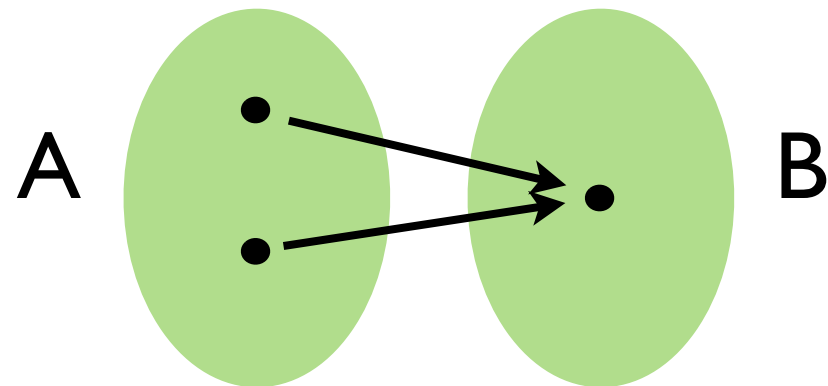
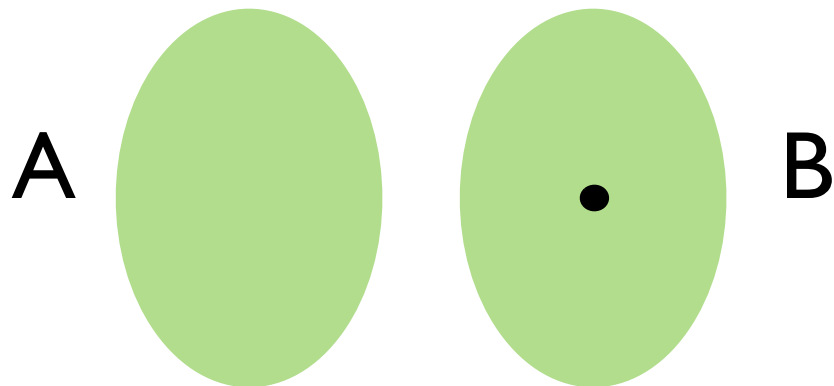
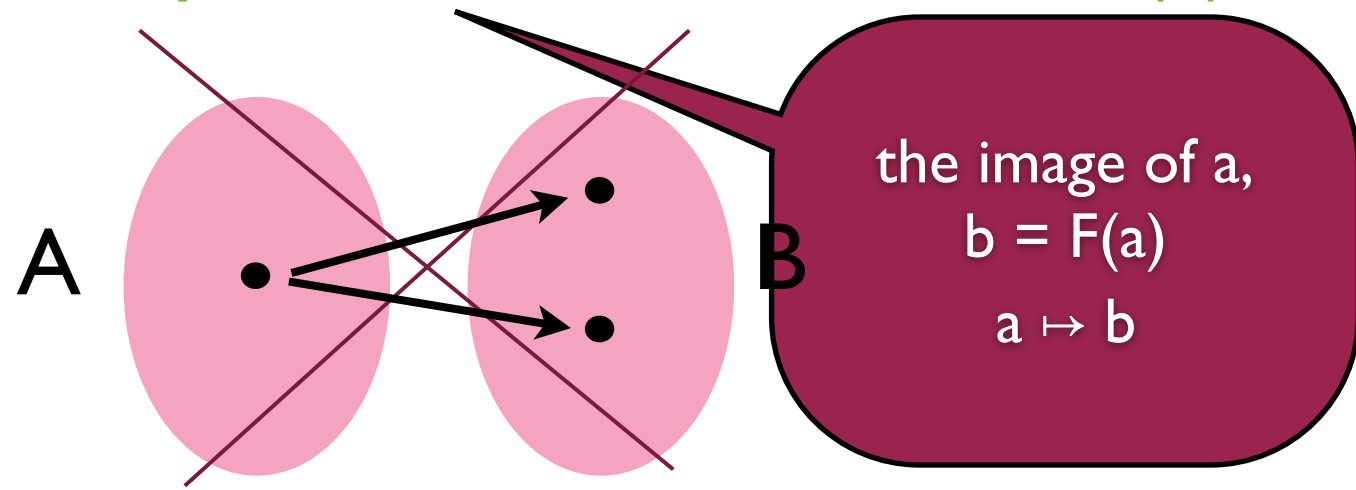
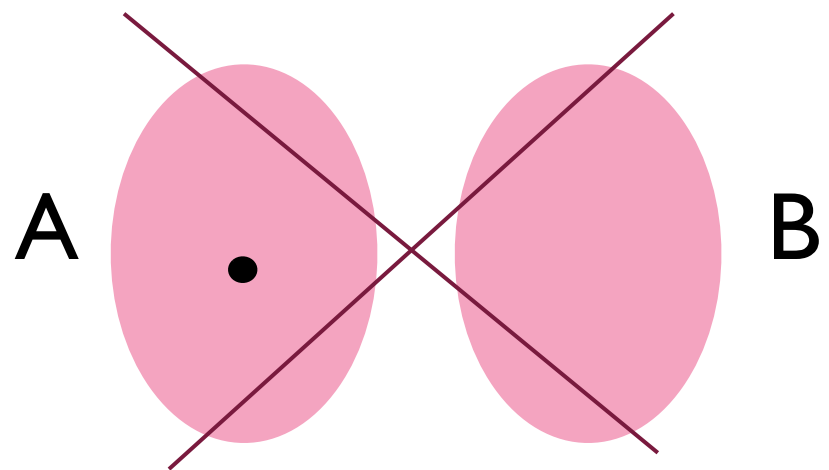


Functions, mappings

Def. If A and B are sets, then F is a function (mapping, *Abbildung*) from A to B , notation $F: A \longrightarrow B$ iff
for every $a \in A$, there exists a unique $b \in B$ such that $b = F(a)$.



$\{(a, F(a)) \mid a \in A\}$ is the **graph** of the function F

Functions, mappings

When $f: A \longrightarrow B$ then $\text{dom } f = A$ and $\text{cod } f = B$

domain of F
(Definitionsbereich)

codomain of F
(Wertebereich)

Let $f: A \longrightarrow B$ and $A' \subseteq A$.

The image (**Bild**) of A' is the set $f(A') = \{f(a) \mid a \in A'\} \subseteq B$.

$$f(A') = \{b \in B \mid \text{there is an } a \in A' \text{ with } b = f(a)\}$$

if $a \in A'$, then $f(a) \in f(A')$

So f extends to a function $f: \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$, the image-function.

Functions, mappings

Let $f: A \longrightarrow B$ and $B' \subseteq B$.

The inverse image (**Urbild**) of B' is the set

$$f^{-1}(B') = \{a \mid f(a) \in B'\} \subseteq A.$$


$$a \in f^{-1}(B') \quad \text{iff} \quad f(a) \in B'$$

Again the inverse image induces a function $f^{-1}: \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$, the inverse-image-function.

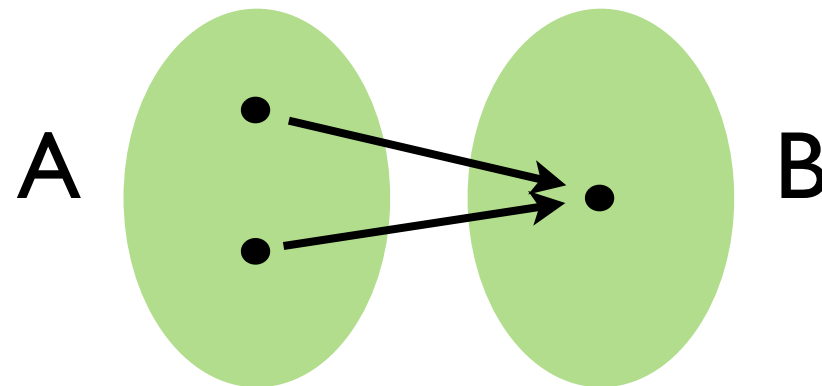
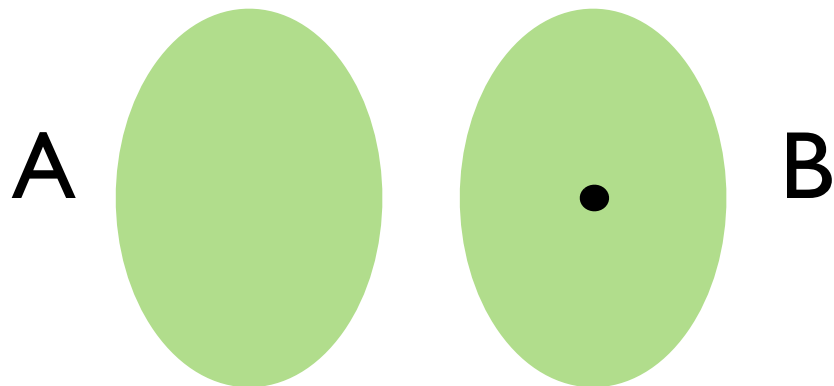
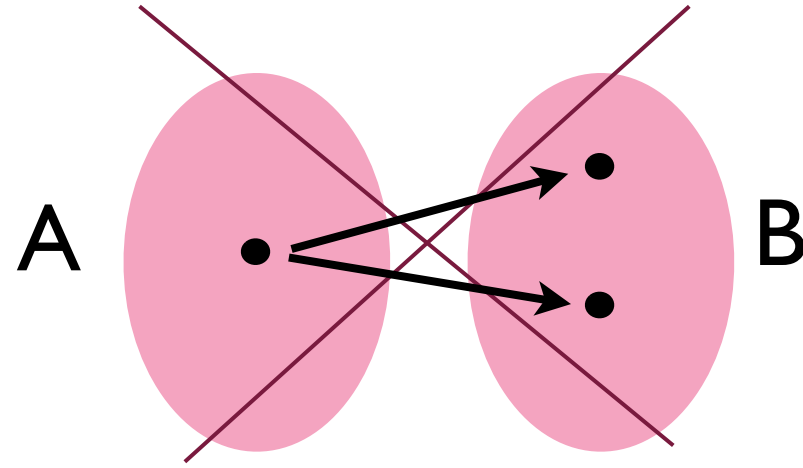
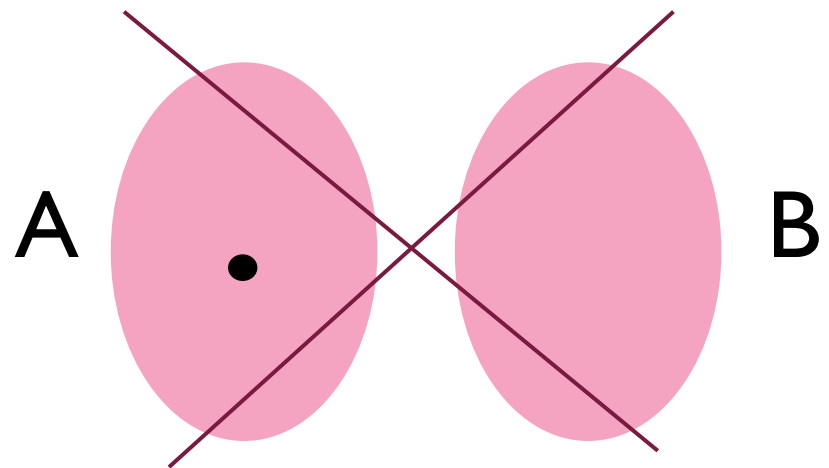
Lemma FI: Let $f: A \longrightarrow B$, $A' \subseteq A$, and $B' \subseteq B$. Then

$$A' \subseteq f^{-1}(f(A')) \quad \text{and} \quad f(f^{-1}(B')) \subseteq B'$$

(in general no more than this holds)

Recall...

Def. If A and B are sets, then F is a function (mapping, *Abbildung*) from A to B , notation $F: A \longrightarrow B$ iff
for every $a \in A$, there exists a unique $b \in B$ such that $b = F(a)$.



Special functions

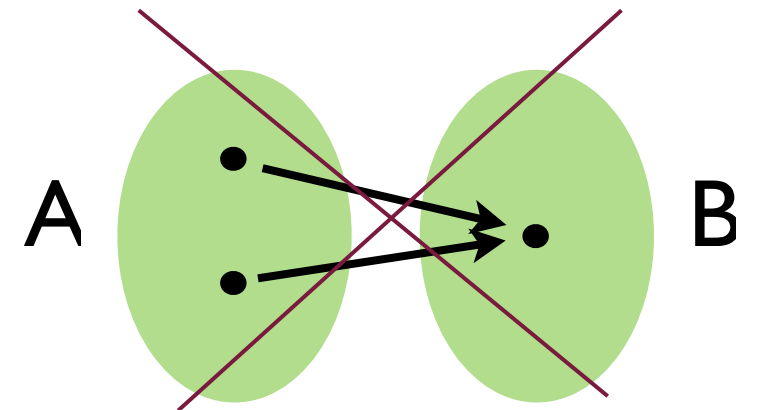
The number of ingoing arrows for a function can be 0, 1, or more. Based on this, we distinguish some special functions.



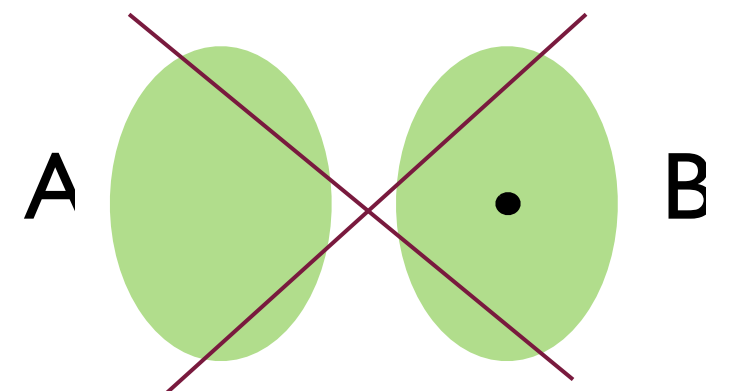
exactly one incoming arrow (injection + surjection) **bijection**

Special functions

Def. A function $f:A \longrightarrow B$ is injective iff
for all $a, b \in A$, if $f(a) = f(b)$ then $a = b$.



Def. A function $f:A \longrightarrow B$ is surjective iff
for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.



Def. A function $f:A \longrightarrow B$ is bijective iff
for all $b \in B$, there exists **unique** $a \in A$ with $f(a) = b$.

Simple characterisations

Lemma I: A function $f:A \longrightarrow B$ is injective iff
for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

at most one incoming arrow
injection

Lemma S: A function $f:A \longrightarrow B$ is surjective iff
 $|f^{-1}(\{b\})| \geq 1$ for all $b \in B$ iff
 $f(A) = B$.

at least one incoming arrow
surjection

Lemma B: A function $f:A \longrightarrow B$ is bijective iff
 $|f^{-1}(\{b\})| = 1$ for all $b \in B$ iff
 f is both injective and surjective.

exactly one incoming arrow
bijection

Some properties

Lemma I2: Let $f:A \longrightarrow B$ be injective and let $A' \subseteq A$. Then
 $f(x) \in f(A')$ iff $x \in A'$.

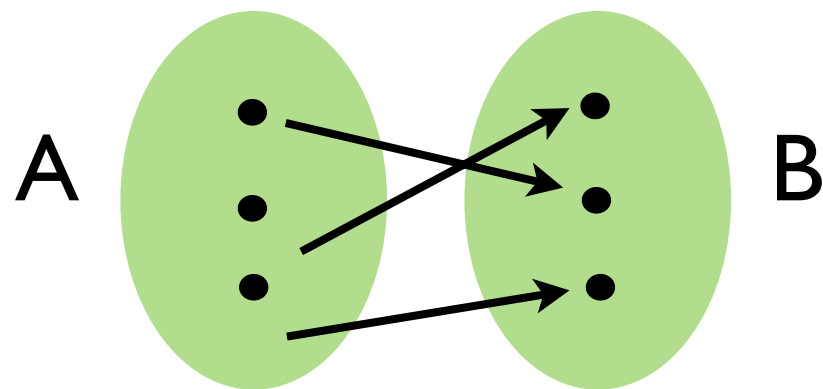
if holds always!

Prop. I3: Let $f:A \longrightarrow B$ be injective and let $A' \subseteq A$. Then
 $f^{-1}(f(A')) = A'$.

Prop. S2: Let $f:A \longrightarrow B$ be surjective and let $B' \subseteq B$. Then
 $f(f^{-1}(B')) = B'$.

Inverse function

Let $f:A \longrightarrow B$ be a **bijection**



well defined only if f is bijective!

Def. The inverse function $f^{-1}: B \longrightarrow A$ is defined as

$$f^{-1}(b) = a \text{ iff } f(a) = b, \quad b \in B.$$

Lemma B2: The inverse function f^{-1} for a bijection f is bijective.

Function composition

Let $f:A \longrightarrow B$ and $g:B \longrightarrow C$

Function composition

Let $f:A \longrightarrow B$ and $g:B \longrightarrow C$

“after”

$g \circ f : A \longrightarrow B \longrightarrow C$

Def. The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by
$$g \circ f (a) = g(f(a)), \text{ for } a \in A.$$

Lemma I4: Let $f:A \longrightarrow B$ and $g:B \longrightarrow C$ be injective. Then
 $g \circ f$ is injective.

Lemma S3: Let $f:A \longrightarrow B$ and $g:B \longrightarrow C$ be surjective. Then
 $g \circ f$ is surjective.

Corollary B2: Let $f:A \longrightarrow B$ and $g:B \longrightarrow C$ be bijective. Then so is $g \circ f$.

A characterization of bijections

Theorem B3: A function $f: A \longrightarrow B$ is bijective iff
there exists a function $g: B \longrightarrow A$ with
 $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.



$\text{id}_A: A \longrightarrow A,$
 $\text{id}_A(a) = a, \text{ for all } a \in A$

Equality of functions

Let $f:A \longrightarrow B$ and $g:C \longrightarrow D$

Def. The functions $f:A \longrightarrow B$ and $g:C \longrightarrow D$ are equal iff

(1) $A = C$

(2) $B = D$

(3) for all $a \in A$, $f(a) = g(a)$.

$\text{dom } f = \text{dom } g$

$\text{cod } f = \text{cod } g$