Transformada de Laplace

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A transformada e anti-transformada de Laplace.

$$x(t) \overset{\mathscr{L}}{\Longleftrightarrow} X(s)$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

A transformada de Laplace.

$$x(t) \overset{\mathscr{L}}{\Longleftrightarrow} X(s)$$

$$X(s) = \int_{0_-}^{\infty} x(t)e^{-st}dt$$

Circuito RC

Um sistema LIT

Para o circuito abaixo conhecemos as equações fundamentais.

$$v_R(t) = R\,i(t) \tag{1}$$

$$i_C(t) = C\frac{d\,v_c(t)}{dt} \tag{2}$$

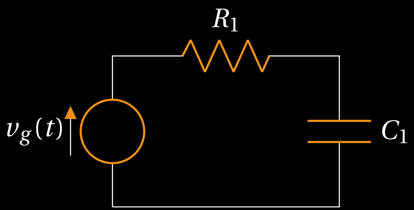


Figura 1: Circuito RC.

Modelagem no tempo

$$
\begin{aligned}
i_C(t) &= \quad C\frac{d\,v_c(t)}{d\,t} \\
v_R(t) &= \quad RC\frac{d\,v_c(t)}{d\,t} \\
v_c(t) &= \quad v_G(t) - RC\frac{d\,v_c(t)}{d\,t}
\end{aligned}
$$

Modelagem na frequência

$$
\begin{aligned}
v_c(t) &= v_G(t) - RC\frac{d\,v_c(t)}{dt} \\
& v_c(t) \overset{\mathscr{L}}{\Longleftrightarrow} V_C(s) \\
V_C(s) &= V_G(s) - RC\,s V_C(s) \\
V_C(s) + RC\,s V_C(s) &= V_G(s) \\
\frac{V_C(s)}{V_G(s)} &= \frac{1}{sRC + 1}
\end{aligned}
$$

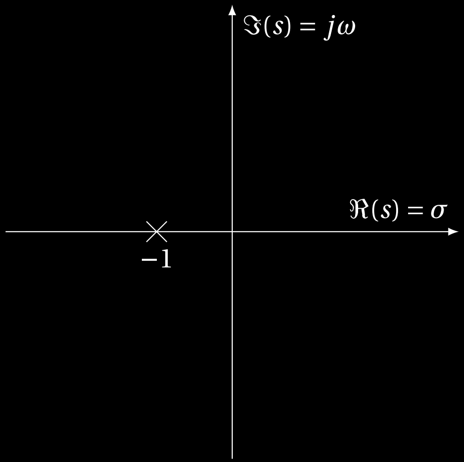


Figura 2: Considerando $R = 1\Omega$ e $C = 1F$.

Resposta impulsiva

$$\underbrace{\frac{V_C(s)}{V_G(s)}}_{\texttt{Ganho}} = \underbrace{\frac{1}{sRC+1}}_{\texttt{Função de transferência}}$$

$$V_C(s) \overset{\mathscr{L}^{-1}}{\Longleftrightarrow} v_c(t)$$

$$v_G(t) = \delta(t) \overset{\mathscr{L}}{\Longleftrightarrow} V_G(s) = 1$$

$$v_C(t) = e^{-t}, t \geq 0$$

Resposta impulsiva

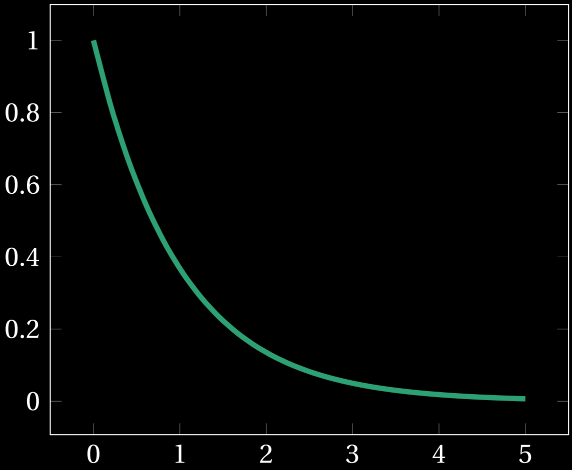


Figura 3: Considerando $R = 1\Omega$ e $C = 1F$.

Circuito RLC

$$v_R(t) = \quad R i(t) \tag{3}$$

$$i_C(t) = \quad C \frac{d\, v_c(t)}{dt} \tag{4}$$

$$v_L(t) = \quad L \frac{d\, i_L(t)}{dt} \tag{5}$$

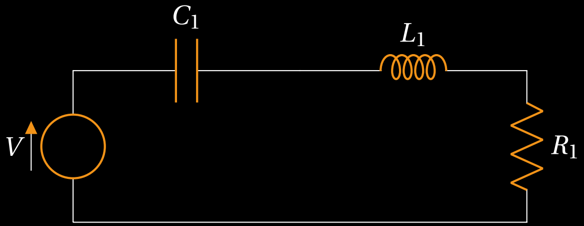


Figura 4: Circuito RLC.

Modelagem no tempo

$$v_L(t) = L\frac{d\, C\frac{d\, v_c(t)}{d\,t}}{d\,t} \tag{6}$$

$$v_L(t) = LC\frac{d^2\, v_c(t)}{d\,t^2} \tag{7}$$

$$\underbrace{RC\frac{d\, v_c(t)}{d\,t}}_{v_R(t)} = v_G(t) - \underbrace{LC\frac{d^2\, v_c(t)}{d\,t^2}}_{v_L(t)} - v_C(t) \tag{8}$$

Modelagem na frequência

$$\underbrace{RC\frac{d\ v_c(t)}{dt}}_{v_R(t)} = v_G(t) - \underbrace{LC\frac{d^2\ v_c(t)}{dt^2}}_{v_L(t)} - v_C(t) \tag{9}$$

$$v_c(t) \overset{\mathscr{L}}{\Longleftrightarrow} V_C(s) \tag{10}$$

$$RCsV_c(s) = V_G(s) - LCs^2V_c(s) - V_C(s) \tag{11}$$

$$\underbrace{\frac{V_C(s)}{V_G(s)}}_{\texttt{Ganho}} = \frac{1}{\underbrace{s^2LC + sRC + 1}_{\texttt{Função de transferência}}} \tag{12}$$

Plano S

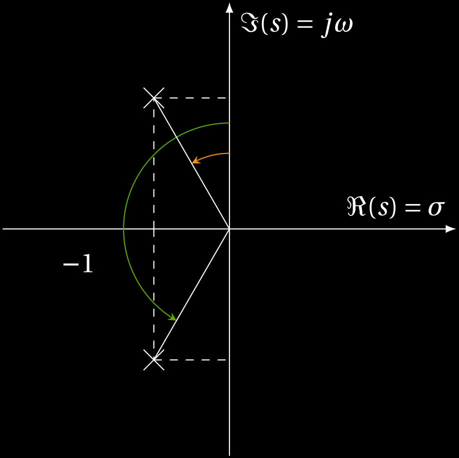


Figura 5: Considerando $R = 1\Omega$, $L = 1H$ e $C = 1F$.

Expandindo em frações parciais

$$\frac{V_C(s)}{V_G(s)} = \frac{1}{s^2+s+1} \tag{13}$$

$$\underbrace{\frac{V_C(s)}{V_G(s)}}_{\texttt{Ganho}} = \underbrace{\frac{-0.5774j}{s+0.5-j0.866} + \frac{0.5774j}{s+0.5+j0.866}}_{\texttt{Função de transferência}} \tag{14}$$

Resposta impulsiva

$$\underbrace{\frac{V_C(s)}{V_G(s)}}_{\texttt{Ganho}} = \underbrace{\frac{-0.5774j}{s+0.5-j0.866} + \frac{0.5774j}{s+0.5+j0.866}}_{\texttt{Função de transferência}}$$

$$V_C(s) \overset{\mathscr{L}^{-1}}{\Longleftrightarrow} v_c(t)$$

$$v_G(t) = \delta(t) \overset{\mathscr{L}}{\Longleftrightarrow} V_G(s) = 1$$

$$v_C(t) = -0.5774je^{-0.5t+j0.866t} + 0.5774je^{-0.5t-j0.866t}$$

$$v_C(t) = -0.5774je^{-0.5t}(e^{j0.866t} - e^{-j0.866t})$$

$$v_C(t) = 1.1548e^{-0.5t}sen(0.866t)\texttt{,}\ t \geq 0$$

Resposta impulsiva

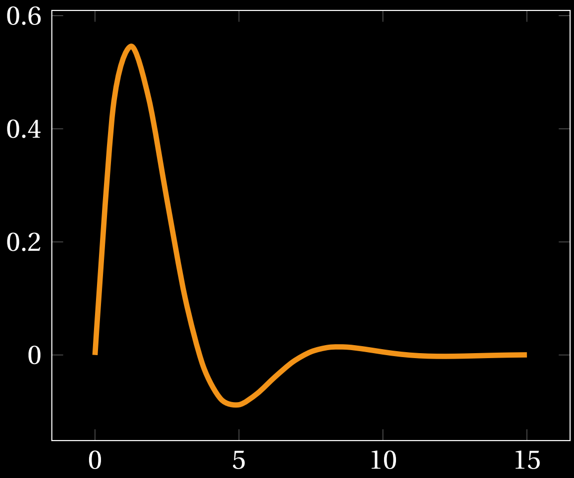


Figura 6: Considerando $R = 1\Omega$, $L = 1H$ e $C = 1F$.

Tensão no resistor - R

$$v_R(t) = \quad R i_C(t) \tag{15}$$

$$i_C(t) = \quad C \frac{d\, v_C(t)}{dt} \tag{16}$$

$$v_R(t) = \quad RC \frac{d\, v_C(t)}{dt} \tag{17}$$

Tensão no resistor - R

$$v_R(t) = RC\frac{d\,v_C(t)}{dt}$$

$$v_R(t) \overset{\mathcal{L}}{\Longleftrightarrow} V_R(s)$$

$$V_R(s) = RCsV_C(s)$$

$$\frac{V_C(s)}{V_G(s)} = \frac{1}{s^2LC + sRC + 1}$$

$$\underbrace{\frac{V_R(s)}{V_G(s)}}_{\textit{Ganho}} = \underbrace{\frac{RCs}{s^2LC + sRC + 1}}_{\textit{Função de transferência}}$$

Resposta impulsiva

$$\underbrace{\frac{V_R(s)}{V_G(s)}}_{\texttt{Ganho}} = \underbrace{\frac{0.5 + j0.2887}{s + 0.5 + j0.866} + \frac{0.5 - j0.2887}{s + 0.5 + j0.866}}_{\texttt{Função de transferência}}$$

$$V_R(s) \stackrel{\mathscr{L}^{-1}}{\Longleftrightarrow} v_r(t)$$

$$v_G(t) = \delta(t) \stackrel{\mathscr{L}}{\Longleftrightarrow} V_G(s) = 1$$

$$\begin{aligned} v_R(t) = {} & 0.5e^{-0.5t}(e^{-j0.866t} + e^{+j0.866t}) \\ + {} & j0.2887e^{-0.5t}(e^{-j0.866t} - e^{+j0.866t}) \\ v_R(t) = {} & e^{-0.5t}cos(0.866t) \\ + {} & 0.5774e^{-0.5t}sin(0.866t)\texttt{,}\ t \geq 0 \end{aligned}$$

Resposta impulsiva - no resitor

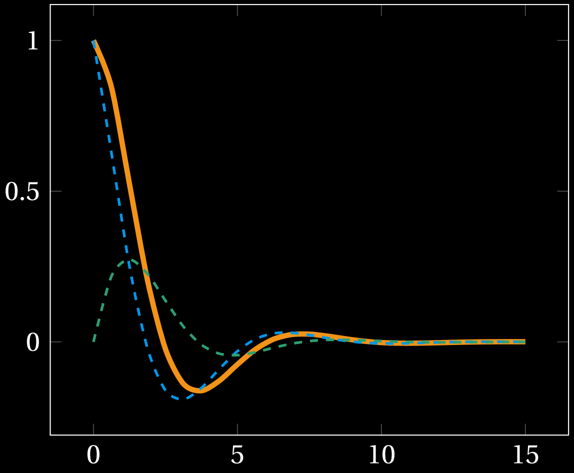


Figura 7: Considerando $R = 1\Omega$, $L = 1H$ e $C = 1F$.

Resposta impulsiva - $R = 0\Omega$

$$\frac{V_C(s)}{V_G(s)} = \frac{1}{s^2+1}$$

$$\underbrace{\frac{V_R(s)}{V_G(s)}}_{\text{Ganho}} = \underbrace{\frac{-0.5j}{s+j} + \frac{0.5j}{s-j}}_{\text{Função de transferência}}$$

$$V_C(s) \overset{\mathscr{L}^{-1}}{\Longleftrightarrow} v_c(t)$$

$$v_G(t) = \delta(t) \overset{\mathscr{L}}{\Longleftrightarrow} V_G(s) = 1$$

$$v_c(t) = -0.5j(e^{-jt} - e^{+jt})$$

$$v_R(t) = sen(t), t \geq 0$$

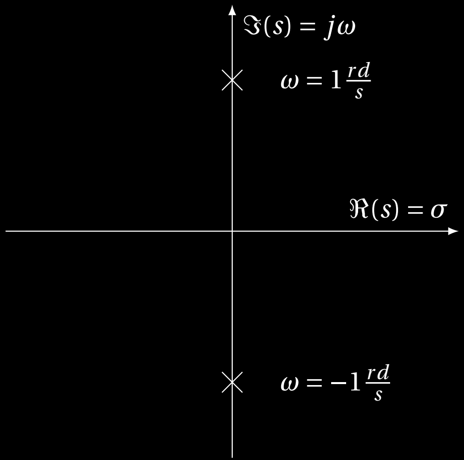


Figura 8: Considerando $R = 1\Omega$, $L = 1H$ e $C = 1F$.

Resposta impulsiva - no capacitor

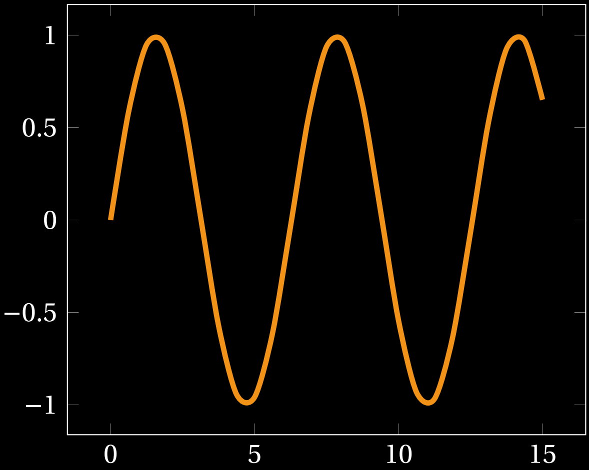


Figura 9: Considerando $R = 0\Omega$, $L = 1H$ e $C = 1F$.

Pólos e zeros

Pólos - ressonâncias - Tensão no capacitor

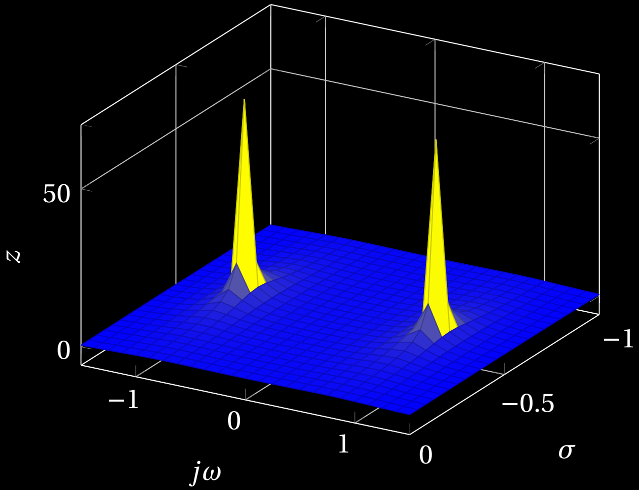


Figura 10: Considerando $R = 1\Omega$, $L = 1H$ e $C = 1F$.

Pólos - ressonâncias - Tensão no capacitor

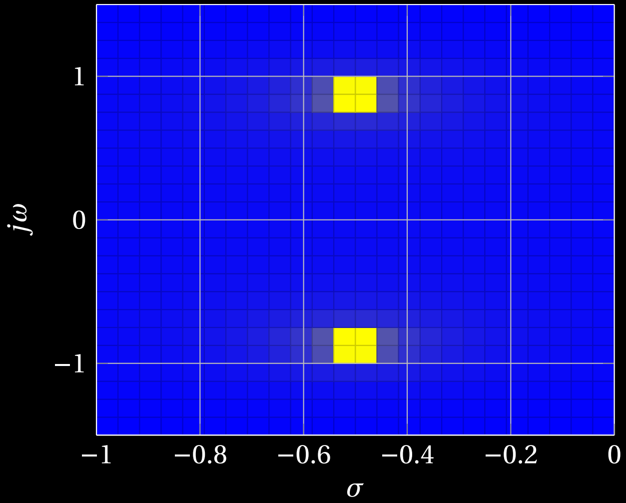


Figura 11: Considerando $R = 1\Omega$, $L = 1H$ e $C = 1F$.

Pólos - ressonâncias - Tensão no resitor

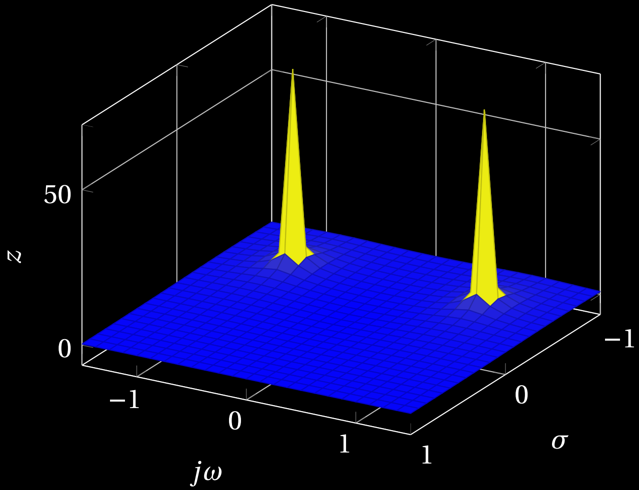
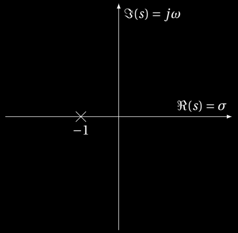
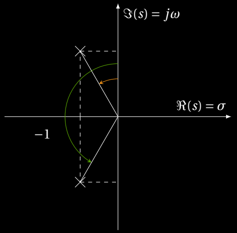
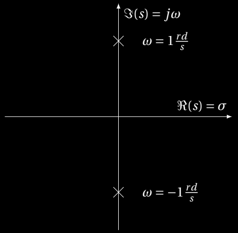


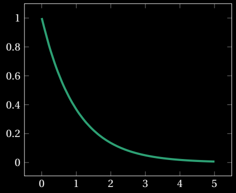
Figura 12: Considerando $R = 1\Omega$, $L = 1H$ e $C = 1F$.

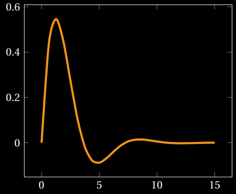
Conclusão

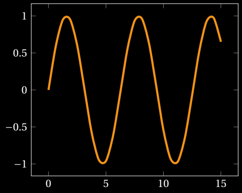












$$\frac{V_C(s)}{V_G(s)} = \frac{1}{s+1}$$

$$\frac{V_C(s)}{V_G(s)} = \frac{1}{s^2+s+1}$$

$$\frac{V_C(s)}{V_G(s)} = \frac{1}{s^2+1}$$