Math 502AB - Lecture 6

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1 Lecture - Part 1 (HW review not in LaTeX)

Theorem:

Let X and Y be two random variables such that all of their *moments* exist. Then:

1. If X and Y have a bounded support, then

$$F_x(u) = F_Y(u) \quad \forall u \quad \text{iff} \quad E(X^r) = E(Y^r) \quad \forall r = 0, 1, 2, ...$$

2. If the MGF exists and $M_X(t) = M_Y(t)$ $\forall t$ in a neighborhood of zero, then $F_X(u) = F_Y(u)$ for all u

2 Lecture - Part 2

Bounded support is important in (1) of the previous theorem.

$$x_1 \sim f_1(x) = \frac{1}{\sqrt{2\pi}x} e^{-(\log x)^2/2}, \quad x \geq 0$$

$$x_2 \sim f_2(x) = f_1(x)[1 + \sin(2\pi \log x)], \quad x \geq 0$$

$$E(x_1^r) = E(x_2^r), \quad r = 0, 1, 2, ...$$

$$\int_0^\infty x^r \frac{1}{\sqrt{2\pi}x} e^{-(\log x)^2/2} dx = \int_0^\infty x^r (1 + \sin 2\pi \log x) f_1(x) dx$$

The idea is that these two integrals are the same *for all values* of r. If the support is *not* bounded, and all of the moments are the same, this **does not mean** the density is the same.

2.1 Convergence in Distribution

Definition:

Let $X_1, X_2, ...$ be a sequence of random variables with corresponding *cdf*'s $F_{X_1}, F_{X_2}, ...$ Furthermore, let X be a random variable with *cdf* F_X. We say that X_i's **converge in distribution** to X if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

or every point x at which F_X is continuous.

2.1.1 Example:

1. Consider a sequence of random variables X_1, X_2,... with corresponding *cdf*'s:

$$F_X(x) = 1 - \left(1 + \frac{x}{n}\right)^{-n} \qquad x > 0$$

$$\lim_{n \to \infty} F_X(n) = 1 - e^{-x} \quad x > 0$$

$$X_i \xrightarrow{D} X \sim exp(1)$$

Theorem:

If $X_1, X_2, ...$ is a sequence of random variables with *mgf*'s $M_{X_1}(t), M_{X_1}(t), ...$ respectively and if

$$\lim_{n \to \infty} M_{X_n}(t) = M_X(t)$$

for all t in a neighbohood of zero, where $M_X(t)$ is the *mgf* of a random variable X, then $X_n \xrightarrow{D} X$

One identity that is useful a lot is:

$$\lim_{n \to \infty} \left(1 + \frac{a_n}{n}\right)^n = e^a \quad \text{if} \lim_{n \to \infty} a_n = a$$

Example:

Consider the *mgf* of $binomial(n, p)$ [i.e. $X_n \sim binomial(n, p)$]

$$\begin{aligned} M_{X_n}(t) &= \left[pe^t + (1-p)\right]^n \\ &= \left[1 + \frac{1}{n}(e^t - 1)np\right]^n \end{aligned}$$

Suppose $np \to \lambda$, as $n \to \infty$, then:

$$\lim_{n \to \infty} M_{X_n}(t) = e^{\lambda(e^t - 1)}$$

$$\Rightarrow X_n \to X \sim Poisson(\lambda)$$

Example: 38(b)

$Y = 2px$, $p \downarrow 0$

$$M_Y(t) = E(e^{2tpx}) = M_X(2tp)$$
$$= \left(\frac{p}{1-(1-p)e^{2pt}}\right)^r \xrightarrow[p\downarrow 0]{WTS} \left(\frac{1}{1-2t}\right)^r$$

$$\lim_{p\to 0}\left[\frac{p}{1-(1-p)e^{2pt}}\right] = \lim_{p\to 0}\left[\frac{p}{1-(1-p)[1+2pt+\frac{(2pt)^2}{2!}+\mathcal{O}(p^2)}\right]$$
$$= \lim_{p\to 0}\left[\frac{p}{1-1+p-2pt+2p^2t-\frac{(2pt)^2}{2}+\mathcal{O}(p^2)}\right]$$
$$= \lim_{p\to 0}\left[\frac{p}{p[1-2t+\mathcal{O}(p)]}\right] = \frac{1}{1-2t}$$

Theorem:

If $Y = aX + b$, then:

$$M_Y(t) = e^{bt}M_X(at)$$

Proof:

$$M_Y(t) = E(e^{tY}) = E(p^{t(ax+b)}) = e^{tb}E(e^{atx}) = e^{tb}M_X(at)$$

Leibnitz' Rule: If $f(x,\theta), a(\theta), b(\theta)$ are differentiable function with respect to θ, then:

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)dx = f(b(\theta),\theta)\frac{d}{d\theta}b(\theta) - f(a(\theta),\theta)\frac{d}{d\theta}a(\theta)$$
$$+\int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial\theta}f(x,\theta)dx$$

2.2 Chapter 3: Common Families of Distributions

1. **Discrete Uniform**

$$f(X = x|N) = \frac{1}{N} \quad x = 1, 2, ..., N$$

To come up with the expectation and variance, there are two identities which are useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$E(X) = \sum_{x=1}^{N} X\frac{1}{N} = \frac{N+1}{2}, \quad Var(X) = \frac{(N+1)(N-1)}{2}$$

2. **Bernoulli Random Variable** $X \sim Bernoulli(p)$

$$f(x=0|p) = 1-p \quad f(x=1|p) = p$$

$$E(X) = p, \quad Var(X) = p(1-p)$$

3. **Binomial Random Variable**: Let $X_1, X_2, ..., X_n \overset{iid}{\sim} Bernoulli(p)$

$$X = \sum_{i=1}^{n} x_i \sim Binomial(n,p)$$

$$f(x|n,p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, ..., n$$

$$E(X) = np, \quad Var(X) = np(1-p), \quad M_X(t) = [pe^t + (1-p)]^n$$

4. **Poisson Distribution:** $X \sim Poisson(\lambda)$

$$f(x|\lambda) = e^{-\lambda}\frac{\lambda^x}{x!} \quad x = 0, 1, 2, ...$$

$$E(X) = \lambda, \quad Var(X) = \lambda$$

5. **Hypergeometric Distribution:** Suppose that you have M red balls and $N-M$ white balls (total of N balls). Say we want to select K balls without replacement. If X is the number of red balls, then:

$$f(x|N,M,K) = \frac{\binom{M}{X}\binom{N-M}{K-X}}{\binom{N}{K}}$$

$$x = 0, ..., K, \quad M \geq X, \quad N-M \geq K-X$$

If this was done with replacement, it would be *binomial*. **Without** replacement it is *hypergeometric*