Now, why is this called a predicate lifting?

Because it maps a predicate on $W$

$$S \in \mathcal{P}W \quad \text{i.e.} \quad S \subseteq W$$

to a predicate $\lambda(S) \in \mathcal{P}FW$ on $FW$.

**Definition** [Predicate lifting]

Let $F: \underline{Sets} \to \underline{Sets}$ be a functor. An $n$-ary predicate lifting for $F$ is a set-indexed family of functions

$$\lambda_X : \mathcal{P}(X)^n \to \mathcal{P}(FX) \quad \text{such that}$$

for any function $f: X \to Y$ the following diagram commutes

$$\begin{tikzcd}
\mathcal{P}(X)^n \arrow[r, "\lambda_X"] & \mathcal{P}(FX) \\
\mathcal{P}(Y)^n \arrow[u, "(f^{-1})^n"] \arrow[r, "\lambda_Y"] & \mathcal{P}(FY) \arrow[u, "(Ff)^{-1}"']
\end{tikzcd}$$

Here $f^{-1}: \mathcal{P}(Y) \to \mathcal{P}(X)$ is the inverse-image function given by $f^{-1}(Y') = \{x \in X \mid f(x) \in Y'\}$ for $Y' \subseteq Y$.

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In categorical terms: A predicate lifting is a natural transformation $\lambda: \mathcal{P}^n \Rightarrow \mathcal{P}F$ where $\mathcal{P}$ here denotes the contravariant powerset functor (which at some point we denoted by $2^{(-)}$).

Hence $\mathcal{P}f$ is exactly $f^{-1}$.

↳ Here we use $\mathcal{P}$ to follow the notation of Pattinson.