

WEEK 7, Task 4

Show via a calculation that:

$$\forall k [P : Q \vee R] \stackrel{\text{val}}{=} \forall k [P \wedge \neg Q : R]$$

We present two solutions.

Solution I: This is a generic solution applicable whenever we want to show that two universally quantified formulas are equivalent.

We have

$$\forall k [P : Q \vee R] \stackrel{\text{val}}{=} \forall k [T : P \Rightarrow Q \vee R]$$

{Domain weakening}

and

$$\forall k [P \wedge \neg Q : R] \stackrel{\text{val}}{=} \forall k [T : P \wedge \neg Q \Rightarrow R]$$

{Domain weakening}

Next we show that $P \Rightarrow Q \vee R \stackrel{\text{val}}{=} P \wedge \neg Q \Rightarrow R$ and the result follows from an application of Leibniz:

We have

$$P \Rightarrow Q \vee R \stackrel{\text{val}}{=} \neg P \vee Q \vee R \quad (3)$$

{Implication}

and

$$P \wedge \neg Q \Rightarrow R \stackrel{\text{val}}{=} \neg(P \wedge \neg Q) \vee R$$

{Implication}

$$\stackrel{\text{val}}{=} \neg P \vee \neg \neg Q \vee R$$

{De Morgan}

$$\stackrel{\text{val}}{=} \neg P \vee Q \vee R \quad (4)$$

{Double negation}

Plugging (1), (2), (3), and (4) together, we get the proof that we wanted, as follows.

$$\begin{aligned}
 \forall x [P: Q \vee R] &\stackrel{\text{val}}{=} \forall x [T: P \Rightarrow Q \vee R] \\
 (1) \\
 &\stackrel{\text{Leibniz} + \text{val}}{=} \forall x [T: \neg P \vee Q \vee R] \\
 (3) \\
 &\stackrel{\text{Leibniz} + \text{val}}{=} \forall x [T: P \wedge \neg Q \Rightarrow R] \\
 (4) \\
 &\stackrel{\text{val}}{=} \forall x [P \wedge \neg Q: R]. \\
 (2)
 \end{aligned}$$

[Of course we could expand all steps here, in particular those of (4), to get the full proof.]

Solution 2: (Less structured/methodical but shorter)
We start from the right-hand side, since it suggests possible simplification.

$$\begin{aligned}
 \forall x [P \wedge \neg Q: R] &\stackrel{\text{val}}{=} \forall x [P: \neg Q \Rightarrow R] \\
 &\quad \text{Subst. + \{Domain weakening\}} \\
 &\stackrel{\text{val}}{=} \forall x [P: \neg \neg Q \vee R] \\
 &\quad \text{Leibniz + \{implication\}} \\
 &\stackrel{\text{val}}{=} \forall x [P: Q \vee R] \\
 &\quad \text{Leibniz + \{double negation\}}
 \end{aligned}$$

REMARK:

Note that we used exactly the same standard equivalences in both solutions.

Moreover, we actually used the same method in both solutions, we only did not weaken the domain completely in Solution 2, which leads to a shorter proof.
Also, the method is implicit in Solution 2.

Week 7, Task 6

Show with a counter example that

$$\exists x [P:Q] \wedge \exists x [P:R] \stackrel{val}{\neq} \exists x [P:Q \wedge R]$$

Intuitively, the formula on the left-hand side states that there exists an element in the domain described by P that satisfies Q AND there exists an element (possibly another one) in the same domain that satisfies R . The formula on the right-hand side is stronger. It states that there is one element in this domain that satisfies both P and Q .

It is not difficult to come up with a counter example. One very simple one is the following:

Let P be the predicate $x \in \{0,1\}$.

Q the predicate $x=0$, and

R the predicate $x=1$.

Then the left-hand side is the proposition

$$\exists x [x \in \{0,1\} : x=0] \wedge \exists x [x \in \{0,1\} : x=1]$$

which is a true proposition.

The right-hand side is the proposition

$$\exists x [x \in \{0,1\} : x=0 \wedge x=1]$$

which is a false proposition ($0 \neq 1$).