Coalgebra for Computer Scientists

Exam, 19.7.2012

Task 1. (10 points) Let F be a functor on a category $\mathbf{C}$. Show that $\mathbf{CoAlg}(F)$ is a category, the category of F-coalgebras and coalgebra homomorphisms.

Task 2. (20 points) Let $(M, +, 0)$ be a monoid. Show that the following definition provides a functor $\mathcal{M}$ on **Sets**. On objects,

$$\mathcal{M}(X) = \{\varphi\colon X \to M \mid \mathrm{supp}(\varphi) \text{ is finite }\}$$

where $\mathrm{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}$.

On functions, for $f\colon X \to Y$, the map $\mathcal{M}(f)\colon \mathcal{M}(X) \to \mathcal{M}(Y)$ is given by

$$\mathcal{M}(f)(\varphi) = \lambda y. \sum_{x \in f^{-1}(\{y\})} \varphi(x)$$

for $\varphi \in \mathcal{M}(X)$.

The functor $\mathcal{M}$ is called the multiset functor, in particular if the monoid is the monoid of natural numbers $(\mathbb{N}, +, 0)$). Why is the requirement of finite support necessary? Why does M need to be a monoid?

Task 3. (20 points) Let F and G be functors on **Sets**. Let $\tau\colon F \Rightarrow G$ be a natural transformation, i.e., a set-indexed collection of maps τ_X for $X \in \mathbf{Sets}$ satisfying $\tau_Y \circ Ff = Gf \circ \tau_X$ for any function $f\colon X \to Y$.
Show that $T_\tau\colon \mathbf{CoAlg}(F) \to \mathbf{CoAlg}(G)$ given by

$$T_\tau(c\colon X \to FX) = (\tau_X \circ c\colon X \to GX)$$

on objects and $T_\tau(h) = h$ on morphisms, is a functor. Show that T_τ preserves bisimilarity, behavioral equivalence, and final coalgebra semantics (in case the final coalgebras exist), i.e., if $s \equiv t$ in $c\colon X \to FX$, then $s \equiv t$ in $T_\tau(c\colon X \to FX)$ as well, for each of the mentioned semantics in place of $\equiv$.

Task 4. (10 points) Let $c\colon X \to A \times X + 1$ be a coalgebra of the sequence functor $F(-) = A \times (-) + 1$ for $A = \{a\}$ with states $\{x, y\}$ and $c(x) = (a, y)$, $c(y) = *$. Show that $x \not\sim y$ where $\sim$ denotes the bisimilarity equivalence on the coalgebra c.