Monads leave traces

Ana Sokolova

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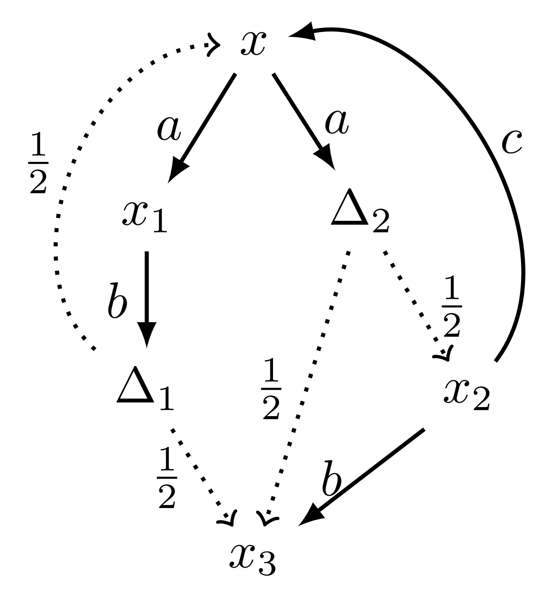

FAU Erlangen 8.2.22

What is a trace ?

I will talk about:

1. The absolute basics of coalgebra

2. Approaches for trace semantics

3. …and an observation :-)

I will talk about:

Mathematical framework based on category theory for state-based systems semantics

1. The absolute basics of coalgebra

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1. The absolute basics of coalgebra

for nondeterministic/ probabilistic systems

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I will talk about:

Mathematical framework based on category theory for state-based systems semantics

1. The absolute basics of coalgebra

for nondeterministic/ probabilistic systems

systems with algebraic effects

2. Approaches for trace semantics

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Mathematical framework based on category theory for state-based systems semantics

for nondeterministic/ probabilistic systems

systems with algebraic effects

the crucial role of monads

I will talk about:

1. The absolute basics of coalgebra

2. Approaches for trace semantics

3. …and an observation :-)

Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

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$$X \xrightarrow{c} FX$$

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states

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object in the base category ***C***

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Functor on **Sets**

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form a category too

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type

Functor on **Sets**

form a category too

generic notion of behavioural equivalence
(bisimilarity)
$\approx$

Examples

LTS

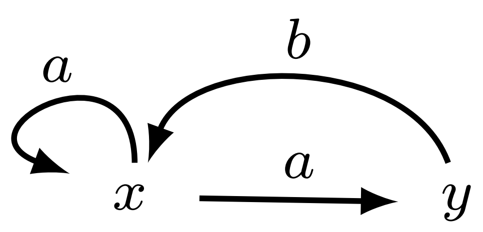
$X \to (\mathcal{P}X)^A \cong \mathcal{P}(A \times X)$

$$\begin{tikzcd} x \arrow[loop left, "a"] \arrow[r, "a"] & y \arrow[l, bend right, "b"'] \end{tikzcd}$$

Examples

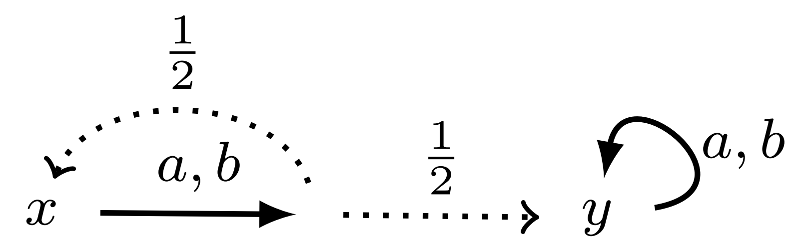
LTS

$X \to (\mathcal{P}X)^A \cong \mathcal{P}(A \times X)$



PTS

$X \to (\mathcal{D}X)^A$



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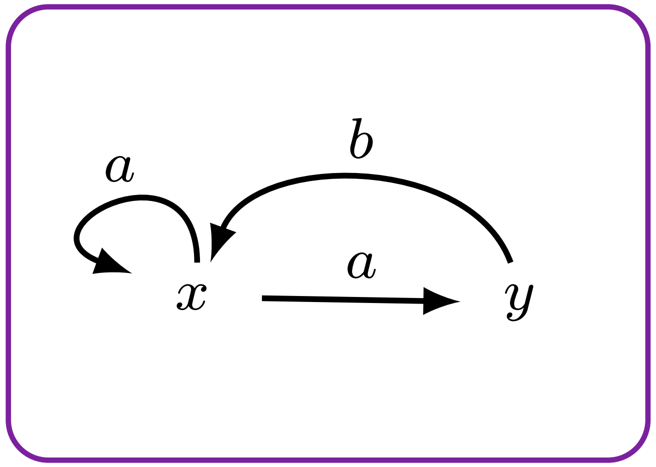

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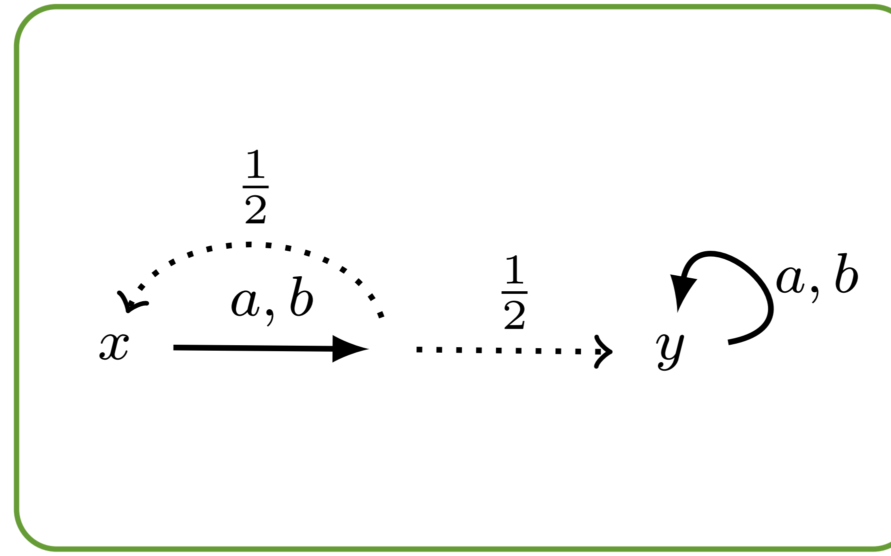

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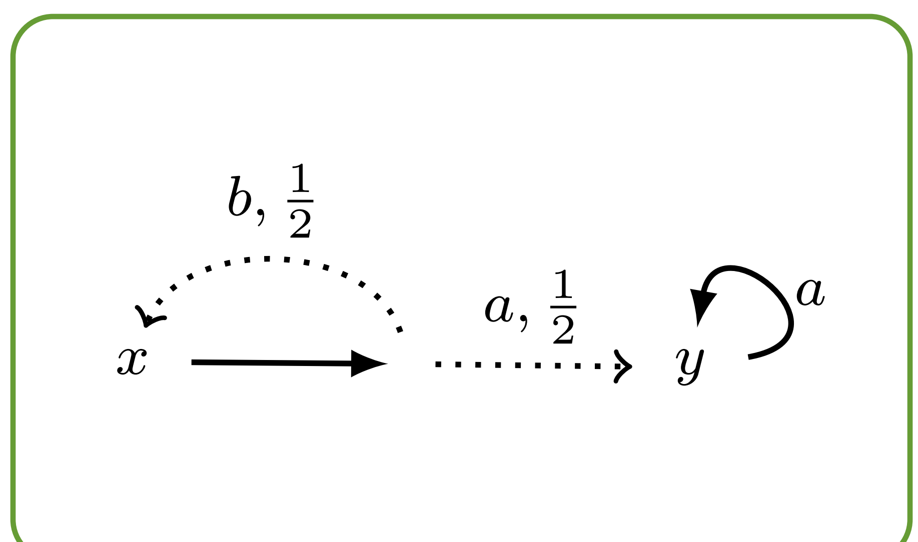

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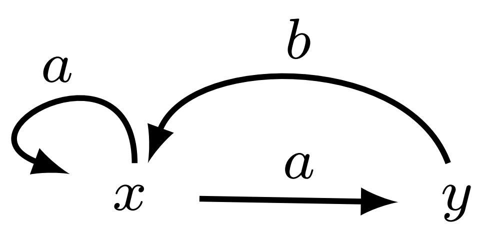

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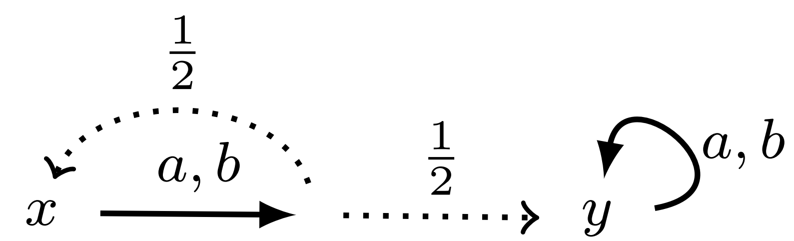

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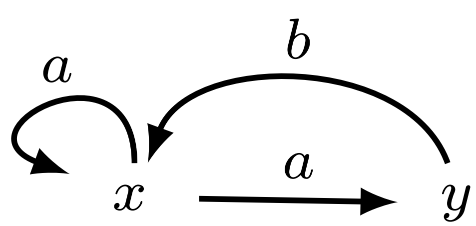

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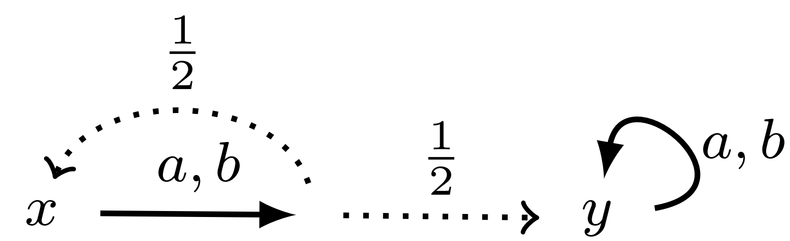
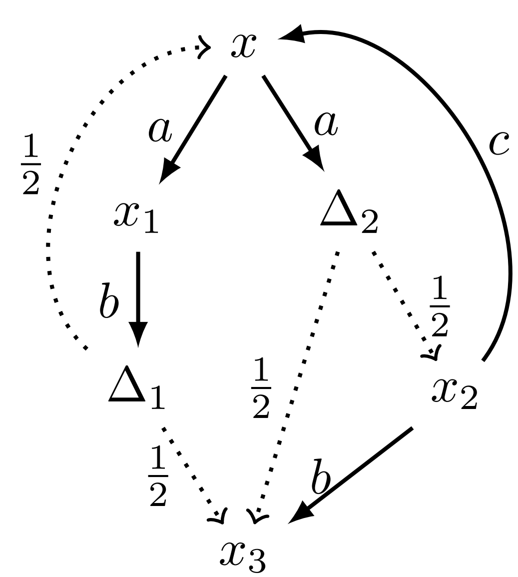


NPLTS

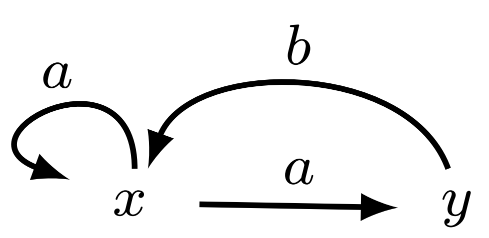
$X \to (\mathcal{P}\mathcal{D}X)^A$



Examples

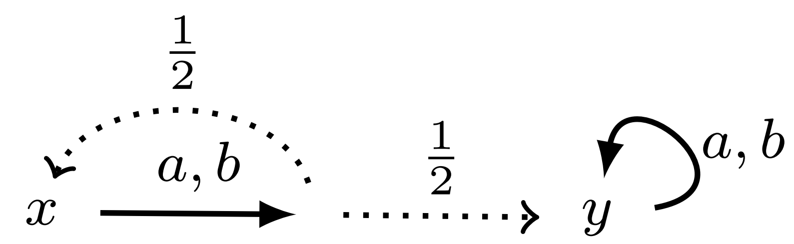
LTS

$X \to (\mathcal{P}X)^A$



PTS

$X \to (\mathcal{D}X)^A$



NPLTS

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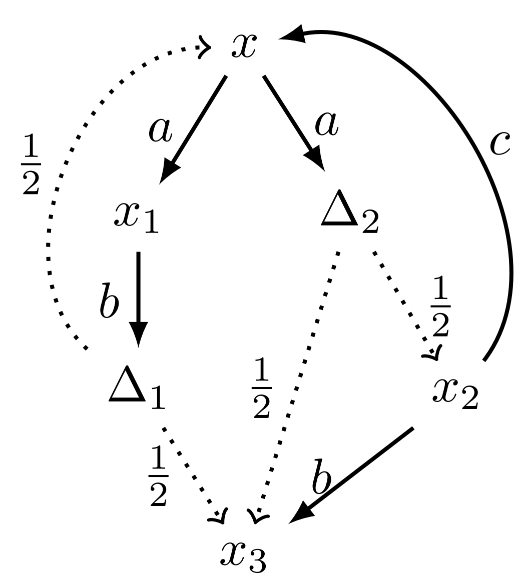
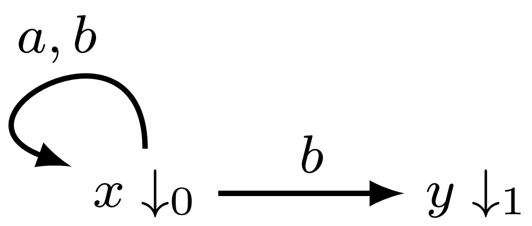


systems with nondeterminism and probability

Examples

NFA

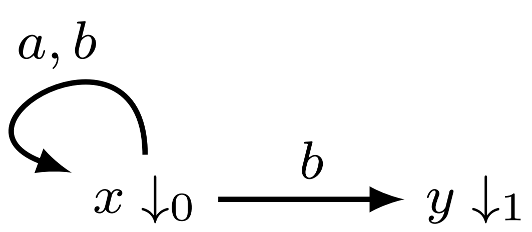
$X \to 2 \times (\mathcal{P}X)^A$



Examples

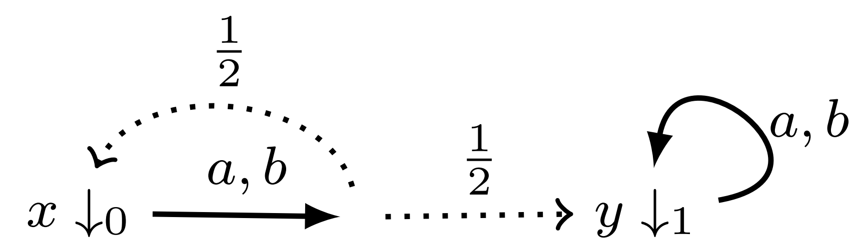
NFA

$X \to 2 \times (\mathcal{P}X)^A$



Rabin PA

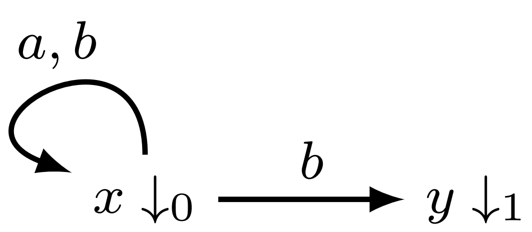
$X \to [0,1] \times (\mathcal{D}X)^A$



Examples

NFA

$X \to 2 \times (\mathcal{P}X)^A$



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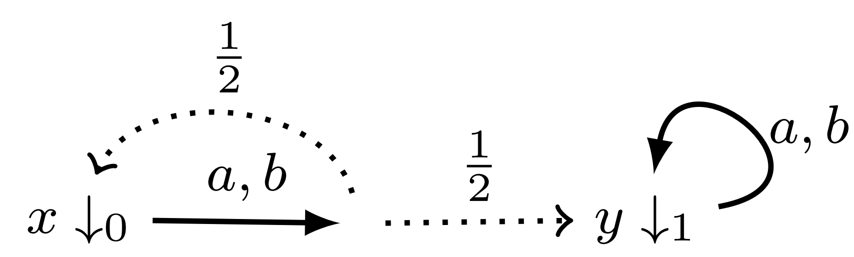
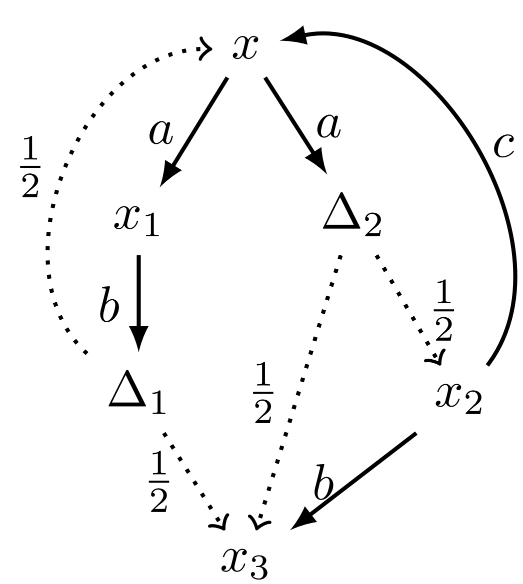


Simple NPA

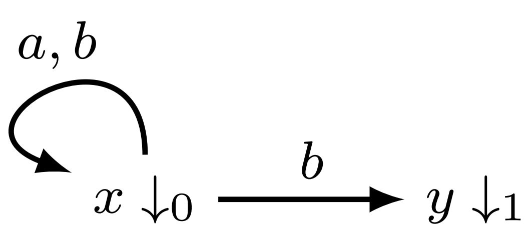
$X \to ? \times (\mathcal{P}\mathcal{D}X)^A$



Examples

NFA

$X \to 2 \times (\mathcal{P}X)^A$



Simple NPA

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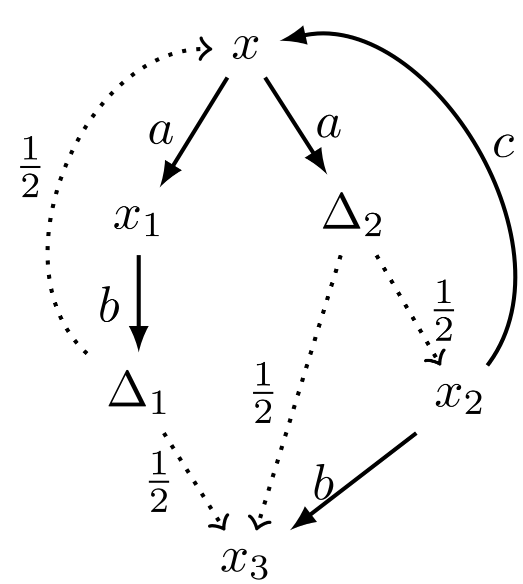

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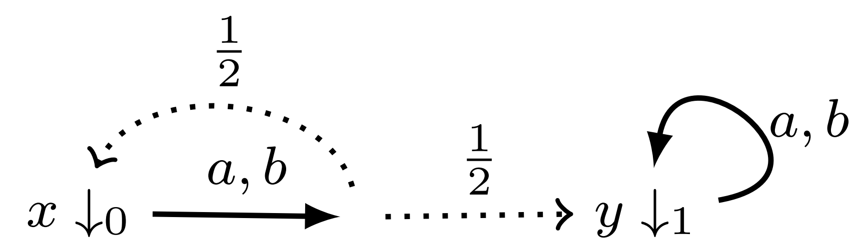


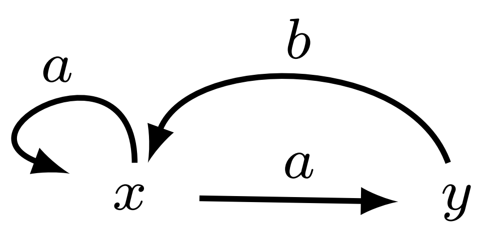
automata = systems with observations

What is a trace ?

What is a trace ?

LTS

$X \to (\mathcal{P}X)^A$

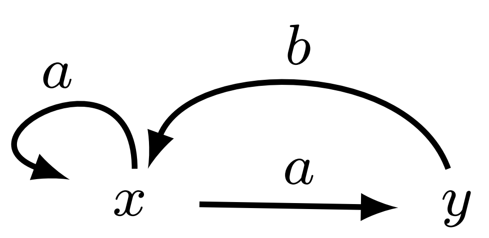


What is a trace ?

a word

LTS

$X \to (\mathcal{P}X)^A$



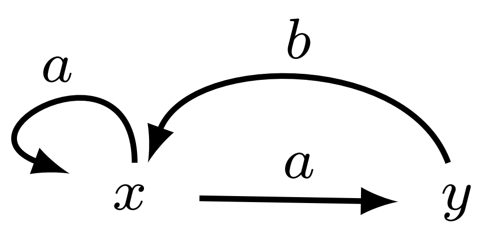
What is a trace ?

a word

a set of words

LTS

$X \to (\mathcal{P}X)^A$



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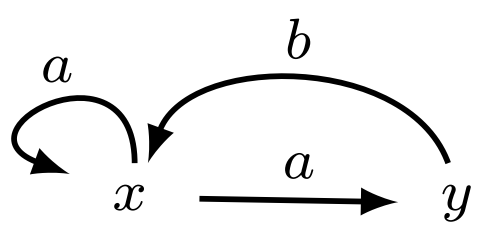


a word

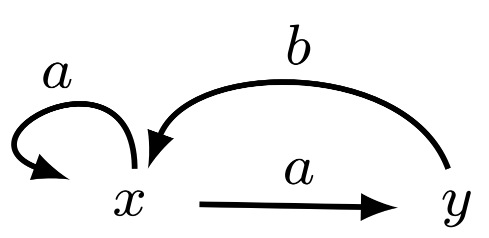
a set of words

for each state

What is a trace ?

LTS

$X \to (\mathcal{P}X)^A$



$$tr\colon X \to \mathcal{P}(A^*)$$

a word

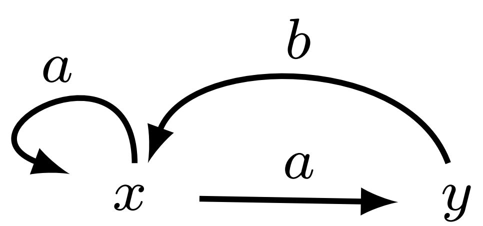
a set of words

for each state

What is a trace ?

LTS

$X \to (\mathcal{P}X)^A$



$tr\colon X \to \mathcal{P}(A^*)$

$a, aa, aaba \in tr(x)$
$b, ba, baa \in tr(y)$

a word

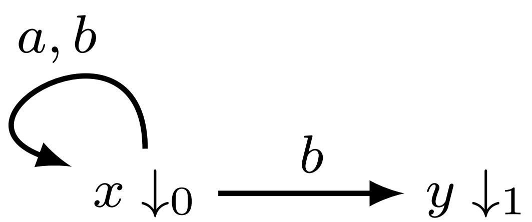
a set of words

for each state

What is a trace ?

NFA

$X \to 2 \times (\mathcal{P}X)^A$



What is a trace ?

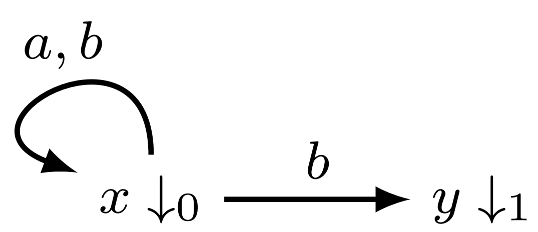
a word

the language

for each state

NFA

$X \to 2 \times (\mathcal{P}X)^A$



What is a trace ?

NFA

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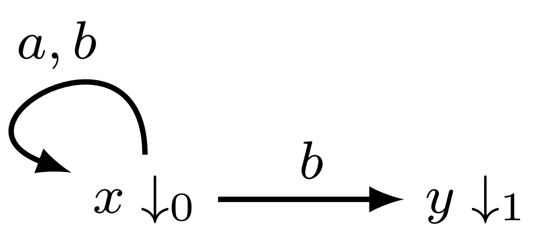


$$tr \colon X \to \mathcal{P}(A^*) \cong 2^{A^*}$$

a word

the language

for each state

What is a trace ?

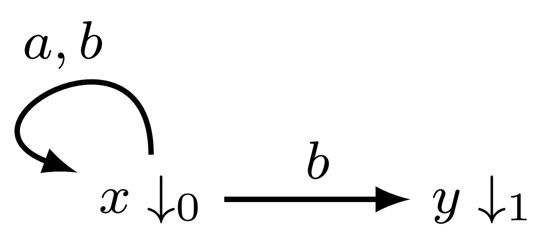
a word

the language

for each state

NFA

X → 2 x ($\mathcal{P}$X)^A



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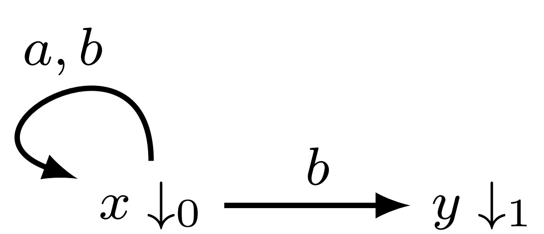
What is a trace ?

NFA

$X \to 2 \times (\mathcal{P}X)^A$



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a word

the language

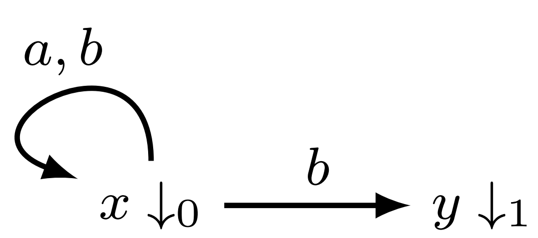
for each state

NFA = LTS with observations in {0,1}

What is a trace ?

NFA

$X \to 2 \times (\mathcal{P}X)^A$



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a word

the language

for each state

NFA = LTS with observations in {0,1}

LTS = NFA in which all states are final

What is a trace ?

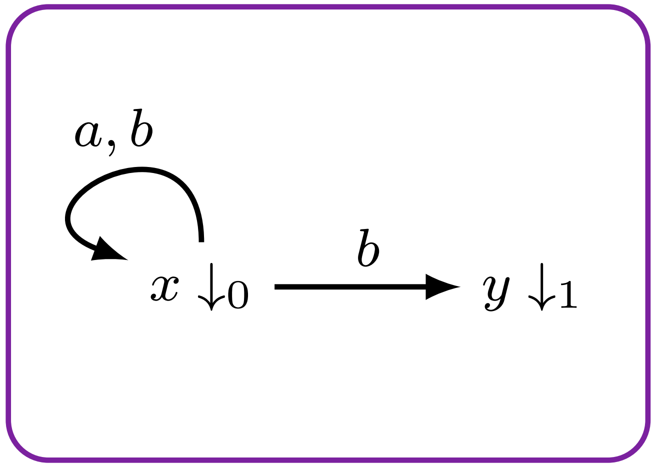
a word

the language

for each state

NFA

$X \to 2 \times (\mathcal{P}X)^A$



$tr\colon X \to \mathcal{P}(A^*) \cong 2^{A^*}$

$ab, bb, aab \in tr(x)$

$\mathrm{tr}(x) = (a \cup b)^* b = \{w \in \{a, b\}^* \mid w \text{ ends with a } b\}$

NFA = LTS with observations in {0,1}

LTS = NFA in which all states are final

Trace semantics coalgebraically?

NFA / LTS

Two ideas:

(1) unfold branching + transitions on words

(2) trace = bisimilarity after determinisation

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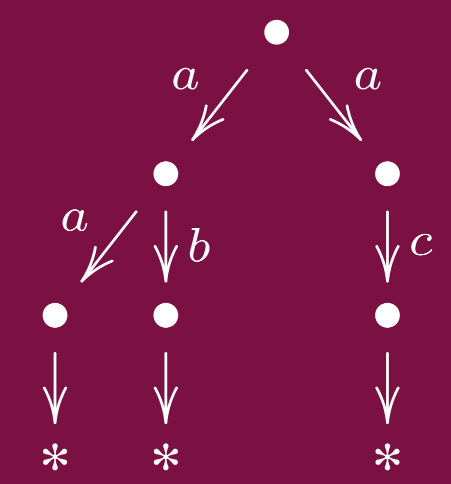

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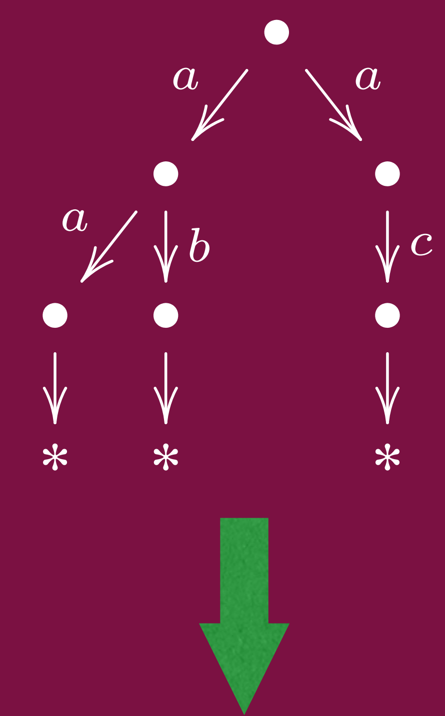

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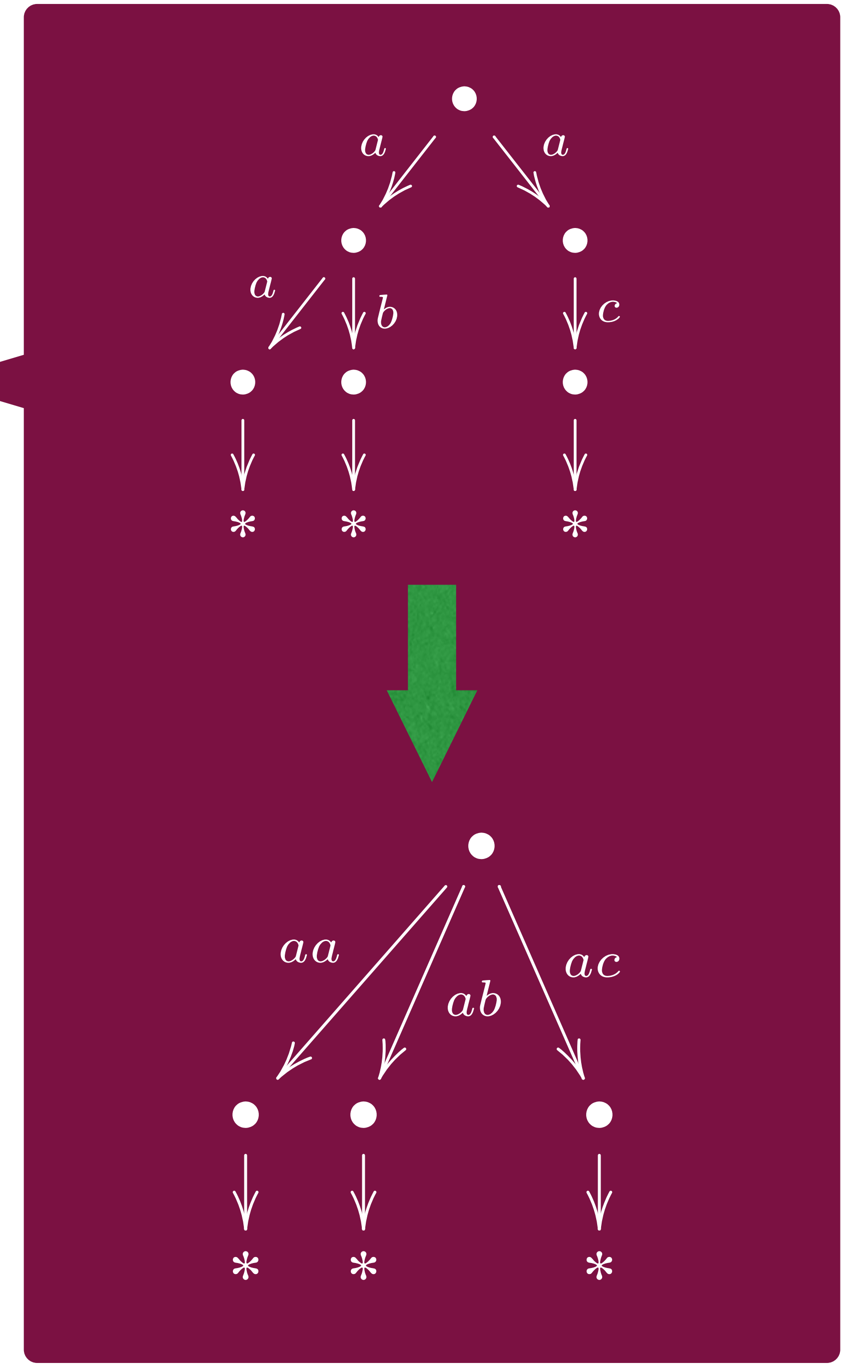

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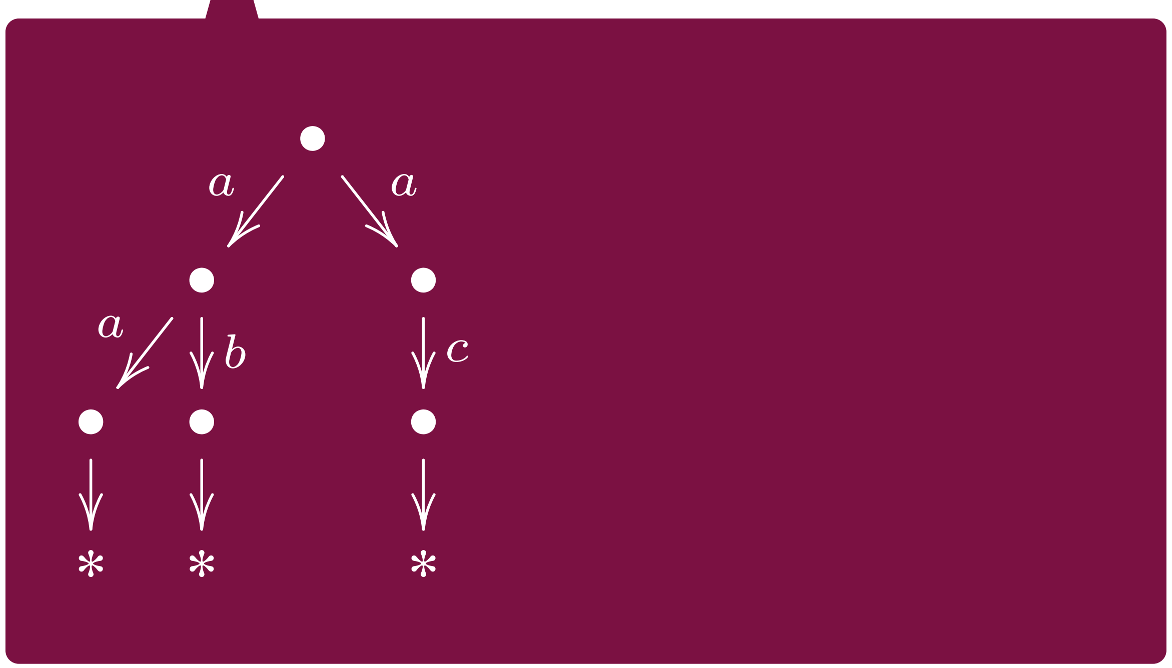

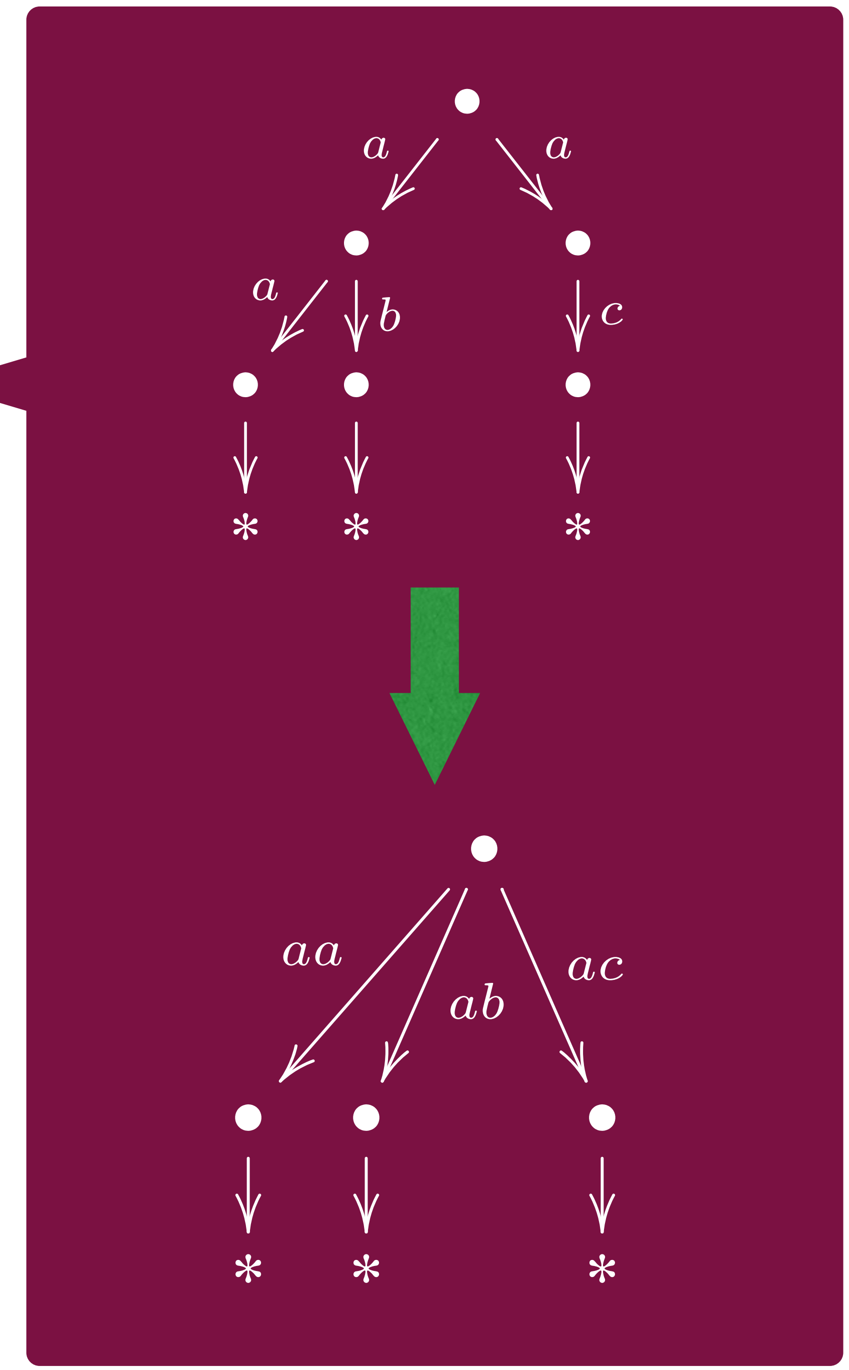

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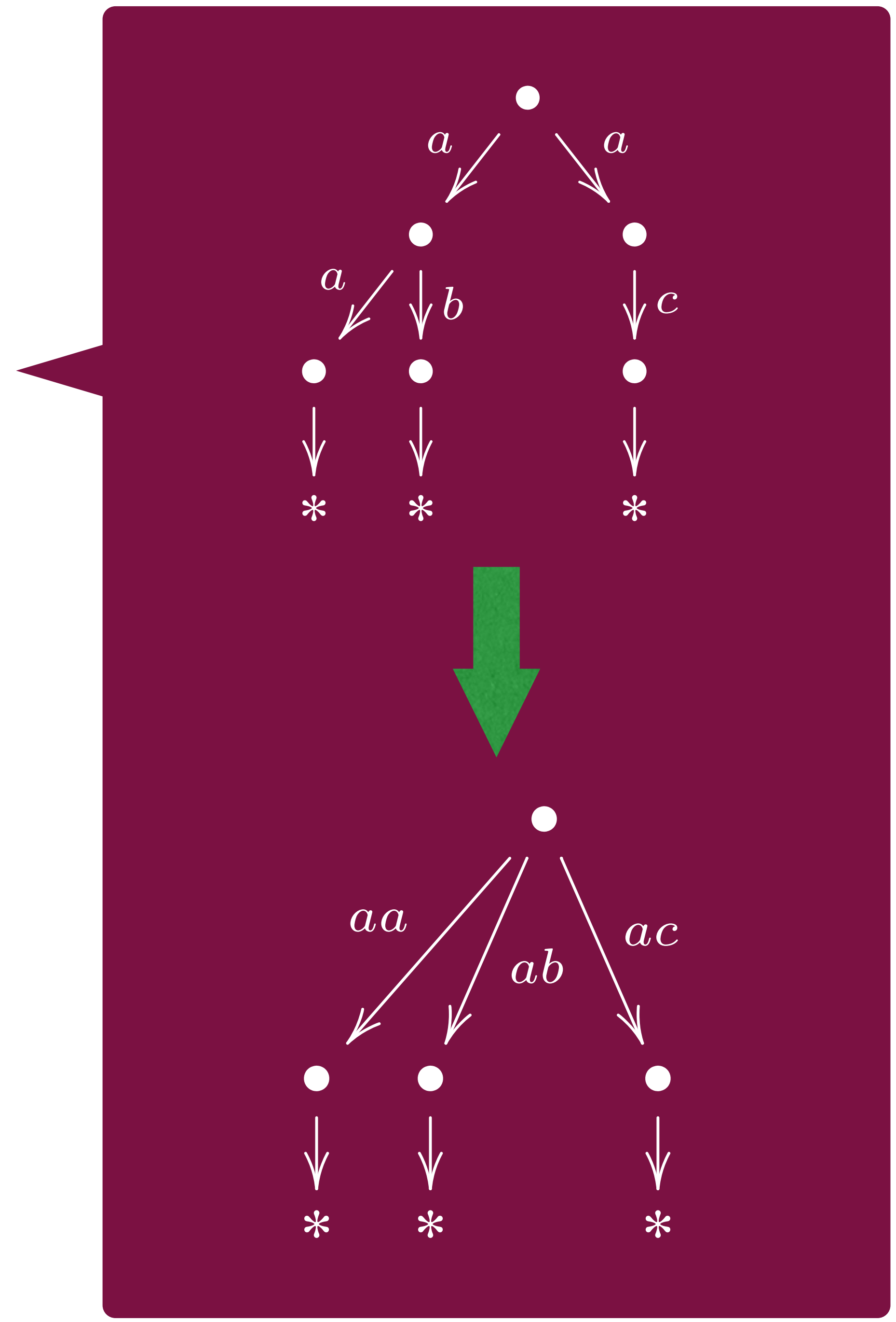
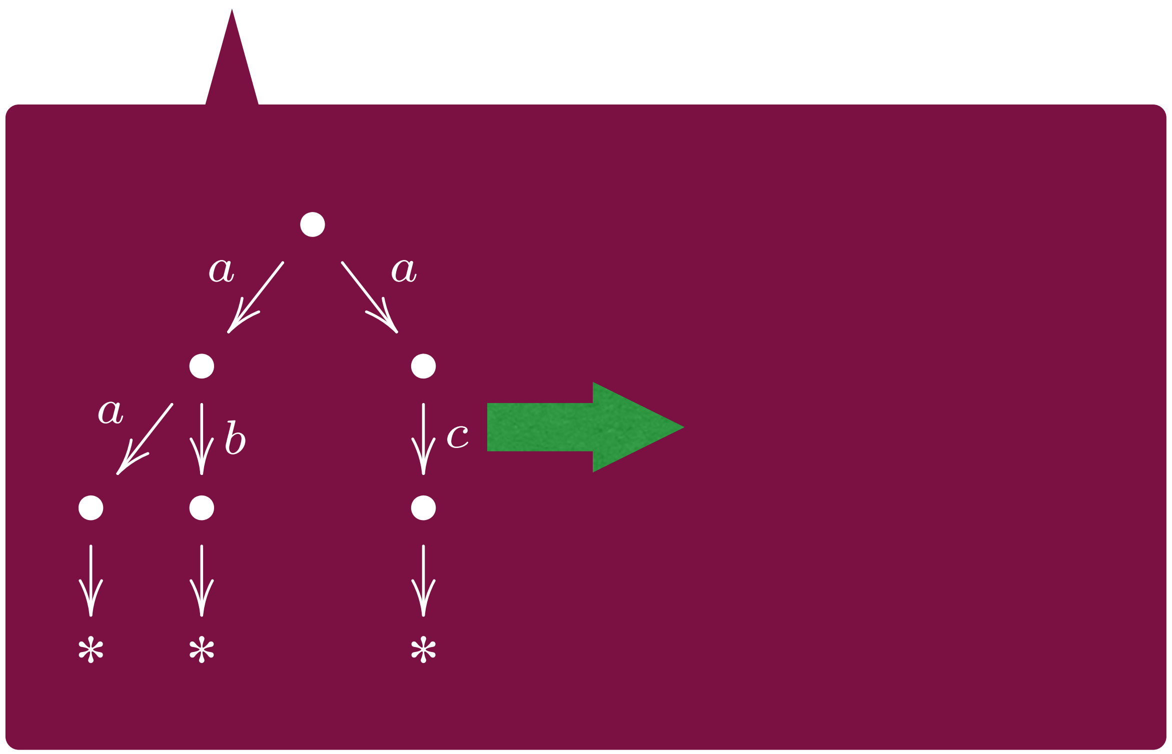


(2) trace = bisimilarity after determinisation



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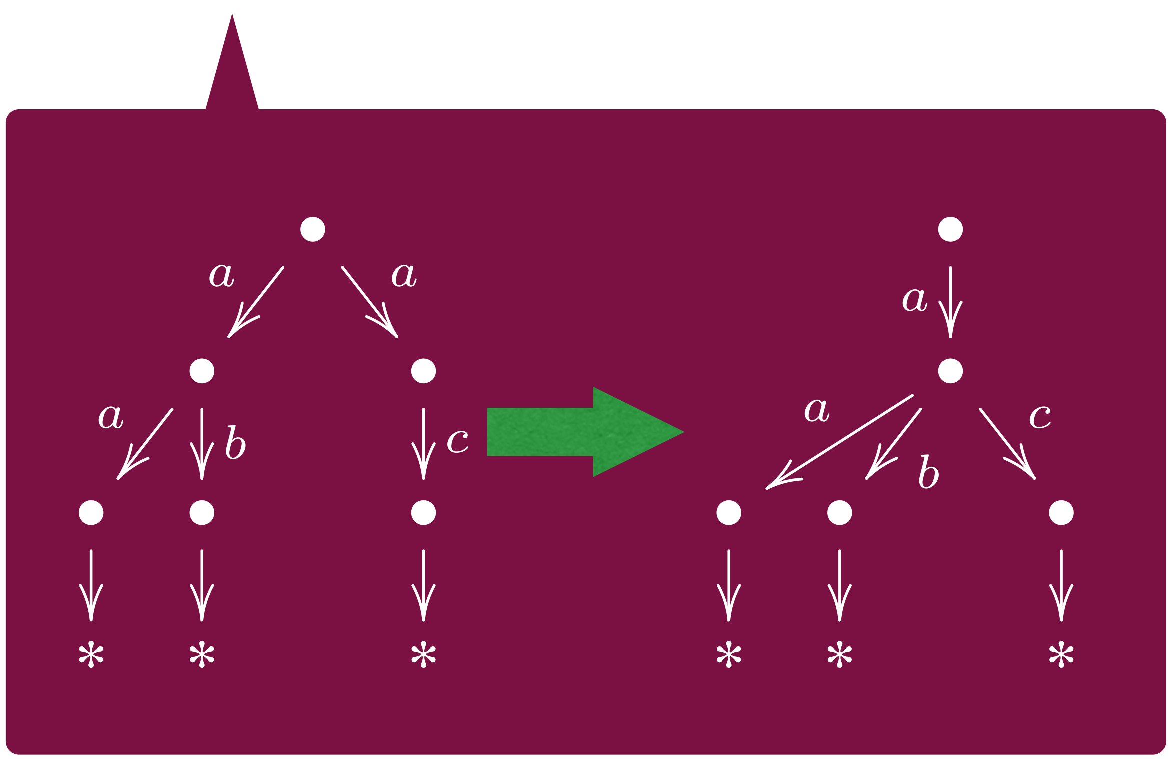

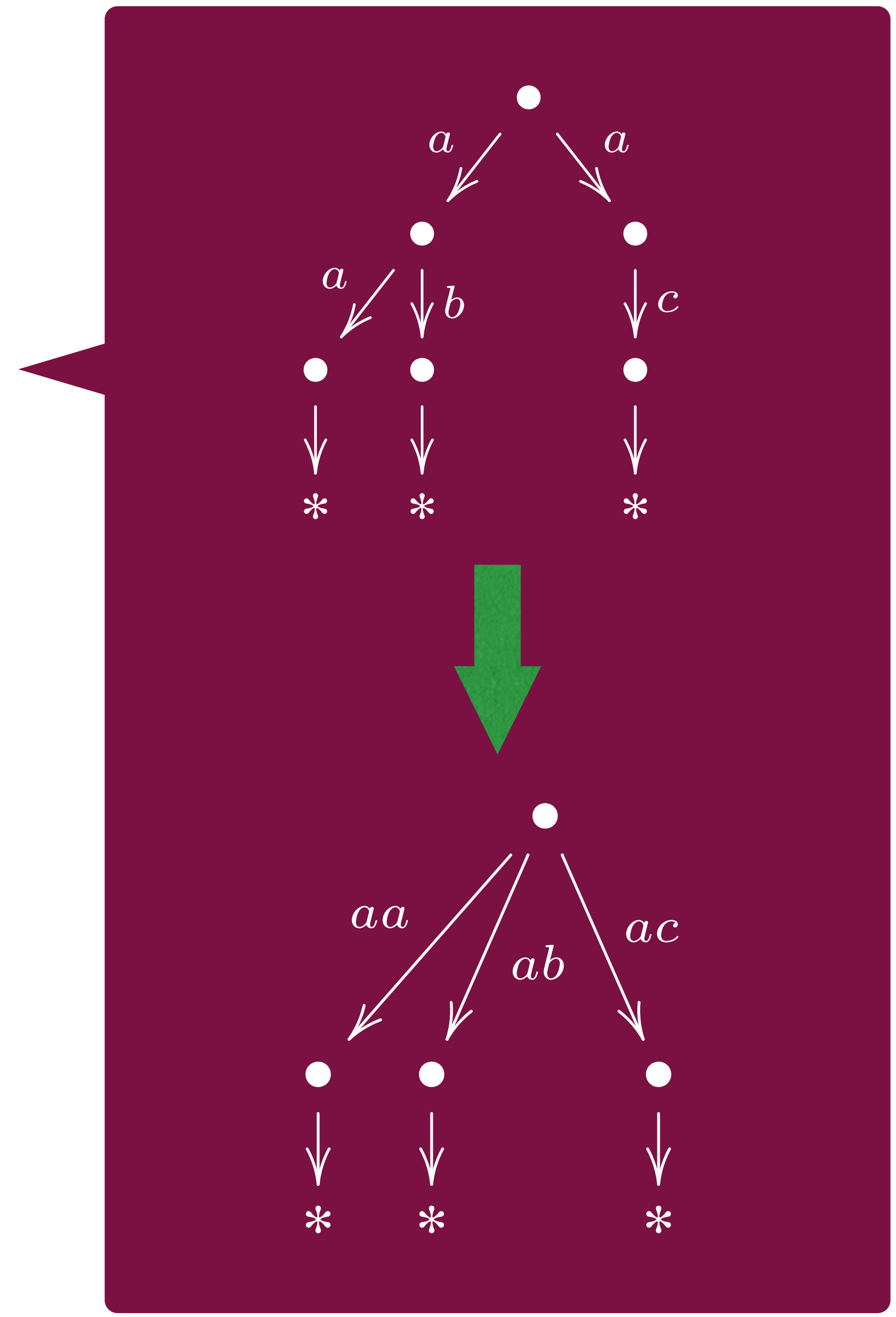

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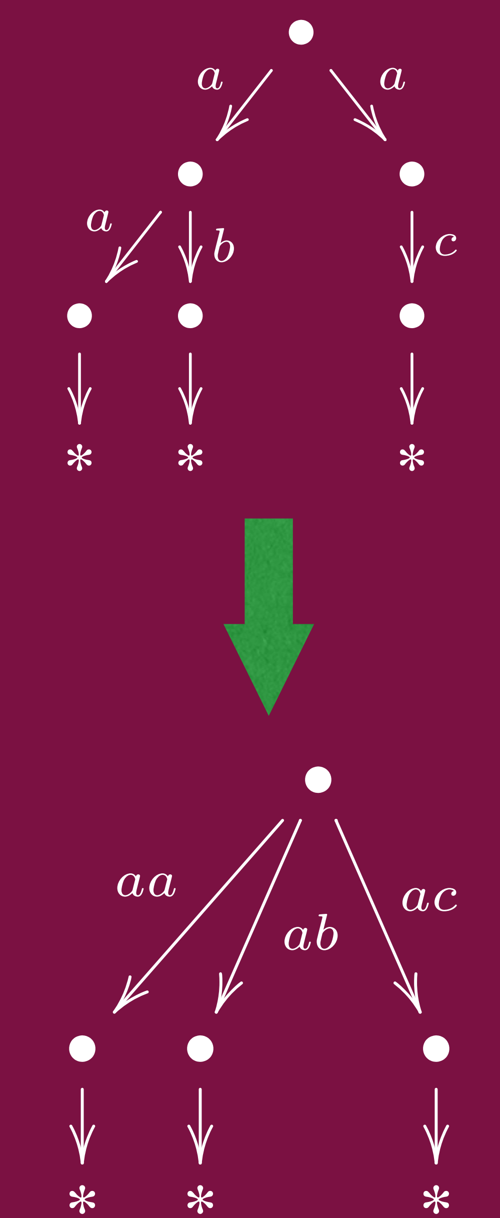
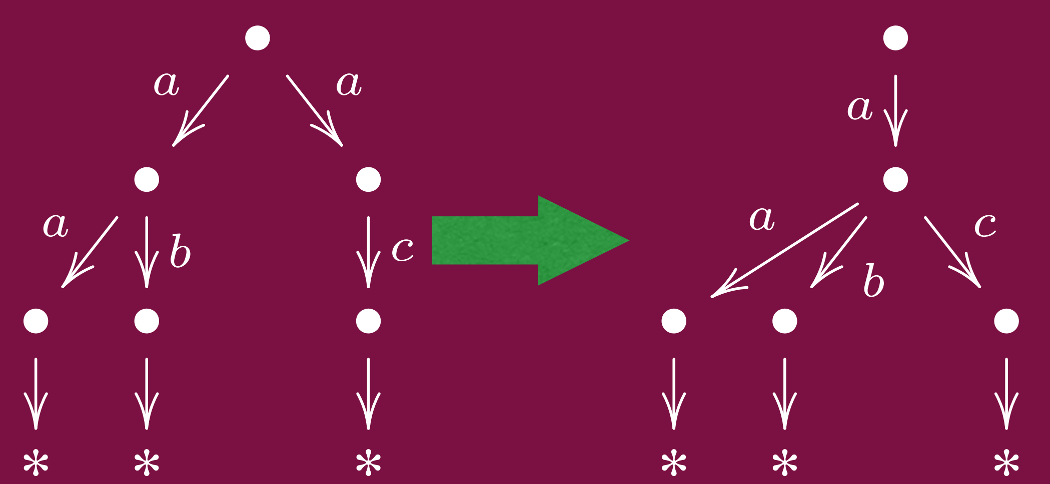


(2) trace = bisimilarity after determinisation

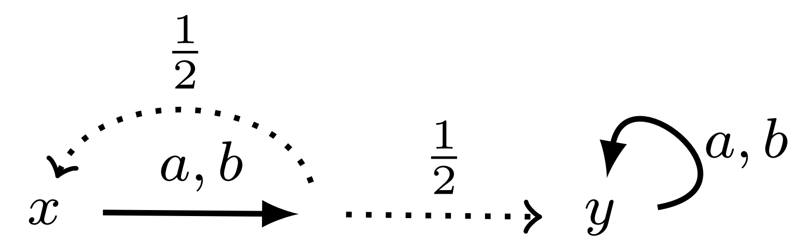


monads !

What is a trace ?

PTS

$X \to (\mathcal{D}X)^A$

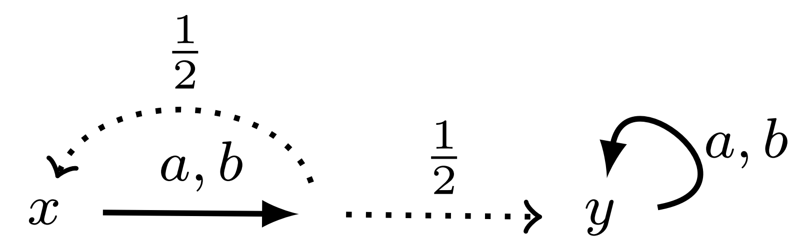


What is a trace ?

a word?

PTS

$X \to (\mathcal{D}X)^A$



What is a trace ?

a word?

a set of probabilistic words?

PTS

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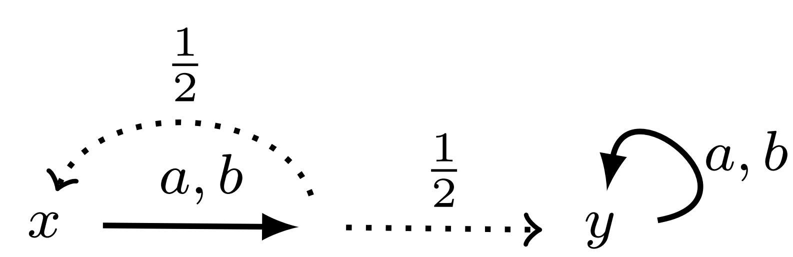


What is a trace ?

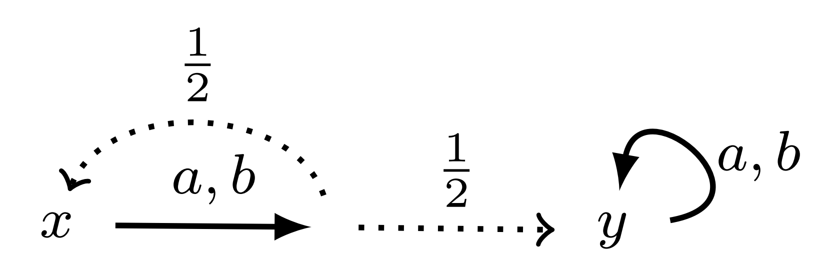
a word?

a set of probabilistic words?

for each state

PTS

$X \to (\mathcal{D}X)^A$



What is a trace ?

a word?

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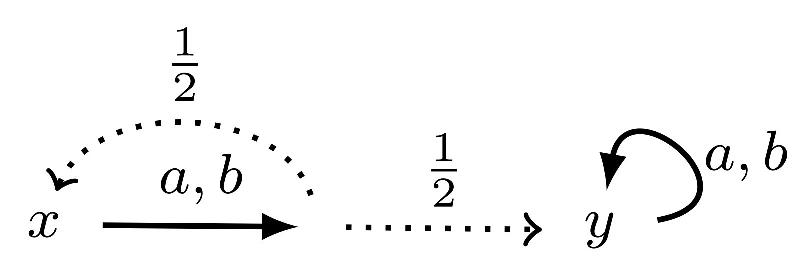
for each state

$$tr\colon X \to [0,1]^{A^*}$$

PTS

$X \to (\mathcal{D}X)^A$



What is a trace ?

a word?

a set of probabilistic words?

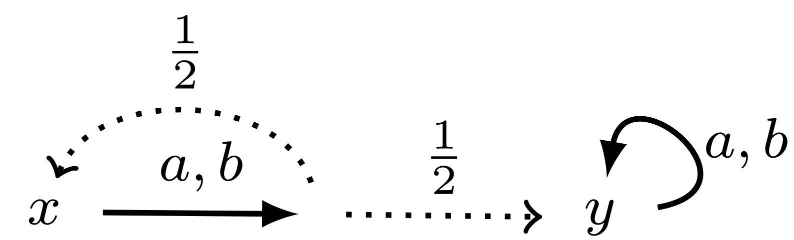
for each state

PTS

$X \to (\mathcal{D}X)^A$

$tr\colon X \to [0,1]^{A^*}$

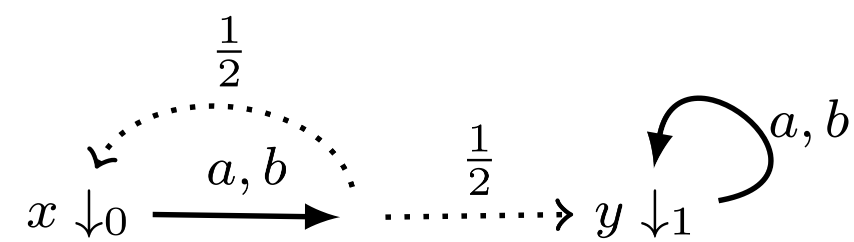
$tr(x) : aa \mapsto 1, ab \mapsto 1, \ldots$



What is a trace ?

Rabin PA

$X \to [0,1] \times (\mathcal{D}X)^A$



What is a trace ?

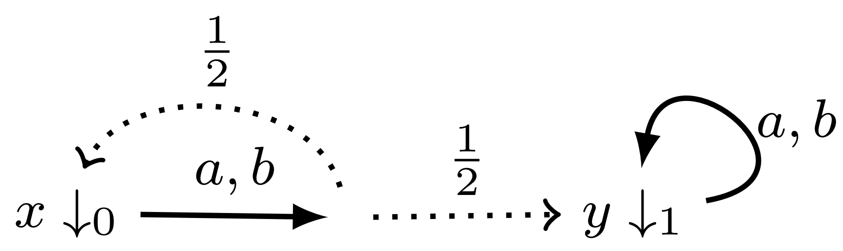
a word

a probabilistic language

for each state

Rabin PA

X → [0,1] x ($\mathcal{D}$X)^A



What is a trace ?

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$tr\colon X \to [0,1]^{A^*}$

Rabin PA

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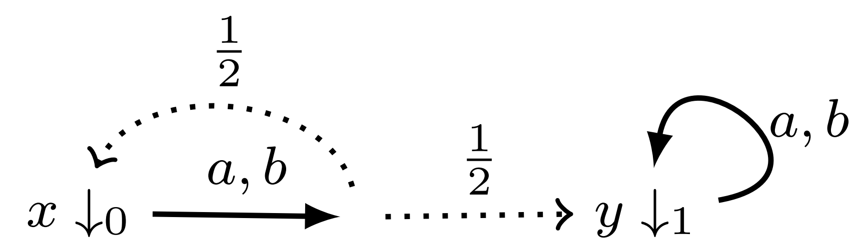


What is a trace ?

a word

a probabilistic language

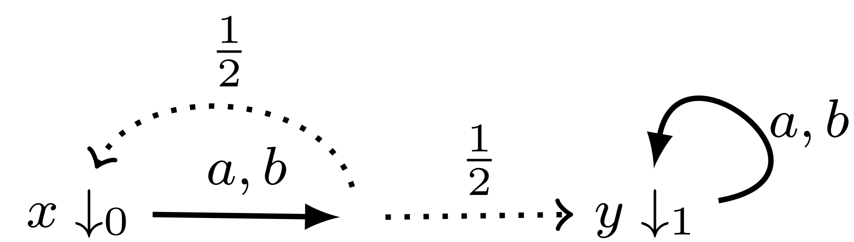
for each state

Rabin PA

$X \to [0,1] \times (\mathcal{D}X)^A$

$tr : X \to [0,1]^{A^*}$

$\mathrm{tr}(x) = \left(a \mapsto \frac{1}{2}, aa \mapsto \frac{3}{4}, \ldots\right)$

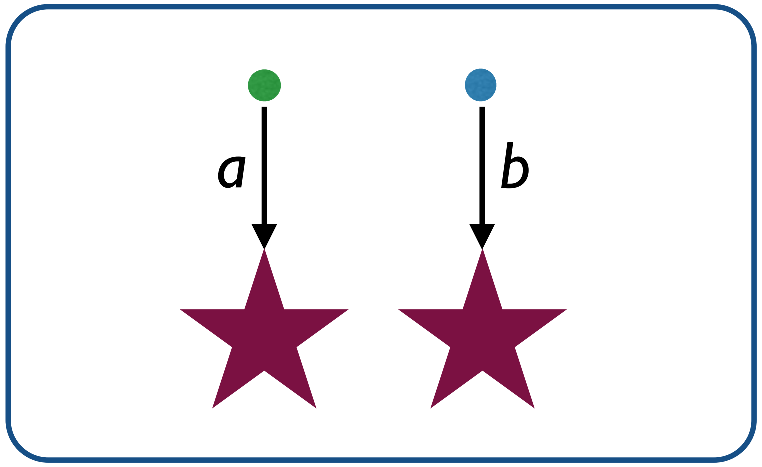


In general

In general

Systems

$X \to (MX)^A$



In general

Systems

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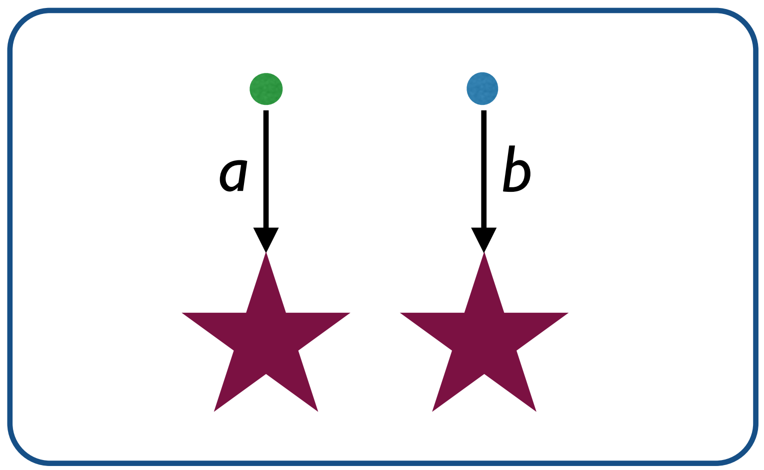
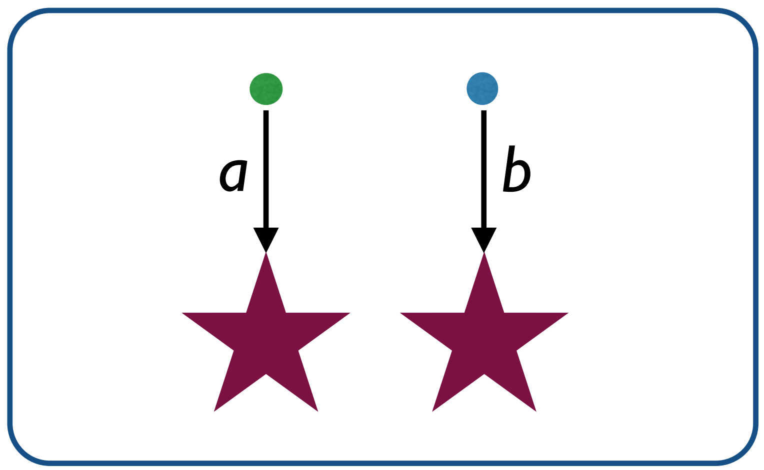


with M-effects

In general

Systems

$X \to (MX)^A$



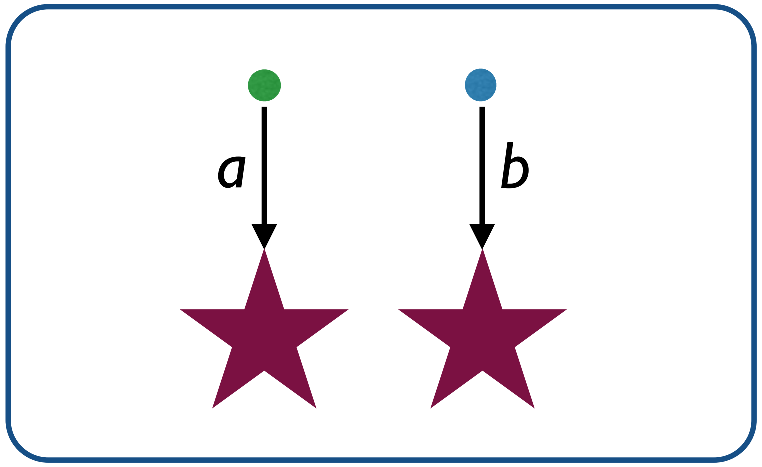
with M-effects

for a monad M

In general

Systems

$X \to (MX)^A$



with M-effects

for a monad M

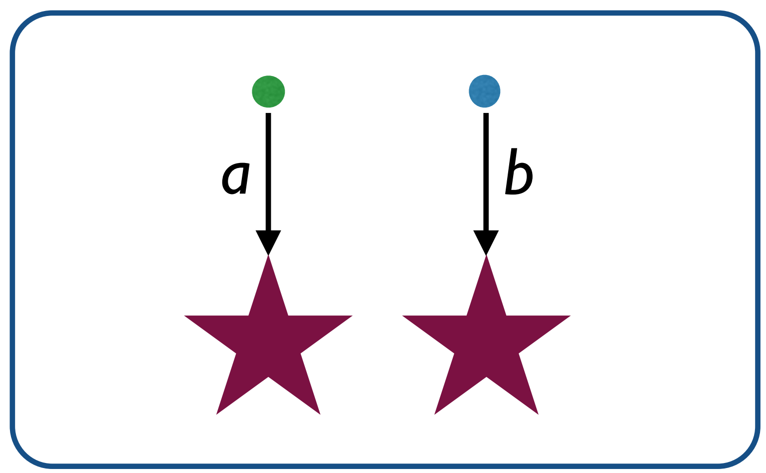
$\mu\colon MM \Rightarrow M$

$\eta\colon Id \Rightarrow M$

In general

Systems

$X \to (MX)^A$



with M-effects

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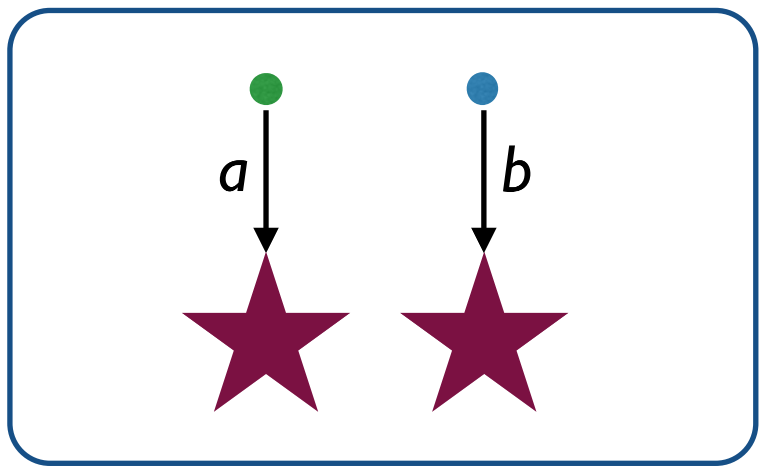
providing algebraic effects

In general

Systems

$X \to (MX)^A$



with M-effects

for a monad M

$\mu\colon MM \Rightarrow M$

$\eta\colon Id \Rightarrow M$

providing algebraic effects

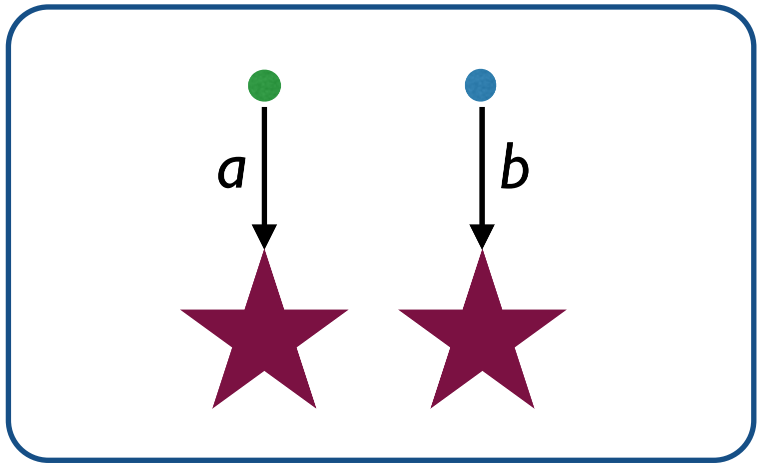
we write $x \xrightarrow{a} t_x$

In general

In general

Automata

$X \to O \times (MX)^A$

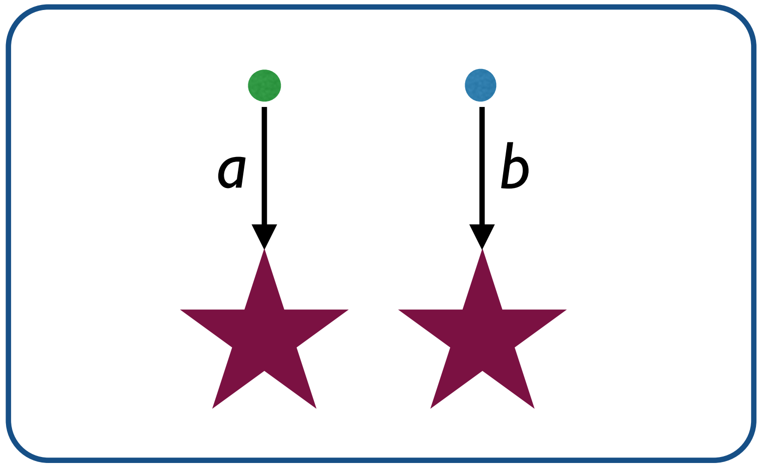


In general

Automata

with observations in O

$X \to O \times (MX)^A$



In general

Automata

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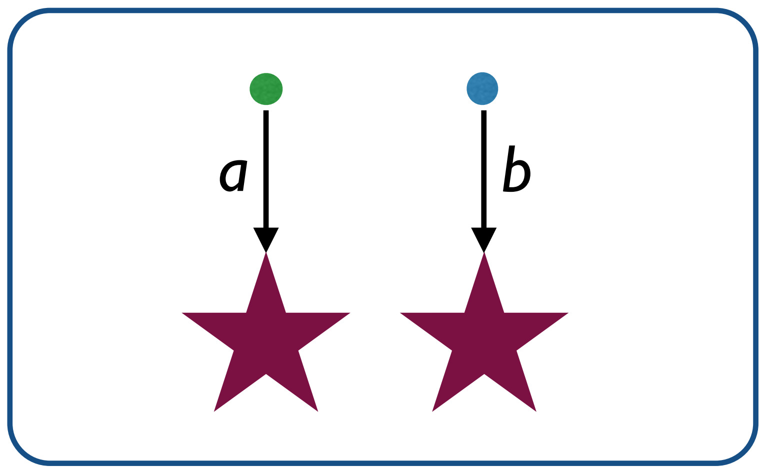


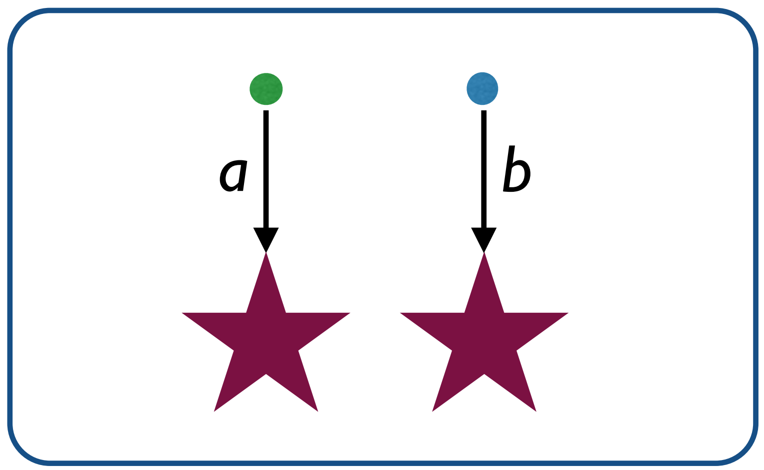
and M-effects

In general

Automata

with observations in O

$X \to O \times (MX)^A$



and M-effects

for a monad M

In general

Automata

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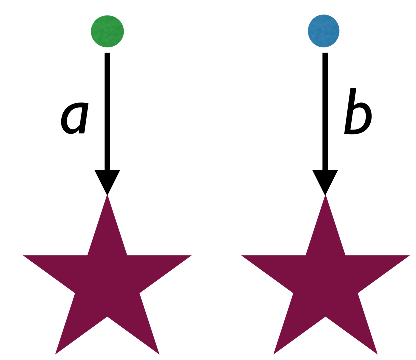

and M-effects

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In general

Automata

with observations in O

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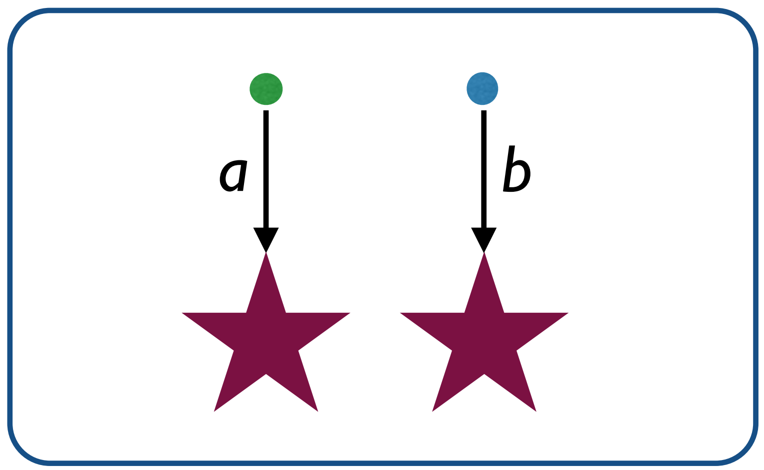

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In general

Automata

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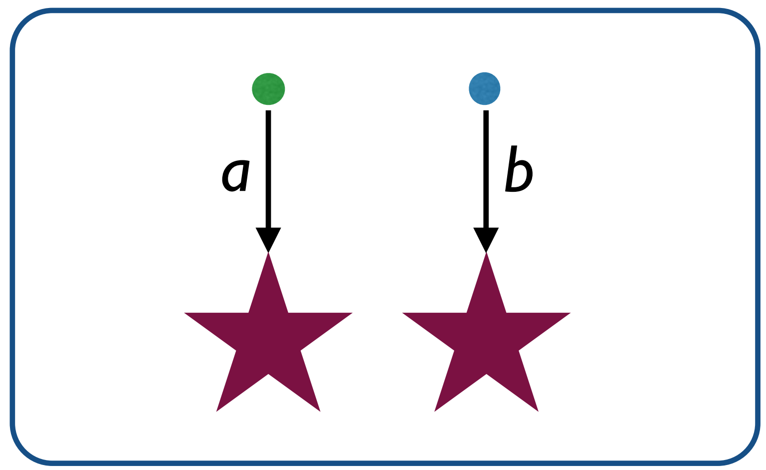

and M-effects

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we write $x{\downarrow}o,\ x \xrightarrow{a} t_x$

In general

Automata

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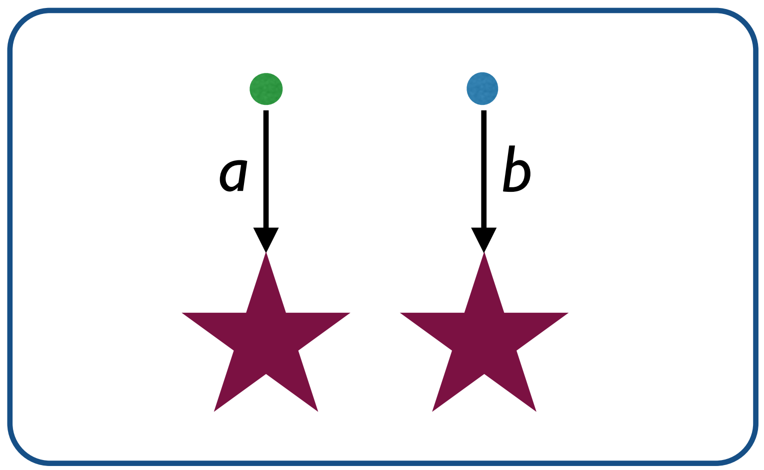

and M-effects

for a monad M

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providing algebraic effects

we write $x{\downarrow}o, \;\; x \xrightarrow{a} t_x$

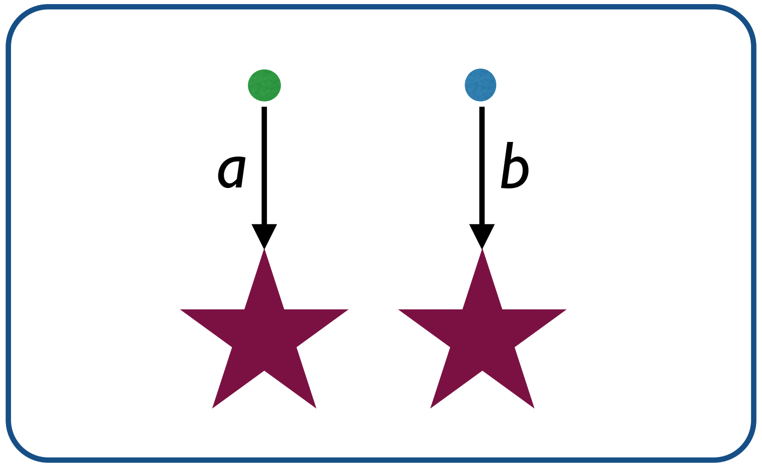
$tr \colon X \to O^{A^*}$

Canonical observations

Canonical observations

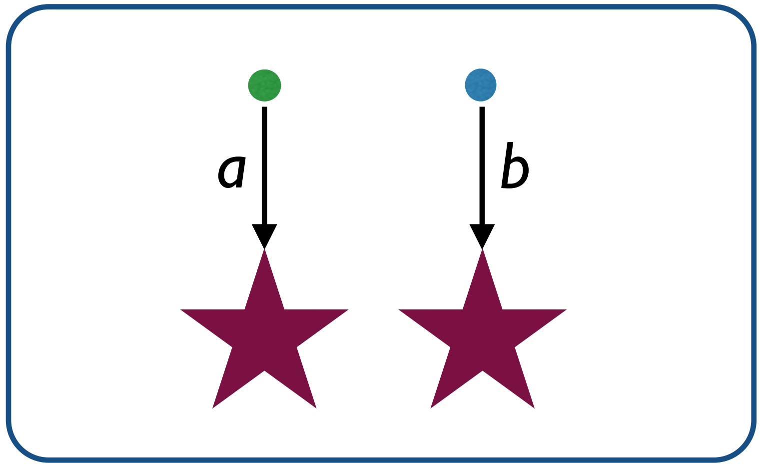
Systems

$X \to (MX)^A$



Automata

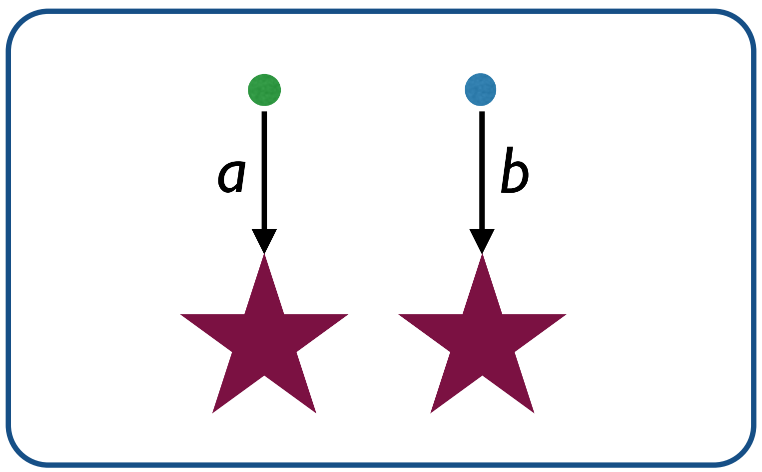
$X \to O \times (MX)^A$



Canonical observations

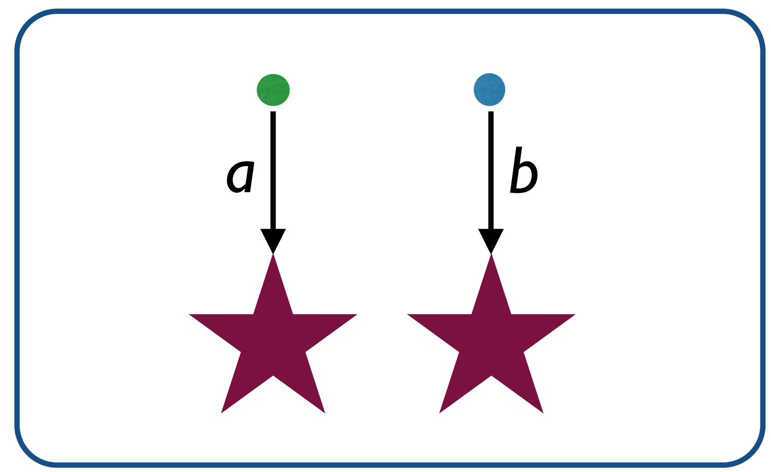
Systems

$X \to (MX)^A$



Automata

$X \to O \times (MX)^A$

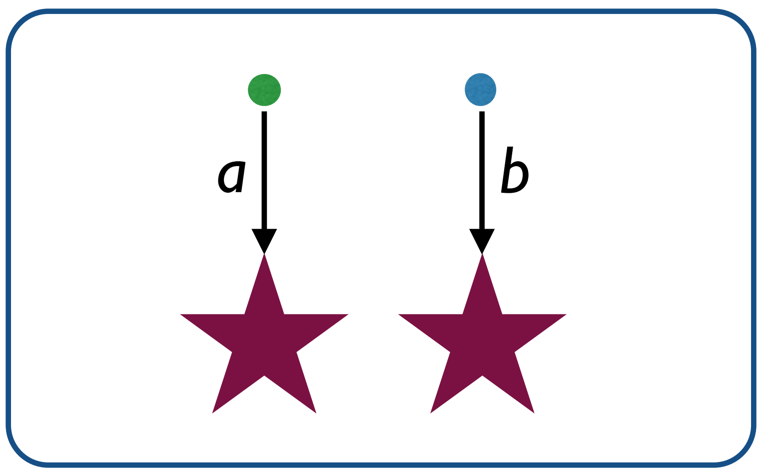


Automata = Systems with observations in O

Canonical observations

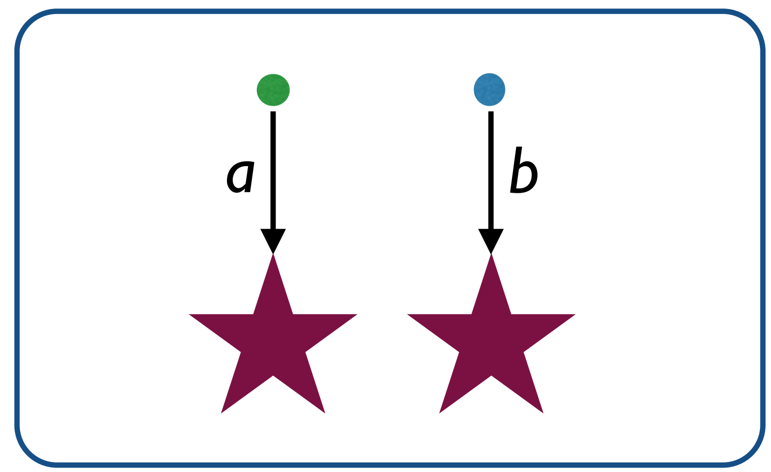
Systems

$X \to (MX)^A$



Automata

$X \to O \times (MX)^A$



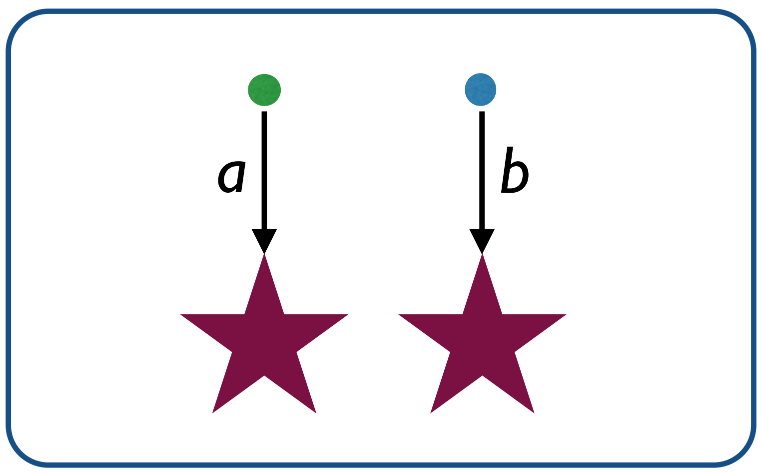
Automata = Systems with observations in O

Systems = Automata with canonical observations

Canonical observations

Systems

$X \to (MX)^A$



Automata

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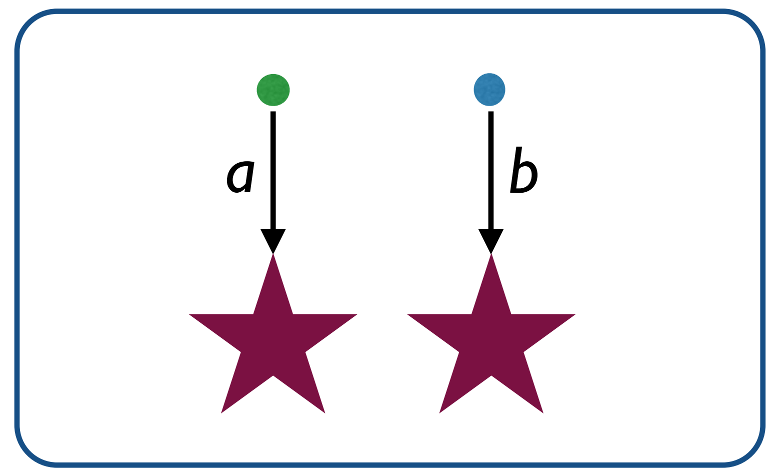

Automata = Systems with observations in O

Systems = Automata with canonical observations

$$X \xrightarrow{!} 1 \xrightarrow{\eta} M1$$

In our examples

NFA

$X \to 2 \times (\mathcal{P}X)^A$

Rabin PA

$X \to [0,1] \times (\mathcal{D}X)^A$

Simple PA

$X \to ? \times (\underline{\mathcal{PD}}X)^A$

In our examples

NFA

$X \to 2 \times (\mathcal{P}X)^A$

$M = \mathcal{P}$
for nondeterminism

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Powerset, subsets

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Powerset, subsets

Rabin PA

$X \to [0,1] \times (\mathcal{D}X)^A$

$M = \mathcal{D}$
for probability

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Powerset, subsets

Rabin PA

$X \to [0,1] \times (\mathcal{D}X)^A$

$M = \mathcal{D}$
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(Sub)Distributions

Simple PA

$X \to\ ? \times (\underline{\mathcal{P}\mathcal{D}}X)^A$

In our examples

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$X \to 2 \times (\mathcal{P}X)^A$

$M = \mathcal{P}$
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In our examples

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(Sub)Distributions

Simple PA

$X \to ? \times (\underline{\mathcal{PD}}X)^A$

$M = \underline{\mathcal{PD}}$
for nondeterminism
and probability

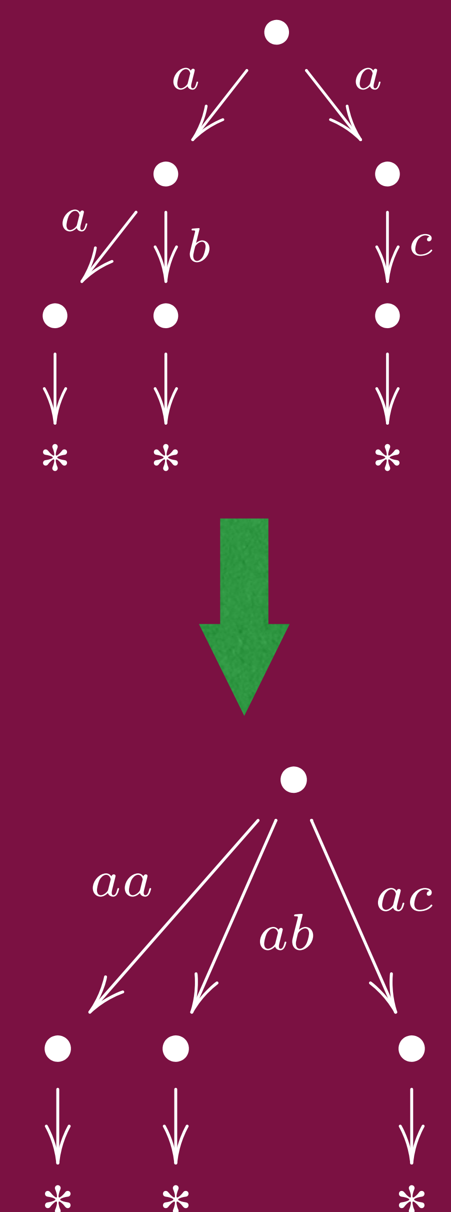
Convex (n.e., f.g.)
subsets of distributions

Trace semantics coalgebraically?

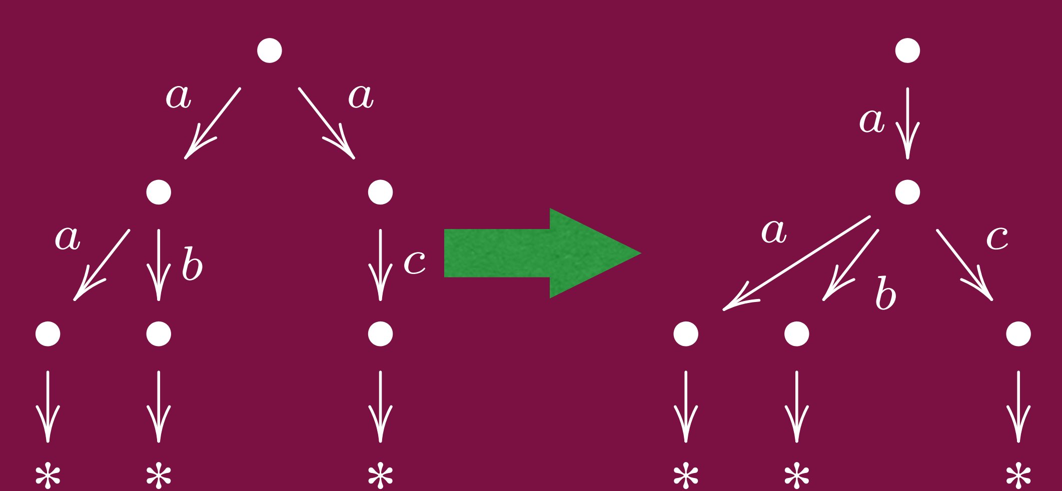
NFA / LTS

Two ideas:

(1) unfold branching + transitions on words



(2) trace = bisimilarity after determinisation



monads !

Trace semantics coalgebraically

Trace semantics coalgebraically

Two approaches:

(1) modelling in a Kleisli category

(2) modelling in an Eilenberg-Moore category

Trace semantics coalgebraically

Two approaches:

(1) modelling in a Kleisli category

free algebras of a monad

(2) modelling in an Eilenberg-Moore category

Trace semantics coalgebraically

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algebras of a monad M

Trace semantics coalgebraically

Hasuo, Jacobs, S. LMCS '07

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(1) and (2) are related

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Jacobs, Silva, S. JCSS'15

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Trace semantics coalgebraically

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Hasuo, Jacobs, S. LMCS '07

$$X \to MFX \to MFMFX \to MMFFX \to MFFX \to \ldots$$

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Trace semantics coalgebraically

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(2) modelling in an Eilenberg-Moore category

Silva, Bonchi, Bonsangue, Rutten FSTTCS'10

$$X \to FMX, \quad MX \to MFMX \to FMMX \to FMX$$

Trace semantics coalgebraically

Two approaches:

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$$X \to MFX \to MFMFX \to MMFFX \to MFFX \to \ldots$$

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Hasuo, Jacobs, S. LMCS '07

Silva, Bonchi, Bonsangue, Rutten FSTTCS'10

Trace semantics is about iteration!

Traces via determinisation

Traces via determinisation

Automaton with M-effects

$X \rightarrow O \times (MX)^A$

Traces via determinisation

Automaton with M-effects

$X \to O \times (MX)^A$

Traces via determinisation

Automaton with M-effects

$X \rightarrow O \times (MX)^A$

Determinisation

$MX \rightarrow O \times (MX)^A$

Traces via determinisation

Automaton with M-effects

$X \to O \times (MX)^A$

Determinisation

$MX \to O \times (MX)^A$

trace = bisimilarity after determinisation

Traces via determinisation

Automaton with M-effects

$X \to O \times (MX)^A$

Determinisation

$MX \to O \times (MX)^A$

trace = bisimilarity after determinisation

Algebras for M

Traces via determinisation

Automaton with M-effects

$X \to O \times (MX)^A$

Determinisation

$MX \to O \times (MX)^A$

trace = bisimilarity after determinisation

Algebras for M

ideally we have a presentation

Traces via determinisation

Automaton with M-effects

$X \to O \times (MX)^A$

O has to be an M-algebra !

Determinisation

$MX \to O \times (MX)^A$

trace = bisimilarity after determinisation

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$MX \to O \times (MX)^A$

MX carries the free M-algebra

trace = bisimilarity after determinisation

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Algebras for M

ideally we have a presentation

trace = bisimilarity after determinisation

$\mathrm{tr}\colon X \to O^{A^*}$

Traces via determinisation

Automaton with M-effects

$X \to O \times (MX)^A$

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Determinisation

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MX carries the free M-algebra

Algebras for M

ideally we have a presentation

trace = bisimilarity after determinisation

$\mathrm{tr}\colon X \to O^{A^*}$

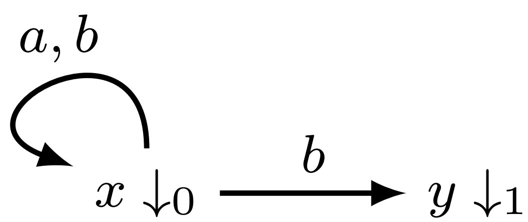
$$\mathrm{tr}(x)(a_1 a_2 \ldots a_n) = o \;\Leftrightarrow\; x \xrightarrow{a_1} t_1 \xrightarrow{a_2} t_2 \cdots t_{n-1} \xrightarrow{a_n} t_n \;\wedge\; t_n{\downarrow}o$$

Traces via determinisation

Traces via determinisation

NFA

$X \rightarrow 2 \times (\mathcal{P}X)^A$



Traces via determinisation

NFA

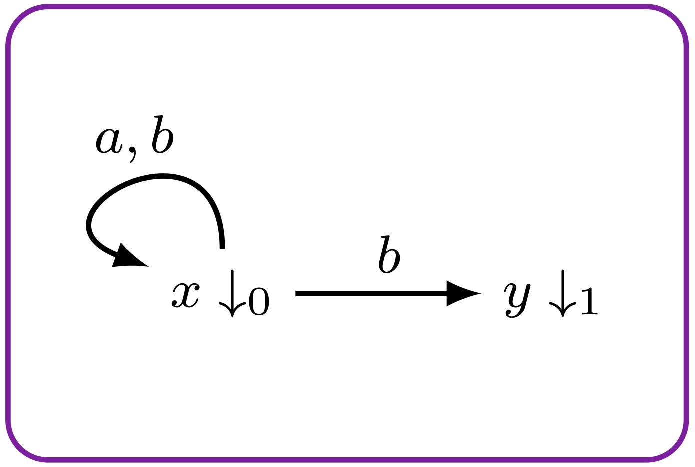
$X \to 2 \times (\mathcal{P}X)^A$

$$\begin{tikzcd} x\downarrow_0 \arrow[loop left, "a,b"] \arrow[r, "b"] & y\downarrow_1 \end{tikzcd}$$

Traces via determinisation

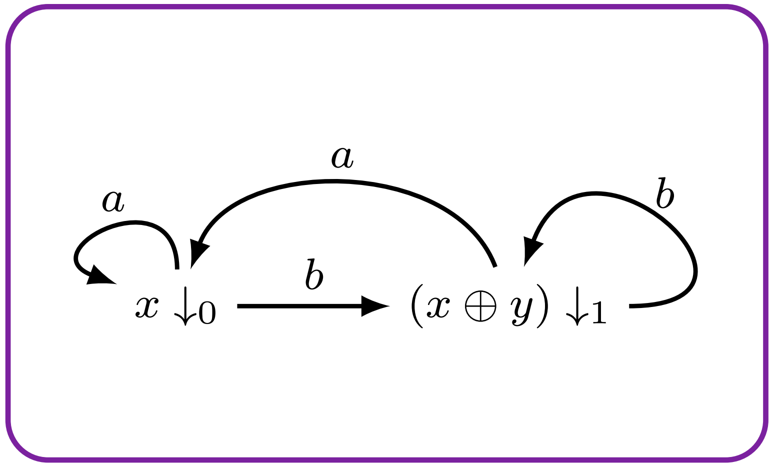
NFA

$X \to 2 \times (\mathcal{P}X)^A$



DFA

$\mathcal{P}X \to 2 \times (\mathcal{P}X)^A$



Traces via determinisation

NFA

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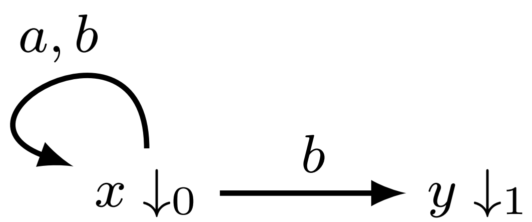

DFA

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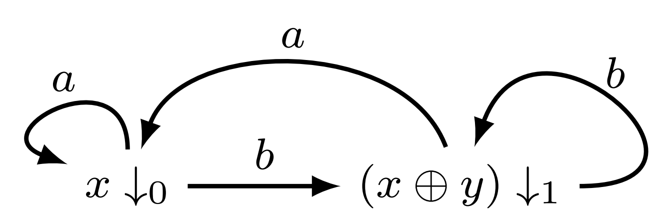


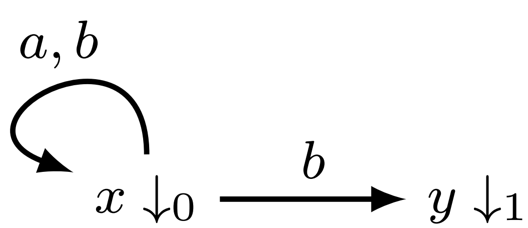
$$\frac{x \xrightarrow{a} t_x,\; y \xrightarrow{a} t_y}{x \oplus y \xrightarrow{a} t_x \oplus t_y}$$

$$\frac{x{\downarrow}o_x,\; y{\downarrow}o_y}{x \oplus y \downarrow o_x \oplus o_y}$$

Traces via determinisation

NFA

$X \to 2 \times (\mathcal{P}X)^A$



DFA

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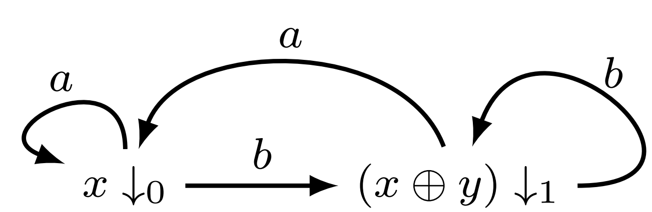


$$\frac{x \xrightarrow{a} t_x,\; y \xrightarrow{a} t_y}{x \oplus y \xrightarrow{a} t_x \oplus t_y}$$

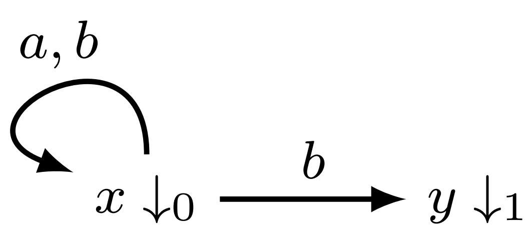
$$\frac{x{\downarrow}o_x,\; y{\downarrow}o_y}{x \oplus y \downarrow o_x \oplus o_y}$$

Algebras for $\mathcal{P}$

Traces via determinisation

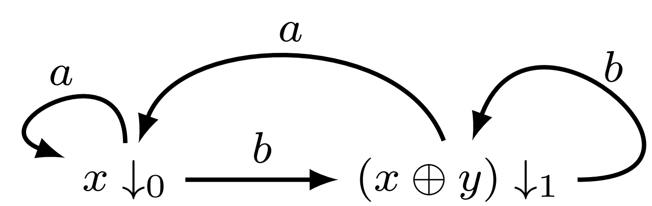
NFA

$X \to 2 \times (\mathcal{P}X)^A$



DFA

$\mathcal{P}X \to 2 \times (\mathcal{P}X)^A$



$$\frac{x \xrightarrow{a} t_x,\ y \xrightarrow{a} t_y}{x \oplus y \xrightarrow{a} t_x \oplus t_y}$$

$$\frac{x{\downarrow}o_x,\ y{\downarrow}o_y}{x \oplus y \downarrow o_x \oplus o_y}$$

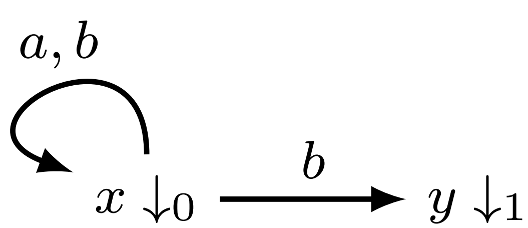
Algebras for $\mathcal{P}$

join semilattices with bottom

Traces via determinisation

NFA

$X \to 2 \times (\mathcal{P}X)^A$



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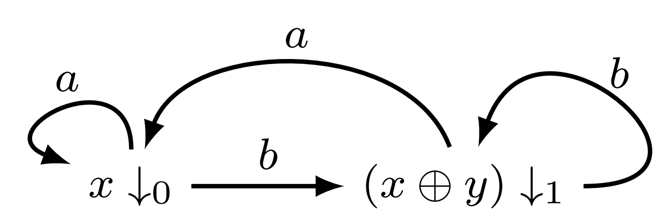

$$\frac{x \xrightarrow{a} t_x,\; y \xrightarrow{a} t_y}{x \oplus y \xrightarrow{a} t_x \oplus t_y}$$

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Algebras for $\mathcal{P}$

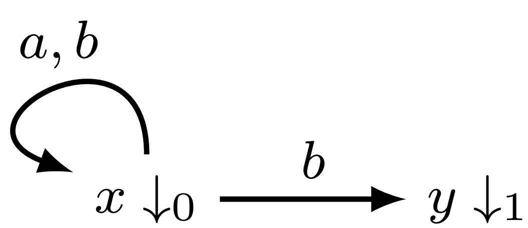
finite powerset !

join semilattices with bottom

Traces via determinisation

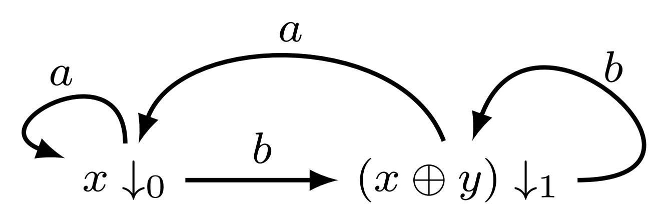
NFA

$X \to 2 \times (\mathcal{P}X)^A$



DFA

$\mathcal{P}X \to 2 \times (\mathcal{P}X)^A$



$$\frac{x \xrightarrow{a} t_x,\ y \xrightarrow{a} t_y}{x \oplus y \xrightarrow{a} t_x \oplus t_y}$$

$$\frac{x{\downarrow}o_x,\ y{\downarrow}o_y}{x \oplus y \downarrow o_x \oplus o_y}$$

Algebras for $\mathcal{P}$

finite powerset !

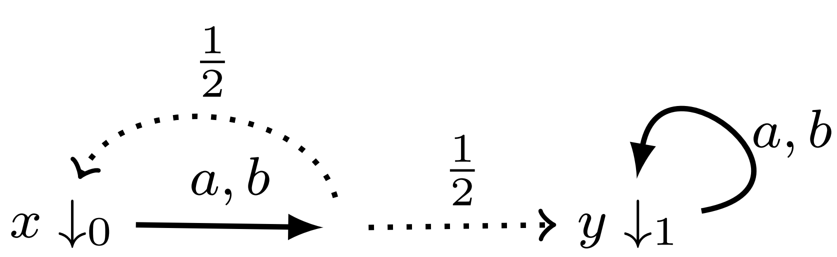
join semilattices with bottom

$2 = \mathcal{P}1$

Traces via determinisation

Rabin PA

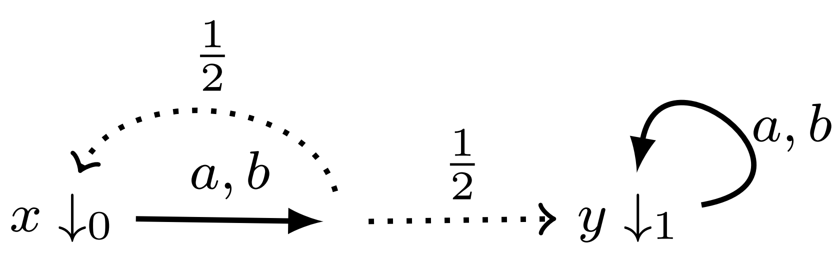
$X \to [0,1] \times (\mathcal{D}X)^A$



Traces via determinisation

Rabin PA

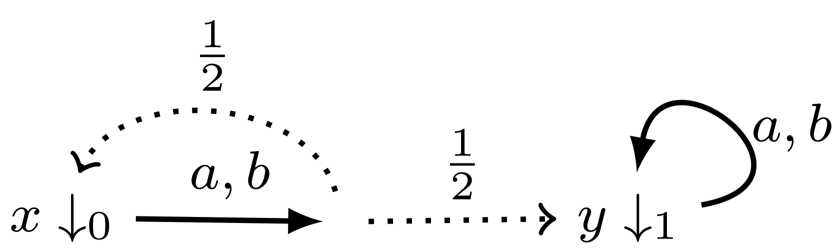
$X \to [0,1] \times (\mathcal{D}X)^A$



Traces via determinisation

Rabin PA

$X \to [0,1] \times (\mathcal{D}X)^A$



DPA

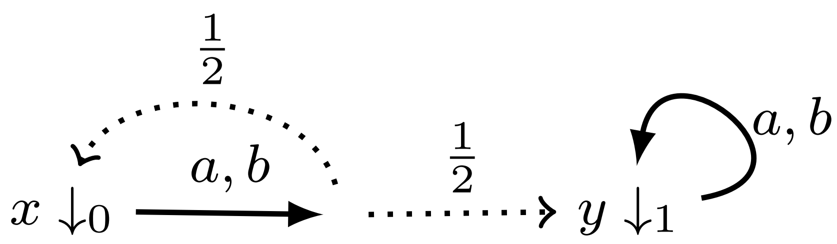
$\mathcal{D}X \to [0,1] \times (\mathcal{D}X)^A$

$$\begin{array}{c} x \downarrow_0 \\ \Big\downarrow a,b \\ (x +_{\frac{1}{2}} y) \downarrow_{\frac{1}{2}} \\ \Big\downarrow a,b \\ (x +_{\frac{1}{4}} y) \downarrow_{\frac{3}{4}} \\ \Big\downarrow a,b \\ \vdots \end{array}$$

Traces via determinisation

Rabin PA

$X \to [0,1] \times (\mathcal{D}X)^A$



DPA

$\mathcal{D}X \to [0,1] \times (\mathcal{D}X)^A$

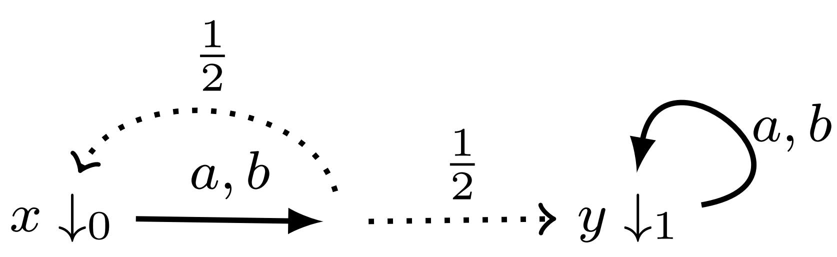
$$
\begin{array}{c}
x \downarrow_0 \\
a,b \big\downarrow \\
(x +_{\frac{1}{2}} y) \downarrow_{\frac{1}{2}} \\
a,b \big\downarrow \\
(x +_{\frac{1}{4}} y) \downarrow_{\frac{3}{4}} \\
a,b \big\downarrow \\
\vdots
\end{array}
$$

Algebras for $\mathcal{D}$

Traces via determinisation

Rabin PA

$X \to [0,1] \times (\mathcal{D}X)^A$



DPA

$\mathcal{D}X \to [0,1] \times (\mathcal{D}X)^A$

$$x \downarrow_0$$
$$\Big\downarrow a, b$$
$$(x +_{\frac{1}{2}} y) \downarrow_{\frac{1}{2}}$$
$$\Big\downarrow a, b$$
$$(x +_{\frac{1}{4}} y) \downarrow_{\frac{3}{4}}$$
$$\Big\downarrow a, b$$
$$\vdots$$

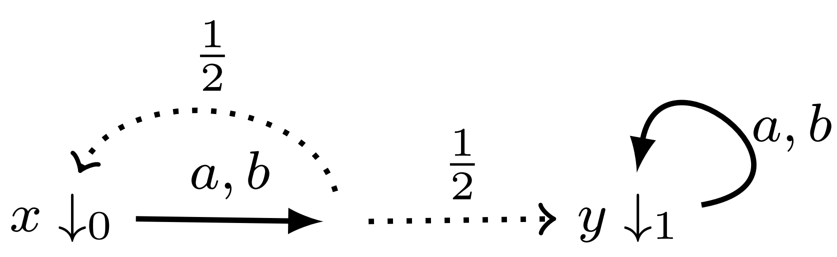
Algebras for $\mathcal{D}$

positive convex algebras

Traces via determinisation

Rabin PA

$X \to [0,1] \times (\mathcal{D}X)^A$



DPA

$\mathcal{D}X \to [0,1] \times (\mathcal{D}X)^A$

$$\begin{array}{c} x \downarrow_0 \\ \Big\downarrow a, b \\ (x +_{\frac{1}{2}} y) \downarrow_{\frac{1}{2}} \\ \Big\downarrow a, b \\ (x +_{\frac{1}{4}} y) \downarrow_{\frac{3}{4}} \\ \Big\downarrow a, b \\ \vdots \end{array}$$

Algebras for $\mathcal{D}$

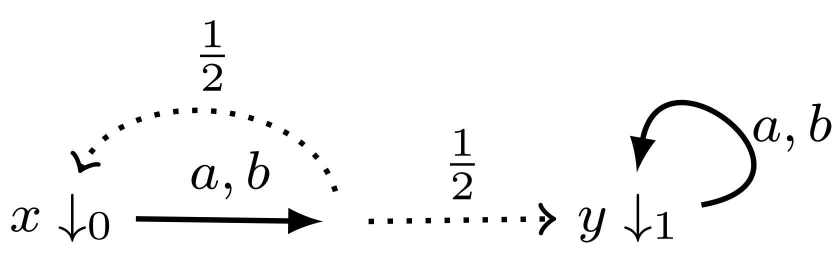
positive convex algebras

finitely supported subdistributions!

Traces via determinisation

Rabin PA

X → [0,1] x $(\mathcal{D}X)^A$



DPA

$\mathcal{D}$X → [0,1] x $(\mathcal{D}X)^A$

$$
\begin{array}{c}
x \downarrow_0 \\
a, b \big\downarrow \\
(x +_{\frac{1}{2}} y) \downarrow_{\frac{1}{2}} \\
a, b \big\downarrow \\
(x +_{\frac{1}{4}} y) \downarrow_{\frac{3}{4}} \\
a, b \big\downarrow \\
\vdots
\end{array}
$$

positive convex algebras

Algebras for $\mathcal{D}$

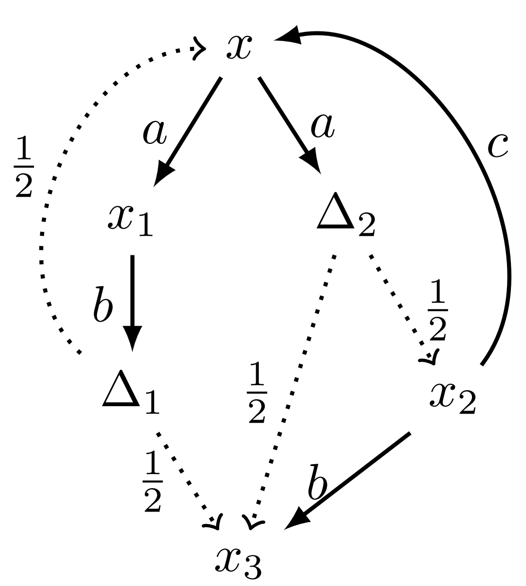
finitely supported subdistributions!

[0,1] = $\mathcal{D}$1

Traces via determinisation

Simple NPA

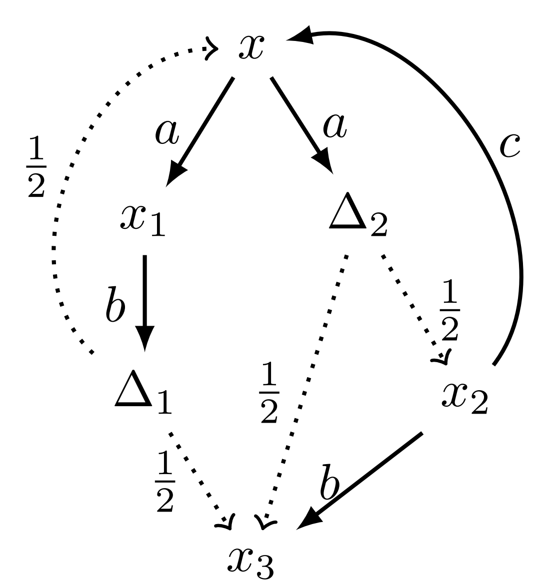
X → ? x ($\mathcal{PD}$X)^A



Traces via determinisation

Simple NPA

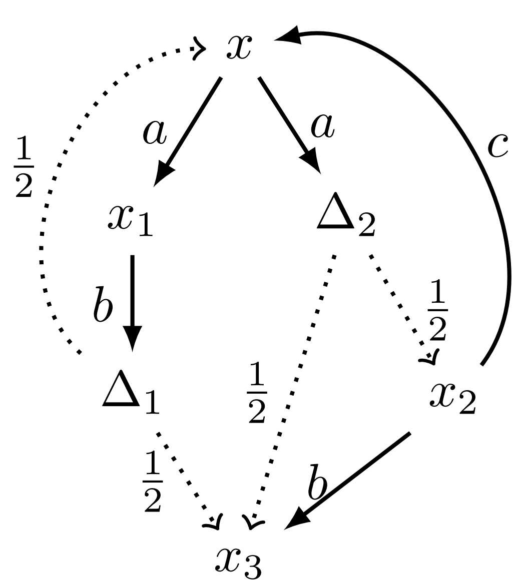
$X \to ? \times (\underline{\mathcal{PD}}X)^A$



Traces via determinisation

Simple NPA

$X \to ? \times (\underline{\mathcal{PD}}X)^A$



DNPA

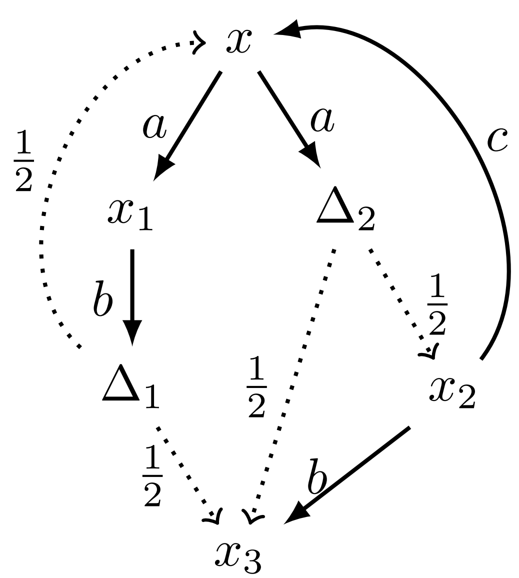
$\underline{\mathcal{PD}}X \to ? \times (\underline{\mathcal{PD}}X)^A$

$$x_1 \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2)$$

Traces via determinisation

Simple NPA

$X \to ? \times (\underline{\mathcal{PD}}X)^A$



DNPA

$\underline{\mathcal{PD}}X \to ? \times (\underline{\mathcal{PD}}X)^A$

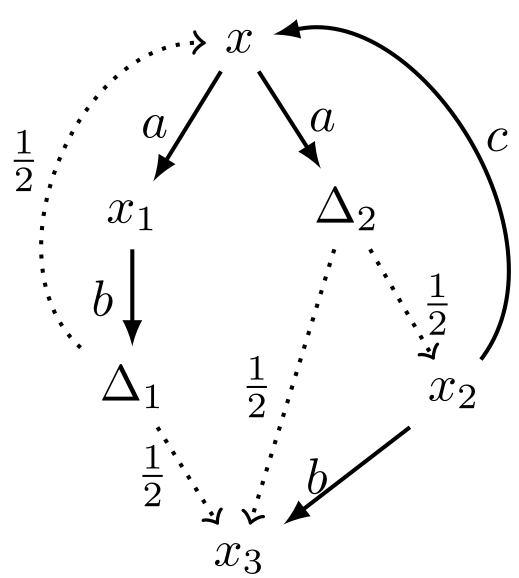
$$x_1 \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2)$$

Algebras for $\underline{\mathcal{PD}}$

Traces via determinisation

Simple NPA

$X \to ? \times (\mathcal{PD}X)^A$



DNPA

$\mathcal{PD}X \to ? \times (\mathcal{PD}X)^A$

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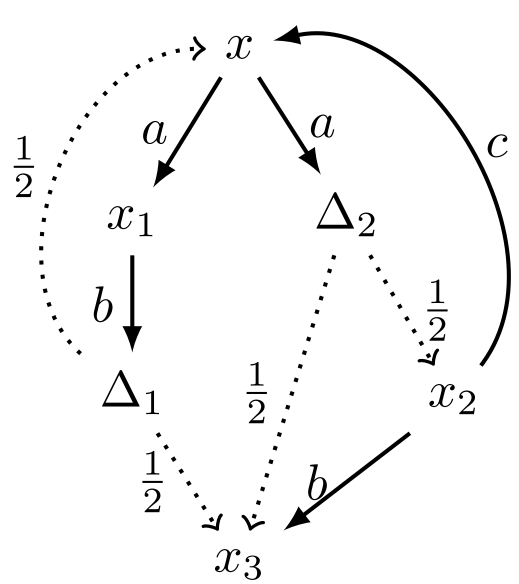
Algebras for $\mathcal{PD}$

convex semilattices

Traces via determinisation

Simple NPA

$X \to ? \times (\mathcal{P}\mathcal{D}X)^A$



DNPA

$\mathcal{P}\mathcal{D}X \to ? \times (\mathcal{P}\mathcal{D}X)^A$

$$x_1 \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2)$$

Algebras for $\mathcal{P}\mathcal{D}$

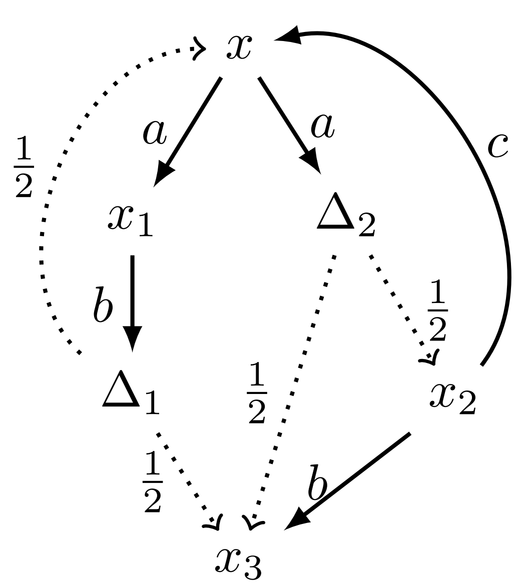
convex semilattices

finitely generated convex sets of distr…

Traces via determinisation

Simple NPA

$X \to ? \times (\underline{\mathcal{PD}}X)^A$



DNPA

$\underline{\mathcal{PD}}X \to ? \times (\underline{\mathcal{PD}}X)^A$

$$x_1 \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2)$$

$? = \underline{\mathcal{PD}}\,1$

Algebras for $\underline{\mathcal{PD}}$

convex semilattices

finitely generated convex sets of distr…

Presentation for $\underline{\mathcal{PD}}$

Algebras for $\underline{\mathcal{PD}}$

convex semilattices

finitely generated convex sets of distr…

Presentation for $\mathcal{PD}$

Algebras for $\mathcal{PD}$

convex semilattices

finitely generated convex sets of distr…

Bonchi, S., Vignudelli '19

Presentation for $\mathcal{PD}$

Algebras for $\mathcal{PD}$

convex semilattices

finitely generated convex sets of distr…

Bonchi, S., Vignudelli ‘19

$$\mathbb{A} = (A, \oplus, +_p)$$

Presentation for $\mathcal{PD}$

Algebras for $\mathcal{PD}$

convex semilattices

Bonchi, S., Vignudelli '19

finitely generated convex sets of distr…

$$\mathbb{A} = (A, \oplus, +_p)$$

$p \in (0,1)$

Presentation for $\mathcal{PD}$

Algebras for $\mathcal{PD}$

convex semilattices

Bonchi, S., Vignudelli '19

finitely generated convex sets of distr…

$$\mathbb{A} = (A, \oplus, +_p)$$

$p \in (0,1)$

$$(x \oplus y) \oplus z \stackrel{(A)}{=} x \oplus (y \oplus z)$$
$$x \oplus y \stackrel{(C)}{=} y \oplus x$$
$$x \oplus x \stackrel{(I)}{=} x$$

$$(x +_q y) +_p z \stackrel{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$$
$$x +_p y \stackrel{(C_p)}{=} y +_{1-p} x$$
$$x +_p x \stackrel{(I_p)}{=} x$$

$$(x \oplus y) +_p z \stackrel{(D)}{=} (x +_p z) \oplus (y +_p z)$$

Presentation for $\mathcal{PD}$

Algebras for $\mathcal{PD}$

convex semilattices

Bonchi, S., Vignudelli '19

finitely generated convex sets of distr…

$$\mathbb{A} = (A, \oplus, +_p)$$

$p \in (0,1)$

semilattice

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Presentation for $\mathcal{PD}$

Algebras for $\mathcal{PD}$

convex semilattices

Bonchi, S., Vignudelli '19

finitely generated convex sets of distr…

$$\mathbb{A} = (A, \oplus, +_p)$$

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$$\begin{aligned} (x \oplus y) \oplus z &\overset{(A)}{=} x \oplus (y \oplus z) \\ x \oplus y &\overset{(C)}{=} y \oplus x \\ x \oplus x &\overset{(I)}{=} x \end{aligned}$$

semilattice

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convex algebra

$$(x \oplus y) +_p z \overset{(D)}{=} (x +_p z) \oplus (y +_p z)$$

Presentation for $\mathcal{PD}$

Algebras for $\mathcal{PD}$

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Bonchi, S., Vignudelli ‘19

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semilattice

S., Woracek ’15, ’17, ‘18

$$(x +_q y) +_p z \overset{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$$
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convex algebra

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Presentation for $\mathcal{PD}$

Algebras for $\mathcal{PD}$

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Bonchi, S., Vignudelli '19

finitely generated convex sets of distr…

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x \oplus y &\stackrel{(C)}{=} y \oplus x\\
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\end{aligned}$$

semilattice

S., Woracek '15, '17, '18

$$\begin{aligned}
(x +_q y) +_p z &\stackrel{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)\\
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\end{aligned}$$

convex algebra

$$(x \oplus y) +_p z \stackrel{(D)}{=} (x +_p z) \oplus (y +_p z)$$

distributivity

The observation:

What if it is all just monads ?

$$X \to MX$$

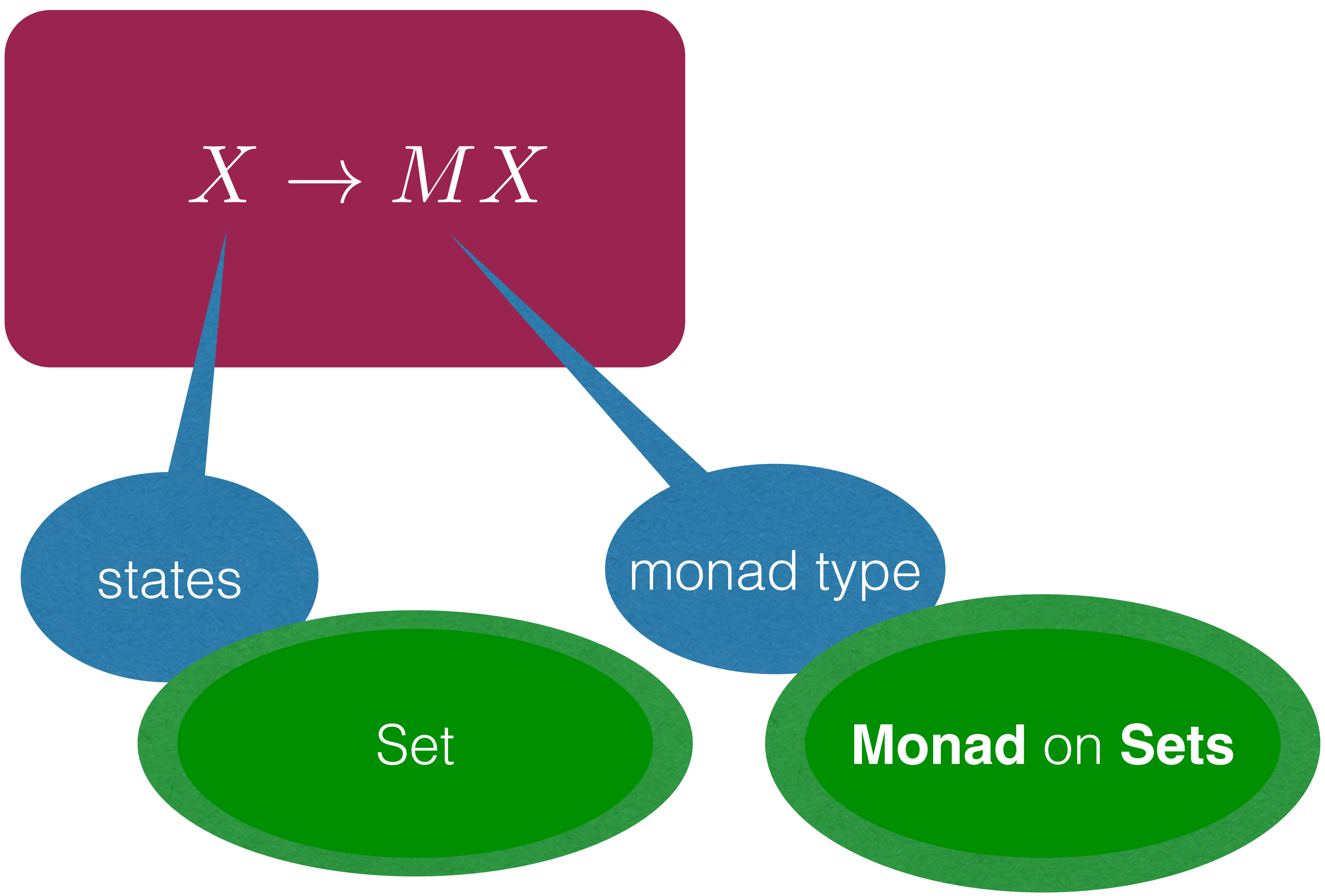
states

Set

Monad on **Sets**

The observation:

What if it is all just monads ?



The observation:

What if it is all just monads ?

$X \to MX$

states

monad type

Set

Monad on **Sets**

generic notion of **trace semantics**

The observation:

What if it is all just monads ?

$X \to MX$

generic notion of **trace semantics**

states

monad type

Set

Monad on **Sets**

$\mu \colon MM \Rightarrow M$

$\eta \colon Id \Rightarrow M$

Generic Traces

Generic Traces

$c: X \to MX, \;\; c^{\#}: MX \to MMX \to MX$

Generic Traces

c: X → MX, c#: MX → MMX → MX

$$X \xrightarrow{!} 1 \xrightarrow{\eta} M1$$

Generic Traces

$c\colon X \to MX, \quad c^{\#}\colon MX \to MMX \to MX$

$X \xrightarrow{!} 1 \xrightarrow{\eta} M1$

$tr\colon X \to (\mathbb{N} \to M1)$

Generic Traces

c: X → MX, c#: MX → MMX → MX

$X \xrightarrow{!} 1 \xrightarrow{\eta} M1$

$tr\colon X \to (\mathbb{N} \to M1)$

$tr\colon \mathbb{N} \to (X \to M1)$

Generic Traces

c: X → MX, c#: MX → MMX → MX

$$X \xrightarrow{!} 1 \xrightarrow{\eta} M1$$

$$tr\colon X \to (\mathbb{N} \to M1)$$

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$$tr(0) = \eta \circ !, \quad tr(n+1) = M(!) \circ (c^{\#})^n \circ c$$

Generic Traces

c: X → MX, c#: MX → MMX → MX

$X \xrightarrow{!} 1 \xrightarrow{\eta} M1$

$tr\colon X \to (\mathbb{N} \to M1)$

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$$tr(0) = \eta \circ !, \quad tr(n+1) = M(!) \circ (c^{\#})^n \circ c$$

graded semantics

Kurz, Pattinson, Schröder,…

Generic Traces

c: X → MX, $c^\#$: MX → MMX → MX

$$X \xrightarrow{!} 1 \xrightarrow{\eta} M1$$

$$tr\colon X \to (\mathbb{N} \to M1)$$

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$$tr(0) = \eta \circ !, \quad tr(n+1) = M(!) \circ (c^{\#})^n \circ c$$

graded semantics

Kurz, Pattinson, Schröder,…

WiP :-)

Goy, S., Petrisan


Thank You