Math 502AB - Lecture 4

Dr. Jamshidian

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1 Lecture - Part 1

1.1 PMF and PDF's (Section 1.6)

Definition:

A *probability mass function* (**pmf**) for a discrete random variable X is defined by:

$$f_X(x) = P(X = x) \quad \forall x$$

Example: Given $X \sim binomial(n, p)$, then it's *pmf* is:

$$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, 2, ..., n$$

Definition:

For a *continuous random variable*, the **probability density function** (*pdf*) is a function $f_X(x)$ such that:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

Theorem:

A function $f_X(x)$ is a *pdf* (or *pmf*) of a random variable X **iff**, over the entire sample space:

1. $f_X(x) \geq 0 \quad \forall x$
2. **Discrete:** $\sum f_X(x) = 1$
3. **Continuous:** $\int f_X(x)dx = 1$

Proof:

($\Rightarrow$): Suppose that $f_X(x)$ is a density.

1. $f_X(x) = F'(x) \geq 0$ since $F(x)$ is *non-decreasing*
2. $$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)dt \\ \lim_{x\to\infty} F_X(x) &= \lim_{x\to\infty} \int_{-\infty}^{x} f_X(t)dt \\ 1 &= \int_{-\infty}^{\infty} f_X(t)dt \end{aligned}$$

($\Leftarrow$): Hint on this proof:

If we have properties (1) and (2), we need to show that f is a density, namely the function

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

satisfies the three required properties of a *cdf.*

1.2 Transformations and Expectations (Section 2.1)

1.2.1 Distributions of a function of a Random Variable

If X is a random variabble, then $Y = g(X)$ is also a random variable, where g is some function. The idea here is, suppose you know the distribution of X, and you want to find the distribution of $g(X)$. How would you do it?

Hint:

- If X is a discrete random variable, obtain the *pmf* of Y
- If X is a continuous random variable, start with obtaining the *cdf* of Y

Examples:

1. **(Negative Binomial)** Consider independent Bernoulli experiments. Let X = the number of trials required to get r successes, when the probability of success for each trial is p. Obtain the *pmf* for X.

$$\begin{aligned} P(X = r) &= p^r \\ P(X = r+1) &= \binom{r}{r-1} p^r(1-p) \\ P(X = r+2) &= \binom{r+1}{r-1} p^r(1-p)^2 \end{aligned}$$

So, we have the *pmf* as:

$$f_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \quad x = r, r+1, ...$$

Our book considers the *negative binomial* as Y = the number of failures before the r^{th} success. In this case, Y is related to X by

$$Y = X - r$$

2. (Gamma Distribution) Consider a random variable X with density:

$$f_X(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x \geq 0; \alpha, \beta > 0$$

Where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

is said to have a gamma distribution with parameters α and β. We write $X \sim gamma(\alpha, \beta)$.

Problem: Let $Y = cX$, where $c > 0$. Obtain the *pdf* of Y

For $y \geq 0$:

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(cX \leq y) \\
&= P\left(X \leq \frac{y}{c}\right) = \int_0^{y/c} f_X(x|\alpha, \beta) dx \\
\frac{d}{dy} F_Y(y) &= \frac{d}{dy} \int_0^{y/c} f_X(x|\alpha, \beta) dx \\
&= f_X\left(\frac{y}{c}|\alpha, \beta\right) \cdot \frac{1}{c}
\end{aligned}$$

So, then we have:

$$\begin{aligned}
f_Y(y) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \left(\frac{y}{c}\right)^{\alpha-1} e^{-y/c\beta} \cdot \frac{1}{c} \\
&= \frac{1}{\Gamma(\alpha)(\beta c)^\alpha} y^{\alpha-1} e^{-y/\beta c} \sim gamma(\alpha, \beta c)
\end{aligned}$$

3. Let $X \sim N(\mu, \sigma^2)$ with density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-0.5\left(\frac{x-\mu}{\sigma}\right)^2}$$

And let $Z \sim N(0, 1)$. Determine the distribution of $Y = Z^2$

Solution: For $y \geq 0$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(Z^2 \leq y) \\ &= P(-\sqrt{y} \leq Z \leq \sqrt{y}) \\ &= F_Z(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

But integrating that would be complicated. So what do we do? We take the derivative of both sides:

$$\begin{aligned} f_Y(y) &= F'_Y(y) \\ &= \frac{d}{dy}[F_Z(\sqrt{y}) - F_Z(-\sqrt{y})] \\ &= f_Z(\sqrt{y}) \cdot \left(\frac{1}{2}y^{-0.5}\right) + f_Z(-\sqrt{y}) \cdot \left(\frac{1}{2}y^{-0.5}\right) \\ &= y^{-0.5} f_Z(\sqrt{y}) \\ &= \frac{1}{\sqrt{2\pi}} e^{-0.5y} \cdot y^{-0.5} \\ &= \frac{1}{2^{0.5}\Gamma(0.5)} e^{-0.5y} \cdot y^{-0.5} \sim gamma(\alpha = 0.5, \beta = 2) \end{aligned}$$

2 Lecture - Part 2

Theorem:

Let X be a continuous random variable, and let $Y = g(X)$ where g is a differentiable function, and it is strictly monotone in the support of X. Then, Y has the density:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

Theorem:

Let X have a continuous *cdf* $F_X(x)$. Then the random variable $Y = F_X(x) \sim unif(0, 1)$.

Proof:

Need to show that

$$F_Y(y) = \begin{cases} y & 0 < y < 1 \\ 0 & y \leq 0 \\ 1 & y \geq 1 \end{cases}$$

For $0 \leq y \leq 1$:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(F_X(x) \leq y) \\ &= P(X \leq F_X^{-1}(y)), \text{ if } F_X \text{ is strictly increasing} \\ &= F_X(F_X^{-1}(y)) = y \end{aligned}$$

But the *cdf* doesn't have to be strictly increasing. However, this theorem still holds. Define:

$$F_X^{-1}(y) = inf\,\{x : F_X(x) \geq y\}$$

This makes the inverse function uniquely defined, since the probability of x being in the bounds where the *cdf* might be flat is zero.

2.1 Expected Value of a Random Variable (Section 2.2)

Definition:

Let X be a random variable. If X is a continuous random variable with pdf $f(x)$ and

$$\int_{-\infty}^{\infty} |x| f_X(x) dx < \infty$$

then the expectation of X is:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

If X is a discrete random variable with *pmf* $f_X(x)$ and

$$\sum_{\text{all } x} |x| f_X(x) < \infty$$

then the expectation of X is:

$$\sum_{\text{all } x} x f_X(x) < \infty$$

2.1.1 Discrete Case

Consider a series of the form:

$$\sum_{n=1}^{\infty} a_n$$

A series is convergent if the sequence of its partial sums converges. That is $\{S_n\}_{n=1}^{\infty}$ converges where $S_n = \sum_{i=1}^{n} a_i$.

The above series is **absolutely convergent** if $\sum_{i=1}^{n} |a_n|$ converges. A sequence is **conditionally convergent** if the sequence $\sum_{i=1}^{n} a_i$ converges, but the sequence $\sum_{i=1}^{n} |a_i|$ does not.

Theorem:

If the series $\sum_{n=1}^{\infty} a_n$ **converges conditionally**, then for any given real number c, there is a rearrangement of the terms a_n such that

$$\sum_{i=1}^{\infty} a_i = c$$

Example:

Consider the *harmonic series*:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + ... = \ln 2$$

If we rearrange the series as follows:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + ... = 3 \ln 2$$

2.1.2 Continuous Case

Consider the following example where the density

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2} \qquad -\infty < x < \infty$$

We then have that:

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

But note that the following integral does not exist:

$$\int_{-\infty}^{\infty} \frac{1}{\pi} \cdot \frac{x}{1+x^2} dx = 1$$

Why? Note that:

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx = \lim_{r \to \infty} \int_{-r}^{r} \frac{x}{1+x^2} dx = 0, \text{ because odd function}$$

Written in another form, we have:

$$\lim_{r \to \infty} \int_{-r}^{kr} \frac{x}{1+x^2} dx = \lim_{r \to \infty} \frac{1}{2} \log \left(\frac{1+k^2 r^2}{1+r^2} \right) = \log(k)$$

To know that a solution to this integral exists, you have to take $|x|$ in the numerator.