var(X) = \sigma^2 = E(X-\mu)^2$$

$$= \frac{(1-3.5)^2 + (2-3.5)^2 + \dots + (6-3.5)^2}{6}$$

$$= \frac{17.5}{6} \approx 2.9167.$$

After applying Chebyshev's inequality, we get–

$$P(|X-3.5| \geq 2.5) \leq \frac{2.9167}{(2.5)^2} \approx 0.4667$$

$$\Rightarrow P(|X-\mu| > 2.5) < 0.47.$$

Actual Probability calculation–

$|X-\mu| > 2.5 \Rightarrow |X-3.5| > 2.5$

i.e. $X-3.5 > 2.5$ or $X-3.5 < -2.5$.

This simplifies to $X > 6$ or $X < 1$.

For a fair die, the possible outcomes are–

$\{1, 2, 3, 4, 5, 6\}$.

Thus, $P(X>6) = 0$ or $P(X<1) = 0$

$\Rightarrow P(|X-\mu| > 2.5) = 0$. [Proved].

17. Let X be the grade of a student in the quiz.
The avg. grade is given as – 70%, so $E(X) = 0.70$.
The propⁿ of students who scored at least 60% is sought, so the threshold is $a = 0.60$.
After applying Markov's inequality, we get –

$$P(X \geq 0.60) \leq \frac{0.70}{0.60} = \frac{7}{6}. \text{ (Ans.)}$$

∴ The upper bound of the propⁿ of students who scored at least 60%.

18. Let X be the no. of heads obtained in 20 coin flips. $X \sim Bin\left(20, \frac{1}{2}\right)$.

$E(X) = 10$, $Var(X) = 20 \times \frac{1}{4} = 5$.

Prob. that head will come at least 16 times

$$= P(X \geq 16) \leq \frac{\cancel{10}\,5}{\cancel{16}\,8} = 0.625 \text{ (Ans)}$$

19. $X \sim Bin\left(100, \frac{1}{2}\right)$; $n = 100$, $p = \frac{1}{2}$.

$E(X) = 50$, $Var(X) = \sigma^2 = 25$.

The event 'at least 60 heads or at most 40 heads' can be expressed as $|X - 50| \geq 10$

$\Rightarrow k = 10$.

∴ By Chebyshev's inequality,

$$P(|X-50| \geq 10) \leq \frac{25}{10^2} = 0.25. \text{ (Ans.)}$$

20. Let X be a r.v. with pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad , \quad -\infty < x < \infty . \Rightarrow X \sim N(0,1).$$

Define $$Y = \begin{cases} 1 & , X < 0 \\ 0 & , X = 0 \\ -1 & , X > 0. \end{cases}$$

$$P(Y=1) = P(X<0) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{2}.$$

$$P(Y=0) = P(X=0) = 0.$$

For a continuous variable, X, the prob. of taking any single specific value is 0.

$$P(Y=-1) = P(X>0) = \frac{1}{2} \quad [\because X \text{ is symmetric}].$$

$\therefore$ The distn. of Y is given by its p.m.f. –

$$P(Y=1) = \frac{1}{2}, \quad P(Y=0) = 0, \quad P(Y=-1) = \frac{1}{2}.$$

Thus, Y is a discrete random variable.

(Ans)

21. Given: $f(x) = \begin{cases} 6x(1-x) & , 0 < x < 1 \\ 0 & , o.w. \end{cases}$

$$Y = X^2(3-2X)$$

$$\Rightarrow Y = 3X^2 - 2X^3$$

$$\Rightarrow \frac{dY}{dX} = 6X - 6X^2$$

$$\therefore |J| = \left|\frac{dX}{dY}\right|^{-1} = \frac{1}{6x(1-x)}.$$

$$f(y) = 6x(1-x)\,|J| = 1 \quad , \quad 0 < y < 1.$$

The pdf of Y is given by—

$$f(y) = \begin{cases} 1 & , 0 < y < 1 \\ 0 & , o.w. \end{cases} \quad \text{(Ans)}$$

22. Given: $f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} & , -\infty < x < \infty. \end{cases}$

Define $Y_1 = |X|$ and $Y_2 = \begin{cases} X & , \text{if } |X| > c \\ 0 & , o.w. \end{cases}$

$$F_{Y_1}(y) = P(Y_1 \leq y) = P(|X| \leq y)$$
$$= P(-y \leq X \leq y) = \int_{-y}^{y} f(x)\,dx$$
$$= 2\int_0^y f(x)\,dx$$
$$= 2\int_0^y \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \quad , \quad y \geq 0.$$

$F_{Y_1}(y) = 0$, for $y < 0$ [$\because Y_1 = |X|$ must be non-negative.]

$$f_{Y_1}(y) = \frac{d}{dy}\left(2\int_0^y \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx\right) \quad , \text{for } y > 0.$$

By the Fundamental Theorem of Calculus,

$$f_{Y_1}(y) = 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \text{ for } y > 0.$$

$$f_{Y_1}(y) = 0 \quad , \text{ for } y \leq 0. \quad \text{(Ans.)}$$

$$F_{Y_2}(y) = P(Y_2 \leq y) = P(X \leq y \text{ and } X < -c)$$
$$= P(X \leq y) \text{ for } y < -c.$$

For $-c \leq y < 0$, $F_{Y_2}(y) = \int_{-\infty}^{c} f(x)\, dx$.

For $0 \leq y < c$,

$$F_{Y_2}(y) = \int_{-\infty}^{-c} f(x)\, dx + \int_{-c}^{c} f(x)\, dx = \int_{-\infty}^{+c} f(x)\, dx.$$

For $y \geq c$,

$$F_{Y_2}(y) = \int_{-\infty}^{-c} f(x)\, dx + \int_{-c}^{c} f(x)\, dx + \int_c^y f(x)\, dx$$
$$= \int_{-\infty}^{y} f(x)\, dx.$$

$$f_{Y_2}(y) = \frac{d}{dy}\int_{-\infty}^{y} f(x)\, dx = f(y) \quad , y < -c$$

$f_{Y_2}(y) = 0$, as the cdf is constant in $-c \le y < c$ and $y \neq 0$.

$f_{Y_2}(y) = \frac{d}{dy} \int_{-\infty}^{y} f(x)\,dx = f(y)$, $y \geq c$.

There is a point mass at $y = 0$ with prob.,

$P(Y_2 = 0) = P(|X| \le c) = \int_{-c}^{c} f(x)\,dx$.

$\therefore$ Y_2 is a mixed random variable with a continuous part and a discrete part. (Ans.)

Q3. Properties of a Cumulative Dist$^{\underline{n}}$ F$^{\underline{n}}$ (CDF) –

i) Right - Continuity of F –

Let $F(x)$ be a cdf and consider a sequence x_n such that $x_n \to x$ as $n \to \infty$.

Define the events $E_n = (-\infty, x_n]$. This forms a decreasing sequence of events, as $E_{n+1} \subseteq E_n$.

The limit of this sequence of events is

$$E = \bigcap_{n=1}^{\infty} E_n = (-\infty, x].$$

By the result for a decreasing sequence of events, $P(E) = \lim_{n\to\infty} P(E_n)$.

In terms of the c.d.f., $P(E) = F(x)$ and $P(E_n) = F(x_n)$.

Thus, $F(x) = \lim_{n\to\infty} \cancel{E}\, F(x_n)$.

ii) $F(+\infty) = 1$ and $F(-\infty) = 0$.

(*) To show $F(+\infty) = 1$, consider an increasing sequence $x_n \to \infty$ as $n \to \infty$.

Define the events $E_n = (-\infty, x_n]$. This forms an increasing sequence of events, as $E_n \subseteq E_{n-1}$.

The limit of this sequence of events is

$$E = \bigcup_{n=1}^{\infty} E_n = (-\infty, \infty) = \Omega.$$

By the result for an increasing sequence of events, $P(E) = \lim_{n\to\infty} P(E_n)$.

Since $P(\Omega) = 1$ and $P(E_n) = F(x_n)$, it follows that $1 = \lim_{n\to\infty} F(x_n) = F(\infty)$.

(*) To show $F(-\infty) = 0$, consider a decreasing sequence $x_n \to -\infty$ as $n \to \infty$.

Define the events $E_n = (-\infty, x_n]$. This forms a decreasing sequence of events.

The limit of this sequence of events is

$$E = \bigcap_{n=1}^{\infty} E_n = \phi.$$

By the result for a decreasing sequence of events, $P(E) = \lim_{n\to\infty} P(E_n)$.

Since $P(\phi) = 0$ and $P(E_n) = F(x_n)$, it follows that $0 = \lim_{n\to\infty} F(x_n) = F(-\infty)$. [Proved].