Compositionality for Markov Reward Chains with Fast and Silent Transitions

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Abstract

A parallel composition is defined for Markov reward chains with stochastic discontinuity, fast, and silent transitions. In this setting, compositionality with respect to the relevant aggregation preorders is established. For Markov reward chains with fast transitions the preorders are τ-lumping and τ-reduction. Discontinuous Markov reward chains are 'limits' of Markov reward chains with fast transitions, and have related notions of lumping and reduction. Markov reward chains with silent transitions are equivalence classes of Markov reward chains with fast transitions and are equipped with the lifted preorders $\tau_{\sim}$-lumping and $\tau_{\sim}$-reduction. In total, six compositionality results are presented. Additionally, the parallel operators are related by a continuity result.

Keywords: Markov reward chains, fast transitions, silent transitions, parallel composition, aggregation

1 Introduction

Compositionality is a central issue in the theory of concurrent processes. Discussing compositionality requires three ingredients: (1) a class of processes or models; (2) an operation to compose processes; and (3) a notion of behaviour,

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usually given by a semantic preorder or equivalence relation on the class of processes. For the purpose of this paper, we will have semantic preorders and the parallel composition as operation. Therefore, the compositionality result can be stated as

$$\mathsf{P}_1 \geqslant \overline{\mathsf{P}}_1,\ \mathsf{P}_2 \geqslant \overline{\mathsf{P}}_2 \implies \mathsf{P}_1 \parallel \mathsf{P}_2 \geqslant \overline{\mathsf{P}}_1 \parallel \overline{\mathsf{P}}_2\,,$$

where P_1, P_2, $\overline{\mathsf{P}}_1$, and $\overline{\mathsf{P}}_2$ are arbitrary processes and $\parallel$ and $\geqslant$ denote their parallel composition and the semantic preorder relation, respectively. Hence, compositionality enables the narrowing of a parallel composition by composing simplifications of its components, thus avoiding the construction of the actual parallel system. In this paper, we study compositionality for augmented types of Markov chains.

Homogeneous continuous-time Markov chains, Markov chains for short, are among the most important and wide-spread analytical performance models. A Markov chain is given by a graph with nodes representing states and outgoing arrows labelled by exponential rates determining the stochastic behavior of each state. An initial probability vector indicates which states may act as starting ones. Markov chains often come equipped with rewards that are used to measure their performance, such as throughput, utilization, etc. (cf. [1]). In this paper, we focus on state rewards only, and we refer to a Markov chain with rewards as a Markov reward chain. Transition (impulse) rewards [1] can similarly be dealt with. A state reward is a number associated to a state, representing the rate at which gain is received while the process resides in the state.

To cope with the ever growing complexity of the systems, several performance modeling techniques have been developed to support the compositional generation of Markov reward chains. This includes stochastic process algebras [2,3], (generalized) stochastic Petri nets [4,5], probabilistic I/O automata [6,7], stochastic automata networks [8], etc. The compositional modeling enables composing a bigger system from several smaller components. The size of the state space of the resulting system is in the range of the product of the sizes of the constituent state spaces. Hence, compositional modeling usually suffers from state space explosion.

In the process of compositional modeling, performance evaluation techniques produce intermediate constructs that are typically extensions of Markov chains featuring transitions with communication labels [2–8]. In the final modeling phase, all labels are discarded and communication transitions are assigned instantaneous behavior. Previous work [9–11] gave an account of handling these models by using Markov reward chains with fast transitions and Markov reward chains with silent transitions. The former present extensions of the standard Markov reward chains with transitions decorated with a real-valued linear parameter and in the later the real-valued linear parameter is not specified. To capture the intuition that the labeled transitions are instantaneous, a limit for the parameter to infinity is taken. The resulting process is a generalization of the standard Markov reward chain that can perform infinitely many transitions in a finite amount of time. This model was initially studied in [12,13] without rewards, and it is called a (stochastically) discontinuous Markov reward chain. The process exhibits stochastic discontinuity and is often considered pathological. However, as shown in [13,14,5], it proves very useful for the explanation of results.

Here, we consider discontinuous Markov reward chains, Markov reward chains with fast transitions, and Markov reward chains with silent transitions. These three models are intimately related: Markov reward chains with fast and silent transitions are used for modeling, but some notions for these processes are expressed asymptotically in terms of discontinuous Markov reward chains. A limiting process of a Markov reward chain with fast transitions is a discontinuous Markov reward chain; a Markov reward chain with silent transitions is identified with an equivalence class of a relation $\sim$ on Markov reward chains with fast transitions relating chains with the 'same shape of fast transitions'. We define parallel composition of all models in vein of standard Markov reward chains [15] using Kronecker products and sums.

As already mentioned, compositional modeling may lead to state space explosion. Current analytical and numerical methods can efficiently handle Markov reward chains with millions of states [16,17]. However, they only alleviate the problem and many real world problems still cannot be feasibly solved. Several aggregation techniques have been proposed to reduce the state space of Markov reward chains. Ordinary lumping is the most prominent one [18,15]. The method partitions the state space into partition classes. In each class, the states exhibit equivalent behavior for transiting to other classes, i.e., the cumulative probability of transiting to another class is the same for every state of the class. If non-trivial lumping exists, i.e., at least one partition contains more than one state, then the method produces a smaller Markov chain that retains the performance characteristics of the original one. For example, the expected reward rate at a given time is the same for the original as for the reduced, so-called lumped, process. Another lumping-based method is exact lumping [19,15]. This method requires that each partition class of states has the same cumulative probability of transiting to every state of another class and, moreover, each state in the class has the same initial probability. The gain of exact lumping is that the probabilities of the original process can be computed for a special class of initial probability vectors by using the lumped Markov reward chain only.

A preliminary treatment of relational properties of lumping-based aggregations of Markov chains has been given in [20]. It has been shown that the notion of exact lumping is not transitive, i.e., there are processes which have exactly lumped versions that can be non-trivially exactly lumped again, but the original process cannot be exactly lumped directly to the resulting process. On the other hand, ordinary lumping of Markov reward chains is transitive and, moreover, it has a property of strict confluence. Strict confluence means that whenever a process can be lumped using two different partitions, there is always a smaller process to which the lumped processes can lump to. Coming back to our models of interest, ordinary lumping is defined for discontinuous Markov reward chains in [9–11]. Also, so-called τ-lumping is proposed for Markov reward chains with fast transitions in [9–11]. The two methods are in agreement and the situation can be pictured as follows:

$$
\begin{tikzcd}
\begin{array}{c}\text{Markov reward chain}\\ \text{with fast transitions}\end{array} \arrow[r, "\tau\text{-lumping}"] \arrow[d, "\text{limit}"'] & \begin{array}{c}\tau\text{-lumped}\\ \text{Markov reward chain}\\ \text{with fast transitions}\end{array} \arrow[d, "\text{limit}"] \\
\begin{array}{c}\text{discontinuous}\\ \text{Markov reward chain}\end{array} \arrow[r, "\begin{array}{c}\text{ordinary}\\ \text{lumping}\end{array}"] & \begin{array}{c}\text{lumped discontinuous}\\ \text{Markov reward chain.}\end{array}
\end{tikzcd}
$$

For Markov reward chains with silent transitions, a lifting of τ-lumping to the $\sim$-equivalence classes is proposed, referred to as $\tau_\sim$-lumping [9–11]. The lifting idea is justified if the τ-lumped processes do not depend on the choice of the representative Markov reward chain with fast transitions, depicted in the following figure.

$$
\begin{tikzcd}
\begin{array}{c}\text{Markov reward chain}\\ \text{with fast transitions}\end{array} \arrow[r, phantom, "\sim"] \arrow[d, "\tau\text{-lumping}"'] & \begin{array}{c}\text{Markov reward chain}\\ \text{with fast transitions}\end{array} \arrow[d, "\tau\text{-lumping}"] \\
\begin{array}{c}\tau\text{-lumped Markov reward chain}\\ \text{with fast transitions}\end{array} \arrow[r, phantom, "\sim"] & \begin{array}{c}\tau\text{-lumped Markov reward chain}\\ \text{with fast transitions.}\end{array}
\end{tikzcd}
$$

In addition, [10,11] study an aggregation method by reduction that eliminates the stochastic discontinuity and reduces a discontinuous Markov reward chain to a Markov reward chain. The reduction method is an extension of a well-known method in perturbation theory [21,22,13]. Its advantage is the ability to split states. The lumping method, in contrast, provides more flexibility: also states that do not exhibit discontinuous behavior can be aggregated. The reduction-based aggregation straightforwardly extends to τ-reduction of Markov reward chains with fast transitions [10,11]. Therefore, we have the following situation:

$$\begin{tikzcd}
\begin{array}{c}\text{Markov reward chain}\\ \text{with fast transitions}\end{array} \arrow[d, "\text{limit}"'] \arrow[dr, "\tau\text{-reduction}"] & \\
\begin{array}{c}\text{discontinuous}\\ \text{Markov reward chain}\end{array} \arrow[r, "\text{reduction}"] & \text{Markov reward chain.}
\end{tikzcd}$$

In the case of Markov reward chains with silent transitions, a direct lifting of the τ-reduction to equivalence classes does not aggregate many processes, as most of the time the reduced process depends on the actual fast transitions [10,11]. In an attempt to remedy the effect of the fast transitions we combine τ-reduction and standard ordinary lumping for Markov reward chains to obtain $\tau_\sim$-reduction as depicted below. [1]

$$\begin{tikzcd}
\begin{array}{c}\text{Markov reward chain}\\ \text{with fast transitions}\end{array} \arrow[d, "\tau\text{-reduction}"] \arrow[ddr, bend right=60, "\tau_\sim\text{-reduction}"'] & \sim & \begin{array}{c}\text{Markov reward chain}\\ \text{with fast transitions}\end{array} \arrow[d, "\tau\text{-reduction}"'] \arrow[ddl, bend left=60, "\tau_\sim\text{-reduction}"] \\
\begin{array}{c}\text{Markov}\\ \text{reward chain}\end{array} \arrow[dr, "\begin{array}{c}\text{ordinary}\\ \text{lumping}\end{array}"'] & \sim & \begin{array}{c}\text{Markov}\\ \text{reward chain}\end{array} \arrow[dl, "\begin{array}{c}\text{ordinary}\\ \text{lumping}\end{array}"] \\
& \begin{array}{c}\text{Markov}\\ \text{reward chain.}\end{array} &
\end{tikzcd}$$

Both the lumping-based and the reduction-based aggregation method induce semantic relations. Namely, for two processes P and $\overline{\mathsf{P}}$ we say that $\mathsf{P} \geqslant \overline{\mathsf{P}}$ if $\overline{\mathsf{P}}$ is an aggregated version of P. As already mentioned, compositionality is very important as it allows us to aggregate the smaller parallel components first, and then combine them into the aggregated complete system. We show that the relations induced by the lumping and reduction methods are indeed preorders, i.e., reflexive and transitive relations. Having all the ingredients in place, we show the compositionality of the aggregation preorders with respect to the defined parallel composition(s). We also show continuity of the parallel composition(s). In short, the parallel operators preserve the diagrams above.

The structure of the rest of the paper is as follows. We start by defining the three types of Markovian models in Section 2. Section 3 and Section 4 focus on the aggregation methods based on lumping and reduction for each of

[1] The method is called total $\tau_\sim$-reduction in [10,11], since there more $\tau_\sim$-reduction methods are considered.

the models, respectively. In Section 5, we show that the aggregation methods define preorders on the models. Section 6 contains the main results of the paper: compositionality of the new parallel operators for each type of Markov chains with respect to both aggregation preorders. Section 7 wraps up with conclusions.

Notation All vectors are column vectors if not indicated otherwise. By $\mathbf{1}^n$ we denote the vector of n 1's; by $\mathbf{0}^{n\times m}$ the $n \times m$ zero matrix; by I^n the $n \times n$ identity matrix. We omit the dimensions n and m when they are clear from the context. By $A[i, j]$ we denote an element of the matrix $A \in \mathbb{R}^{m\times n}$ assuming $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. We write $A \geqslant 0$ when all elements of A are non-negative. The matrix A is called stochastic if $A \geqslant 0$ and $A \cdot \mathbf{1} = \mathbf{1}$. By A^{T} we denote the transpose of A.

Let $\mathcal{S}$ be a finite set. A set $\mathcal{P} = \{S_1, \ldots, S_N\}$ of N subsets of $\mathcal{S}$ is called a partition of $\mathcal{S}$ if $\mathcal{S} = S_1 \cup \ldots \cup S_N$, $S_i \neq \emptyset$ and $S_i \cap S_j = \emptyset$ for all i, j, with $i \neq j$. The partitions $\left\{\mathcal{S}\right\}$ and $\Delta = \left\{\{i\} \mid i \in \mathcal{S}\right\}$ are the trivial partitions. Let $\mathcal{P}_1 = \{S_1, \ldots, S_N\}$ be a partition of $\mathcal{S}$ and $\mathcal{P}_2 = \{T_1, \ldots, T_M\}$, in turn, a partition of $\mathcal{P}_1$. The composition $\mathcal{P}_1 \circ \mathcal{P}_2$ of the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ is a partition of $\mathcal{S}$, given by $\mathcal{P}_1 \circ \mathcal{P}_2 = \{\, U_1, \ldots, U_M \,\}$, where $U_i = \bigcup_{C\in T_i} C$.

2 Markovian Models

In this section we introduce the Markovian models studied in this paper: discontinuous Markov reward chains as generalizations of standard Markov reward chains where infinitely many transitions can be performed in a finite amount of time; Markov reward chains with fast transitions as Markov reward chains parameterized by a real variable τ; and Markov reward chains with silent transitions as equivalence classes of Markov reward chains with fast transitions with the same structure and unspecified 'speeds' of the fast transitions. The fast transitions explicitly model stochastic behavior, while the silent transitions model non-deterministic internal steps.

2.1 Discontinuous Markov Reward Chains

In the standard theory (cf. [23,24,1]) Markov chains are assumed to be stochastically continuous. This means that when $t \to 0$, the probability of the process occupying at time t the same state as at time 0 is 1. As we include instantaneous transitions in our theory [13], this requirement must be dropped.

Therefore, we work in the more general setting of discontinuous Markov chains originating from [12].

A discontinuous Markov reward chain is a time-homogeneous finite-state stochastic process with an associated (state) reward structure that satisfies the Markov property. It is completely determined by: (1) a stochastic initial probability row vector that gives the starting probabilities of the process for each state, (2) a transition matrix function $P : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ that defines the stochastic behavior of the transitions at time $t > 0$, and (3) a state reward rate vector that associates a number to each state representing the gain of the process while spending time in the state. The transition matrix function gives a stochastic matrix $P(t)$ at any time $t > 0$, and has the property $P(t+s) = P(t)\cdot P(s)$ [23,24]. It has a convenient characterization independent of time [13,25], which allows for the following equivalent definition.

Definition 1 *A discontinuous Markov reward chain* D *is a quadruple* $\mathsf{D} = (\sigma, \Pi, Q, \rho)$*, where* σ *is a stochastic initial probability row vector,* ρ *is a state reward vector and* $\Pi \in \mathbb{R}^{n\times n}$ *and* $Q \in \mathbb{R}^{n\times n}$ *satisfy the following six conditions: (1)* $\Pi \geqslant 0$*, (2)* $\Pi \cdot \mathbf{1} = \mathbf{1}$*, (3)* $\Pi^2 = \Pi$*, (4)* $\Pi Q = Q\Pi = Q$*, (5)* $Q \cdot \mathbf{1} = \mathbf{0}$*, and (6)* $Q + c\Pi \geqslant 0$*, for some* $c \geqslant 0$*. The matrix function* $P(t) = \Pi e^{Qt}$ *is the transition matrix of* D*.*

We note that the transition matrix uniquely determines the matrices Π and Q as given in Definition 1. It is continuous at zero if and only if $\Pi = I$. In this case, Q is a standard generator matrix [13,9]. Otherwise, the matrix Q might contain negative non-diagonal entries. We note that, unlike for standard Markov reward chains, a meaningful graphical representation of discontinuous Markov reward chains when $\Pi \neq I$ is not common. The intuition behind the matrix Π is that $\Pi[i,j]$ denotes the probability that a process occupies two states via an instantaneous transition. Therefore, in case of no instantaneous transitions, i.e., when $\Pi = I$, we get a standard (stochastically continuous) Markov reward chain denoted by $\mathsf{M} = (\sigma, Q, \rho)$.

For every discontinuous Markov reward chain $\mathsf{D} = (\sigma, \Pi, Q, \rho)$, Π gets the following 'ergodic' form after a suitable renumbering of states [13], viz.

$$\Pi = \begin{pmatrix} \Pi_1 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \dots & \Pi_M & \mathbf{0} \\ \overline{\Pi}_1 & \dots & \overline{\Pi}_M & \mathbf{0} \end{pmatrix} \qquad L = \begin{pmatrix} \mu_1 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \dots & \mu_M & \mathbf{0} \end{pmatrix} \qquad R = \begin{pmatrix} \mathbf{1} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{1} \\ \delta_1 & \dots & \delta_M \end{pmatrix}$$

where for all $1 \leqslant k \leqslant M$, $\Pi_k = \mathbf{1} \cdot \mu_k$ and $\overline{\Pi}_k = \delta_k \cdot \mu_k$ for a row vector $\mu_k > 0$ such that $\mu_k \cdot \mathbf{1} = 1$ and a vector $\delta_k \geqslant 0$ such that $\sum_{k=1}^{M} \delta_k = \mathbf{1}$. Then the pair of matrices (L, R) depicted above forms a canonical product decomposition of Π (cf. Section 4.1), which is needed for the definition of the reduction-based method of aggregation.

The new numbering induces a partition $\mathcal{E} = \{E_1, \ldots, E_M, T\}$ of the state space $\mathcal{S} = \{1, \ldots, n\}$, where $E_1, \ldots, E_M$ are the ergodic classes, determined by $\Pi_1, \ldots, \Pi_M$, respectively, and T is the class of transient states, determined by any $\overline{\Pi}_i$, $1 \leqslant i \leqslant M$. The partition $\mathcal{E}$ is called the ergodic partition. For every ergodic class E_k, the vector μ_k is the vector of ergodic probabilities. If an ergodic class E_k contains exactly one state, then $\mu_k = (\,1\,)$ and the state is called regular. The vector δ_k contains the trapping probabilities from transient states to the ergodic class E_k.

We next discuss the behavior of a discontinuous Markov reward chain $\mathsf{D} = (\sigma, \Pi, Q, \rho)$. It starts in a state with a probability given by the initial probability vector σ. In an ergodic class with multiple states the process spends a non-zero amount of time switching rapidly (infinitely many times) among the states. The probability that it is found in a specific state of the class is given by the vector of ergodic probabilities. The time the process spends in the class is exponentially distributed and determined by the matrix Q. In an ergodic class with a single state the row of Q corresponding to that state has the form of a row in a generator matrix, and $Q[i,j]$ for $i \neq j$ is interpreted as the rate from i to j. In a transient state the process spends no time (with probability one) and it immediately becomes trapped in some ergodic class. The process in $i \in T$ can be trapped in E_k if and only if the trapping probability $\delta_k[i] > 0$.

The expected reward (rate) at time $t > 0$, notation $R(t)$, is obtained as $R(t) = \sigma P(t)\rho$. It is required in the calculation of the expected accumulated reward up to time t, given by $\int_0^t R(s)ds$. We have that the expected reward remains unchanged if the reward vector ρ is replaced by $\Pi\rho$. To see this, we use that $P(t) = P(t)\Pi$ (cf. [13,11]), so $\sigma P(t)\Pi\rho = \sigma P(t)\rho = R(t)$. Intuitively, the reward in a transient state can be replaced by the sum of the rewards of the ergodic states that it can get trapped in as the process gains no reward while transiting through transient states. The reward of an ergodic state is the sum of the rewards of all states inside its ergodic class weighted according to their ergodic probabilities. This alternative representation of the reward vector alleviates the presentation of some aggregation methods in later sections.

2.2 Markov Reward Chains with Fast Transitions

A Markov reward chain with fast transitions is obtained by adding parameterized, so-called fast, transitions to a standard Markov reward chain. The remaining standard transitions are referred to as slow. The behavior of a Markov reward chain with fast transitions is determined by two generator matrices S

and F, which represent the rates of the slow transitions and the rates (called speeds) of the fast transitions, respectively.

Definition 2 *A Markov reward chain with fast transitions* $\mathsf{F} = (\sigma, S, F, \rho)$ *is a function assigning to each* $\tau > 0$*, the parameterized Markov reward chain*

$$\mathsf{M}_\tau = (\sigma, S + \tau F, \rho)$$

where $\sigma \in \mathbb{R}^{1\times n}$ *is an initial probability vector,* $S, F \in \mathbb{R}^{n\times n}$ *are two generator matrices, and* $\rho \in \mathbb{R}^{n\times 1}$ *is the reward vector.*

By taking the limit when $\tau \to \infty$, fast transitions become instantaneous. Then, a Markov reward chain with fast transitions behaves as a discontinuous Markov reward chain [13,9–11].

Definition 3 *Let* $\mathsf{F} = (\sigma, S, F, \rho)$ *be a Markov reward chain with fast transitions. The discontinuous Markov reward chain* $\mathsf{D} = (\sigma, \Pi, Q, \Pi\rho)$ *is the limit of* F*, where the matrix* Π *is the so-called ergodic projection at zero of* F*, that is* $\Pi = \lim_{t\to\infty} e^{Ft}$*, and* $Q = \Pi S \Pi$*. We write* $\mathsf{F} \to_\infty \mathsf{D}$*.*

The initial probability vector is not affected by the limit construction. We will later motivate the choice of using the reward vector $\Pi\rho$ instead of just ρ. In addition, we define the ergodic partition of a Markov reward chain with fast transitions to be the ergodic partition of its limit discontinuous Markov reward chain.

We depict Markov reward chains with fast transitions as in Fig. 1. The initial probabilities are depicted left above, and the reward rates right above each state. Here, a, b, and c are speeds, whereas λ, μ, ν, and ξ are rates of slow transitions. As in the definition, τ denotes the real parameter.

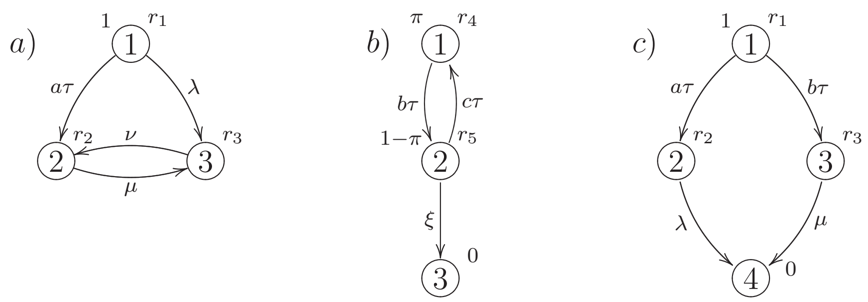

Fig. 1. Markov reward chains with fast transitions

As an example, the limit of the Markov reward chain with fast transitions in Fig. 1b) is given by:

$$\sigma = \begin{pmatrix} \pi & 1-\pi & 0 \end{pmatrix} \ \Pi = \begin{pmatrix} p & q & 0 \\ p & q & 0 \\ 0 & 0 & 1 \end{pmatrix} \ Q = \begin{pmatrix} -pq\xi & -q^2\xi & q\xi \\ -pq\xi & -q^2\xi & q\xi \\ 0 & 0 & 0 \end{pmatrix} \ \rho = \begin{pmatrix} pr_4 + qr_5 \\ pr_4 + qr_5 \\ 0 \end{pmatrix},$$

where $p = \frac{c}{b+c}$ and $q = \frac{b}{b+c}$. States 1 and 2 in Fig. 1b) form an ergodic class and state 3 is regular.

2.3 Markov Reward Chains with Silent Transitions

We define a Markov reward chain with silent transitions as a Markov reward chain with fast transitions in which the speeds of the fast transitions are left unspecified. To abstract away from the speeds of the fast transitions we introduce a suitable equivalence relation on Markov reward chains with fast transitions that is induced by the following equivalence relation of matrices.

Definition 4 *Two matrices $A, B \in \mathbb{R}^{n\times n}$ have the same shape (also called grammar), notation $A \sim B$, if and only if they have zeros on the same positions. That is,*

$$A \sim B \iff (\forall i,j)\,(A[i,j] = 0 \iff B[i,j] = 0)\,.$$

It is obvious that $\sim$ is an equivalence on matrices of the same order. The abstraction from speeds is achieved by identifying generator matrices of fast transitions with the same shape. Thus, silent transitions are modeled by equivalence classes of $\sim$.

Definition 5 *A Markov reward chain with silent transitions S is a quadruple $\mathsf{S} = (\sigma, S, \mathcal{F}, \rho)$ where $\mathcal{F}$ is an equivalence class of $\sim$ and, for every $F \in \mathcal{F}$, $\mathsf{F} = (\sigma, S, F, \rho)$ is a Markov reward chain with fast transitions.*

We write $\mathsf{F} \in \mathsf{S}$ if $\mathsf{S} = (\sigma, S, \mathcal{F}, \rho)$, and $\mathsf{F} = (\sigma, S, F, \rho)$ with $F \in \mathcal{F}$. Furthermore, we lift the relation $\sim$ to Markov reward chains with fast transitions and write $\mathsf{F} \sim \mathsf{F}'$ if $\mathsf{F}, \mathsf{F}' \in \mathsf{S}$. The notion of an ergodic partition is speed independent, i.e., if $\mathsf{F} \sim \mathsf{F}'$, then they have the same ergodic partition, since the ergodic partition depends only on the existence of fast transitions, but not on the actual speeds. Hence we can define: The ergodic partition of a Markov reward chain with silent transitions S is the ergodic partition of any Markov reward chain with fast transitions F such that $\mathsf{F} \in \mathsf{S}$.

We depict Markov reward chains with silent transitions as in Fig. 2, i.e., by omitting the speeds of the fast transitions. The depicted Markov reward chains with silent transitions are induced by the Markov reward chains with fast transitions in Fig. 1.

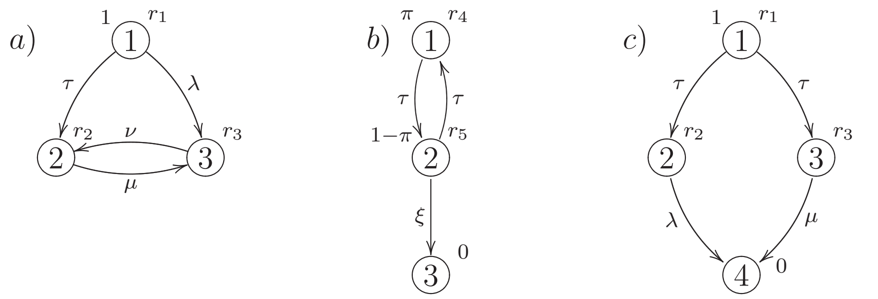

Fig. 2. Markov reward chains with silent transitions

In Fig. 2, τ can be understood as a label of internal action transitions, as it is common in transition system modeling and process algebra [26,27]. In this way we formalize the notion of performance analysis for Markov reward chains with non-deterministic internal steps.

3 Aggregation by Lumping

In this section we introduce lumping methods for the Markovian models of the previous section following [9–11]. First, we generalize ordinary lumping of [18] to discontinuous Markov reward chains. Then, we define τ-lumping for Markov reward chains with fast transitions based on ordinary lumping of discontinuous Markov reward chains. Finally, we lift the τ-lumping to $\tau_\sim$-lumping of Markov reward chains with silent transitions.

We define aggregation by lumping in terms of matrices. Every partition $\mathcal{P} = \{C_1, \ldots, C_N\}$ of $\mathcal{S} = \{1, \ldots, n\}$ can be associated with a so-called collector matrix $V \in \mathbb{R}^{n \times N}$ defined by $V[i,k] = 0$ if $i \notin C_k$, $V[i,k] = 1$ if $i \in C_k$, and vice versa. The k-th column of V has 1's for elements corresponding to states in C_k and has 0's otherwise. Note that $V \cdot \mathbf{1} = \mathbf{1}$. A distributor matrix $U \in \mathbb{R}^{N \times n}$ for $\mathcal{P}$ is defined as a matrix $U \geqslant 0$, such that $UV = I^N$. To satisfy these conditions, the elements of the k-th row of U, which correspond to states in the class C_k, sum up to one, whereas the other elements of the row are 0.

3.1 Ordinary Lumping

An ordinary lumping is a partition of the state space of a discontinuous Markov reward chain into classes such that the states that are lumped together have equivalent behavior for transiting to other classes and, additionally, have the same reward.

Definition 6 *A partition $\mathcal{L}$ of $\{1, \ldots, n\}$ is an ordinary lumping, or lumping for short, of a discontinuous Markov reward chain $\mathsf{D} = (\sigma, \Pi, Q, \rho)$ if and only if the following hold: (1) $VU\Pi V = \Pi V$, (2) $VUQV = QV$, and (3) $VU\rho = \rho$, where V is the collector matrix and U is any distributor matrix for $\mathcal{L}$.*

The lumping conditions only require that the rows of ΠV (respectively QV and ρ) that correspond to the states of the same partition class are equal. We have the following property [9–11].

Proposition 7 *Let $\mathsf{D} = (\sigma, \Pi, Q, \rho)$ be a discontinuous Markov reward chain and let $\mathcal{L}$ be its ordinary lumping. Define (1) $\overline{\sigma} = \sigma V$, (2) $\overline{\Pi} = U\Pi V$, (3) $\overline{Q} = UQV$ and (4) $\overline{\rho} = U\rho$, for the collector matrix V of $\mathcal{L}$ and any distributor U. Then $\overline{\mathsf{D}} = (\overline{\sigma}, \overline{\Pi}, \overline{Q}, \overline{\rho})$ is a discontinuous Markov reward chain. Moreover, $\overline{P}(t) = UP(t)V$ where $\overline{P}(t)$ and $P(t)$ are the transition matrices of $\overline{\mathsf{D}}$ and D, respectively.* □

Definition 8 *If the conditions of Proposition 7 are satisfied, then $\mathsf{D} = (\sigma, \Pi, Q, \rho)$ lumps to $\overline{\mathsf{D}} = (\overline{\sigma}, \overline{\Pi}, \overline{Q}, \overline{\rho})$, called the lumped discontinuous Markov reward chain with respect to $\mathcal{L}$. We write $\mathsf{D} \xrightarrow{\mathcal{L}} \overline{\mathsf{D}}$.*

It can readily be seen that neither the definition of a lumping, nor the definition of the lumped process depends on the choice of a distributor matrix U. For example, if $VUQV = QV$, then $VU'QV = VU'VUQV = VUQV = QV$, for any other distributor U'. In the continuous case, when $\Pi = I$ we have $\overline{\Pi} = I$, so $\overline{Q}$ is a generator matrix and our notion of ordinary lumping coincides with the standard definition [18,28]. The expected reward is preserved by ordinary lumping, since:

$$\overline{R}(t) = \sigma VUP(t)VU\rho = \sigma P(t)VU\rho = \sigma P(t)\rho = R(t).$$

Similarly, as in [18], one can show that other performance measures are also preserved by lumping.

3.2 τ-Lumping

The notion of τ-lumping is based on ordinary lumping for discontinuous Markov reward chains. The aim is that the limit of a τ-lumped Markov reward chain with fast transitions is an ordinary lumped version of the limit of the original Markov reward chain with fast transitions.

Definition 9 *A partition $\mathcal{L}$ of the state space of a Markov reward chain with* *fast transitions* F *is called a* τ*-lumping, if it is an ordinary lumping of its limiting discontinuous Markov reward chain* D *with* $\mathsf{F} \to_\infty \mathsf{D}$.

Note that since we defined the reward of the limit by $\Pi\rho$, a τ-lumping may identify states with different rewards.

Like for ordinary lumping, we define the τ-lumped process by multiplying σ, S, F and ρ with a collector matrix and a distributor matrix. However, unlike for ordinary lumping, not all distributors are allowed. Following [9–11], we provide a class of special distributors, called τ-distributors, that yield a τ-lumped process.

Definition 10 *Let* $\mathsf{D} = (\sigma, \Pi, Q, \rho)$ *be a discontinuous Markov reward chain. Let* V *be a collector corresponding to a partition of the state space of this chain. A matrix* W *is a* τ*-distributor for* V *if and only if (1) it is a distributor for* V*, (2)* $\Pi V W \Pi = \Pi V W$*, and (3) the entries of* W *corresponding to states in classes of transient states are positive.*
A τ*-distributor for a partition of a Markov reward chain with fast transitions is any* τ*-distributor for the same partition of its limiting discontinuous Markov reward chain.*

Remark 11 An alternative, explicit definition of the τ-distributors can be found in [9–11]. We note here that the class of τ-distributors given by Definition 10 depends on two sets of parameters. Namely, after suitable renumbering, any τ-distributor W can be written as $W = \left(\begin{smallmatrix} W(\alpha) & \mathbf{0} \\ \mathbf{0} & W(\beta) \end{smallmatrix}\right)$, where $W(\alpha)$ is a distributor for the classes containing ergodic states and $W(\beta)$ is a distributor for the classes of transient states. As the notation suggests, the distributor $W(\alpha)$ depends on a set of parameters α and the distributor $W(\beta)$ is determined by a set of parameters β. To explicitly state this dependence we may write $W_{\alpha,\beta}$ for a τ-distributor depending on the parameter sets α and β. By the alternative definition of τ-distributors we can also establish the existence of a τ-distributor for any τ-lumping.

Having defined τ-distributors, we can now explicitly define a τ-lumped process.

Definition 12 *Let* $\mathsf{F} = (\sigma, S, F, \rho)$ *and let* $\mathcal{L}$ *be a lumping with a collector matrix* V*, and a corresponding* τ*-distributor* W*. The* τ*-lumped Markov reward chain with fast transitions* $\overline{\mathsf{F}} = (\overline{\sigma}, \overline{S}, \overline{F}, \overline{\rho})$ *is defined as* $\overline{\sigma} = \sigma V$, $\overline{S} = WSV$, $\overline{F} = WFV$, $\overline{\rho} = W\rho$*. We say that* F τ*-lumps to* $\overline{\mathsf{F}}$ *with respect to* W *and write* $\mathsf{F} \overset{\mathcal{L}}{\rightsquigarrow}_W \overline{\mathsf{F}}$*. We write* $\mathsf{F} \overset{\mathcal{L}}{\rightsquigarrow} \overline{\mathsf{F}}$ *if* $\mathsf{F} \overset{\mathcal{L}}{\rightsquigarrow}_W \overline{\mathsf{F}}$ *for some* τ*-distributor* W*.*

In general, when lumping F using a collector V and a distributor U, USV and UFV are not uniquely determined, i.e., they depend on the choice of the distributor. The restriction to τ-distributors does not change this. Subsequently, the τ-lumped process depends on the choice of the τ-distributor. In order to make the τ-distributor used explicit, we sometimes write $\mathsf{F} \overset{\mathcal{L}}{\rightsquigarrow}_{\alpha,\beta} \overline{\mathsf{F}}$ in order to emphasize the parameter sets such that $W = W_{\alpha,\beta}$.

The motivation for restricting to τ-distributors, despite that they do not ensure a unique τ-lumped process, is that all τ-lumped processes are equivalent in the limit. This is stated in the following proposition, which gives the precise connection of ordinary lumping and τ-lumping.

Proposition 13 ([9]) *The following diagram commutes*

$$\begin{tikzcd} \mathsf{F} \arrow[r, rightsquigarrow, "\mathcal{L}"] \arrow[d, "\infty"'] & \overline{\mathsf{F}} \arrow[d, "\infty"] \\ \mathsf{D} \arrow[r, "\mathcal{L}"] & \overline{\mathsf{D}} \end{tikzcd}$$

that is, if $\mathsf{F} \overset{\mathcal{L}}{\rightsquigarrow} \overline{\mathsf{F}} \to_\infty \overline{\mathsf{D}}$ *and if* $\mathsf{F} \to_\infty \mathsf{D} \overset{\mathcal{L}}{\to} \overline{\mathsf{D}}'$*, then* $\overline{\mathsf{D}} = \overline{\mathsf{D}}'$*, for* F *and* $\overline{\mathsf{F}}$ *Markov reward chains with fast transitions, and* D*,* $\overline{\mathsf{D}}$*, and* $\overline{\mathsf{D}}'$ *discontinuous Markov reward chains.* □

Moreover, the τ-lumped processes that originate from the same Markov reward chain with fast transitions become exactly the same, once all fast transitions are eliminated [10,11].

3.3 $\tau_\sim$-Lumping

We lift τ-lumping to equivalence classes of $\sim$ to obtain $\tau_\sim$*-lumping* for Markov reward chains with silent transitions. Intuitively, a partition is a $\tau_\sim$-lumping of S, if it is a τ-lumping for every $\mathsf{F} \in \mathsf{S}$ and, moreover, the limit of the τ-lumped process of F does not depend on the parameters chosen for the τ-distributor. Recall that the parameter set α affects ergodic states, whereas the parameter set β affects only transient states.

Definition 14 *Let* S *be a Markov reward chain with silent transitions and let* $\mathcal{L}$ *be its partition. Then* $\mathcal{L}$ *is a* $\tau_\sim$*-lumping if and only if it is a* τ*-lumping for every Markov reward chain with fast transitions* $\mathsf{F} \in \mathsf{S}$ *and, moreover, for every* $\mathsf{F}, \mathsf{F}' \in \mathsf{S}$ *if* $\mathsf{F} \overset{\mathcal{L}}{\rightsquigarrow}_{\alpha,\beta} \overline{\mathsf{F}}$ *and* $\mathsf{F}' \overset{\mathcal{L}}{\rightsquigarrow}_{\alpha',\beta} \overline{\mathsf{F}}'$*, then* $\overline{\mathsf{F}} \sim \overline{\mathsf{F}}'$*.*

The motivation behind the use of the same parameter set β in Definition 14 is that there may be slow transitions originating from transient states which will depend on β in the lumped process. If we do not restrict to the same parameter set β, then $\tau_\sim$-lumpings will only exist in rare cases in which transient states have no slow transitions. We refer to [10,11] for details.

Now we can define a $\tau_\sim$-lumped process which is unique for a given $\tau_\sim$-lumping $\mathcal{L}$ and a parameter set β.

Definition 15 *Let* S *be a Markov reward chain with silent transitions and* $\mathcal{L}$ *its* $\tau_\sim$*-lumping. Let* $\mathsf{F} \in \mathsf{S}$ *be such that* $\mathsf{F} \overset{\mathcal{L}}{\rightsquigarrow}_{\alpha,\beta} \overline{\mathsf{F}}$ *and let* $\overline{\mathsf{S}}$ *be the Markov reward chain with silent transitions with* $\overline{\mathsf{F}} \in \overline{\mathsf{S}}$*. Then* S $\tau_\sim$*-lumps to* $\overline{\mathsf{S}}$*, with respect to* $\mathcal{L}$ *and* β*, notation* $\mathsf{S} \overset{\mathcal{L}}{\rightsquigarrow}_\beta \overline{\mathsf{S}}$*. We write* $\mathsf{S} \overset{\mathcal{L}}{\rightsquigarrow} \overline{\mathsf{S}}$ *if* $\mathsf{S} \overset{\mathcal{L}}{\rightsquigarrow}_\beta \overline{\mathsf{S}}$ *for some parameter set* β*.*

4 Aggregation by Reduction

Reduction is a specific aggregation method for transforming a discontinuous Markov chain into a standard Markov chain, originally studied in [21,22,13]. Extended to reward processes, the method reduces a discontinuous Markov reward chain to a Markov reward chain by eliminating instantaneous states, while retaining the behavior of the regular states. In the same spirit, we define reduction methods that reduce Markov reward chains with fast and silent transitions to Markov reward chains following [10,11], called τ-reduction and $\tau_\sim$-reduction, respectively.

4.1 Reduction

The reduction-based aggregation method masks the stochastic discontinuity of a discontinuous Markov reward chain and transforms it into a Markov reward chain [21,13,10,11]. The underlying idea is to abstract away from the behavior of individual states in an ergodic class. The method is based on the notion of a canonical product decomposition.

Definition 16 *Let* $\mathsf{D} = (\sigma, \Pi, Q, \rho)$ *and assume that* $\mathrm{rank}(\Pi) = M$*, i.e., that there are* M *ergodic classes. A canonical product decomposition of* Π *is a pair of matrices* (L, R) *with* $L \in \mathbb{R}^{M\times n}$ *and* $R \in \mathbb{R}^{n\times M}$ *such that* $L \geqslant 0$*,* $R \geqslant 0$*,* $\mathrm{rank}(L) = \mathrm{rank}(R) = M$*,* $L \cdot \mathbf{1} = \mathbf{1}$*, and* $\Pi = RL$*.*

A canonical product decomposition always exists and it can be constructed from the ergodic form of Π (see page 7). Moreover, it can be shown that any other canonical product decomposition is permutation equivalent to this one. Since a canonical product decomposition (L, R) of Π is a full-rank decomposition, and since Π is idempotent, we also have that $LR = I^M$. Thus, we have $L\Pi = LRL = L$ and $\Pi R = RLR = R$. Next, we present the reduction method.

Definition 17 *For a discontinuous Markov reward chain* $\mathsf{D} = (\sigma, \Pi, Q, \rho)$*, the reduced Markov reward chain* $\mathsf{M} = (\overline{\sigma}, \overline{Q}, \overline{\rho})$ *is given by* $\overline{\sigma} = \sigma R$*,* $\overline{Q} = LQR$ *and* $\overline{\rho} = L\rho$*, where* (L, R) *is a canonical product decomposition of* Π*. We write*

$\mathsf{D} \rightarrow_r \mathsf{M}$.

If $\overline{P}(t)$ and $P(t)$ are the transition matrices of the reduced and the original chain, respectively, then one can show that $\overline{P}(t) = LP(t)R$. See [13,22].

The reduced process is unique up to a permutation of the states, since the canonical product decomposition is. The states of the reduced process are given by the ergodic classes of the original process, while the transient states are 'ignored'. Intuitively, the transient states are split probabilistically between the ergodic classes according to their trapping probabilities. In case a transient state is also an initial state, its initial probability is split according to its trapping probabilities. The reward rate is calculated as the sum of the individual reward rates of the states of the ergodic class weighted by their ergodic probabilities. Like lumping, the reduction also preserves the expected reward rate at time t:

$$\overline{R}(t) = \sigma RLP(t)RL\rho = \sigma\Pi P(t)\Pi\rho = \sigma P(t)\rho = R(t).$$

In case the original process has no stochastic discontinuity, i.e., $\Pi = I$, the reduced process is equal to the original.

4.2 τ-Reduction

We now define a reduction-based aggregation method called τ-reduction. It aggregates a Markov reward chain with fast transitions to an asymptotically equivalent Markov reward chain.

Definition 18 *A Markov reward chain with fast transitions* $\mathsf{F} = (\sigma, S, F, \rho)$ *τ-reduces to the Markov reward chain* $\mathsf{M} = (\overline{\sigma}, \overline{Q}, \overline{\rho})$*, given by (1)* $\overline{\sigma} = \sigma R$*, (2)* $\overline{Q} = LSR$*, and (3)* $\overline{\rho} = L\rho$*, where* $\mathsf{F} \rightarrow_\infty (\sigma, \Pi, Q, \Pi\rho)$ *and* (L, R) *is a canonical product decomposition of* Π*. When* F *τ-reduces to* M*, we write* $\mathsf{F} \rightsquigarrow_r \mathsf{M}$.

The following simple property relates τ-reduction to reduction. It holds since $LQR = L\Pi S\Pi R = LSR$ and $L\Pi\rho = L\rho$.

Proposition 19 *The following diagram commutes*

$$\begin{tikzcd} \mathsf{F} \arrow[d, "\infty"'] \arrow[dr, rightsquigarrow, "r"] & \\ \mathsf{D} \arrow[r, "r"'] & \mathsf{M} \end{tikzcd}$$

that is, if $\mathsf{F} \rightsquigarrow_r \mathsf{M}$ *and* $\mathsf{F} \rightarrow_\infty \mathsf{D} \rightarrow_r \mathsf{M}'$*, then* $\mathsf{M} = \mathsf{M}'$*, for* F *a Markov reward chain with fast transitions,* D *a discontinuous Markov reward chain and* M *and* M' *(continuous) Markov reward chains.* □

By combining τ-reduction with ordinary lumping of Markov reward chains, we can eliminate the effect of the speeds and obtain a reduction-like aggregation method for Markov reward chains with silent transitions. Here, we refer to this method as $\tau_\sim$-reduction.

One could define a reduction-based method for Markov reward chains with silent transitions by saying that a Markov reward chain with silent transitions S reduces to a Markov reward chain M if all Markov reward chains with fast transitions $\mathsf{F} \in \mathsf{S}$ τ-reduce to M. However, such is not an efficient reduction method as it is applicable only in a few special cases when all Markov reward chains with fast transitions in a $\sim$-equivalence class τ-reduce to the same Markov reward chain [10,11]. For this reason we combine τ-reduction and lumping.

Similarly as for $\tau_\sim$-lumping, the result of the $\tau_\sim$-reduction should not depend on the representative Markov reward chain with fast transitions. Therefore, a Markov reward chain with silent transitions can be $\tau_\sim$-reduced if all Markov reward chains with fast transitions in its equivalence class τ-reduce to Markov reward chains that can be ordinary lumped to the same Markov reward chain, as depicted below:

$$
\begin{tikzcd}
\mathsf{F} \arrow[r, rightsquigarrow, "r"'] & \mathsf{M} \arrow[dr, "\mathcal{L}"] & \\
\sim & & \overline{\mathsf{M}} \\
\mathsf{F}' \arrow[r, rightsquigarrow, "r"'] & \mathsf{M}' \arrow[ur, "\mathcal{L}"] &
\end{tikzcd}
$$

Definition 20 *Let S be a Markov reward chain with silent transitions, let $\mathcal{E} = \{E_1, \ldots, E_M, T\}$ be its ergodic partition, and $\mathcal{L}$ a partition of $\{E_1, \ldots, E_M\}$. Then S can be $\tau_\sim$-reduced according to $\mathcal{L}$ if and only if there exists a Markov reward chain $\overline{\mathsf{M}}$, such that for every $\mathsf{F} \in \mathsf{S}$, we have that $\mathsf{F} \rightsquigarrow_r \mathsf{M} \xrightarrow{\mathcal{L}} \overline{\mathsf{M}}$ for some Markov reward chain M. We write $\mathsf{S} \overset{\mathcal{L}}{\rightsquigarrow}_r \overline{\mathsf{M}}$. We may also write $\mathsf{S} \rightsquigarrow_r \overline{\mathsf{M}}$ if a partition $\mathcal{L}$ exists such that $\mathsf{S} \overset{\mathcal{L}}{\rightsquigarrow}_r \overline{\mathsf{M}}$.*

We note that both $\tau_\sim$-lumping and $\tau_\sim$-reduction produce the same process when all silent transitions are eliminated, cf. [10,11] for details.

5 Relational Properties

We investigate the relational properties of the lumping-based aggregation methods. For ordinary lumping, the combination of transitivity and strict confluence ensures that iterative application yields a uniquely determined process.

In the case of τ-lumping, by Proposition 13, only the limit of the final reduced process is uniquely determined, unless the final process contains no fast transitions. Similarly, for $\tau_{\sim}$-lumping the reduced process is uniquely determined only if it does not contain any silent transitions.

There is no need to investigate the relational properties of the reduction-based methods, since they act in one step (no iteration is possible), in a unique way, between different types of models.

First, we investigate the properties of the relation $\geqslant$ on discontinuous Markov reward chains defined by

$$\mathsf{D}_1 \geqslant \mathsf{D}_2 \iff (\exists \mathcal{L})\, \mathsf{D}_1 \stackrel{\mathcal{L}}{\to} \mathsf{D}_2 \,.$$

The above relation is clearly reflexive, since the trivial partition Δ is always an ordinary lumping, i.e., $\mathsf{D} \stackrel{\Delta}{\to} \mathsf{D}$ for any discontinuous Markov reward chain D. Transitivity enables replacement of repeated application of ordinary lumping by a single application using an ordinary lumping that is a composition of the individual lumpings.

Theorem 21 *Let* D *be a discontinuous Markov reward chain such that* $\mathsf{D} \stackrel{\mathcal{L}}{\to} \overline{\mathsf{D}}$ *and* $\overline{\mathsf{D}} \stackrel{\overline{\mathcal{L}}}{\to} \overline{\overline{\mathsf{D}}}$. *Then* $\mathsf{D} \stackrel{\mathcal{L}\circ\overline{\mathcal{L}}}{\to} \overline{\overline{\mathsf{D}}}$.

PROOF. Let $\mathsf{D} = (\sigma, \Pi, Q, \rho)$, $\overline{\mathsf{D}} = (\overline{\sigma}, \overline{\Pi}, \overline{Q}, \overline{\rho})$, and $\overline{\overline{\mathsf{D}}} = (\overline{\overline{\sigma}}, \overline{\overline{\Pi}}, \overline{\overline{Q}}, \overline{\overline{\rho}})$. Let V and $\overline{V}$ denote the collector matrices for $\mathcal{L}$ and $\overline{\mathcal{L}}$, respectively. The collector matrix for $\mathcal{L}\circ\overline{\mathcal{L}}$ is $V\overline{V}$. The following lumping conditions hold: $VU\Pi V = \Pi V$, $VUQV = QV$ and $VU\rho = \rho$. Also $\overline{\Pi} = U\Pi V$, $\overline{Q} = UQV$ and $\overline{\rho} = U\rho$ for any distributor U for V. Similarly, it holds that: $\overline{V}\,\overline{U}\,\overline{\Pi}\,\overline{V} = \overline{\Pi}\,\overline{V}$, $\overline{V}\,\overline{U}\,\overline{Q}\,\overline{V} = \overline{Q}\,\overline{V}$ and $\overline{V}\,\overline{U}\,\overline{\rho} = \overline{\rho}$. Moreover $\overline{\overline{\Pi}} = \overline{U}\,\overline{\Pi}\,\overline{V}$, $\overline{\overline{Q}} = \overline{U}\,\overline{Q}\,\overline{V}$ and $\overline{\overline{\rho}} = \overline{U}\,\overline{\rho}$ for any distributor $\overline{U}$ for $\overline{V}$.

The iterative application of the ordinary lumping method can be replaced by the ordinary lumping given by the partition $\mathcal{L}\circ\overline{\mathcal{L}}$, that corresponds to the collector matrix $\overline{\overline{V}} = V\overline{V}$. A corresponding distributor is $\overline{\overline{U}} = \overline{U}\,U$, because $\overline{\overline{U}}\,\overline{\overline{V}} = \overline{U}UV\overline{V} = I$. That the partition is indeed an ordinary lumping follows from: $\overline{\overline{V}}\,\overline{\overline{U}}\,\Pi\,\overline{\overline{V}} = V\,\overline{V}\,\overline{U}\,U\Pi V\,\overline{V} = V\,\overline{V}\,\overline{U}\,\overline{\Pi}\,\overline{V} = V\,\overline{\Pi}\,\overline{V} = VU\Pi V\,\overline{V} = \Pi V\,\overline{V} = \Pi\overline{\overline{V}}$. Similarly, one gets the condition for Q, and $\overline{\overline{V}}\,\overline{\overline{U}}\,\rho = V\,\overline{V}\,\overline{U}\,U\rho = V\,\overline{V}\,\overline{U}\,\overline{\rho} = V\,\overline{\rho} = VU\rho = \rho$. □

The relation $\geqslant$ on Markov reward chains with fast transitions, defined by

$$\mathsf{F}_1 \geqslant \mathsf{F}_2 \iff (\exists \mathcal{L})\, \mathsf{F}_1 \stackrel{\mathcal{L}}{\rightsquigarrow} \mathsf{F}_2$$

is a preorder as well. It is reflexive via the trivial lumping Δ. The following theorem shows the transitivity of the τ-lumping relation.

Theorem 22 *Let* F *be a Markov reward chain with fast transitions, such that* $\mathsf{F} \overset{\mathcal{L}}{\rightsquigarrow} \overline{\mathsf{F}}$ *and* $\overline{\mathsf{F}} \overset{\overline{\mathcal{L}}}{\rightsquigarrow} \overline{\overline{\mathsf{F}}}$*. Then* $\mathsf{F} \overset{\mathcal{L}\circ\overline{\mathcal{L}}}{\rightsquigarrow} \overline{\overline{\mathsf{F}}}$*.*

PROOF. Let $\mathsf{F} = (\sigma, F, S, \rho)$ and $\overline{\mathsf{F}} = (\overline{\sigma}, \overline{F}, \overline{S}, \overline{\rho})$. Denote by V and $\overline{V}$ the collector matrices for $\mathcal{L}$ and $\overline{\mathcal{L}}$, respectively. The collector matrix for $\mathcal{L} \circ \overline{\mathcal{L}}$ is then $\overline{\overline{V}} = V\overline{V}$. Let W and $\overline{W}$ be the corresponding τ-distributors used for $\mathsf{F} \overset{\mathcal{L}}{\rightsquigarrow} \overline{\mathsf{F}}$ and $\overline{\mathsf{F}} \overset{\overline{\mathcal{L}}}{\rightsquigarrow} \overline{\overline{\mathsf{F}}}$, respectively. Since τ-lumping is defined in terms of ordinary lumping, it is sufficient to show that $\overline{\overline{W}} = \overline{W}W$ is a τ-distributor. From Theorem 21 it is a distributor. The condition requiring positive entries corresponding to transient states that lump only with other transient states, can be checked using the explicit description of τ-distributors [11]. It remains to verify the third condition.

Let Π and $\overline{\Pi}$ be the ergodic projections of F and $\overline{F}$. Then, $\Pi V W \Pi = \Pi V W$ and $\overline{\Pi}\,\overline{V}\,\overline{W}\,\overline{\Pi} = \overline{\Pi}\,\overline{V}\,\overline{W}$. We have that:

$$\begin{aligned}
\Pi\,\overline{\overline{V}}\,\overline{\overline{W}}\,\Pi &= \Pi V\,\overline{V}\,\overline{W}\,W\Pi = VW\Pi V\,\overline{V}\,\overline{W}W\Pi = V\,\overline{\Pi}\,\overline{V}\,\overline{W}\,W\Pi \\
&= V\,\overline{\Pi}\,\overline{V}\,\overline{W}\,\overline{\Pi}\,W\Pi = V\overline{\Pi}\,\overline{V}\,\overline{W}\,W\Pi V W\Pi = V\,\overline{\Pi}\,\overline{V}\,\overline{W}\,W\Pi V W \\
&= \text{(the same derivation steps backwards)} \\
&= \Pi V\,\overline{V}\,\overline{W}\,W = \Pi\,\overline{\overline{V}}\,\overline{\overline{W}}. \qquad \square
\end{aligned}$$

Similarly, $\tau_\sim$-lumping induces a preorder on Markov reward chains with silent transitions defined by

$$\mathsf{S}_1 \geqslant \mathsf{S}_2 \iff (\exists\mathcal{L})\,\mathsf{S}_1 \overset{\mathcal{L}}{\rightsquigarrow} \mathsf{S}_2.$$

Reflexivity again holds due to the trivial partition Δ, while transitivity is a direct consequence of Theorem 22 and the definition of $\tau_\sim$-lumping, Definition 14. Thus, we have the following theorem.

Theorem 23 *Let* S *be a Markov reward chain with silent transitions. Suppose* $\mathsf{S} \overset{\mathcal{L}}{\rightsquigarrow} \overline{\mathsf{S}}$ *and* $\overline{\mathsf{S}} \overset{\overline{\mathcal{L}}}{\rightsquigarrow} \overline{\overline{\mathsf{S}}}$*. Then* $\mathsf{S} \overset{\mathcal{L}\circ\overline{\mathcal{L}}}{\rightsquigarrow} \overline{\overline{\mathsf{S}}}$*.* $\square$

The lumping preorders also have the strict confluence property. In case of lumping this means that if $\mathsf{P} \overset{\mathcal{L}_1}{\to} \mathsf{P}_1$ and $\mathsf{P} \overset{\mathcal{L}_2}{\to} \mathsf{P}_2$, then there exist two partitions $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ such that $\mathsf{P}_1 \overset{\mathcal{L}_1\circ\overline{\mathcal{L}}_1}{\to} \overline{\mathsf{P}}$ and $\mathsf{P}_2 \overset{\mathcal{L}_2\circ\overline{\mathcal{L}}_2}{\to} \overline{\mathsf{P}}$. One can prove the strict confluence property by adapting the proof for Markov reward chains, e.g., from [20].

6 Parallel Composition and Compositionality

In this section we define parallel composition for each of the models, and prove the compositionality results. The definitions are based on Kronecker products and sums, as for standard Markov reward chains [15,29]. The intuition behind this is that the Kronecker sum represents interleaving, whereas the Kronecker product represents synchronization. Let us first recall the definition of Kronecker product and sum.

Definition 24 *Let $A \in \mathbb{R}^{n_1 \times n_2}$ and $B \in \mathbb{R}^{m_1 \times m_2}$. The* Kronecker product *of A and B is a matrix $(A \otimes B) \in \mathbb{R}^{n_1 m_1 \times n_2 m_2}$ defined by*

$$(A \otimes B)[(i-1)m_1 + k, (j-1)m_2 + \ell] = A[i,j]B[k,\ell]$$

for $1 \leqslant i \leqslant n_1$, $1 \leqslant j \leqslant n_2$, $1 \leqslant k \leqslant m_1$ and $1 \leqslant \ell \leqslant m_2$.

The Kronecker sum of two square matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$ is a matrix $(A \oplus B) \in \mathbb{R}^{nm \times nm}$ defined by $A \oplus B = A \otimes I^m + I^n \otimes B$.

Next, we list some basic properties of the Kronecker product and sum [30].

Proposition 25 *The following equations hold:*

(1) $(A \otimes B)(C \otimes D) = AC \otimes BD$,
(2) $(A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D$,
(3) $c(A \otimes B) = (cA \otimes B) = (A \otimes cB)$,
(4) $c(A \oplus B) = (cA \oplus cB)$,
(5) $e^{A \oplus B} = e^A \otimes e^B$,
(6) $\operatorname{rank}(A \otimes B) = \operatorname{rank}(A) \operatorname{rank}(B)$. □

We also need the notion of a Kronecker product of two partitions. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two partitions with corresponding collector matrices V_1 and V_2, respectively. Then $\mathcal{L}_1 \otimes \mathcal{L}_2$ denotes the partition corresponding to the collector matrix $V_1 \otimes V_2$.

6.1 Composing Discontinuous Markov Reward Chains

First, we present the definition of parallel composition of discontinuous Markov reward chains. The intuition is that 'rates' interleave, and the probabilities of the instantaneous transitions synchronize, i.e., they are independent.

Definition 26 *Let $\mathsf{D}_1 = (\sigma_1, \Pi_1, Q_1, \rho_1)$ and $\mathsf{D}_2 = (\sigma_2, \Pi_2, Q_2, \rho_2)$ be discon-*

tinuous Markov reward chains. Their parallel composition is defined as:

$$\mathsf{D}_1 \parallel \mathsf{D}_2 = (\sigma_1 \otimes \sigma_2, \Pi_1 \otimes \Pi_2, Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2, \rho_1 \otimes \mathbf{1}^{|\rho_2|} + \mathbf{1}^{|\rho_1|} \otimes \rho_2).$$

The following theorem shows that the parallel composition of two discontinuous Markov reward chains is well defined.

Theorem 27 *Let* D_1 *and* D_2 *be two discontinuous Markov reward chains. Then* $\mathsf{D}_1 \parallel \mathsf{D}_2$ *is a discontinuous Markov reward chain.*

PROOF. Let $\mathsf{D}_1 = (\sigma_1, \Pi_1, Q_1, \rho_1)$ and $\mathsf{D}_2 = (\sigma_2, \Pi_2, Q_2, \rho_2)$. The initial probability vector $\sigma_1 \otimes \sigma_2$ is a stochastic vector and the reward vector is well defined. Using Proposition 25(1)-(3), it is easy to check that the matrices $\Pi_1 \otimes \Pi_2$ and $Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2$ satisfy the conditions of Definition 1, i.e., (1) $(\Pi_1 \otimes \Pi_2) \geq 0$, (2) $(\Pi_1 \otimes \Pi_2) \cdot \mathbf{1} = \mathbf{1}$, (3) $(\Pi_1 \otimes \Pi_2)^2 = \Pi_1 \otimes \Pi_2$, (4) $(\Pi_1 \otimes \Pi_2) \cdot (Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2) = (Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2) \cdot (\Pi_1 \otimes \Pi_2) = Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2$, (5) $(Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2) \cdot \mathbf{1} = \mathbf{0}$, and (6) $Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2 + (c_1 + c_2) \cdot (\Pi_1 \otimes \Pi_2) = (Q_1 + c_1\Pi_1) \otimes \Pi_2 + \Pi_1 \otimes (Q_2 + c_2\Pi_2) \geq 0$ for $c_1, c_2 \geq 0$ such that $Q_1 + c_1\Pi_1 \geq 0$ and $Q_2 + c_2\Pi_2 \geq 0$. □

In the special case, when both discontinuous Markov reward chains are continuous, their parallel composition is again a Markov reward chain as defined in [15]. Moreover, the following property shows that the parallel composition of two discontinuous Markov reward chains has a transition matrix that is the Kronecker product of the individual transition matrices, corresponding to the intuition that the Kronecker product represents synchronization. This justifies the definition of the parallel composition.

Theorem 28 *Let* D_1 *and* D_2 *be two discontinuous Markov reward chains with transition matrices* $P_1(t)$ *and* $P_2(t)$*, respectively. Then the transition matrix of* $\mathsf{D}_1 \parallel \mathsf{D}_2$ *is given by* $P_1(t) \otimes P_2(t)$*.*

PROOF. Let $\mathsf{D}_1 = (\sigma_1, \Pi_1, Q_1, \rho_1)$ and $\mathsf{D}_2 = (\sigma_2, \Pi_2, Q_2, \rho_2)$. As the matrices $Q_1 \otimes \Pi_2$ and $\Pi_1 \otimes Q_2$ commute, and $P_i(t)\Pi_i = \Pi_i P_i(t) = P_i(t)$, we derive:

$$\begin{aligned}
&(\Pi_1 \otimes \Pi_2)\, e^{(Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2)t} \\
&= (\Pi_1 \otimes \Pi_2)(e^{(Q_1 \otimes \Pi_2)t} e^{(\Pi_1 \otimes Q_2)t}) \\
&= (\Pi_1 \otimes \Pi_2)(\textstyle\sum_{n=0}^{\infty} (Q_1 \otimes \Pi_2)^n t^n/n!)(\sum_{n=0}^{\infty} (\Pi_1 \otimes Q_2)^n t^n/n!) \\
&= (\Pi_1 \otimes \Pi_2)(I \otimes I + \textstyle\sum_{n=1}^{\infty} (Q_1 \otimes \Pi_2)^n t^n/n!)(I \otimes I + \sum_{n=1}^{\infty} (\Pi_1 \otimes Q_2)^n t^n/n!) \\
&= (\Pi_1 \otimes \Pi_2)(I \otimes I + \textstyle\sum_{n=1}^{\infty} (Q_1^n \otimes \Pi_2^n) t^n/n!)(I \otimes I + \sum_{n=1}^{\infty} (\Pi_1^n \otimes Q_2^n) t^n/n!) \\
&= (\Pi_1 \otimes \Pi_2)(I \otimes I + \textstyle\sum_{n=1}^{\infty} (Q_1^n \otimes \Pi_2) t^n/n!)(I \otimes I + \sum_{n=1}^{\infty} (\Pi_1 \otimes Q_2^n) t^n/n!) \\
&= (\Pi_1 \otimes \Pi_2)(I \otimes I + (\textstyle\sum_{n=1}^{\infty} Q_1^n t^n/n!) \otimes \Pi_2)(I \otimes I + \Pi_1 \otimes \sum_{n=1}^{\infty} Q_2^n t^n/n!) \\
&= (\Pi_1 \otimes \Pi_2)(I \otimes I + (e^{Q_1 t} - I) \otimes \Pi_2)(I \otimes I + \Pi_1 \otimes (e^{Q_2 t} - I)) \\
&= (\Pi_1 \otimes \Pi_2)(I \otimes I + e^{Q_1 t} \otimes \Pi_2 - I \otimes \Pi_2)(I \otimes I + \Pi_1 \otimes e^{Q_2 t} - \Pi_1 \otimes I)
\end{aligned}$$

$$
\begin{aligned}
&= (\Pi_1 \otimes \Pi_2 + P_1(t) \otimes \Pi_2 - \Pi_1 \otimes \Pi_2)(I \otimes I + \Pi_1 \otimes e^{Q_2 t} - \Pi_1 \otimes I) \\
&= (P_1(t) \otimes \Pi_2)(I \otimes I + \Pi_1 \otimes e^{Q_2 t} - \Pi_1 \otimes I) \\
&= (P_1(t) \otimes \Pi_2 + P_1(t) \otimes P_2(t) - P_1(t) \otimes \Pi_2) \\
&= P_1(t) \otimes P_2(t). \qquad \square
\end{aligned}
$$

Remark 29 We can motivate Definition 26 using an another perspective. By the standard probabilistic (i.e., non-matrix) representation of discontinuous Markov reward chain the same notion can be obtained by the following analysis. Let $\{X(t) \mid t \geqslant 0\}$ and $\{Y(t) \mid t \geqslant 0\}$ be two discontinuous Markov reward chains defined on state spaces S_X and S_Y respectively. Their parallel composition can be defined as the stochastic process $\{(X \parallel Y)(t) \mid t \geqslant 0\}$ with the state space $S_X \times S_Y$, such that $(X \parallel Y)(t) = (x, y)$ if and only if $X(t) = x$ and $Y(t) = y$. One can show that this process is again a discontinuous Markov reward chain with transition matrix equal to the Kronecker product of the transition matrices of $\{X(t) \mid t \geqslant 0\}$ and $\{Y(t) \mid t \geqslant 0\}$. It is known that the matrices Π and Q characterizing a transition matrix $P(t)$ are obtained as $\Pi = \lim_{t\to 0} P(t)$ and $Q = \lim_{h\to 0}(P(h) - \Pi)/h$ [13]. Applying this result on the transition matrix of $\{(X \parallel Y)(t) \mid t \geqslant 0\}$ and using the definition of $(X \parallel Y)(0)$ we obtain the first three components of the quadruple from Definition 26. The reward vector for the parallel composition encodes the assumption that the reward rate in (x, y) is the sum of the reward rates in x and y.

It is easy to see that the expected reward of the parallel composition is the sum of the expected rewards of the components. Using Proposition 25(1) and (2) we have $(\sigma_1 \otimes \sigma_2)(P_1(t) \otimes P_2(t))(\rho_1 \otimes \mathbf{1} + \mathbf{1} \otimes \rho_2) = \sigma_1 P_1(t)\rho_1 \otimes \sigma_1 P_1(t)\mathbf{1} + \sigma_2 P_2(t)\mathbf{1} \otimes \sigma_2 P_2(t)\rho_2 = R_1(t) \otimes 1 + 1 \otimes R_2(t) = R_1(t) + R_2(t)$.

The following theorem shows that both lumping and reduction are compositional with respect to the parallel composition of discontinuous Markov reward chains.

Theorem 30 *If* $\mathsf{D}_1 \overset{\mathcal{L}_1}{\to} \overline{\mathsf{D}}_1$ *and* $\mathsf{D}_2 \overset{\mathcal{L}_2}{\to} \overline{\mathsf{D}}_2$*, then* $\mathsf{D}_1 \parallel \mathsf{D}_2 \overset{\mathcal{L}_1 \otimes \mathcal{L}_2}{\longrightarrow} \overline{\mathsf{D}}_1 \parallel \overline{\mathsf{D}}_2$*. Also, if* $\mathsf{D}_1 \to_r \mathsf{M}_1$ *and* $\mathsf{D}_2 \to_r \mathsf{M}_2$*, then* $\mathsf{D}_1 \parallel \mathsf{D}_2 \to_r \mathsf{M}_1 \parallel \mathsf{M}_2$.

PROOF. Let $\mathsf{D}_1 = (\sigma_1, \Pi_1, Q_1, \rho_1)$, $\overline{\mathsf{D}}_1 = (\overline{\sigma}_1, \overline{\Pi}_1, \overline{Q}_1, \overline{\rho}_1)$, $\mathsf{D}_2 = (\sigma_2, \Pi_2, Q_2, \rho_2)$, and $\overline{\mathsf{D}}_2 = (\overline{\sigma}_2, \overline{\Pi}_2, \overline{Q}_2, \overline{\rho}_2)$. We first prove the compositionality of lumping. We show that $\mathcal{L}_1 \otimes \mathcal{L}_2$ is an ordinary lumping of

$$\mathsf{D}_1 \parallel \mathsf{D}_2 = (\sigma_1 \otimes \sigma_2, \Pi_1 \otimes \Pi_2, Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2, \rho_1 \otimes \mathbf{1} + \mathbf{1} \otimes \rho_2).$$

Let U_1, U_2, and $U_1 \otimes U_2$ be distributors and V_1, V_2, and $V_1 \otimes V_2$ be the collectors for $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_1 \otimes \mathcal{L}_2$, respectively. By using the lumping conditions and Proposition 25(1) and (2) we have that

$$
\begin{aligned}
(V_1 \otimes V_2)(U_1 \otimes U_2)(\Pi_1 \otimes \Pi_2)(V_1 \otimes V_2) &= (V_1 U_1 \Pi_1 V_1 \otimes V_2 U_2 \Pi_2 V_2) \\
&= (\Pi_1 V_1 \otimes \Pi_2 V_2) \\
&= (\Pi_1 \otimes \Pi_2)(V_1 \otimes V_2)
\end{aligned}
$$

$$
\begin{aligned}
&(V_1 \otimes V_2)(U_1 \otimes U_2)(Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2)(V_1 \otimes V_2) \\
&\qquad = V_1 U_1 Q_1 V_1 \otimes V_2 U_2 \Pi_2 V_2 + V_1 U_1 \Pi_1 V_1 \otimes V_2 U_2 Q_2 V_2 \\
&\qquad = Q_1 V_1 \otimes \Pi_2 V_2 + \Pi_1 V_1 \otimes Q_2 V_2 \\
&\qquad = (Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2)(V_1 \otimes V_2)
\end{aligned}
$$

$$
\begin{aligned}
(V_1 \otimes V_2)(U_1 \otimes U_2)(\rho_1 \otimes \mathbf{1} + \mathbf{1} \otimes \rho_2) &= (V_1 U_1 \rho_1 \otimes V_2 U_2 \mathbf{1} + V_1 U_1 \mathbf{1} \otimes V_2 U_2 \rho_2) \\
&= \rho_1 \otimes \mathbf{1} + \mathbf{1} \otimes \rho_2.
\end{aligned}
$$

Next, we prove that the lumped parallel composition is the parallel composition of the lumped components. We easily get, by Proposition 25(1) and (2),

$$
\begin{aligned}
&(U_1 \otimes U_2)(\Pi_1 \otimes \Pi_2)(V_1 \otimes V_2) = \overline{\Pi}_1 \otimes \overline{\Pi}_2 \text{ and} \\
&(U_1 \otimes U_2)(Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2)(V_1 \otimes V_2) = \overline{Q}_1 \otimes \overline{\Pi}_2 + \overline{\Pi}_1 \otimes \overline{Q}_2.
\end{aligned}
$$

Next, we consider reduction. Let $\Pi_1 = R_1 L_1$ and $\Pi_2 = R_2 L_2$ be some canonical product decompositions. Put $L = L_1 \otimes L_2$ and $R = R_1 \otimes R_2$. Note that $L \geq 0$ and $R \geq 0$ because $L_1, L_2, R_1, R_2 \geq 0$. We also have $L \cdot \mathbf{1} = (L_1 \otimes L_2) \cdot (\mathbf{1} \otimes \mathbf{1}) = L_1 \cdot \mathbf{1} \otimes L_2 \cdot \mathbf{1} = \mathbf{1} \otimes \mathbf{1} = \mathbf{1}$. Since $\mathrm{rank}(A \otimes B) = \mathrm{rank}(A) \cdot \mathrm{rank}(B)$ by Proposition 25(6), we get that (L, R) is a canonical product decomposition of $\Pi = \Pi_1 \otimes \Pi_2$. Reducing $\mathsf{D}_1 \parallel \mathsf{D}_2$ using the canonical product decomposition (L, R) gives us $\mathsf{M}_1 \parallel \mathsf{M}_2$. □

6.2 Composing Markov Reward Chains with Fast Transitions

We now present the definition of the parallel composition of Markov reward chains with fast transitions. It comprises Kronecker sums of the generator matrices, i.e., interleaving of the rates for both slow and fast transitions.

Definition 31 *Let $\mathsf{F}_1 = (\sigma_1, S_1, F_1, \rho_1)$ and $\mathsf{F}_2 = (\sigma_2, S_2, F_2, \rho_2)$ be two Markov reward chains with fast transitions. Then their parallel composition is defined as*

$$
\mathsf{F}_1 \parallel \mathsf{F}_2 = (\sigma_1 \otimes \sigma_2, S_1 \oplus S_2, F_1 \oplus F_2, \rho_1 \otimes \mathbf{1} + \mathbf{1} \otimes \rho_2).
$$

It is not difficult to see that the parallel composition of Markov reward chains with fast transitions is well defined. In Fig. 3 we present an example of parallel composition of two Markov reward chains with fast transitions: 3c) is the parallel composition of 3a) and 3b), the same Markov reward chains with fast transitions from the example in Fig. 1. We note that for the sake of readability the rewards of 3c) are omitted. They are given by the vector

$$(r_1 + r_4, r_1 + r_5, r_1, r_2 + r_4, r_2 + r_5, r_2, r_3 + r_4, r_3 + r_5, r_3)\,.$$

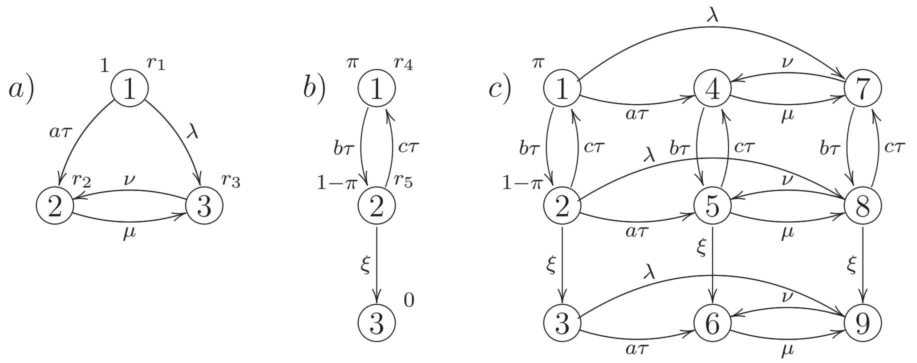

Fig. 3. Parallel composition of Markov reward chains with fast transitions

Having defined parallel composition for both models, we show how they are related: the limit of the parallel composition of two Markov reward chains with fast transitions is the parallel composition of the limits of the components (that are discontinuous Markov reward chains). Hence, a continuity property of the parallel composition holds as stated in the next result.

Theorem 32 *Let* $\mathsf{F}_1 \to_\infty \mathsf{D}_1$ *and* $\mathsf{F}_2 \to_\infty \mathsf{D}_2$*. Then* $\mathsf{F}_1 \parallel \mathsf{F}_2 \to_\infty \mathsf{D}_1 \parallel \mathsf{D}_2$.

PROOF. Let $\mathsf{F}_1 = (\sigma_1, S_1, F_1, \rho_1)$ and $\mathsf{F}_2 = (\sigma_2, S_2, F_2, \rho_2)$, and let their corresponding limits be $\mathsf{D}_1 = (\sigma_1, \Pi_1, Q_1, \Pi_1\rho_1)$ and $\mathsf{D}_2 = (\sigma_2, \Pi_2, Q_2, \Pi_2\rho_2)$. Using Proposition 25(4) and (5) we get that $\Pi_1 \otimes \Pi_2$ is the ergodic projection of $F_1 \oplus F_2$, i.e. $\lim_{t\to\infty} e^{(F_1 \oplus F_2)t} = \Pi_1 \otimes \Pi_2$. As before, using the distributivity of the Kronecker product and the fact that Π_1 is a stochastic matrix, we derive $Q_1 \otimes \Pi_2 + \Pi_2 \otimes Q_1 = (\Pi_1 \otimes \Pi_2)(S_1 \oplus S_2)(\Pi_1 \otimes \Pi_2)$ and $(\Pi_1 \otimes \Pi_2)(\rho_1 \otimes \mathbf{1} + \mathbf{1} \otimes \rho_2) = \Pi_1\rho_1 \otimes \mathbf{1} + \mathbf{1} \otimes \Pi_2\rho_2$. $\square$

Next we show that τ-lumping and τ-reduction are compositional as well, with respect to the parallel composition of Markov reward chains with fast transitions.

Theorem 33 *If* $\mathsf{F}_1 \overset{\mathcal{L}_1}{\rightsquigarrow} \overline{\mathsf{F}}_1$ *and* $\mathsf{F}_2 \overset{\mathcal{L}_2}{\rightsquigarrow} \overline{\mathsf{F}}_2$*, then* $\mathsf{F}_1 \| \mathsf{F}_2 \overset{\mathcal{L}_1 \otimes \mathcal{L}_2}{\rightsquigarrow} \overline{\mathsf{F}}_1 \| \overline{\mathsf{F}}_2$*. Also, if* $\mathsf{F}_1 \rightsquigarrow_r \mathsf{M}_1$ *and* $\mathsf{F}_2 \rightsquigarrow_r \mathsf{M}_2$*, then* $\mathsf{F}_1 \parallel \mathsf{F}_2 \rightsquigarrow_r \mathsf{M}_1 \parallel \mathsf{M}_2$*.*

PROOF. Let $\mathsf{F}_1 = (\sigma_1, S_1, F_1, \rho_1)$, $\mathsf{F}_2 = (\sigma_2, S_2, F_2, \rho_2)$, $\overline{\mathsf{F}}_1 = (\overline{\sigma}_1, \overline{S}_1, \overline{F}_1, \overline{\rho}_1)$, and $\overline{\mathsf{F}}_2 = (\overline{\sigma}_2, \overline{S}_2, \overline{F}_2, \overline{\rho}_2)$. By Theorem 30 and the continuity result Theorem 32, we get that $\mathcal{L}_1 \otimes \mathcal{L}_2$ is a τ-lumping for $\mathsf{F}_1 \parallel \mathsf{F}_2$. Let W_1 and W_2 be the τ-distributors used for the τ-lumped processes in the assumption, respectively. By Definition 10, Theorem 32, and Definition 26 for the parallel composition of discontinuous Markov reward chains, we have that $W_1 \otimes W_2$ is a τ-distributor for $\mathsf{F}_1 \parallel \mathsf{F}_2$. The τ-lumped process corresponding to $W_1 \otimes W_2$ is exactly $\overline{\mathsf{F}}_1 \parallel \overline{\mathsf{F}}_2$.

We next show the compositionality of τ-reduction. Let $\Pi_1 = R_1 L_1$ and $\Pi_2 = R_2 L_2$ be the canonical product decompositions of $\Pi_1 = \lim_{t\to\infty} e^{F_1 t}$ and $\Pi_2 = \lim_{t\to\infty} e^{F_2 t}$, respectively. Put $L = L_1 \otimes L_2$ and $R = R_1 \otimes R_2$. Then (L, R) is a canonical product decomposition of $\Pi = \Pi_1 \otimes \Pi_2$, as in the proof of Theorem 30. This canonical product decomposition applied to $\mathsf{F}_1 \| \mathsf{F}_2$ produces $\mathsf{M}_1 \parallel \mathsf{M}_2$ as the τ-reduced process. $\square$

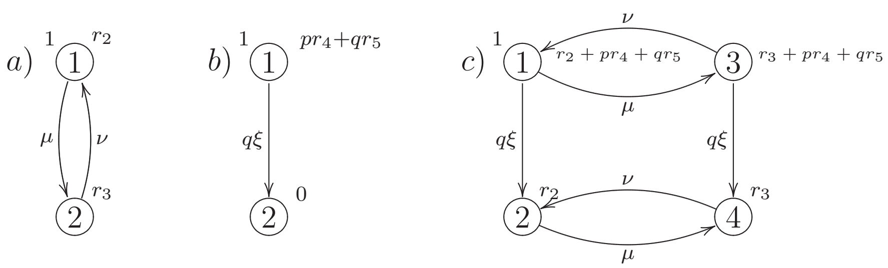

Fig. 4. Aggregated Markov reward chains with fast transitions

In Fig. 4 we depict the aggregated versions of the Markov reward chains with fast transitions from Fig. 3. The Markov reward chain with fast transitions in 4c) is the parallel composition of the Markov reward chains with fast transitions in 4a) and 4b) with $p = \frac{c}{b+c}$ and $q = \frac{b}{b+c}$. The aggregated versions 4a), 4b), and 4c) can be obtained from 3a), 3b), and 3c), respectively, by either applying τ-reduction or τ-lumping. See [10,11] for more details on the relationship between lumping-based and reduction-based aggregation methods. The τ-lumpings used are $\{\{1,2\},\{3\}\}$ for 3a) and 3b), and $\{\{1,2,4,5\},\{3,6\},\{7,8\},\{9\}\}$ for 3c). By Theorem 33, we have that the Markov reward chain in 4c), is in fact the parallel composition of the chains in 4a) and 4b).

6.3 Composing Markov Reward Chains with Silent Transitions

We define the parallel composition of two Markov reward chains with silent transitions via the equivalence class of the parallel composition of the representative Markov reward chains with fast transitions.

Definition 34 *Let* $\mathsf{S}_1 = (\sigma_1, S_1, \mathcal{F}_1, \rho_1)$ *and* $\mathsf{S}_2 = (\sigma_2, S_2, \mathcal{F}_2, \rho_2)$ *be two Markov reward chains with silent transitions. Then their parallel composition is defined as:*

$$\mathsf{S}_1 \parallel \mathsf{S}_2 = (\sigma_1 \otimes \sigma_2, S_1 \oplus S_2, \mathcal{F}_1 \oplus \mathcal{F}_2, \rho_1 \otimes \mathbf{1} + \mathbf{1} \otimes \rho_2),$$

where $\mathcal{F}_1 \oplus \mathcal{F}_2$ *denotes the equivalence class of* $F_1 \oplus F_2$ *with respect to* $\sim$*, for some* $F_1 \in \mathcal{F}_1$ *and* $F_2 \in \mathcal{F}_2$*.*

The parallel composition of Markov reward chains with silent transitions is well defined as the Kronecker sum respects the equivalence $\sim$. Next we state the compositionality result for $\tau_\sim$-lumping and $\tau_\sim$-reduction. It is a direct consequence of Theorem 33 for compositionality of τ-lumping and τ-reduction, and compositionality of ordinary lumping for standard Markov reward chain as a special case of Theorem 30.

Theorem 35 *Let* S_1 *and* S_2 *be two Markov reward chains with silent transitions. If* $\mathsf{S}_1 \overset{\mathcal{L}_1}{\rightsquigarrow} \overline{\mathsf{S}}_1$ *and* $\mathsf{S}_2 \overset{\mathcal{L}_2}{\rightsquigarrow} \overline{\mathsf{S}}_2$*, then* $\mathsf{S}_1 \parallel \mathsf{S}_2 \overset{\mathcal{L}_1 \otimes \mathcal{L}_2}{\rightsquigarrow} \overline{\mathsf{S}}_1 \parallel \overline{\mathsf{S}}_2$*. Also, if* $\mathsf{S}_1 \overset{\mathcal{L}_1}{\rightsquigarrow}_r \mathsf{M}_1$ *and* $\mathsf{S}_2 \overset{\mathcal{L}_2}{\rightsquigarrow}_r \mathsf{M}_2$*, then* $\mathsf{S}_1 \parallel \mathsf{S}_2 \overset{\mathcal{L}_1 \otimes \mathcal{L}_2}{\rightsquigarrow}_r \mathsf{M}_1 \parallel \mathsf{M}_2$*.* □

7 Conclusion

We considered three types of performance models. Markov reward chains with fast transitions are our central model used for analyzing systems with stochastic and instantaneous probabilistic transitions. Their limits are the discontinuous Markov reward chains. Their quotients are the Markov reward chains with silent transitions which can be used for the analysis of systems with stochastic transitions and non-deterministic (internal) τ steps.

For each type of models, we presented two aggregation methods: lumping and reduction for discontinuous Markov reward chains, τ-lumping and τ-reduction for Markov reward chains with fast transitions, and $\tau_\sim$-lumping and $\tau_\sim$-reduction for Markov reward chains with silent transitions. In short, the contributions of the paper are:

- A definition of parallel composition of discontinuous Markov reward chains, Markov reward chains with fast transitions, and Markov reward chains with silent transitions allowing for compositional modeling.
- Identification of preorder properties of the aggregation methods for all types of models.
- Compositionality theorems for each type of models and each corresponding aggregation preorder, and a continuity property of the parallel compositions.

$$\begin{tikzcd}
\mathsf{S}_1 \arrow[r, squiggly, "\mathcal{L}_1"] \arrow[d, phantom, "\Cup"] & \overline{\mathsf{S}}_1 \arrow[d, phantom, "\Cup"] & \mathsf{S}_2 \arrow[r, squiggly, "\mathcal{L}_2"] \arrow[d, phantom, "\Cup"] & \overline{\mathsf{S}}_2 \arrow[d, phantom, "\Cup"] & & \mathsf{S}_1 \parallel \mathsf{S}_2 \arrow[r, squiggly, "\mathcal{L}_1\otimes\mathcal{L}_2"] \arrow[d, phantom, "\Cup"] & \overline{\mathsf{S}}_1 \parallel \overline{\mathsf{S}}_2 \arrow[d, phantom, "\Cup"] \\
\mathsf{F}_1 \arrow[r, squiggly, "\mathcal{L}_1"] \arrow[d, "\infty"'] & \overline{\mathsf{F}}_1 \arrow[d, "\infty"] & \mathsf{F}_2 \arrow[r, squiggly, "\mathcal{L}_2"] \arrow[d, "\infty"'] & \overline{\mathsf{F}}_2 \arrow[d, "\infty"] & \Longrightarrow & \mathsf{F}_1 \parallel \mathsf{F}_2 \arrow[r, squiggly, "\mathcal{L}_1\otimes\mathcal{L}_2"] \arrow[d, "\infty"'] & \overline{\mathsf{F}}_1 \parallel \overline{\mathsf{F}}_2 \arrow[d, "\infty"] \\
\mathsf{D}_1 \arrow[r, "\mathcal{L}_1"] & \overline{\mathsf{D}}_1 & \mathsf{D}_2 \arrow[r, "\mathcal{L}_2"] & \overline{\mathsf{D}}_2 & & \mathsf{D}_1 \parallel \mathsf{D}_2 \arrow[r, "\mathcal{L}_1\otimes\mathcal{L}_2"] & \overline{\mathsf{D}}_1 \parallel \overline{\mathsf{D}}_2
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{S}_1 \arrow[d, phantom, "\Cup"] \arrow[ddrr, squiggly, "\mathcal{L}_1", "r"'] & & & \mathsf{S}_2 \arrow[d, phantom, "\Cup"] \arrow[ddrr, squiggly, "\mathcal{L}_2", "r"'] & & & & \mathsf{S}_1 \parallel \mathsf{S}_2 \arrow[d, phantom, "\Cup"] \arrow[ddrr, squiggly, "\mathcal{L}_1\otimes\mathcal{L}_2", "r"'] & & \\
\mathsf{F}_1 \arrow[d, "\infty"'] \arrow[dr, squiggly, "r"] & & & \mathsf{F}_2 \arrow[d, "\infty"'] \arrow[dr, squiggly, "r"] & & & \Longrightarrow & \mathsf{F}_1 \parallel \mathsf{F}_2 \arrow[d, "\infty"'] \arrow[dr, squiggly, "r"] & & \\
\mathsf{D}_1 \arrow[r, "r"'] & \mathsf{M}_1 \arrow[r, "\mathcal{L}_1"] & \overline{\mathsf{M}}_1 & \mathsf{D}_2 \arrow[r, "r"'] & \mathsf{M}_2 \arrow[r, "\mathcal{L}_2"] & \overline{\mathsf{M}}_2 & & \mathsf{D}_1 \parallel \mathsf{D}_2 \arrow[r, "r"'] & \mathsf{M}_1 \parallel \mathsf{M}_2 \arrow[r, "\mathcal{L}_1\otimes\mathcal{L}_2"] & \overline{\mathsf{M}}_1 \parallel \overline{\mathsf{M}}_2
\end{tikzcd}$$

Fig. 5. Summary compositionality results

The results on compositionality are summarized by Fig. 5 which is justified by the Theorems 21–35, as well as by Proposition 13 and Proposition 19.

Further work focusses on the analysis of models that combine stochastic transitions and (non-internal) action labeled transitions, so that in addition to interleaving, synchronization can also be expressed.

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