Math 502AB - Lecture 13

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1 Lecture - Part 1

1.0.1 Theorem: Conditional Variance Identity

$$Var(Y) = Var(E[Y|X]) + E[Var(Y|X)]$$

Where:

$$\begin{aligned} Var(E[Y|X]) &= E\left(E[Y|X]\right)^2 - \left(E(E[Y|X])\right)^2 \\ E[Var(Y|X)] &= E\left[E(Y^2|X) - (E[Y|X])^2\right] \end{aligned}$$

Proof:

$$\begin{aligned} Var[E(Y|X)] &= E[E(Y|X)^2] - E[Y]^2 \\ E[Var(Y|X)] &= E[E(Y^2|X)] - E[E(Y|X)]^2 \\ &= E[Y^2] - E[E(Y|X)]^2 \end{aligned}$$

If we add these two together, we get $Var(Y)$.

Example:

Consider the Beta-binomial distribution. That is, consider:

$$\begin{aligned} X|P &\sim binomial(n, \mathcal{P}) \\ \mathcal{P} &\sim Beta(\alpha, \beta) \end{aligned}$$

Find the variance of X.

$$\begin{aligned} Var(X) &= Var[E(X|\mathcal{P})] + E[Var(X|\mathcal{P})] \\ &= Var[n\mathcal{P}] + E[n\mathcal{P}(1-\mathcal{P})] \\ &= n^2 Var(\mathcal{P}) + nE[\mathcal{P}] - nE[\mathcal{P}^2] \\ &= \cdots \end{aligned}$$

Example

This is **Problem 4.33** in the book. Say we have N insects, where each insect lays X_i **(iid)** eggs where $X_i \perp\!\!\!\perp N$. We want to find H, the distribution of the *total number of eggs*.

$$\begin{aligned} N &\sim Poisson(\lambda) \\ X_i &\sim logarithmic(p) \\ P(X_i = x) &= \frac{-(1-p)^x}{x \log p} \quad x = 1, 2, \dots \\ H &= X_1 + \cdots + X_N | N \end{aligned}$$

To find H, we want to find the *moment generating function*:

$$M_H(t) = E[e^{tH}] = E_N[E(e^{tH}|N)]$$

We note:

$$\begin{aligned} E[e^{tH}|N] &= E[e^{t(x_1 + \dots + x_N)}|N] \\ &= \prod_{i=1}^{N} E[e^{tx_i}|N] \\ &= [E(e^{tx_1}|N)]^N \\ E(e^{tx_1}|N) &= \sum_{x=1}^{\infty} e^{tx} \frac{-(1-p)^x}{x \log p} \\ &= -\frac{1}{\log p} \left[-\log\left(1 - e^t(1-p)\right) \right] \end{aligned}$$

So, we have:

$$\begin{aligned} E[e^{tH}] &= E[E(e^{tH}|N)] \\ &= E_N \left(\frac{\log(1 - e^t(1-p)}{\log p} \right)^N \\ &= \sum_{n=0}^{\infty} \left(\frac{\log(1 - e^t(1-p)}{\log p} \right)^n \frac{e^{-\lambda}\lambda^n}{n!} \\ &= \text{... (algebra)} \\ &= \left(\frac{p}{1 - e^{t(1-p)}} \right)^{-\lambda/\log p} \\ H &\sim Neg.Binom. \left(r = -\frac{\lambda}{\log p}, p \right) \end{aligned}$$