Compressive Sensing and Application to MRI

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1 Part 1: Denoising 1D Signals

1.1 Generating the signal

a) The original sparse signal with n = 128 and k = 5 is generated (Figure 1).
b) The original sparse signal x is plotted
c) After that, a random Gaussian noise is added to it (Figure 2).
d) The noisy signal is plotted.

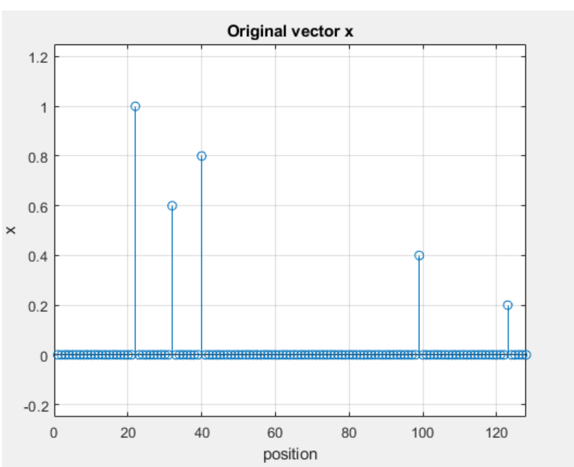

Figure 1: Original signal x

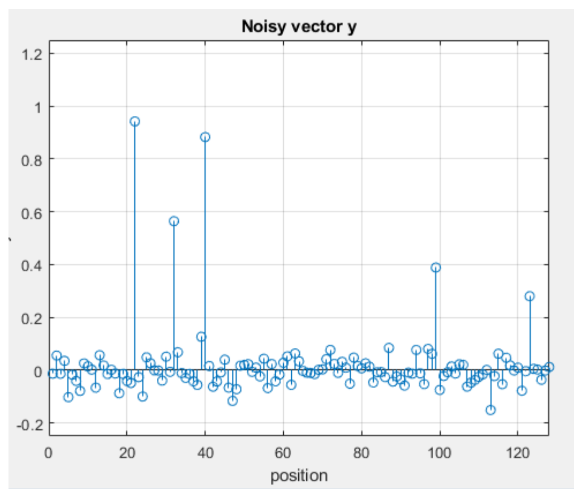

Figure 2: Noisy signal y

e) Regularisation with Tichonov penalty can be used in order to denoise the signal, solving $\lambda \geq 0$.

$$\hat{x} = \underset{z}{\text{argmin}} \frac{1}{2}||x-y||_2^2 + \lambda \frac{1}{2}||z||_2 \tag{1}$$

We are going to show that the solution for this problem is:

$$\hat{x} = \frac{1}{1+\lambda} * y \tag{2}$$

$$\frac{1}{2}(x-y)^2 + \lambda \frac{1}{2} z^2 \tag{3}$$

$$\frac{df}{dz} = z - y + \lambda z = 0 \tag{4}$$

$$z - \lambda z = y \tag{5}$$

$$z(1-\lambda) = y \tag{6}$$

$$z = \frac{1}{1+\lambda} * y \tag{7}$$

f)Compute the estimate using the formula (7) with values of lambda in {0.01, 0.05, 0.1, 0.2}
Is the solution sparse? : As we can see from Figure 3 the solution is NOT sparse - the noise is NOT filtered out.

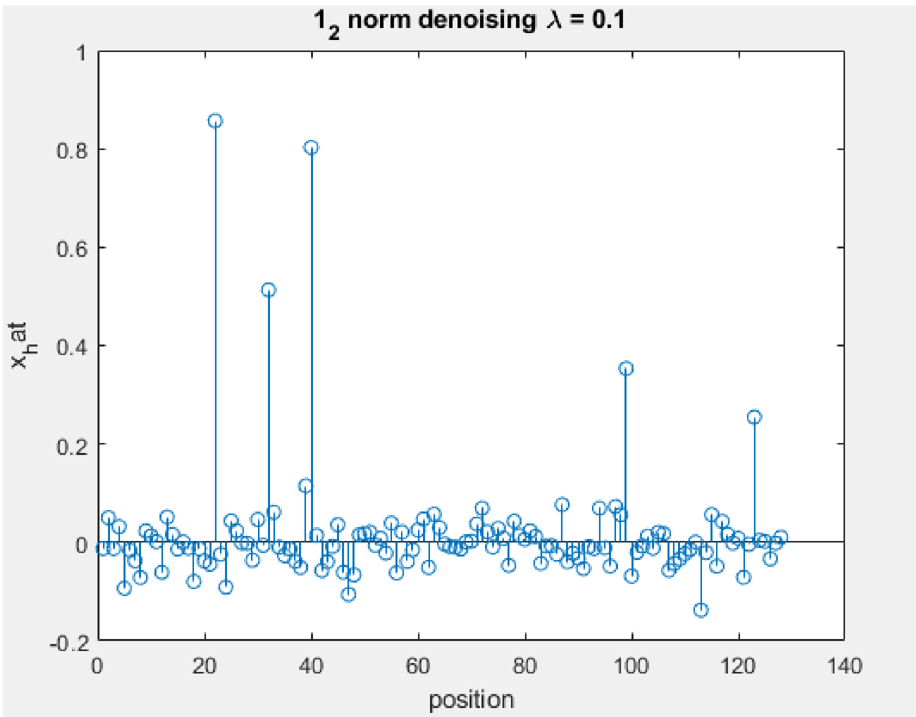

Figure 3: l2 norm denoising

g) Next, we will solve for $\lambda \geq 0$:

$$\hat{x} = \underset{z}{\operatorname{argmin}} \frac{1}{2}||x-y||_2^2 + \lambda \frac{1}{2}||z||_1 \tag{8}$$

For the soft thresholding approach, values for both positive and negative coefficients are being "shrinked" towards zero, in contrary to hard thresholding which either keeps or removes values of coefficients.

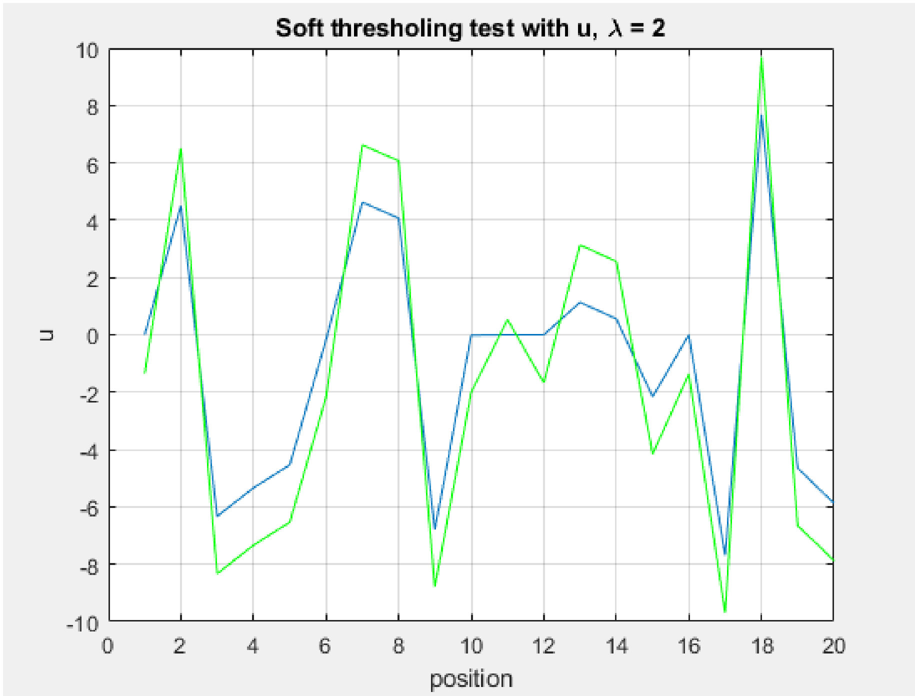

Figure 4: Soft thresholding for u vector. The original signal is green while the blue line represents the recovered signal.

Figure 4 represents the result when applying Soft Thresholding function to a prototype signal containing 20 values between -10 and 10. It can be noticed that the recovered signal (blue line) presents the expected behaviour - it looks slightly shrinked compared to the original (green).

What is the advantage of generating figure title with title([Soft thresholing test with u, lambda = num2str(lambdau)]) command?
In comparison to title(Soft thresholing test with u, lambda = lambda), the chosen approach inserts the numerical value of lambda into the title. The chosen approach is better given the fact that, apart from displaying relevant information in the title of the figure, it also allows much more flexiblity. When the value of the lambda variable is changed, the title changes automatically.

What happens when the value if y is small compared to the value of lambda? What happens when Y is too large compared to the value of lambda?
When Y is small compared to lambda, real signal values are treated as noise, therefore, being tuned down in the reconstruction of the signal => we will recover less from the original signal.

When Y is too large compared to lambda, then Soft Thresholding algorithm recovers the signal while including a lot of noise => the algorithm will not provide a clear reconstruction of the signal.

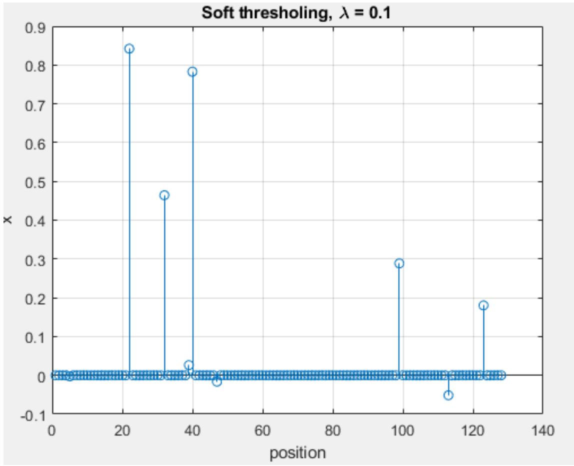

Figure 5: Recovered signal with $\lambda = 0.1$

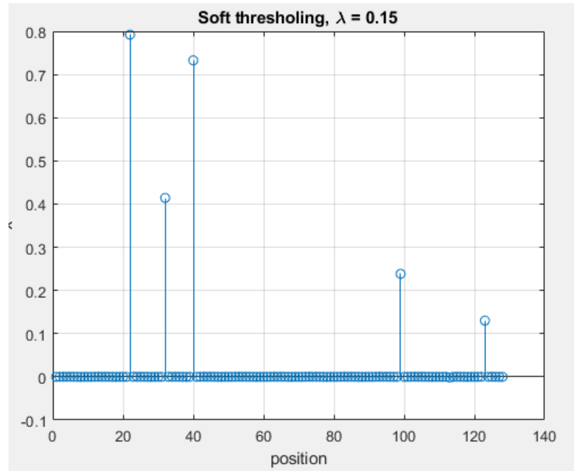

Figure 6: Recovered signal with $\lambda = 0.15$

h) Applying SoftThresholding to the noisy signal y.

Is the solution sparse?

Figure 5 presents the signal recovered with the SoftThresholding algorithm for lambda = 0.1. In our case the recovered signal is almost sparse, while for lambda 0.15 we can notice that the recovered signal is fully sparse. Moreover, all the peaks from the original signal are successfully recovered.

i) **Here, a closed form solution for the estimate is written while in the course there is not closed form expression for the estimate** because in this case, our problem does NOT contain the A dictionary, which was a fat matrix, leading to an undetermined set of equations.In this particular case, solution is straight forward, given by calculating the first derivative in z and finding z values for which the function is minimum.

2 Part 2: Random Frequency Domain Sampling and Aliasing

Now we explore the connection between compressive sensing and denoising
a) First, we generate a sparse vector with n = 128 and k = 5 non zero coefficients. b) Generate graph of the original singal.

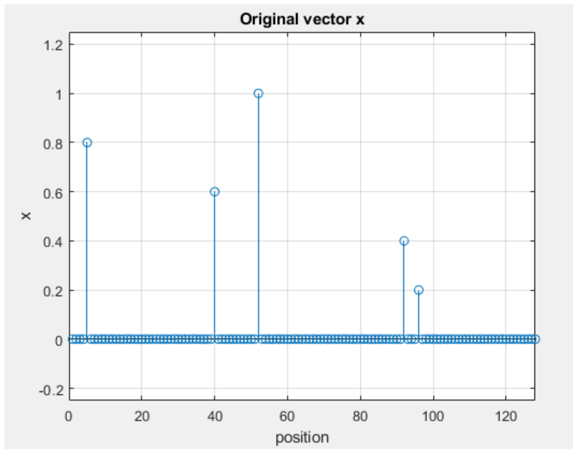

Figure 7: Original sparse signal x.

c) Compute the centered Fourier transform

$$X = Fx \tag{9}$$

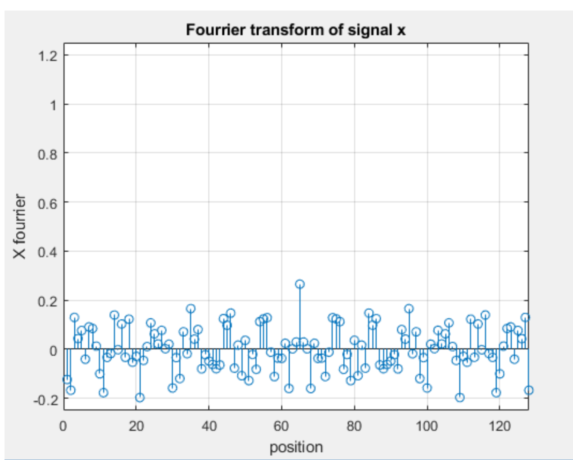

Figure 8: Centered Fourrier transform of original vector x.

d) Compute inverse Fourrier transform of the EQUISPACED undersampling.

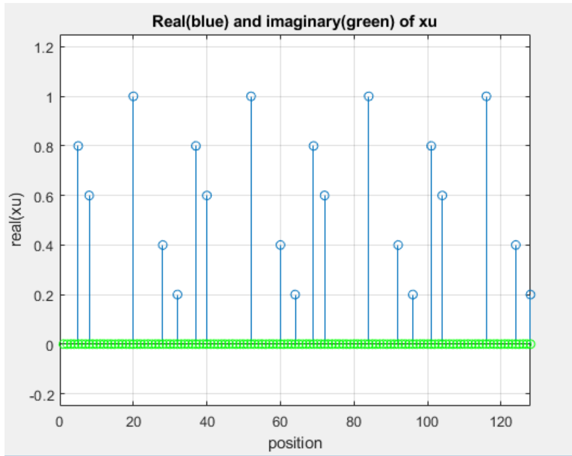

Figure 9: Inverse Fourrier transform of the equispaced undersampling.

e) **This is the minimum l2 solution of the Fourrier Transform?**
Yes, because similarily to the previous part, the inverse fourrier transform is the equivalent of removing columns from your sensing matrix, so we now have more equations than unknowns and therefore no solution. Since we have no solution, the solution is the one that minimises the error.
Will we be able to reconstruct the original signal from the result?
Not in the case of EQUISPACED sampling, because of the priodical natrue of the equispaced unersampling. We don't know the true nature of the non 0 coefficients.

f) Comupte inverse Fourrier transform of the RANDOM undersampling.

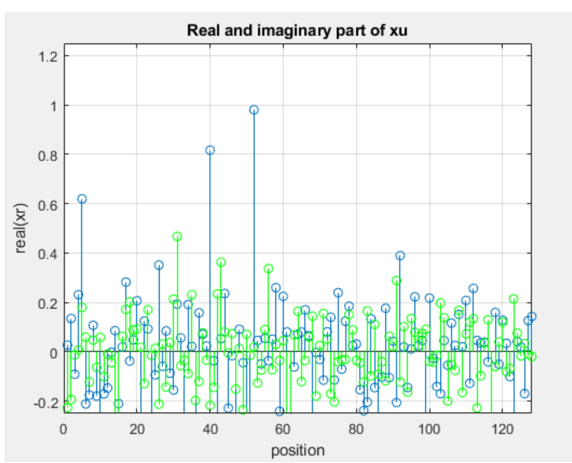

Figure 10: Inverse Fourrier transform of the random undersampling.

g) **Will we be able to reconstruct the signal from the result? How does this resemble the denoising problem?**
Yes, we are able to reconstruct the signal from the result.The solution is the minimum of the

l2, there is no true solution, but it is a solution that minimizes the aliasing errors. Fourrier tranform introduces aliasing , which looks like the Gaussian noise from Part 1, making the problem look like a denoising one.

3 Part 3: Reconstruction from Randomly Sampled Frequency Domain Data

a) First, we generate a sparse vector with n = 128 and k = 5 non zero coefficients. b) Generate graph of the original singal.

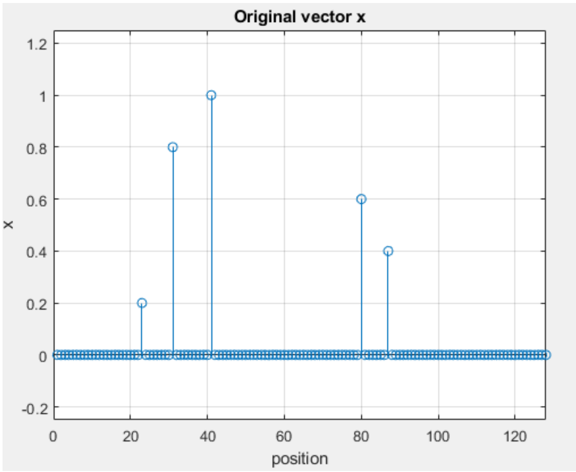

Figure 11: Original signal

c) Compute the Fourrier transform and randomly undersample X (this leads to Y)

$$X = Fx \tag{10}$$

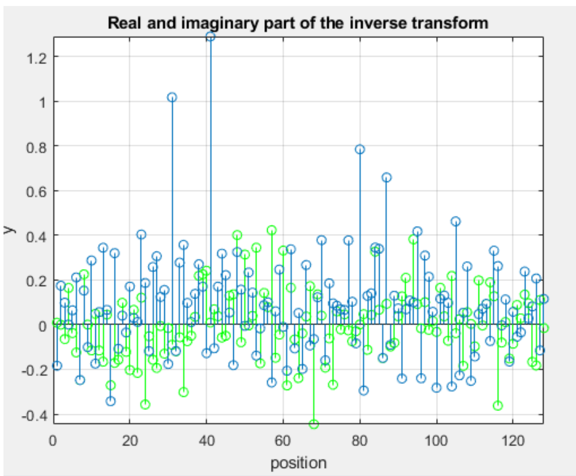

Figure 12: Inverse Fourrier transform of the random undersampling

d) Solving the problem for ($\lambda \geq 0$), similarily to part 1.

$$\hat{x} = \underset{z}{\text{argmin}} \frac{1}{2}||F_u z - y||_2^2 + \lambda \frac{1}{2}||z||_2 \tag{11}$$

e) In order to solve the problem, we implement the POCS algorithm iteratively (100 iterations) for values of lambda in $\{0.01, 0.05, 0.1\}$.

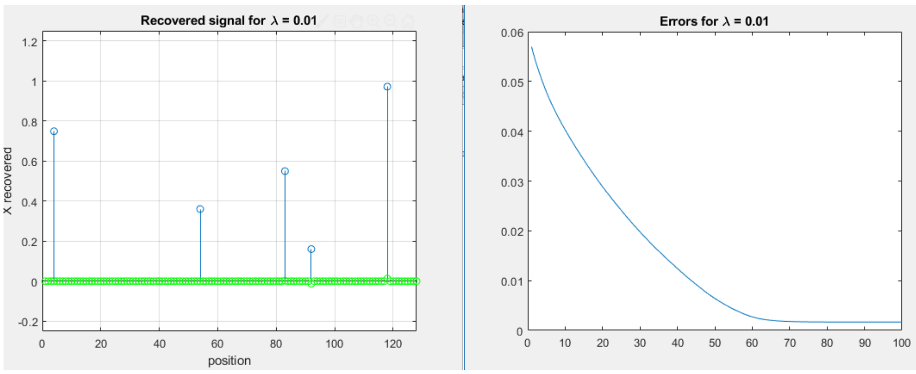

Figure 13

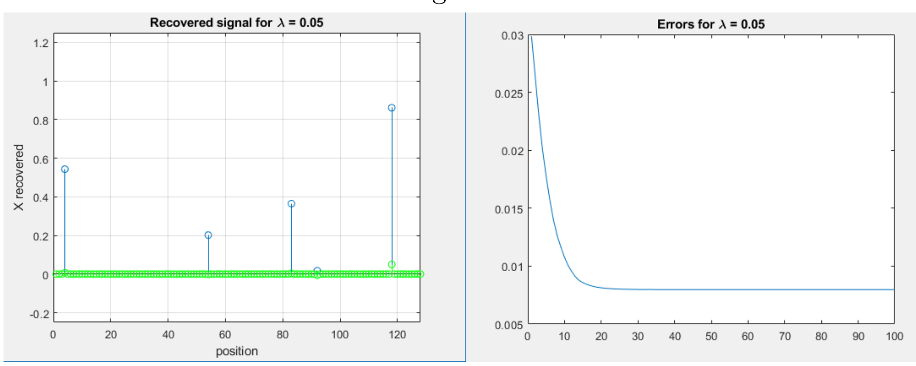

Figure 14

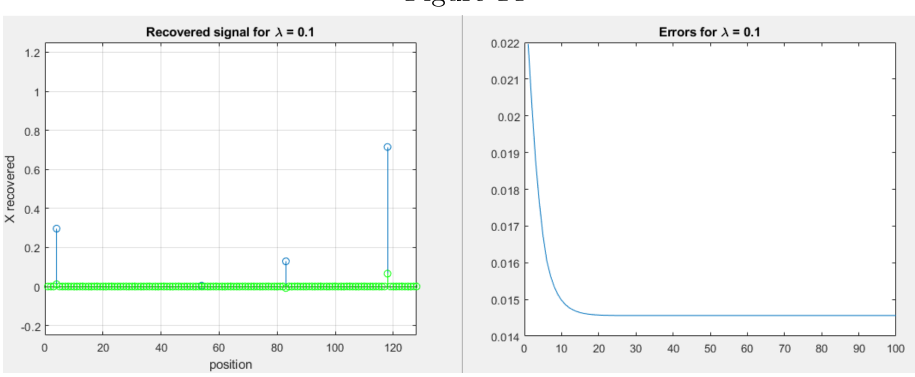

Figure 15

f) **Comment on the errors**

For the lowest values of lambda (0.01) we can notice that the peaks are fully recovered, with an evolution of errors which converges abruptly after 60 iteration to a value below 0.01.

For a value of lambda = 0.05, some of the peaks are tuned down and the value of errors decreases abruptly after 20 iterations to a value almost equal to 0.01.

For a value of lambda = 0.1, we observed that a part of the peaks from the original signal are filtered out, and the errors converge to 0.15 after 20 iterations.

The best value of lambda for the recovery of the signal, which is also refletcted in the evolution of errors is lambda = 0.01.

g) The algorithm is repeated for the equispaced undersampling. Similariy to part 2, we can see that the signal cannot be recovered because of periodicity. We don't know where the true positive coefficients are.

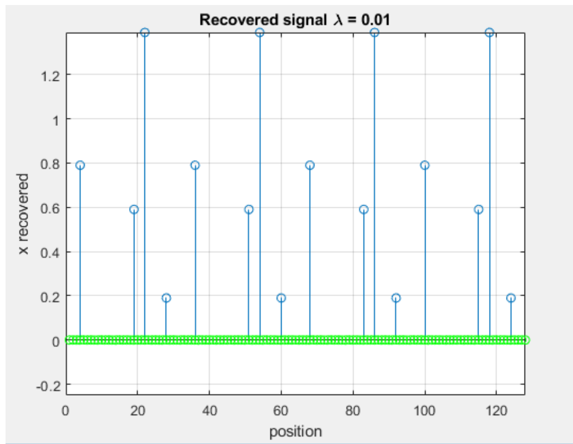

Figure 16: Recovered equispaced signal

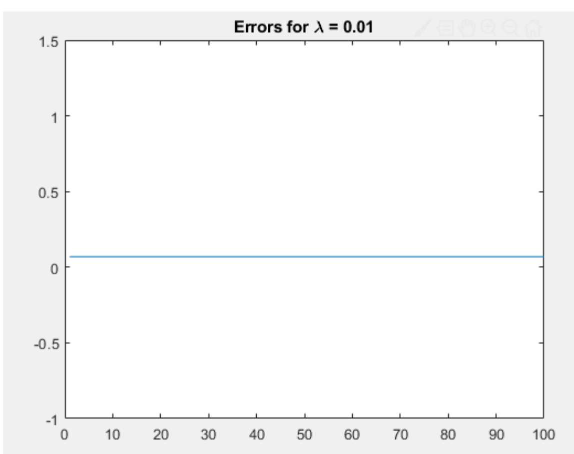

Figure 17

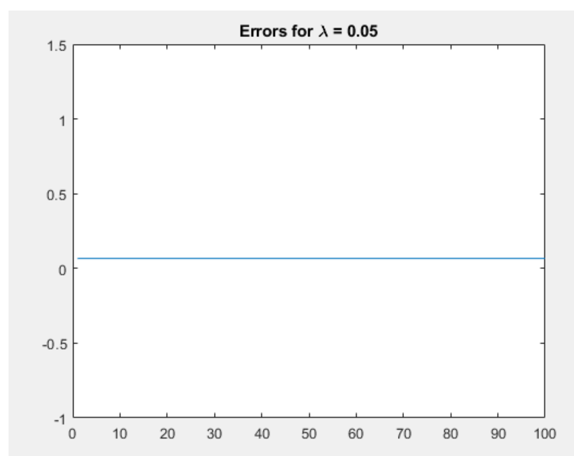

Figure 18

For the equispace undersampling, we notice a strange behavior of errors for lambda for 0.01 and 0.05, both of them being constant over the iterations and almost close to 0. But this is not to say that our recovery is good. This is only due to the periodicity of the equispaced undersampling.

4 Part 4: MRI Image Reconstruction

In this part, the reconstruction algorithm for a Brain MRI is implemented and explored. The reconstruction of an image from a sparse signal is similar to the problem explored in the previous parts, the difference is that we studied 1D sparse signals before, and now we are tackling the 2D problem.
a) Load the MR image.

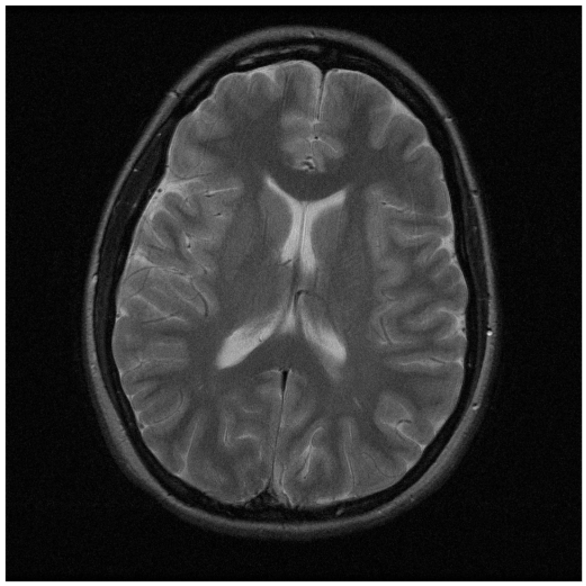

Figure 19: Real Brain MRI

Figure 1 presents the initial Brain Scan that we will try to recover.

4.1 Random sub sampling in the frequency domain

b) Two sub sampling patterns are explored: the Uniform sub sampling (mask unif) and Variable Density Random Sampling (mask vardens).

Figure 21: Upper: output images and the difference between the sample values and the values from the initial signal for Uniform subsampling
Bottom figure: presents the same output but for Variable Density Random Sampling.

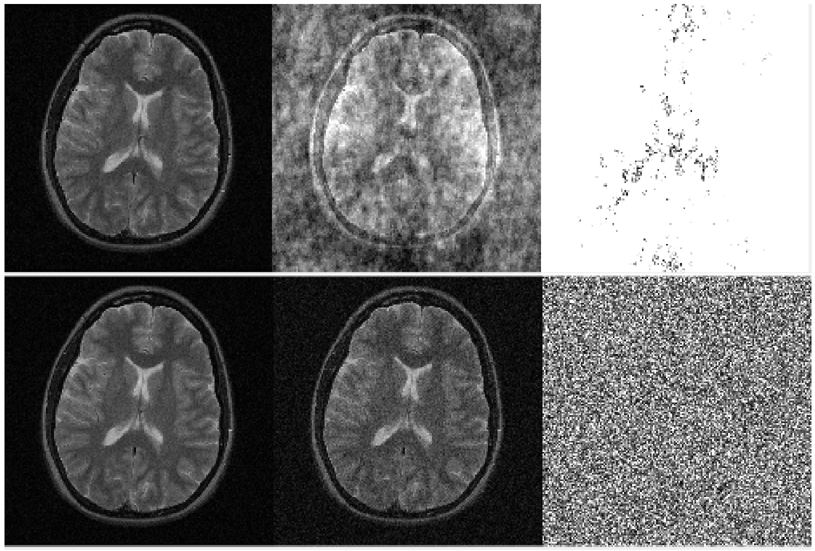

Figure 21: Uniform(upper) and Variable Density Random Sampling (bottom)

d) As it can be noticed from the figures, for Uniform sampling, the aliasing error is not random and the errors are larger in some specific regions of the image, while the recovered images looks blurry (noisy).

For Variable Density Sampling, the aliasing error looks like white noise, and the recovered image (center) is more similar to the original one.

With these facts, we can draw a similar conclusion as in Part 2, where we discovered that random undersampling (in our case Variable Density Sampling) turned the ill condition problem into a sparse signal denoising problem, enabling us to eventually recover the exact sparse signal.

What would you reccomend? As a conclusion, given all the facts presented above, the Vardens sampling is recommended.

4.2 MRI in the Wavelet domain

e) By applying the Wavelet transform on the MR image, we notice that in the wavelet domain the image is even sparser.

f) In order to reconstruct the image, we need to solve the following l1 problem:

$$\hat{x} = \underset{z}{\operatorname{argmin}} \frac{1}{2}||F_{vs} * im - im_{vs}||_2^2 + \lambda \frac{1}{2}||W * im||_1 \tag{12}$$

z = image estimate
Fvs = Fourrier Transform for the Vardens Undersampling
im vs = inverse of the fourrier transform taking into account variable density random sampling.
We are looking for a z that is very similar to our image.
In practice, we will perform the Soft Thresholding problem in the Wavelet domain and then apply the inverse Wavelet transform.

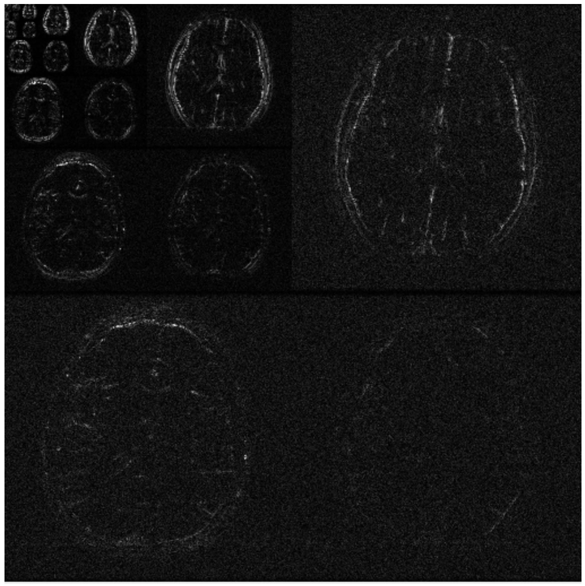

Figure 22: MR image in the Wavelet domain

Before doing that, we want to get an intuition of a good value for lambda by vizualising the coefficients that are kept after thresholding.

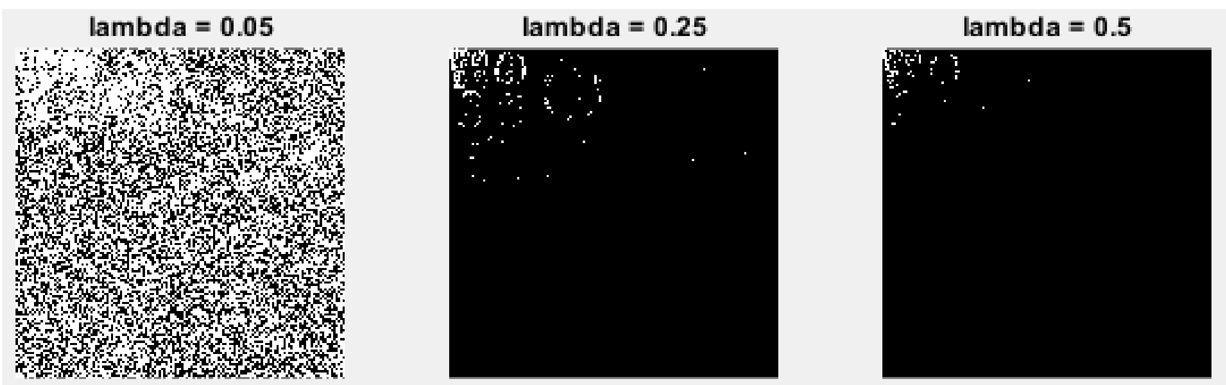

Figure 23: Coefficients kept for different values of lambda

We notice that the larger the value of lambda for thresholding, the more coefficients form the signal are set to 0, which can be noticed in the right image of Figure 5, where lambda is equal to 0.5. By taking a value for lambda that is too large, we risk treating the relevant information as noise, therefore throwing away valuable information.

On the other hand, taking a lambda value that is too low leads to taking into account coefficients that are actually noise. From trying multiple values for lambda, we've concluded that a value for lambda equal to 0.25 gives us the best trade off in terms of keeping relevant information.

4.3 Implementing POCS algorithm

By implementing the POCS algorithm for Uniform Undersampling, we notice that for a value of lambda = 0.25, the error starts at a value of 0.141 and keeps decreasing to 0.1375 after 20 iterations. However, as forseen, the recovered image is noisy.

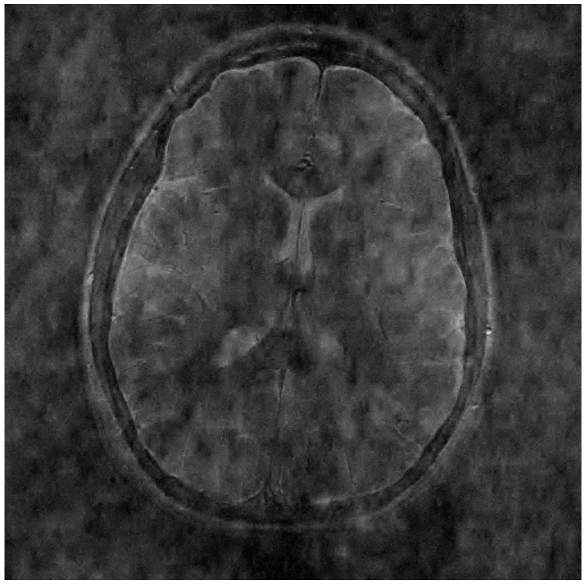

Figure 24: POCS recovered image for Uniform undersampling

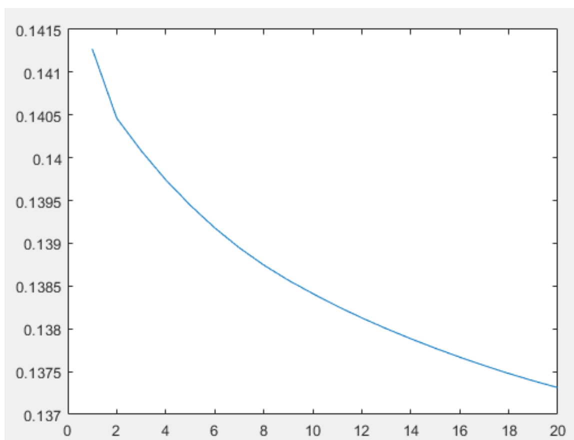

Figure 25: POCS errors for Uniform undersampling

By implementing the POCS algorithm for Vardens Undersampling, we notice that for a value of lambda = 0.25, the error starts at a value of 0.0291, which is better than Uniform, and converges to to 0.296 after 6 iterations, and the recovered image is close to the original. However, by choosing a smaller lambda value = 0.025, we notice a different behaviour in error evolution, which is presented in Figure 28. For lambda = 0.025, the errors start at 0.3 and converge to 0.27 after 6 iterations (there are no visible changes in the recovered MRI).

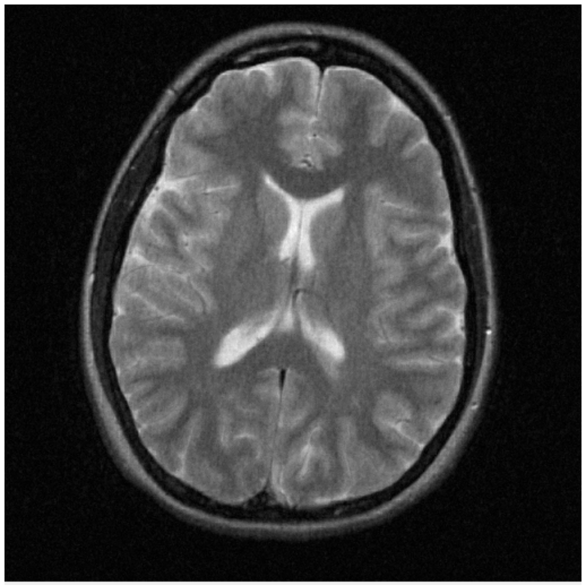

Figure 26: POCS recovered image for Vardens undersampling

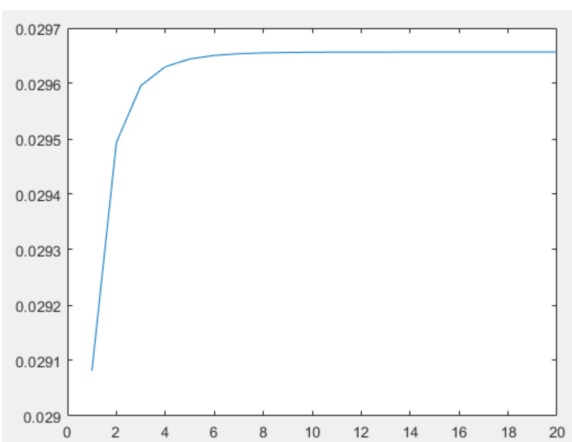

Figure 27: POCS errors for Vardens undersampling, lambda = 0.25

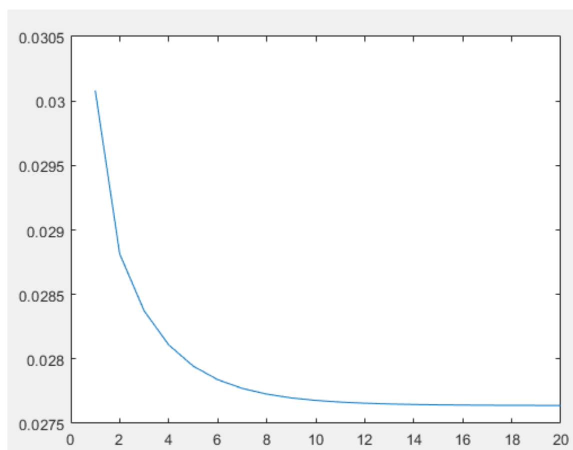

Figure 28: POCS errors for Vardens undersampling, lambda = 0.025

As a conclusion, compressive sensing MRI presents the same challanges as the problem presented in Part 3. Random sampling corresponding in this case to Vardens sampling is still the best solution, and a smaller value of lambda for the POCS algorithm leads to a better evoultion of errors over iterations.