Math 502AB - Lecture 8

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1 Homework Review

- **Problem 14:** We want to show:

$$\sum_{x=1}^{\infty} \frac{-(1-p)^x}{x \log p} = 1$$

We have seen Taylor expansion before:

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+...$$

Our goals is to show that the Taylor expansion for $\log p$ is:

$$\log p = \sum_{x=1}^{\infty} \frac{-(1-p)^x}{x}$$

That is:

$$\begin{aligned}\log p &= (p-1) - \frac{(p-1)^2}{2!} + \frac{2(p-1)^2}{3!} - \frac{6(p-1)^4}{4!} + ...\\ &= \sum_{x=1}^{\infty} \frac{(-1)^{x+1}}{x}(p-1)^x\end{aligned}$$

To solve, consider the odd and even sums.

To find E[X] we have:

$$\begin{aligned}E[X] = \sum_{x=1}^{\infty} \frac{-x(1-p)^x}{x \log p} &= \frac{-1}{\log p}\sum_{x=1}^{\infty}(1-p)^x\\ &= \frac{-1}{\log p}\left(\frac{1-p}{p}\right)\end{aligned}$$

$$E[X^2] = \sum_{x=1}^{\infty} \frac{-x^2(1-p)^x}{x \log p} = \frac{-1}{\log p}\sum_{x=1}^{\infty} x(1-p)^x$$

Recall that:

$$x \sim geometric(p)$$
$$f_X(x) = (1-p)^{x-1}p$$
$$E[X] = \frac{1}{p}$$
$$\sum_{x=1}^{\infty} xp(1-p)^{x-1} = \frac{1}{p}$$

So we have:

$$E[X^2] = -\frac{1}{\log p}\frac{1}{p}(1-p)\sum_{x=1}^{\infty} xp(1-p)^{x-1} = -\frac{1}{\log p}\frac{(1-p)}{p^2}$$

- **Problem 13b:**

$$P(X_T = x) = \frac{\binom{r+x-1}{x}p^r(1-p)^x}{1-p^r} \quad x = 1, 2, ...$$

$$\begin{aligned} E[x(x-1)] &= \frac{1}{1-p^r}\sum_{x=1}^{\infty} x(x-1)\frac{(r+x-1)!}{x!(r-1)!}p^r(1-p)^x \\ &= \frac{1}{1-p^r}\sum_{x=2}^{\infty}\frac{(r+x-1)!}{(x-2)!(r-1)!}p^r(1-p)^x \end{aligned}$$

This looks like the *pmf* of the negative binomial. To get there, we do a *change in variables* letting $y = x - 2$

$$= \frac{1}{1-p^r}\sum_{y=0}^{\infty}\frac{(r+y+1)!}{y!(r-1)!} \tag{1}$$

The negative binomial *pmf* is:

$$f_X(x) = \binom{r+x-1}{x}p^r(1-p)^x \quad x = 0, 1, ...$$

So, to make it this way in (1), we need to include the terms up to $r+1$:

$$= \frac{r(r+1)}{1-p^r}\sum_{y=0}^{\infty}\binom{r+y+1}{y}p^r(1-p)^{y+2} \tag{2}$$

Note that we need the *pmf* for our new distibution:

$$X \sim NB(r+2, p)$$
$$f_X(x) = \binom{r+x+1}{x}p^{r+2}(1-p)^x$$

Going back to (2), we can simplify to find our answer:

$$E[X(X-1)] = E[X^2] - E[X] = \frac{(1-p)^2 r(r+1)}{p^2(1-p)^r} \sum_{y=0}^{\infty} \binom{r+y+1}{y} p^{r+1}(1-p)^y$$

Solving directly: If we consider

$$E[X^2] = \sum_{x=1}^{\infty} x^2 \frac{\binom{r+x-1}{x} p^r (1-p)^x}{1-p^r}$$

We can note that the top term is the *pmf* for the negative binomial, so we can factor out the denominator and we are done.

- **Problem 20:** Consider

$$f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \quad x \geq 0$$

And $Y = g(x)$ for $Y = X^2$

$$\begin{aligned} F_Y(y) = P(Y \leq y) &= P(X^2 \leq Y) \\ &= P(0 \leq X \leq \sqrt{Y}) \\ &F_X(\sqrt{y}) - F_X(0) \end{aligned}$$

$$f_Y(y) = f_X(\sqrt{y}) \left(\frac{1}{2} y^{-1/2} \right)$$

$$\begin{aligned} F_Y(y) &= \frac{2}{\sqrt{2\pi}} e^{-y/2} \frac{1}{2} (y^{-1/2}) \\ &= \frac{1}{\sqrt{2}\Gamma(\frac{1}{2})} y^{-1/2} e^{-y/2} \\ Y &\sim gamma\left(\frac{1}{2}, 2\right) \end{aligned}$$

- **Problem 12:** Done during office hours

- **Problem 6:** $X \sim binomial(n = 2000, p = 0.01)$

  To approximate it:

$$X \sim N(\mu = 2000(0.01), \sigma^2 = 2000(0.01)(0.99)$$

- **Problem 15:** If two functions have *moment generating functions*, you can show convergence of the distributions by showing convergence in the *mgf* themselves

$$\left(1 + \frac{a_r}{r}\right) \to e^a$$
$$\text{If: } a_r \to a$$

2 Lecture - Part 2: Continuous Distributions, Continued

2.1 Section 3.4 - Exponential Family

These distributions have very important properties that have many implications, as we will learn.

A family of *pdf*s (or *pmf*s) is called **an exponential family** if it can be expressed as

$$f(x|\overrightarrow{\theta}) = h(x)c(\overrightarrow{\theta})exp\left\{\sum_{i=1}^{k} w_i\left(\overrightarrow{\theta}\right)t_i(x)\right\}$$

Where $\overrightarrow{\theta} = (\theta_1, \theta_2, ..., \theta_p)$ is a vector of parameters $h(x), t_1(x), t_2(x), ..., t_k(x)$ are functions of x not involving $\overrightarrow{\theta}$, and $c(\overrightarrow{\theta})$ and $w_i(\overrightarrow{\theta})$ are functions of $\overrightarrow{\theta}$ not involving x.

2.1.1 Examples:

1. Let $X \sim binomial(n, p)$ with p being the only parameter. For a given n, it is expnential family:

$$\begin{aligned} f(x|p) &= \binom{n}{x} p^x (1-p)^{n-x} \\ &= \binom{n}{x} \left(\frac{p}{1-p}\right)^x (1-p)^n \\ &= \binom{n}{x} (1-p)^n exp\left\{x \log\left(\frac{p}{1-p}\right)\right\} \end{aligned}$$

In this example, we have:

$$\begin{aligned} h(x) &= \binom{n}{x} \\ c(\theta) &= (1-p)^n \\ t_1(x) &= x \\ w_1(x) &= \log\left(\frac{p}{1-p}\right) \end{aligned}$$

2. Let $X \sim gamma(\alpha, \beta)$

$$\begin{aligned}
f_X(x|\alpha,\beta) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{x} exp\left\{\alpha \log x\right\} e^{-x/\beta} \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{x} exp\left\{\alpha \log x - \frac{1}{\beta} x\right\}
\end{aligned}$$

3. Consider X with pdf

$$f_X(x|\theta) = \begin{cases} \theta e^{\theta^2} e^{-x/\theta} & x > \theta > 0 \\ 0 & x \le \theta \end{cases}$$

Is this an exponential family? No, X is dependent on θ!

2.1.2 Theorem

If X is a *pdf* (or *pmf*) from an exponential family of distributions, then:

1. $$E\left[\sum_{i=1}^{k} \frac{\partial w_i(\overrightarrow{\theta})}{\partial(\theta_j)} t_i(X)\right] = \frac{\partial}{\partial \theta_j} \log c(\overrightarrow{\theta})$$

2. $$Var\left[\sum_{i=1}^{k} \frac{\partial w_i(\overrightarrow{\theta})}{\partial(\theta_j)} t_i(X)\right] = \frac{\partial^2}{\partial \theta_j^2} \log c(\overrightarrow{\theta}) - E\left[\sum_{i=1}^{k} \frac{\partial^2 w_i(\overrightarrow{\theta})}{\partial \theta_j^2} t_i(X)\right]$$

Proof:
Since we have an exponential density, we write:

$$1 = \int_{-\infty}^{\infty} h(x)c(\theta) exp\left[\sum_{i=1}^{k} w_i(\theta)t_i(x)\right] dx$$

We will denote the *exp* term as $g(\theta, x)$. We need to take the derivative of everything with respect to θ_j.

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty} \frac{\partial}{\partial \theta_j} \left[h(x)c(\theta)g(\theta,x)\right] dx \\
&= \int_{-\infty}^{\infty} h(x) \left[\left(\frac{\partial}{\partial \theta_j} c(\theta)\right) g(\theta,x) + c(\theta)\left(\frac{\partial}{\partial \theta_j} g(\theta,x)\right)\right] dx \\
&= (*) \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial \theta_j} \log c(\theta)\right] c(\theta)h(x)g(\theta,x)dx + (**) \int_{-\infty}^{\infty} h(x)c(\theta)\frac{\partial}{\partial \theta_j} g(\theta,x)dx
\end{aligned}$$

Note that:

$$\frac{\partial}{\partial\theta_j}g(\theta,x) = \left[\frac{\partial}{\partial\theta_j}\sum_{i=1}^{k} w_i(\theta)t_i(x)\right] g(\theta,x)$$

$(*)$: We see that the density was the right hand side, so we have it equal to $\frac{\partial}{\partial\theta_j}\log c(\theta)$

$(**)$:

$$E\left[\frac{\partial}{\partial\theta_j}\sum_{i=1}^{k} w_i(\theta)t_i(x)\right]$$

Thus, since we have shown $(*)+(**)=0$, we have proven the identity holds.

Example: Recall that, if $X \sim \mathit{binomial}(n,p)$, then:

$$h(x) = \binom{n}{x}$$
$$c(p) = (1-p)^n$$
$$w_i(\theta) = \log\frac{p}{1-p}$$
$$t_1(x) = x$$

So, let's compute the expected value:

$$\frac{\partial w_1}{\partial p} = \frac{\partial}{\partial p}\left(\log\frac{p}{1-p}\right)$$
$$E\left[\frac{\partial}{\partial p}\log\frac{p}{1-p}X\right] = \frac{1}{p(1-p)}E[X]$$

$$\frac{-\partial}{\partial p}\log(1-p)^n = \frac{n}{1-p}$$

Now, we set them equal:

$$\frac{1}{p(1-p)}E(X) = \frac{n}{1-p}$$
$$\Rightarrow E(X) = np$$