Math 502AB - Lecture 10

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1 Lecture - Part 1

1.1 Identities

Examples

1. Consider $X \sim Poisson(\lambda)$. Then, we have:

$$P(X = x+1) = \frac{\lambda}{x+1} P(X = x)$$

To prove something like this, it isn't very difficult:

$$\begin{aligned} P(X = x+1) &= \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!} \\ &= \frac{\lambda}{x+1} \cdot \frac{e^{-\lambda}\lambda^{x}}{x!} \\ &= \frac{\lambda}{x+1} \cdot P(X = x) \end{aligned}$$

2. Consisder $X \sim gamma(\alpha, \beta)$ with $\alpha > 1$. Then for any a and b:

$$P(a < X < b) = \beta \left(f(a|\alpha,\beta) - f(b|\alpha,\beta) \right) + P(a < Y < b)$$

Where $Y \sim gamma(\alpha - 1, \beta)$.
Proof:

$$\begin{aligned} P(a < X < b) &= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_a^b e^{-x/\beta} x^{\alpha-1} dx \\ \text{Let } u = x^{\alpha-1} &\quad dv = e^{-x/\beta} dx \\ \Rightarrow du = (\alpha-1)x^{\alpha-2} dx &\quad v = -\beta e^{-x/\beta} \end{aligned}$$

Then we have:

$$\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\left\{-\beta x^{-\alpha-1}e^{-x/\beta}\Big|_a^b+\int_a^b(\alpha-1)x^{\alpha-2}\beta e^{-x/\beta}dx\right\}$$
$$=\beta\left\{f(a|\alpha,\beta)-f(b|\alpha,\beta)\right\}+\int_a^b\frac{1}{\beta^{\alpha-1}\Gamma(\alpha-1)}x^{\alpha-2}e^{-x/\beta}dx$$
$$=\beta\left(f(a|\alpha,\beta)-f(b|\alpha,\beta)\right)+P(a<Y<b)$$

1.2 Stein's Lemma

Let $Z \sim N(0,1)$ and $g(x)$ be a differentiable function that satisfies $E|g'(z)|<\infty$. Then:

$$E[z\cdot g(z)]=E[g'(z)]$$

Proof:

We have:

$$f(z)=\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\Rightarrow f'(z)=\frac{1}{\sqrt{2\pi}}-ze^{-z^2/2}=-zf(z)$$

$$E[g'(z)]=\int_{-\infty}^{\infty}g'(z)f_Z(z)dz$$
$$=\int_{-\infty}^{0}g'(z)f_Z(z)dz\ (1^*)\ +\int_0^{\infty}g'(z)f_Z(z)dz\ (2^*)$$

(2*):

$$\int_0^{\infty}g'(z)\left[\int_{-\infty}^{z}f'_Z(t)dt\right]dz=\int_0^{\infty}\int_{-\infty}^{z}g'(z)f'_Z(t)dtdz$$
$$=\int_0^{\infty}\int_{-\infty}^{z}-tf_Z(t)g'(z)dtdz$$
$$=\int_{-\infty}^{0}\int_0^{\infty}-g'(z)tf_Z(t)dzdt+\int_0^{\infty}\int_t^{\infty}-g'(z)tf_Z(t)dzdt$$
$$=\int_0^{\infty}tf_Z(t)\left[\int_0^{\infty}g'(z)dz\right]dt-\int_0^{\infty}tf_Z(t)g'(z)dtdz$$
$$=\int_0^{\infty}tf_Z(t)\left[\int_0^{\infty}g'(z)dz-\int_t^{\infty}g'(t)dz\right]dt$$
$$=\int_0^{\infty}tf_Z(t)\left[\int_0^{t}g'(z)dz\right]dt$$
$$=\int_0^{\infty}tf_Z(t)\left[g(t)-g(0)\right]dt$$

(2*): Similarly, it can be shown that

$$\int_{-\infty}^{0} g'(z) f_Z(z) dz = \int_{-\infty}^{0} t f_Z(t)[g(t) - g(0)] dt$$

Combining **(1*)** and **(2*)** we get:

$$\begin{aligned}\int_{-\infty}^{\infty} t[g(t) - g(0)] f_Z(t) dt &= E[Zg(Z) - Zg(0)] \\ &= E[Zg(Z)] - g(0)E[Z] \\ &= E[zg(z)]\end{aligned}$$

1.3 Stein's Lemma (Case: $N(\mu, \sigma^2)$)

Let $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$. By the lemma that we just proved, we have:

$$E\left[\frac{X-\mu}{\sigma} g\left(\frac{X-\mu}{\sigma}\right)\right] = E\left[g'\left(\frac{X-\mu}{\sigma}\right)\right]$$

Let $h(x) = g\left(\frac{x-\mu}{\sigma}\right)$ and $h'(x) = \frac{1}{\sigma} g'\left(\frac{x-\mu}{\sigma}\right)$. This implies that:

$$E\left[\left(\frac{X-\mu}{\sigma}\right) h(X)\right] = E\left[\sigma h'(x)\right]$$

And, equivalently:

$$E\left[(X-\mu)h(X)\right] = E\left[\sigma^2 h'(x)\right]$$

Example:

Consider obtaining higher order moments of $X \sim N(\mu, \sigma^2)$. We know that $E(X) = \mu$, $E(X^2) = \mu^2 + \sigma^2$. But what if we wanted to find $E(X^3)$?

Let $h(x) = x^2$. Then:

$$\begin{aligned}E[(X-\mu)X^2] &= E[\sigma^2(2X)] \\ E[X^3] - \mu E[X^2] &= 2\sigma^2 E(X) \\ \Rightarrow E[X^3] &= \mu(\mu^2 + \sigma^2) + 2\sigma^2\mu\end{aligned}$$

Theorem:

Let $X_p \sim \chi^2_{(p)}$. Then:

$$E[h(X_p)] = pE\left[\frac{h(X_{p+2})}{X_{p+2}}\right]$$

Proof:

$$
\begin{aligned}
E[h(X_p)] &= \int_0^\infty \frac{h(x)}{\Gamma\left(\frac{p}{2}\right)2^{p/2}} x^{\frac{p}{2}-1}e^{-x/2}dx \\
&= \frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}} \int_0^\infty x^{\frac{p}{2}-1}e^{-x/2}h(x)\cdot\frac{x}{x}dx \\
&= \frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}} \int_0^\infty \frac{h(x)}{x}x^{p/2}e^{-x/2}dx
\end{aligned}
$$

We recognize that The right side is the *kernel* of a $\chi^2_{(p+2)}$ distribution, since $\frac{p}{2} = \frac{p+2}{2} - 1$. So we then have:

$$
\begin{aligned}
\Gamma\left(\frac{p}{2}\right) &= \frac{\frac{p}{2}\Gamma\left(\frac{p}{2}\right)2^{p/2}}{p/2} \\
&= \frac{\Gamma\left(\frac{p}{2}+1\right)2^{\frac{p}{2}+1}}{p} = \frac{\Gamma\left(\frac{p+2}{2}\right)2^{\frac{p+2}{2}}}{p}
\end{aligned}
$$

Example:

Consider $X_p \sim \chi^2_{(p)}$. Let $h(x) = x$. Now, using the above theorem, we have:

$$E[X_p] = pE\left[\frac{X_{p+2}}{X_{p+2}}\right] = p$$

Let's calculate the variance. We will let $h(x) = x^2$, and, by the theorem we have:

$$
\begin{aligned}
E[X_p^2] = pE\left[\frac{X_{p+2}^2}{X_{p+2}}\right] = pE[x_{p+2}] = p(p+2) \\
\Rightarrow Var(X_p) = E[X_p^2] - E[X_p]^2 = p(p+2) - p^2 = 2p
\end{aligned}
$$

1.4 Discrete Case Identities

Poisson (Hwang Theorem)

Suppose that $X \sim poisson(\lambda)$ and $g(x)$ is a function such that $-\infty < E[g(X)] < \infty$ and $-\infty < g(-1) < \infty$. Then we have:

$$E[\lambda g(X)] = E[Xg(X-1)]$$

Proof:

We first note that:

$$E[\lambda g(X)] = \sum_{x=0}^{\infty} \lambda g(x) \frac{e^{-\lambda}\lambda^x}{x!}$$

Letting $y = x + 1$, we get $x = y - 1$:

$$\sum_{y=1}^{\infty} \lambda g(y-1) \frac{e^{-\lambda}\lambda^{y-1}}{(y-1)!}$$

Using the same trick as the other theorem, we multiply by $\frac{y}{y}$ to get:

$$\begin{aligned}
&\sum_{y=1}^{\infty} \lambda y g(y-1) \frac{e^{-\lambda}\lambda^{y-1}}{y(y-1)!} \\
&= \sum_{y=1}^{\infty} y g(y-1) \frac{e^{-\lambda}\lambda^{y}}{y!} \\
&= \sum_{y=0}^{\infty} y g(y-1) \frac{e^{-\lambda}\lambda^{y}}{y!} = E[Yg(Y-1)]
\end{aligned}$$

Example:

Consider $X \sim poisson(\lambda)$. Then we know that $E[X] = \lambda$ and $E[X^2] = \lambda + \lambda^2$. Now, suppose that we let $g(x) = x^2$. Then we have:

$$\begin{aligned}
E[\lambda X^2] &= E[X(X-1)^2] \\
\lambda E[X^2] &= E[X(X^2 - 2X + 1)] \\
E[X^3] &= \lambda E[X^2] + 2E[X^2] - E[X] \\
&= (\lambda + 2)(\lambda + \lambda^2) - \lambda
\end{aligned}$$