Cardinality

Def.

Two sets A and B have the same cardinality (are equinumerous) if there is a bijection $f:A \rightarrow B$. Notation A ~ B, or |A| = |B|.

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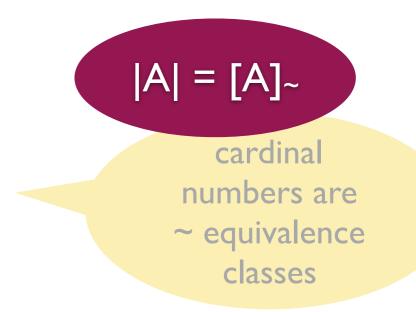
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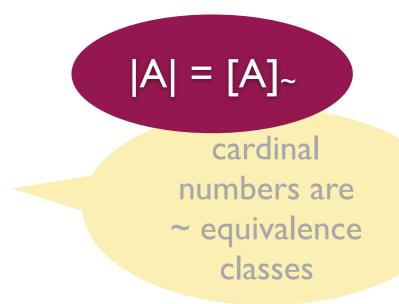
cardinal
numbers are
~ equivalence
classes

Theorem (Cantor)

If $|A| \le |B|$ and $|B| \le |A|$, then |A| = |B|.



Let A and B be two disjoint sets. Then $|A| + |B| = |A \cup B|$.



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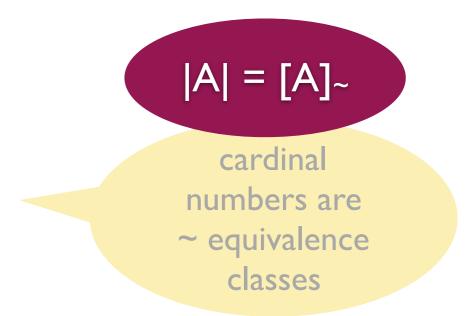
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Note: $2 = |\{0,1\}|$



We write \mathbb{N}_k for the set $\{0,1,...,k-1\}$. Then $\mathbb{N}_0 = \emptyset$.

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The operations on cardinals when restricted to finite cardinals coincide with the operations on natural numbers!

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E.g. If |A| = k and |B| = mfor some k,m $\in \mathbb{N}$ then $|AxB| = k \cdot m$

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Time for a video!

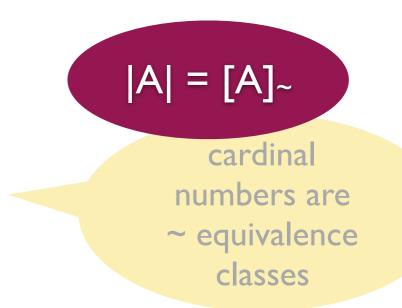
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Hilbert's infinite hotel :-)

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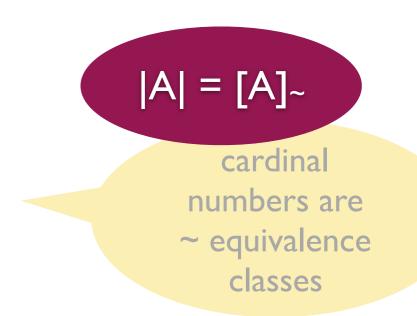
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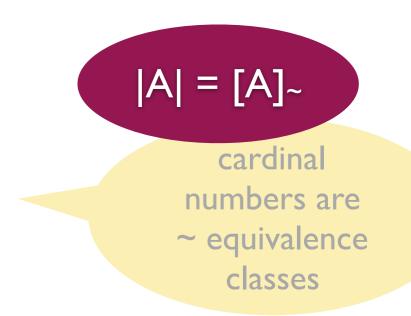
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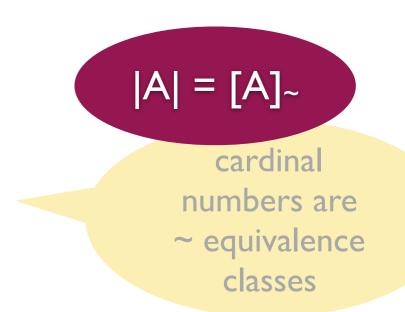
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Prop. \mathbb{R} is uncountable.

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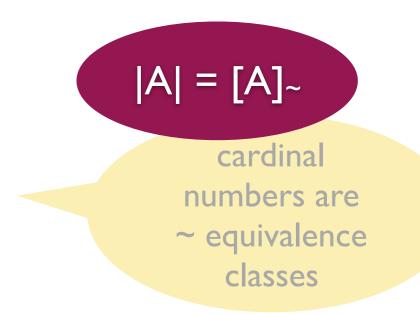
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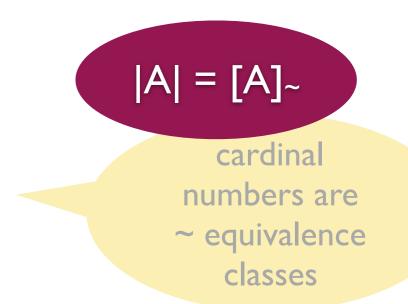
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Hence, for every cardinal there is a larger one.

