Strengthening
and
weakening

We had

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{val}{=} Q$, iff

(1) Always when P has truth value 1,
also Q has truth value 1, and

(2) Always when Q has truth value 1,
also P has truth value 1.

if we relax this, we get strengthening

Strengthening

Definition: The abstract proposition P is stronger than Q, notation $P \overset{val}{\models} Q$, iff

~~(1)~~ Always when P has truth value 1, also Q has truth value 1~~, and~~

~~(2) Always when Q has truth value 1, also P has truth value 1.~~

Q is weaker than P

Strengthening

Definition: The abstract proposition P is stronger than Q,
notation $P \models^{val} Q$, iff
always when P has truth value 1,
also Q has truth value 1.

always when P is true,
Q is also true

Q is weaker
than P

Properties

Lemma E1: $P \overset{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Lemma EW1: $P \overset{val}{=} Q$ iff $P \overset{val}{\models} Q$ and $Q \overset{val}{\models} P$.

Lemma W2: $P \overset{val}{\models} P$

Lemma W3: If $P \overset{val}{\models} Q$ and $Q \overset{val}{\models} R$ then $P \overset{val}{\models} R$

Lemma W4: $P \overset{val}{\models} Q$ iff $P \Rightarrow Q$ is a tautology.

Standard Weakenings

and-or-weakening

$$P \wedge Q \overset{val}{\models} P$$

$$P \overset{val}{\models} P \vee Q$$

Extremes

$$\mathrm{F} \overset{val}{\models} P$$

$$P \overset{val}{\models} \mathrm{T}$$

Calculating with weakenings
(the use of standard weakenings)

Substitution

just holds

Simple

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P] \overset{val}{\models} \psi[\xi/P]}$$

Sequential

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{\models} \psi[\xi/P][\eta/Q]}$$

EVERY occurrence of P is substituted!

Simultaneous

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{\models} \psi[\xi/P, \eta/Q]}$$

The rule of Leibniz

does not hold for weakening!

Leibniz

$$\frac{\phi \overset{val}{\models} \psi}{C[\phi] \overset{val}{\models} C[\psi]}$$

formula that has ϕ as a sub formula

The rule of Leibniz

does not hold for weakening!

Leibniz

$$\frac{\phi \overset{val}{\models} \psi}{C[\phi] \overset{val}{\models} C[\psi]}$$

Monotonicity

$$\frac{P \overset{val}{\models} Q}{P \wedge R \overset{val}{\models} Q \wedge R}$$

$$\frac{P \overset{val}{\models} Q}{P \vee R \overset{val}{\models} Q \vee R}$$

Predicate logic

Limitations of propositional logic

Propositional logic only allows us to reason about completed statements about things, not about the things themselves.

Example

Some chicken cannot fly
All chicken are birds

Some birds cannot fly

this reasoning can not be expressed in propositional logic

Example

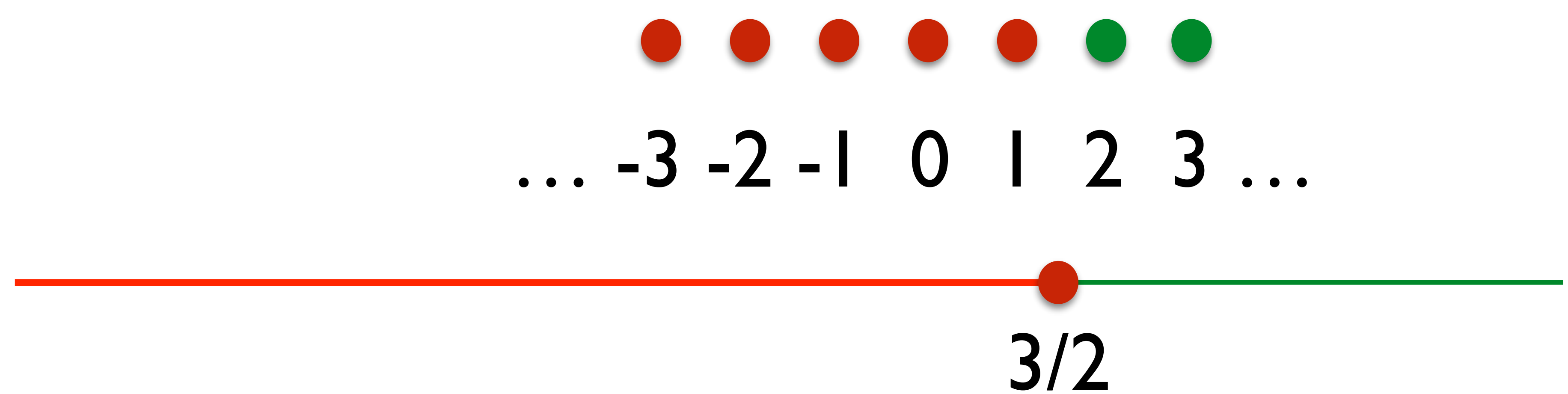
Every player except the winner looses a match

Unary predicate (example)

Consider the statement $2m>3$.

a unary relation

Whether this statement is true or false depends on the value of m (and on the domain of values).



on $\mathbb{Z}$

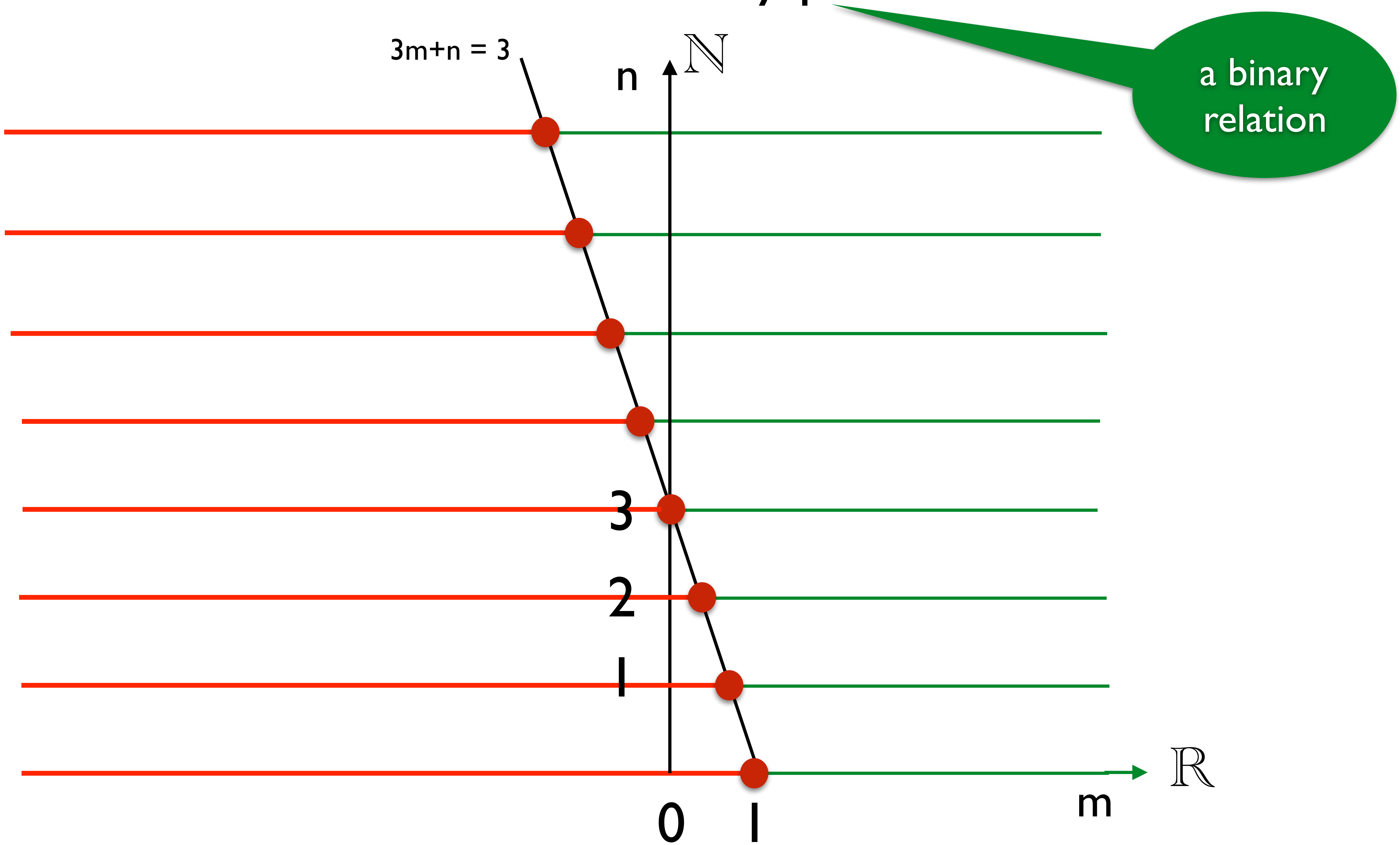
on $\mathbb{R}$

Note: $2m > 3 \overset{\text{val}}{=} m > 3/2$ on $\mathbb{Z}$ and $\mathbb{R}$

$2m > 3 \overset{\text{val}}{=} m \geq 2$ on $\mathbb{Z}$ but not on $\mathbb{R}$

Binary predicate (example)

The statement $3m+n > 3$ is a binary predicate on $\mathbb{R}$ x $\mathbb{N}$.



Predicates

In general, an n-ary predicate is an n-ary relation.

If it is on a domain D, then it's a relation $P(x_1, .., x_n) \subseteq D^n$ or equivalently a function $P: D^n \rightarrow \{0,1\}$.

true for certain values of the variables

$2m>3$

We can turn a predicate, into a proposition in three ways:

1. By assigning values to the variables.
2. By universal quantification.
3. By existential quantification.

for $m=2$

$2 \cdot 2 > 3$

is a true proposition

Universal quantification

The unary predicate $2m > 3$ on $\mathbb{Z}$ can be turned into a proposition by universal quantification:

false, e.g. for m = 1

For all m in $\mathbb{Z}$, $2m > 3$

universal quantifier

standard (!) notation:

$\forall x\ (P(x) \Rightarrow Q(x))$

$\forall x.\ P(x) \Rightarrow Q(x)$

Notation: $\forall_m\ [\underbrace{m \in \mathbb{Z}}_{\text{domain (predicate)}} : \underbrace{2m > 3}_{\text{predicate}}]$

In general: $\forall_x\ [P(x) : Q(x)]$ for "all x satisfying P satisfy Q"

Existential quantification

The unary predicate $2m > 3$ on $\mathbb{Z}$ can also be turned into a proposition by existential quantification:

true, e.g. m =2

There exists m in $\mathbb{Z}$, $2m > 3$

existential quantifier

Notation: $\exists_m\ [\underbrace{m \in \mathbb{Z}}_{\text{domain (predicate)}} : \underbrace{2m > 3}_{\text{predicate}}]$

standard (!) notation:

$\exists x\ (P(x) \wedge Q(x))$

$\exists x.\ P(x) \wedge Q(x)$

In general: $\exists_x\ [P(x) : Q(x)]$ for
"there exists x satisfying P that satisfies Q"

Quantification

The binary predicate $3m+n > 3$ on $\mathbb{R} \times \mathbb{N}$ can also be turned into a proposition by quantification:

in 8 possible ways

One way is: $\exists_m [m \in \mathbb{R} : \forall_n [n \in \mathbb{N} : 3m + n > 3]]$

unary predicate

binary predicate

proposition, nullary predicate

standard (!) notation:

$\exists m \, (m \in \mathbb{R} \wedge \forall n \, (n \in \mathbb{N} \Rightarrow 3m+n>3))$

Notation

also for $\exists$

We write $\forall_x$ [P] for $\forall_x$ [T : P]

We also write $\exists_{m,} \forall_n$ [$(m,n) \in \mathbb{R} \times \mathbb{N} : 3m + n > 3$]

for $\exists_m$ [$m \in \mathbb{R} : \forall_n$ [$n \in \mathbb{N} : 3m + n > 3$]]

And even $\exists_{m,n}$ [$(m,n) \in \mathbb{R} \times \mathbb{N} : 3m + n > 3$]

for $\exists_m$ [$m \in \mathbb{R} : \exists_n$ [$n \in \mathbb{N} : 3m + n > 3$]]

but only for the same quantifier!

Quantification - task

Thanks to Bas Luttik

Let P be the set of all tennis players.
Let $w \in P$ be the winner.

For $p, q \in P$, write $p \neq q$ for "p and q are different players" .

Let M be the set of all matches.
For $p \in P$ and $m \in M$, write $L(p,m)$ for
"player p loses match m".

Write the following sentence as a formula with predicates and quantifiers:

Every player except the winner loses a match.

Equivalences with quantifiers

Renaming bound variables

Bound variables

$$\forall_x[P{:}Q] \stackrel{val}{=} \forall_y[P[y/x]{:}Q[y/x]]$$

$$\exists_x[P{:}Q] \stackrel{val}{=} \exists_y[P[y/x]{:}Q[y/x]]$$

if y does not occur in
P or Q (not even in ∀y, ∃y)

Domain splitting

Examples:

$$\forall_x[x \leqslant 1 \vee x \geqslant 5 \colon x^2 - 6x + 5 \geqslant 0]$$
$$\overset{val}{=} \forall_x[x \leqslant 1 \colon x^2 - 6x + 5 \geqslant 0] \wedge \forall_x[x \geqslant 5 \colon x^2 - 6x + 5 \geqslant 0]$$

$$\exists_k[0 \leqslant k \leqslant n : k^2 \leqslant 10]$$
$$\overset{val}{=} \exists_k[0 \leqslant k \leqslant n-1 \vee k = n : k^2 \leqslant 10]$$
$$\overset{val}{=} \exists_k[0 \leqslant k \leqslant n-1 : k^2 \leqslant 10] \vee \exists_k[k = n : k^2 \leqslant 10]$$

Domain splitting

$$\forall_x[P \vee Q : R] \overset{val}{=} \forall_x[P : R] \wedge \forall_x[Q : R]$$
$$\exists_x[P \vee Q : R] \overset{val}{=} \exists_x[P : R] \vee \exists_x[Q : R]$$