# BAREM DE CORECTARE PENTRU TS4 / I-A

## Subiectul 1

Abordarea subiectului, cu excluderea reproducerii enunțului ........ 1p

Formularea corectă a problemei de extrem condiționat:

$$\min\left\{\sqrt{x^2+(y-1)^2} \,/\, (x,y)\in\mathbb{R}^2,\ x^2=4y\right\}(=\alpha) \quad (1) \text{ ------ } 2p$$

Problema mai comod de tratat:

$$\min\left\{x^2+(y-1)^2 \,/\, (x,y)\in\mathbb{R}^2,\ x^2=4y\right\}(=\beta) \quad (2) \text{ ------ } 1p$$

Legătura dintre $\alpha$ și $\beta$ : $\alpha=\sqrt{\beta}$ ------ (3) ------ 1p

$f(x,y)=x^2+(y-1)^2$, $g(x,y)=x^2-4y \Rightarrow$ Lagrangeanul în cauză:

$$L(x,y;\lambda)\equiv f(x,y)+\lambda g(x,y)=x^2+(y-1)^2+\lambda(x^2-4y),\ \forall (x,y)\in\mathbb{R}^2,\ \forall\lambda\in\mathbb{R} \quad (4) \text{ -- } 2p$$

Problema de extrem necondiționat corespunzătoare problemei (2):

$$\min\left\{L(x,y;\lambda) \,/\, (x,y,\lambda)\in\mathbb{R}^3\right\} \text{ --- } (5) \text{ ------ } 1p$$

Sistemul punctelor critice pentru (5):

$$\begin{cases}\dfrac{\partial L}{\partial x}(x,y;\lambda)\equiv 2x(1+\lambda)=0,\\ \dfrac{\partial L}{\partial y}(x,y;\lambda)\equiv 2(y-1-2\lambda)=0,\\ \dfrac{\partial L}{\partial \lambda}(x,y;\lambda)\equiv x^2-4y=0,\ (x,y,\lambda)\in\mathbb{R}^3\end{cases} \quad (6) \text{ ------ } 1p$$

Soluția sistemului (6) : $x\equiv x_0=0,\ y\equiv y_0=0,\ \lambda\equiv\lambda_0=-\frac{1}{2}$ ... (7) ------ 2p

$$L(x,y;\lambda_0)\overset{not.}{=}L_0(x,y)\equiv\tfrac{1}{2}x^2+y^2+1,\ \forall (x,y)\in\mathbb{R}^2 \text{ --- } (8) \text{ ------ } 1p$$

$$(8)\Rightarrow d^2L_0(x_0,y_0)=\left\langle H_{L_0}(x_0,y_0)\begin{pmatrix}dx\\dy\end{pmatrix},\begin{pmatrix}dx\\dy\end{pmatrix}\right\rangle_c=\left\langle\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{pmatrix}dx\\dy\end{pmatrix},\begin{pmatrix}dx\\dy\end{pmatrix}\right\rangle_c=(dx)^2+2(dy)^2 \text{ -- } (9) \text{ -- } 3p$$

$$g(x,y)=0\Rightarrow dg(x_0,y_0)\equiv 2x_0dx-4dy=-4dy=0 \text{ -- } (10) \text{ ------ } 1p$$

$$(9)+(10)\Rightarrow d^2L_0(x_0,y_0)\Big|_{dg(x_0,y_0)=0}=(dx)^2>0 \text{ -- } (11) \text{ ------ } 1p$$

$(11)\Rightarrow (x_0,y_0)=(0,0)$ - punct de minim pentru (2) --- (12) ------ 1p

$(12)\Rightarrow \beta=1 \xrightarrow{(3)} \alpha=1$ --- (13) ------ 1p

$(13)\Rightarrow$ Concluzia: distanța căutată este egală cu 1 ------ 1p

Total: 20 de puncte

Subiectul 2

Abordarea, fără reproducerea enunțului: ______ 1p

Utilizarea seriei lui Maclaurin (valabilitatea acestei utilizări): (1) ______ 1p

$(1) \Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \ \forall x \in V \in \mathcal{V}(0)$ ______ (2) 2p

$x_1 = \frac{-1-\sqrt{5}}{2},\ x_2 = \frac{-1+\sqrt{5}}{2} \Rightarrow \frac{2x+1}{1-x-x^2} = \frac{1}{x_2-x_1}\left(\frac{2x_1+1}{x-x_1} - \frac{2x_2+1}{x-x_2}\right), \ \forall x \in \mathbb{R} \setminus \{x_1, x_2\}$ (3) 2p

$(3) \Rightarrow f(x) = \frac{\sqrt{5}}{5}\left(\frac{2x_1+1}{x-x_1} - \frac{2x_2+1}{x-x_2}\right), \ \forall x \in \mathbb{R} \setminus \{x_1, x_2\}$ ______ (4) 1p

$\left(\frac{1}{x-x_k}\right)^{(n)} = \frac{(-1)^n n!}{(x-x_k)^{n+1}}, \ \forall x \in \mathbb{R} \setminus \{x_k\},\ n \in \mathbb{N},\ k=1,2$ ______ (5) 2p

$(4)+(5) \Rightarrow f^{(n)}(x) = \frac{\sqrt{5}(-1)^n n!}{5}\left(\frac{2x_1+1}{(x-x_1)^{n+1}} - \frac{2x_2+1}{(x-x_2)^{n+1}}\right), \ \forall x \in \mathbb{R} \setminus \{x_1, x_2\},\ n \in \mathbb{N}$ ______ (6) 1p

$(6) \Rightarrow f^{(n)}(0) = \frac{\sqrt{5}(-1)^n n!}{5}\left(\frac{2x_1+1}{(-1)^{n+1} x_1^{n+1}} - \frac{2x_2+1}{(-1)^{n+1} x_2^{n+1}}\right) = \frac{\sqrt{5}\, n!}{5}\left(\frac{2x_2+1}{x_2^{n+1}} - \frac{2x_1+1}{x_1^{n+1}}\right), \ \forall n \in \mathbb{N}$ (7) 1p

$(7) \Rightarrow \frac{f^{(n)}(0)}{n!} = \frac{\sqrt{5}}{5}\left(\frac{\sqrt{5}}{(\sqrt{5}-1)^{n+1}} + \frac{\sqrt{5}}{(\sqrt{5}+1)^{n+1}}(-1)^{n+1}\right)2^{n+1}, \ \forall n \in \mathbb{N}$ ______ (8) 1p

$(8) \Rightarrow \frac{f^{(n)}(0)}{n!} = \frac{1}{2^{n+1}}\left[(\sqrt{5}+1)^{n+1} + (1-\sqrt{5})^{n+1}\right], \ \forall n \in \mathbb{N}$ ______ (9) 1p

$(1)+(9) \Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}\left[(\sqrt{5}+1)^{n+1} + (1-\sqrt{5})^{n+1}\right] x^n, \ \forall x \in V$ (10) 1p

$\exists\, \ell = \lim_{n\to\infty} \frac{\frac{1}{2^{n+2}}\left|(1+\sqrt{5})^{n+2} + (1-\sqrt{5})^{n+2}\right|}{\frac{1}{2^{n+1}}\left|(\sqrt{5}+1)^{n+1} + (1-\sqrt{5})^{n+1}\right|} = \frac{\sqrt{5}+1}{2}$ ______ (11) 1p

$(11) \Rightarrow \exists\, r = \frac{1}{\ell} = \frac{\sqrt{5}-1}{2}$ = raza de convergență a seriei de puteri (10) ______ (12) 1p

$\pm r \notin A_{cp}$ (mulțimea de convergență punctuală a seriei (10)) ______ (13) 1p

$(12)+(13) \overset{(2)}{\Rightarrow} V = A_{cp} = (-r, r) = \left(\frac{1-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}\right)$ ______ (14) ______ 2p

$(10)+(14) \Rightarrow$ Concluzia: $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}\left[(\sqrt{5}+1)^{n+1} + (1-\sqrt{5})^{n+1}\right] x^n, \ \forall x \in \left(\frac{1-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}\right)$ ... 1p

Subiectul 3

Abordarea efectivă a subiectului ............ 1p

$\int_0^\infty e^{-tx}\frac{\sin x}{x}dx$ : integrală improprie, cu parametru $(t\in\mathbb{R}_+)$ (1) .. 1p

$\exists \lim_{\substack{x\to 0\\ x>0}}\left(e^{-tx}\frac{\sin x}{x}\right)=1 \Rightarrow$ integrala nu este improprie de speța a II-a (2) .. 1p

$\exists \lim_{x\to\infty}\left(e^{-tx}\frac{|\sin x|}{x}\right)=0 \Rightarrow$ integrala improprie (de speța I-a) are criteriul necesar de convergență îndeplinit ......... 1p

Aplicarea criteriului în $\beta \Rightarrow \exists \beta>1$ (de exemplu, $\beta=2$), astfel încât $\exists \lim_{x\to\infty}\left(x^\beta e^{-tx}\frac{|\sin x|}{x}\right)=0\in[0,+\infty) \Rightarrow$ integrala converge (3) . 2p

$(3)\Rightarrow \exists F(t)=\int_0^\infty e^{-tx}\frac{\sin x}{x}dx,\ \forall t\in[0,\infty)\equiv\mathbb{R}_+$ (4) ........ 1p

$(4)\Rightarrow F(t)=\lim_{b\to\infty}\int_0^b e^{-tx}\frac{\sin x}{x}dx,\ \forall t\in\mathbb{R}_+$ ... (5) ........ 1p

($\exists \int_0^b e^{-tx}\frac{\sin x}{x}dx,\ \forall b\in\mathbb{R}_+^*$, deoarece $e^{-tx}\frac{\sin x}{x}$ este continuă în raport cu $x$, $\forall t\in\mathbb{R}_+$, pe $(0,b]$.) (*) ........ 1p

$\frac{d}{dt}\left(\int_0^b e^{-tx}\frac{\sin x}{x}dx\right)=-\int_0^b e^{-tx}\sin x\,dx=-\frac{1}{1+t^2}+\frac{te^{-tb}\sin b}{1+t^2}+\frac{e^{-bt}\cos b}{1+t^2}$ (6) ... 3p

$(5)+(6)\Rightarrow \exists F'(t)=-\frac{1}{1+t^2},\ \forall t\in\mathbb{R}_+$ ........ (7) – 1p

$(7)\Rightarrow F(t)-F(0)=-\operatorname{arctg} t,\ \forall t\in\mathbb{R}_+$ ........ (8) – 2p

$F(0)=\int_0^\infty\frac{\sin x}{x}dx=\frac{\pi}{2}$ ........ (9) .. 1p

(Justificare pentru (9) : $\int_0^\infty\left(\int_0^\infty e^{-tx}\sin x\,dx\right)dt=\int_0^\infty\left(\int_0^\infty e^{-tx}dt\right)\sin x\,dx\Rightarrow$
$\Rightarrow\int_0^\infty\frac{1}{1+t^2}dt=\int_0^\infty\frac{\sin x}{x}dx\Rightarrow\frac{\pi}{2}=\operatorname{arctg} t\Big|_0^\infty=\int_0^\infty\frac{\sin x}{x}dx$) ........ 2p

$(10) \Rightarrow \int_0^\infty x^{-1} e^{-x} \sin x \, dx = F(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ ............ 1p

Total: 20 de puncte

## Subiectul 4

Abordarea subiectului ............ 1p

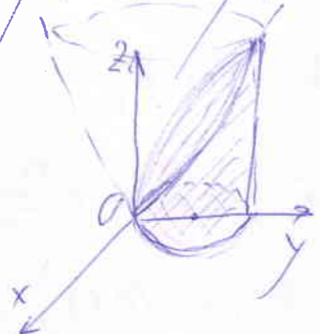
Formula: $V = \iiint_A dx\,dy\,dz$ ... (1) ............ 2p

$A = \{(x,y,z) \in \mathbb{R}^3 \,/\, x^2 + y^2 - 2y \leq 0,\ 0 \leq z \leq \sqrt{x^2+y^2}\}$ .. (2) ... 3p

$B = \{(x,y) \in \mathbb{R}^2 \,/\, x^2+y^2-2y \leq 0\} = \{(x,y) \in \mathbb{R}^2 \,/\, x^2 + (y-1)^2 \leq 1\} =$

$=$ discul de rază 1 și cu centrul în punctul $(0,1)$ . (3) ... 2p

Reprezentarea grafică a corpului A: ............ 3p




$(1)+(2)+(3) \Rightarrow V = \iint_B \left( \int_0^{\sqrt{x^2+y^2}} dz \right) dx\,dy = \iint_B \sqrt{x^2+y^2}\, dx\,dy = \iint_{x^2+(y-1)^2 \leq 1} \sqrt{x^2+y^2}\, dx\,dy$ (4) .. 2p

$(x,y) \xrightarrow[y = r\sin\alpha]{x = r\cos\alpha} (r,\alpha) \Rightarrow B \rightsquigarrow \widetilde{B} = \{(r,\alpha) \in \mathbb{R}_+ \times [0, 2\pi] \,/\, r \leq 2\sin\alpha\}$ .. (5) ... 2p

$(5) \overset{(4)}{\Rightarrow} V = \int_0^{\pi/2} \int_0^{2\sin\alpha} \sqrt{r^2 \underbrace{(\sin^2\alpha + \cos^2\alpha)}_{1}}\, r\,dr\,d\alpha = \int_0^{\pi/2} \left( \int_0^{2\sin\alpha} r^2 dr \right) d\alpha = \frac{1}{3} \int_0^{\pi/2} 8 \sin^3\alpha\, d\alpha =$

$= \frac{8}{3} \int_0^{\pi/2} \sin^3\alpha\, d\alpha = \frac{8}{3} \int_0^{\pi/2} (1 - \cos^2\alpha)\, d(-\cos\alpha) = \frac{8}{3} \left( \frac{\cos^4\alpha}{4} \Big|_0^{\pi/2} - \frac{\cos^2\alpha}{2} \Big|_0^{\pi/2} \right) =$

$= \frac{8}{3} \left( -\frac{1}{4} + \frac{1}{2} \right) = \frac{2}{3}$ ............ (6) 4p