# Semantics of Probabilistic Automata via Coalgebra


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ENS DE LYON


Women in Logic @ LICS'19

probabilistic automata

# The different natures of PA

probabilistic automata

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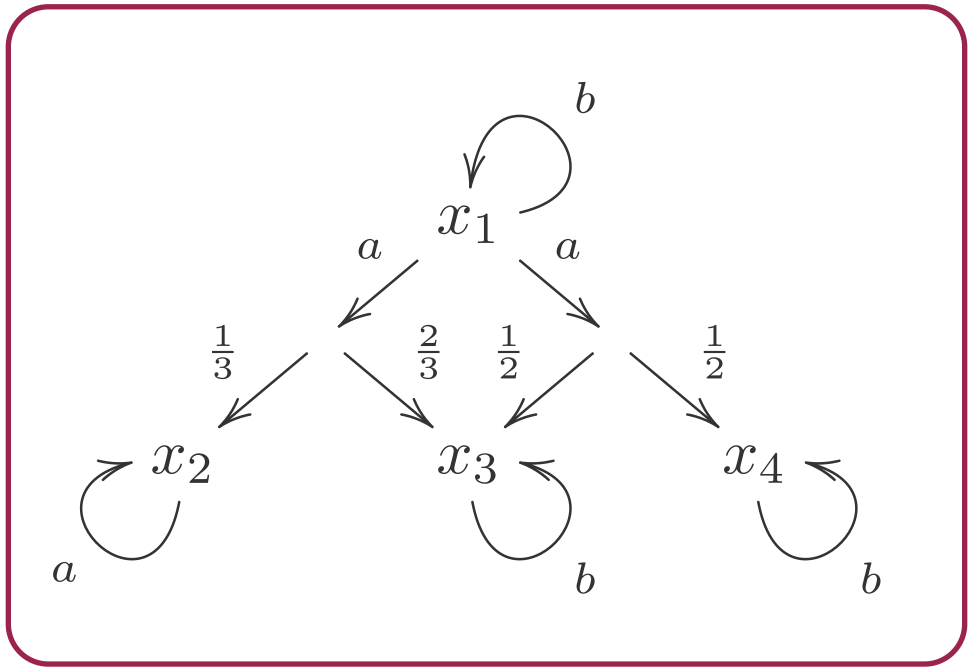
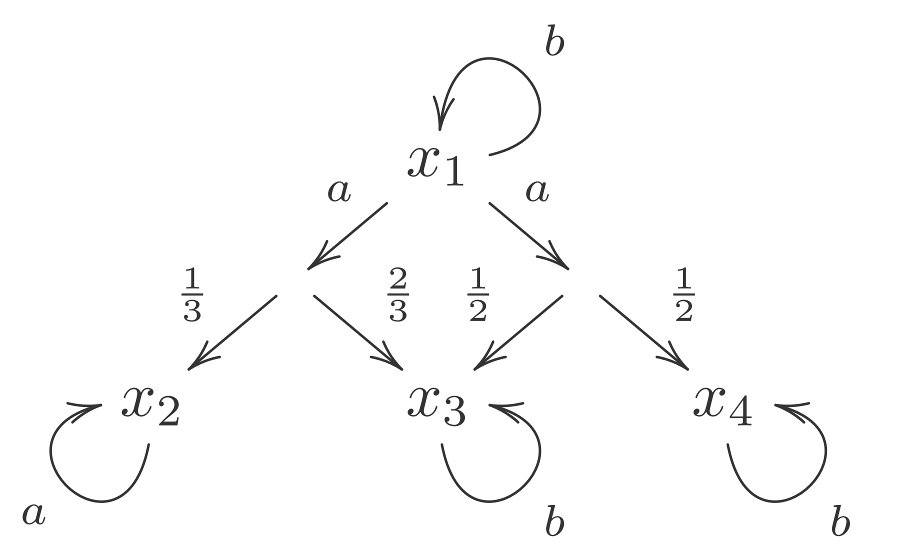


probabilistic automata

# The different natures of PA

$$X \xrightarrow{c} (\mathcal{P}\mathcal{D}X)^A$$



probabilistic automata

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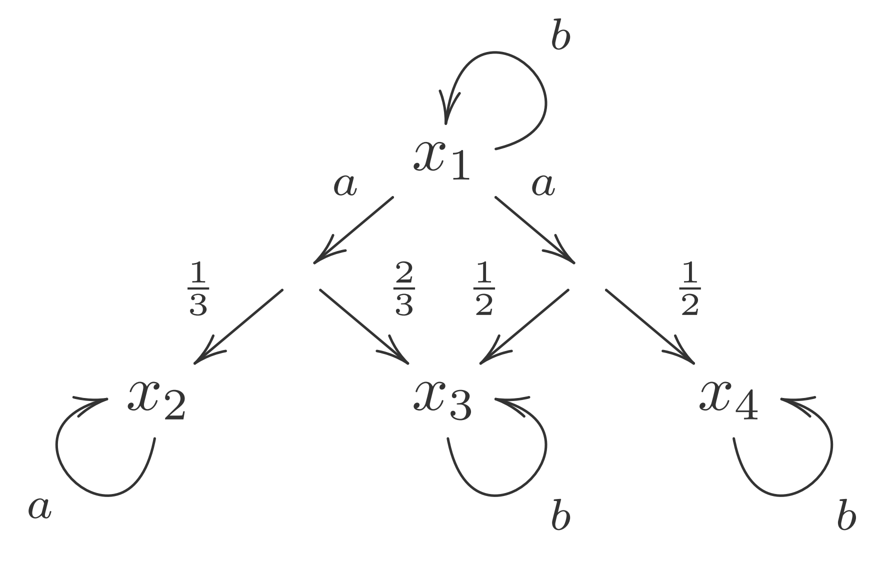

we write $s \xrightarrow{a} \mu$

for $\mu \in c(s)(a)$

# Probabilistic Automata

Can be given different semantics:

1. Bisimilarity
2. Convex bisimilarity
3. Distribution bisimilarity
4. Trace semantics

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trace and testing theory

[Bonchi, S., Vignudelli LICS'19]

# Bisimilarity

An equivalence relation $R$ on the PA $c\colon X \to (\mathcal{P}\mathcal{D}X)^A$ is a bisimulation iff whenever $(s,t) \in R$ for all $a \in A$ and $\mu \in \mathcal{D}X$

$$s \xrightarrow{a} \mu \Longrightarrow \exists \nu \in \mathcal{D}X.\ t \xrightarrow{a} \nu \wedge \mu \equiv_R \nu$$

where $\mu \equiv_R \nu$ iff $\mu[C] = \nu[C]$ for all $R$-equivalence classes $C$, with $\mu[C] = \sum_{x \in C} \mu(x)$.

Bisimilarity on $c\colon X \to (\mathcal{P}\mathcal{D}X)^A$, denoted by $\sim$, is the largest bisimulation.

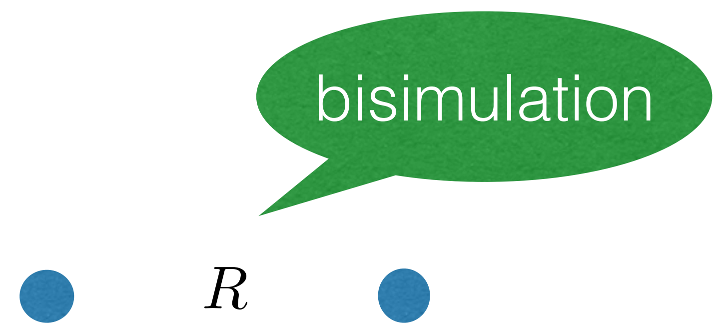
# Bisimilarity

# Bisimilarity

bisimulation

$R$

# Bisimilarity



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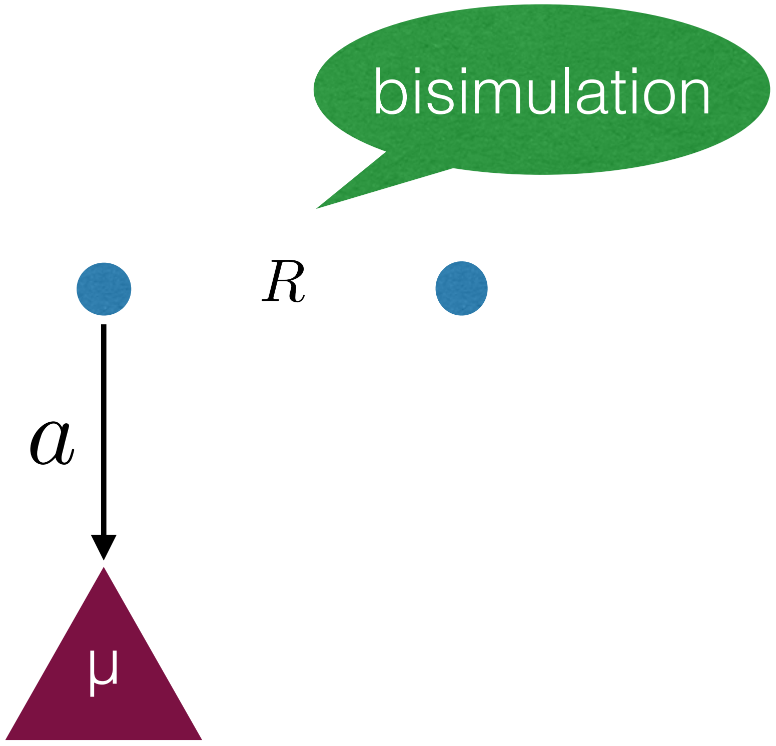

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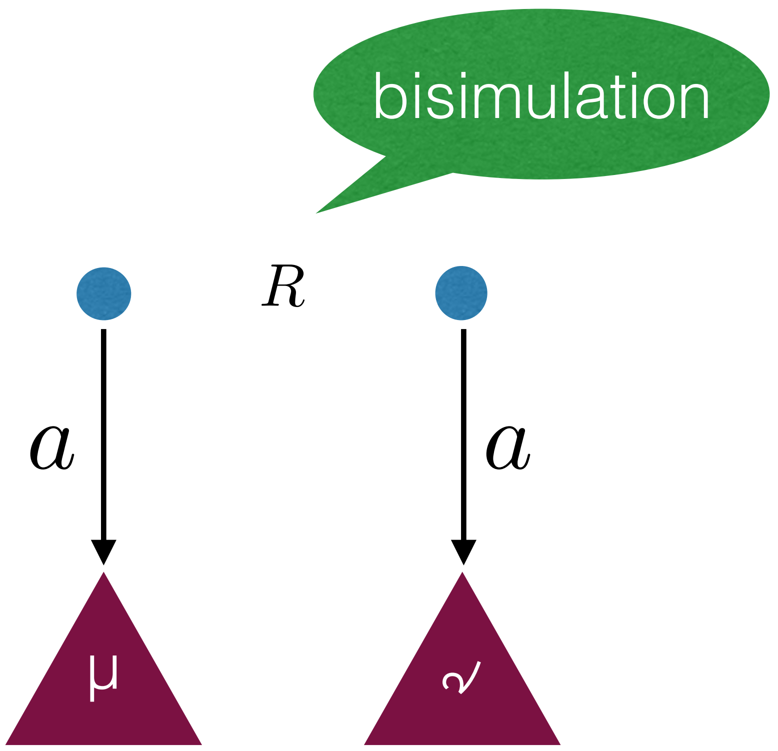

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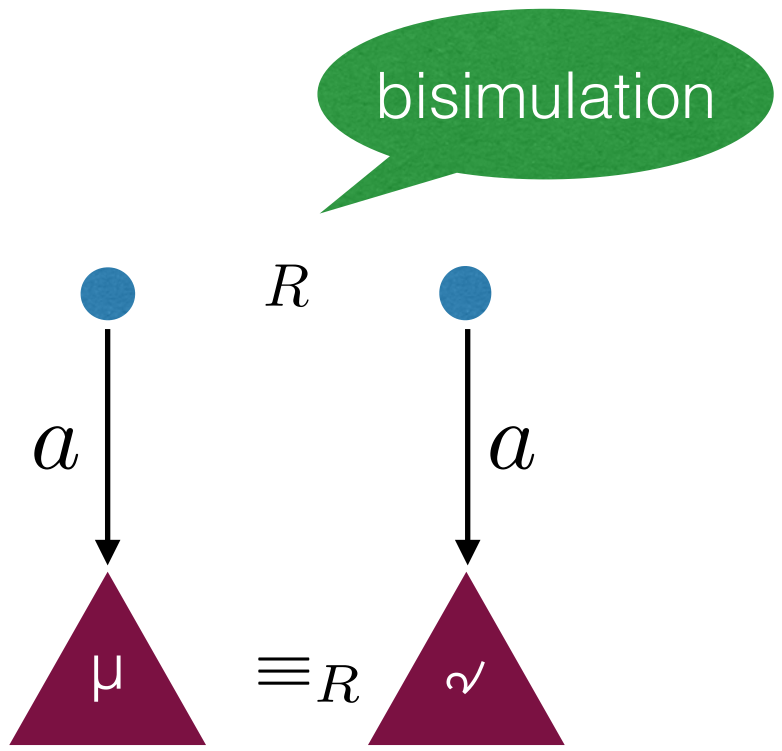

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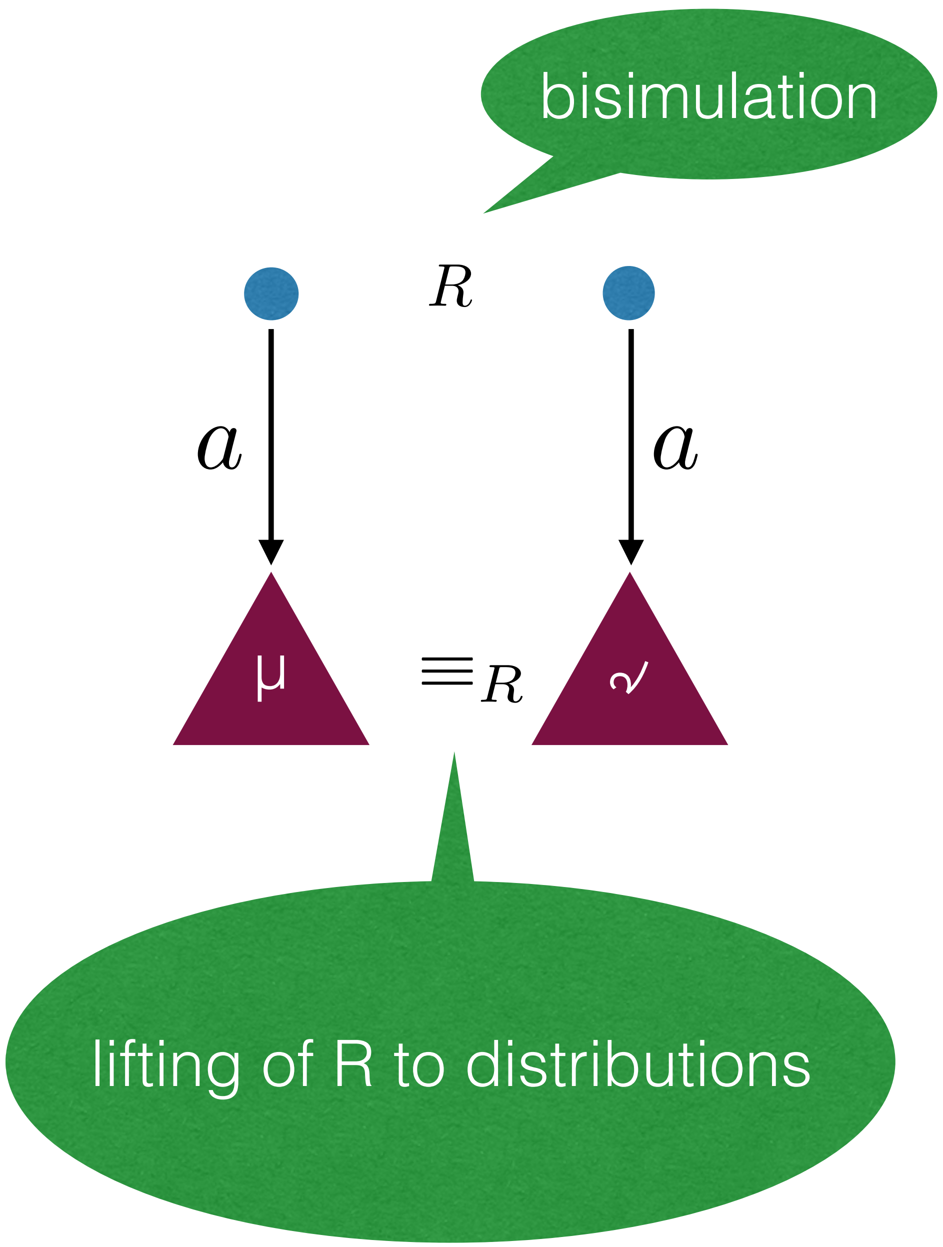

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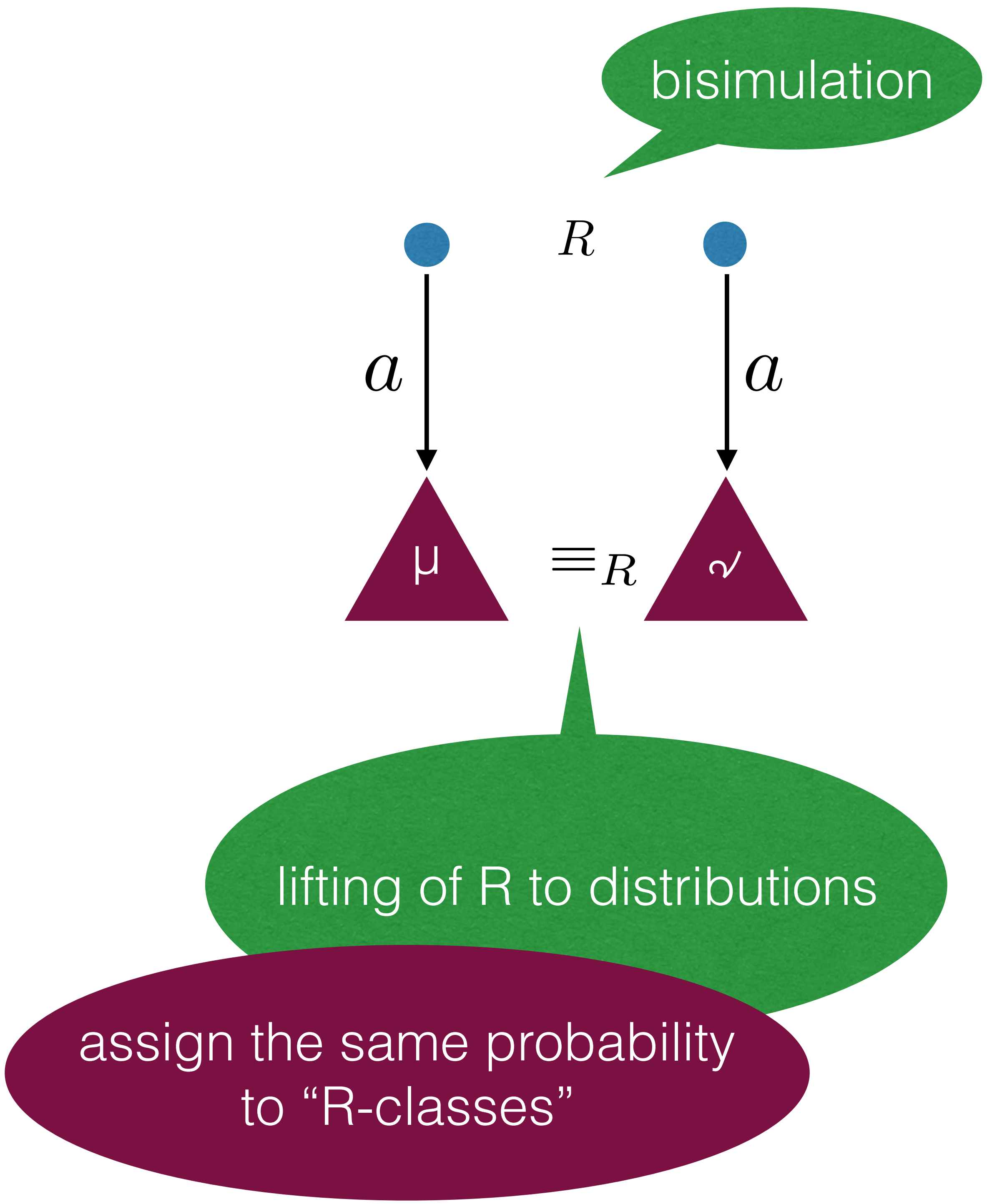
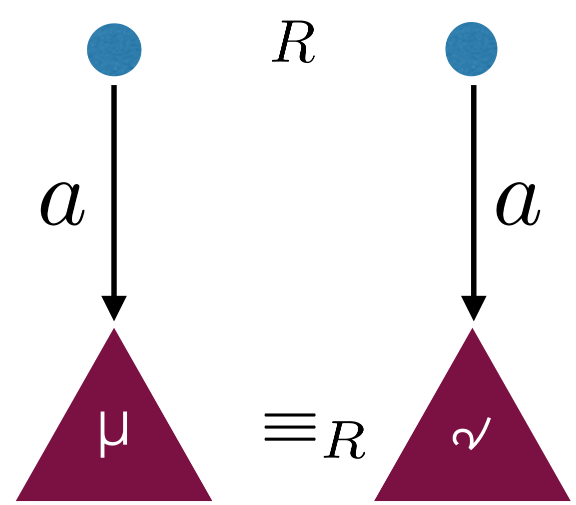


# Bisimilarity

$\sim$ largest bisimulation

bisimulation




lifting of R to distributions

assign the same probability to "R-classes"

# Convex bisimilarity

An equivalence relation $R \subseteq X \times X$ is a convex bisimulation of the PA $c\colon X \to (\mathcal{P}\mathcal{D}X)^A$ iff whenever $(x, y) \in R$, for all $a \in A$ and $\mu \in \mathcal{D}X$

$$x \xrightarrow{a} \mu \quad \Rightarrow \quad \exists \nu.\, \mu \equiv_R \nu \wedge \nu = \sum_{i=1}^{n} p_i \nu_i \wedge y \xrightarrow{a} \nu_i.$$

Convex bisimilarity on $c\colon X \to (\mathcal{P}\mathcal{D}X)^A$, denoted by $\sim_c$, is the largest bisimulation.

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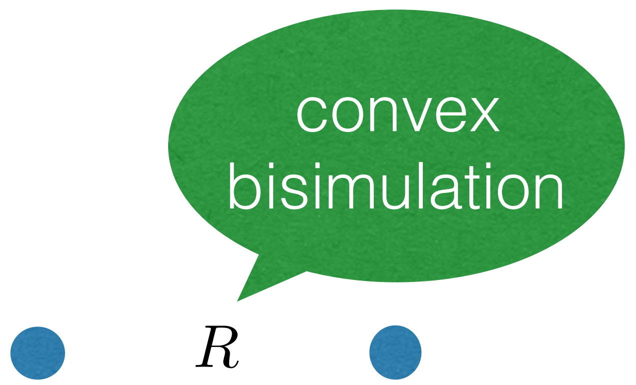

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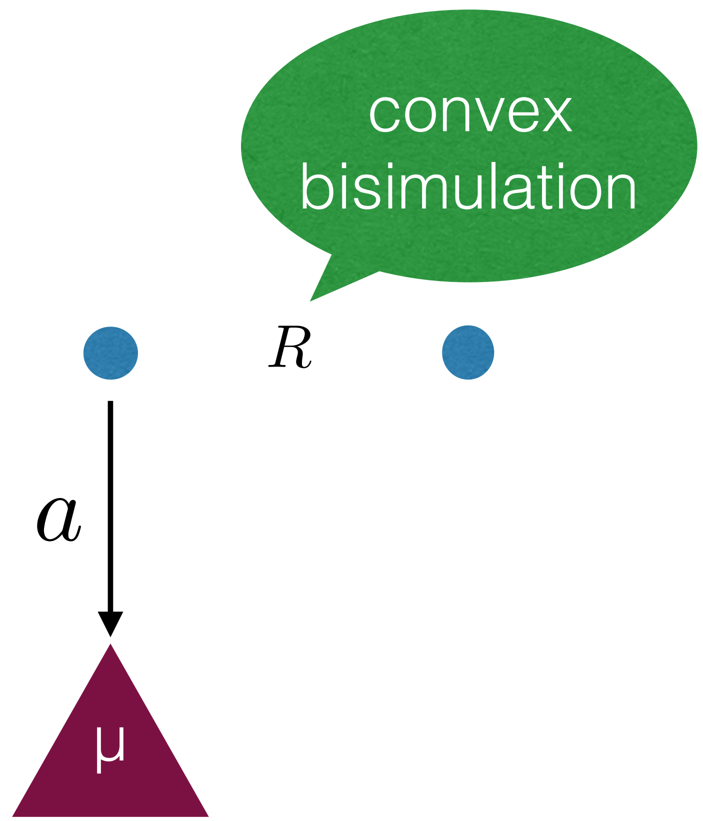

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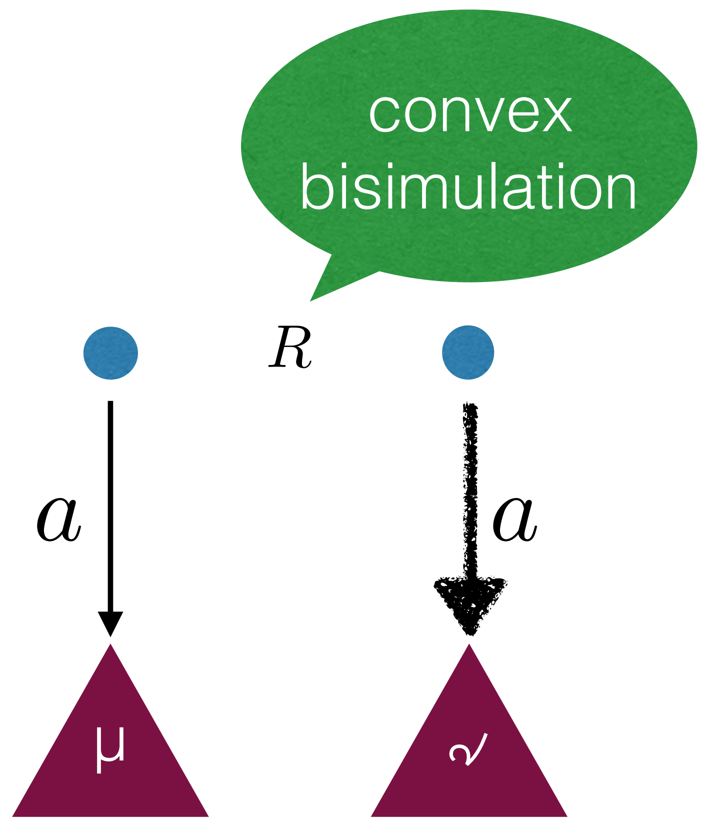

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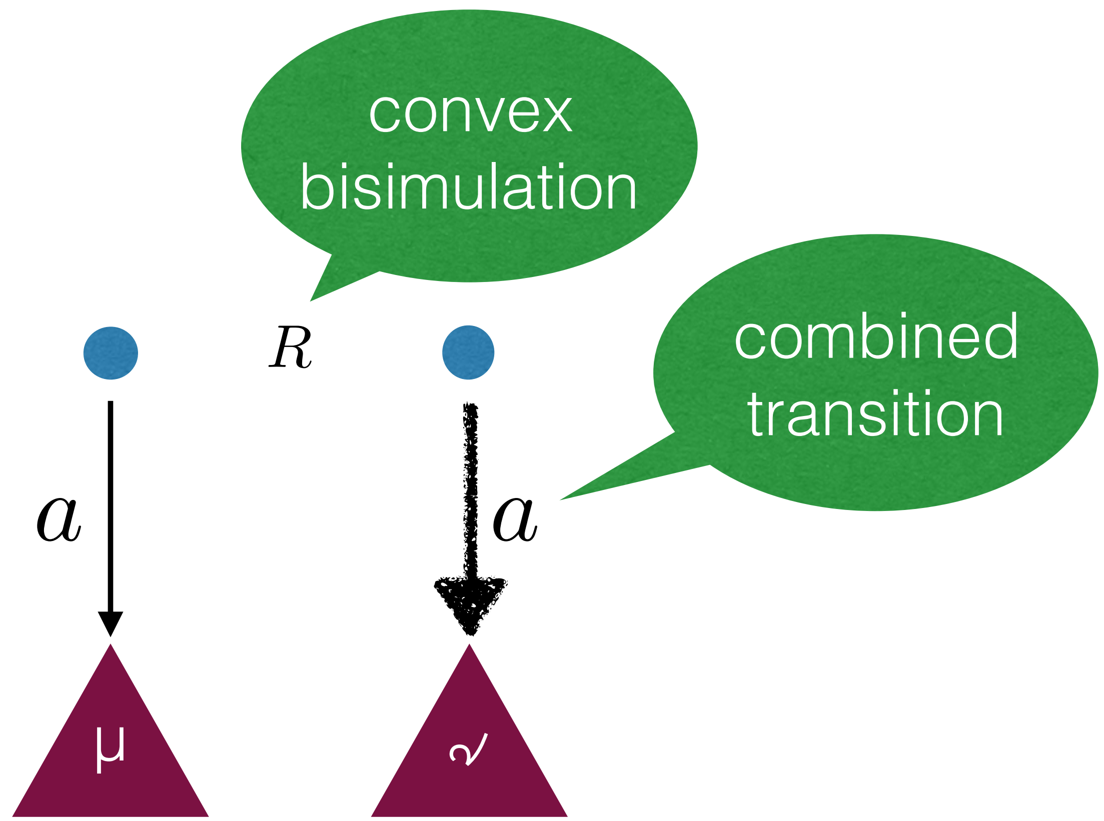

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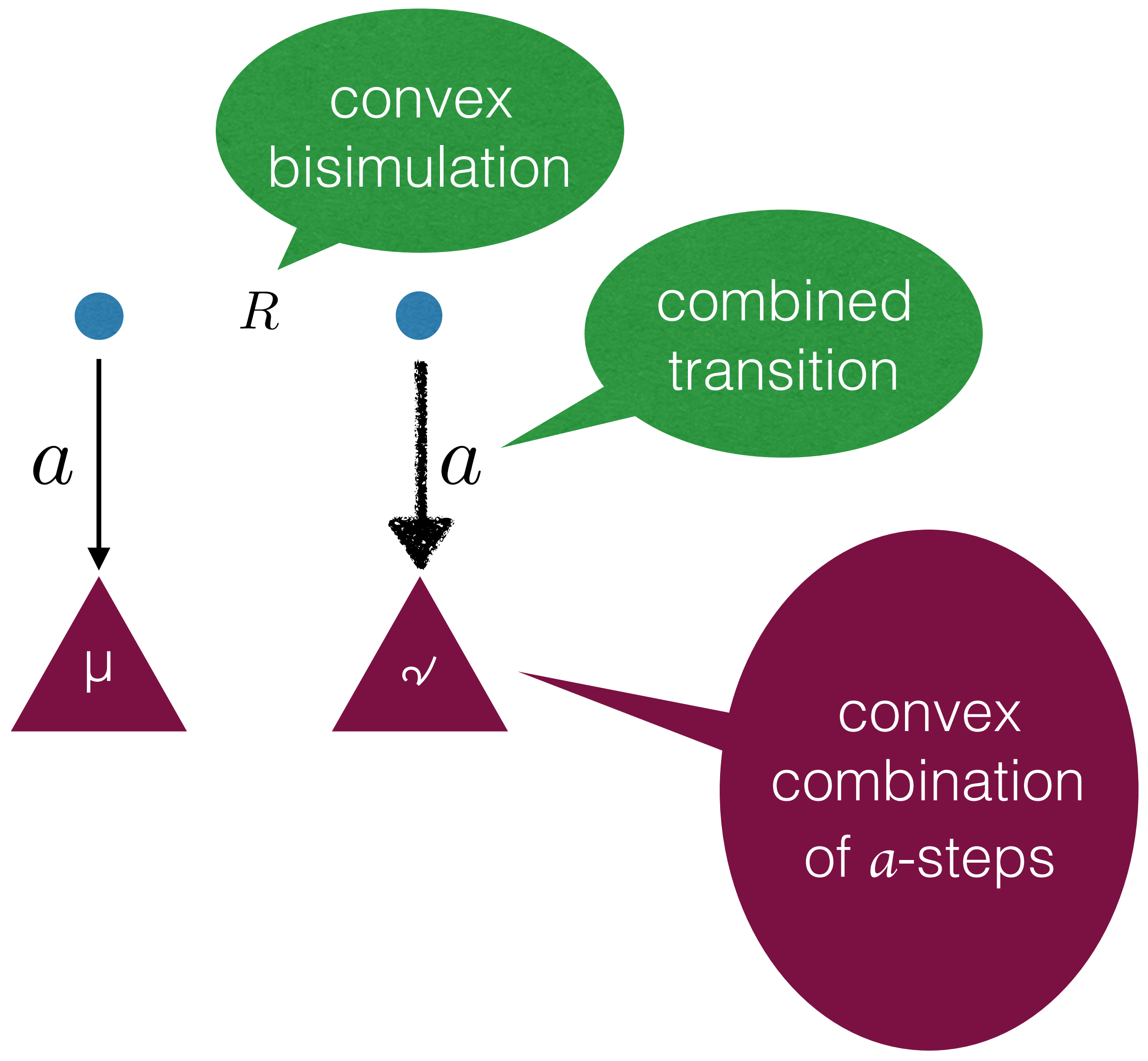

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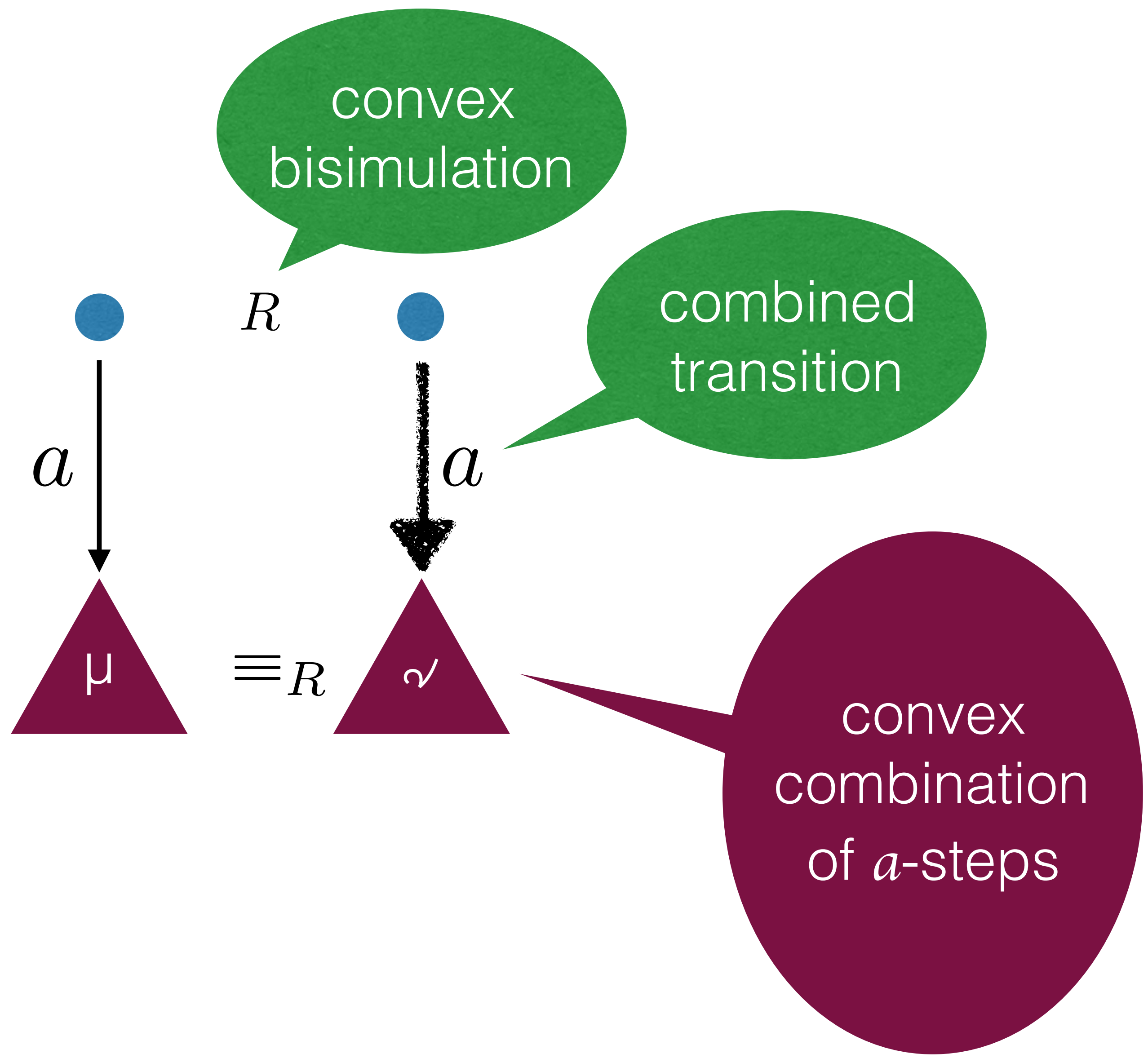

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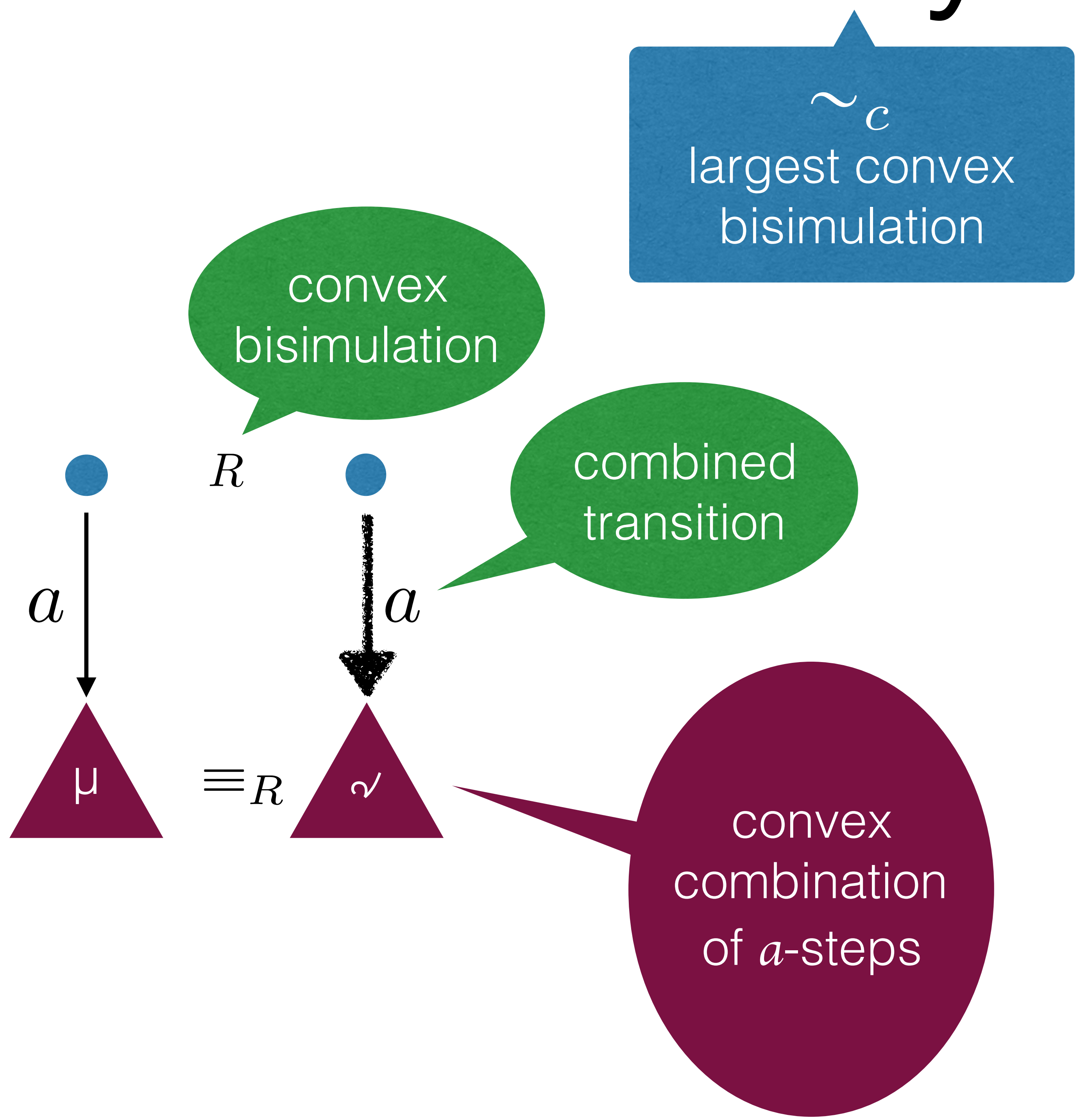

# Distribution bisimilarity

An equivalence relation $R$ on the carrier of the belief-state transformer $c\colon \mathcal{D}X \to (\mathcal{P}\mathcal{D}X)^A$ is a distribution bisimulation iff whenever $(\mu, \nu) \in R$ for all $a \in A$

$$\mu \xrightarrow{a} \mu' \Longrightarrow \exists \nu' \in \mathcal{D}X.\ \nu \xrightarrow{a} \nu' \wedge (\mu', \nu') \in R.$$

Distribution bisimilarity on $c\colon \mathcal{D}X \to (\mathcal{P}\mathcal{D}X)^A$, denoted by $\sim_d$, is the largest distribution bisimulation.

# Distribution bisimilarity

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distribution bisimulation

$R$

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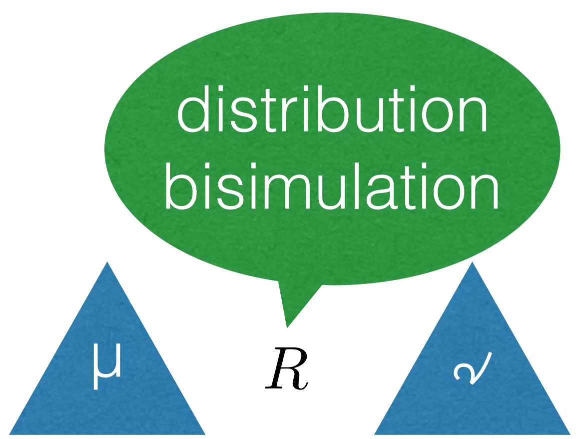

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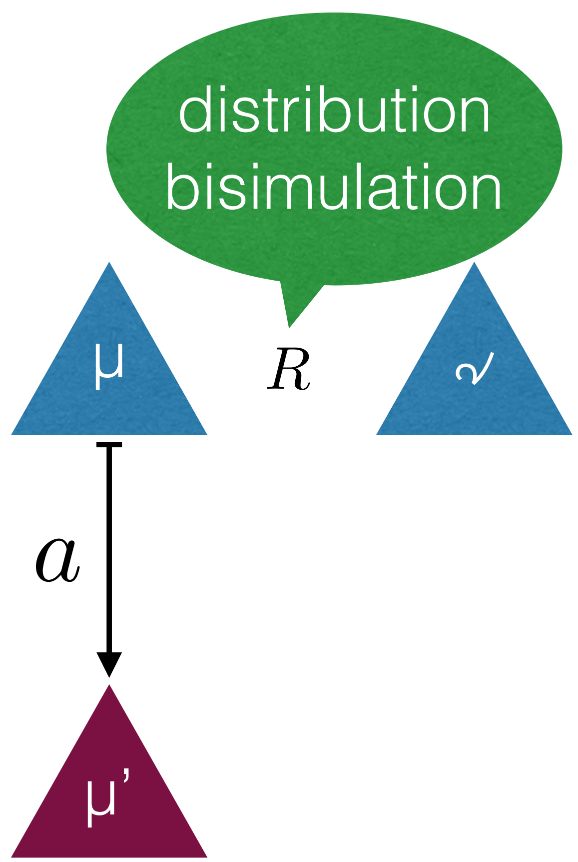

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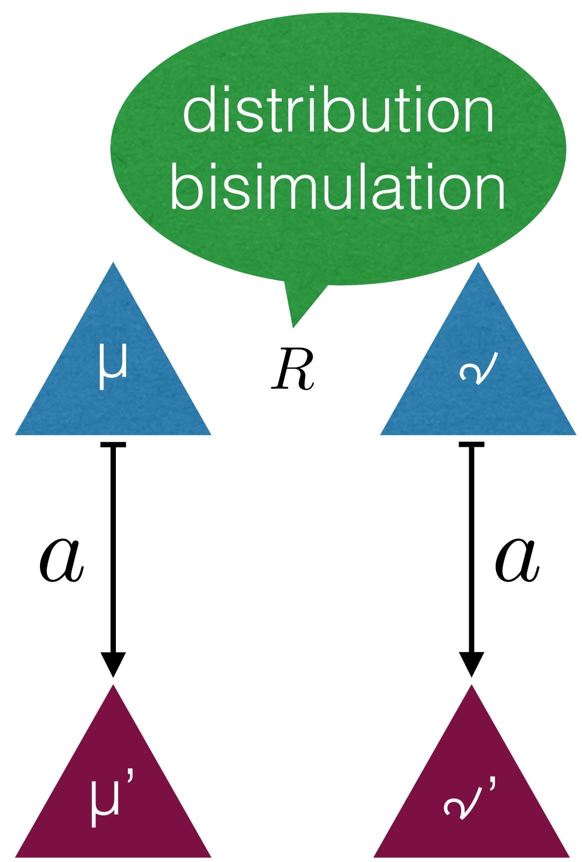

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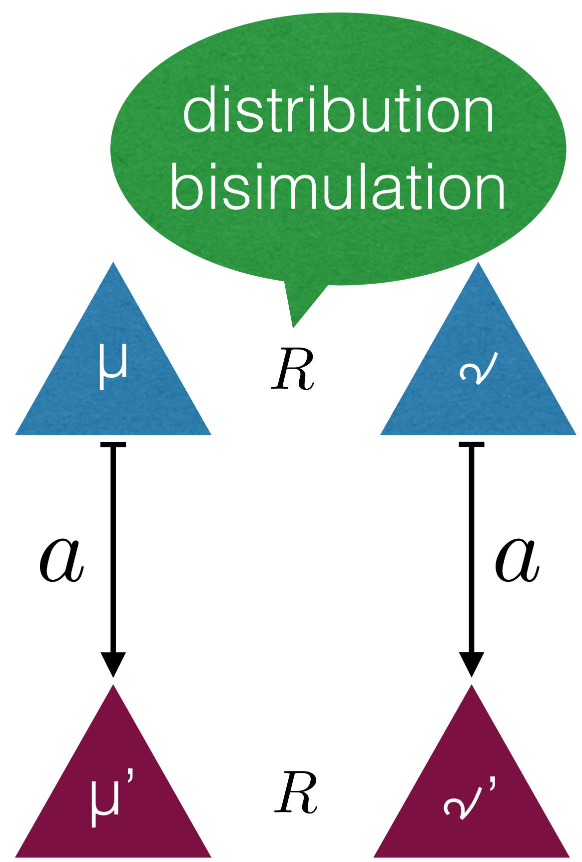

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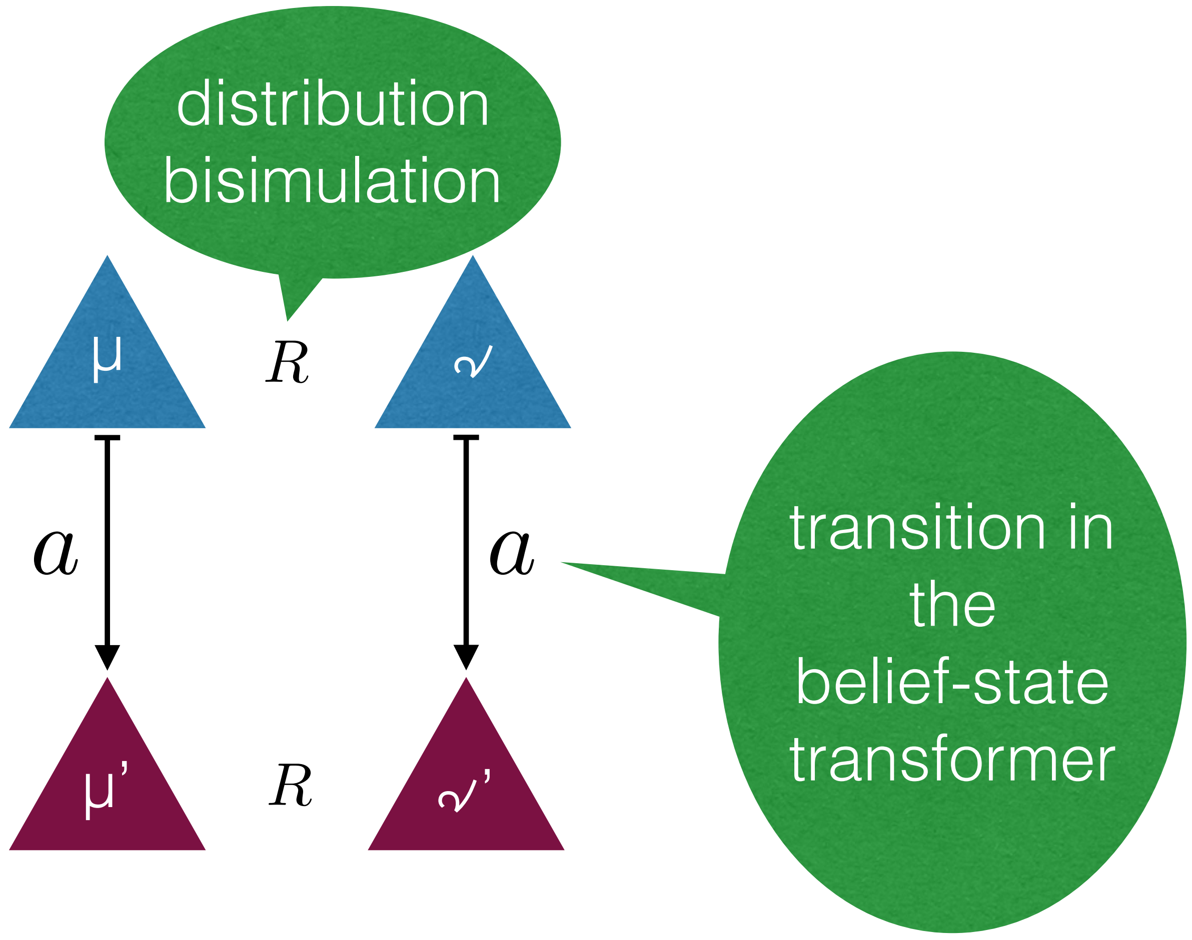

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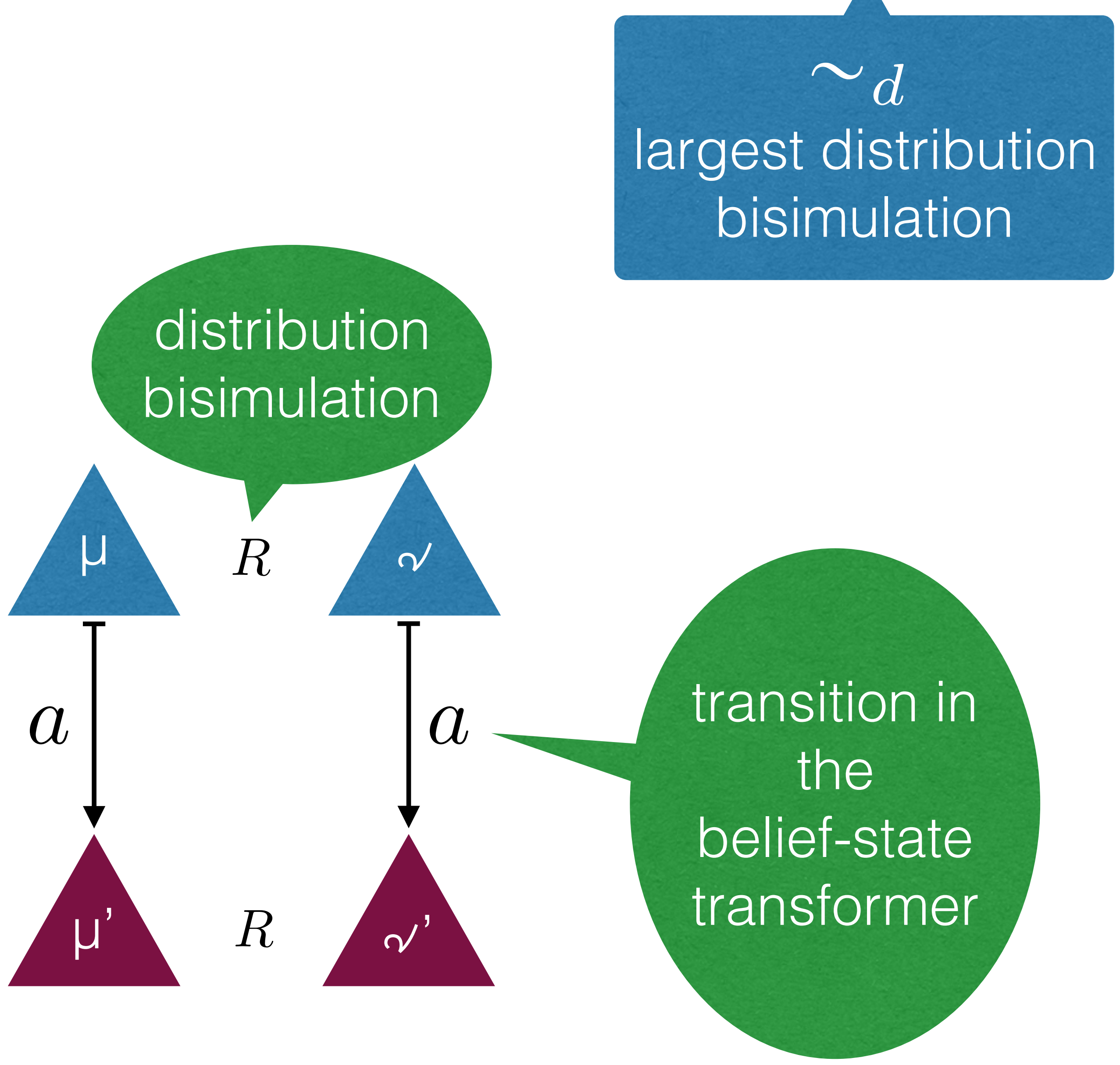

# Distribution bisimilarity

$\sim_d$
largest distribution bisimulation

distribution bisimulation

$$\begin{tikzcd} \mu \arrow[r, phantom, "R"] \arrow[d, "a"'] & \nu \arrow[d, "a"] \\ \mu' \arrow[r, phantom, "R"] & \nu' \end{tikzcd}$$

transition in the belief-state transformer

$\sim_d$
is LTS bisimilarity on the belief-state transformer

# Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

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$$\mathbf{CoAlg}_{\mathbf{C}}(F)$$

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form a category too

$\mathbf{CoAlg}_{\mathbf{C}}(F)$

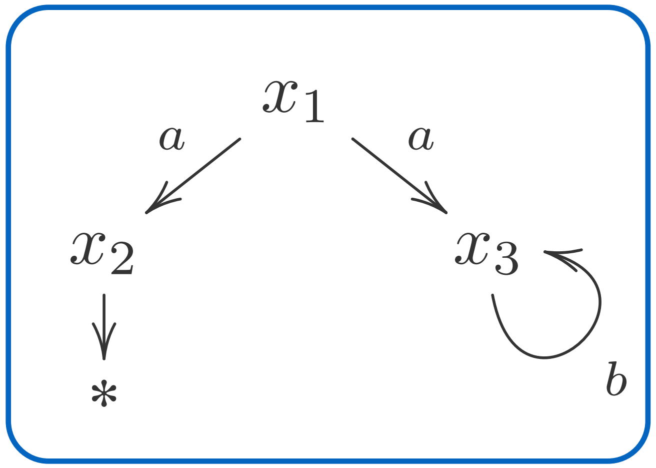
generic notion of behavioural equivalence

$\approx$

# Examples

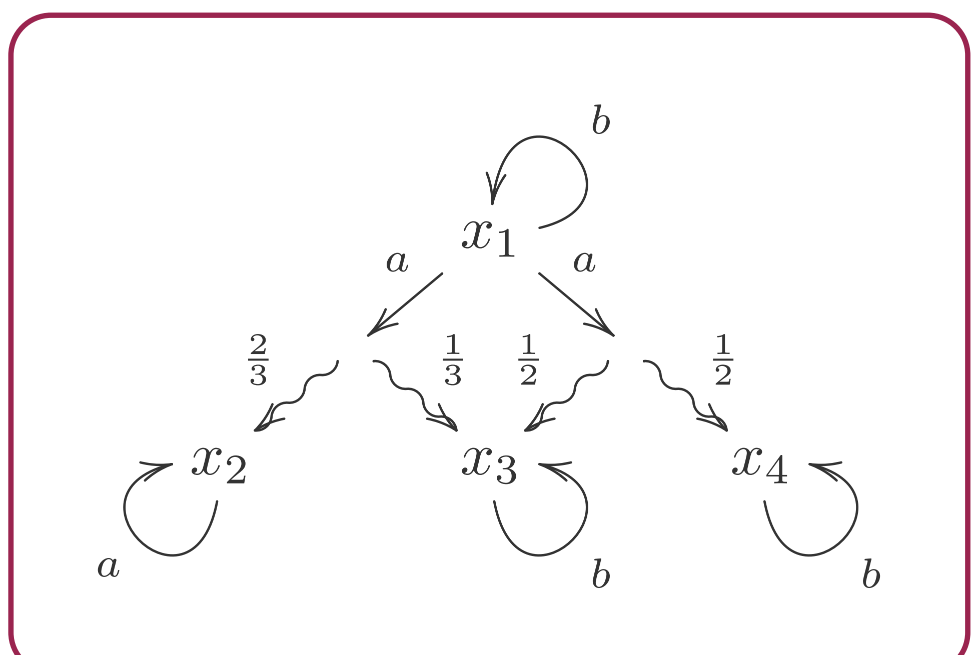
## NFA

$X \to 2 \times (\mathcal{P}X)^A$



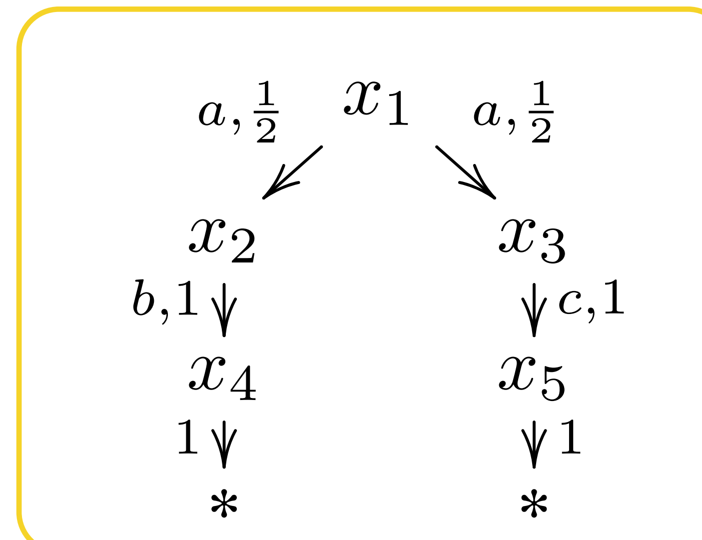

## PA

$X \to (\mathcal{P}\mathcal{D}X)^A$




## Generative PTS

$X \to \mathcal{D}(1 + A \times X)$



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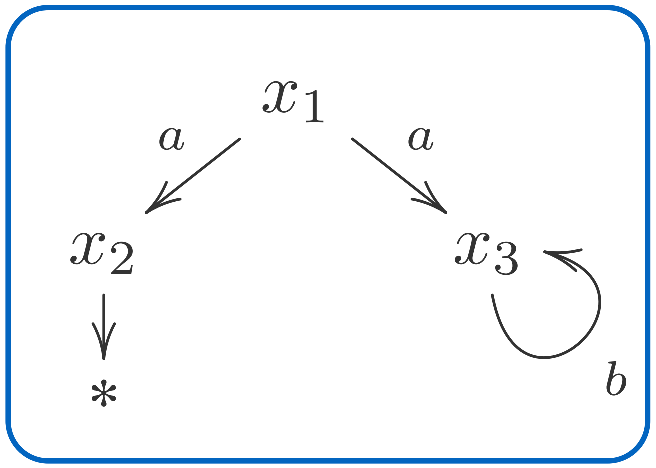


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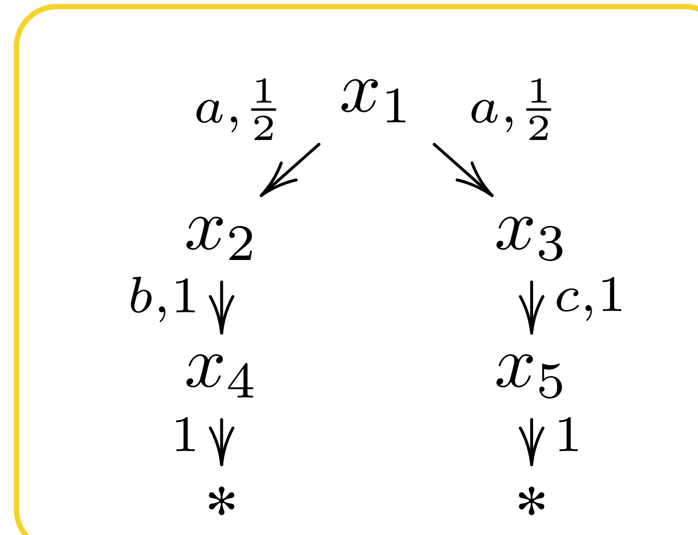


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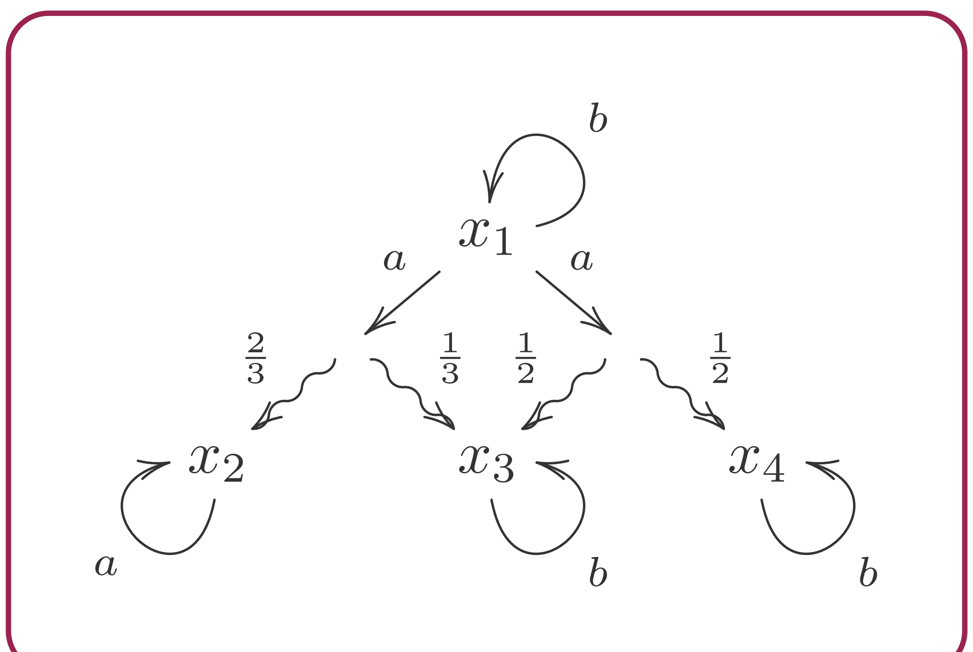


all on
**Sets**

# PA coalgebraically

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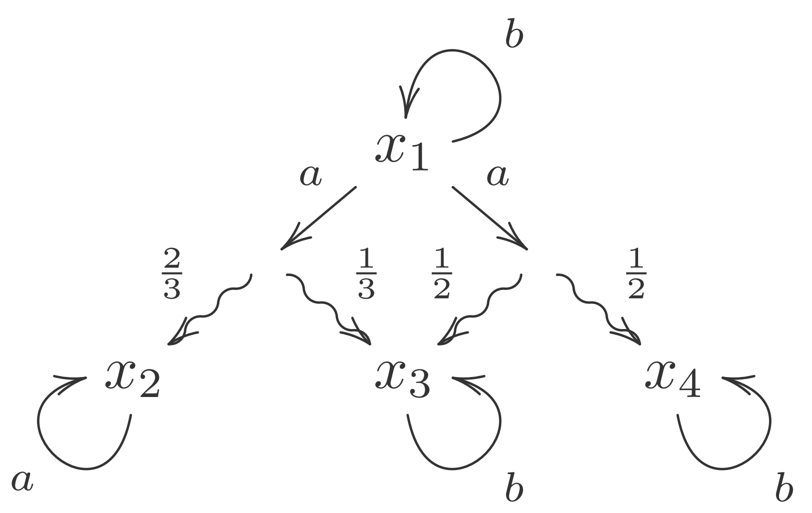

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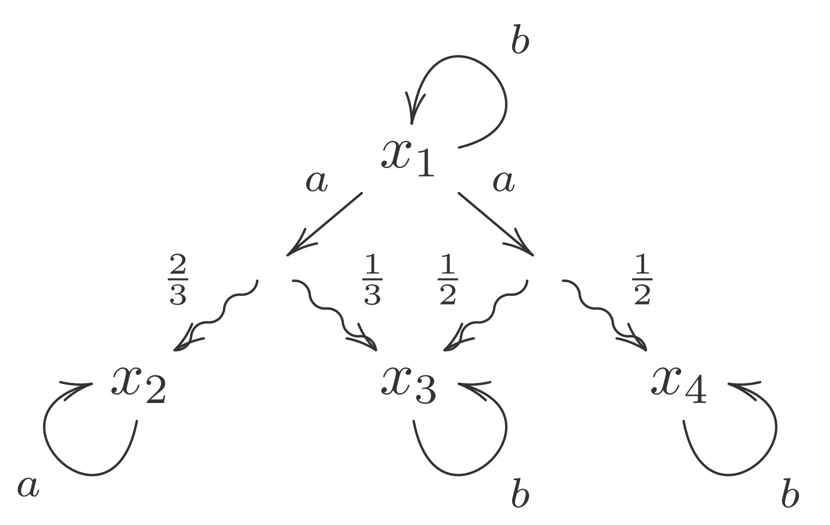


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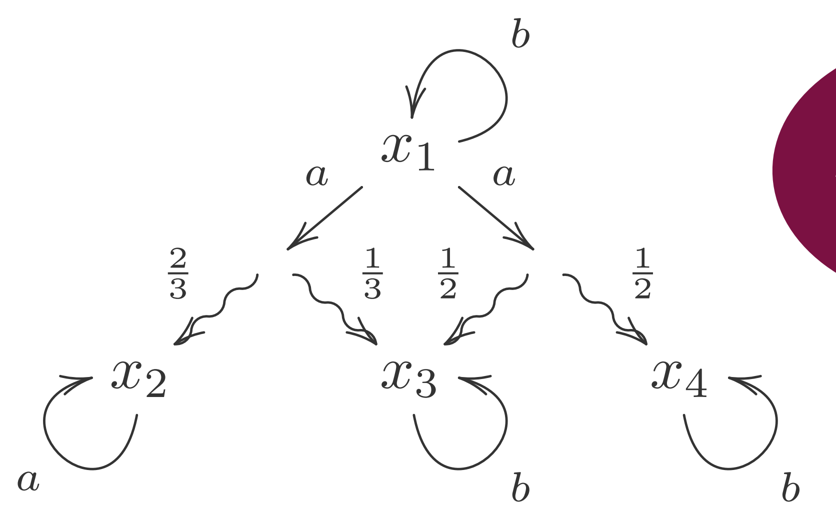


on **Sets**

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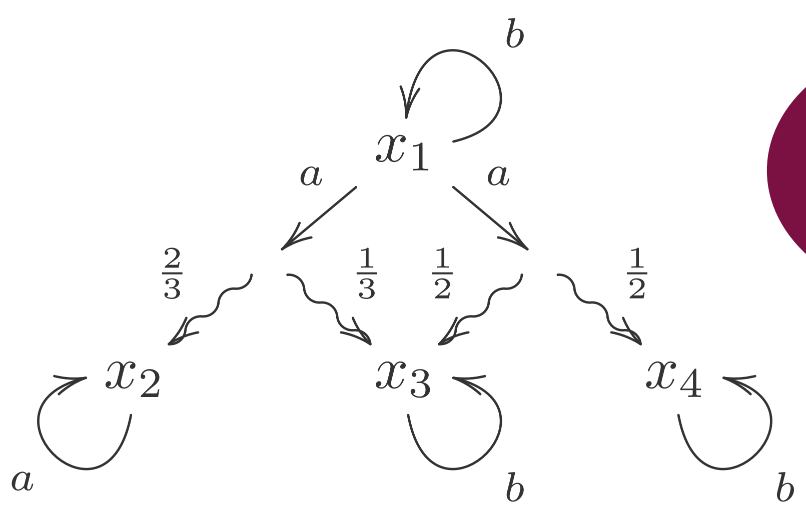


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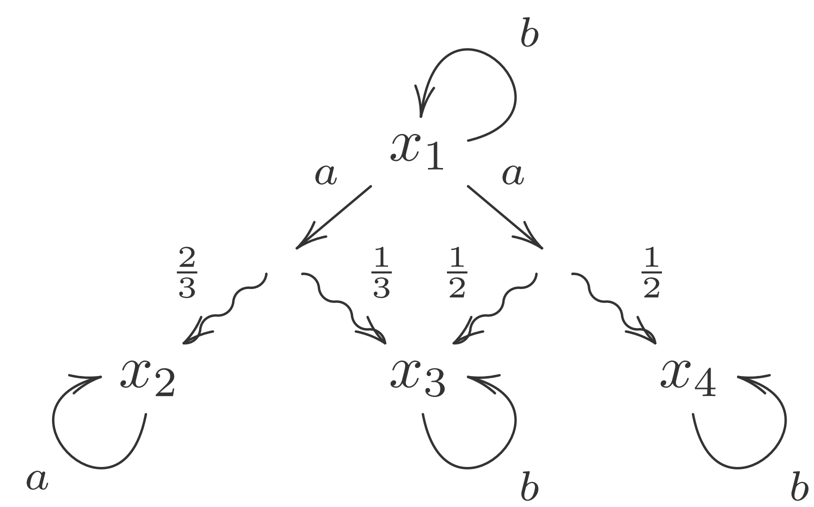
$X \to (\mathcal{C}X)^A$



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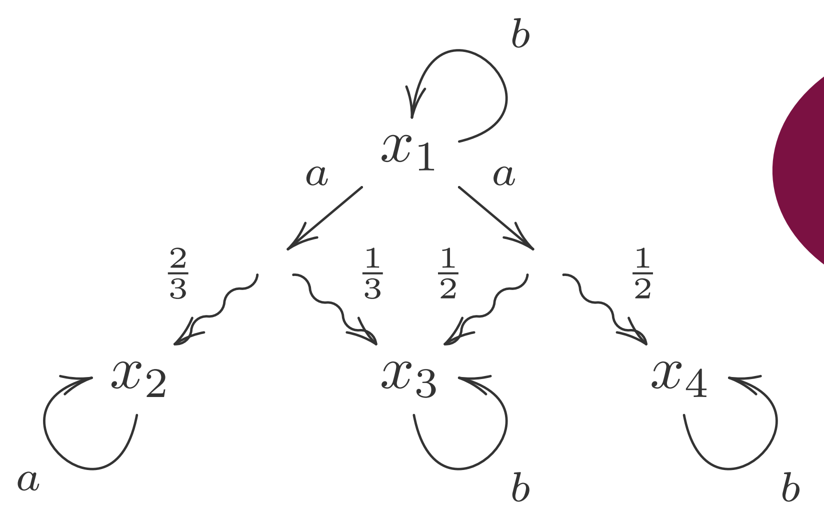


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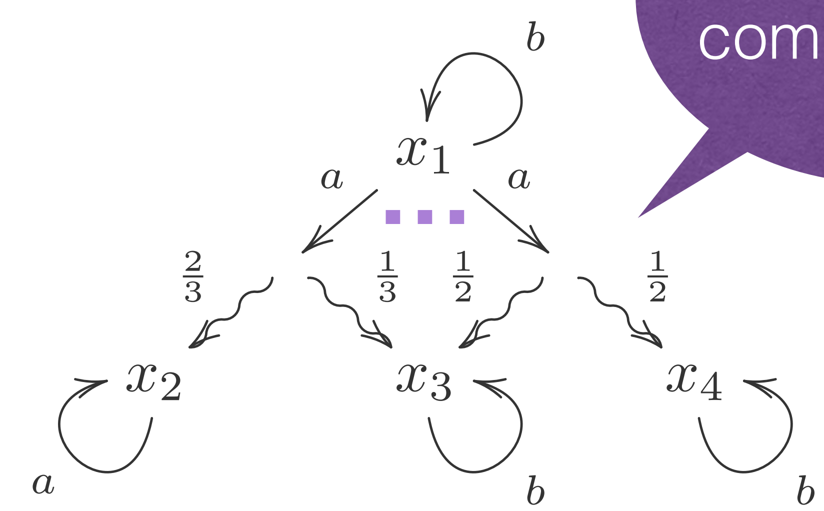


and all convex combinations

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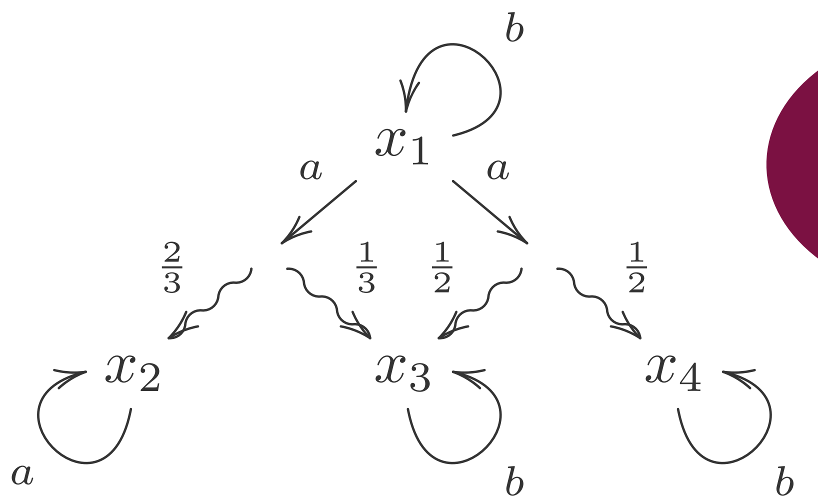


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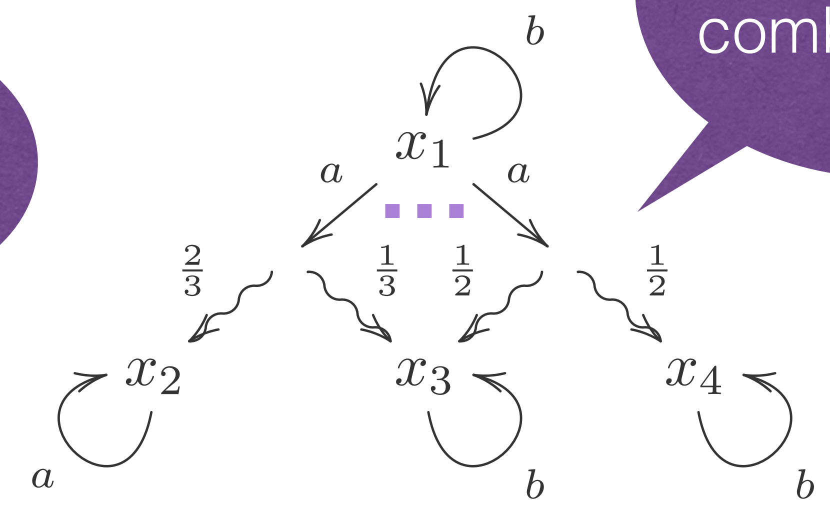

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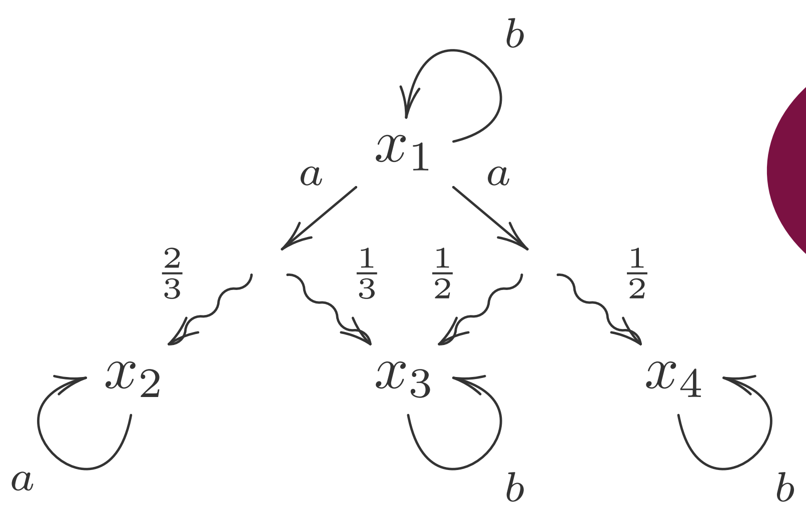


on **Sets**

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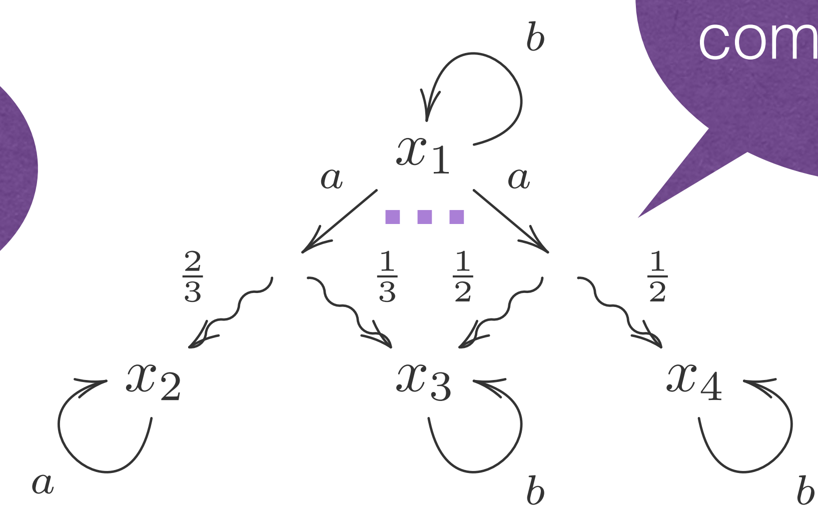
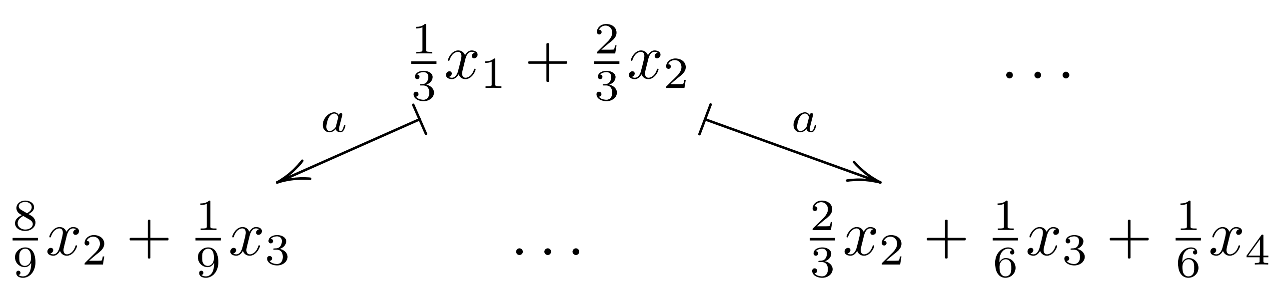



and all convex combinations

$X \to (\mathcal{P}_c X+1)^A$



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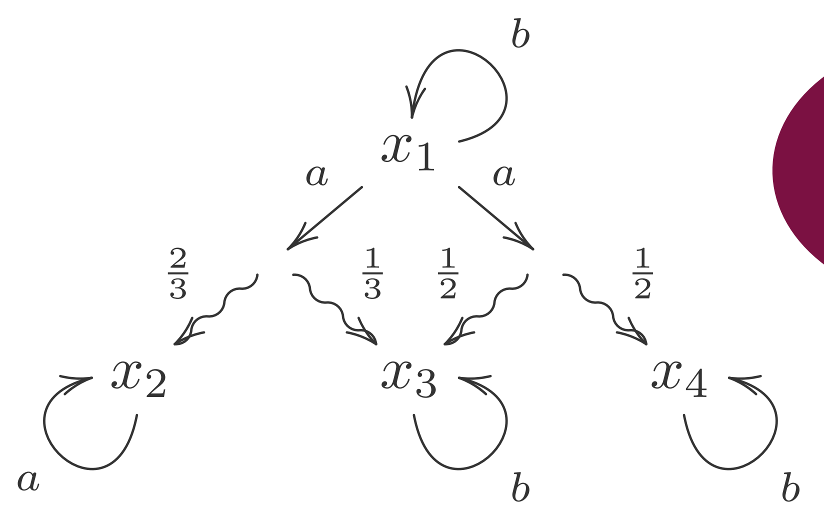


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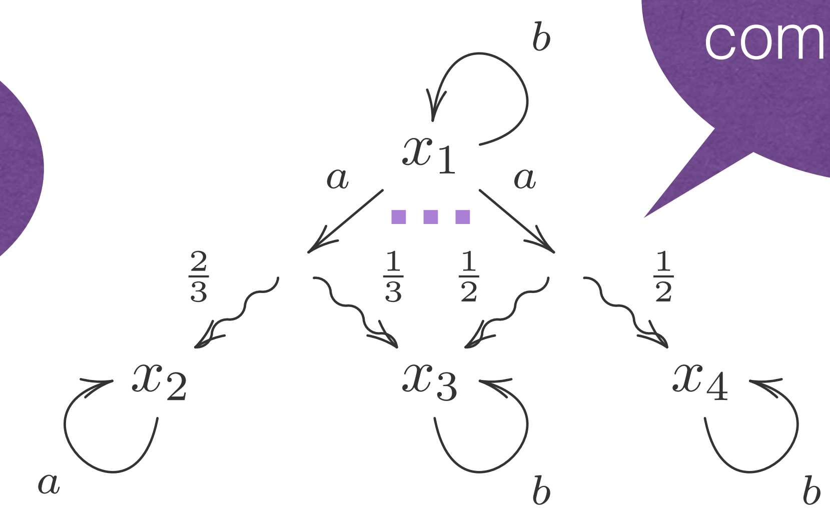


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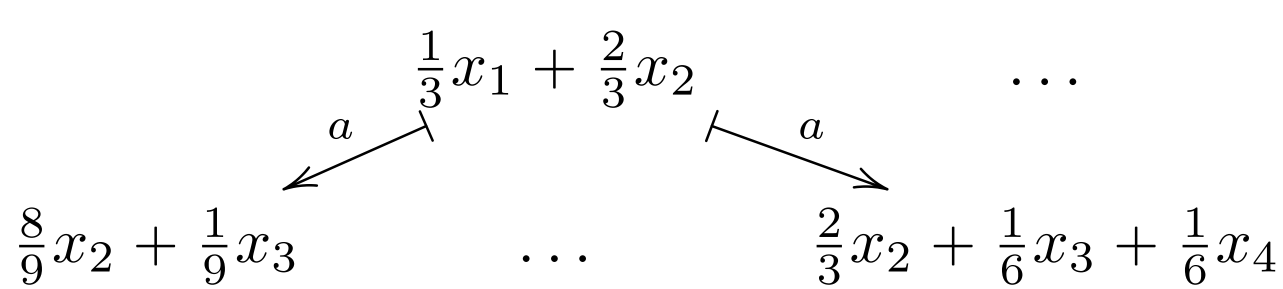


on
convex
algebras

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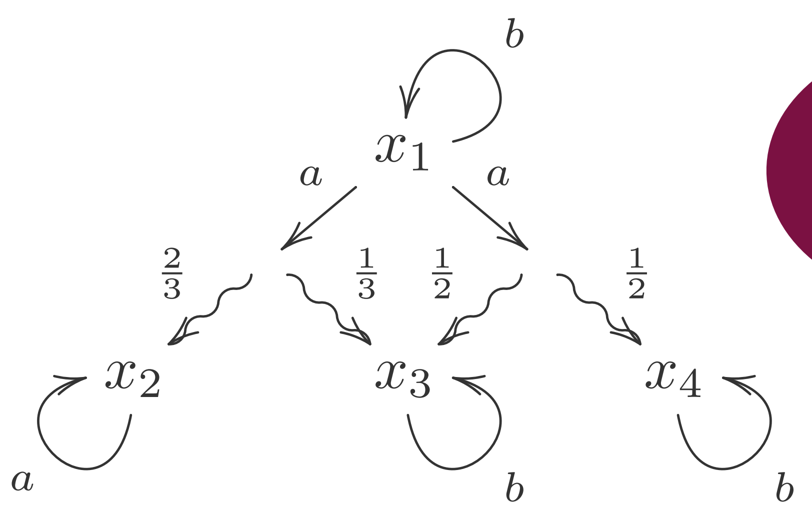


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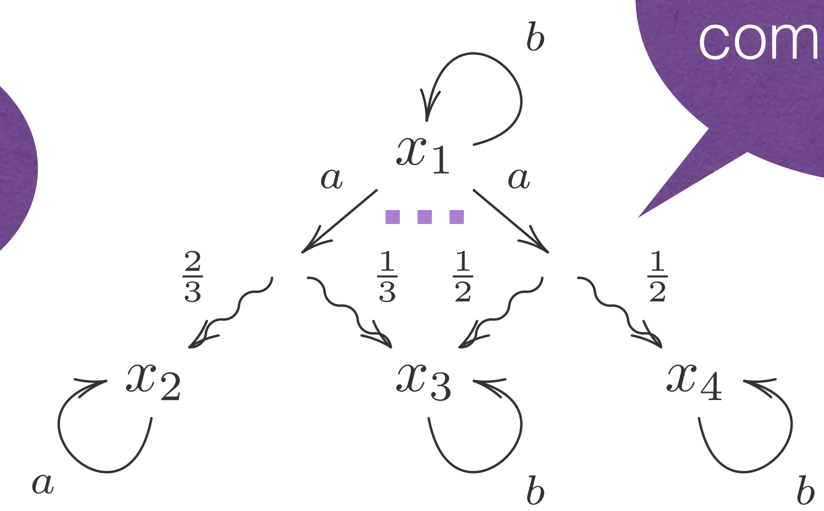



$X \to (\mathcal{P}_c X+1)^A$

$$\frac{1}{3}x_1 + \frac{2}{3}x_2 \qquad \ldots$$

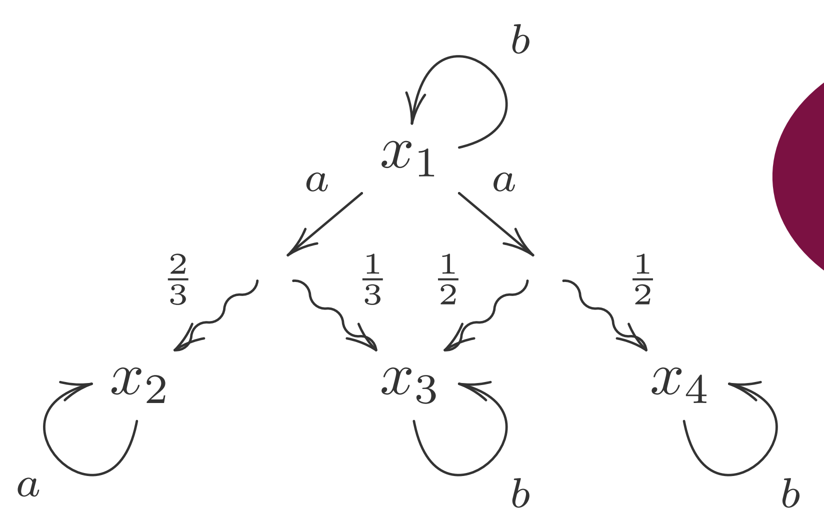
$$\frac{8}{9}x_2 + \frac{1}{9}x_3 \xleftarrow{a} \quad \ldots \quad \xrightarrow{a} \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4$$

on convex algebras

$\mathcal{EM}(\mathcal{D})$

# PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}X)^A$



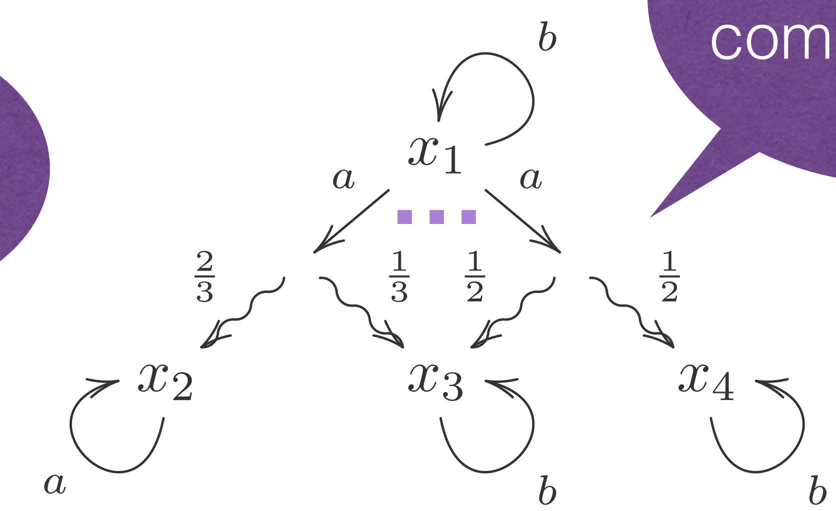

on **Sets**

$\sim \; = \; \approx$

$\sim_c \; = \; \approx$

$X \to (\mathcal{C}X)^A$

and all convex combinations




$X \to (\mathcal{P}_c X+1)^A$

$$\begin{tikzcd} & \frac{1}{3}x_1+\frac{2}{3}x_2 \arrow[dl, "a"'] \arrow[dr, "a"] & & \dots \\ \frac{8}{9}x_2+\frac{1}{9}x_3 & \dots & \frac{2}{3}x_2+\frac{1}{6}x_3+\frac{1}{6}x_4 & \end{tikzcd}$$

on convex algebras

$\mathcal{EM}(\mathcal{D})$

$\sim_d \; = \; \approx$

$\mathcal{EM}(\mathcal{D})$

# Convex algebras

- algebras

$$(A, \sum_{i=1}^{n} p_i(-)_i)$$

- convex (affine) maps

$$h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)$$

$\mathcal{EM}(\mathcal{D})$

# Convex algebras

- algebras $\boxed{(A, \sum_{i=1}^{n} p_i(-)_i)}$ $\quad p_i \in [0,1], \sum_{i=1}^{n} p_i = 1$

- convex (affine) maps $\boxed{h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)}$

$\mathcal{EM}(\mathcal{D})$

# Convex algebras

infinitely many finitary operations

- algebras

$$\boxed{(A, \sum_{i=1}^{n} p_i(-)_i)} \qquad p_i \in [0,1], \sum_{i=1}^{n} p_i = 1$$

- convex (affine) maps

$$\boxed{h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)}$$

$\mathcal{EM}(\mathcal{D})$

# Convex algebras

infinitely many finitary operations

convex combinations

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$\mathcal{EM}(\mathcal{D})$

# Convex algebras

infinitely many finitary operations

convex combinations

- algebras $\boxed{(A, \sum_{i=1}^{n} p_i(-)_i)}$ $\quad p_i \in [0,1], \sum_{i=1}^{n} p_i = 1$

- convex (affine) maps $\boxed{h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)}$

satisfying

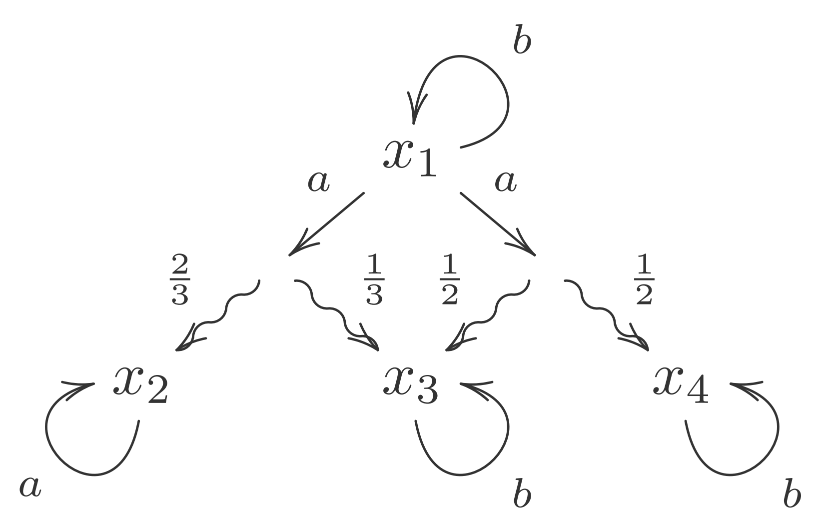
- Projection $\quad \sum_{i=1}^{n} p_i a_i = a_k, \quad p_k = 1$
- Barycenter $\quad \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} p_{i,j} a_j\right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} p_i p_{i,j}\right) a_j$

# Belief-state transformer

# Belief-state transformer

PA

$X \to (\mathcal{P}\mathcal{D}X)^A$



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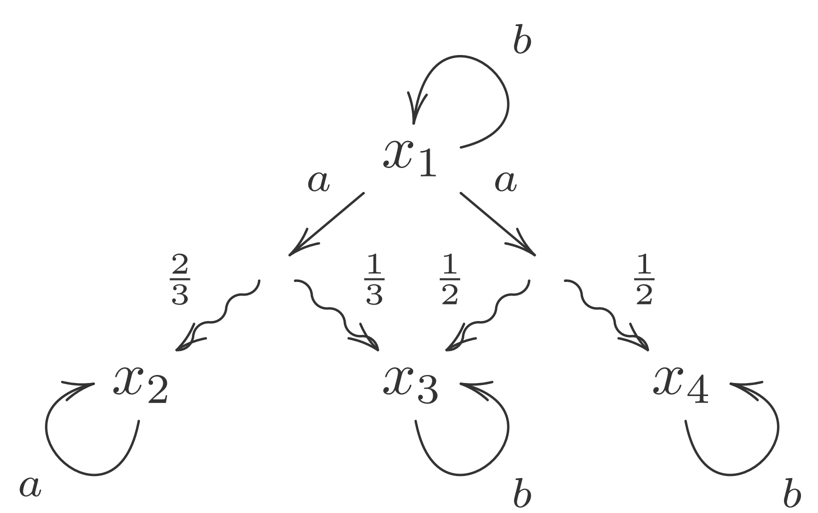

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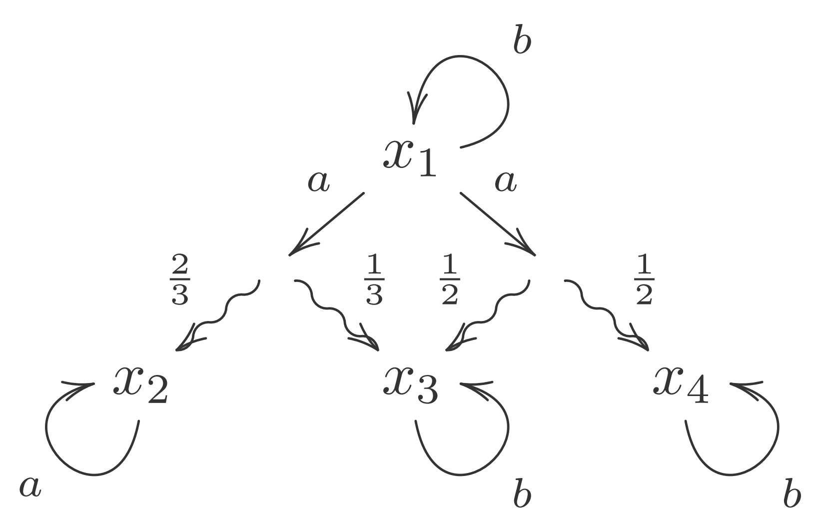
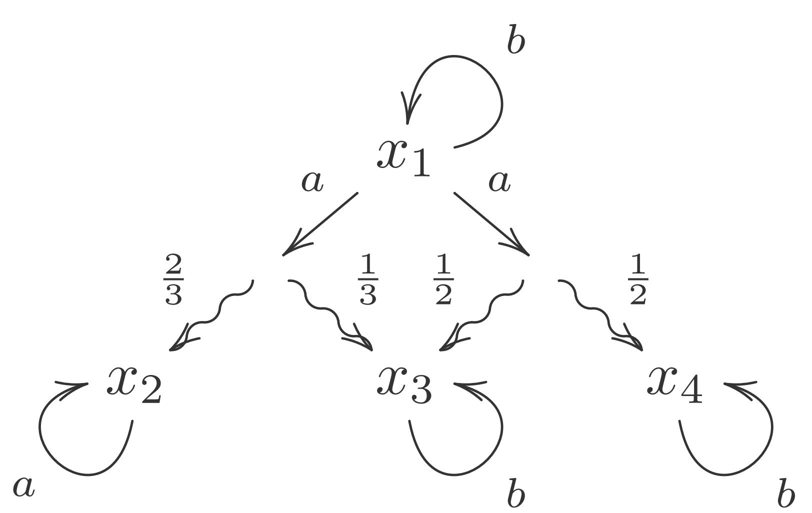



$$\begin{tikzcd}
& \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & \cdots \\
\frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 & \cdots & \frac{8}{9}x_2 + \frac{1}{9}x_3
\end{tikzcd}$$

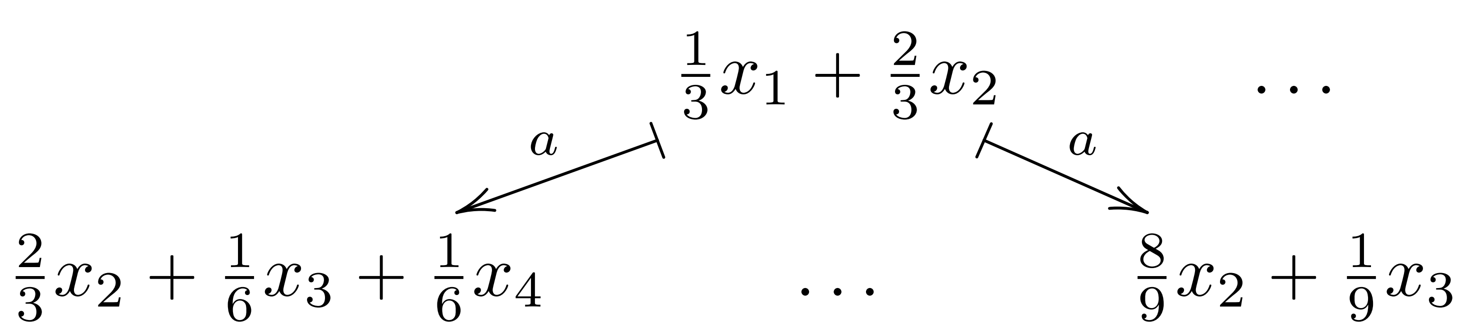
# Belief-state transformer

PA

$X \to (\mathcal{PD}X)^A$



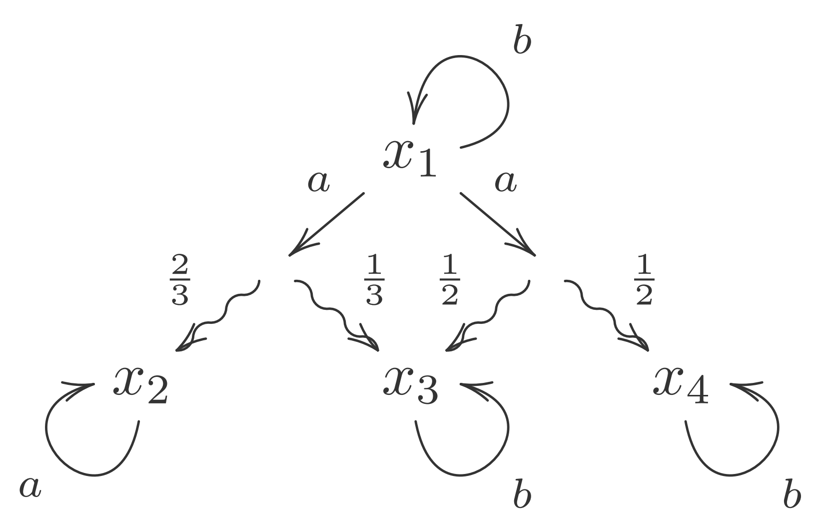

belief-state transformer



# Belief-state transformer

PA

$X \to (\mathcal{PD}X)^A$




belief-state transformer

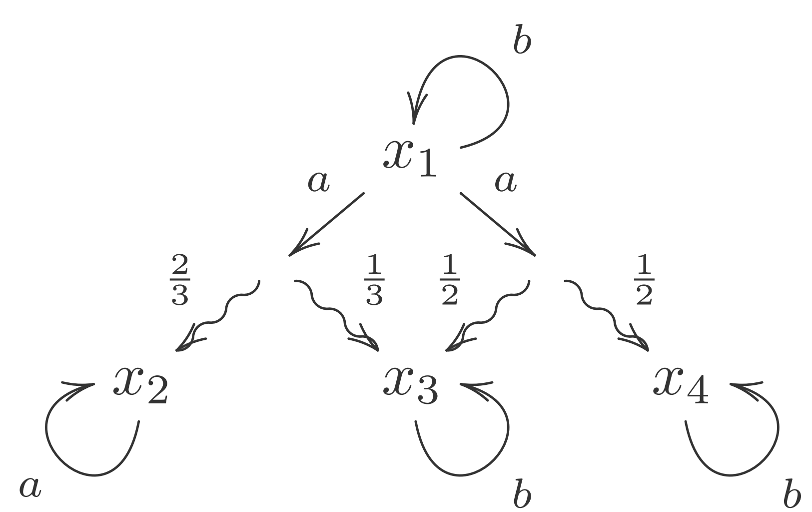
belief state

$$\begin{tikzcd}[column sep=small] & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & \ldots \\ \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 & \ldots & \frac{8}{9}x_2 + \frac{1}{9}x_3 \end{tikzcd}$$

# Belief-state transformer

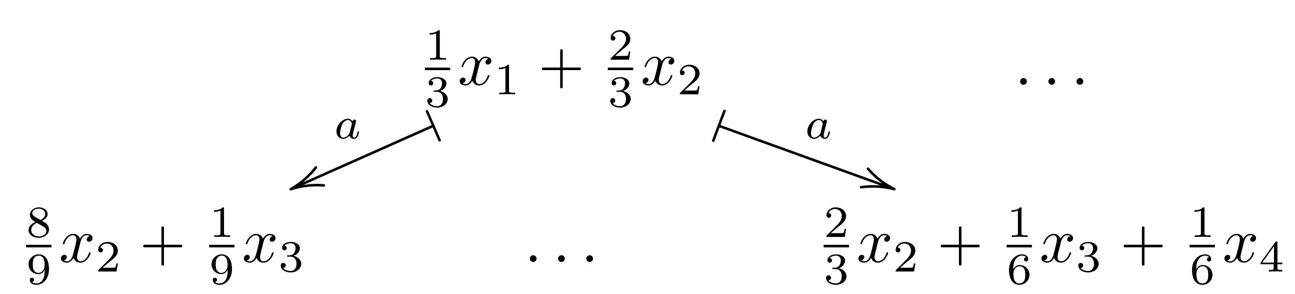
PA

$X \to (\mathcal{P}\mathcal{D}X)^A$

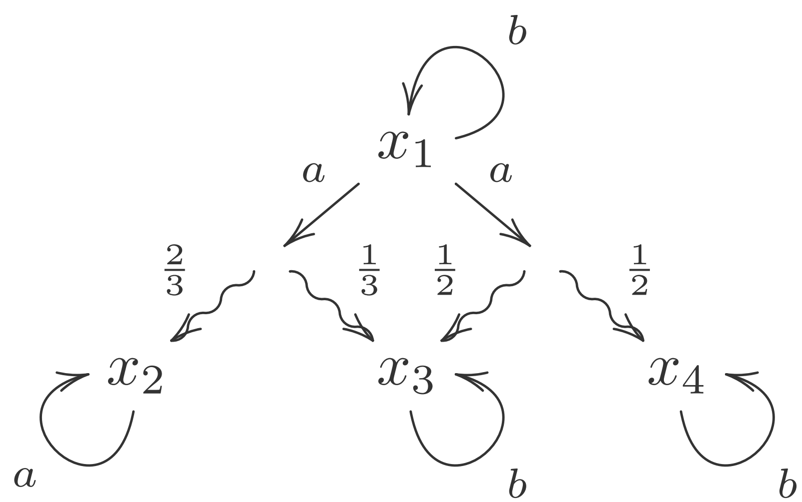



how does it emerge?

what is it?

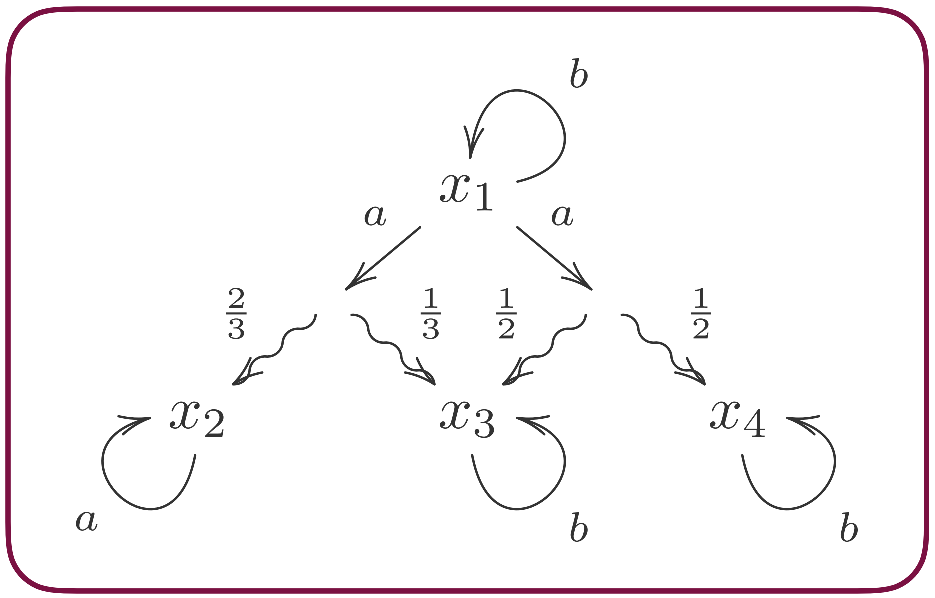


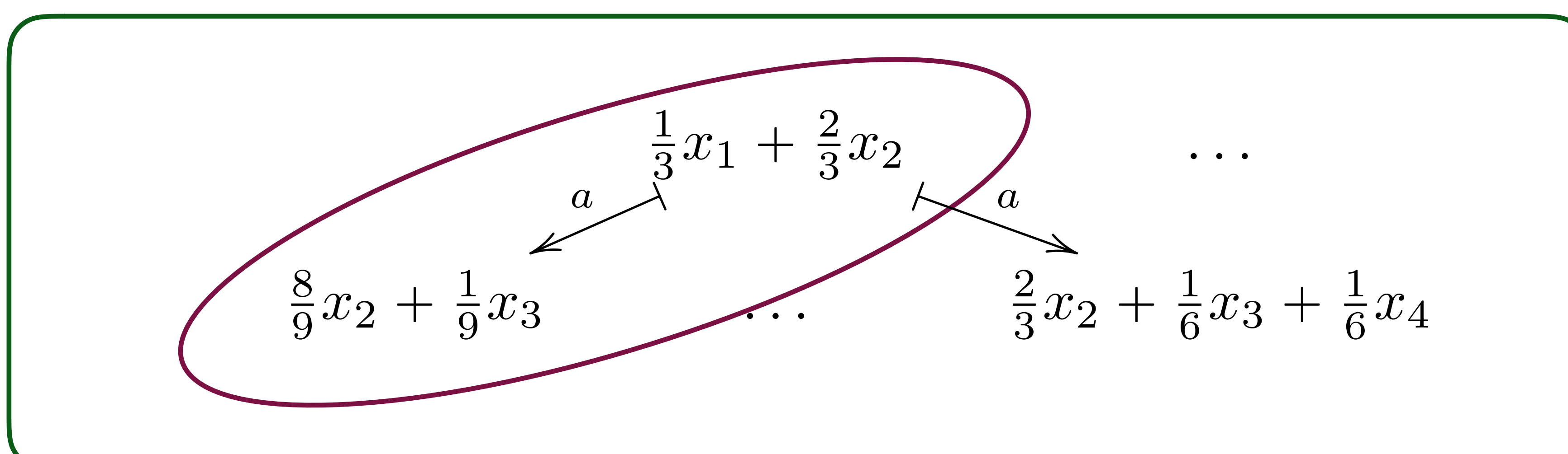
# Belief-state transformer



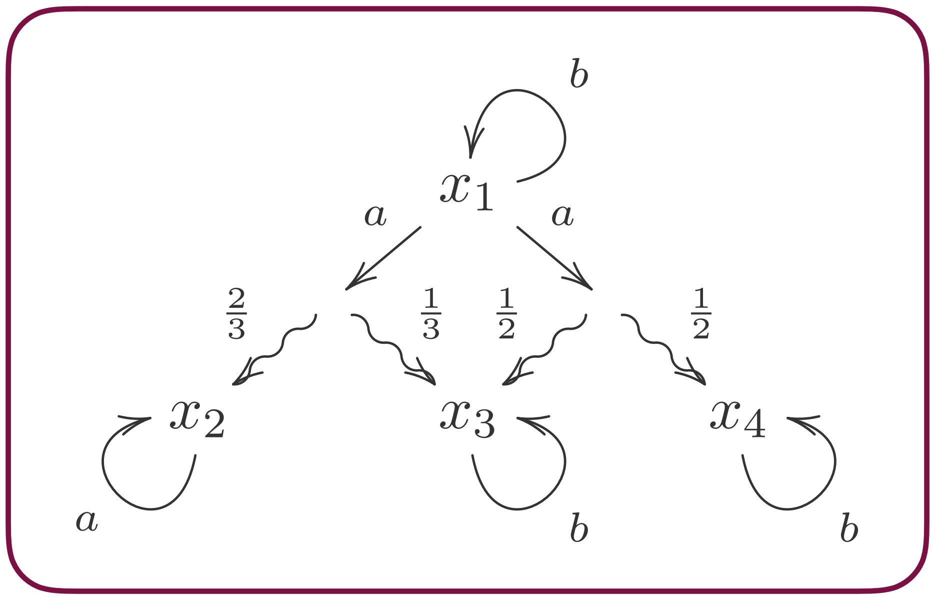

$$\begin{tikzcd} & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & & \cdots \\ \frac{8}{9}x_2 + \frac{1}{9}x_3 & \cdots & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 & \end{tikzcd}$$

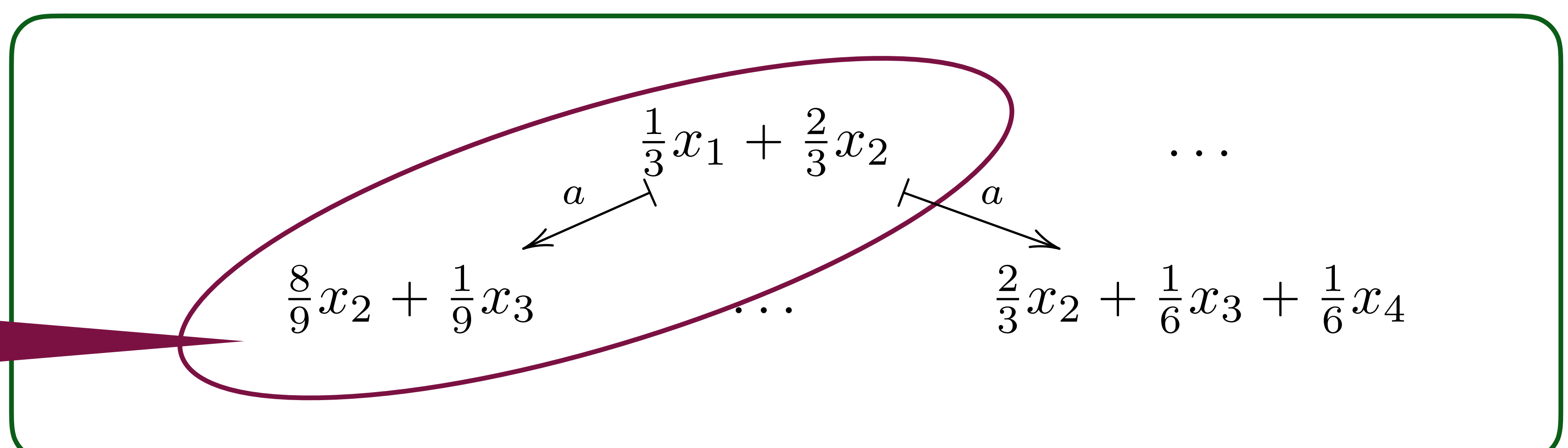
# Belief-state transformer



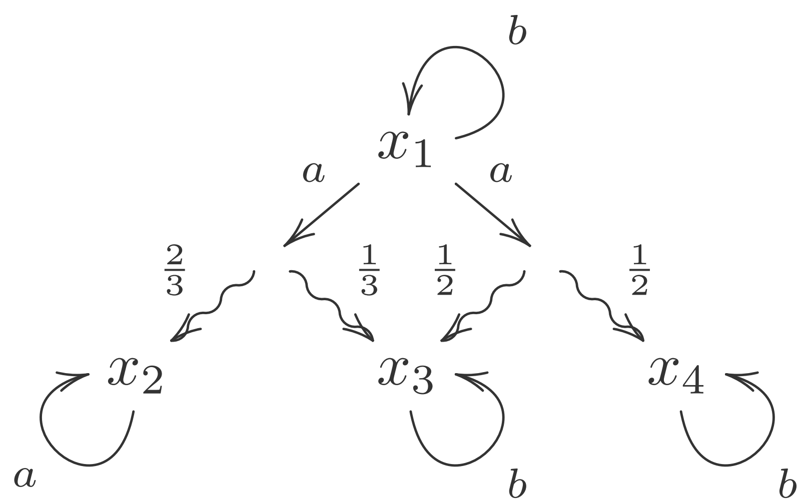
# Belief-state transformer



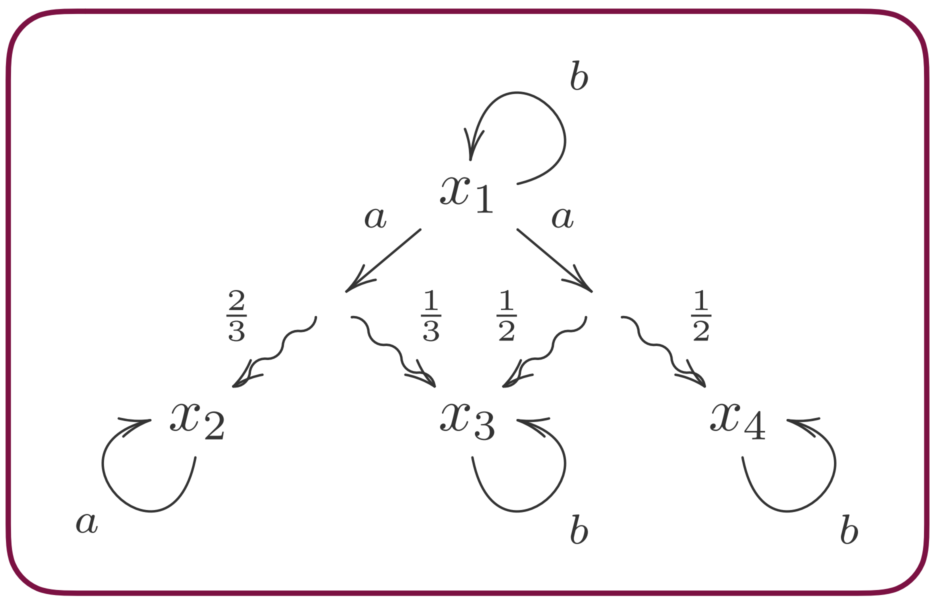

$$\frac{1}{3}\left(\frac{2}{3}x_2 + \frac{1}{3}x_3\right) + \frac{2}{3}(1x_2)$$

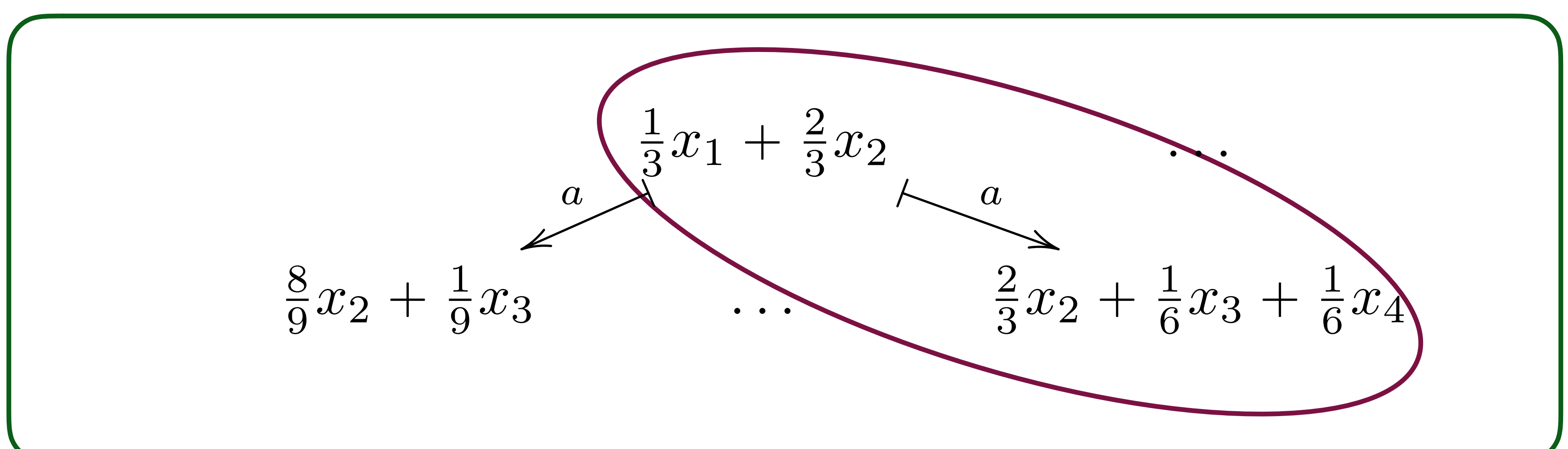
# Belief-state transformer



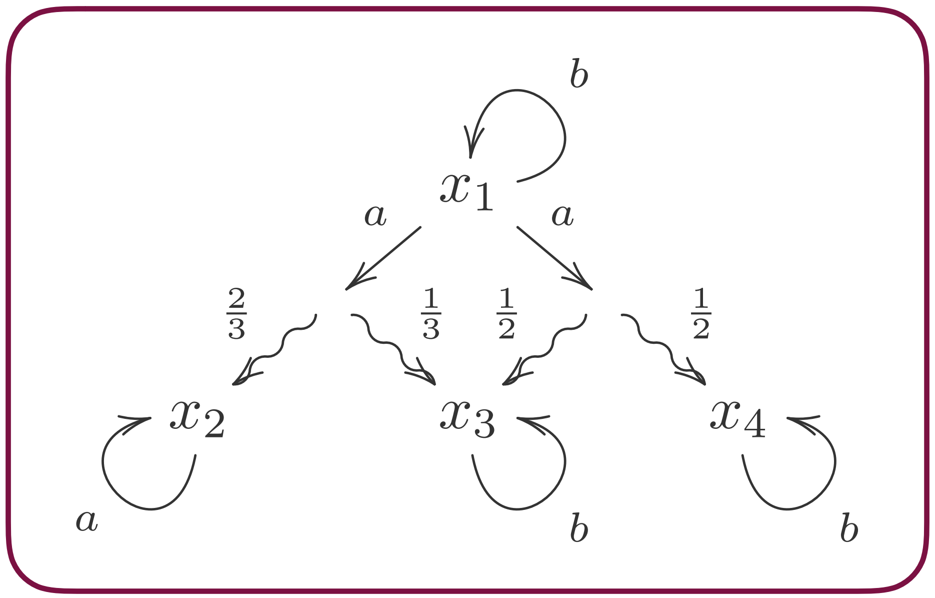

$$
\begin{tikzcd}
 & & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dll, mapsto, "a"'] \arrow[drr, mapsto, "a"] & & \ldots \\
\frac{8}{9}x_2 + \frac{1}{9}x_3 & & \ldots & & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4
\end{tikzcd}
$$

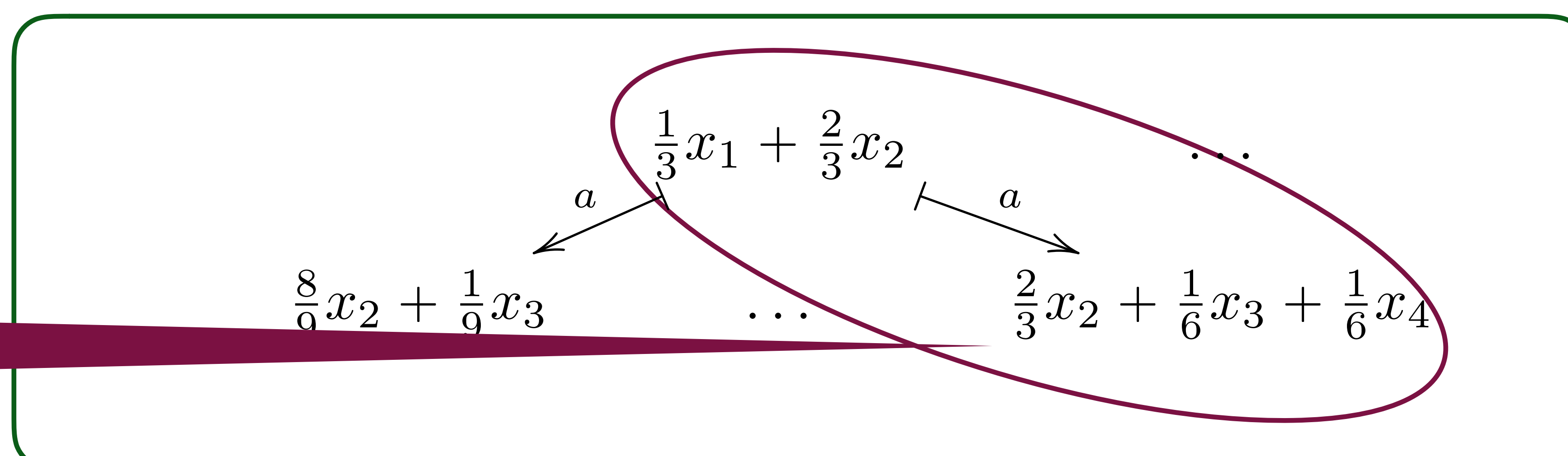
# Belief-state transformer



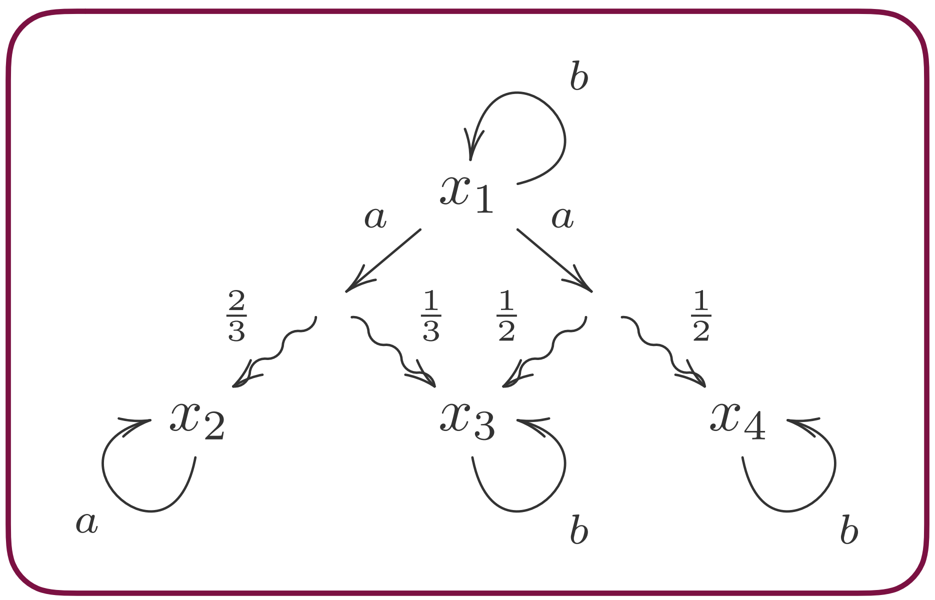
# Belief-state transformer

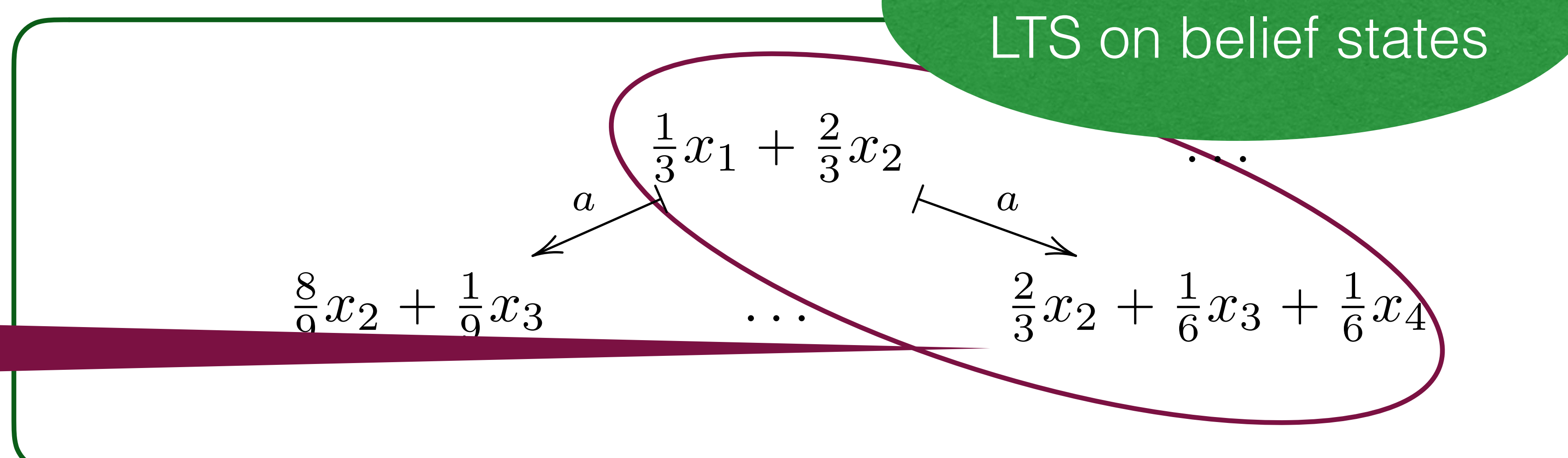



$$\frac{1}{3}\left(\frac{1}{2}x_3 + \frac{1}{2}x_4\right) + \frac{2}{3}(1x_2)$$

# Belief-state transformer



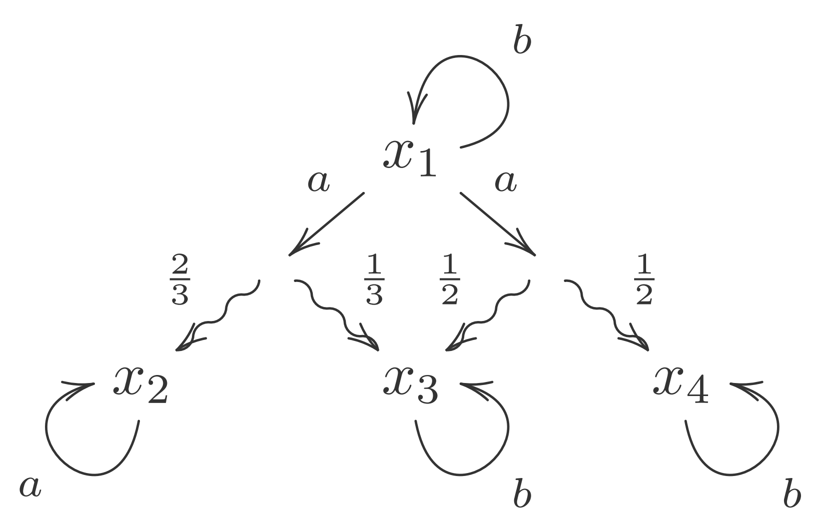

very infinite
LTS on belief states

$$\frac{1}{3}\left(\frac{1}{2}x_3 + \frac{1}{2}x_4\right) + \frac{2}{3}(1x_2)$$

# Belief-state transformer

PA

$X \to (\mathcal{P}\mathcal{D}X)^A$




how does it emerge?

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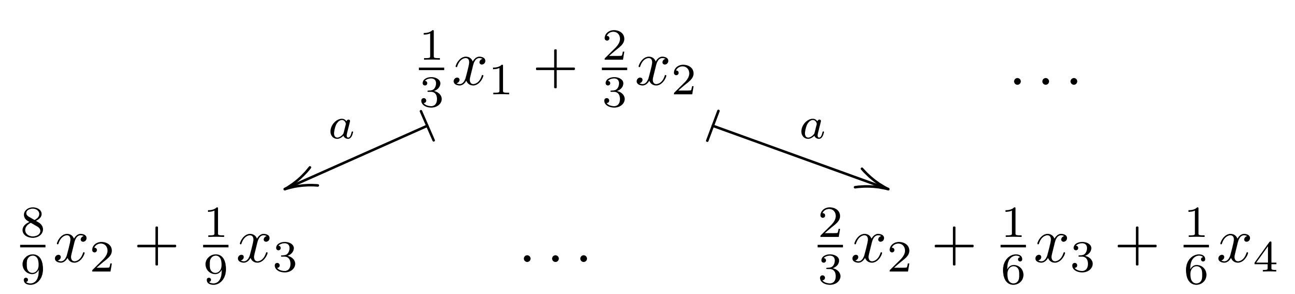
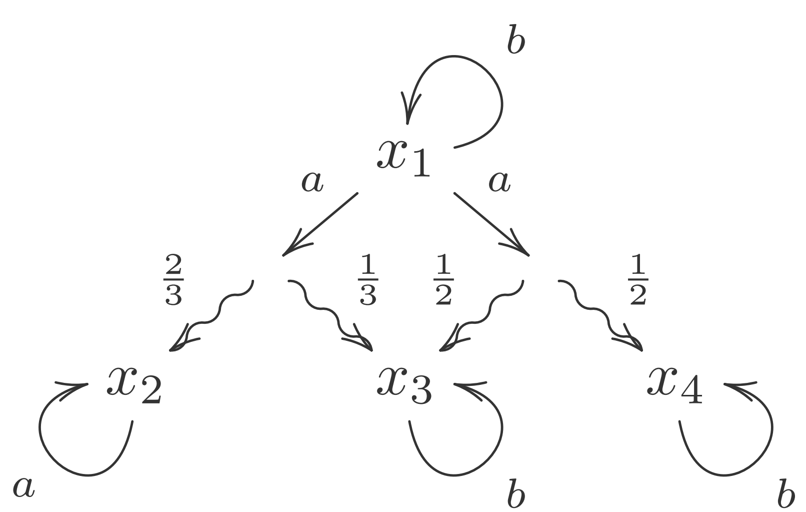


# Belief-state transformer

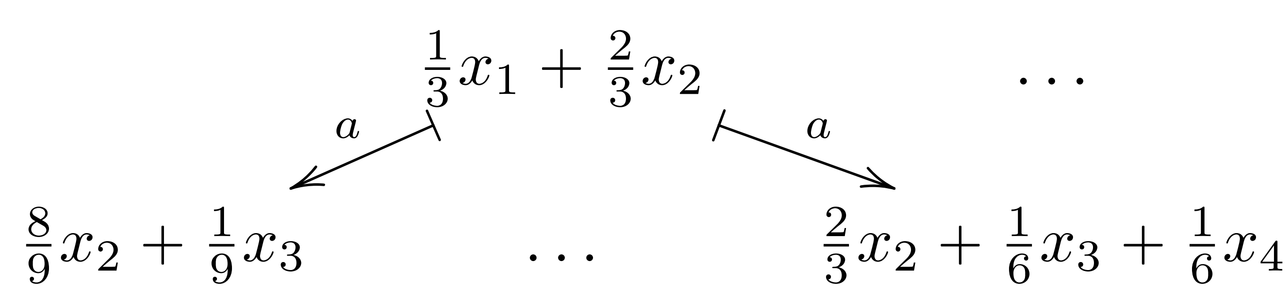
PA

$X \to (\mathcal{PD}X)^A$




how does it emerge?

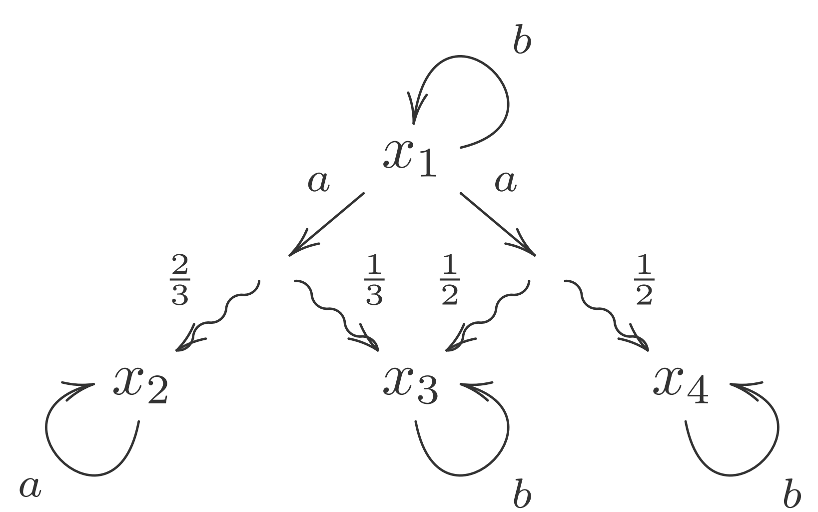
coalgebra over free convex algebra



# Belief-state transformer

## PA

$X \to (\mathcal{P}\mathcal{D}X)^A$




via a generalised determinisation

coalgebra over free convex algebra

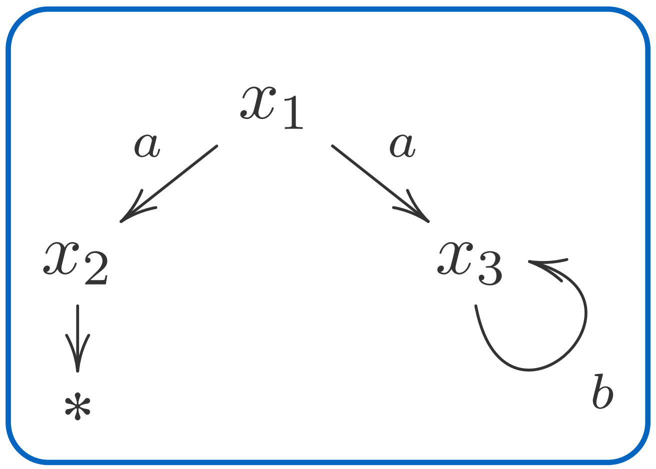
$$\begin{tikzcd} & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & \dots \\ \frac{8}{9}x_2 + \frac{1}{9}x_3 & \dots & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \end{tikzcd}$$

# Determinisations

# Determinisations

NFA

$X \rightarrow 2 \times (\mathcal{P}X)^A$



# Determinisations

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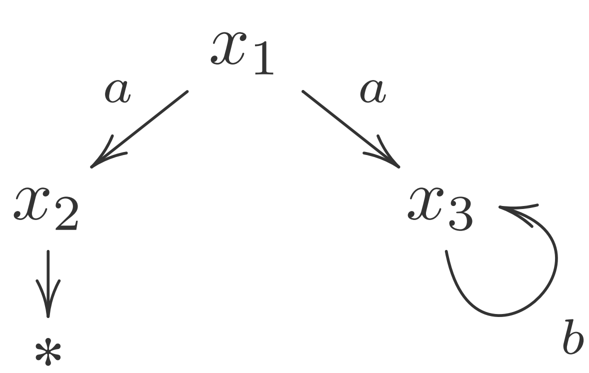
$X \to 2 \times (\mathcal{P}X)^A$



# Determinisations

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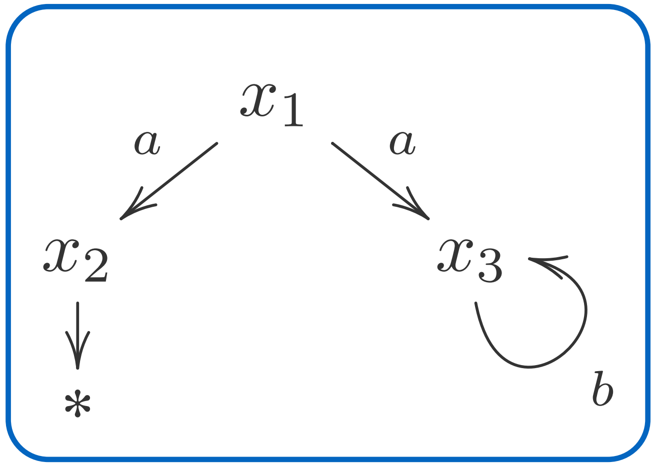


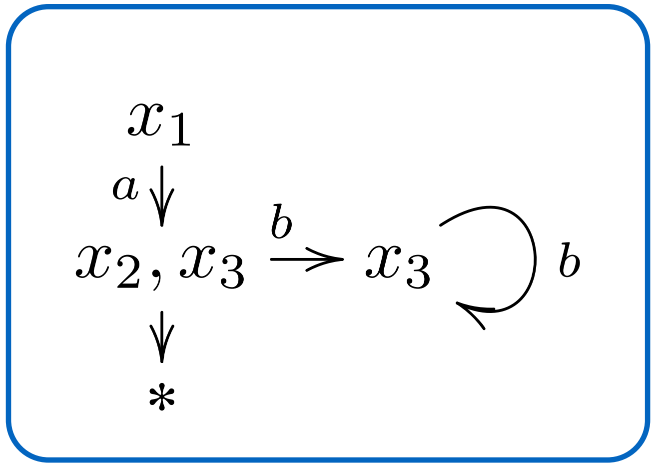

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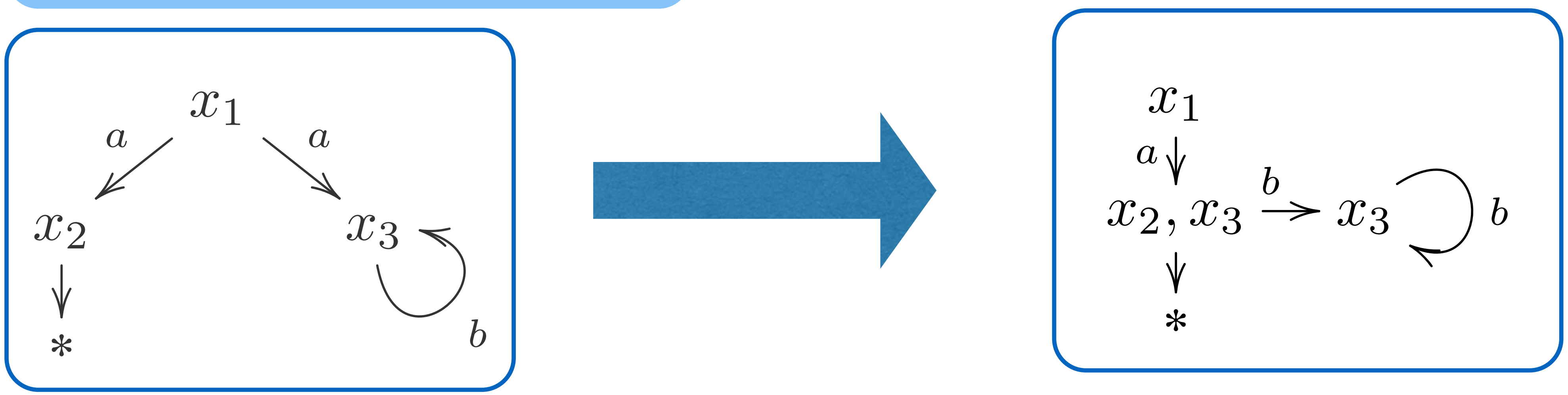



[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

# Determinisations

# Determinisations

Generative PTS

$X \to \mathcal{D}(1 + A \times X)$

$$
\begin{tikzcd}
 & x_1 \arrow[dl, "{a,\frac{1}{2}}"'] \arrow[dr, "{a,\frac{1}{2}}"] & \\
x_2 \arrow[d, "{b,1}"'] & & x_3 \arrow[d, "{c,1}"] \\
x_4 \arrow[d, "1"'] & & x_5 \arrow[d, "1"] \\
* & & *
\end{tikzcd}
$$

# Determinisations

Generative PTS

$$X \to \mathcal{D}(1 + A \times X)$$

$$\begin{tikzcd}
 & x_1 \arrow[dl, "{a, \frac{1}{2}}"'] \arrow[dr, "{a, \frac{1}{2}}"] & \\
x_2 \arrow[d, Rightarrow, "{b, 1}"'] & & x_3 \arrow[d, Rightarrow, "{c, 1}"] \\
x_4 \arrow[d, Rightarrow, "1"'] & & x_5 \arrow[d, Rightarrow, "1"] \\
* & & *
\end{tikzcd}$$

# Determinisations

## Generative PTS

$X \to \mathcal{D}(1 + A \times X)$

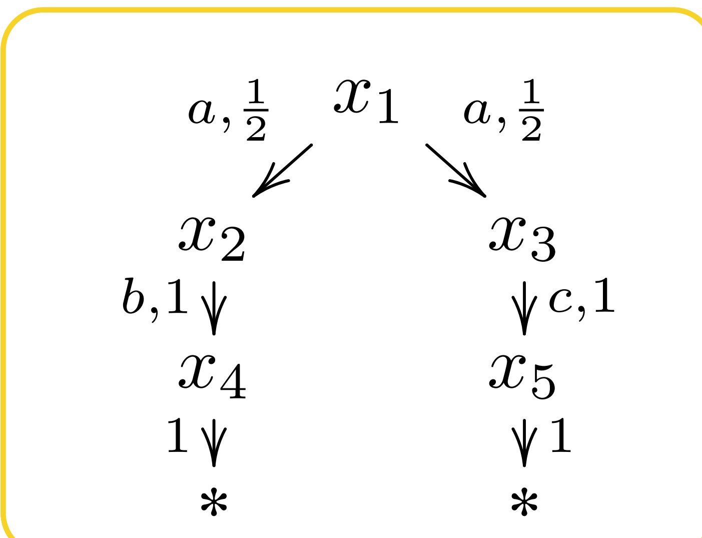
$$
\begin{tikzcd}
 & x_1 \arrow[dl, "{a,\frac{1}{2}}"'] \arrow[dr, "{a,\frac{1}{2}}"] & \\
x_2 \arrow[d, "{b,1}"'] & & x_3 \arrow[d, "{c,1}"] \\
x_4 \arrow[d, "1"'] & & x_5 \arrow[d, "1"] \\
* & & *
\end{tikzcd}
$$

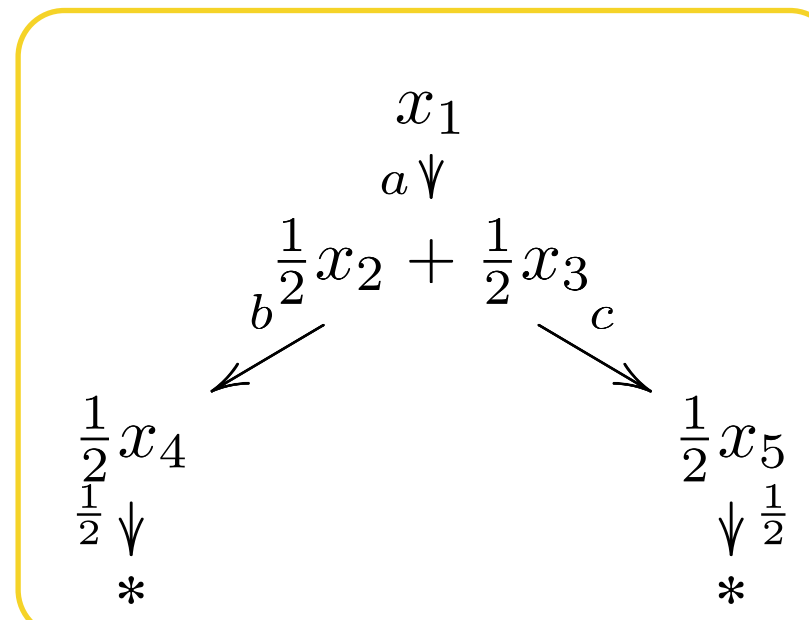
$$
\begin{tikzcd}
 & x_1 \arrow[d, "a"'] & \\
 & \frac{1}{2}x_2 + \frac{1}{2}x_3 \arrow[dl, "b"'] \arrow[dr, "c"] & \\
\frac{1}{2}x_4 \arrow[d, "\frac{1}{2}"'] & & \frac{1}{2}x_5 \arrow[d, "\frac{1}{2}"] \\
* & & *
\end{tikzcd}
$$

# Determinisations

## Generative PTS

$X \rightarrow \mathcal{D}(1 + A \times X)$




[Silva, S. MFPS'11]

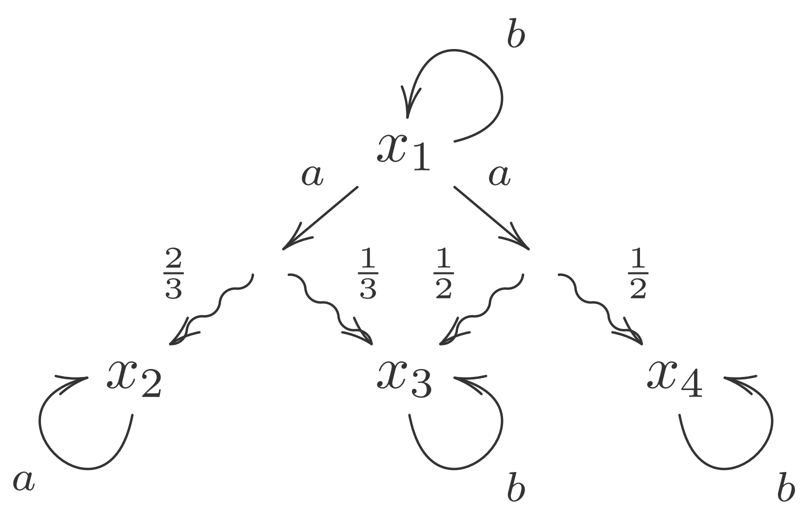
[Jacobs, Silva, S. JCSS'15]

# Determinisations

# Determinisations

PA

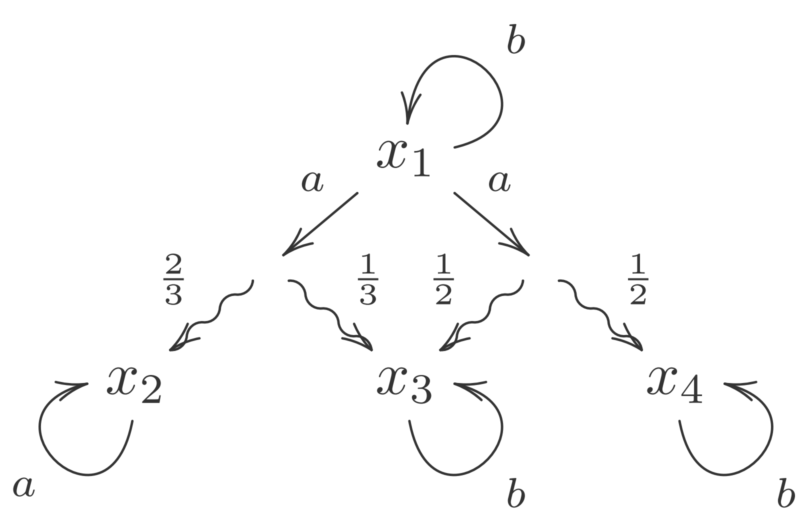
$X \to (\mathcal{P}\mathcal{D}X)^A$



# Determinisations

PA

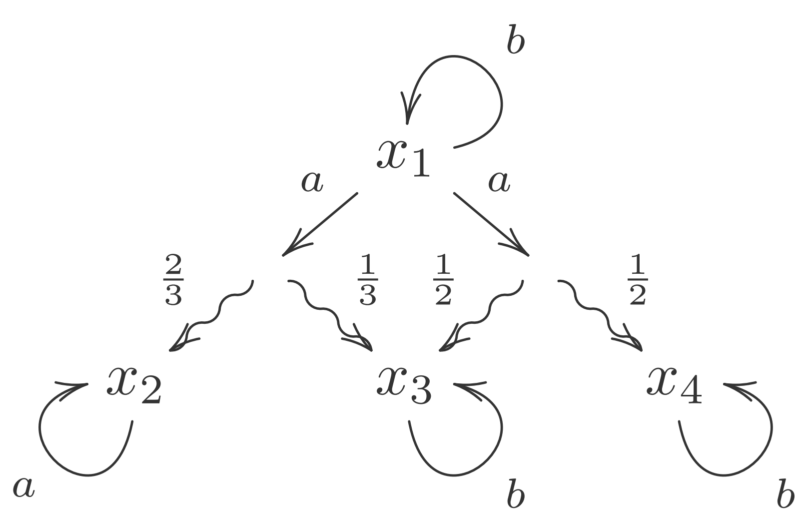
$X \to (\mathcal{PD}X)^A$



# Determinisations

## PA

$X \to (\mathcal{P}\mathcal{D}X)^A$



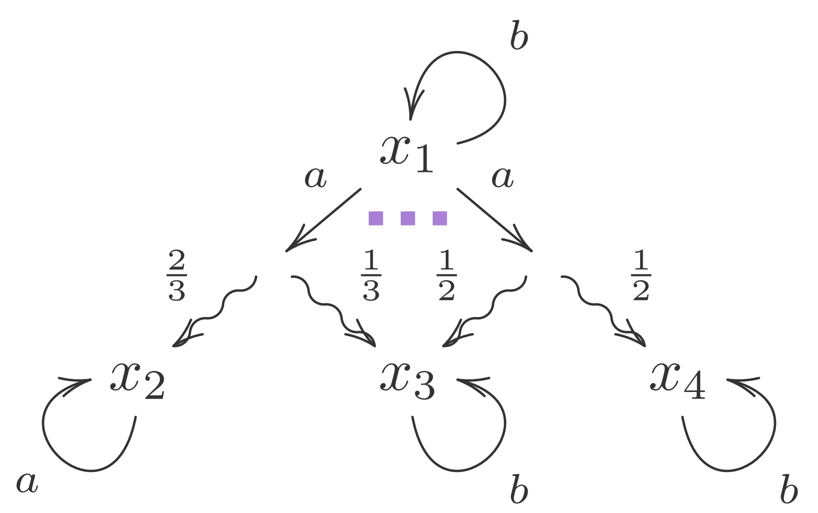

belief-state transformer

$$
\begin{tikzcd}
 & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & \cdots \\
\frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 & \cdots & \frac{8}{9}x_2 + \frac{1}{9}x_3
\end{tikzcd}
$$

# Determinisations

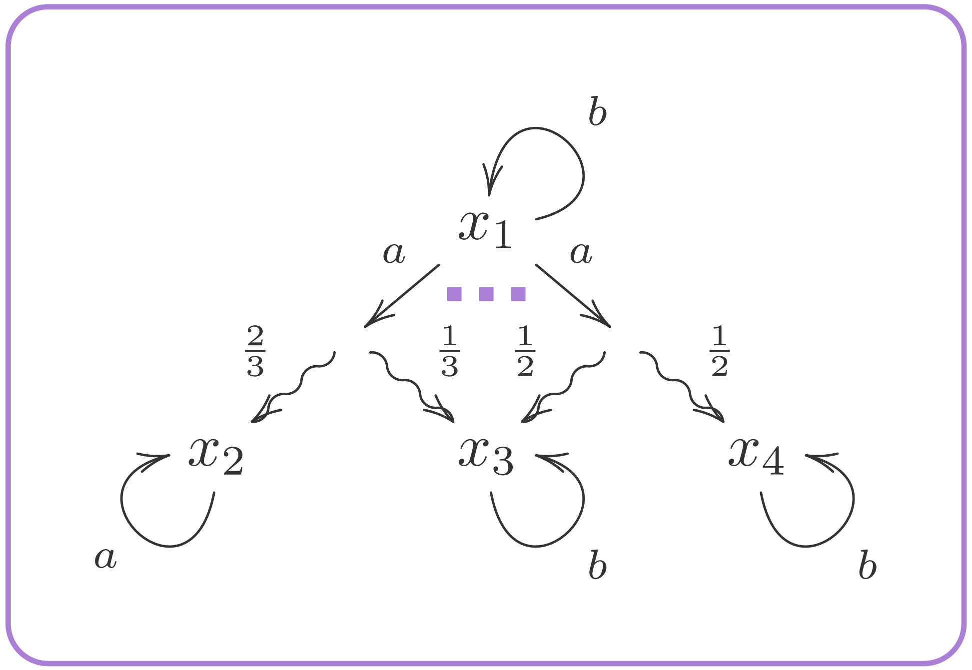
# Determinisations

$X \to (\mathscr{C}X)^A$



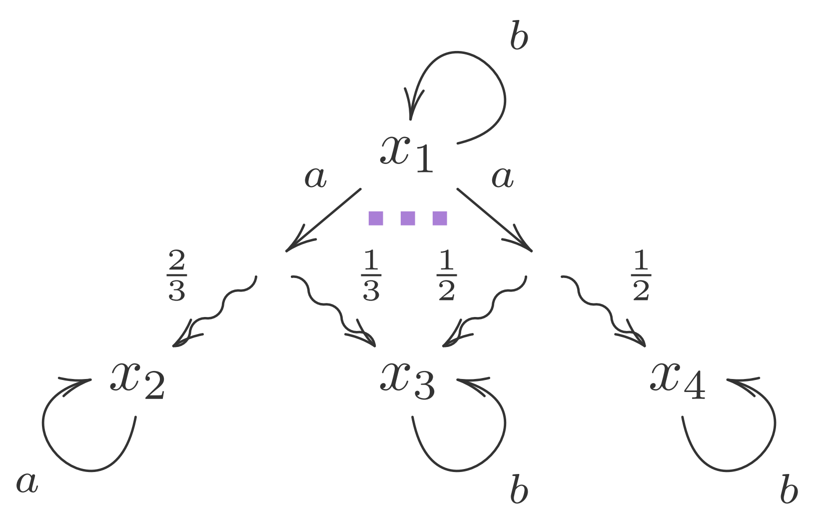
# Determinisations

$X \to (\mathcal{C}X)^A$




# Determinisations

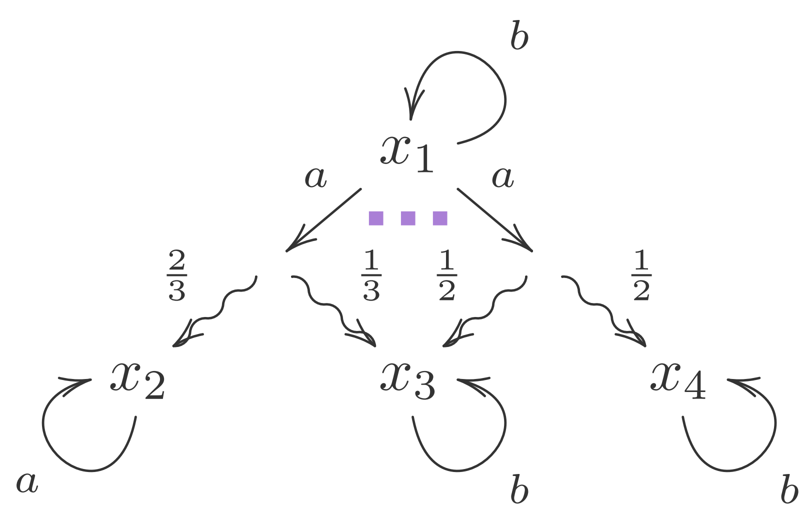
$X \to (\mathcal{C}X)^A$




$$x_1 \xrightarrow{a} \left(\tfrac{2}{3}x_2 + \tfrac{1}{3}x_3\right) \oplus \left(\tfrac{1}{2}x_3 + \tfrac{1}{2}x_4\right)$$

# Determinisations

$X \to (\mathcal{C}X)^A$

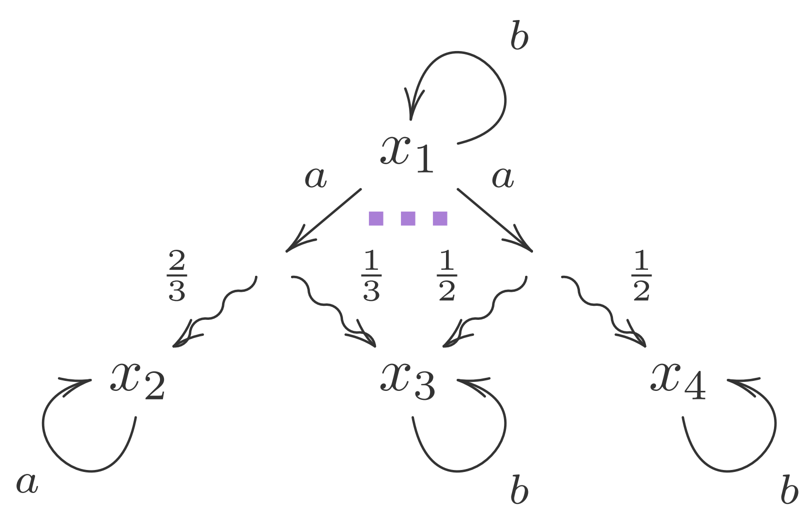


$$\begin{array}{c} x_1 \\ a \big\downarrow \\ \left(\frac{2}{3}x_2 + \frac{1}{3}x_3\right) \oplus \left(\frac{1}{2}x_3 + \frac{1}{2}x_4\right) \end{array}$$

LTS on a convex semilattice

# Determinisations

$X \to (\mathcal{C}X)^A$



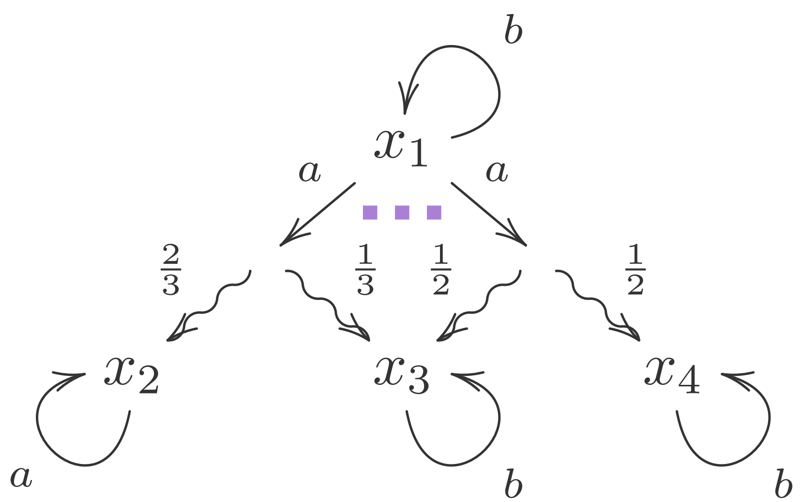

$$x_1 \xrightarrow{a} \left(\tfrac{2}{3}x_2 + \tfrac{1}{3}x_3\right) \oplus \left(\tfrac{1}{2}x_3 + \tfrac{1}{2}x_4\right)$$

Theory of traces for PA
Wednesday@LICS

LTS on a convex semilattice

# Determinisations

$X \to (\mathcal{C}X)^A$




$$x_1 \xrightarrow{a} \left(\tfrac{2}{3}x_2 + \tfrac{1}{3}x_3\right) \oplus \left(\tfrac{1}{2}x_3 + \tfrac{1}{2}x_4\right)$$

LTS on a convex semilattice

Theory of traces for PA
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Thank You !

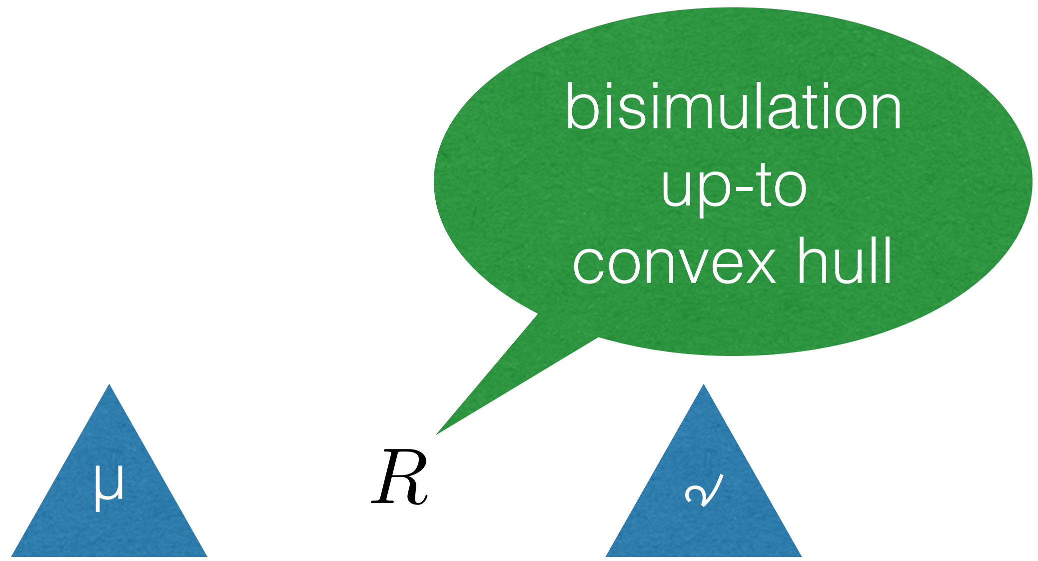
# Coinductive proof method for distribution bisimilarity

# Coinductive proof method for distribution bisimilarity

bisimulation up-to convex hull

$R$

# Coinductive proof method for distribution bisimilarity



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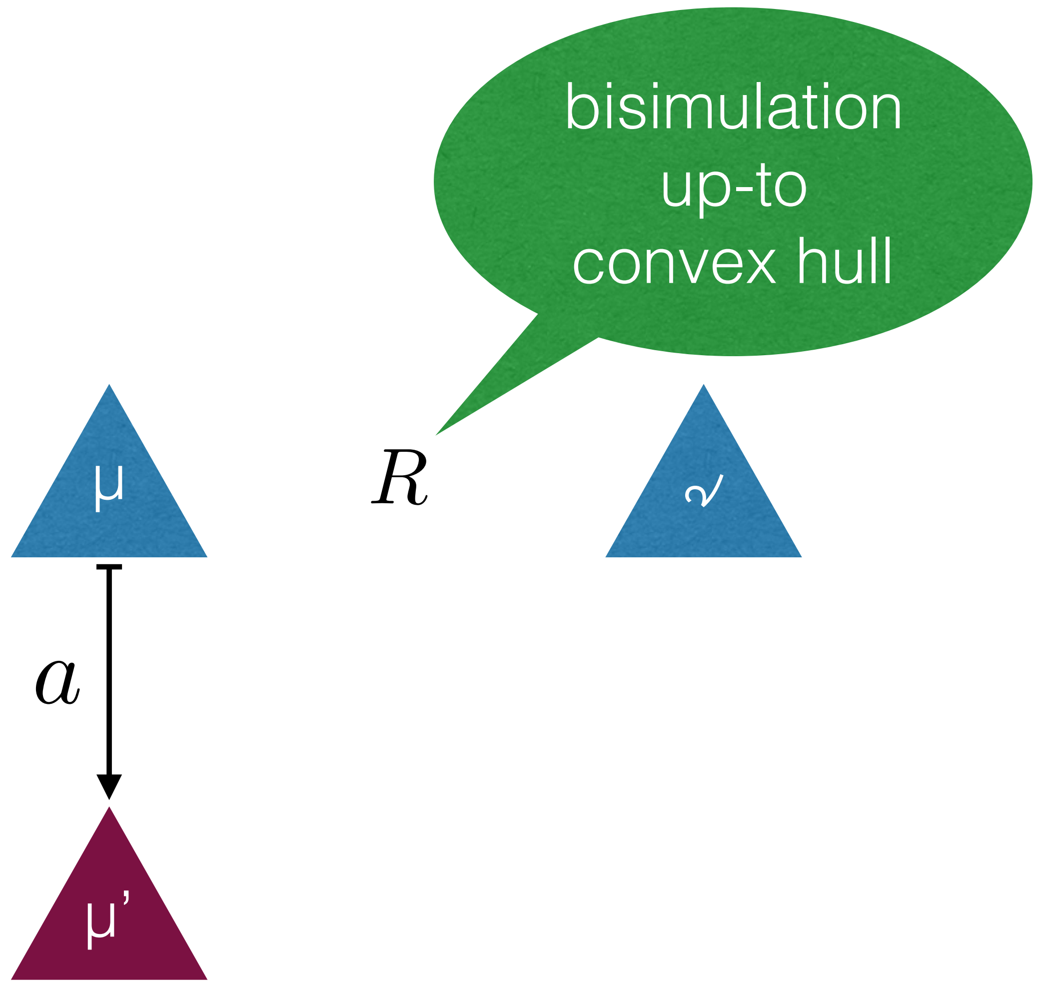

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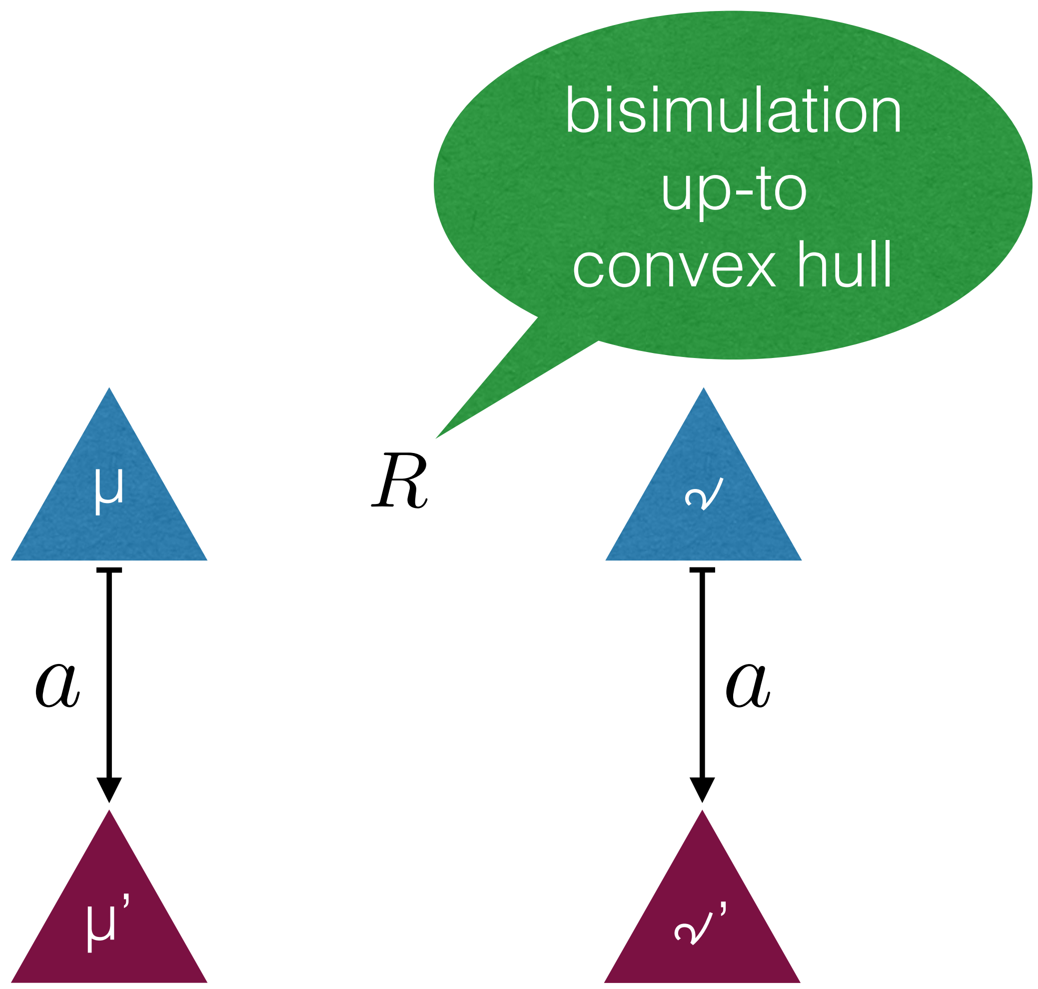

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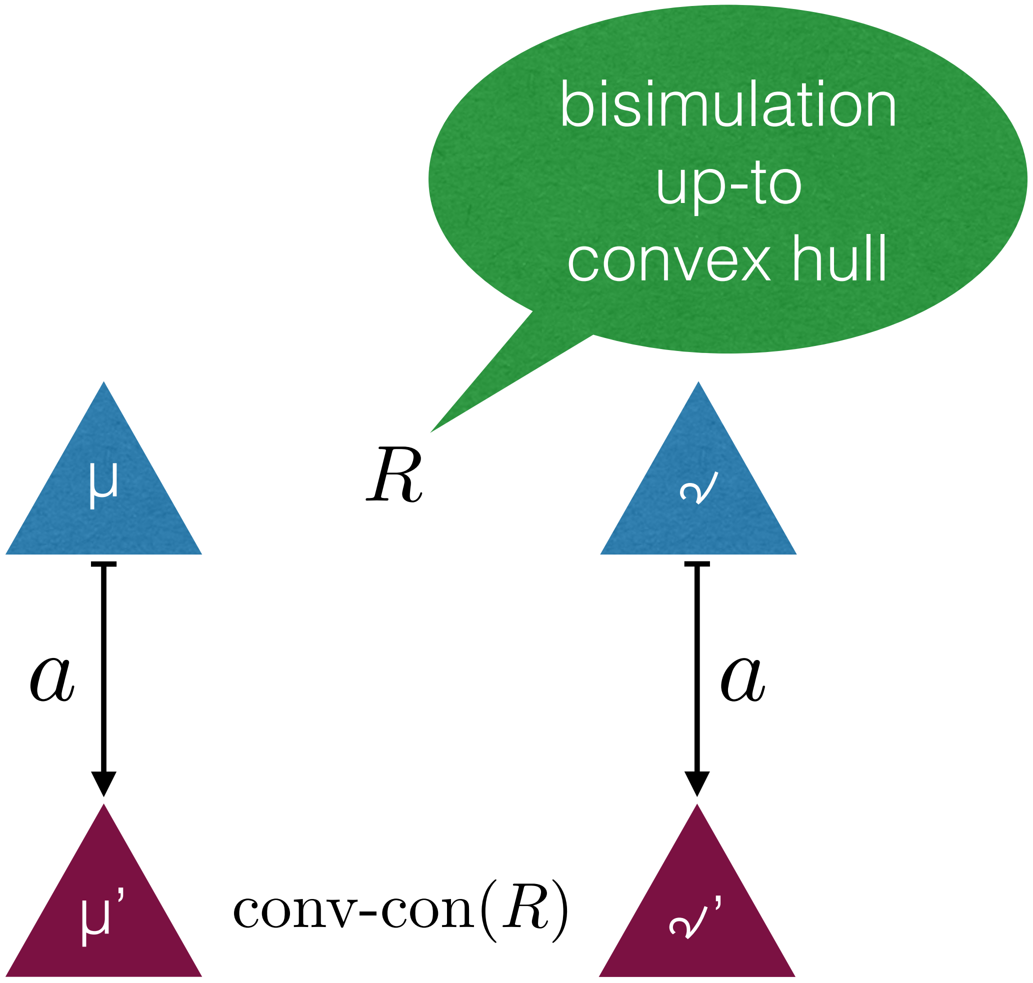

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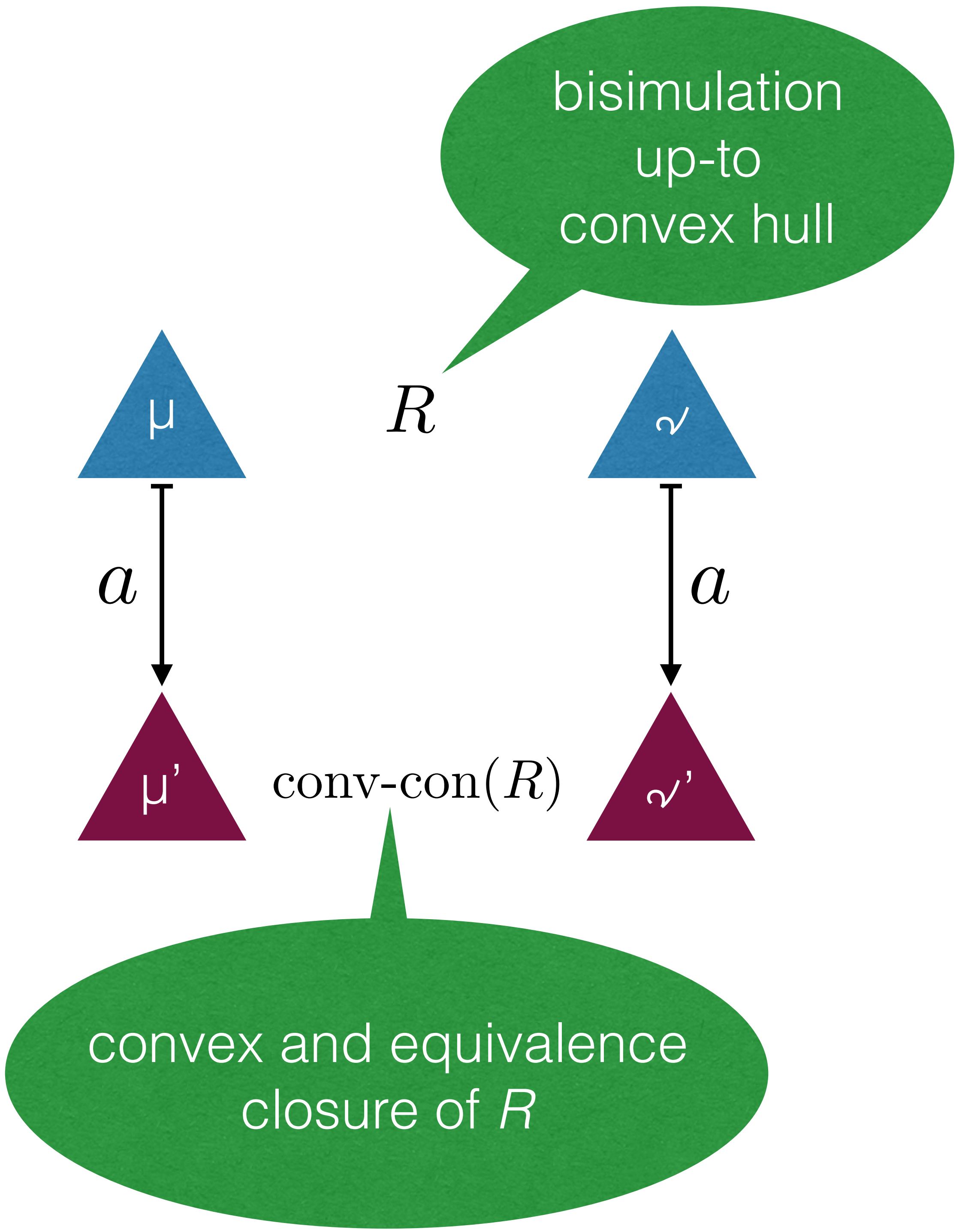
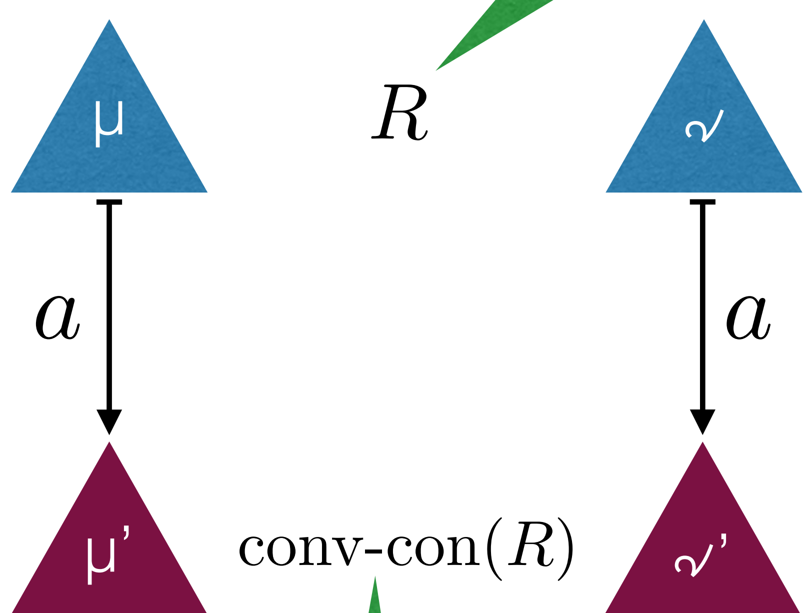


# Coinductive proof method for distribution bisimilarity

bisimulation up-to convex hull



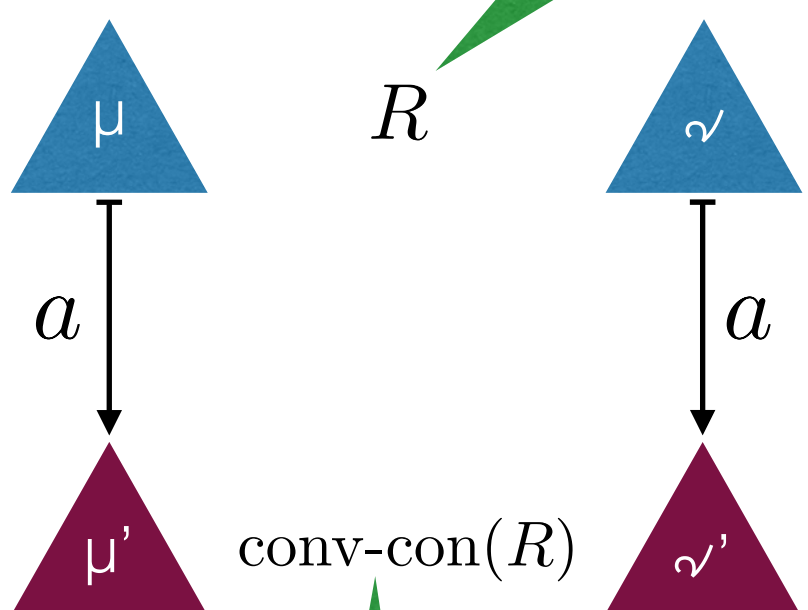

convex and equivalence closure of *R*

to prove μ $\sim_d$ ν

it suffices to find a bisimulation up-to convex hull *R* with μ *R* ν

# Coinductive proof method for distribution bisimilarity

bisimulation up-to convex hull




convex and equivalence closure of $R$

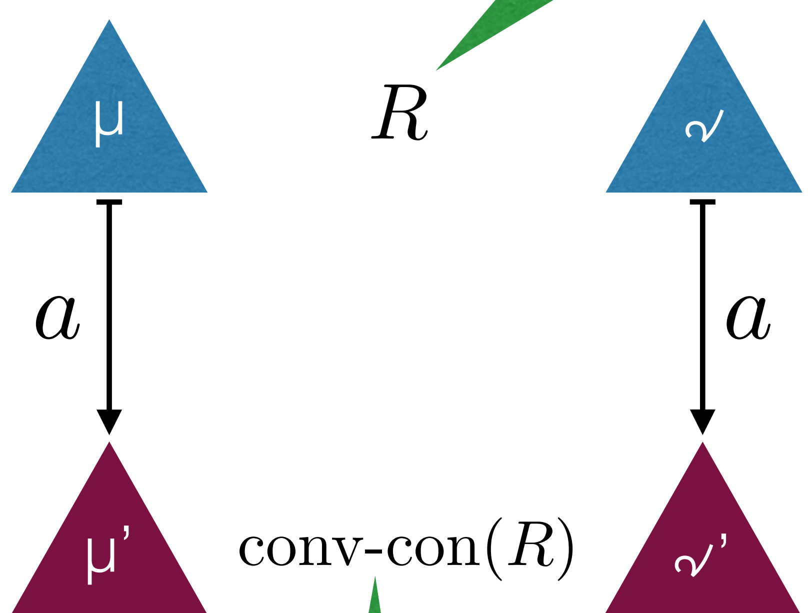
to prove $\mu \sim_d \nu$

it suffices to find a bisimulation up-to convex hull $R$ with $\mu\ R\ \nu$

there always exists a finite one!

# Coinductive proof method for distribution bisimilarity

bisimulation up-to convex hull

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to prove $\mu \sim_d \nu$ it suffices to find a bisimulation up-to convex hull $R$ with $\mu \; R \; \nu$

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[S., Woracek JPAA’15]