Termination in Convex Sets of Distributions

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CALCO 2017, Ljubljana, 14.6.2017

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Convex Algebras

infinitely many finitary operations

convex combinations

binary ones "suffice"

- algebras $\boxed{(A, \sum_{i=1}^{n} p_i(-)_i)}$ $\quad p_i \in [0,1], \sum_{i=1}^{n} p_i = 1$

- convex (affine) maps

$$\boxed{h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)}$$

satisfying

- Projection $\quad \sum_{i=1}^{n} p_i a_i = a_k, \quad p_k = 1$
- Barycenter $\quad \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} p_{i,j} a_j\right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} p_i p_{i,j}\right) a_j$

Eilenberg-Moore Algebras

convex algebras abstractly

$\mathcal{EM}(\mathcal{D})$

- objects

$$\begin{tikzcd} \mathcal{D}A \arrow[d, "a"] \\ A \end{tikzcd}$$

satisfying

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, equal, "a"] & \mathcal{D}A \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}\mathcal{D}A \arrow[r, "\mu"] \arrow[d, "\mathcal{D}a"'] & \mathcal{D}A \arrow[d, "a"] \\ \mathcal{D}A \arrow[r, "a"] & A \end{tikzcd}$$

- morphisms

$$\begin{tikzcd} \mathcal{D}A \arrow[d, "a"] \\ A \end{tikzcd} \xrightarrow{h} \begin{tikzcd} \mathcal{D}B \arrow[d, "b"] \\ B \end{tikzcd}$$

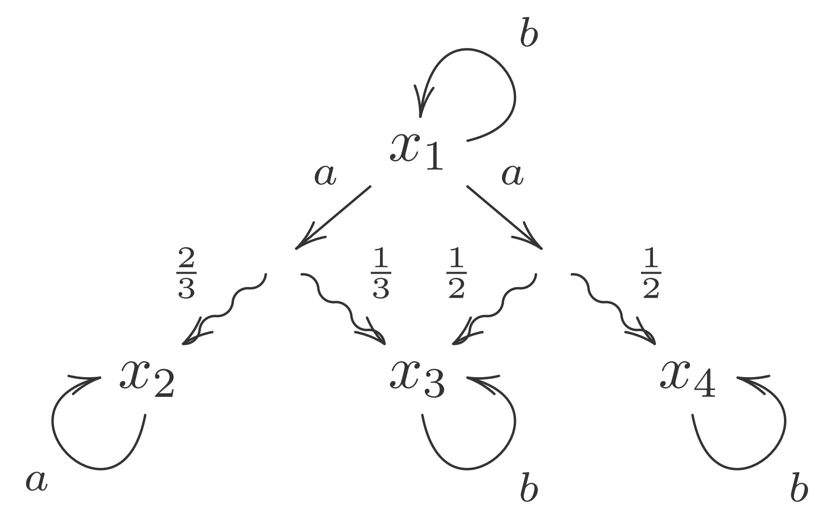
$$\begin{tikzcd} \mathcal{D}A \arrow[r, "\mathcal{D}h"] \arrow[d, "a"'] & \mathcal{D}B \arrow[d, "b"] \\ A \arrow[r, "h"] & B \end{tikzcd}$$

Probabilistic Automata

without termination

belief-state transformers

coalgebras on $\mathcal{EM}(\mathcal{D})$



possible behaviour

$$\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 \;\downarrow a\; \tfrac{2}{3}x_2 + \tfrac{1}{6}x_3 + \tfrac{1}{6}x_4$$

$$\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 \;\downarrow a\; \tfrac{8}{9}x_2 + \tfrac{1}{9}x_3$$

termination?

$$\tfrac{1}{3}x_1 + \tfrac{2}{3}x_3 \;\downarrow a\; ?$$

Probabilistic Automata

without termination

belief-state transformers

coalgebras on $\mathcal{EM}(\mathcal{D})$ with carriers free algebras

$$\mathcal{D}\mathcal{D}S \xrightarrow{\mu} \mathcal{D}S$$

free algebra

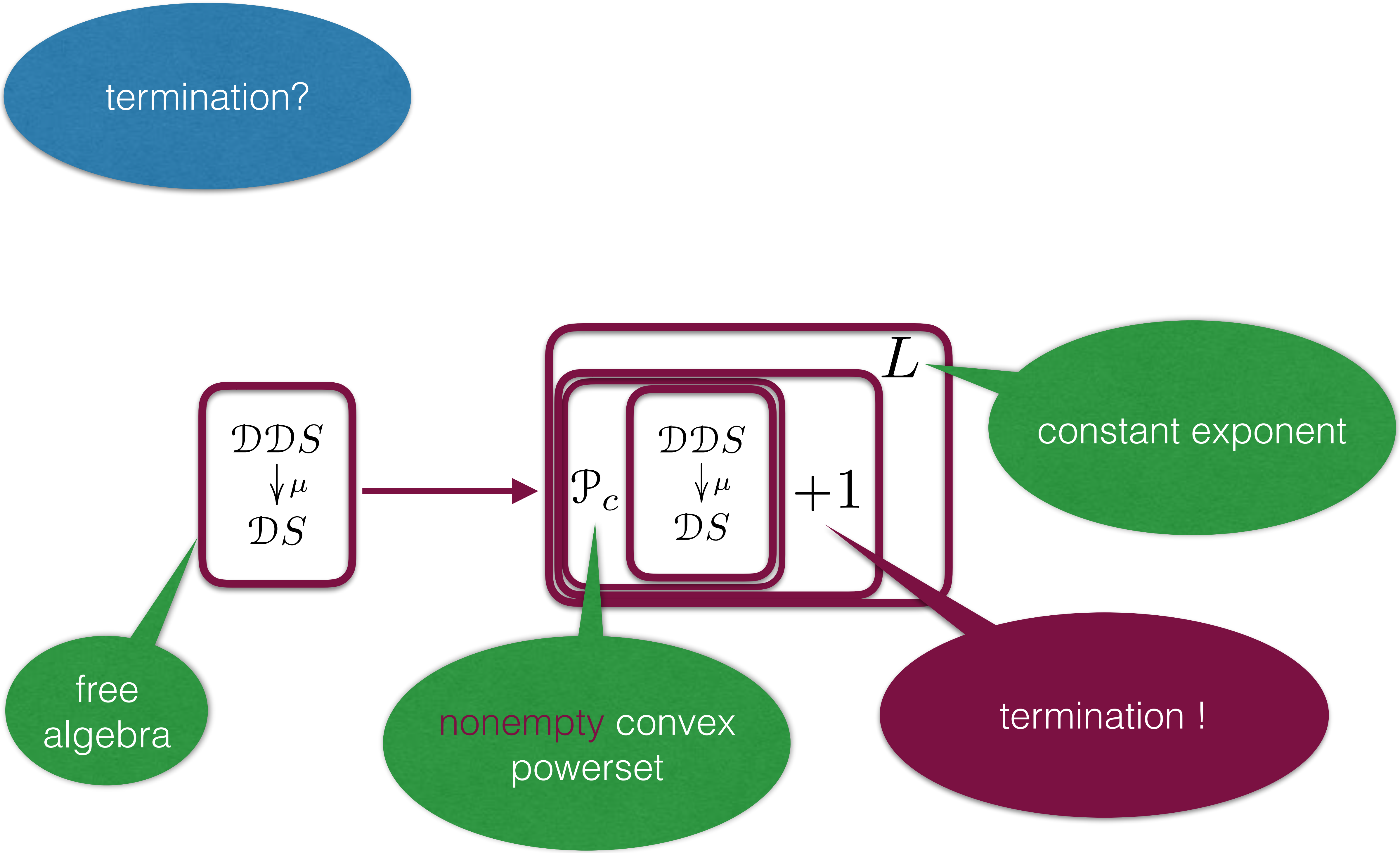
$$L\,\mathcal{P}_c\,\mathcal{D}\mathcal{D}S \xrightarrow{\mu} L\,\mathcal{P}_c\,\mathcal{D}S$$

constant exponent

nonempty convex powerset

[Bonchi Silva S. '17]

Probabilistic Automata

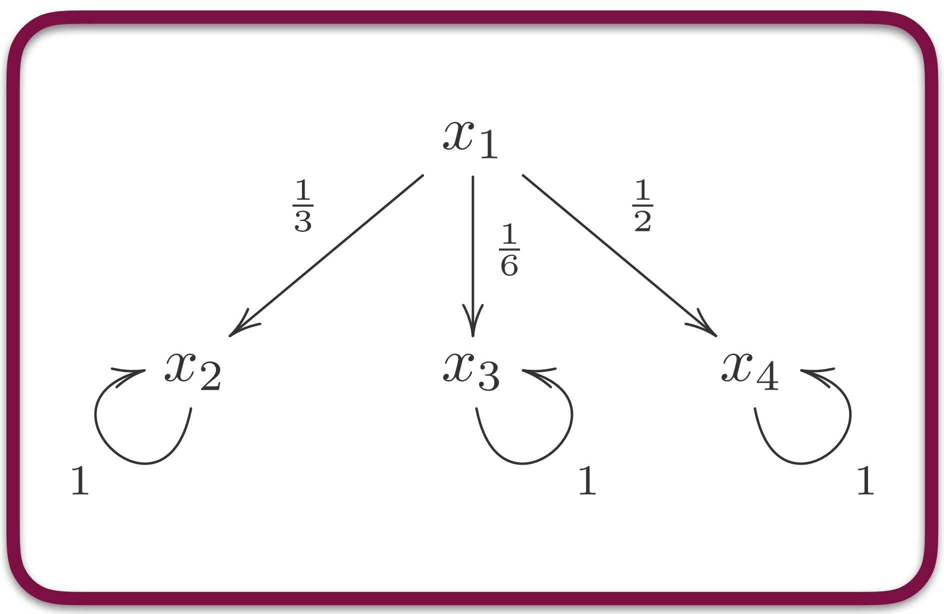


Markov Chains

no termination

belief-state transformers

coalgebras on $\mathcal{EM}(\mathcal{D})$



possible behaviour

$$\frac{1}{2}x_1 + \frac{1}{2}x_2$$
$$\downarrow$$
$$\frac{2}{3}x_2 + \frac{1}{12}x_3 + \frac{1}{4}x_4$$

Markov Chains

no termination

belief-state transformers

coalgebras on $\mathcal{EM}(\mathcal{D})$ with carriers free algebras

$$\begin{array}{c}\mathcal{DDS}\\ \downarrow \mu\\ \mathcal{DS}\end{array} \longrightarrow \begin{array}{c}\mathcal{DDS}\\ \downarrow \mu\\ \mathcal{DS}\end{array}$$

free algebra

for the identity functor

termination?

Markov Chains

termination?

$$\begin{array}{c}\mathcal{D}\mathcal{D}S\\ \big\downarrow{\scriptstyle\mu}\\ \mathcal{D}S\end{array} \longrightarrow \begin{array}{c}\mathcal{D}\mathcal{D}S\\ \big\downarrow{\scriptstyle\mu}\\ \mathcal{D}S\end{array} +1$$

free algebra

identity functor

termination !

The Problem

- Given a convex algebra, is it possible to extend it by a single element?

YES!

- If yes, what are all the possible ways?

there are many possible ways

we can give full description for…

single naturally functorial way

The Cases of Interest★

we can give full description for:

- Free convex algebras $\mathbb{D}_S = \boxed{\begin{array}{c} \mathcal{D}\mathcal{D}S \\ \downarrow \mu \\ \mathcal{D}S \end{array}}$

- Convex subsets of convex subsets★ of a vector space

nonempty

$\mathcal{P}_c\mathbb{D}$

convex subset of a vector space

in particular

$$\mathcal{P}_c\mathbb{D}_S = \mathcal{P}_c \boxed{\begin{array}{c} \mathcal{D}\mathcal{D}S \\ \downarrow \mu \\ \mathcal{D}S \end{array}}$$

Free convex algebras

carrier
$X_* = X + 1 = X \cup \{*\}$

Possible extensions $\mathbb{X}_*$ are:

- the black-hole extension

$$px + (1-p)* = *$$

- * imitates a point $w \in X$

$$px + (1-p)* = px + (1-p)w$$

- * imitates one of the extremal points $s \in S$ on all other points, and adheres this point

$$px + (1-p)* = px + (1-p)s, \quad x \neq s$$

$$ps + (1-p)* = *$$

these are all extensions!

Functoriality

Given a functor $F \colon \mathcal{EM}(\mathcal{D}) \to \mathcal{EM}(\mathcal{D})$ with

- $\mathbb{X} \leqslant F\mathbb{X}$
- $F\mathbb{X}$ has carrier $X + 1 = X \cup \{*\}$
- $$\begin{tikzcd} F\mathbb{X} \arrow[r, "Ff"] & F\mathbb{Y} \\ \mathbb{X} \arrow[u, "\iota_X"] \arrow[r, "f"'] & \mathbb{Y} \arrow[u, "\iota_Y"'] \end{tikzcd}$$

unique / single
functorial
extension

Then $F\mathbb{X}$ must be
the black-hole extension !

$\mathbb{X} = \mathcal{P}_c\mathbb{D}$

Convex subsets of…

Possible extensions $\mathbb{X}_*$ are:

- the black-hole extension
- * imitates a "point" $C \in \mathcal{P}_c(\mathrm{Vis}(D))$
- * imitates one of the extremal "points" of $\mathbb{D}$ on all other points, and adheres this point
- * imitates $C \in \mathcal{P}_c(\mathrm{Vis}(D))$ on P and adheres $X \backslash P$

visibility hull

these are all extensions if D - D is linearly bounded!

$|C| \geqslant 2$

$\mathrm{conv}\{A \in X \mid A \not\multimap\!\bullet C\} \subseteq P \neq X$

prime ideal

$\mathbb{X} = \mathcal{P}_c\mathbb{D}$

$D \subseteq V$

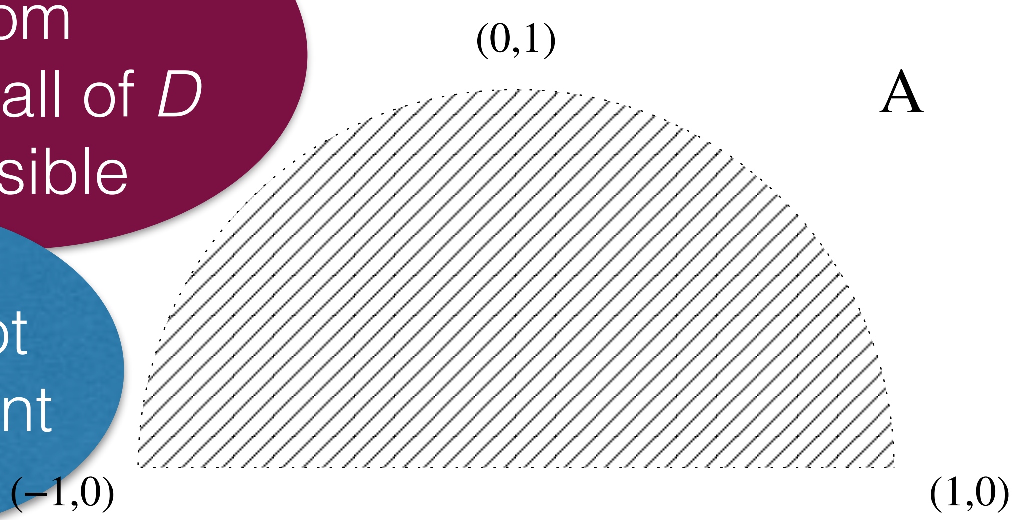
vector space

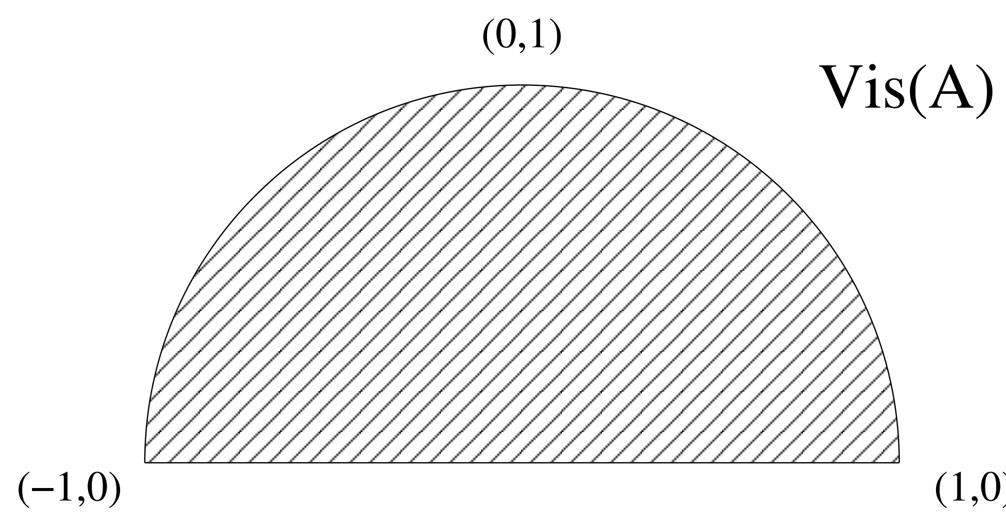
Visibility hull

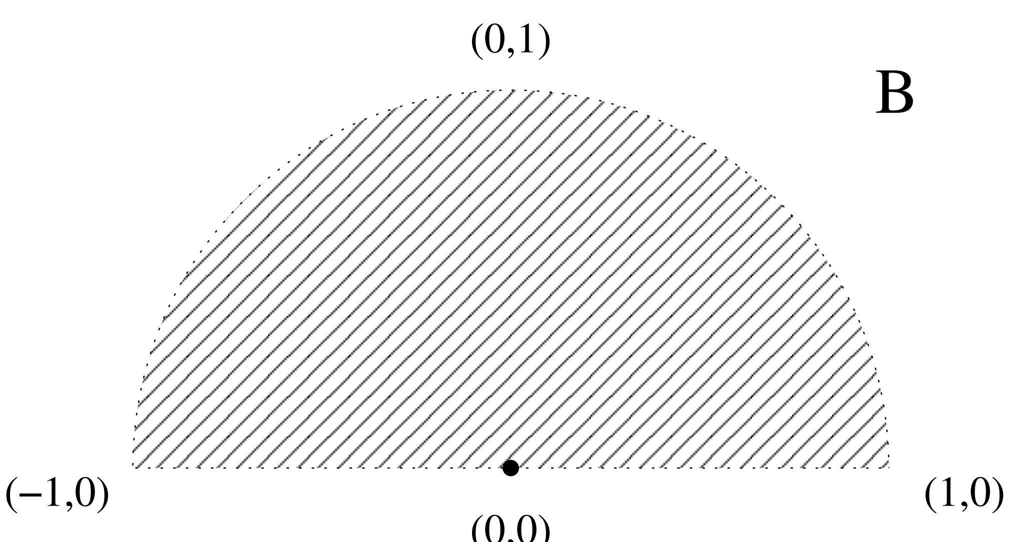
$\mathrm{Vis}(D) = \{v \in V \mid \forall d \in D.\ \forall p \in (0,1).\ pv + (1-p)d \in D\}$

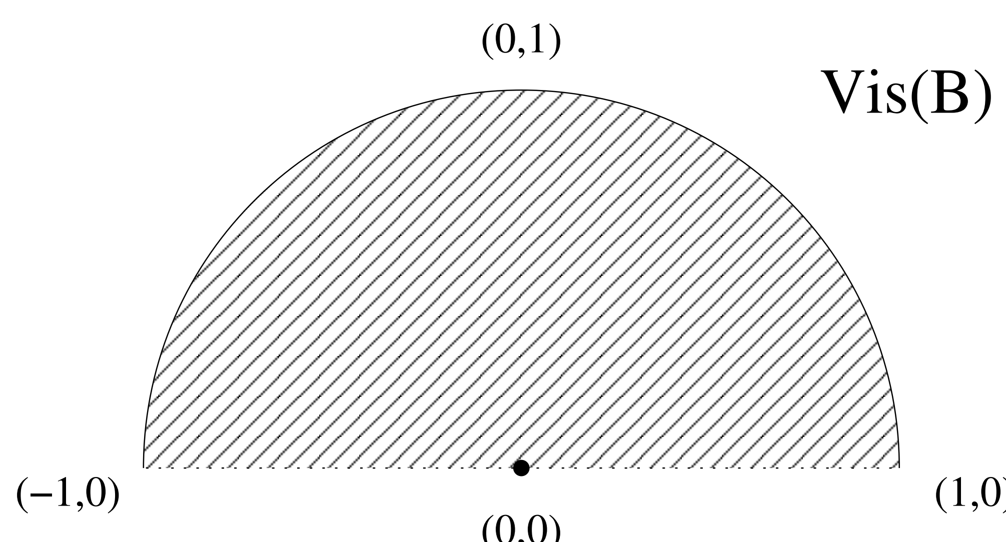
all points from which all of *D* is visible

$V \backslash D$ is not transparent









$\mathbb{X} = \mathcal{P}_c\mathbb{D}$

Convex subsets of…

Possible extensions $\mathbb{X}_*$ are:

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visibility hull

these are all extensions if D - D is linearly bounded!

$|C| \geqslant 2$

$\mathrm{conv}\{A \in X \mid A \mathrel{\circ\!\!\not\!\!-\!\!\bullet} C\} \subseteq P \neq X$

prime ideal

somewhat implicit description

Summing-up

Thank You!

- Convex algebras are important for the semantics of probabilistic systems

- We looked at one-point extensions, for termination.

  Every convex algebra can be extended by a single point

- What are all the possible ways?

  there are many possible ways

  we can give full description for…

  single naturally functorial way