The structure of natural numbers

is helpful for proving properties
$\forall n[n \in \mathbb{N}: P(n)]$

The structure of natural numbers

On natural numbers we can define a notion of a successor, a mapping

s: $\mathbb{N} \to \mathbb{N}$

by $s(n) = n+1$

The successor mapping imposes a structure on the set that enables us to count:

1) there is a starting natural number 0
2) for every natural number n, there is a next natural number $s(n) = n+1$.

(Some) Peano Axioms

Important properties

(1) Different natural numbers have different successors:

$\forall n,m \; [n,m \in \mathbb{N} : s(m) = s(n) \Rightarrow m = n]$

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s is injective!

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stated positively

s is injective!

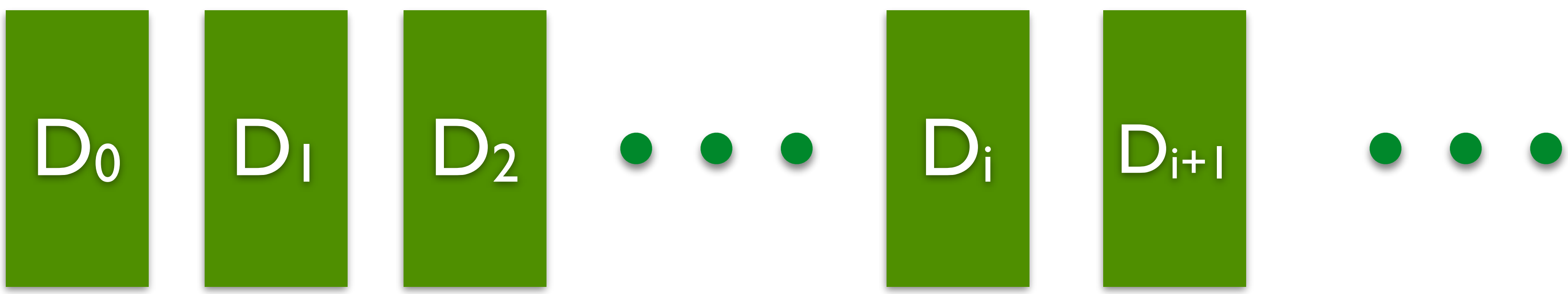
(2) 0 is not a successor: $\forall n\ [n \in \mathbb{N} : \neg\ (s(n) = 0)\ ]$

(3) All natural numbers except 0 are successors:

$\forall n[n \in \mathbb{N} \wedge \neg(n = 0) : \exists m[m \in \mathbb{N} :\ n = s(m)]$

There is more to it - induction

Imagine an infinite sequence of dominos



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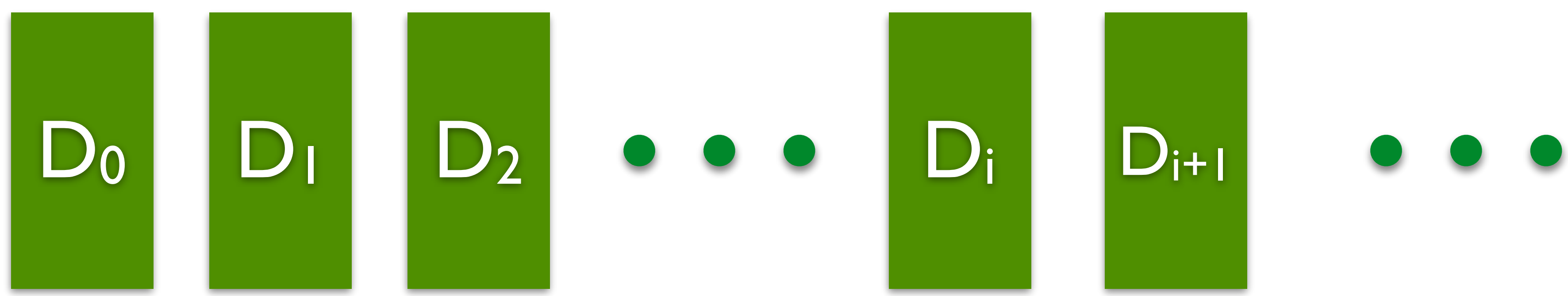

If we know that

1. D_0 falls
2. The dominos are close enough together so that if D_i falls, then D_{i+1} falls (for all $i \in \mathbb{N}$)

Then we can conclude that every domino D_n ($n \in \mathbb{N}$) falls!

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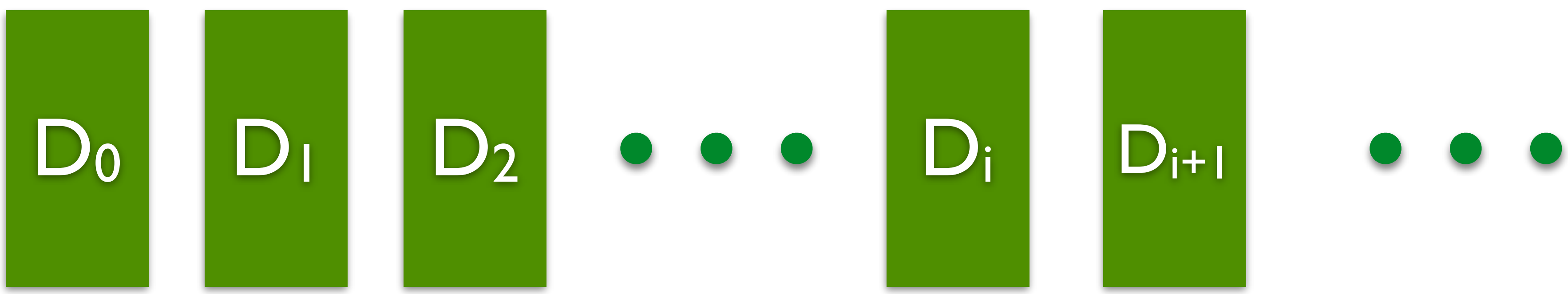

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induction

Induction

$$P(0) \wedge \forall i\ [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n\ [n \in \mathbb{N} : P(n)]$$

Induction

P - unary predicate over $\mathbb{N}$

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Variant of the Peano Axiom:

Let $K \subseteq \mathbb{N}$ have the property that

(a) $0 \in K$ and

(b) for all $n \in \mathbb{N}$, $n \in K \Rightarrow (n+1) \in K$.

Then $K = \mathbb{N}$.

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. . .

(m) $P(0)$

{Assume}

(k) **var** i; $i \in \mathbb{N}$

(k+1) $P(i)$

. . .

(l-1) $P(i+1)$

{$\Rightarrow$-intro on (k+1) and (l-1)}

(l) $P(i) \Rightarrow P(i+1)$

{$\forall$-intro on (k) and (l)}

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(l+2) $\forall n[n \in \mathbb{N} : P(n)]$

Induction

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Basis

induction hypothesis

Induction step

Inductive definitions

Inductive proof: truth is passed on

Inductive definition: construction is passed on

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well defined by induction

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Example

The sequence of real numbers $(a_i \mid i \in \mathbb{N})$ is defined inductively by

$$a_0 = 2$$
$$a_{i+1} = 2a_i - 1$$

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2	3	5	9	17	...

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Conjecture

For all $n \in \mathbb{N}$ it holds that

$$a_n = 2^n + 1$$

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Definition of $(a_i \mid i \in \mathbb{N})$ with strong induction

a_n is defined via $a_0, .., a_{n-1}$

Cardinality

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Theorem (Cantor)

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Operations on cardinals

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Prop. Let A be a set. Then $|\mathcal{P}(A)| = 2^{|A|}$.

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Note: $2 = |\{0,1\}|$

Finite sets, finite cardinals

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Finite sets, finite cardinals

We write $\mathbb{N}_k$ for the set $\{0,1,\ldots,k-1\}$. Then $\mathbb{N}_0 = \varnothing$.

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Def. A set A is finite if and only if $|A| = |\mathbb{N}_k|$, for some $k \in \mathbb{N}$. We write then $|A| = k$.

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The operations on cardinals when restricted to finite cardinals coincide with the operations on natural numbers!
This justifies the notation.

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E.g. If $|A| = k$ and $|B| = m$ for some $k,m \in \mathbb{N}$

then $|A \times B| = k \cdot m$

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Infinite, countable and uncountable sets

Time for a video!

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Hilbert's
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Prop.
$\mathbb{N}$ is countable.
$\mathbb{Z}$ is countable.
$\mathbb{Q}$ is countable.

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cardinal numbers are $\sim$ equivalence classes

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Prop.
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Infinite, countable and uncountable sets

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We write c for $|\mathbb{R}|$

Cardinals are unbounded

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Theorem (Cantor)

For every set A we have $|A| < |\mathcal{P}(A)|$.

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Theorem (Cantor)

For every set A we have $|A| < |\mathcal{P}(A)|$.

Hence, for every cardinal there is a larger one.