Differential Equation(NMCC203)
IIT (ISM) Dhanbad

UNIT 3 | Tutorial

1 Topics

Second order boundary value problems, Self-adjoint eigen value problems, Sturm-Liouville Systems.
Stability of a Linear and Nonlinear Systems.

1. Find the eigen values and eigen functions of the given boundary value problems.

   (a) Dirichlet Problem: $(x^3y')' + \lambda xy = 0$; $y(1) = 0$, $y(e) = 0$.
   $Ans : y_n(x) = (1/x) \times sin(n\pi \, log \, x)$; $\lambda_n = n^2\pi^2$, $n \in \mathbb{N}$.

   (b) Neumann Problem: $y'' + \lambda^2 y = 0$; $y'(0) = 0 = y'(L)$.
   $Ans : y_n = cos(\frac{n\pi x}{L})$; $\lambda_n = \frac{n\pi}{L}$, $n \in \mathbb{N}$.

   (c) Periodic Boundary value Problem: $y'' + \lambda^2 y = 0$; $y(-L) = 0 = y(L)$, $y'(-L) = 0 = y'(L)$.
   $Ans : y_n = \{1, cos(\frac{n\pi x}{L}), sin(\frac{n\pi x}{L})\}$; $\lambda_n = \frac{n\pi}{L}$, $n \in \mathbb{N}$.

2. Show that the functions $sin(\pi x)$, $sin(2\pi x)$, $sin(3\pi x), \cdots$ form an orthogonal set on the interval $-1 \leq x < 1$ and obtain the corresponding orthonormal set.
   $Ans : sin(\pi x), sin(2\pi x), sin(3\pi x), \cdots$.

3. Find the eigenvalues and eigenfunctions of $[xy'(x)]' + (\frac{\lambda}{x})y(x) = 0$, $y'(1) = y'(e^{2\pi}) = 0$.
   $Ans : y_n(x) = cos\left((\frac{n}{2}) \times logx\right), \lambda_n = \frac{n^2}{4}$, for $n = 1, 2, 3, \cdots$; and $y(x) = 1$ for $\lambda = 0$.

4. Given that $f_1(x) = a_0,\ f_2(x) = b_0 + b_1x,\ and\, f_3(x) = c_0 + c_1x + c_2x^2$. Determine the constants a_0, b_0, c_0, b_1, c_1, and c_2 so that the given functions form an orthonormal set on the interval $-1 \le x \le 1$.
Ans : a_0, b_1, c_0 are arbitrary real values and $b_0 = 0, c_1 = 0, c_2 = -3c_0$.

5. The set of real numbers λ for which the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0$$

has a non-trivial solution is
(a) $(-\infty, 0)$ (b) $\{\sqrt{n} : n \in \mathbb{N}\}$ (c) $\{n^2 : n \in \mathbb{N}\}$✓ (d) $\mathbb{R}$.

6. Find the eigenvalue and eigenfunction of $y''+\lambda y = 0, \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) + 2y'(1) = 0$.
Ans : $\lambda_n = \mu_n^2, \quad \mu_n$ are the positive roots of $\tan(\mu) = -2\mu$ and $y_n(x) = \sin(\mu_n x), \quad n = 1, 2, 3, \ldots$

7. For the following system of ODEs,

$$\frac{\mathbf{d}x}{\mathbf{d}t} = x(3 - 2x - 2y); \quad \frac{\mathbf{d}y}{\mathbf{d}t} = y(2 - 2x - y)$$

find out all the critical points and the critical point $(0, 2)$ is
(a) a stable spiral (b) an unstable spiral (c) a stable node ✓ (d) an unstable node.
Ans : $(0, 0)$, $(0, 2)$, $(\frac{3}{2}, 0)$, $(\frac{3}{2}, 2)$.

8. Consider the first order system of linear differential equations

$$\frac{dX}{dt} = AX; \quad A = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(a) The coefficient matrix A has a repeated eigenvalue $\lambda = 1$. ✓
(b) The only one linearly independent eigenvector is $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. ✓
(c) The general solution of the ODE is

$$(aX_1 + bX_2)e^t, \quad X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ X_2 = \begin{bmatrix} t \\ \frac{1}{2} - t \end{bmatrix},$$

where a, b are arbitrary constants. ✓
(d) The vectors X_1 and X_2 in option (c) given above are linearly independent. ✓

9. What is the nature of the critical point $(0, 0)$ of the following system of equation,
$x'(t) = x - 2y + y^2\, sin(x), \quad y'(t) = 2x - 2y - 3y\, cos(y^2)$.
Ans : Unstable saddle point.