WEEK 8, PS

Solutions of some exercises

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(1) Show that the following abstract proposition is a contingency (i.e., neither a tautology nor a contradiction)

$$((a \Leftrightarrow b) \Rightarrow (\neg a \vee c)) \vee d \vee (e \wedge T)$$

Advice: Do not make a full truth table.

An abstract proposition is a tautology if for all possible values of the proposition variables, the truth value of the abstract proposition is 1 (T, true).

It is a ~~contingen~~ contradiction if for all possible values of the proposition variables the truth value is 0 (F, false).

In order to show that an abstract proposition is a contingency, we need an assignment of values to the proposition variables that make the proposition true (not a contradiction) and an assignment of values to the proposition variables that make it false (not a tautology).

Of course, for this we do not need a full truth table, and in this concrete case it would be unwise to make one, since there are 5 variables so $2^5 = 32$ rows in the truth table.

We look at the structure of

$e \equiv ((a \Leftrightarrow b) \Rightarrow (\neg a \vee c)) \vee d \vee (e \wedge T)$.

It is a disjunction of three sub-formulas, so if e.g. $\underline{d=1}$, then independently of the values of the other variables (say e.g. $a=b=c=d=1$) our proposition e gets the value 1, and hence it is not a contradiction.

Showing that it is not a tautology requires more thinking. We need values for a, b, c, d, e, so that e gets value 0. Since $e \equiv e_1 \vee d \vee e_2$ with $e_1 \equiv ((a \Leftrightarrow b) \Rightarrow (\neg a \vee c))$, $e_2 \equiv e \wedge T$, this is only possible if $\underline{e_1 = d = e_2 = 0}$

Hence $\boxed{d=0}$. From $e_2 \equiv e \wedge T = 0$, we conclude that it must be that $\boxed{e=0}$.

Finally, from $e_1 = 0$, it must be that $(a \Leftrightarrow b) = 1$ and $(\neg a \vee c) = 0$.

From this second condition, we get (since it is a disjunction) that

$\neg a = 0$ and $\boxed{c = 0}$

so, $\boxed{a = 1}$.

Now, having set $a = 1$, since $(a \Rightarrow b) = 1$, it must be that $\boxed{b = 1}$

Hence, for the following assignment of values the proposition variables:

$a = b = 1, \ c = d = e = 0$

φ has truth value 0, which shows that is not a tautology.

(2) Prove with a calculation that the following two formulas are comparable (i.e., one is stronger that the other or vice-versa).

$$P \Rightarrow ((Q \Rightarrow R) \wedge (Q \vee R)) \quad \text{and} \quad (\neg P \Rightarrow Q) \Rightarrow R.$$

A proof by calculations that two formulas e_1, e_2 are equivalent is a sequence of equivalent formulas

$$\varphi_1 \overset{val}{\underset{\{Arg\,1\}}{=}} \varphi_2 \overset{val}{\underset{\{Arg\,2\}}{=}} \cdots \overset{val}{\underset{\{Arg\,n-1\}}{=}} \varphi_n$$

where $\varphi_1 = e_1$, $\varphi_n = e_2$ and each pair of equivalent formulas $\varphi_i \overset{val}{\underset{\{Arg.\,i\}}{=}} \varphi_{i+1}$

is obtained by some of the standard equivalence by substitution and/or Leibnitz.

It is also possible that $\varphi_1 = e_2$, $\varphi_n = e_1$ (proof from right to left), or that we prove

$$e_1 \overset{val}{=} e \quad \text{and hence} \quad e_1 \overset{val}{=} e_2$$

$$e_2 \overset{val}{=} e$$

(by transforming both the left-hand side and the right-hand side to the same formula)

A proof by calculations that a formula e_1 is stronger than e_2 $(e_1 \overset{val}{\models} e_2)$

is a sequence of the form

$$\psi_1 \overset{val}{\underset{\{Arg.1\}}{=}} / \models^{val} \psi_2 \overset{val}{\underset{\{Arg.2\}}{=}} / \models^{val} \psi_3 \cdots \psi_{n-1} \overset{val}{=} / \models^{val}_{\{Arg.\, n-1\}} \psi_n$$

where $\psi_1 = e_1$, $\psi_n = e_2$, and each weakening

$$\psi_i \models^{val}_{\{Arg.\, i\}} \psi_{i+1}$$

is obtained by some of the standard weakenings using substitution [but in general not Leibnitz → excep for Monotonicity].

In our particular example, we first simplify both formulas to equivalent formulas.

We have

$$(\neg P \Rightarrow Q) \Rightarrow R \overset{val}{\underset{\{Implication\}}{=}} (\neg\neg P \vee Q) \Rightarrow R$$

$$\overset{val}{\underset{\{Double\ Neg.\}}{=}} (P \vee Q) \Rightarrow R$$

$$\overset{val}{\underset{\{Implication\}}{=}} \neg(P \vee Q) \vee R$$

$$\overset{val}{\underset{\{De\ Morgan\}}{=}} (\neg P \wedge \neg Q) \vee R$$

$$\overset{val}{\underset{\{distributivity\}}{=}} (\neg P \vee R) \wedge (\neg Q \vee R)$$

and

$$P \Rightarrow ((Q \Rightarrow R) \wedge (Q \vee R)) \overset{val}{=} \{\text{Implication}\}$$

$$P \Rightarrow ((\neg Q \vee R) \wedge (Q \vee R)) \overset{val}{=} \{\text{Implication}\}$$

$$\neg P \vee ((\neg Q \vee R) \wedge (Q \vee R)) \overset{val}{=} \{\text{Distributivity}\}$$

$$\neg P \vee ((\neg Q \wedge Q) \vee R) \overset{val}{=} \{\text{Contradiction}\}$$

$$\neg P \vee F \vee R \overset{val}{=} \{\text{T/F-elimination}\}$$

$$\neg P \vee R$$

Hence, we have

$$(\neg P \Rightarrow Q) \Rightarrow R \overset{val}{=} (\neg P \vee R) \wedge (\neg Q \vee R) \quad (1)$$

and

$$P \Rightarrow ((Q \Rightarrow R) \wedge (Q \vee R)) \overset{val}{=} \neg P \vee R \quad (2)$$

Now from the standard weakening $\{\wedge\text{-}\vee\text{-weakening}\}$

$$P \wedge Q \overset{val}{\models} P$$

with substitution, we get that

$$(\neg P \vee R) \wedge (\neg Q \vee R) \overset{val}{\models} \neg P \vee R$$

and hence, by (1) and (2) we conclude:

$$(\neg P \Rightarrow Q) \Rightarrow R \overset{val}{\models} P \Rightarrow ((Q \Rightarrow R) \wedge (Q \vee R))$$

So, indeed, the formulas are comparable.

(3) Show with a counter example that

$$\exists k\,[P \vee Q : R] \overset{val}{\neq} \neg \forall k\,[Q : \neg R]$$

We will first rewrite both sides to equivalent (simpler) formulas.
└ or more intuitive

We have $\neg \forall k\,[Q : \neg R] \underset{\{\text{DeMorgan}\}}{\overset{val}{=}} \exists k\,[Q : R]$

and $\exists k\,[P \vee Q : R] \underset{\{\text{Domain splitting}\}}{\overset{val}{=}} \exists k\,[P : R] \vee \exists k\,[Q : R]$

Now, in order to show that

$$\exists k\,[P \vee Q : R] \overset{val}{\neq} \neg \forall k\,[Q : \neg R]$$

it is enough to show that

$$\exists k\,[P : R] \vee \exists k\,[Q : R] \overset{val}{\neq} \exists k\,[Q : R]$$

It should be clear that these two are not equivalent, as an instance of $P \vee Q \overset{val}{\neq} Q$ (*)

(if P is true and Q false in (*), then the left-hand side is true but the right-hand side false)

Hence we need an example in which
↳ of predicates P, Q, R

such that $\exists k\,[P : R]$ is true, but $\exists k\,[Q : R]$ is false.

Consider $P: k \in \mathbb{Z}$

$Q: k \in \mathbb{N}$

$R: k < 0$

Then $\exists k\,[P : R]$ is the proposition

$\exists k\,[k \in \mathbb{Z} : k < 0]$ which is true

and $\exists k\,[Q : R]$ is the proposition

$\exists k\,[k \in \mathbb{N} : k < 0]$ which is not true

And we have found our counter example showing that

$$\exists k\,[P \vee Q : R] \overset{val}{\neq} \neg \forall k\,[Q : \neg R].$$

(4) Write the following sentence (in quotes below) - 9
as a formula with connectives and quantifiers.
You may use that $\mathbb{P}$ denotes the set of all prime numbers.

"Every even natural number greater than 4 is the sum of two prime numbers"

We define two predicates Even(n) for $n \in \mathbb{N}$ and sum-two-primes(n) for $n \in \mathbb{N}$ as

$$\text{Even}(n): \ \exists k [k \in \mathbb{N} : n = 2k]$$

$$\text{sum-two-primes}(n): \ \exists p_1, p_2 [p_1, p_2 \in \mathbb{P} : n = p_1 + p_2]$$

The required formula is then

$$\forall n [n \in \mathbb{N} \land \text{Even}(n) \land n > 4 : \text{sum-two-primes}(n)]$$

i.e.

$$\forall n [n \in \mathbb{N} \land \exists k [k \in \mathbb{N} : n = 2k] \land n > 4 : \exists p_1, p_2 [p_1, p_2 \in \mathbb{P} : n = p_1 + p_2]]$$

(or any formula equivalent to it).

(5) Check whether the proposition

$$A \cap B \subseteq C \Rightarrow A \cup B \subseteq C$$

holds for all sets A, B, C. If so, then give a proof if not, then give a counter example.

The proposition does not hold.

For example, for

$A = \{0\}$

$B = \{0,1,2\}$

$C = \{0,1\}$

We have $A \cup B = B \not\subseteq C$, $A \cap B = A \subseteq C$

i.e. $A \cap B \subseteq C$ is true since $\{0\} \cap \{0,1,2\} = \{0\} \subseteq \{0,1\}$

but $A \cup B \subseteq C$ is not since $\{0\} \cup \{0,1,2\} = \{0,1,2\} \not\subseteq \{0,1\}$

(e.g. $2 \notin \{0,1\}$)

and so the implication is false for this choice of sets A, B, C.