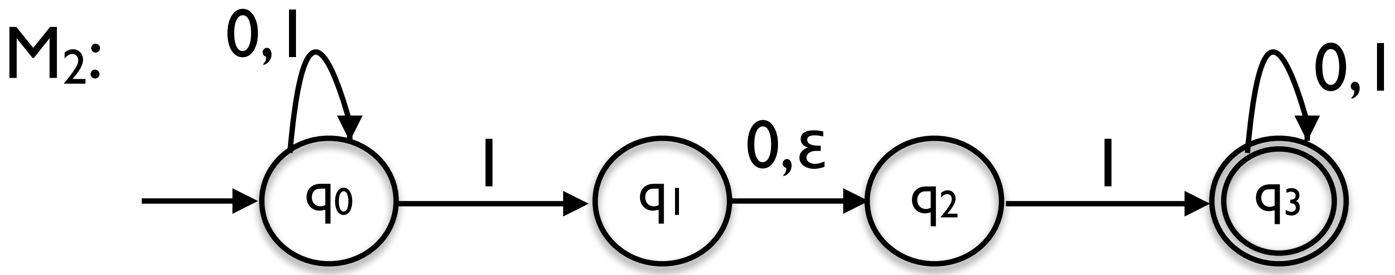
Nondeterministic Automata (NFA)

Informal example

$\Sigma = \{0,1\}$



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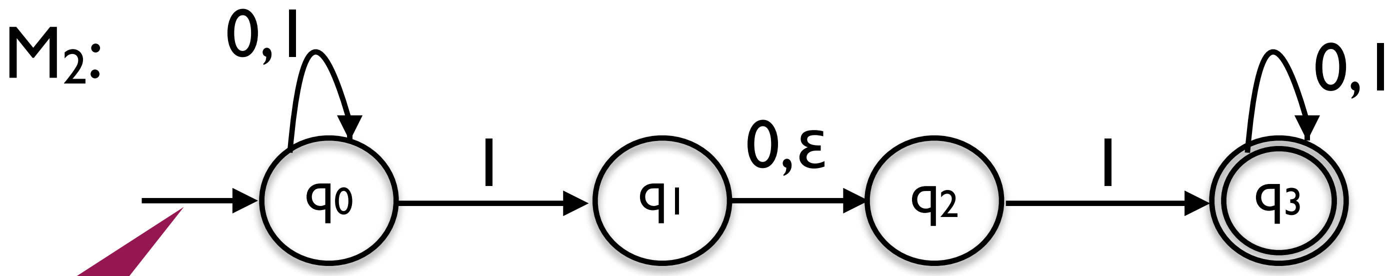

sources of nondeterminism

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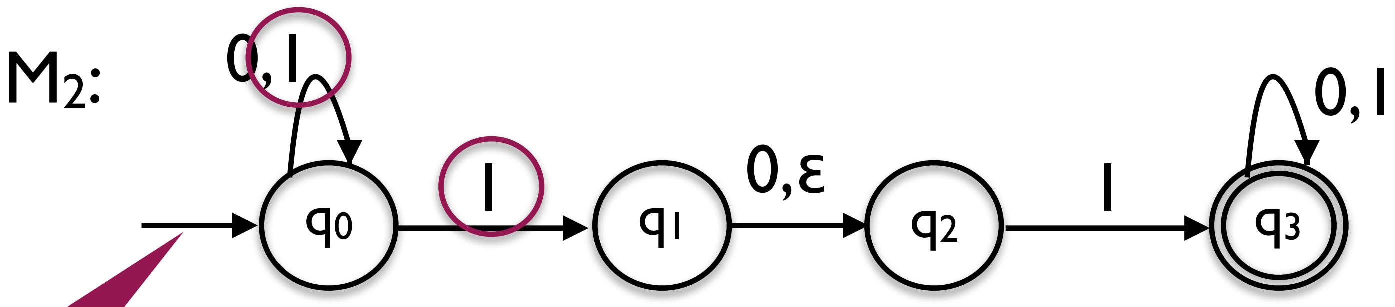

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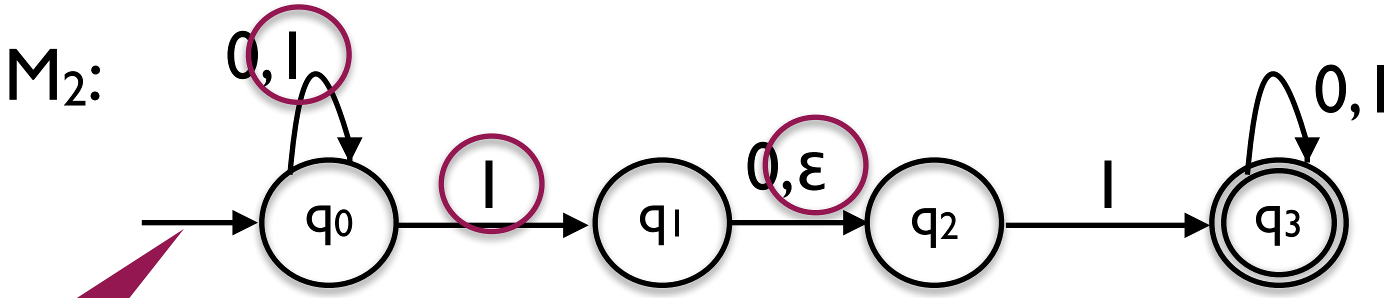

sources of nondeterminism

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no 1 transition

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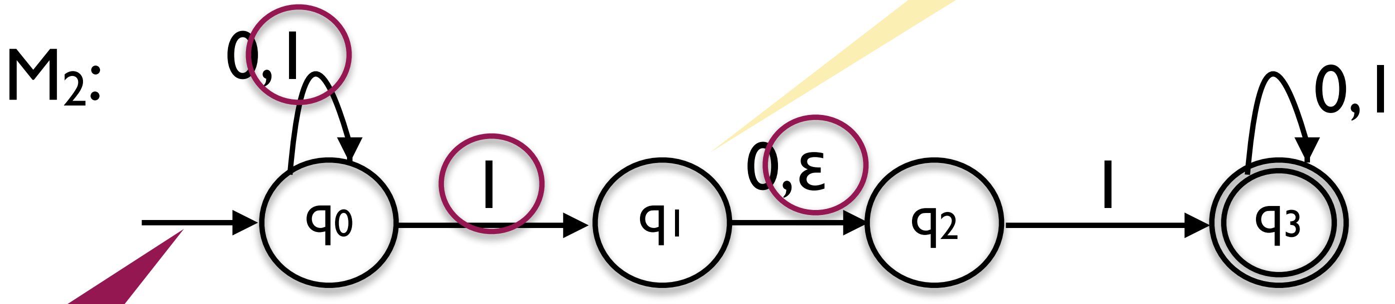

sources of nondeterminism

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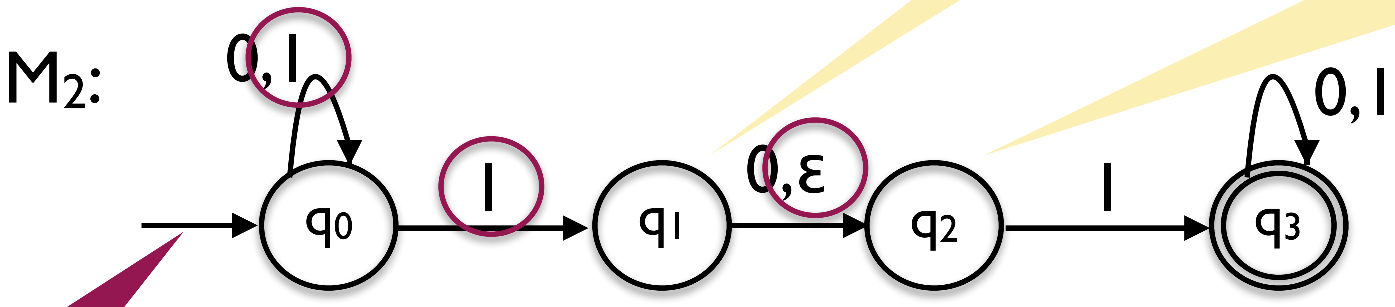

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sources of nondeterminism

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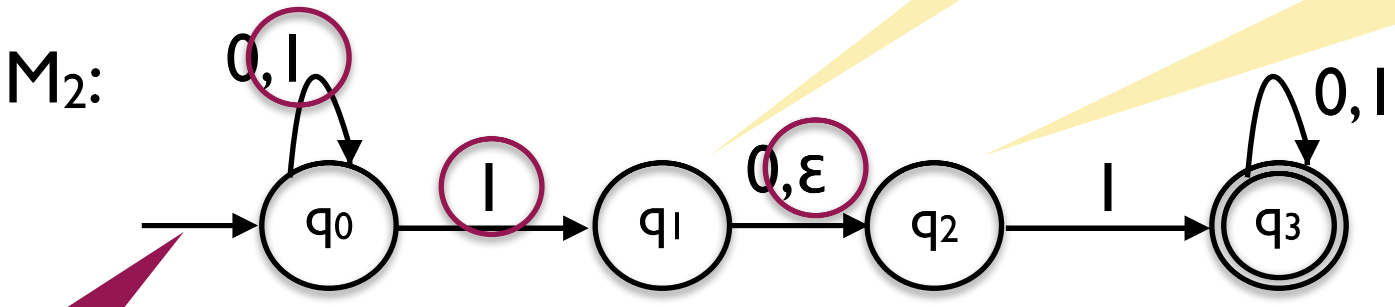

no 1 transition

no 0 transition

sources of nondeterminism

Accepts a word iff there exists an accepting run

NFA

Definition

A nondeterministic automaton M is a tuple M = $(Q, \Sigma, \delta, q_0, F)$ where

Q is a finite set of states
Σ is a finite alphabet
$\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function
q_0 is the initial state, $q_0 \in Q$
F is a set of final states, $F \subseteq Q$

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$\sum$ is a finite alphabet
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In the example M

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$\Sigma = \{0, 1\}$ $F = \{q_3\}$

$\delta(q_0, 0) = \{q_0\}$

$\delta(q_0, 1) = \{q_0, q_1\}$

$\delta(q_0, \varepsilon) = \varnothing$

…..

NFA

The extended transition function

NFA

The extended transition function

Given an NFA M = $(Q, \Sigma, \delta, q_0, F)$ we can extend $\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ to

$\delta^*: Q \times \Sigma^* \longrightarrow \mathcal{P}(Q)$

inductively, by:

$\delta^*(q, \varepsilon) = E(q)$ and $\delta^*(q,wa) = E(\bigcup_{q' \in \delta^*(q,w)} \delta(q', a))$

NFA

$E(q) = \{q' \mid q' = q \vee \exists n \in \mathbb{N}^{+}. \exists q_0, .., q_n \in Q. q_0 = q, q_n = q', q_{i+1} \in \delta(q_i, \varepsilon),$ for $i = 0, .., n-1\}$

The extended transition function

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ε-closure of q, all states reachable by ε-transitions from q

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NFA

ε-closure of q, all states reachable by ε-transitions from q

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$E(X) = \bigcup_{x \in X} E(x)$

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Definition

The language recognised / accepted by a nondeterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ is

$$L(M) = \{w \in \Sigma^* \mid \delta^*(q_0, w) \cap F \neq \emptyset\}$$

ε-closure of q, all states reachable by ε-transitions from q

NFA

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Definition

The language recognised / accepted by a
automaton M = $(Q, \Sigma, \delta, q_0, F)$ is

$L(M_2) = \{u101w \mid u,w \in \{0,1\}^*\}$
$\cup$
$\{u11w \mid u,w \in \{0,1\}^*\}$

$$L(M) = \{w \in \Sigma^* \mid \delta^*(q_0, w) \cap F \neq \emptyset\}$$

Equivalence of automata

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Two automata M_1 and M_2 are equivalent if $L(M_1) = L(M_2)$

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Proof via the "powerset construction" / determinization

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Proof via the "powerset construction" / determinization

Corollary

A language is regular iff it is recognised by a NFA

Closure under regular operations

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Theorem C1

The class of regular languages is closed under union

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Theorem C2

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Theorem C4

The class of regular languages is closed under Kleene star

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Theorem C4

The class of regular languages is closed under Kleene star

Now we can prove these too

Regular expressions

Definition

finite representation of infinite languages

Regular expressions

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finite representation of infinite languages

Regular expressions

Definition

Let ∑ be an alphabet. The following are regular expressions

1. a for $a \in \Sigma$
2. ε
3. $\varnothing$
4. $(R_1 \cup R_2)$ for R_1, R_2 regular expressions
5. $(R_1 \cdot R_2)$ for R_1, R_2 regular expressions
6. $(R_1)^*$ for R_1 regular expression

finite representation of infinite languages

Regular expressions

inductive

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finite representation of infinite languages

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example:
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corresponding languages

$$L(a) = \{a\}$$
$$L(\varepsilon) = \{\varepsilon\}$$
$$L(\varnothing) = \varnothing$$
$$L(R_1 \cup R_2) = L(R_1) \cup L(R_2)$$
$$L(R_1 \cdot R_2) = L(R_1) \cdot L(R_2)$$
$$L(R_1^*) = L(R_1)^*$$

Equivalence of regular expressions and regular languages

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Theorem (Kleene)

A language is regular (i.e., recognised by a finite automaton) iff it is the language of a regular expression.

Equivalence of regular expressions and regular languages

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A language is regular (i.e., recognised by a finite automaton) iff it is the language of a regular expression.

Proof $\Leftarrow$ easy, as the constructions for the closure properties,

$\Rightarrow$ not so easy, we'll skip it for now…

Nonregular languages

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Theorem (Pumping Lemma)

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every long enough word of a
regular language can be pumped

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Nonregular languages

every long enough word of a regular language can be pumped

Theorem (Pumping Lemma)

If L is a regular language, then there is a number p ∈ $\mathbb{N}$ (the pumping length) such that for any w ∈ L with |w| ≥ p, there exist x, y, z ∈ Σ^* such that w = xyz and

1. $xy^iz \in L$, for all $i \in \mathbb{N}$
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Example "corollary"

$L = \{ 0^n1^n \mid n \in \mathbb{N}\}$ is nonregular.

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Note the logical structure!