Congruences of convex algebras (for CA, PCA, TCA, f.p. = f.g.)

Ana Sokolova — University of Salzburg
Harald Woracek — TU Vienna

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CA = convex algebras

- Variety of algebras of type

$$\mathcal{T}_{\mathrm{ca}} := \left\{ (p_i)_{i=1}^n \in \mathbb{R}^n \mid \; n \in \mathbb{N}^+, p_1, \dots, p_n \geq 0, \sum_{i=1}^n p_i = 1 \right\}.$$

- two axioms

$$f_{(\delta_{ij})_{i=1}^n}(x_1, \dots, x_n) = x_j, \quad n \in \mathbb{N}^+, \; j = 1, \dots, n,$$

$$f_{(p_i)_{i=1}^n}\left(f_{(p_{1j})_{j=1}^m}(x_1, \dots, x_m), \dots, f_{(p_{nj})_{j=1}^m}(x_1, \dots, x_m)\right) = \\ = f_{(\sum_{i=1}^n p_i p_{ij})_{j=1}^m}(x_1, \dots, x_m)$$

CA = convex algebras

PCA - $\leq$

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TCA

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CA $\supseteq$ PCA $\supseteq$ TCA

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(P)CA - EM algebras for the (sub)distribution monad

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(P)CA - EM algebras for the (sub)distribution monad

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Finitely generated = quotients of free
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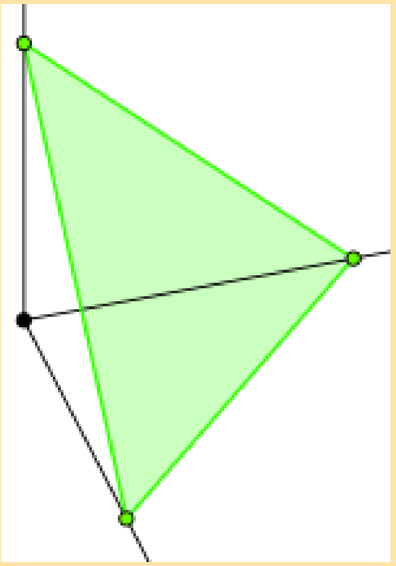
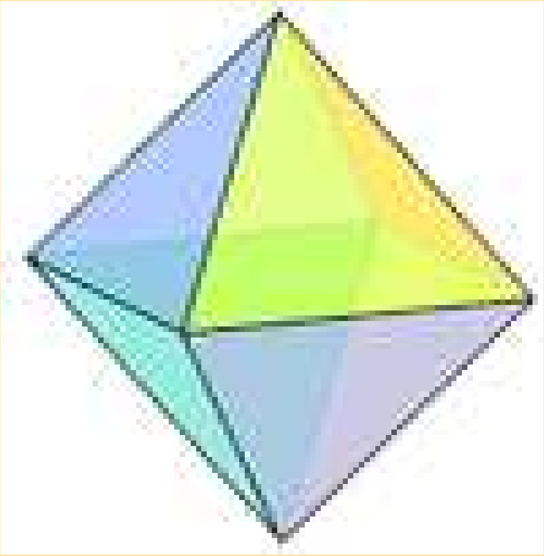
for CA, PCA, and TCA all congruences are f.g., hence f.p.=f.g.

Free CA, PCA, TCA

CA	PCA	TCA
	tetrahedron	
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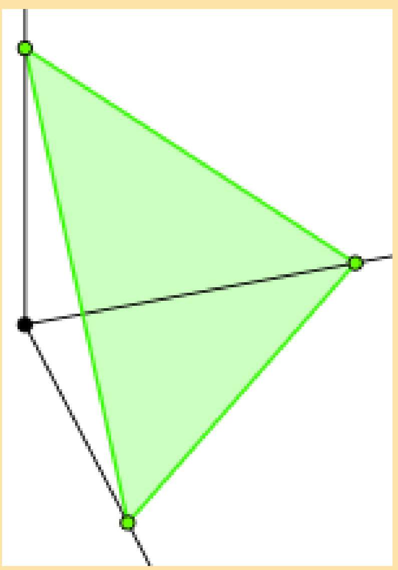
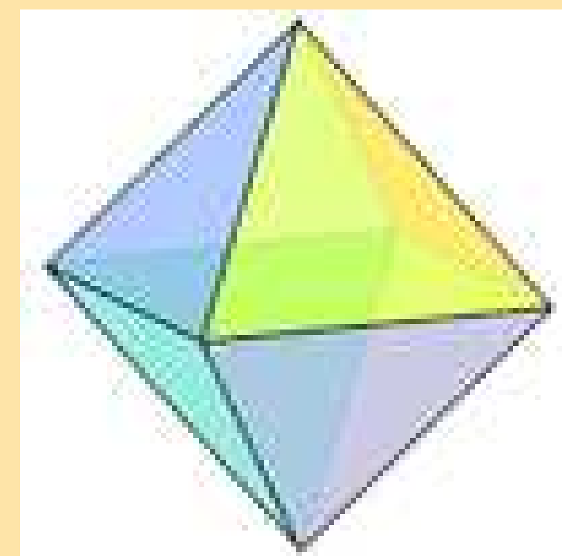
	tetrahedron	
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Polytopes

Congruences on polytopes

convex hull

- $K \subseteq \mathbb{R}^n$ is a polytope if $K = \mathrm{co}\, Y$ for a finite set Y

  equivalently

  K is convex, compact, with finitely many extremal points

  $K = \mathrm{co}\,(\mathrm{ext}\, K)$

$$\mathrm{co}\, Y := \Big\{ \sum_{y \in Y} \lambda_y y \mid \lambda_y \in [0,1], \sum_{y \in Y} \lambda_y = 1 \Big\}$$

- Every polytope is a CA/PCA

- An equivalence on K is a CA/PCA congruence iff it is a convex set (in $\mathbb{R}^n \times \mathbb{R}^n$)

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How do they look like?

Example

$Y = \{0,1\}$, $K = \text{co } Y$ in $\mathbb{R}$

$K = [0,1]$

K has exactly 5 congruences:

all finitely generated

$$
\begin{aligned}
\Theta_1 &= \Delta\\
\Theta_2 &= \{(0,0),(1,1)\} \cup (0,1) \times (0,1)\\
\Theta_3 &= \{(0,0)\} \cup (0,1] \times (0,1]\\
\Theta_4 &= [0,1) \times [0,1) \cup \{(1,1)\}, \text{ and}\\
\Theta_5 &= [0,1] \times [0,1].
\end{aligned}
$$

Four important `guys'

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join-semilattice of vertices

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$\varphi_\Theta : V_K \to \operatorname{Sub} \mathbb{R}^n$

$$\varphi_\Theta(Y) = \operatorname{span}\left\{x_2 - x_1 \mid x_1, x_2 \in \breve{\operatorname{co}}\, Y,\, x_1 \Theta x_2\right\}, \quad Y \in V_K.$$

interior of the convex hull

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graph $\mathcal{G}_\Theta$

$$\{Y_1, Y_2\} \in E_\Theta \iff \Theta \cap (\breve{\mathrm{co}}\, Y_1 \times \breve{\mathrm{co}}\, Y_2) \neq \emptyset, \quad Y_1, Y_2 \in V_K.$$

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$\approx_\Theta$

connectivity equivalence

interior of the convex hull

Forward theorem

(i) *The map* φ_Θ *is monotone.*

(ii) *Let* $\mathcal{C}$ *be a component of the graph* $\mathcal{G}_\Theta$. *Then* $\mathcal{C}$ *contains a largest element with respect to inclusion. Denoting this largest element by* $Y(\mathcal{C})$, *we have* $\{Y, Y(\mathcal{C})\} \in E_\Theta$, $Y \in \mathcal{C}$.

(iii) *The relation* $\approx_\Theta$ *is a congruence of the join-semilattice* V_K.

(iv) *Let* $\mathcal{C}$ *be a component of* $\mathcal{G}_\Theta$ *and* $Y(\mathcal{C})$ *its largest element. Then*

$$\varphi_\Theta(Y) = \varphi_\Theta(Y(\mathcal{C})) \cap \operatorname{dir} Y \qquad \textit{for } Y \in \mathcal{C},$$

$$\left[\breve{\operatorname{co}}\, Y + \varphi_\Theta(Y(\mathcal{C}))\right] \cap \breve{\operatorname{co}}\, Y(\mathcal{C}) \neq \emptyset \qquad \textit{for } Y \in \mathcal{C}.$$

Set

$$Z(\mathcal{C}) := \bigcup_{Y \in \mathcal{C}} \breve{\operatorname{co}}\, Y,$$

then the congruence Θ *can be recovered from* φ_Θ *and* $\mathcal{G}_\Theta$ *as*

$$\Theta = \bigcup_{\substack{\mathcal{C} \textit{ component} \\ \textit{of } \mathcal{G}_\Theta}} \left\{(x_1, x_2) \in Z(\mathcal{C}) \times Z(\mathcal{C}) :\ x_2 - x_1 \in \varphi_\Theta(Y(\mathcal{C}))\right\}.$$

Backward theorem

Let K be a polytope in $\mathbb{R}^n$. Let $\sim$ be a congruence relation of the join-semilattice V_K with the property that each congruence class $\mathcal{C}$ of $\sim$ contains a largest element, say $Y(\mathcal{C})$. Moreover, let

$$\varphi\colon \{Y(\mathcal{C}) \mid \mathcal{C} \text{ class of } \sim\} \longrightarrow \operatorname{Sub}\mathbb{R}^n$$

be a monotone map such that, for each class $\mathcal{C}$ of $\sim$,

$$\varphi(Y(\mathcal{C})) \subseteq \operatorname{dir} Y(\mathcal{C}), \quad \left[\breve{\mathrm{co}}\, Y + \varphi(Y(\mathcal{C}))\right] \cap \breve{\mathrm{co}}\, Y(\mathcal{C}) \neq \emptyset \quad \text{for } Y \in \mathcal{C}.$$

Then there exists a unique congruence $\Theta \in \operatorname{Con}_{\mathsf{CA}} K$ such that

$$\approx_\Theta = \sim, \quad \varphi_\Theta(Y(\mathcal{C})) = \varphi(Y(\mathcal{C})) \quad \text{for } \mathcal{C} \text{ a class of } \sim .$$

This congruence Θ can be computed from $\sim$ and φ by means of the formula

$$\Theta = \bigcup_{\substack{\mathcal{C} \text{ class} \\ \text{of } \sim}} \left\{(x_1, x_2) \in Z(\mathcal{C}) \times Z(\mathcal{C}) :\, x_2 - x_1 \in \varphi(Y(\mathcal{C}))\right\},$$

where again $Z(\mathcal{C}) := \bigcup_{Y \in \mathcal{C}} \breve{\mathrm{co}}\, Y$. Its associated function φ_Θ is given as

$$\varphi_\Theta(Y) = \varphi\big(Y(\mathcal{C})\big) \cap \operatorname{dir} Y \qquad \text{for } Y \in \mathcal{C},$$

and the set of edges E_Θ of its associated graph $\mathcal{G}_\Theta$ is given as

$$\{Y_1, Y_2\} \in E_\Theta \iff \Big(Y_1 \sim Y_2 \wedge \left[\breve{\mathrm{co}}\, Y_1 + \varphi\big(Y([Y_1]_\sim)\big)\right] \cap \breve{\mathrm{co}}\, Y_2 \neq \emptyset\Big)$$

where $[Y_1]_\sim$ denotes the equivalence class of Y_1.

Order theorem

$$\Theta_1 \subseteq \Theta_2 \quad \Longleftrightarrow \quad \left(E_{\Theta_1} \subseteq E_{\Theta_2} \wedge \varphi_{\Theta_1} \leq \varphi_{\Theta_2}\right)$$

And so we fully know the congruence lattice on K

And all are finitely generated

as congruences

But there are always some that are not finitely generated as subalgebras of K x K

Hence in CA,PCA,TCA
f.g. algebras are not closed under kernel pairs!

Conclusions

We saw:

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* And many other things...

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THANK YOU