Coalgebra Course 28.6.2012

The very basics of coalgebraic modal logics

First, we look at some concrete modal logics.

The logic K — the mother of all modal logics.

$$\varphi, \psi ::= \underbrace{\overset{\overset{AP}{\wedge}}{p} \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi}_{\text{standard propositional logic}} \mid \underbrace{\Box\varphi \mid \Diamond\varphi}_{\text{two modalities}}$$ — syntax of formulas

The semantics is defined on a given Kripke structure.

A Kripke structure is a coalgebra of type

$$c : X \to \mathcal{P}(X) \times \mathcal{P}(AP)$$

where AP is a constant set of atomic propositions (the one from the syntax). Hence the functor is $\underline{\mathcal{P}(-) \times \mathcal{P}(AP)}$

Remark: Atomic propositions are not so important when it comes to the coalgebraic treatment, so one could focus only on Kripke frames which are just $\mathcal{P}$-coalgebras like $c : X \to \mathcal{P}(X)$.

Given a Kripke structure (for now with atomic propositions)

$$c : W \to \mathcal{P}(W) \times \mathcal{P}(AP)$$

the semantics of a K-formula is given as follows

(here $w \in W$ is a state of the Kripke structure) ⟶

Note that usually logicians use W for a set of states (rather than S or X) because they like to call states "worlds", so W is the set of worlds (and a formula is true in some world ☺)

For K, we have

$$
\left.\begin{array}{l}
w \models p, \text{ for } p \in AP \quad \text{iff} \quad p \in \pi_2(c(w)) \\
w \not\models \perp \\
w \models \neg\varphi \quad \text{iff} \quad w \not\models \varphi \\
w \models \varphi \wedge \psi \quad \text{iff} \quad w \models \varphi \text{ and } w \models \psi
\end{array}\right\} \text{standard for the propositional formulas}
$$

$$w \models \Box\varphi \quad \text{iff} \quad \forall w' \in W. (\underbrace{w \to w'}_{w' \in \pi_1(c(w))} \Rightarrow w' \models \varphi)$$

$$w \models \Diamond\varphi \quad \text{iff} \quad \exists w' \in W. (w \to w' \wedge w' \models \varphi)$$

Having defined the satisfiability relation $\models$, we can also define truth-sets, so

$$[\![\varphi]\!] = \{w \in W \mid w \models \varphi\}$$

Hennessy-Milner Logic

SYNTAX:

$$\varphi, \psi ::= \overset{AP}{p} \mid \perp \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_a \varphi \mid \Diamond_a \psi \quad , \text{ for } a \in A$$

The semantics is given in terms of LTS (with atomic propositions labelling the states) so

$$c : X \to \underbrace{\mathcal{P}(X)^A}_{\text{more convenient modelling for modal logics}} \times \mathcal{P}(AP)$$

we can forget about these for a while

$$\cong \mathcal{P}(A \times X)$$

The semantics (interpretation) of propositional formulas is as usual.
The modalities talk about a-successors.

- Given a fixed LTS (with AP)

$$c : W \to \mathcal{P}(W)^A \times \mathcal{P}(AP)$$

and a state $w \in W$, we have

$$w \models \Box_a \varphi \text{ iff } \forall w' \in W. (\underbrace{w \xrightarrow{a} w'}_{w' \in \pi_1(c(w))(a)} \Rightarrow w' \models \varphi)$$

$$w \models \Diamond_a \varphi \text{ iff } \exists w' \in W. (w \xrightarrow{a} w' \wedge w' \models \varphi)$$

Probabilistic Modal logic

- Syntax

$$\varphi, \psi ::= p \mid \bot \mid \neg \varphi \mid \varphi \wedge \psi \mid L_r \varphi \qquad \text{for } \underline{r \in \mathbb{Q} \cap [0,1]}$$

$p \in AP$

probabilistic modalities

The intended meaning of L_r is
"with probability at least r ..."
(I did not mention but the intended meaning of
$\Box$ is "all successors ..."
$\Diamond$ is "there exists a successor .."
$\Box_a$ is "for all a-successors"
$\Diamond_a$ is "there exists an a-successor")

Probabilistic modal logic is interpreted over so-called probabilistic frames
[Markov chains (with atomic propositions as labels on states)]

as mentioned before, we can forget about those

Probabilistic frames are coalgebras

$$c : X \to \mathcal{D}(X)$$

if also atomic propositions then

$$c : X \to \mathcal{D}(X) \times \mathcal{P}(AP)$$

finitely supported

Where $\mathcal{D}$ is the probability distribution functor
(on Sets - all this is on Sets)

given on objects as

$$\mathcal{D}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1 \text{ , } supp(\mu) \text{ is finite}\}$$

where for $\mu : X \to [0,1]$, the support set of μ is defined as

$$Supp(\mu) = \{x \in X \mid \mu(x) > 0\}$$

and on functions, for $f : X \to Y$, $\mathcal{D}f : \mathcal{D}(X) \to \mathcal{D}(Y)$

$$\mathcal{D}f(\mu) = \lambda y . \sum_{x : f(x) = y} \mu(x)$$

Now given a probabilistic frame $c : W \to \mathcal{D}(W) \times \mathcal{P}(AP)$
satisfiability is defined as (for $w \in W$)

$$w \models L_r \varphi \quad \text{iff} \quad \underbrace{\sum_{w' : w' \models \varphi} \pi_1(c(w))(w')}_{\text{The probability of reaching the set } [\![\varphi]\!] \text{ from state } w.} \geq r$$

There are many other examples - Graded modal Logic, Coalition Logic, ...
(see the lecture notes of Dirk Pattinson on the web)

Why modal logics?
- for verification
- simply theoretically, the theory of logics and modal logics is huge

One thing that made Hennessy-Milner logic famous is the property called "expressivity" or nowadays just "Hennessy-Milner property" which says that the logic exactly characterizes bisimilarity (for LTS (with atomic propositions))

Let's see what this property is

Given a logic on some kind of transition systems (frames, actually coalgebras) it induces logical equivalence on the set of states defined as

$$w \equiv u \quad \text{iff} \quad \underbrace{\{\varphi \mid w \models \varphi\} = \{\varphi \mid u \models \varphi\}}_{w \text{ and } u \text{ satisfy the same formulas.}}$$

Theorem [Hennessy-Milner]
For Hennessy-Milner logic on ~~finitely~~ ~~branching~~ image finite LTSs (coalgebras of type $c : X \to \mathcal{P}_f(X)^A \; (\times \mathcal{P}(AP))$

finite powerset

logical equivalence coincides with bisimilarity.

Similar results exist (are of interest) for many (all?) modal logics

(Certainly for K and for Probabilistic modal logic)

REMARK: If no atomic propositions, then for K and for Prob. mod. logic this is a trivial property since it is not difficult to see that

$$\sim \; = \; \equiv \; = \; \nabla$$

where $\nabla = W \times W$ is the largest equivalence on the set of states (everything is bisimilar, everything is logically equivalent). But when adding labels/atomic-propositions, things get more interesting.

[Important result of Desharnais, Panangaden, et al. shows that negation-free prob. modal logic is expressive for bisimilarity of probabilistic systems]

One of the most developed lines of research in coalgebra deals with modal logics

(see Computer Journal article by Cîrstea et.al)
54(1): 31-41 (2011)
"Modal logics are coalgebraic"

There are different approaches and plenty of results and papers. We focus only a bit on the nowadays standard approach and only on "Hennessy-Milner property"

and here, in the coalgebraic setting behavioral equivalence is more handy than bisimilarity

why it was discovered at all

Main intuitive idea: Given $c : X \to FX$

A modalized formula
$\heartsuit \varphi$ asserts a property of successor states.

φ .... interpreted over states
$\heartsuit\varphi$ ... interpreted over successors in FX

Hence, a unary modal operator $\heartsuit$ → interpreted over $\boxed{c : W \to FW}$
maps a $\underbrace{\text{property}}_{\text{why is this a property? Think of } [\![\varphi]\!] \text{ or } \underline{\text{unary predicate in general}}}$ $S \subseteq W$ of states, to a
property $\lambda(S) \subseteq FW$ of successors.

So, in order to interpret $\heartsuit$ in a coalgebra $c : W \to FW$
we need a λ such that
$$w \models \heartsuit\varphi \quad \text{iff} \quad c(w) \in \lambda([\![\varphi]\!])$$
--> so, we see that powerset becomes very important

In other words to interpret modal logics over F-coalgebras
we need to associate a predicate lifting
$$\lambda_W : \mathcal{P}W \to \mathcal{P}FW$$
to each unary modal operator of the logic, uniformly in
W.
(for n-ary modal operator we'd need
$$\lambda_W : (\mathcal{P}W)^n \to \mathcal{P}FW.$$

Therefore a unary predicate lifting is simply a natural transformation $\lambda: \mathcal{P} \Rightarrow \mathcal{P}F$ with $\mathcal{P}$ contravariant powerset, i.e. a set-indexed family of maps

$$\lambda_X : \mathcal{P}X \to \mathcal{P}FX \quad \text{that satisfies}$$

$$\begin{array}{ccc} \mathcal{P}X & \xrightarrow{\lambda_X} & \mathcal{P}FX \\ {\scriptstyle f^{-1}}\uparrow & & \uparrow{\scriptstyle (Ff)^{-1}} \\ \mathcal{P}Y & \xrightarrow{\lambda_Y} & \mathcal{P}FY \end{array}$$

for any $f: X \to Y$.

[The contravariance of $\mathcal{P}$ is necessary in order to show that model semantics is invariant under coalgebra homomorphism which then leads the Hennessy-Milner property]

We will see that in all our examples (and in all other as well) indeed interpretation happens via predicate lifting. So if we want to interpret a unary $\heartsuit$ and we have $\lambda_\heartsuit = [\![\heartsuit]\!]$, the satisfiability relation is given by

$$\boxed{w \models \heartsuit e \quad \text{iff} \quad e(w) \in [\![\heartsuit]\!]_W([\![e]\!])}$$

$$\Updownarrow$$

$$w \in [\![\heartsuit e]\!]$$

Here come the examples

(we forget about AP now)

K on Kripke frames

$c : W \to \mathcal{P}W$

Here we had

$w \models \Box \varphi \iff$ all elements of "the" successor of w satisfy φ

In general in these examples it helps to think about "the" successor as an element of FW, if $c : W \to FW$.

$\iff$ "the" successor $c(w)$ is a subset of $[\![\varphi]\!]$

$\iff c(w) \in \{B \subseteq W \mid B \subseteq [\![\varphi]\!]\}$

$\iff c(w) \in [\![\Box]\!]_W([\![\varphi]\!])$

this point of view gives us a predicate lifting

where

$[\![\Box]\!]_W : \mathcal{P}W \to \mathcal{P}\mathcal{P}W$

$S \xmapsto{[\![\Box]\!]_W} \{B \subseteq W \mid B \subseteq S\}$

it is direct to check that this is indeed a predicate lifting.

Similarly, for the diamond

$w \models \Diamond \varphi \iff c(w) \in [\![\Diamond]\!]_W([\![\varphi]\!])$

for the predicate lifting $[\![\Diamond]\!]_W : \mathcal{P}W \to \mathcal{P}\mathcal{P}W$

$S \xmapsto{[\![\Diamond]\!]_W} \{B \subseteq W \mid S \cap B \neq \emptyset\}$

Why? $w \models \Diamond \varphi \iff$ there is an element of "the" successor of w that satisfies φ

$\iff$ "the" successor $c(w)$ of w has a non-empty intersection with $[\![\varphi]\!]$

$\iff c(w) \in \{B \subseteq W \mid [\![\varphi]\!] \cap B \neq \emptyset\}$

$\iff c(w) \in [\![\Diamond]\!]_W([\![\varphi]\!])$

Hennessy-Milner logic over LTS

$$c: W \to \mathcal{P}^A(W)$$

$$w \models \Box_a \varphi \iff c(w) \in [\![\Box_a]\!]_W([\![\varphi]\!])$$

where now $[\![\Box_a]\!]_W : \mathcal{P}W \to \mathcal{P}\mathcal{P}^A(W)$

$$S \xmapsto{[\![\Box]\!]_W} \{ f : A \to \mathcal{P}(W) \mid f(a) \subseteq S \}$$

(the correspondence is similar and quite obvious, convince yourself that it is really the case)

Probabilistic logic over probabilistic frames

We want $[\![L_r]\!]_W$ (given a prob. frame $c: W \to \mathcal{D}W$)

So that $w \models L_r \varphi \iff c(w) \in [\![L_r]\!]_W([\![\varphi]\!])$.

We know $w \models L_r \varphi \iff \sum_{w' \in [\![\varphi]\!]} c(w)(w') \geq r$

$$\iff c(w) \in \{ \mu \in \mathcal{D}W \mid \mu([\![\varphi]\!]) \geq r \}$$

this is a usual notation $\mu(S) = \sum_{w \in S} \mu(w)$ for $S \subseteq W$.

So a predicate lifting offers itself :-)

$$[\![L_r]\!]_W : \mathcal{P}W \to \mathcal{P}\mathcal{D}W$$

$$S \xmapsto{[\![L_r]\!]_W} \{ \mu \in \mathcal{D}W \mid \mu([\![\varphi]\!]) \geq r \} \text{ for } S \subseteq W$$

That's the whole point:

MODAL LOGICS VIA PREDICATE LIFTINGS

(can also be done modularly ⇝ for modularly defined functors)

Now when does a Hennessy-Milner property hold?
(for a given logic over given type of coalgebras)

It holds under two conditions: ⇝ we can't expect to hold just any time

(1) A condition on F that limits the branching degree of the models (coalgebras in question) — The functor F needs to be finitary (see Def. 2.3.7)

So (1) corresponds to the original image-finite condition of Hennessy & Milner

[This D is finitary (finite support), Pf is finitary...]

(2) A completeness condition on the set of modal operators that ensures that one doesn't miss observable behavior. "There are enough modal operators"

(see Def. 2.3.1 for this)

Then the general coalgebraic H-M-theorem is

Thm. 2.3.10 ⇝ if interested and time, please read Chapter 1 & Chapter 2 from the notes by Pattinson (containing also the proof). You already know most of Ch. 1, but there are interesting examples. But watch out for typos, which is why I also wrote these notes. THANK! CHEERS! Ana ☺