

Logic

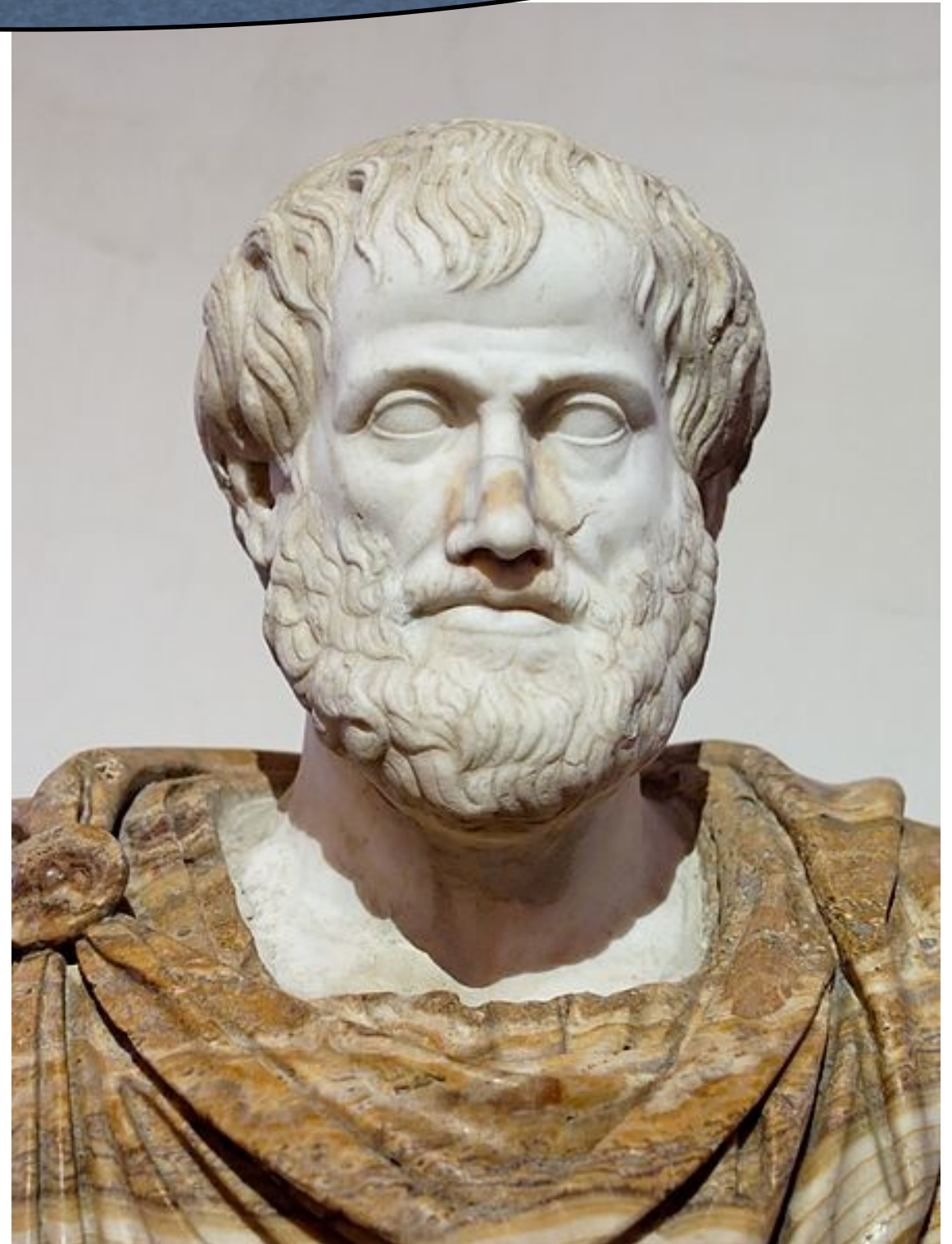
Logic = study of correct reasoning

In the beginning

Aristotle +/- 350 B.C.

Organon

19 syllogisms



Barbara syllogism

only later called so,
in the Middle Ages

All K's are L's
All L's are M's

All K's are M's

from the two
premises

one can
always conclude the
conclusion

independent of what the parameters K,L,M are

Logic (Logos, Greek for word, understanding, reason) deals with general reasoning laws in the form of formulas with parameters.

Propositions

Def. A proposition (**Aussage**) is a grammatically correct sentence that is either true or false.

Connectives

- \wedge for “and”
- \vee for “or”
- \neg for “not”
- \Rightarrow for “if .. then” or “implies”
- \Leftrightarrow for “if and only if”

logic deals with patterns!
what matters are not particular
propositions but the way in
which (abstract) propositions
are combined and what follows
from them

Abstract propositions

Definition

Basis Propositional variables are abstract propositions.

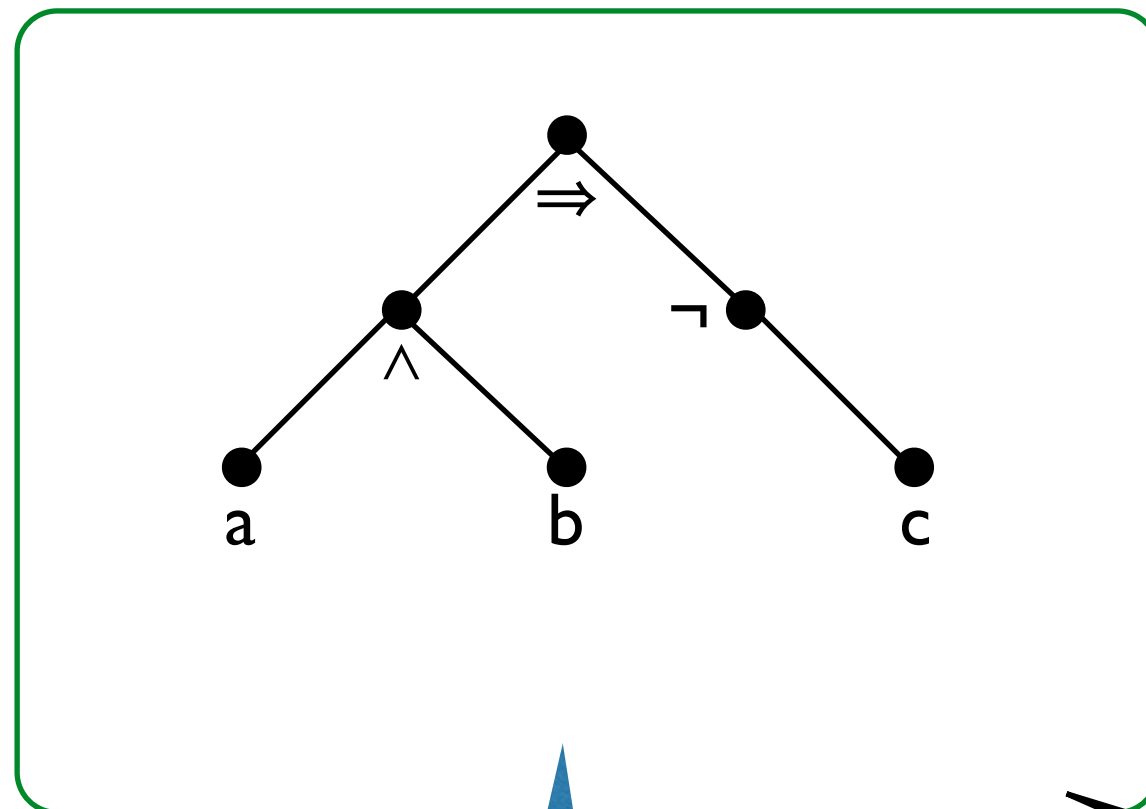
Step (Case 1) If P is an abstract proposition, then so is $(\neg P)$.

Step (Case 2) If P and Q are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.



a recursive/inductive
definition

...and their structure



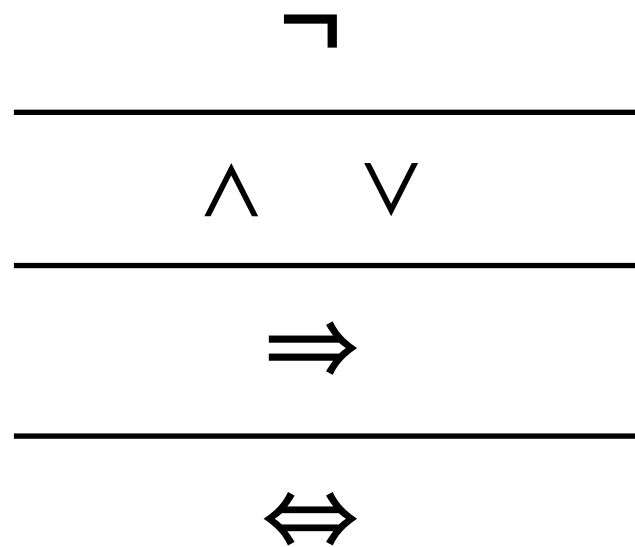
↑
building

↓
decomposing

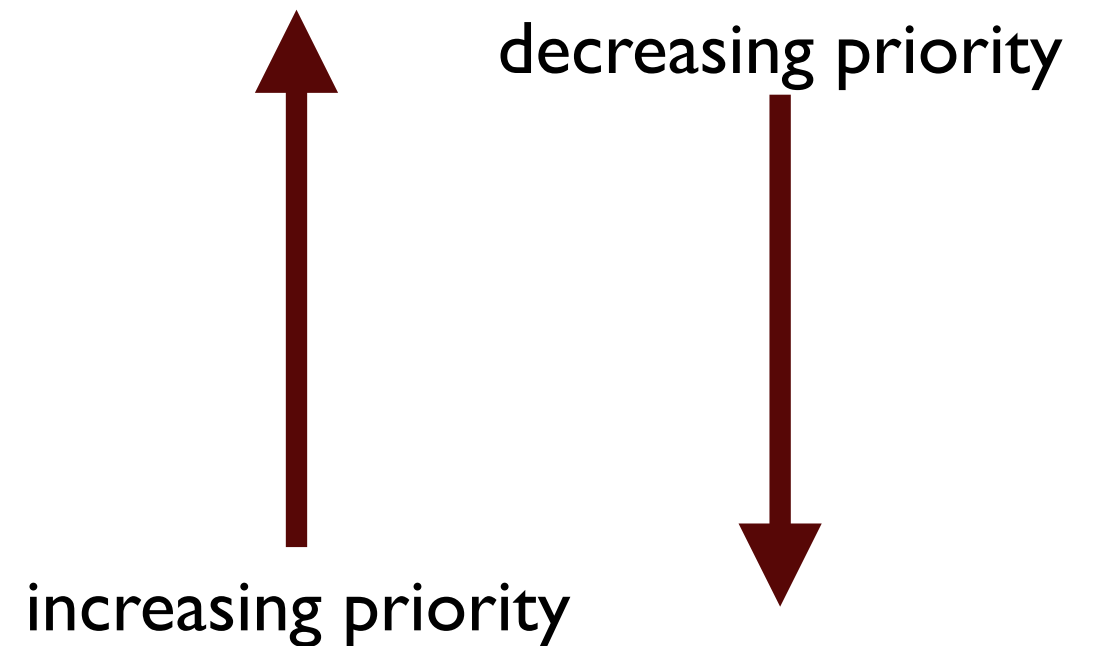
the tree of
 $((a \wedge b) \Rightarrow (\neg c))$

tree representation
(no need of
parenthesis)

Dropping parenthesis



priority schema
(top binds the most)



Example: $((a \wedge b) \Rightarrow (\neg c))$
becomes
 $a \wedge b \Rightarrow \neg c$

Truth tables

Conjunction

P	Q	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

only true when both
P and Q are true

Truth tables

Disjunction

P	Q	$P \vee Q$
0	0	0
0	1	1
1	0	1
1	1	1

true when either P
or Q or both are
true

Truth tables

Negation

unary connective

P	$\neg P$
0	1
1	0

true when P
is false

Truth tables

Implication

needs more attention

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

only false when P is true and Q is false

Truth tables

Bi-implication

$$P \Leftrightarrow Q$$

$$\text{is } (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

true when P and Q
have the same truth
value

Truth-functions

Def. A **truth-function** or **Boolean function** is a function

$$f: \{0,1\}^n \longrightarrow \{0,1\}$$

a_1, \dots, a_n are the variables in P (and more) ordered in a sequence

Property: Every abstract proposition $P(a_1, \dots, a_n)$ induces a truth-function.

by its inductive structure, using the truth tables

Notation in the book...

$$\left\{ \begin{array}{l} a, b \\ (0,0) \longmapsto 0 \\ (0,1) \longmapsto 1 \\ (1,0) \longmapsto 0 \\ (1,1) \longmapsto 1 \end{array} \right.$$

$$P(a,b): (a \wedge b) \vee b$$

Truth-functions

a_1, \dots, a_n are the variables in P (and more) ordered in a sequence

Property: Every abstract proposition $P(a_1, \dots, a_n)$ with ordered and specified variables induces a truth-function.

Note:

The sequence of specified variables matters!

$P(a,b,c): (a \wedge b) \vee b$

induces

a, b, c

$(0,0,0) \mapsto 0$

$(0,0,1) \mapsto 0$

$(0,1,0) \mapsto 1$

$(0,1,1) \mapsto 1$

$(1,0,0) \mapsto 0$

$(1,0,1) \mapsto 0$

$(1,1,0) \mapsto 1$

$(1,1,1) \mapsto 1$

Equivalence of propositions

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{\text{val}}{=} Q$, iff they induce the same truth-function

on any sequence containing their common variables

Property: The relation $\stackrel{\text{val}}{=}$ is an equivalence on the set of all abstract propositions

i.e., for all abstract propositions P, Q, R ,
(1) $P \stackrel{\text{val}}{=} P$; (2) if $P \stackrel{\text{val}}{=} Q$, then $Q \stackrel{\text{val}}{=} P$; and
(3) if $P \stackrel{\text{val}}{=} Q$ and $Q \stackrel{\text{val}}{=} R$, then $P \stackrel{\text{val}}{=} R$

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0				
0	1				
1	0				
1	1				

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1			
0	1	1			
1	0	0			
1	1	0			

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1		
0	1	1	0		
1	0	0	1		
1	1	0	0		

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1	0	
0	1	1	0	0	
1	0	0	1	0	
1	1	0	0	0	

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1	0	0
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1	0	0
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

Their truth values are the same, so they are equivalent

$$b \wedge \neg b \stackrel{val}{=} c \wedge \neg c$$

Tautologies and contradictions

Def. An abstract proposition P is a **tautology** iff its truth-function is constant 1.

all tautologies are equivalent

Def. An abstract proposition P is a **contradiction** iff its truth-function is constant 0.

all contradictions are equivalent

but not all contingencies!

Def. An abstract proposition P is a **contingency** iff it is neither a tautology nor a contradiction.

Abstract propositions

Definition

Basis (Case 1) T and F are abstract propositions.

Basis (Case 2) Propositional variables are abstract propositions.

Step (Case 1) If P is an abstract proposition, then so is $(\neg P)$.

Step (Case 2) If P and Q are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

a recursive/inductive
definition

Propositional Logic

Standard Equivalences

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

$$P \Rightarrow Q \stackrel{val}{\neq} Q \Rightarrow P$$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0	1	1	0

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0		

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0	0	1

Idempotence and Double Negation

Idempotence

$$P \wedge P \stackrel{val}{=} P$$

$$P \vee P \stackrel{val}{=} P$$

$$P \Rightarrow P \stackrel{val}{\neq} P$$

$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Idempotence and Double Negation

Idempotence

$$P \wedge P \stackrel{val}{=} P$$

$$P \vee P \stackrel{val}{=} P$$

$$P \Rightarrow P \stackrel{val}{\neq} P$$

$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Double negation

$$\neg\neg P \stackrel{val}{=} P$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

T/F - elimination

$$P \wedge T \stackrel{val}{=} P$$

$$P \wedge F \stackrel{val}{=} F$$

$$P \vee T \stackrel{val}{=} T$$

$$P \vee F \stackrel{val}{=} P$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

T/F - elimination

$$P \wedge T \stackrel{val}{=} P$$

$$P \wedge F \stackrel{val}{=} F$$

$$P \vee T \stackrel{val}{=} T$$

$$P \vee F \stackrel{val}{=} P$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \stackrel{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \stackrel{val}{=} (P \vee Q) \wedge (P \vee R)$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \stackrel{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \stackrel{val}{=} (P \vee Q) \wedge (P \vee R)$$



De Morgan

$$\neg(P \wedge Q) \stackrel{val}{=} \neg P \vee \neg Q$$

$$\neg(P \vee Q) \stackrel{val}{=} \neg P \wedge \neg Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \vee Q$$

$$P \vee Q \stackrel{val}{=} \neg P \Rightarrow Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \vee Q$$

$$P \vee Q \stackrel{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \stackrel{val}{=} \neg Q \Rightarrow \neg P$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \vee Q$$

$$P \vee Q \stackrel{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \stackrel{val}{=} \neg Q \Rightarrow \neg P$$

$$P \Rightarrow Q \stackrel{val}{\neq} \neg P \Rightarrow \neg Q$$

common
mistake!

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \stackrel{val}{=}$$

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \stackrel{val}{=} T$$

Calculating with equivalent propositions (the use of standard equivalences)

Recall...

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{\text{val}}{=} Q$, iff they induce the same truth-function

on any sequence containing their common variables

Property: The relation $\stackrel{\text{val}}{=}$ is an equivalence on the set of all abstract propositions.

i.e., for all abstract propositions P, Q, R ,
(1) $P \stackrel{\text{val}}{=} P$; (2) if $P \stackrel{\text{val}}{=} Q$, then $Q \stackrel{\text{val}}{=} P$; and
(3) if $P \stackrel{\text{val}}{=} Q$ and $Q \stackrel{\text{val}}{=} R$, then $P \stackrel{\text{val}}{=} R$

Substitution

meta rule

Simple

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]}$$

Sequential

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P][\eta/Q] \stackrel{val}{=} \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P, \eta/Q] \stackrel{val}{=} \psi[\xi/P, \eta/Q]}$$

EVERY
occurrence of P
is substituted!

The rule of Leibniz

Leibniz

$$\phi \stackrel{val}{=} \psi$$

$$C[\phi] \stackrel{val}{=} C[\psi]$$

meta rule

formula that has
 ϕ as a sub formula

single
occurrence is
replaced!

Strengthening
and
weakening

We had

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{\text{val}}{=} Q$, iff

- (1) Always when P has truth value 1, also Q has truth value 1, and
- (2) Always when Q has truth value 1, also P has truth value 1.



if we relax this,
we get
strengthening

Strengthening

Definition: The abstract proposition P is stronger than Q , notation $P \models^{\text{val}} Q$, iff

- ~~(1) Always when P has truth value 1, also Q has truth value 1, and~~
- ~~(2) Always when Q has truth value 1, also P has truth value 1.~~

Q is weaker than P

Strengthening

Definition: The abstract proposition P is stronger than Q ,
notation $P \models^{\text{val}} Q$, iff
always when P has truth value 1,
also Q has truth value 1.

always when P is true,
 Q is also true

Q is weaker
than P

Properties

Lemma EI: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Lemma EW1: $P \stackrel{val}{=} Q$ iff $P \stackrel{val}{\models} Q$ and $Q \stackrel{val}{\models} P$.

Lemma W2: $P \stackrel{val}{\models} P$

Lemma W3: If $P \stackrel{val}{\models} Q$ and $Q \stackrel{val}{\models} R$ then $P \stackrel{val}{\models} R$

Lemma W4: $P \stackrel{val}{\models} Q$ iff $P \Rightarrow Q$ is a tautology.

Standard Weakenings

and-or-weakening

$$P \wedge Q \stackrel{val}{\models} P$$

$$P \stackrel{val}{\models} P \vee Q$$

Extremes

$$\text{F} \stackrel{val}{\models} P$$

$$P \stackrel{val}{\models} \text{T}$$

Calculating with weakenings
(the use of standard weakenings)

Substitution

just holds

Simple

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]}$$

Sequential

$$\frac{\phi \stackrel{val}{\models} \psi}{\phi[\xi/P][\eta/Q] \stackrel{val}{\models} \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\phi \stackrel{val}{\models} \psi}{\phi[\xi/P, \eta/Q] \stackrel{val}{\models} \psi[\xi/P, \eta/Q]}$$

EVERY
occurrence of P
is substituted!

The rule of Leibnitz

Leibnitz

$$\phi \stackrel{val}{=} \psi$$

$$C[\phi] \stackrel{val}{=} C[\psi]$$

does not hold
for weakening!

formula that has
 ϕ as a sub formula

The rule of Leibniz

does not hold
for weakening!

Leibniz

$$\frac{\phi \stackrel{val}{\models} \psi}{C[\phi] \stackrel{val}{\models} C[\psi]}$$

Monotonicity

$$\frac{P \stackrel{val}{\models} Q}{P \wedge R \stackrel{val}{\models} Q \wedge R}$$

$$\frac{P \stackrel{val}{\models} Q}{P \vee R \stackrel{val}{\models} Q \vee R}$$