The structure of natural numbers

is helpful for proving properties

$\forall n[n \in \mathbb{N}: P(n)]$

The structure of natural numbers

On natural numbers we can define a notion of a successor, a mapping

s: $\mathbb{N} \to \mathbb{N}$

by $s(n) = n+1$

The successor mapping imposes a structure on the set that enables us to count:

1) there is a starting natural number 0
2) for every natural number n, there is a next natural number $s(n) = n+1$.

Cardinality

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Theorem (Cantor)

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Operations on cardinals

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Def. Let A and B be two disjoint sets. Then $|A| + |B| = |A \cup B|$.

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Note: $2 = |\{0,1\}|$

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Finite sets, finite cardinals

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We write $\mathbb{N}_k$ for the set $\{0,1,\ldots,k\text{-}1\}$. Then $\mathbb{N}_0 = \varnothing$.

We will also write k for $|\mathbb{N}_k|$.

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Def. A set A is finite if and only if $|A| = |\mathbb{N}_k|$, for some $k \in \mathbb{N}$. We write then $|A| = k$.

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The operations on cardinals when restricted to finite cardinals coincide with the operations on natural numbers!
This justifies the notation.

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E.g. If $|A| = k$ and $|B| = m$ for some $k,m \in \mathbb{N}$ then $|A \times B| = k \cdot m$

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Infinite, countable and uncountable sets

Time for a video!

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Prop. $\mathbb{R}$ is uncountable.

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Cardinals are unbounded

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Theorem (Cantor)

For every set A we have $|A| < |\mathcal{P}(A)|$.

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Hence, for every cardinal there is a larger one.