BAREM DETALIAT DE CORECTARE

pentru TS2 la "Matematică" / I1A & I1Xa

(seria 2016 - 2017 / 24.11.2016)

15 puncte - bonusul de participare la TS2

25 de puncte - subiectul 1

$i)$ Abordarea chestiunii .... 1 punct

$\mathcal{A} = \{a \in \mathbb{R} \mid f(a) \text{ are sens}\} = \left\{ a \in \mathbb{R} \mid \sum_{n=1}^{\infty} \frac{(arctg(3n+1) - arctg(3n-2))^a}{16n^2+8n-3} \ (C) \right\}$ (1.1) .... 1 punct

$arctg(3n+1) > arctg(3n-2), \forall n \in \mathbb{N}^* \Longrightarrow$

$\Longrightarrow \quad x_n = \frac{(arctg(3n+1) - arctg(3n-2))^a}{16n^2+8n-3} > 0, \forall n \in \mathbb{N}^*, \forall a \in \mathbb{R}$ (1.2) .... 1 punct

(1.2) $\Longrightarrow \sum_{n=1}^{\infty} x_n$ este o serie cu termeni pozitivi (1.3) .... 1 punct

$arctg(3n+1) - arctg(3n-2) = arctg\frac{3}{9n^2-3n-1}, \forall n \in \mathbb{N}^*$ (1.4) .... 1 punct

$$(1.4) \Longrightarrow \exists \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\left(arctg\frac{3}{9n^2-3n-1}\right)^a}{16n^2+8n-3} = \begin{cases} 3^a \lim\limits_{n\to\infty} \frac{(9n^2-3n-1)^{-a}}{16n^2+8n-3}, & a < 0 \\ 0, & a \geq 0 \end{cases} =$$

$$= \begin{cases} \infty, & a < -1 \\ \frac{3}{16}, & a = -1 \\ 0, & a > -1 \end{cases} \quad (1.5)$$ .... 3 puncte

(1.5) $\Longrightarrow \sum_{n=1}^{\infty} x_n \ (D), \forall a \in (-\infty, -1]$ şi

criteriul necesar de convergenţă = îndeplinit, $\forall a \in (-1, \infty)$ (1.6) .... 2 puncte

$\forall a \in \mathbb{R}, \exists \lim_{n\to\infty} \frac{x_n}{n^{-2(a+1)}} = \frac{3^{-a}}{16} \in (0, \infty)$ (1.7) .... 2 puncte

(1.3) + (1.6) (pentru $a \in (-1, \infty)$) + (1.7) $\Longrightarrow \sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} \frac{1}{n^{2(a+1)}}$ (1.8) .... 1 punct

$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \ (D)$, $\forall \alpha \in (-\infty, 1]$ şi $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \ (C)$, $\forall \alpha \in (1, \infty)$ (1.9) .... 2 puncte

(1.8) + (1.9) $\Longrightarrow \sum_{n=1}^{\infty} x_n$ (D), $\forall a \in (-\infty, -\frac{1}{2}]$ şi $\sum_{n=1}^{\infty} x_n \ (C), \forall a \in (-\frac{1}{2}, \infty)$ (1.10) .... 1 punct

(1.10) $\Longrightarrow$ concluzia : $\mathcal{A} = (-\frac{1}{2}, \infty)$ (1.11) .... 1 punct

Total: 17 puncte

$ii)$ Abordarea chestiunii .... 1 punct

(1.11) $\Longrightarrow 0 \in \mathcal{A} \Longrightarrow f(0)$ are sens (1.12) .... 1 punct

(1.1) + (1.12) $\Longrightarrow f(0) = \sum_{n=1}^{\infty} \frac{1}{16n^2+8n-3}$ (1.13) .... 1 punct

$16n^2 + 8n - 3 = (4n-1)(4n+3), \forall n \in \mathbb{N}^*$ (1.14) .... 1 punct

(1.14) $\Longrightarrow \frac{1}{16n^2+8n-3} = \frac{1}{4}\left(\frac{1}{4n-1} - \frac{1}{4n+3}\right), \forall n \in \mathbb{N}^*$ (1.15) .... 1 punct

$(1.15) \Longrightarrow\ S_n = \sum\limits_{k=1}^{n} \frac{1}{16k^2+8k-3} = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{4n+3}\right),\ \forall n \in \mathbb{N}^*$ (1.16) ....1 punct

$(1.16) \Longrightarrow \exists\ S = \sum\limits_{n=1}^{\infty} \frac{1}{16n^2+8n-3} = \lim\limits_{n\to\infty} S_n = \frac{1}{12}$ (1.17) ....1 punct

(1.17) $\Longrightarrow$ concluzia : $f(0) = \frac{1}{12}$ ....1 punct

Total: 8 puncte

25 de puncte - subiectul 2

$j)$ Abordarea chestiunii ....1 punct

M = mulţimea soluţiilor $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

ale sistemului algebric liniar şi omogen (S) $\begin{cases} x_1 - x_3 + x_4 = 0 \\ 2x_1 - 3x_2 - x_3 = 0 \\ 3x_2 - x_3 + 2x_4 = 0 \end{cases}$ (2.1) ....1 punct

$rang \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & -3 & -1 & 0 \\ 0 & 3 & -1 & 2 \end{pmatrix} = 2$ (2.2) ....1 punct

(2.1) + (2.2) $\Longrightarrow$ (S) = sistem compatibil, dublu-nedeterminat (2.3) ....1 punct

$(2.3) \Longrightarrow\ M = \{\ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = x_3 - x_4,\ x_2 = \frac{1}{3}x_3 - \frac{2}{3}x_4\}$ (2.4) ....1 punct

$(2.4) \Longrightarrow\ M = \left\{(\alpha - \beta, \frac{1}{3}\alpha - \frac{2}{3}\beta, \alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\right\}$ (2.5) ....1 punct

$(2.5) \Longrightarrow\ M = Lin(\left\{(1, \frac{1}{3}, 1, 0),\ (-1, -\frac{2}{3}, 0, 1)\right\})$ (2.6) ....2 puncte

(2.6) $\Longrightarrow$ M = subspaţiu liniar al lui $\mathbb{R}^4$, generat de $\mathrm{u} = (1, \frac{1}{3}, 1, 0)$ şi $\mathrm{v} = (-1, -\frac{2}{3}, 0, 1)$ (2.7) ....1 punct

$\widehat{(\mathrm{u}, \mathrm{v})} = \arccos \frac{\langle \mathrm{u},\mathrm{v}\rangle_c}{\sqrt{\langle \mathrm{u},\mathrm{u}\rangle_c}\sqrt{\langle \mathrm{v},\mathrm{v}\rangle_c}} = \arccos\left(-\sqrt{\frac{11}{38}}\right)$ (2.8) ....2 puncte

$(2.8) \Longrightarrow \widehat{(\mathrm{u}, \mathrm{v})} \in \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$ (2.9) ....1 punct

(2.9) $\Longrightarrow$ concluzia este conformă cerinţei din enunţ ....1 punct

Total: 13 puncte

$jj)$ Abordarea chestiunii ....1 punct

$rang \begin{pmatrix} 1 & \frac{1}{3} & 1 & 0 \\ -1 & -\frac{2}{3} & 0 & 1 \end{pmatrix} = 2 \overset{(2.7)}{\Longrightarrow}$ u şi v sunt vectori liniar independenţi (2.10) ....1 punct

$(2.10) \Longrightarrow Sp(M) = Sp(\{\mathrm{u}, \mathrm{v}\}) = Lin(\{\mathrm{u}, \mathrm{v}\})$ (2.11) ....1 punct

$(2.11) \Longrightarrow\ (Sp(M))^{\perp} = \{\mathrm{y} \in \mathbb{R}^4 \mid \langle \mathrm{y}, \mathrm{u}\rangle_c = \langle \mathrm{y}, \mathrm{v}\rangle_c = 0\}$ (2.12) ....1 punct

$(2.12) \Longrightarrow\ (Sp(M))^{\perp} = \{\mathrm{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 \mid\ y_1 + \frac{1}{3}y_2 + y_3 = y_1 + \frac{2}{3}y_2 - y_4 = 0\}$ (2.13) .1 punct

$(2.13) \Longrightarrow\ (Sp(M))^{\perp} = \{\mathrm{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 \mid y_3 = -\ y_1 - \frac{1}{3}y_2,\ y_4 = y_1 + \frac{2}{3}y_2\} =$

$= \{y_1(1, 0, -1, 1) + y_2(0, 1, -\frac{1}{3}, \frac{2}{3}) \mid\ y_1, y_2 \in \mathbb{R}\}$ (2.14) ....1 punct

$rang \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} = 2 \overset{(2.14)}{\Longrightarrow} \left\{(1, 0, -1, 1),\ (0, 1, -\frac{1}{3}, \frac{2}{3})\right\} =$

= bază pentru $(Sp(M))^{\perp}$ (2.15) ....2 puncte

(2.15) $\Longrightarrow$ Se iau: $\mathrm{w} = (1, 0, -1, 1)$, $\mathrm{z} = (0, 1, -\frac{1}{3}, \frac{2}{3}) + \gamma(1, 0, -1, 1)$,

cu $\gamma \in \mathbb{R}$, aşa încât $\langle \mathrm{z}, \mathrm{w}\rangle_c = 0$ (2.16) ....1 punct

$(2.16) \Longrightarrow\ \gamma = -\frac{\langle (0,1,-\frac{1}{3},\frac{2}{3}),(1,0,-1,1)\rangle_c}{\|(1,0,-1,1)\|^2} = -\frac{1}{3}$ (2.17) ....1 punct

$(2.16) + (2.17) \Longrightarrow \mathrm{z} = (-\frac{1}{3}, 1, 0, \frac{1}{3})\ \perp \mathrm{w}$ (2.18) ....1 punct

(2.18) $\Longrightarrow$ concluzia: $\{\mathrm{w}, \mathrm{z}\}$ = bază ortogonală pentru $(Sp(M))^{\perp}$ ....1 punct

Total: 12 puncte

20 de puncte - subiectul 3

$l)$ Abordarea chestiunii ....1 punct

$\forall A$ = *mulţime frontieră* în $(X, \tau) \Longleftrightarrow \overline{X \setminus A} = X$ (3.1) ....2 puncte

$\forall A, B \in \mathcal{P}(X),\ B \subseteq A \Longrightarrow X \setminus A \subseteq X \setminus B$ (3.2) ....2 puncte

$\forall U, V \in \mathcal{P}(X),\ U \subseteq V \implies \overline{U} \subseteq \overline{V}$ (3.3) .... 2 puncte
$(3.2) + (3.3) \implies \overline{X \setminus A} \subseteq \overline{X \setminus B}$ (3.4) .... 1 punct
$(3.4) + (3.1) \implies \overline{X \setminus B} = X$ (3.5) .... 1 punct
$(3.5) + (3.1) \implies \forall A =$ *mulţime frontieră* în $(X, \tau) \overset{\forall B \in \mathcal{P}(A)}{\implies} B =$ *mulţime frontieră* în (X, τ) 1 punct
Total: 10 puncte

$ll)$ Abordarea chestiunii .... 1 punct
$\forall A \in \mathcal{P}(X)$, cu $\overline{X \setminus A} = X$ şi $\forall D \in \tau,\ D \neq \varnothing$ (3.6) $\implies D \cap (X \setminus A) \neq \varnothing$ (trebuie dovedit) 1 punct
Prin reducere la absurd, presupunere: $D \cap (X \setminus A) = \varnothing$ (3.7) .... 2 puncte
$(3.7) \implies D \subseteq A$ (3.8) .... 1 punct
$(3.6) + (3.8) \implies X \setminus A \subseteq X \setminus D \implies X = \overline{X \setminus A} \subseteq \overline{X \setminus D} = X \setminus D$ (3.9) .... 3 puncte
$(3.9) \implies D = \varnothing$, în contradicţie cu (3.6) (3.10) .... 1 punct
$(3.10) \implies$ concluzia: $D \cap (X \setminus A) \neq \varnothing, \forall D \in \tau,\ D \neq \varnothing$ .... 1 punct
Total: 10 puncte

15 puncte - subiectul 4

$v)$ Abordarea chestiunii .... 1 punct
$\forall \lambda =$ valoare proprie a lui $T \in \mathcal{L}(\mathbb{R}^3) \Leftrightarrow \exists u \in \mathbb{R}^3 \setminus \{\theta_{\mathbb{R}^3}\},\ T(u) = \lambda u$ (4.1) .... 1 punct
Prin reducere la absurd, presupunere: $\lambda = 0$ (4.2) .... 1 punct
$(4.1) + (4.2) \implies T(u) = \theta_{\mathbb{R}^3}$ (4.3) .... 1 punct
$(4.3) \implies T^*(T(u)) = \theta_{\mathbb{R}^3} \overset{(T^* \circ T = I)}{\implies} u = \theta_{\mathbb{R}^3}$, în contradicţie cu $u \in \mathbb{R}^3 \setminus \{\theta_{\mathbb{R}^3}\}$ (4.4) .... 2 puncte
$(4.4) \implies$ concluzia: $\lambda \neq 0$ (4.5) .... 1 punct
Total: 7 puncte

$vv)$ Abordarea chestiunii .... 1 punct
$\forall u \in Ker(T - \lambda I) \setminus \{\theta_{\mathbb{R}^3}\} \implies T(u) = \lambda u$ (4.6) .... 1 punct

$(4.6) \implies T^*(T(u)) = \lambda T^*(u)$ (4.7) .... 2 puncte
$(4.7) \overset{(T^* \circ T = I)}{\implies} u = \lambda T^*(u)$ (4.8) .... 2 puncte
$(4.8) \overset{(4.5)}{\implies} T^*(u) = \frac{1}{\lambda} u$ (4.9) .... 1 punct
$(4.9) \implies$ concluzia: $u \in Ker(T^* - \frac{1}{\lambda} I) \setminus \{\theta_{\mathbb{R}^3}\}$ .... 1 punct
Total: 8 puncte

Precizări: a) Sunt luate în consideraţie, punctându-se în mod echivalent, şi alte soluţionări decât cele sugerate de prezentul barem.
b) Nota acordată întregii teme se stabileşte prin împărţirea la 10 a punctajului total obţinut.

F. Iacob / 23.11.2016