Sound and Complete Axiomatization of Trace Semantics for Probabilistic Transition Systems

Alexandra Silva and Ana Sokolova
CWI and University of Salzburg

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We will discuss

- history
- probabilistic transition systems
- (finite) trace semantics
- the sound and complete axiomatization
- in a coalgebraic setting

generative PTS,
fully probabilistic,
with labels

History

- For LTS Milner '84, JCSS
- expressions for LTS
- Kleene style theorem
- axiomatization
- sound and complete for bisimilarity

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  $P + Q \equiv Q + P,\ P + 0 \equiv P, \mu x.P \equiv P[\mu x.P/x],\ ...$
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- Kleene style theorem
- axiomatization $P + Q \equiv Q + P,\ P + 0 \equiv P, \mu x.P \equiv P[\mu x.P/x], \ldots$
- sound and complete for bisimilarity

$$P \equiv Q \iff P \sim Q$$

Some years later

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  Milner's axioms plus $a.(P+Q) \equiv a.P + a.Q$
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$$P \equiv Q \iff \mathrm{tr}(P) = \mathrm{tr}(Q)$$

Now we do it for PTS

- Expressions/axioms for PTS come in many flavors (mainly for bisimilarity)
  we build on Silva, Bonchi, Bonsangue, Rutten '09/'10

- Trace semantics for PTS also exists in variants
  we build on Hasuo, Jacobs, S. '06/'07

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expressions, Kleene style theorem, sound and complete axiomatization for bisimilarity
(generic approach applicable to a large class of systems)

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same expressions, one more axiom, sound and complete for trace semantics

generic coalgebraic approach applicable to PTS

Probabilistic transition systems

PTS here are generative, labelled, with explicit termination

$$\boxed{X \to \mathcal{D}_\omega(1 + A \times X)}$$

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Example:

$$\begin{tikzcd}
 & \bullet \arrow[dl, "{a,\frac{1}{2}}"'] \arrow[dr, "{a,\frac{1}{4}}"] & \\
\bullet \arrow[d, "{b,\frac{1}{3}}"'] & & \bullet \arrow[d, "{c,\frac{1}{2}}"] \\
\bullet \arrow[d, "1"'] & & \bullet \arrow[d, "1"] \\
* & & *
\end{tikzcd}$$

Coalgebra basics

Category $\mathbf{C}$, functor F, category of coalgebras:

$\mathbf{Coalg}_F$

Objects: $X \xrightarrow{c} FX$

Arrows:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "c"'] & Y \arrow[d, "d"] \\ FX \arrow[r, "Fh"] & FY \end{tikzcd}$$

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Final coalgebra semantics:

$$\begin{tikzcd} X \arrow[r, dashed, "\exists!\,\mathrm{beh}"] \arrow[d, "c"'] & \Omega_F \arrow[d, "\cong"] \\ FX \arrow[r, "F\,\mathrm{beh}"] & F\Omega_F \end{tikzcd}$$

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bisimilarity in $\mathbf{Sets}$
(for wpp functors)
trace semantics in $\mathcal{K}\ell(T)$
(for TF-coalgebras)

Traces for PTS

Coalgebraic trace semantics [HJS'06/'07] applies to PTS

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trace-map as final coalgebra map in $\mathcal{K}\ell(\mathcal{D})$

$$\begin{tikzcd} X \arrow[r, "\mathrm{tr}"] \arrow[d, "c"'] & A^* \arrow[d, "\cong"] \\ 1 + A \times X \arrow[r] & 1 + A \times A^* \end{tikzcd}$$

$\mathrm{tr} : X \to \mathcal{D}(A^*)$ in **Sets**

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$\mathrm{tr}: X \to \mathcal{D}(A^*)$ in **Sets**

It instantiates to finite trace distribution:

$$\begin{aligned}
\mathrm{tr}(x)(\varepsilon) &= c(x)(*) \\
\mathrm{tr}(x)(aw) &= \textstyle\sum_{x' \in X} c(x)(a, x') \cdot \mathrm{tr}(x)(w)
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$\mathrm{tr}: X \to \mathcal{D}(A^*)$ in **Sets**

$$\begin{tikzcd} & x_1 \arrow[dl, "{a, \frac{1}{2}}"'] \arrow[dr, "{a, \frac{1}{4}}"] & \\ x_2 \arrow[d, "{b, \frac{1}{3}}"'] & & x_3 \arrow[d, "{c, \frac{1}{2}}"] \\ x_4 \arrow[d, "1"'] & & x_5 \arrow[d, "1"] \\ * & & * \end{tikzcd}$$

$$\mathrm{tr}(x_1)(ab) = \frac{1}{6}$$

$$\mathrm{tr}(x_1)(ac) = \frac{1}{8}$$

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Are the same ones as for bisimilarity

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$$\mathsf{E} ::= \bigoplus_{i \in I} p_i \cdot \mathsf{F}_i \mid \mu x.\mathsf{E}^g \mid x \qquad (p_i \in [0,1], \sum_{i \in I} p_i \leqslant 1)$$

$$\mathsf{E}^g ::= \bigoplus_{i \in I} p_i \cdot \mathsf{F}_i \mid \mu x.\mathsf{E}^g \qquad (p_i \in [0,1], \sum_{i \in I} p_i \leqslant 1)$$

$$\mathsf{F}_i ::= * \mid a \cdot \mathsf{E}$$

Expressions for PTS

Are the same ones as for bisimilarity

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carry a scalar product

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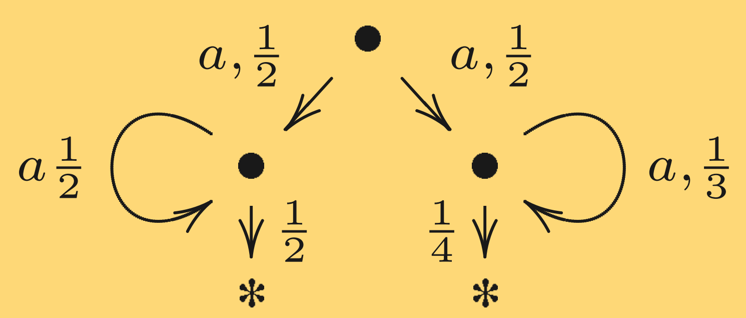

$$\frac{1}{2} \cdot a \cdot \mu x. \left(\frac{1}{2} \cdot a \cdot x \oplus \frac{1}{2} \cdot * \right)$$
$$\oplus \frac{1}{4} \cdot a \cdot \mu y. \left(\frac{1}{3} \cdot a \cdot y \oplus \frac{1}{2} \cdot * \right)$$

Expressions for PTS

expressions behave! (Kleene-style theorem)

carry a scalar product

Are the same ones as for bisimilarity

Silva, Bonchi, Bonsanque, Rutten '09/'10

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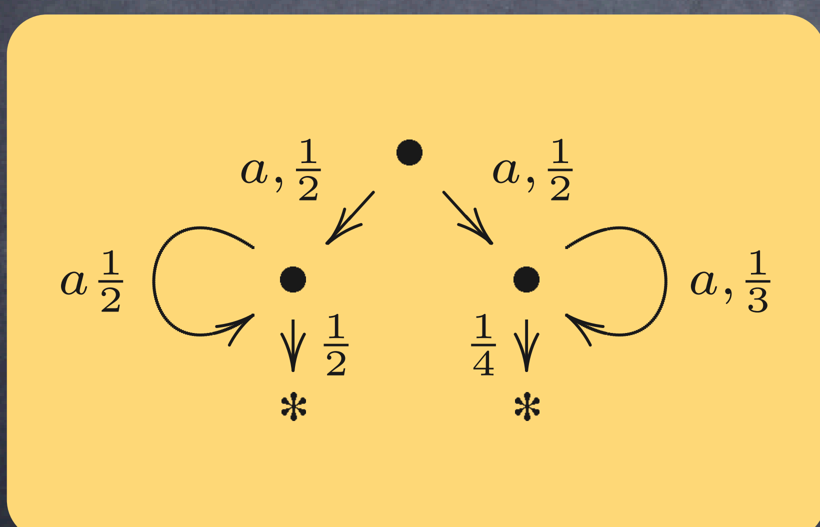

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Axioms

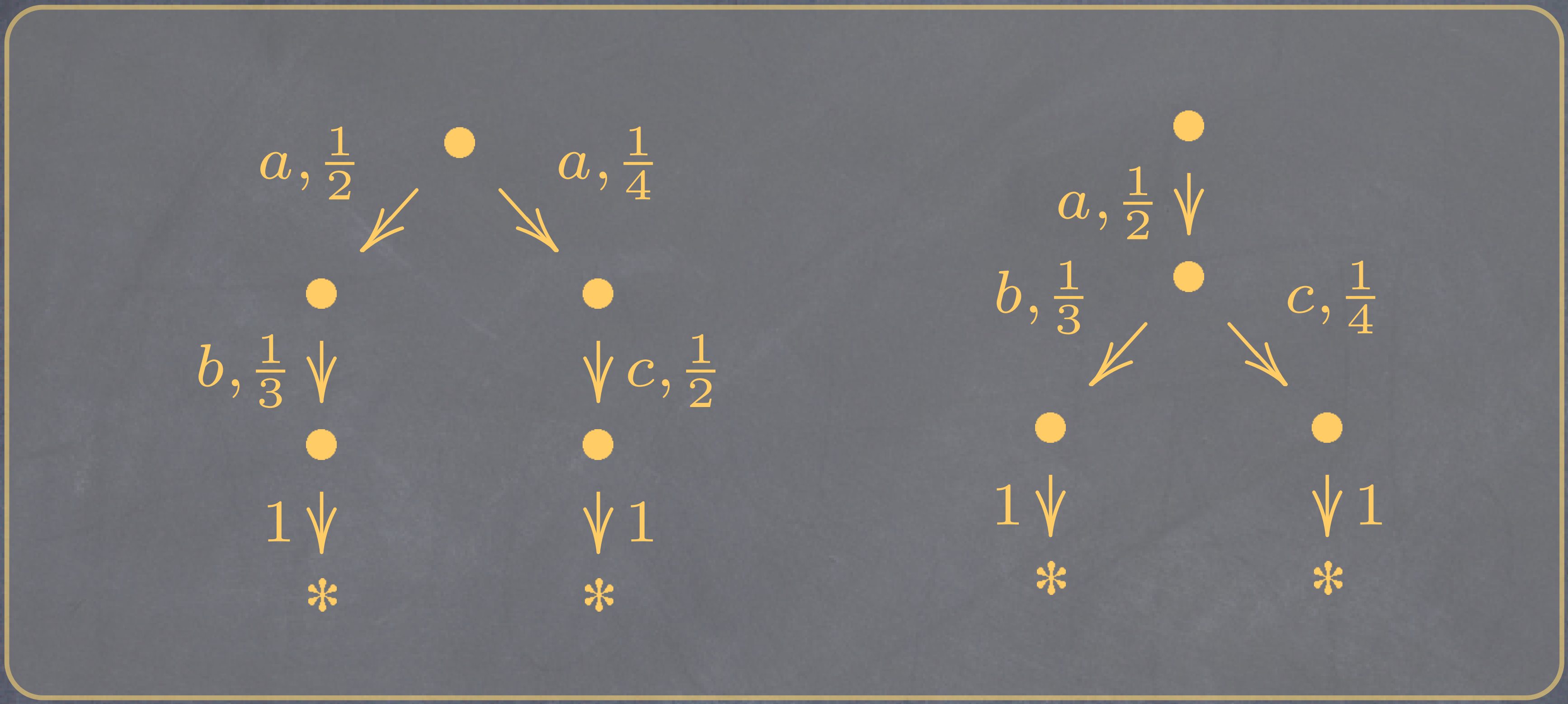
bisimilarity

$$
\begin{array}{llll}
\mathsf{E}_1 \oplus (\mathsf{E}_2 \oplus \mathsf{E}_3) & \equiv & (\mathsf{E}_1 \oplus \mathsf{E}_2) \oplus \mathsf{E}_3 & (A) \\
\mathsf{E}_1 \oplus \mathsf{E}_2 & \equiv & \mathsf{E}_2 \oplus \mathsf{E}_1 & (C) \\
\mathsf{E} \oplus \underline{\emptyset} & \equiv & \mathsf{E} & (E) \\
\mu x.\mathsf{E} & \equiv & \mathsf{E}[\mu x.\mathsf{E}/x] & (FP) \\
\gamma[\mathsf{E}/x] \equiv \mathsf{E} & \Rightarrow & \mu x.\gamma \equiv \mathsf{E} & (UFP) \\
\mu x.\mathsf{E} & \equiv & \mu y.\mathsf{E}[y/x] \ \text{ if } y \text{ is not free in } \mathsf{E} & (\alpha - equiv) \\
\mathsf{E}_1 \equiv \mathsf{E}_2 & \Rightarrow & \mathsf{E}[\mathsf{E}_1/x] \equiv \mathsf{E}[\mathsf{E}_2/x] & (Cong) \\
0 \cdot \mathsf{E} & \equiv & \underline{\emptyset} & (Z) \\
p \cdot \mathsf{E} \oplus p' \cdot \mathsf{E} & \equiv & (p + p') \cdot \mathsf{E} & (S)
\end{array}
$$

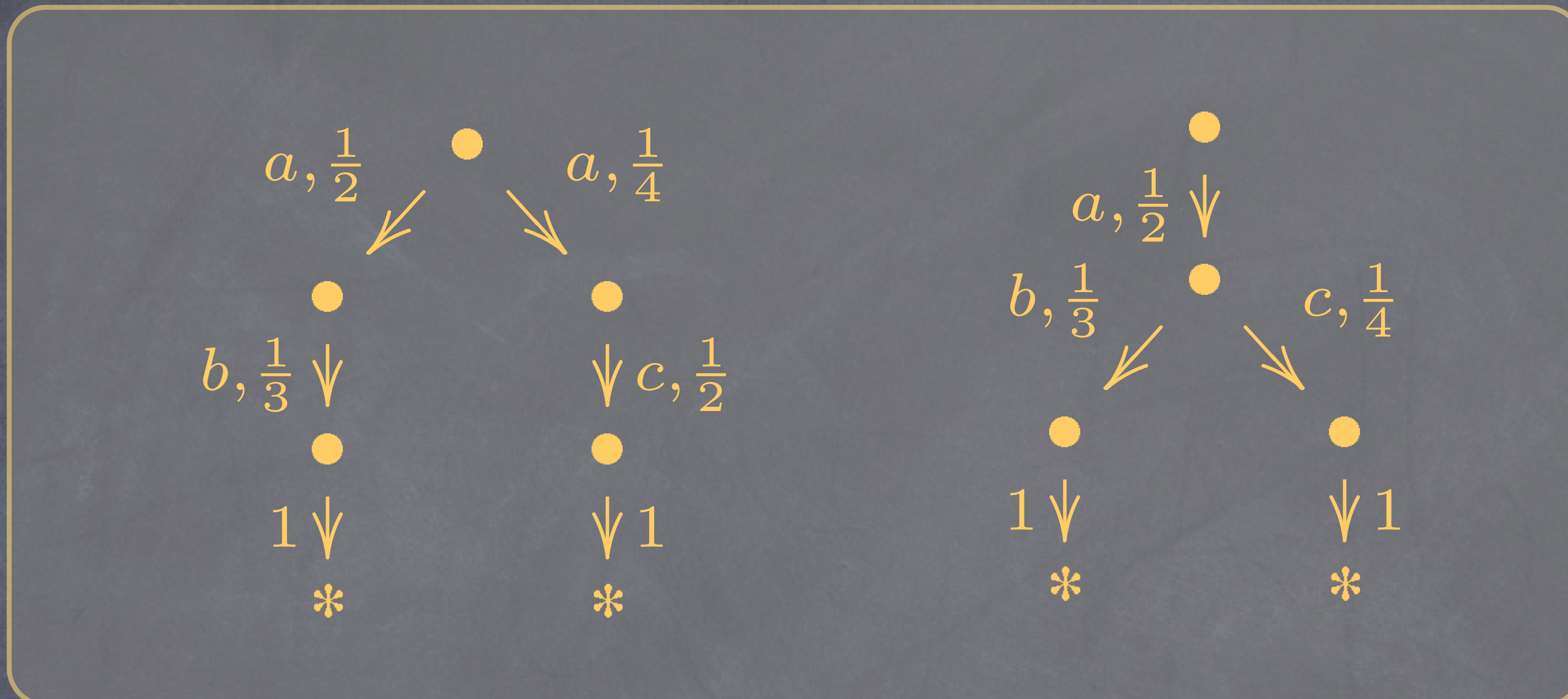
$$
p \cdot a \cdot (p_1\mathsf{E}_1 \oplus p_2\mathsf{E}_2) \equiv p_1 \cdot a \cdot p\mathsf{E}_1 \oplus p_2 \cdot a \cdot p\mathsf{E}_2 \quad (D)
$$

trace

Example

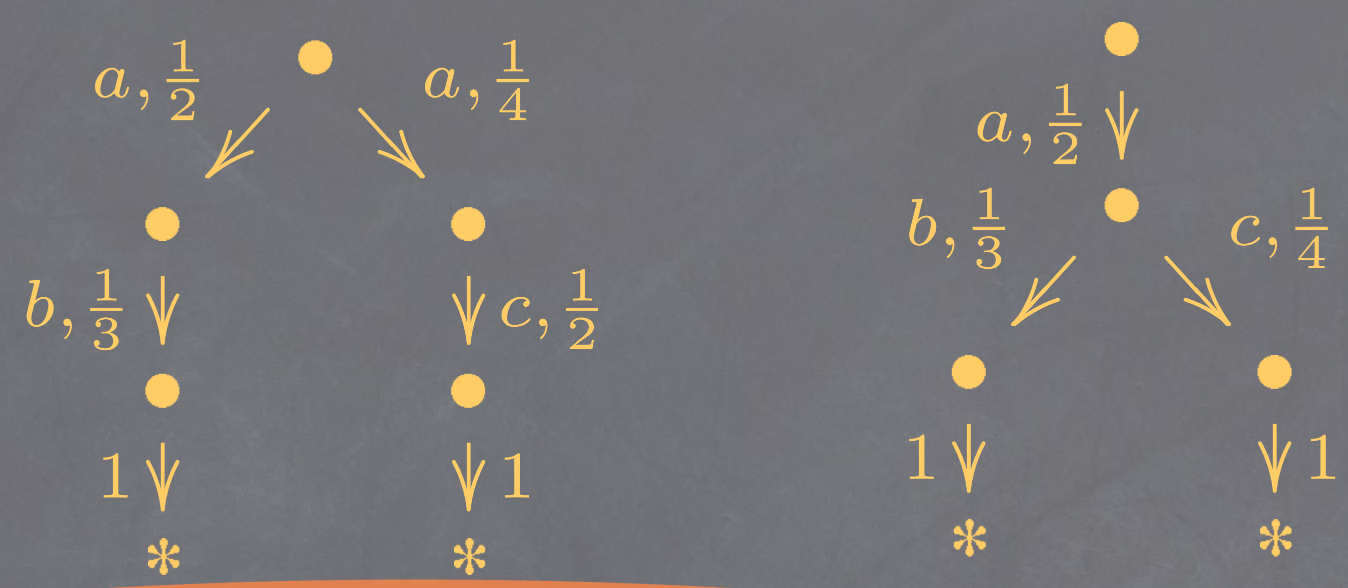

$a, \frac{1}{2}$
$a, \frac{1}{4}$
$b, \frac{1}{3}$
$c, \frac{1}{2}$
1
1
*
*
$a, \frac{1}{2}$
$b, \frac{1}{3}$
$c, \frac{1}{4}$
1
1
*
*

Example



$$
\begin{aligned}
\left(\frac{1}{2}\cdot a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *\right)\oplus\left(\frac{1}{4}\cdot a\cdot\frac{1}{2}\cdot c\cdot 1\cdot *\right) &\overset{(D)}{\equiv} \frac{1}{2}\cdot a\cdot\left(\frac{1}{2}\left(\frac{2}{3}\cdot b\cdot 1\cdot *\right)\oplus\frac{1}{4}(1\cdot c\cdot 1\cdot *)\right)\\
&= \frac{1}{2}\cdot a\cdot\left(\frac{1}{3}\cdot b\cdot 1\cdot *\oplus\frac{1}{4}\cdot c\cdot 1\cdot *\right)
\end{aligned}
$$

Example



$$\frac{1}{3}\cdot b\cdot 1\cdot * = \frac{1}{2}\left(\frac{2}{3}\cdot b\cdot 1\cdot *\right),\ \frac{1}{2}\cdot c\cdot 1\cdot * = \frac{1}{2}(1\cdot c\cdot 1\cdot *)$$

$$\begin{aligned}\left(\frac{1}{2}\cdot a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *\right)\oplus\left(\frac{1}{4}\cdot a\cdot\frac{1}{2}\cdot c\cdot 1\cdot *\right) &\overset{(D)}{\equiv} \frac{1}{2}\cdot a\cdot\left(\frac{1}{2}\left(\frac{2}{3}\cdot b\cdot 1\cdot *\right)\oplus\frac{1}{4}(1\cdot c\cdot 1\cdot *)\right)\\ &= \frac{1}{2}\cdot a\cdot\left(\frac{1}{3}\cdot b\cdot 1\cdot *\oplus\frac{1}{4}\cdot c\cdot 1\cdot *\right)\end{aligned}$$

Soundness and Completeness

Find an injective map $out_{\equiv}$ with $\mathrm{tr} = out_{\equiv} \circ [-]$

Soundness and Completeness

canonical map to $\equiv$ - classes

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Soundness and Completeness

canonical map to $\equiv$ - classes

Find an injective map $out_{\equiv}$ with $\mathrm{tr} = out_{\equiv} \circ [-]$

Soundness

$$
\begin{array}{ll}
 & \mathsf{E}_1 \equiv \mathsf{E}_2 \\
\Leftrightarrow & [\mathsf{E}_1] = [\mathsf{E}_2] \\
\overset{(*)}{\Rightarrow} & out_{\equiv}([\mathsf{E}_1]) = out_{\equiv}([\mathsf{E}_2]) \\
\overset{(\triangle)}{\Leftrightarrow} & tr(\mathsf{E}_1) = tr(\mathsf{E}_2)
\end{array}
$$

Completeness

$$
\begin{array}{ll}
 & tr(\mathsf{E}_1) = tr(\mathsf{E}_2) \\
\overset{(\triangle)}{\Leftrightarrow} & out_{\equiv}([\mathsf{E}_1]) = out_{\equiv}([\mathsf{E}_2]) \\
\overset{(\heartsuit)}{\Rightarrow} & [\mathsf{E}_1] = [\mathsf{E}_2] \\
\Leftrightarrow & \mathsf{E}_1 \equiv \mathsf{E}_2
\end{array}
$$

Soundness and Completeness

canonical map to $\equiv$ - classes

Find an injective map $out_\equiv$ with $\mathrm{tr} = out_\equiv \circ [-]$

Soundness

$$\begin{array}{ll} & E_1 \equiv E_2 \\ \Leftrightarrow & [E_1] = [E_2] \\ \overset{(*)}{\Rightarrow} & out_\equiv([E_1]) = out_\equiv([E_2]) \\ \overset{(\triangle)}{\Leftrightarrow} & tr(E_1) = tr(E_2) \end{array}$$

Completeness

$$\begin{array}{ll} & tr(E_1) = tr(E_2) \\ \overset{(\triangle)}{\Leftrightarrow} & out_\equiv([E_1]) = out_\equiv([E_2]) \\ \overset{(\heartsuit)}{\Rightarrow} & [E_1] = [E_2] \\ \Leftrightarrow & E_1 \equiv E_2 \end{array}$$

$(*)$ - existence of $out_\equiv$ $\quad(\triangle)$ - $\mathrm{tr} = out_\equiv \circ [-]$ $\quad(\heartsuit)$ - injectivity

How to get out?

Bisimilarity case, F-coalgebras

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$$\begin{tikzcd} X \arrow[r, "{[-]_\sim}"] \arrow[d, "c"'] & \mathsf{Exp}/_\sim \arrow[rr, bend left, "\mathrm{beh}"] \arrow[d, "c_\sim"] & & \Omega_F \arrow[d, "\cong"] \\ FX \arrow[r, "{F[-]_\sim}"] & F\mathsf{Exp}/_\sim \arrow[rr, bend right, "F\,\mathrm{beh}"'] & & F\Omega_F \end{tikzcd}$$

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final in Coalg_F

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FX \arrow[r, "{F[-]_\sim}"] & F\mathsf{Exp}/_\sim \arrow[r, two heads, "Fe"] \arrow[rr, bend right, "F\,\mathrm{beh}"'] & FI \arrow[r, tail, "Fm"] & F\Omega_F
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$$

final in $\mathbf{Coalg}_F$

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\end{tikzcd}$$

Hence, e is iso, and out = beh is injective

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\end{tikzcd}
$$

Hence, e is iso, and out = beh is injective

Silva et al. '08/'09/'10, Jacobs'06

there are also algebras around

A way out for traces?

- It is tough to work in Kleisli categories
- Factorization ?
- So we find a way to stay in $\mathbf{Sets}$ or rather in $\mathbf{Sets}^{\mathcal{D}_\omega}$
- A way out - determinization

Determinization of PTS

PTS example

$$
\begin{tikzcd}[column sep=small]
 & x_1 \arrow[dl, "{a,\frac{1}{2}}"'] \arrow[dr, "{a,\frac{1}{4}}"] & \\
x_2 \arrow[d, "{b,\frac{1}{3}}"'] & & x_3 \arrow[d, "{c,\frac{1}{2}}"'] \\
x_4 \arrow[d, "1"'] & & x_5 \arrow[d, "1"] \\
* & & *
\end{tikzcd}
$$

Its determinization

a G-coalgebra

$$
\begin{tikzcd}[column sep=small]
 & x_1 \arrow[d, "a"] & \\
 & \frac{1}{2}x_2 + \frac{1}{4}x_3 \arrow[dl, "b"'] \arrow[dr, "c"] & \\
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Determinization of PTS

PTS example

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$$GX = [0,1] \times X^A$$

$$\begin{tikzcd}
X \arrow[r, "out"] \arrow[d, "c"'] & {[0,1]^{A^*}} \arrow[d, "\cong"] \\
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Determinization of PTS

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final G-coalgebra

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$\mathbf{Sets}^{\mathcal{D}_\omega}$

$$GX = [0,1] \times X^A$$

Its determinization

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Sets$^{\mathcal{D}_\omega}$

$$GX = [0,1] \times X^A$$

$$
\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, "c"'] & \mathcal{D}_\omega(X) \arrow[d, "(\delta \circ c)^\#"] \\
\mathcal{D}_\omega(1 + A \times X) \arrow[r, "\delta"] & G\mathcal{D}_\omega(X)
\end{tikzcd}
$$

Its determinization

a G-coalgebra

$$
\begin{tikzcd}
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A way out for traces

Trace case, almost G-coalgebras on $\mathbf{Sets}^{\mathcal{D}_\omega}$

PCA
Eilenberg-Moore
algebras

A way out for traces

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$$
\begin{tikzcd}
\mathsf{Exp}/_{\equiv} \arrow[r, bend left, "out"] \arrow[d, "d"] & \Omega_{\hat{G}} \arrow[d, "\cong"] \\
\hat{G}\mathsf{Exp}/_{\equiv} \arrow[r, bend right, "\hat{G}out"] & \hat{G}\Omega_{\hat{G}}
\end{tikzcd}
$$

A way out for traces

Trace case, almost G-coalgebras on $\mathbf{Sets}^{\mathcal{D}_\omega}$

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$$
\begin{tikzcd}
\mathsf{Exp}/_{\equiv} \arrow[r, two heads, "e"'] \arrow[rr, bend left, "out"] \arrow[d, "d"] & I \arrow[r, tail, "m"'] \arrow[d, "d_I"] & \Omega_{\hat{G}} \arrow[d, "\cong"] \\
\hat{G}\mathsf{Exp}/_{\equiv} \arrow[r, two heads, "\hat{G}e"] \arrow[rr, bend right, "\hat{G}out"] & \hat{G}I \arrow[r, tail, "\hat{G}m"] & \hat{G}\Omega_{\hat{G}}
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Trace case, almost G-coalgebras on $\mathbf{Sets}^{\mathcal{D}_\omega}$

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A way out for traces

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A way out for traces

Trace case, almost G-coalgebras on $\mathbf{Sets}^{\mathcal{D}_\omega}$

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Hence, e is iso, and out is injective

A way out for traces

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$$

Hence, e is iso, and out is injective

Moreover: $tr = out \circ [-]$

Conclusions

- we present a solution to a concrete problem

  sound and complete axiomatization of traces for PTS

  bisimilarity expressions and axioms plus one new axiom

- in a coalgebraic setting
- it opens many generalization questions...
- all about algebra and coalgebra

Conclusions

- we present a solution to a concrete problem

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Thank you !