Math 502AB - Lecture 14

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1 Lecture - Part 1

1.1 Chapter 5 - Properties of Random Samples

Definition:

The random variables $X_1, ..., X_n$ are called a *random sample of size n* from the population $f(x)$ if $X_1, ..., X_n$ are distributed **independently and identically (iid)** $\sim f(x)$

Example:

Let X = Average weight of a newborn baby $\sim N(7, 1.5)$. A sample will be observations from this population $X_1, ..., X_n \sim X$ (**iid**)

1.1.1 Joint Distribution of Random Sample

The joint distribution of $X_1, ..., X_n$ has information about $X_1, ..., X_n$

$$f_{X_1,...,X_n}(X_1, ..., X_n) = \prod_{i=1}^{n} f(x_i|\theta)$$

Example:

Consider $X_1, ..., X_n \sim poisson(\lambda = 1)$

$$P(X_1 = x_1, ..., X_n = x_n) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$
$$P(X_1 = 0, ..., X_n = 0) = e^{-\lambda n}$$

1.1.2 Sampling from a Finite Population

Suppose that a population has N elements, $\{y_1, y_2, ..., y_N\}$. Let X_1 denote the 1^{st} sample taken and X_2 denote the 2^{nd} sample taken.

$$\begin{aligned}
P(X_1 = y_k) &= \frac{1}{N} \\
P(X_2 = y_k) &= \sum_{i=1}^{N} P(X_2 = y_k | X_1 = y_i) P(X_1 = y_i) \quad i \neq k \\
&= \sum_{i=1}^{N} \left(\frac{1}{N-1} \right) \left(\frac{1}{N} \right) = \frac{1}{N}
\end{aligned}$$

1.2 Statistics

Definition

Given a sample of size n and a sample $X_1, ..., X_n$ from a population, a **statistic** is defined to be a function of $X_1, ..., X_n$ (not including parameters)

$$Y = T(X_1, ..., X_n)$$

Where Y is a random variable, and the distribution of Y is referred to as the sampling distribution

Examples:

1. Sample mean $\overline{x}$
$$\frac{1}{n} \sum_{i=1}^{n} x_i$$
2. Median
3. Sample Variance
$$\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

Theorem:

Let $x_1, ..., x_n$ be any numbers. Then:

$$\min_a = \sum_{i=1}^{n} (x_i - a)^2 = \sum_{i=1}^{n} (x_i - \overline{x})^2$$

Where $\overline{x}$ is defined as above.

Proof:

$$\sum_{i=1}^{n} (x_i - a)^2 = \sum_{i=1}^{n} (x_i - \overline{x} + \overline{x} - a)^2$$

Using the fact that $\sum_{i=1}^{n} (x_i - \overline{x}) = 0$, the result follows (and is left as an exercise to the reader)

Identity:

$$\sum_{i=1}^{n}(x_i - \overline{x})^2 = \sum_{i=1}^{n} x_i^2 - n\overline{x}$$

Lemma:

Let $X_1, ..., X_n$ be **iid** and $g(x)$ be a function such that $E(g(x_i))$ and $Var(g(x_i))$ are finite. Then:

$$E\left[\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right] = E[g(x_i)]$$
$$Var() = 3$$

2 Lecture - Part 2

2.1 Properties of $\overline{X}$ and S^2

Theorem:

Let $X_1, ..., X_n$ be a random sample from a population with $E(X_i) = \mu$ and $var(X_i) = \sigma^2$. Then:

$$E(\overline{X}) = \mu$$
$$Var(\overline{X}) = \frac{\sigma^2}{n}$$
$$E(S^2) = \sigma^2$$

Proof:

$$E(\overline{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu$$

$$Var(\overline{X}) = Var\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} Var(x_i) = \frac{1}{n^2}n\sigma^2$$

$$\begin{aligned}
E(S^2) &= E\left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2\right] \\
&= E\left[\frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right)\right] \\
&= \frac{1}{n-1}\sum_{i=1}^{n} E(x_1^2) - \frac{n}{n-1}E(\overline{x}^2) \\
&= \frac{1}{n-1}\left[n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right] \\
&= \sigma^2
\end{aligned}$$

Using some tedious calculations, it can be shown that:

$$Var(S^2) = \frac{1}{n}\left[\theta_4 - \frac{n-3}{n-1}\theta_2^2\right]$$

Where

$$\theta_4 = E(x_i - \mu)^4 \qquad \theta_2 = E(x_i - \mu)^2 = \sigma^2$$

Under which conditions are $\overline{x}$ *uncorrelated* with S^2?

It can be shown that:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2$$

WLOG, assume that $E(X_i) = 0$ (since shifts do not affect *covariance*).

$$\begin{aligned} Cov(\overline{X}, S^2) &= E[\overline{X}S^2] - E(\overline{X})E(S^2) \\ &= \frac{1}{2n^2(n-1)}E\left[\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}x_k(x_i - x_j)^2\right] \end{aligned}$$

Note that $i = j$ makes this zero

$$\begin{aligned} &\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}E(x_k(x_i - x_j)^2) \\ &= \sum\sum\sum\{i \neq j, i = k \neq j\} + \sum\sum\sum\{i \neq j, j = k \neq i\} \end{aligned}$$

Theorem:

Let $X_1, ..., X_n \sim N(\mu, \sigma^2)$ **iid**. Then:

1. $\overline{X}$ and S^2 are independent
2. $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$
3. $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

Proof:

1. Since shift and scale do not effect independence, WLOG, we assume that $X_1, ..., X_n \sim N(0, 1)$.

$$
\begin{aligned}
S^2 &= \frac{1}{n-1}\left[(x_1-\overline{x})^2\sum_{i=2}^{n}(x_i-\overline{x})^2\right] \\
&= \frac{1}{n-1}\left[\left(\sum_{i=2}^{n}(x_i-\overline{x})\right)^2+\sum_{i=2}^{n}(x_i-\overline{x})^2\right]
\end{aligned}
$$

Since S^2 is a function of $X_2, ..., X_n$, it is sufficient to show that $\overline{X}$ is independent of $(X_2-\overline{X}), ..., (X_n-\overline{X})$.

Let:

$$
\begin{aligned}
Y_1 = \frac{1}{n}\sum_{i=1}^{n}x_i &\Rightarrow X_1 = nY_1 - \sum_{i=1}^{n}x_i \\
&\Rightarrow X_1 = -\sum_{i=2}^{n}Y_i + Y_1 \\
Y_2 = x_2 - \frac{1}{n}\sum x_i &\Rightarrow X_2 = Y_1 + Y_2 \\
&\vdots \\
Y_n = x_n - \frac{1}{n}\sum x_i &\Rightarrow X_n = Y_n + Y_1
\end{aligned}
$$

Yielding the Jacobian:

$$
\mathcal{J}^{-1} = \begin{pmatrix}
1 & -1 & -1 & \cdots & -1 \\
1 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 1
\end{pmatrix}
$$

The Joint density of $X_1, ..., X_n$

$$
\begin{aligned}
f_{X_1,...,X_n}(x_1,...,x_n) &= \frac{1}{(2\pi)^{n/2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}x_1^2\right\} \\
f_{Y_1,...,Y_n}(y_1,...,y_n) &= \frac{n}{(2\pi)^{n/2}}\exp\left\{-\frac{1}{2}\left(y_1-\sum_{i=2}^{n}y_i\right)^2-\frac{1}{2}\sum_{i=2}^{n}(y_i+y_1)^2\right\} \\
&= \frac{n}{(2\pi)^{n/2}}\exp\left\{-\frac{1}{2}ny_1^2\right\}\cdot\exp\left\{-\frac{1}{2}\left[\left(\sum_{i=2}^{n}y_i\right)^2-\sum_{i=2}^{n}y_i^2\right]\right\}
\end{aligned}
$$