Math 502AB - Lecture 24

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1 Lecture - Part 1

1.1 Likelihood Ratio Test, Cont'd

Example:

1. Suppose $X_1, ..., X_n \sim N(\mu, \sigma^2)$ (**iid**)

$$\begin{cases} H_0: & \mu = \mu_0 \quad \mu_0 \text{ is a given number} \\ H_a: & \mu > \mu_0 \end{cases}$$

Obtain the **LRT**:

$$\begin{aligned} \Theta_0 &= \left\{(\mu, \sigma^2): \quad \mu = \mu_0, \quad \sigma^2 > 0\right\} \\ \Theta &= \left\{(\mu, \sigma^2): \quad \mu \geq \mu_0, \quad \sigma^2 > 0\right\} \\ \Theta_1 &= \left\{(\mu, \sigma^2): \quad \mu > \mu_0, \quad \sigma^2 > 0\right\} \\ \mathcal{L}(\mu, \sigma^2) &= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\} \end{aligned}$$

If we maximize the likeilihood in Θ_0, we get:

$$\widetilde{\mu} = \mu_0 \quad \text{and} \quad \widetilde{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2$$

Maximization over Θ:

(a) Case 1: $\mu_0 > \overline{X}$
This implies that $\hat{\mu} = \mu_0$ and we have:

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2$$
$$\Rightarrow \Lambda(X) = 1 \quad \text{will not reject } H_0$$

(b) Case 2: $\mu_0 < \overline{X}$

This implies that $\hat{\mu} = \overline{X}$. We then have our test as:

$$\frac{\exp\left\{-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^n (x_i - \mu_0)^2\right\}}{\exp\left\{-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^n (x_i - \overline{X})^2\right\}} = \frac{\exp\left\{-\frac{n}{2}\right\}}{\exp\left\{-\frac{n}{2}\right\}} = 1$$

$$\Rightarrow \Lambda(X) = \left(\frac{\hat{\sigma}^2}{\widetilde{\sigma}^2}\right)^{n/2} = \left(\frac{\sum(x_i - \overline{X})^2}{\sum(x_i - \mu_0)^2}\right)^{n/2}$$

Which gives us the rejection region:

$$R = \left\{\left(\frac{\sum(x_i - \overline{X})^2}{\sum(x_i - \mu_0)^2}\right)^{n/2} < c\right\}$$

$$\iff \frac{\sum(x_i - \overline{X})^2}{\sum(x_i - \mu_0)^2} < c$$

$$\frac{\sum(x_i - \overline{X})^2}{\sum(x_i - \overline{X})^2 + n(\overline{X} - \mu_0)^2} \leq c$$

$$\frac{\sum(x_i - \overline{X})^2 + n(\overline{X} - \mu_0)^2}{\sum(x_i - \overline{X})^2} \geq c$$

$$\frac{n(\overline{X} - \mu_0)^2}{\sum(x_i - \overline{X})^2} \geq c$$

$$\frac{n(\overline{X} - \mu_0)^2}{S^2} \geq c$$

The reason we want to do this, is because if we divide both sides by a constant, we get a rejection region of the (familiar) form:

$$\frac{(\overline{X} - \mu_0)}{S/\sqrt{n}} \geq c$$

Which is a t-**statistic** with $n-1$ degrees of freedom.

1.1.1 Theorem:

Under some smoothness conditions on the *pdf* from which the data comes, $-2\log\Lambda(X)$ converges in distribution as the sample size increases to a χ^2 distribution with degrees of freedom equal to:

$$\dim(\Theta) - \dim(\Theta_0)$$

Example:

Let $X_1, ..., X_n \sim Poisson(\theta_1)$ and $Y_1, ..., Y_n \sim Poisson(\theta_2)$. Suppose you wish to test the hypothesis:

$$\begin{cases} H_0: & \theta_1 = \theta_2 \\ H_a: & \theta_1 \neq \theta_2 \end{cases}$$

We won't go through the derivation, but we end up with:

$$\Lambda = \frac{\frac{1}{2}(\overline{X}+\overline{Y})^{n(\overline{X}+\overline{Y})}}{\overline{X}^{n\overline{X}}\overline{Y}^{n\overline{Y}}}$$
$$-2\log\Lambda \sim \chi^2_{(1)}$$

1.2 Bayesian Testing Procedure

Let X be a random variable with *pdf* (or *pmf*) $f(X|\theta)$, with $\theta \in \Theta$. Suppose that we are to test:

$$\begin{cases} H_0: & \theta \in \Theta_0 \\ H_a: & \theta \in \Theta_1 \end{cases}$$

Where $\Theta_0 \cap \Theta_1 = \emptyset$. Let $f_\Theta(\theta)$ be the prior, and let $X_1, ..., X_n \sim f(X|\theta)$. Denote the posterior *pdf* (or *pmf*) by $f_{\Theta|X}(\theta|x)$, and compute the probabilities:

$$P(\theta \in \Theta_0|X_1, ..., X_n) \quad \text{and} \quad P(\theta \in \Theta_1|X_1, ..., X_n)$$

Where we refer to these probabilities as "**truth** of H_0" (and H_1), respectively. A simple rule is to accept H_0 if the truth of H_0 is greater than the truth of H_1. Otherwise, accept H_1.

Example:

Let $X_1, ..., X_n \sim Poisson(\theta)$. Test $H_0 : \theta \leq 10$ versus $H_a : \theta > 10$ where we have observed $X_1, ..., X_{20}$ $(n = 20)$ with $\sum_{i=1}^{20} x_i = 177$. Suppose that you expect that θ is around 12. You choose the prior:

$$\theta \sim gamma(\alpha = 10, \beta = 1.2)$$
$$E[\theta] = 12$$
$$Var(\theta) = 14.4$$

This yields the posterior:

$$\theta|X_1, ..., X_n \sim gamma\left(\alpha = 177 + 10, \beta = \frac{1.2}{20(1.2) + 1}\right)$$
$$\sim gamma(187, 0.048)$$
$$E[\theta|X] = 187(0.048) = 8.976$$

So, we must calculate probabilities of the posterior, given the data:

$$
\begin{aligned}
P\left(\theta \leq 10 | \sum x_i = 177\right) &= P[gamma(1.87, 0.048) \leq 10] \\
&= 0.9368 \\
P\left(\theta > 10 | \sum x_i = 177\right) &= 0.0632
\end{aligned}
$$

And, thus, we accept H_0.

1.3 Power and Size of a Test

Given the hypothesis test:

$$
\begin{cases} H_0 : & \theta \in \Theta_0 \\ H_a : & \theta \in \Theta_0^c \end{cases}
$$

We define:

1. **Type 1 Error:** Rejecting H_0 when H_0 is true
2. **Type 2 Error:** Accepting H_0 when H_a is true
3. **Power:** Power is defined as the probability of rejecting H_0 when H_a is true. In other words, *correctly rejecting* H_0

   We can define a **power function** as:

   $$\beta(\theta) = P(\text{rejecting } H_0 | \theta)$$

Definition:

A test is said said to have size α $(0 \leq \alpha \leq 1)$ if:

$$\max_{\theta \in \Theta_0} P(\text{rejecting } H_0 | \theta) = \alpha$$

Definition:

A **level** α test is one for which

$$\max_{\theta \in \Theta_0} P(\text{rejecting } H_0 | \theta) \leq \alpha$$

1.3.1 Examples

1. Suppose $X \sim binomial(5, \theta)$

   $$
   \begin{cases} H_0 : \theta \leq \frac{1}{2} \\ H_a : \theta > \frac{1}{2} \end{cases}
   $$

   Let $R = \{4, 5\}$.

What is the size of the test if:

$$\max_{\theta \leq \frac{1}{2}} P(X = 4, 5|\theta) = \binom{5}{4}\theta^4(1-\theta) + \binom{5}{5}\theta^5 \Rightarrow \alpha = 0.1875$$

What is the size if:

$$\alpha = \max_{\theta \leq \frac{1}{2}} P(X = 5|\theta) = \max_{\theta \leq \frac{1}{2}} \theta^5 = 0.03125.$$

Which results in a *level* 0.05 *test*.

2 Lecture - Part 2

1. Let $X_1, ..., X_n \sim N(\theta, 1)$. Consider the test:

$$\begin{cases} H_0: & \theta \leq \theta_0 \\ H_a: & \theta > \theta_0 \end{cases}$$

We then have the rejection region:

$$R = \left\{ X: \quad \frac{\overline{X} - \theta_0}{1/\sqrt{n}} > c \right\}$$

What's important here is the *power function*. We have:

$$\begin{aligned} \beta(\theta) &= P(\text{rejecting } H_0|\theta) \\ &= P(\sqrt{n}(\overline{X} - \theta_0) > c|\theta) \\ &= P(\sqrt{n}(\overline{X} - \theta + \theta - \theta_0) > c|\theta) \\ &= P(\sqrt{n}(\overline{X} - \theta) + \sqrt{n}(\theta - \theta_0) > c|\theta) \\ &= P(Z \geq c - \sqrt{n}(\theta - \theta_0)|\theta) \end{aligned}$$

(a) Determine c so that you now have a size α test. ($\Theta_0 : \{\theta \leq \theta_0\}$)

$$\max_{\theta \in \Theta_0} P\left(Z \geq c - \sqrt{n}(\theta - \theta_0)|\theta\right)$$

We notice that this probability is an **increasing** function of θ. Thus, since we are maximizing over the region Θ_0, this is maximized at θ_0. Plugging this into the maximization above, we get $P(Z \geq c)$. To get the c such that we have a size alpha test, we just need to calculate $c = Z_\alpha$.

Therefore, we have the rejection region for a test of size α as:

$$R = \left\{X: \quad \sqrt{n}\left(\overline{X} - \theta_0\right) > Z_\alpha\right\}$$

(b) Obtain the power of the test at θ_a.

$$\begin{aligned}
\beta(\theta_a) &= P(\text{rejecting } H_0|\theta = \theta_a) \\
&= P(\sqrt{n}(\overline{X} - \theta_0) > Z_\alpha|\theta = \theta_a) \\
&= P\left(\sqrt{n}(\overline{X} - \theta_a + \theta_a - \theta_0) > Z_\alpha|\theta = \theta_a\right) \\
&= P\left(\sqrt{n}(\overline{X} - \theta_a) > Z_\alpha + \sqrt{n}(\theta_0 - \theta_a)|\theta = \theta_a\right) \\
&= P\left(Z > Z_\alpha + \sqrt{n}(\theta_0 - \theta_a)\right) \\
&= 1 - \Phi\left(Z_\alpha - \sqrt{n}(\theta_a - \theta_0)\right)
\end{aligned}$$

So we see that when α **increases**, the power of the test *decreases*. Conversely, when the **sample size** n **increases**, the power of the test increases.

Given a formula like the one found above, we can also calculate the sample size needed given the θ_a and α by plugging in and solving.

2.1 P-Value

A practical definition of **p-value** is that the **p-value** is a measure of discrepancy between the null hypothesis H_0 and the observed data X.

If $T(X)$ is the test statistic used to test H_0, and the rejection region is of the form

$$\{X : \quad T(X) \leq c\} \tag{1}$$

and we have an observed value $T(X)$, then

$$P - \text{value } = P(T(X) \leq T(x)|H_0 \text{ is true })$$

Suppose that F_T is the *cdf* of $T(X)$. Then we have:

$$P - \text{value } = F_T(T(X)) \text{ , provided } H_0 \text{ is true}$$

Example:

1. Suppose $X_1, ..., X_{25} \sim N(\mu, \sigma^2 = 4)$. Consider the test:

$$\begin{cases} H_0 : & \mu = 77 \\ H_a : & \mu < 77 \end{cases}$$

Suppose $\overline{X} = 76.1$ and $Var(\overline{X}) = \frac{4}{25} = 0.16$. Under H_0:

$$Z = \frac{\overline{X} - 77}{0.4} \sim N(0, 1)$$

Thus, we have:

$$
\begin{aligned}
P-\text{value } &= P\left(\frac{\overline{X}-77}{0.4} < \frac{76.1-77}{0.4}\right) = P(Z < -2.25) \\
&= 0.012
\end{aligned}
$$