Math 502AB - Lecture 11

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1 Lecture - Part 1

1.1 Chapter 4: Jointly Distributed Random Variables

To address the relationships between multiple random variables, we need **joint distributions** of random variables

Definition:

An n-dimensional random vector $\overline{X} = (x_1, ..., x_n)$ where each component is a random variable.

Example

A fair coin is tossed 3 times. Let X be the number of heads on the 1^{st} toss and Y be the total number of heads.

$$\begin{aligned} X &= \text{Number of Heads on First Toss} \\ Y &= \text{Total Number of Heads} \end{aligned}$$

With the probability matrix:

$$\begin{bmatrix} x/y & 0 & 1 & 2 & 3 \\ 0 & \frac{1}{8} & \frac{2}{8} & \frac{1}{8} & 0 \\ 1 & 0 & \frac{1}{8} & \frac{2}{8} & \frac{1}{8} \end{bmatrix}$$

Calculating joint probabilities:

$$\begin{aligned} P(X = 0 \cap Y = 0) &= P(Y = 0 | X = 0) \cdot P(X = 0) \\ &= \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} \end{aligned}$$

You can also use the probability matrix to find values of the **marginal density** by summing over the rows/column probabilities of the other variable. For example:

$$P(Y=2) = \frac{1}{8} + \frac{2}{8} = \frac{3}{8}$$
$$P(X=1) = 0 + \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = \frac{1}{2}$$

1.1.1 Joint Discrete Random Variables

Definition:

Let (X,Y) be a bivariate random variable. Then, the function $f(x,y) = P(X = x, Y = y)$, $\forall x, y$ is called a **joint pmf** of (X,Y).
Notation: Sometimes, $f_{x,y}(x,y)$ is used to indicate the joint distribution

Example:

A bin of five transistors contains two defective units. The transistors are to be continually tested until the two defective units are discovered. Let:

$$\begin{aligned} X &= \text{ Number of tests until the first defective is discovered} \\ Y &= \text{ Number of add'l tests until the second defective is discovered} \end{aligned}$$

Obtain the joint *pmf* of (X,Y)

$$\begin{array}{c|cccc} X/Y & 1 & 2 & 3 & 4 \\ \hline 1 & \left(\tfrac{2}{5}\right)\left(\tfrac{1}{4}\right) & \left(\tfrac{2}{5}\right)\left(\tfrac{1}{4}\right) & \left(\tfrac{2}{5}\right)\left(\tfrac{1}{4}\right) & \left(\tfrac{2}{5}\right)\left(\tfrac{1}{4}\right) \\ 2 & \left(\tfrac{3}{5}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{1}{3}\right) & \left(\tfrac{3}{5}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{1}{3}\right) & \left(\tfrac{3}{5}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{1}{3}\right) & 0 \\ 3 & \left(\tfrac{3}{5}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{2}\right) & \left(\tfrac{3}{5}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{2}\right) & 0 & 0 \\ 4 & \left(\tfrac{3}{5}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{1}{3}\right)(1) & 0 & 0 & 0 \end{array}$$

Examples:

1. On average, what is the total number of tries to discover both defects? In other words, we wish to find $E[X+Y]$. First, we note that:

$$E[g(x,y)] = \sum_{\text{all } (x,y)} g(x,y) f_{x,y}(x,y)$$

Thus, we have:

$$\begin{aligned} E[X+Y] &= \sum_{x=1}^{4}\sum_{y=1}^{4}(x+y)f(x,y) \\ &= \frac{1}{10}(2+3+4+5+3+4+5+4+5+5) \\ &= \frac{1}{10}(40) = 4 \end{aligned}$$

2. What is the probability that the number of additional tests made is equal to 2?

$$P(Y=2) = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{3}{10}$$

Definition:

The *marginal distribution* of Y is:

$$f_Y(y) = \sum_{\text{all } X} P(X=x, Y=y) = \sum_{\text{all } X} f_{x,y}(x,y)$$

Example:

Find the marginal distribution of X from the prior example:

$$\begin{bmatrix} X & 1 & 2 & 3 & 4 \\ & \frac{4}{10} & \frac{3}{10} & \frac{2}{10} & \frac{1}{10} \end{bmatrix}$$

1.1.2 Joint Continuous Random Variables

Definition:

A function $f(x,y)$ is called the *joint pdf* of a continuous random variable (x,y) if $\forall A \subset \mathcal{R}^2$

$$P[(x,y) \in A] = \int_A \int f(x,y)dxdy$$

We then have:

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$$
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

Properties:

$$f(x,y) \geq 0 \quad \forall(x,y)$$
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

Example:

1. Consider (X,Y) with joint pdf:

$$f(x,y) = \begin{cases} 4x^2y + 2y^5 & 0 \leq x \leq 1; 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $P(X+Y \geq 1)$
When doing these kinds of problems, don't focus on the density so much as the **domain** where the density is positive.

$$P(X+Y \geq 1) = \int_0^1 \int_{1-x}^1 (4x^2y + 2y^5)dydx$$

(b) Compute $E[X+Y]$

$$\int_0^1 \int_0^1 (x+y)(4x^2y + 2y^5)dydx$$

(c) Find the *marginal distribution* of X

$$f_X(x) = \int_0^1 (4x^2y + 2y^5)dy$$

2. Consider (X,Y) with joint pdf:

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

(a) Obtain the marginal distribution of Y and X

$$f_Y(y) = \int_0^y \lambda^2 e^{-\lambda y} dx = \lambda^2 y e^{-\lambda y} \quad y \geq 0$$
$$f_X(x) = \int_0^\infty \lambda^2 e^{-\lambda y} dy = \lambda e^{-\lambda x} \quad x \geq 0$$

(b) Find $P(X+Y \leq 1)$
Remember, for this problem it's important to think about the *domain* of the functions.

$$\int_0^{0.5} \int_x^{1-x} \lambda^2 e^{-\lambda y} dydx$$

1.1.3 Cumulative Distribution Function for (X,Y)

We want to find:
$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

In the continuous case:

$$\int_{-\infty}^x \int_{-\infty}^y f_{x,y}(s,t)dtds$$

By the Fundamental Theorem of Calculus, we then have:

$$\frac{\partial^2}{\partial x \partial y} F(x,y) = f_{x,y}(x,y)$$

1.1.4 Conditional Distribution of (X,Y)

If X and Y are **discrete random variables**, then the *conditional pmf* of $X|Y=y$ is:

$$\begin{aligned} f_{X|Y}(x|y) &= P(X=x|Y=y) \\ &= \frac{P(X=x,Y=y)}{P(Y=y)} = \frac{f_{x,y}(x,y)}{f_Y(y)} \end{aligned}$$

Similarly, for the **continuous case**:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \forall f_Y(y) > 0$$

2 Lecture - Part 2

Example:

1. Recall the example of the failed parts from earlier, with joint pmf:

$$\begin{bmatrix} X/Y & 1 & 2 & 3 & 4 \\ 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ 2 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & 0 \\ 3 & \frac{1}{10} & \frac{1}{10} & 0 & 0 \\ 4 & \frac{1}{10} & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} f_{Y|X}(y|1) &= \frac{f_{X,Y}(1,y)}{f_X(1)} = \frac{1/10}{4/10} = \frac{1}{4}, \quad y=1,2,3,4 \\ f_{Y|X}(y|2) &= \frac{f_{X,Y}(2,y)}{f_X(2)} = \frac{1/10}{3/10} = \frac{1}{3}, \quad y=1,2,3 \\ f_{Y|X}(y|3) &= \frac{1}{2}, \quad y=1,2 \\ f_{Y|X}(y|4) &= 1, \quad y=1 \end{aligned}$$

2. Consider the joint density:

$$f_{X,Y}(x,y) = 6e^{-2x}e^{-3y} \quad x>0, y>0$$

Obtain the conditional density $f_{X|Y}(x|y)$.

   (a) First, we do it the hard way:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

To do this, we need to calculate the *marginal distribution of Y*

$$
\begin{aligned}
f_Y(y) = \int_0^\infty 6e^{-2x}e^{-3y}dx &= 6e^{-3y}\int_0^\infty e^{-2x}dx \\
&= 3e^{-3y} \quad y > 0
\end{aligned}
$$

But notice that in the conditional probability, it will be a function of X only. So, in other words, by using the kernel we know that:

$$
\begin{aligned}
f_{X|Y}(x|y) &\propto f_{X,Y}(x,y) \\
&\propto 6e^{-2x}e^{-3y} \\
\Rightarrow f_{X|Y}(x|y) &= 2e^{-2x} \quad x > 0
\end{aligned}
$$

3. Consider the function $f_{X,Y}(x,y) = 24xy$ with $0 < X < 1$, $0 < Y < 1$, and $0 < X + Y < 1$. Obtain $f_{X|Y}(x|y)$.

$$
\begin{aligned}
f_Y(y) &= \int_0^{1-y} 24xydx = 12y(1-y)^2 \\
f_{X|Y}(x|y) &= \frac{24xy}{12y(1-y)^2} = \frac{2x}{(1-y)^2} \quad 0 \leq X \leq 1 - Y
\end{aligned}
$$

Alternate Solution:

$$
\begin{aligned}
f_{X|Y}(x|y) &\propto f_{X,Y}(x,y) \\
&\propto x
\end{aligned}
$$

We can rewrite the joint distribution as:

$$
\begin{aligned}
f_{X,Y}(x,y) &= 24xy \cdot \mathcal{I}_{\{0<x<1\}} \cdot \mathcal{I}_{\{0<y<1\}} \cdot \mathcal{I}_{\{0<x+y<1\}} \\
\Rightarrow f_{X|Y}(x|y) &\propto x \cdot \mathcal{I}_{\{0<x<1\}} \cdot \mathcal{I}_{\{-1\leq x<1-y\}} \\
&= x \cdot \mathcal{I}_{\{0<x<1-y\}}
\end{aligned}
$$

So now we have:

$$
\int_0^{1-y} xdx = \frac{x^2}{2}\bigg|_0^{1-y} = \frac{(1-y)^2}{2}
$$

And you can combine it with the **kernel** and you are done.

2.0.1 Conditional Expectation

We now have:

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$$
$$Var(X|Y=y) = E[X^2|Y=y] + (E[X|Y=y])^2$$

Example:

Considering the above example, find the *conditional variance.*

$$E[X|Y=y] = \int_0^{1-y} x \cdot \frac{2x}{(1-y)^2}dx = \frac{2}{3}(1-y)$$
$$E[X^2|Y=y] = \int_0^{1-y} x^2 \cdot \frac{2x}{(1-y)^2}dx = \frac{1}{2}(1-y)^2$$
$$Var(X|Y=y) = \frac{1}{2}(1-y)^2 - \left[\frac{2}{3}(1-y)\right]^2 = \frac{1}{18}(1-y)^2$$

2.1 Independent Random Variables

Definition: Independent R.V.

Let X, Y have a joint pdf (or pmf) $f(x,y)$. We say that X is independent of Y if:

$$f(x,y) = f_X(x)f_Y(y) \quad \forall X, Y$$

Notation:

If X and Y are independent, we notate it as:

$$X \perp\!\!\!\perp Y$$

It can be shown that if $X \perp\!\!\!\perp Y$, then

$$f_{X|Y}(x|y) = f_X(x) \quad \forall x, y$$

Theorem:

Let (X,Y) have a joint *CDF* F(x,y) with respective marginal CDF's F_X and F_Y. Then

$$X \perp\!\!\!\perp Y \iff F_{X,Y}(x,y) = F_X(x)F_Y(y) \quad \forall x, y$$

Proof: (Continuous)

$(\Leftarrow)$

$$\begin{aligned} F_{X,Y}(x,y) &= F_X(x)F_Y(y) \\ \Rightarrow \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) &= \frac{\partial^2}{\partial x \partial y} F_X(x)F_Y(y) \\ \Rightarrow f_{X,Y}(x,y) &= \frac{\partial}{\partial x} F_X(x) \frac{\partial}{\partial y} F_Y(y) \\ f_{X,Y}(x,y) &= f_X(x)f_Y(y) \end{aligned}$$

$(\Rightarrow)$ Suppose $X \perp\!\!\!\perp Y$, then $f_{X,Y}(x,y) = f_X(x)f_Y(y)$

$$\begin{aligned} F_{X,Y} &= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)dtds \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} f_X(s)f_Y(t)dtds \\ &= \int_{-\infty}^{x} f_X(s)ds \int_{-\infty}^{y} f_Y(t)dt = F_X(x)F_Y(y) \end{aligned}$$

Theorem:

Let (X,Y) have a joint pdf $f(x,y)$. Then $X \perp\!\!\!\perp Y$ **iff** $\exists$ functions $g(x)$ and $h(y)$ such that $f(x,y) = g(x)h(y)\ \forall x, y$

Proof:

$(\Rightarrow)$: Suppose $X \perp\!\!\!\perp Y$, thus let $g(x) = f_X(x)$ and $h(y) = f_Y(y)$.
$(\Leftarrow)$: We need to show that if $f_{X,Y}(x,y) = g(x)h(y)$ for some g and h, then

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

Define:

$$c = \int_{-\infty}^{\infty} g(x)dx \text{ and } d = \int_{-\infty}^{\infty} h(y)dy$$

Then:

$$\begin{aligned} cd &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)dxdy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1 \end{aligned}$$

On the other hand:

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} g(x)h(y)dy = g(x)\int_{-\infty}^{\infty} h(y)dy \\
f_Y(y) &= \int_{-\infty}^{\infty} g(x)h(y)dx = h(y)\int_{-\infty}^{\infty} g(x)dx
\end{aligned}
$$

We note that the values of these two integrals are d and c, respectively. Then:

$$
\begin{aligned}
f_{X,Y}(x,y) = g(x)h(x) &= c \cdot d \cdot g(x) \cdot h(y) \\
&= f_Y(y)f_X(x)
\end{aligned}
$$

Examples:

1. Consider the joint density

   $$f(x,y) = 6e^{-2x}e^{-3y} \quad x > 0; y > 0$$

   Are X and Y independent? **Yes.** There are two separate functions here like we want!

2. Consider the joint density:

   $$f(x,y) = \begin{cases} 24xy & 0 < x < 1; 0 < y < 1; 0 < x + 1 < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Are X and Y independent? **No.** There is an indicator function which is dependent on *both* X and Y