BAREM DE CORECTARE PENTRU TS4 / I-B

Subiectul 1

Abordarea subiectului, fără reproducerea enunțului 1p

Formularea problemei de extrem condiționat:

$$\min\left\{\sqrt{(x-3)^2+(y-2)^2+(z-1)^2}\ /\ (x,y,z)\in\mathbb{R}^3,\ 2x-3y-4z=25\right\}\ (1)\ \text{.......}\ 2p$$

Problema mai ușor de tratat, cu aceleași puncte de extrem:

$$\min\left\{(x-3)^2+(y-2)^2+(z-1)^2\ /\ (x,y,z)\in\mathbb{R}^3,\ 2x-3y-4z=25\right\}\ (2)\ \text{.......}\ 1p$$

Necesitatea utilizării unui singur multiplicator Lagrange (λ) (3) 1p

Lagrangeanul în cauză: $f(x,y,z)=(x-3)^2+(y-2)^2+(z-1)^2$, $g(x,y,z)=2x-3y-4z-25 \Rightarrow$

$$\overset{(3)}{\Rightarrow} L(x,y,z;\lambda)=f(x,y,z)+\lambda g(x,y,z)=$$

$$=(x-3)^2+(y-2)^2+(z-1)^2+\lambda(2x-3y-4z-25),\ \forall (x,y,z)\in\mathbb{R}^3, \lambda\in\mathbb{R}\ (4)\ 2p$$

(4) $\Rightarrow$ Problema de extrem necondiționat corespunzătoare problemei (2):

$$\min\left\{L(x,y,z;\lambda)\ /\ (x,y,z,\lambda)\in\mathbb{R}^4\right\}\ \text{.......}\ (5)\ \text{.......}\ 1p$$

Sistemul punctelor critice pentru (5):

$$\begin{cases}\dfrac{\partial L}{\partial x}(x,y,z;\lambda)\equiv 2(x-3)+2\lambda=0\\ \dfrac{\partial L}{\partial y}(x,y,z;\lambda)\equiv 2(y-2)-3\lambda=0\\ \dfrac{\partial L}{\partial z}(x,y,z;\lambda)\equiv 2(z-1)-4\lambda=0\\ \dfrac{\partial L}{\partial \lambda}(x,y,z;\lambda)\equiv 2x-3y-4z-25=0\end{cases},\ (x,y,z,\lambda)\in\mathbb{R}^4\ (6)\ \text{.......}\ 1p$$

Soluția sistemului (6): $x_0=5,\ y_0=-1,\ z_0=-3,\ \lambda_0=-2$ ... (7) 2p

$$L_0(x,y,z)=L(x,y,z;\lambda_0)=(x-3)^2+(y-2)^2+(z-1)^2-2(2x-3y-4z-25),\ \forall(x,y,z)\in\mathbb{R}^3\ (8)\ \text{...}\ 1p$$

$$(8)\Rightarrow d^2L_0(x_0,y_0,z_0)=\langle H_{L_0}(x_0,y_0,z_0)(dx,dy,dz)^T,(dx,dy,dz)\rangle_c=\langle \mathrm{diag}(2,2,2)(dx,dy,dz),(dx,dy,dz)\rangle_c=$$

$$=2\left[(dx)^2+(dy)^2+(dz)^2\right]\ \text{....}\ (9)\ \text{.......}\ 3p$$

$g(x,y,z)=0 \Rightarrow dg(x_0,y_0,z_0)=0 \Rightarrow dx=\frac{3}{2}dy+2dz$ ... (10) 1p

$(9)+(10) \Rightarrow d^2 L_0(x_0,y_0,z_0)\Big|_{dg(x_0,y_0,z_0)=0} = 2\left[\frac{13}{4}(dy)^2+6\,dy\,dz+5(dz)^2\right] =$

$= 2\left[3(dy+dz)^2+\frac{1}{4}(dy)^2+2(dz)^2\right]$ ...... (11) ———— 1p

$(11) \Rightarrow (x_0,y_0,z_0)=(5,-1,-3)=$ punct de minim pentru problema (2) — (12) — 1p

$(12)+(1)+(2) \Rightarrow (5,-1,-3)=$ punct de minim pentru problema (1) — (13) — 1p

$(13) \Rightarrow$ Concluzia: $(5,-1,-3)$ este punctul care, aparținând planului de ecuație $2x-3y-4z=25$, este cel mai apropiat de punctul $(3,2,1)$, în raport cu distanța euclidiană uzuală 1p

Total: 20 de puncte

Subiectul 2

Abordarea subiectului, fără reproducerea enunțului ———— 1p

$e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n,\ \forall x\in\mathbb{R}$ —— (1) ———— 1p

$(1)\Rightarrow \frac{e^x-1}{x}=\sum_{n=1}^{\infty}\frac{1}{n!}x^{n-1}=\sum_{n=0}^{\infty}\frac{1}{(n+1)!}x^n,\ x\neq 0$ (2) ———— 2p

$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n},\ \forall x\in\mathbb{R}$ ..... (3) ———— 1p

$(3)\Rightarrow \frac{\cos x-1}{x^2}=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n-2}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+2)!}x^{2n},\ \forall x\in\mathbb{R}^*$ ... (4) —— 2p

$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1},\ \forall x\in\mathbb{R}$ ... (5) ———— 1p

$(5)\Rightarrow \frac{\sin x-x}{x^3}=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n-2}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+3)!}x^{2n},\ \forall x\in\mathbb{R}^*$ — (6) —— 2p

$(2)+(4)+(6)\Rightarrow f(x)=\sum_{n=0}^{\infty}\frac{1}{(n+1)!}x^n-\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+2)!}x^{2n}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+3)!}x^{2n},\ \forall x\in\mathbb{R}$ (7) — 2p

$(7)\Rightarrow f(x)=\sum_{n=0}^{\infty}\left(\frac{1}{(2n+1)!}-\frac{(-1)^{n+1}}{(2n+2)!}+\frac{(-1)^{n+1}}{(2n+3)!}\right)x^{2n}+\sum_{n=0}^{\infty}\frac{1}{(2n+2)!}x^{2n+1},\ \forall x\in\mathbb{R}$ — (8) — 2p

Formula (seria) lui Maclaurin $\Rightarrow f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k,\ \forall x\in V\in \mathcal{V}(0)$ ... (9) ... 2p

$(8)+(9)\xRightarrow[(n=1008)]{(k=2017)} \frac{f^{(2017)}(0)}{(2017)!}=\frac{1}{(2018)!}$ ... (10) ... 3p

(10) $\Rightarrow$ Concluzia : $f^{(2017)}(0)=\frac{1}{2018}$ ... 1p

Total: 20 de puncte

Subiectul 3

Abordarea subiectului cu cunoștințe de cauză ... 1p

$\int_0^{\infty}e^{-tx}\frac{\cos x-1}{x}dx$: integrală improprie (de speța I-a), cu parametru $(t\in\mathbb{R}_+^*)$ (1) ... 1p

$\exists \lim\limits_{\substack{x\to 0\\ x>0}}\left(e^{-tx}\frac{\cos x-1}{x}\right)=0 \Rightarrow$ integrala nu este și improprie de speța a II-a (2) ... 1p

$\exists \lim\limits_{x\to\infty}\left(e^{-tx}\frac{|\cos x-1|}{x}\right)=0 \Rightarrow$ criteriul necesar de convergență îndeplinit ... 1p

Aplicarea criteriului în β : $\Rightarrow \exists \beta>1$ (de exemplu, $\beta=2$), astfel încât $\exists \lim\limits_{x\to\infty}\left(x^{\beta}e^{-tx}\frac{|\cos x-1|}{x}\right)=0 \Rightarrow$ integrala vizată este convergentă ... (3) ... 2p

$(1)+(2)+(3)\Rightarrow \exists F(t)=\int_0^{\infty}e^{-tx}\frac{\cos x-1}{x}dx,\ \forall t\in(0,+\infty)=\mathbb{R}_+^*$ ... (4) ... 1p

$f(t,x)=e^{-tx}\cdot\frac{\cos x-1}{x} \Rightarrow f$ = funcție continuă în raport cu $x\in(0,b]$, $\forall b\in\mathbb{R}_+^*$, $\forall t\in\mathbb{R}_+^*$ (uniform în raport cu t) ... (5) ... 1p

$(4)+(5)\Rightarrow F(t)=\lim\limits_{b\to\infty}\int_0^{b}e^{-tx}\cdot\frac{\cos x-1}{x}dx,\ \forall t\in\mathbb{R}_+^*$ ... (6) ... 1p

$\exists \frac{d}{dt}\left(\int_0^b e^{-tx}\frac{\cos x-1}{x}dx\right)=\int_0^b e^{-tx}(1-\cos x)dx,\ \forall t\in\mathbb{R}_+^*,\ b\in\mathbb{R}_+^*$ ... (7) ... 1p

$\int_0^b e^{-tx}(1-\cos x)dx=\frac{1}{t(t^2+1)}-\frac{e^{-tb}\sin b}{t^2+1}-\frac{e^{-tb}\cos b}{t(t^2+1)},\ \forall t,b\in\mathbb{R}_+^*$ ... (8) ... 4p

$(6)+(7)+(8) \Rightarrow \exists F'(t) = \dfrac{1}{t(t^2+1)}, \ \forall t \in \mathbb{R}_+^* \ldots$ (9) — 1p

$(9) \Rightarrow F(t) = \ln \dfrac{t}{\sqrt{1+t^2}}, \ \forall t \in \mathbb{R}_+^* \ldots$ (10) — 2p

$(10) \Rightarrow$ Prima concluzie : $\displaystyle\int_0^\infty e^{-tx} \cdot \frac{\cos x - 1}{x} dx = \ln \frac{t}{\sqrt{t^2+1}}, \ \forall t \in \mathbb{R}_+^*$ (11) 1p

$(11) \xrightarrow{(t=1)}$ A doua concluzie : $\displaystyle\int_0^\infty x^{-1} e^{-x} (\cos x - 1) dx = -\frac{1}{2}\ln 2 \ldots$ 2p

Total: 20 de puncte

Subiectul 4

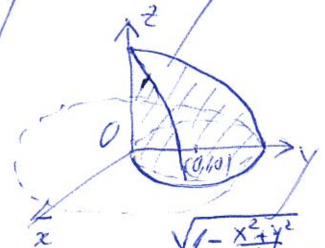
Abordarea subiectului 1p

Formula : $V = \displaystyle\iiint_A dx\,dy\,dz$ ...... (1) — 2p

$A = \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, x^2+y^2-2y \le 0, \ 0 \le z \le \sqrt{1 - \frac{x^2+y^2}{4}} \right\}$ — (2) — 3p

$B = \{ (x,y) \in \mathbb{R}^2 \,/\, x^2+(y-1)^2 \le 1 \} \ldots$ (3) — 2p

Reprezentarea grafică a corpului A:



(4) — 3p

$(1) \sim (4) \Rightarrow V = \displaystyle\iint_B \left(\int_0^{\sqrt{1-\frac{x^2+y^2}{4}}} dz \right) dx\,dy = \iint_{x^2+(y-1)^2 \le 1} \sqrt{1 - \frac{x^2+y^2}{4}}\, dx\,dy$ ... (5) ... 2p

$(x,y) \underset{y=2r\cos\alpha}{\overset{x=2r\sin\alpha}{\longrightarrow}} (r,\alpha) \Rightarrow B \rightsquigarrow \widetilde{B} = \{ (r,\alpha) \in \mathbb{R}_+ \times [0, \frac{\pi}{2}] \,/\, 0 \le r \le \cos\alpha \}$ ... (6) — 2p

$(6)+(5) \Rightarrow V = \displaystyle\iint_{\widetilde{B}} \sqrt{1-r^2}\, r\,dr\,d\alpha = \int_0^{\frac{\pi}{2}} \left(\int_0^{\cos\alpha} r\sqrt{1-r^2}\,dr \right) d\alpha = \frac{3\pi - 4}{18}$ (7) — 4p

$(7) \Rightarrow$ Concluzia : volumul cerut are valoarea $\dfrac{3\pi-4}{18}$ 1p

Total: 20 de puncte