

An introduction to graph analysis and modeling

The Stochastic block model

MSc in Statistics for Smart Data – ENSAI

Autumn semester, 2018

<https://github.com/jchiquet/CourseStatNetwork>



Motivations

Last time: find an underlying organization in a observed network

Spectral or hierachical clustering for network data

~> Not model-based, thus no statistical inference possible

Today: clustering of network based on a probabilistic model of the graph

Become familiar with

- the stochastic block model, a random graph model tailored for clustering vertices,
- the variational EM algorithm used to infer SBM from network data.

hierarchical clustering \leftrightarrow Gaussian mixture models



hierarchical/spectral clustering for network \leftrightarrow Stochastic block model

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Outline

① Background: mixture models and EM

- Mixture models

- Expectation-Maximization algorithm

② The Stochastic Block Model (SBM)

- Some Graphs Models and their limitations

- Mixture of Erdős-Rényi and the SBM

- Inference in SBM with variational EM

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References



Pattern recognition and machine learning,
Christopher Bishop

Chapter 9: Mixture Models and EM

<http://users.isr.ist.utl.pt/~wurmd/Livros/school/>



Models with Hidden Structure with Applications in Biology and
Genomics,

Stéphane Robin

Master MathSV Course

[https:](https://www6.inra.fr/mia-paris/content/download/4587/42934/version/1/file/ModelsHiddenStruct-Biology.pdf)

[//www6.inra.fr/mia-paris/content/download/4587/42934/version/1/file/ModelsHiddenStruct-Biology.pdf](https://www6.inra.fr/mia-paris/content/download/4587/42934/version/1/file/ModelsHiddenStruct-Biology.pdf)



Classification non-supervisées,

É. Lebarbier, T. Mary-Huard

Chapitre 3 - méthode probabiliste: le modèle de mélange

<https://www.agroparistech.fr/IMG/pdf/ClassificationNonSupervisee-AgroParisTech.pdf>

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 - Mixture models
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- 2 The Stochastic Block Model (SBM)

Latent variables models

Definition

A **latent variable model** is a statistical model that relates, for $i = 1, \dots, n$ individuals,

- a set of **manifest** (observed) variables $\mathbf{X} = (X_i, i = 1, \dots, n)$ to
- a set of **latent** (unobserved) variables $\mathbf{Z} = (Z_i, i = 1, \dots, n)$.

Common assumption: conditional independence

$$\mathbb{P}((X_1, \dots, X_n) | (Z_1, \dots, Z_n)) = \prod_{i=1}^n \mathbb{P}(X_i | Z_i).$$

Famous examples

- $(Z_i, i \geq 1)$ is Markov chain: **Markov models**
- Z_i categorical and independent: **mixture models**

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- Z_i categorical and independent: **mixture models**
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Mixture models: the latent variables

When (Z_1, \dots, Z_n) are independent categorical variables, they give a natural (latent) classification of the observations (X_1, \dots, X_n) – or labels.

Notations

Let (Z_1, \dots, Z_n) be *iid* categorical variables with distribution

$$\mathbb{P}(i \in q) = \mathbb{P}(Z_i = q) = \alpha_q, \quad \text{s.t.} \quad \sum_{q=1}^Q \alpha_q = 1.$$

Alternative (equivalent) notation

Let $Z_i = (Z_{i1}, \dots, Z_{iQ})$ be an indicator vector of label for i :

$$\mathbb{P}(i \in q) = \mathbb{P}(Z_{iq} = 1) = \alpha_q, \quad \text{s.t.} \quad \sum_{q=1}^Q \alpha_q = 1.$$

By definition, $Z_i \sim \mathcal{M}(1, \alpha)$, with $\alpha = (\alpha_1, \dots, \alpha_Q)$.

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Mixture models: the manifest variables

A mixture model represents the **presence of subpopulations** within an overall population as follows:

$$\mathbb{P}(X_i) = \sum_{z_i \in \mathcal{Z}_i} \mathbb{P}(X_i, Z_i) = \sum_{Z_i \in \mathcal{Z}_i} \mathbb{P}(X_i | Z_i) \mathbb{P}(Z_i).$$

Conditional distribution of the manifest variables

We assume a **parametric distribution** of X in each subpopulation

$$X_i | \{Z_i = q\} \sim \mathbb{P}_{\theta_q} \quad \left(\Leftrightarrow X_i | \{Z_{iq}\} = 1 \sim \mathbb{P}_{\theta_q} \right)$$

The specificity of each class is handled by $\{\theta_q\}_{q=1}^Q$.

Mixture models: likelihoods

The complete-data likelihood

It is the join distribution of (X_i, Z_i) :

$$\mathbb{P}(X_i, Z_i) = \alpha_{Z_i} \mathbb{P}_{\theta_q}(X_{Z_i})$$

The incomplete-data likelihood

It is the marginal distribution of X_i once Z_i integrated:

$$\mathbb{P}(X_i) = \sum_{q=1}^Q \mathbb{P}(X_i, Z_i = q) = \sum_{q=1}^Q \alpha_q \mathbb{P}_{\theta_q}(X_i)$$

↪ A mixture model is a sum of distributions weighed by the proportion of each subpopulation.

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Intractability of the Likelihood

Maximum Likelihood Estimator

The MLE aims to maximize the (marginal) likelihood of the observations:

$$L(\boldsymbol{\theta}; \mathbf{X}) = \mathbb{P}_{\boldsymbol{\theta}}((X_1, \dots, X_n)) = \int_{\mathbf{Z} \in \mathcal{Z}} \mathbb{P}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{Z}) d\mathbf{Z}$$

Integrations are summation over $\{1, \dots, Q\}$: we have Q^n terms !

Intractable summation

With mixture models, for $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_Q)$ we have

$$\log L(\boldsymbol{\theta}; \mathbf{X}) = \sum_{i=1}^n \log \left\{ \sum_{q=1}^Q \alpha_q \mathbb{P}_{\boldsymbol{\theta}_q}(X_i) \right\}.$$

↪ Direct maximization of the likelihood is impossible in practice

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⇒ Direct maximization of the likelihood is impossible in practice

Bayes decision rule / Maximum *a posteriori*

Principle

Affect an individual i to the subpopulation which is the most likely according to the data:

$$\tau_{iq} = \mathbb{P}(Z_{iq} = 1 | X_i = x_i)$$

This is the **posterior probability** for $i \in q$.

Application of the Bayes Theorem

It is straightforward to show that

$$\tau_{iq} = \frac{\alpha_q \mathbb{P}_{\theta_q}(x_i)}{\sum_{q=1}^Q \alpha_q \mathbb{P}_{\theta_q}(x_i)}$$

Principle of the EM algorithm

If θ were known

...estimating the **posterior probability** $\mathbb{P}(Z_i|\mathbf{X})$ of \mathbf{Z} should be easy

By means of the Bayes decision rule

If \mathbf{Z} were known...

...estimating the **best set of parameter** θ should be easy

This is close to usual maximum likelihood estimation

EM principle

Maximize the marginal likelihood iteratively:

- ① Initialize θ
- ② Compute the probability of \mathbf{Z} given θ
- ③ Get a better θ with the new \mathbf{Z}
- ④ Iterate until convergence

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- 4 Iterate until convergence

Formal algorithm

Initialization: start from a good guess either of \mathbf{Z} or $\boldsymbol{\theta}$, then iterate 1-2

1. Expectation step

Calculate the expected value of the loglikelihood under the current $\boldsymbol{\theta}$

$$Q\left(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)}\right)=\mathbb{E}_{\mathbf{Z}|\mathbf{X};\boldsymbol{\theta}^{(t)}}\left[\log L(\boldsymbol{\theta};\mathbf{X},\mathbf{Z})\right] \quad (\text{needs } \mathbb{P}_{\boldsymbol{\theta}^{(t)}}(\mathbf{Z}|\mathbf{X}))$$

2. Maximization step

Find the parameters that maximize this quantity

$$\boldsymbol{\theta}^{(t+1)}=\arg \max _{\boldsymbol{\theta}} Q\left(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)}\right)$$

Stop when $\left\|\boldsymbol{\theta}^{(t+1)}-\boldsymbol{\theta}^{(t)}\right\|<\varepsilon$ or $\left\|Q^{(t+1)}-Q^{(t)}\right\|<\varepsilon$

(Basic) Convergence analysis

Theorem

At each step of the EM algorithm, the loglikelihood increases. EM thus reaches a local optimum.

Proof.

On board.



Choosing the number of component

Reminder: Bayesian Information Criterion

The BIC is a model selection criterion which penalizes the adjustment to the data by the number of parameter in model \mathcal{M} as follows:

$$\text{BIC}(\mathcal{M}) = \log L(\hat{\boldsymbol{\theta}}; \mathbf{X}) - \frac{1}{2} \log(n) \text{df}(\mathcal{M}).$$

Integrated Classification Criterion

It is an adaptation working with the complete-data likelihood:

$$\begin{aligned} \text{ICL}(\mathcal{M}) &= \log L(\hat{\boldsymbol{\theta}}; \mathbf{X}, \hat{\mathbf{Z}}) + \frac{1}{2} \log(n) \text{df}(\mathcal{M}) \\ &= \text{BIC} - \mathcal{H}(\mathbb{P}(\hat{\mathbf{Z}}|\mathbf{X})), \end{aligned}$$

where the entropy \mathcal{H} measures the separability of the subpopulations.

⇒ We choose $\mathcal{M}(Q)$ that maximizes either BIC or ICL

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Some Graphs Models and their limitations

Mixture of Erdős-Rényi and the SBM

Inference in SBM with variational EM

References



Statistical Analysis of Network Data: Methods and Models

Eric Kolaczyk

Chapters 5 and 6



Mixture model for random graphs, Statistics and Computing

Daudin, Robin, Picard

pbil.univ-lyon1.fr/members/fpicard/franckpicard_fichiers/pdf/DPR08.pdf



Analyse statistique de graphes,

Catherine Matias

Chapitre 4, Section 4

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A mathematical model: Erdős-Rényi graph

Definition

Let $\mathcal{V} = 1, \dots, n$ be a set of fixed vertices. The (simple) Erdős-Rényi model $\mathcal{G}(n, \pi)$ assumes random edges between pairs of nodes with probability π . In other word, the (random) adjacency matrix \mathbf{X} is such that

$$X_{ij} \sim \mathcal{B}(\pi)$$

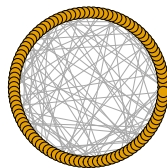
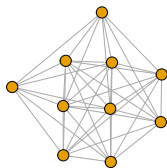
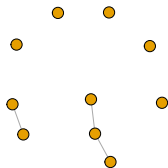
Proposition (degree distribution)

The (random) degree D_i of vertex i follows a binomial distribution:

$$D_i \sim b(n - 1, \pi).$$

Erdős-Rényi - example

```
G1 <- igraph::sample_gnp(10, 0.1)
G2 <- igraph::sample_gnp(10, 0.9)
G3 <- igraph::sample_gnp(100, .02)
par(mfrow=c(1,3))
plot(G1, vertex.label=NA) ; plot(G2, vertex.label=NA)
plot(G3, vertex.label=NA, layout=layout.circle)
```



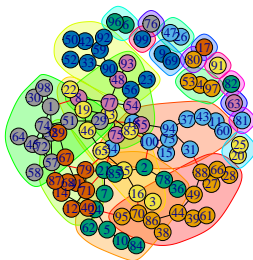
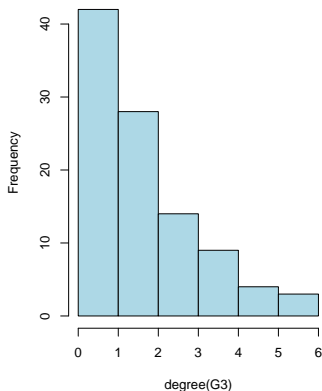
Erdős-Rényy - limitations: very homogeneous

```
average.path.length(G3); diameter(G3)
```

```
## [1] 5.673545
```

```
## [1] 13
```

Histogram of degree(G3)



Mechanism-based model: preferential attachment

The graph is defined dynamically as follows

Definition

Start from a initial graph $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$, then for each time step,

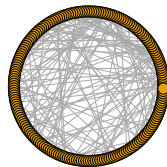
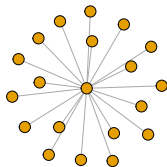
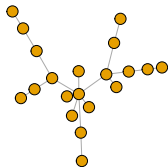
- ① At t a new node V_t is added
- ② V_t is connected to $i \in V_{t-1}$ with probability

$$D_i^\alpha + \text{cst.}$$

\rightsquigarrow Nodes with high degree get more connections thus **richers get richers**

Preferential attachment - example

```
G1 <- igraph::sample_pa(20, 1, directed=FALSE)
G2 <- igraph::sample_pa(20, 5, directed=FALSE)
G3 <- igraph::sample_pa(200, directed=FALSE)
par(mfrow=c(1,3))
plot(G1, vertex.label=NA) ; plot(G2, vertex.label=NA)
plot(G3, vertex.label=NA, layout=layout.circle)
```



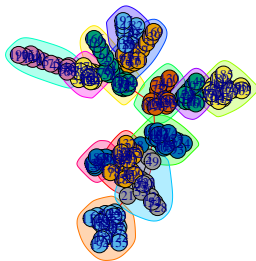
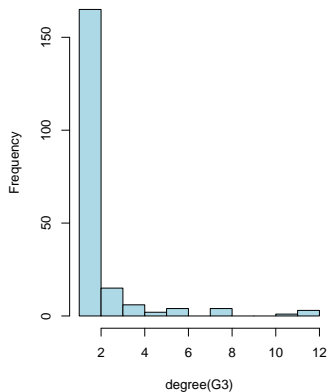
Preferential attachment - limitations

```
average.path.length(G3); diameter(G3)
```

```
## [1] 7.049447
```

```
## [1] 17
```

Histogram of degree(G3)



Limitations

- Erdős-Rényi

The ER model does not fit well real world network

- As can be seen from its degree distribution
- ER is generally too homogeneous

- Preferential attachment

- Is defined through an algorithm so performing statistics is complicated
- Is stucked to the power-law distribution of degrees

The Stochastic Block Model

The SBM¹ generalizes ER in a mixture framework. It provides

- a statistical framework to adjust and interpret the parameters
- a flexible yet simple specification that fits many existing network data

¹Other models exist (e.g. exponential model for random graphs) but less popular.

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Stochastic Block Model: definition

Mixture model point of view: mixture of Erdős-Rényi

Latent structure

Let $\mathcal{V} = \{1, \dots, n\}$ be a fixed set of vertices. We give each $i \in \mathcal{V}$ a **latent label** among a set $\mathcal{Q} = \{1, \dots, Q\}$ such that

- $\alpha_q = \mathbb{P}(i \in q), \quad \sum_q \alpha_q = 1;$
- $Z_{iq} = \mathbf{1}_{\{i \in q\}}$ are independent hidden variables.

The conditional distribution of the edges

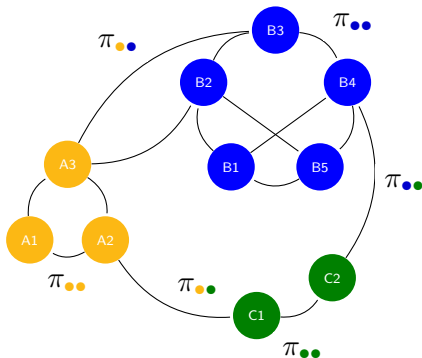
Connexion probabilities depend on the node class belonging:

$$X_{ij} | \{i \in q, j \in \ell\} \sim \mathcal{B}(\pi_{q\ell}) \quad \left(\Leftrightarrow X_{ij} | \{Z_{iq}Z_{j\ell} = 1\} \sim \mathcal{B}(\pi_{q\ell}). \right)$$

The $Q \times Q$ matrix π gives for all couple of labels

$$\pi_{q\ell} = \mathbb{P}(X_{ij} = 1 | i \in q, j \in \ell).$$

Stochastic Block Model: the big picture



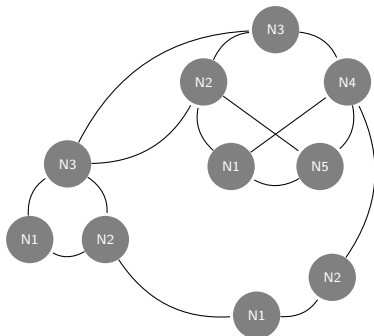
Stochastic Block Model

Let n nodes divided into

- $\mathcal{Q} = \{\bullet, \bullet, \bullet\}$ classes
- $\alpha_{\bullet} = \mathbb{P}(i \in \bullet), \bullet \in \mathcal{Q}, i = 1, \dots, n$
- $\pi_{\bullet\bullet} = \mathbb{P}(i \leftrightarrow j | i \in \bullet, j \in \bullet)$

$$Z_i = \mathbf{1}_{\{i \in \bullet\}} \sim^{\text{iid}} \mathcal{M}(1, \alpha), \quad \forall \bullet \in \mathcal{Q},$$
$$X_{ij} | \{i \in \bullet, j \in \bullet\} \sim^{\text{ind}} \mathcal{B}(\pi_{\bullet\bullet})$$

Stochastic Block Model: unknown parameters



Stochastic Block Model

Let n nodes divided into

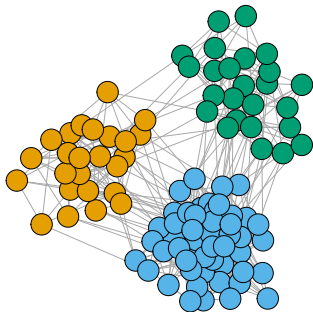
- $\mathcal{Q} = \{\bullet, \bullet, \bullet\}$, $\text{card}(\mathcal{Q})$ known
- $\alpha_{\bullet} = ?$,
- $\pi_{\bullet\bullet} = ?$

$$Z_i = \mathbf{1}_{\{i \in \bullet\}} \sim^{\text{iid}} \mathcal{M}(1, \alpha), \quad \forall \bullet \in \mathcal{Q},$$
$$X_{ij} \mid \{i \in \bullet, j \in \bullet\} \sim^{\text{ind}} \mathcal{B}(\pi_{\bullet\bullet})$$

Stochastic block models – examples of topology

Community network

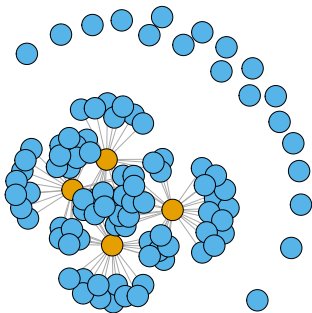
```
pi <- matrix(c(0.3,0.02,0.02,0.02,0.3,0.02,0.02,0.02,0.3),3,3)
communities <- igraph::sample_sbm(100, pi, c(25, 50, 25))
plot(communities, vertex.label=NA, vertex.color = rep(1:3,c(25, 50, 25)))
```



Stochastic block models – examples of topology

Star network

```
pi <- matrix(c(0.05,0.3,0.3,0),2,2)
star <- igraph::sample_sbm(100, pi, c(4, 96))
plot(star, vertex.label=NA, vertex.color = rep(1:2,c(4,96)))
```



Degree distributions

Conditional degree distribution

The conditional degree distribution of a node $i \in q$ is

$$D_i | i \in q \sim \text{b}(n-1, \bar{\pi}) \approx \mathcal{P}(\lambda_q), \quad \bar{\pi}_q = \sum_{\ell=1}^Q \alpha_\ell \pi_{q\ell} \quad \lambda_q = (n-1) \bar{\pi}_q$$

Conditional degree distribution

The degree distribution of a node i can be approximated by a mixture of Poisson distributions:

$$\mathbb{P}(D_i = k) = \sum_{q=1}^Q \alpha_q \exp\{-\lambda_q\} \frac{\lambda_q^k}{k!}$$

Likelihoods

Complete-data loglikelihood

$$\log L(\mathbf{X}, \mathbf{Z}) = \sum_{i,q} Z_{iq} \log \alpha_q + \sum_{i < j, q, \ell} Z_{iq} Z_{j\ell} \log \pi_{q\ell}^{X_{ij}} (1 - \pi_{q\ell})^{1-X_{ij}}.$$

Conditional expectation of the complete-data loglikelihood

$$\mathbb{E}_{\mathbf{Z}|\mathbf{X}}[\log L(\boldsymbol{\theta}; \mathbf{X}, \mathbf{Z})] = \sum_{i,q} \tau_{iq} \log \alpha_q + \sum_{i < j, q, \ell} \eta_{ijq\ell} \log \pi_{q\ell}^{X_{ij}} (1 - \pi_{q\ell})^{1-X_{ij}}$$

where $\tau_{iq}, \eta_{ijq\ell}$ are the posterior probabilities:

- $\tau_{iq} = \mathbb{P}(Z_{iq} = 1 | \mathbf{X}) = \mathbb{E}[Z_{iq} | \mathbf{X}]$.
- $\eta_{ijq\ell} = \mathbb{P}(Z_{iq} Z_{j\ell} = 1 | \mathbf{X}) = \mathbb{E}[Z_{iq} Z_{j\ell} | \mathbf{X}]$.

Outline

① Background: mixture models and EM

② The Stochastic Block Model (SBM)

Some Graphs Models and their limitations

Mixture of Erdős-Rényi and the SBM

Inference in SBM with variational EM

The EM strategy does not apply directly for SBM

Ouch: another intractability problem

- the Z_{iq} are **not independent** in the SBM framework. . .
- we cannot compute $\eta_{ijql} = \mathbb{P}(Z_{iq}Z_{jl} = 1|\mathbf{X}) = \mathbb{E}[Z_{iq}Z_{jl}|\mathbf{X}]$,
- the conditional expectation $Q(\boldsymbol{\theta})$, i.e. the main EM ingredient, is **intractable**.

Solution: mean field approximation

Approximate η_{ijql} by $\tau_{iq}\tau_{jl}$, i.e., **assume independence between Z_{iq}**

\rightsquigarrow This can be formalized in the variational framework

Revisiting the EM algorithm I

Proposition

Consider a distribution \mathbb{Q} for the $\{Z_{iq}\}$. We have

$$\log L(\boldsymbol{\theta}; \mathbf{X}) = \mathbb{E}_{\mathbb{Q}}[\log L(\boldsymbol{\theta}, \mathbf{X}, \mathbf{Z})] + \mathcal{H}(\mathbb{Q}) + \text{KL}(\mathbb{Q} \mid \mathbb{P}(\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta})),$$

where \mathcal{H} is the entropy and $\text{KL}(\cdot|\cdot)$ is the Kullback-Leibler divergence:

$$\mathcal{H}(\mathbb{Q}) = - \sum_z \mathbb{Q}(z) \log \mathbb{Q}(z) = -\mathbb{E}_{\mathbb{Q}}[\log \mathbb{Q}(Z)]$$

$$\mathcal{KL}(\mathbb{Q} \mid \mathbb{P}(\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta})) = \sum_z \mathbb{Q}(z) \log \frac{\mathbb{Q}(z)}{\mathbb{P}(\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta})} = \mathbb{E}_{\mathbb{Q}} \left[\log \frac{\mathbb{Q}(z)}{\mathbb{P}(\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta})} \right]$$

Revisiting the EM algorithm II

Let

$$J(\mathbb{Q}, \boldsymbol{\theta}) \triangleq \mathbb{E}_{\mathbb{Q}} (\log L(\boldsymbol{\theta}; \mathbf{X}, \mathbf{Z})) + \mathcal{H}(\mathbb{Q})$$

The steps in the EM algorithm may be viewed as:

Expectation step : choose \mathbb{Q} to maximize $J(\mathbb{Q}; \boldsymbol{\theta}^{(t)})$

The solution is $\mathbb{P}(\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta}^{(t)})$

Maximization step : choose $\boldsymbol{\theta}$ to maximize $J(\mathbb{Q}^{(t)}; \boldsymbol{\theta})$

The solution maximizes $\mathbb{E}_{\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta}^{(t)}} (\log L(\boldsymbol{\theta}; \mathbf{X}, \mathbf{Z}))$

Variational approximation for SBM

Problem for SBM

$\mathbb{P}(\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta}^{(t)})$ cannot be computed thus the E-step cannot be solved.

Idea

Choose \mathbb{Q} in a class of function so that the E-step can be solved.

Family of distribution that factorizes

We chose \mathbb{Q} so as the Z_{iq} are marginally independents:

$$\mathbb{Q}(\mathbf{Z}) = \prod_{i=1}^n \mathbb{Q}_i(Z_i) = \prod_{i=1}^n \prod_{q=1}^Q \tau_{iq}^{Z_{iq}},$$

where $\tau_{iq} = \mathbb{Q}_i(Z_i = q) = \mathbb{E}Q(Z_{iq})$, with $\sum_q \tau_{iq} = 1$ for all $i = 1, \dots, n$.

Variational EM for SBM: the criterion

Lower bound of the loglikelihood

Since \mathbb{Q} is an approximation of $\mathbb{P}(\mathbf{Z}|\mathbf{X})$, the Kullback-Leibler divergence is non-negative and

$$\log L(\boldsymbol{\theta}; \mathbf{X}) \geq \mathbb{E}_{\mathbb{Q}}[\log L(\boldsymbol{\theta}, \mathbf{X}, \mathbf{Z})] + \mathcal{H}(\mathbb{Q}) = J(\mathbb{Q}, \boldsymbol{\theta}).$$

For the SBM,

$$J(\mathbb{Q}, \boldsymbol{\theta}) = \sum_{i,q} \tau_{iq} \log \alpha_q + \sum_{i < j, q, \ell} \tau_{iq} \tau_{j\ell} \log b(X_{ij}; \pi_{q\ell}) - \sum_{i,q} \tau_{iq} \log(\tau_{iq}),$$

\rightsquigarrow we optimize the loglikelihood lower bound $J(\mathbb{Q}, \boldsymbol{\theta}) = J(\boldsymbol{\tau}, \boldsymbol{\theta})$ in $(\boldsymbol{\tau}, \boldsymbol{\theta})$.

E and M steps for SBM

Variational E-step

Maximizing $J(\boldsymbol{\tau})$ for fixed $\boldsymbol{\theta}$, we find a fixed-point relationship:

$$\hat{\tau}_{iq} \propto \alpha_q \prod_j \prod_{\ell} b(X_{ij}, \pi_{q\ell})^{\hat{\tau}_{j\ell}} \quad (1)$$

M-step

Maximizing $J(\boldsymbol{\theta})$ for fixed $\boldsymbol{\tau}$, we find,

$$\hat{\alpha}_q = \frac{1}{n} \sum_i \hat{\tau}_{iq}, \quad \hat{\pi}_{q\ell} = \frac{\sum_{i \neq j} \hat{\tau}_{iq} \hat{\tau}_{j\ell} X_{ij}}{\sum_{i \neq j} \hat{\tau}_{iq} \hat{\tau}_{j\ell}}. \quad (2)$$

Model selection

We use our lower bound of the loglikelihood to compute an approximation of the ICL

$$\begin{aligned} \text{vICL}(Q) = \mathbb{E}_{\hat{\mathbb{Q}}}[\log L(\hat{\boldsymbol{\theta}}; \mathbf{X}, \mathbf{Z})] \\ - \frac{1}{2} \left(\frac{Q(Q+1)}{2} \log \frac{n(n-1)}{2} + (Q-1) \log(n) \right), \end{aligned}$$

where

$$\mathbb{E}_{\hat{\mathbb{Q}}}[\log L(\hat{\boldsymbol{\theta}}; \mathbf{X}, \mathbf{Z})] = J(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\theta}}) - \mathcal{H}(\hat{\mathbb{Q}}).$$

The variational BIC is just

$$\text{vBIC}(Q) = J(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\theta}}) - \frac{1}{2} \left(\frac{Q(Q+1)}{2} \log \frac{n(n-1)}{2} + (Q-1) \log(n) \right).$$

Example on the French blogosphere I

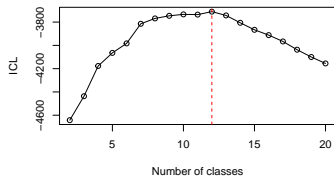
```
library(mixer)
data(blog)
mix.blog <- mixer(x=blog$links,qmin=2,qmax=20)

## Mixer: the adjacency matrix has been transformed in a undirected edge list

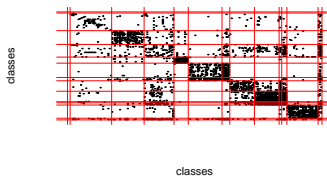
plot(mix.blog)
```

Example on the French blogosphere II

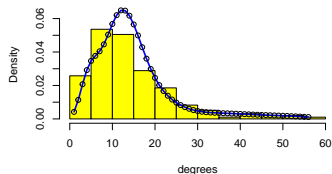
Integrated Classification Likelihood



Reorganized Adjacency matrix



Degree distribution



Inter/intra class probabilities

