Math 502AB - Lecture 2

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1 Lecture - Part 1

Theorem

If $A_1, A_2, ...$ are a sequence of increasing or decreasing events, then:

$$\lim_{n\to\infty} P[A_i] = P\left[\lim_{n\to\infty} A_i\right]$$

Proof

Suppose that $A_1 \subset A_2 \subset ...$ is a sequence of increasing events, then:

$$P\left[\lim_{n\to\infty} A_i\right] = P\left[\bigcup_{i=1}^{\infty} A_i\right] \quad (*)$$

Note that we can't just go straight to our answer "due to countable additivity", because these sets are not disjoint. So, let $B_1 = A_1$, $B_2 = A_2/A_1$, $B_3 = A_3/A_2, ...$ By construction, $B_i \cap B_j = \emptyset$, $\forall i \neq j$ and $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$

Then, going back to $(*)$, we have:

$$\begin{aligned}
P\left[\lim_{n\to\infty} A_i\right] &= P\left[\bigcup_{i=1}^{\infty} B_i\right] = \sum_{i=1}^{\infty} P(B_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) \\
&= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right) \\
&= \lim_{n\to\infty} P(A_n)
\end{aligned}$$

Now, suppose that $A_1 \supset A_2 \supset ...$ is a sequence of decreasing events. Then:

$$
\begin{aligned}
P\left[\lim_{n\to\infty} A_n\right] &= P\left[\bigcap_{i=1}^{\infty} A_i\right] \\
&= 1 - P\left[\bigcap_{i=1}^{\infty} A_i\right]^c \\
&= 1 - P\left[\bigcup_{i=1}^{\infty} A_i^c\right], \text{ by DeMorgan's Law} \\
&= 1 - P\left[\lim_{n\to\infty} A_n^c\right], \text{ since } A_1^c \supset A_2^c \supset \ldots \\
&= 1 - \lim_{n\to\infty} P\left[A_n^c\right], \text{ by first part of proof} \\
&= \lim_{n\to\infty} \left[1 - P(A_n^c)\right] \\
&= \lim_{n\to\infty} P(A_n)
\end{aligned}
$$

1.1 Axiom of Continuity

If $A_1 \supset A_2 \supset \ldots$ is a sequence of decreasing events such that

$$
\lim_{n\to\infty} A_n = \bigcap_{i=1}^{\infty} A_i = \emptyset \quad \Rightarrow \quad \lim_{n\to\infty} P(A_n) = P\left[\lim_{n\to\infty} A_n\right] = P(\emptyset) = 0
$$

Note:

Finite additivity combined with the *axiom of continuity* **imply** countable additivity. Conversely, *countable additivity* implies finite additivity as well as the axiom of continuity (if we accept countable additivity as an axiom)

1.2 Law of Total Probability

Consider a set of events $c_1, c_2, \ldots, c_n$ such that $c_i \cap c_j = \emptyset$ and $\mathcal{S} = \cup_{i=1}^{n} c_i$, then $c_1, \ldots, c_n$ is called a **partition of $\mathcal{S}$**.

Theorem (Law of Total Probability):

If $c_1, c_2, \ldots, c_n$ is a partition of $\mathcal{S}$, and A is an event, then

$$
P(A) = \sum_{i=1}^{n} \frac{P(A \cap c_i)}{P(A|c_i)P(c_i)}
$$

Proof:

$$\begin{aligned} A = A \cap \mathcal{S} = A \cap \left[\bigcup_{i=1}^{n} c_i\right] \\ = \bigcup_{i=1}^{n} [A \cap c_i] \end{aligned}$$

$$P(A) = P\left[\bigcup_{i=1}^{n}(A \cap c_i)\right] = \sum_{i=1}^{n} P(A \cap c_i), \text{ by finite additivity}$$

1.3 Bode's Inequality

Theorem:

For *any* set of events $A_1, A_2, ...$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

Proof:

This is an inequality because the sets are not disjoint (picture a Venn Diagram).

$$\begin{aligned} \text{Let } B_1 &= A_1 \\ B_2 &= A_2/A \\ B_3 &= A_3/[A_1 \cup A_2] \\ &\vdots \\ B_k &= A_k/[A_1 \cup ... \cup A_{k-1}] \end{aligned}$$

By construction, we have:

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

Moreover, since $B_i \subset A_i$:

$$P\left[\bigcup_{i=1}^{\infty} A_i\right] = P\left[\bigcup_{i=1}^{\infty} B_i\right] = \sum_{i=1}^{\infty} P(B_i) \leq \sum_{i=1}^{\infty} P(A_i)$$

1.4 Bonferroni Inequality

Theorem:

Let $A_1, A_2, ..., A_n$ be a set of events. Then:

$$\begin{aligned} P\left[\bigcap_{i=1}^{n} A_i\right] &= 1 - P\left[\bigcap_{i=1}^{n} A_i\right]^c \\ &= 1 - P\left[\bigcap_{i=1}^{n} A_i^c\right] \\ \geq 1 - \sum_{i=1}^{n} P(A_i^c) &= 1 - \sum_{i=1}^{n} (1 - P(A_i)) \\ &= 1 - n + \sum_{i=1}^{n} P(A_i) \end{aligned}$$

2 Lecture - Part 2

Suppose you have n items

- n_1 of which is type 1
- n_2 of which is type 2
- $\vdots$
- n_k of which is type k

The number of arrangements of these items is:

$$\frac{n!}{n_1! n_2! n_3! \cdots n_k!}$$

Examples:

1. Consider the word STATISTICS.

$$\begin{aligned} n &= 10 \\ S &= 3 \\ T &= 3 \\ A &= 1 \quad \Rightarrow \quad \frac{10!}{3!3!1!1!2!} \\ C &= 1 \\ I &= 2 \end{aligned}$$

2. (Ex: 1.2.20 in book) Consider the numbers 2, 4, 9,12. Select 4 numbers, with replacement, from these numbers and take the mean of the selected numbers.

   (a) How many groups of 4 can we select?

   $$\binom{n+r-1}{r} = \binom{4+4-1}{4} = \binom{7}{4} = 35$$

   (b) What proportion of possible selections contain 2 4's and 2 9's?
   We first note that the total number of ways to draw 4 numbers is 4^4. Then we note the number of ways to choose 2 4's and 2 9's. That would be $n = 4$, $n_1 = 2$, $n_2 = 2$:

   $$\frac{4!}{2!2!} = 6 \quad \Rightarrow \quad P(\text{two 4's and two 9's}) = \frac{6}{256}$$

2.1 Conditional Probability

In some cases, we are only concerned in the probability of events given that a specific outcome such as E occurs. In this case, E plays the role of "sample space". Let $P(\cdot)$ be the probability function defined on $\mathcal{S}$ with $P(E) > 0$. Let F be a subset of $\mathcal{S}$, relative to the new sample space E. We denote the probability of F as $P(F|E)$

(a) Since E is the sample space, we have:

$$P(E|E) = 1$$

(b) Since we know that E has occured, we are mainly interested in elements of $E \cap F$ and we have:

$$P(F|E) = P(F \cap E|E)$$

(c) From the relative frequency point of view:

$$\frac{P(E \cap F|E)}{P(E|E)} = \frac{P(E \cap F)}{P(E)} \quad \Rightarrow \quad P(F|E) = \frac{P(E \cap F)}{P(E)}$$

Independence Rule:

If $P(E|F) = P(E)$, then E and F are independent. Recall that E and F are independent **iff** $P(E \cap F) = P(E)P(F)$

2.1.1 Bayes' Rule

Given that $A_1, ..., A_n$ is a partition of $\mathcal{S}$:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

Example:

Consider the prisoner warden question. The warden tells A that B is to be executed. What is the probability that A will be executed?

Let W be the event that the warden says B is to be executed, and let A, B, and C be the events that prisoners A, B, and C are pardoned, respectively. Our question now becomes:

$$P(A|W) = \frac{P(W|A) \cdot P(A)}{P(W)}$$

Now, using Bayes rule, we calculate $P(W)$

$$\begin{aligned} P(W) &= P(W|A)P(A) + P(W|B)P(B) + P(W|C)P(C) \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) = \frac{1}{2} \end{aligned}$$

We then have:

$$P(A|W) = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{3}$$

But note that $P(A|W) = P(A)$, implying the Warden's information is independent from A being pardoned.