Probabilistic Systems Coalgebraically

Ana Sokolova
University of Salzburg

QAIS, University of Minho, Braga, 16.9.2013

We will discuss

- probabilistic systems
- their modelling as coalgebras
- coalgebraic results for probabilistic systems

We will discuss

- probabilistic systems
- their modelling as coalgebras
- coalgebraic results for probabilistic systems

specific results for
probabilistic systems

We will discuss

- probabilistic systems
- their modelling as coalgebras
- coalgebraic results for probabilistic systems

specific results for probabilistic systems

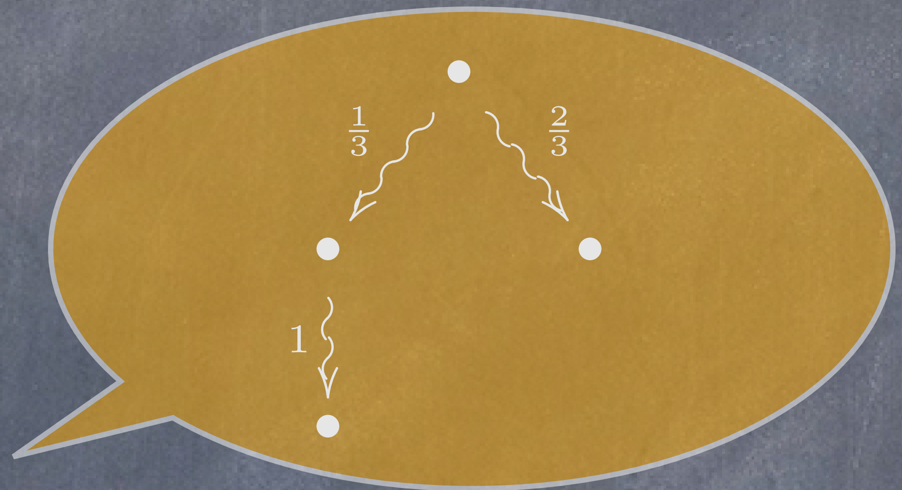
generic results: "probability" is just a parameter

Major distinction

- Discrete systems
  discrete probability distributions
- Continuous systems
  continuous state space/continuous measures

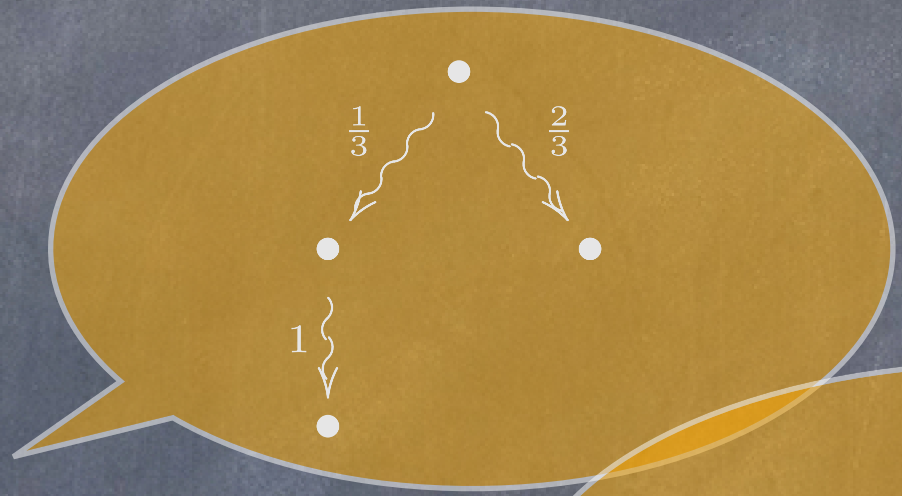
Major distinction

- Discrete systems
  discrete probability distributions

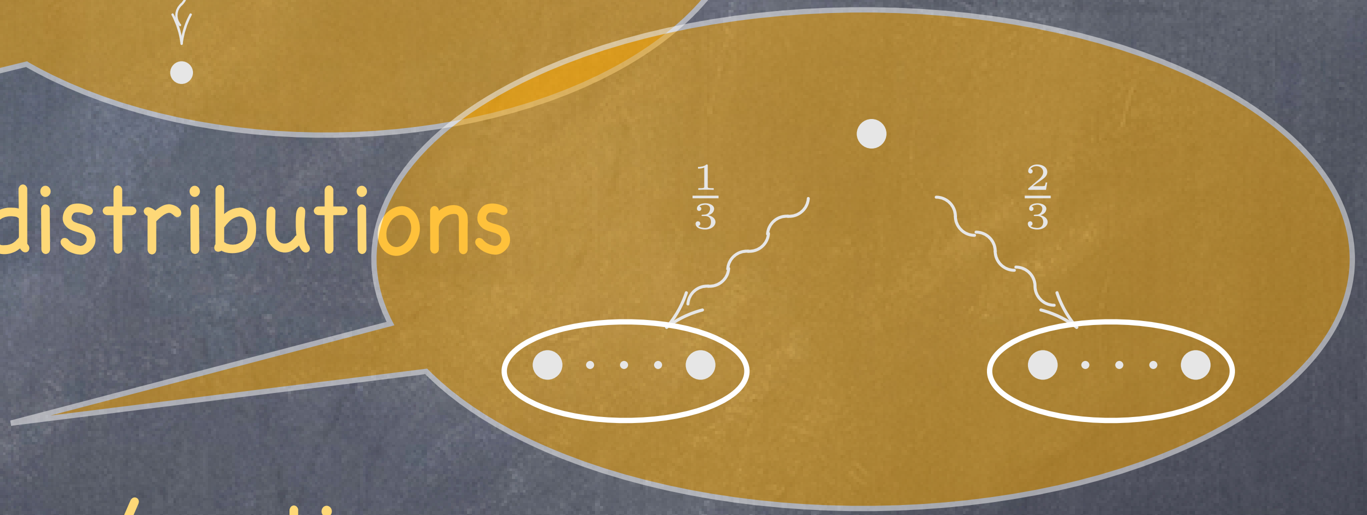


- Continuous systems
  continuous state space/continuous measures

Major distinction



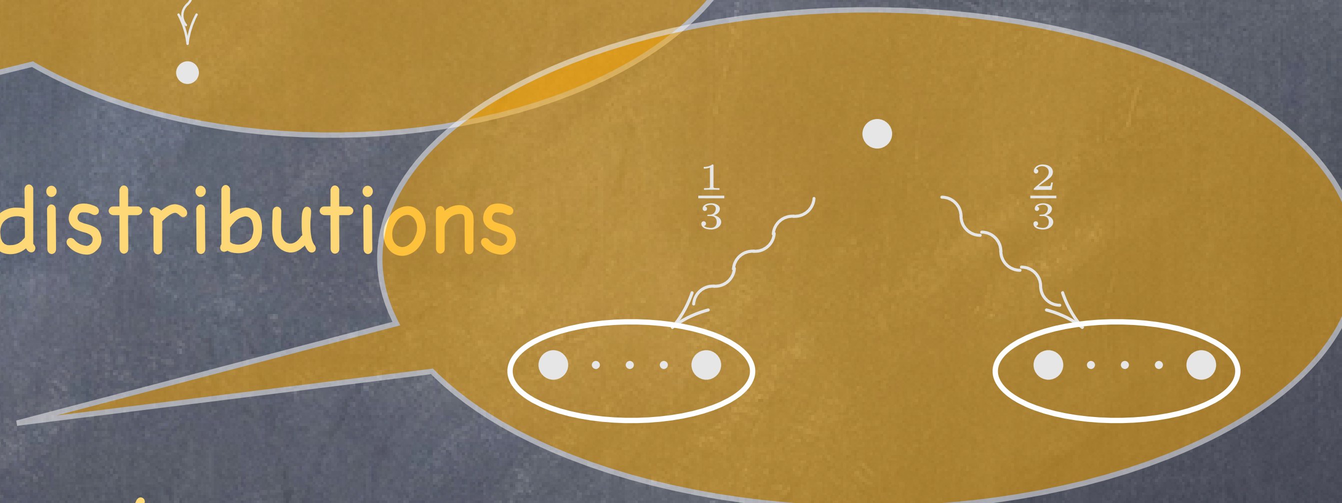
- Discrete systems
  discrete probability distributions



- Continuous systems
  continuous state space/continuous measures

Major distinction

- Discrete systems
  discrete probability distributions



- Continuous systems
  continuous state space/continuous measures

Major distinction

- Discrete systems
  discrete probability distributions
- Continuous systems
  continuous state space/continuous measures

Part 1

Discrete probabilistic systems

Modelling discrete probabilistic systems

Probability distribution functor on **Sets**

$$\mathcal{D}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

for $f : X \to Y$ we have $\mathcal{D}(f) : \mathcal{D}(X) \to \mathcal{D}(Y)$

$$\mathcal{D}(f)(\mu)(y) = \sum_{x \in f^{-1}(y)} \mu(x) = \mu(f^{-1}(y))$$

Modelling discrete probabilistic systems

Probability distribution functor on **Sets**

$$\mathcal{D}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

for $f : X \to Y$ we have $\mathcal{D}(f) : \mathcal{D}(X) \to \mathcal{D}(Y)$

preserves weak pullbacks

$$\mathcal{D}(f)(\mu)(y) = \sum_{x \in f^{-1}(y)} \mu(x) = \mu(f^{-1}(y))$$

Modelling discrete probabilistic systems

Probability distribution functor on **Sets**

$$\mathcal{D}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

and its variants

$$\mathcal{D}_{\leq 1}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) \leq 1\}$$

$$\mathcal{D}_f(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1, \ \mathsf{supp}(\mu) \text{ is finite}\}$$

Modelling discrete probabilistic systems

Probability distribution functor on **Sets**

$$\mathcal{D}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

and its variants

$$\mathcal{D}_{\leq 1}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) \leq 1\}$$

$\{x \in X \mid \mu(x) > 0\}$

$$\mathcal{D}_f(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1, \ \mathsf{supp}(\mu) \text{ is finite}\}$$

Modelling discrete probabilistic systems

Probability distribution functor on **Sets**

$$\mathcal{D}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

and its variants

$$\mathcal{D}_{\leq 1}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) \leq 1\}$$

$\{x \in X \mid \mu(x) > 0\}$

$$\mathcal{D}_f(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1, \text{ supp}(\mu) \text{ is finite}\}$$

has a final coalgebra

Modelling discrete probabilistic systems

Markov chains are $\mathcal{D}$-coalgebras on $\mathbf{Sets}$

$$\boxed{X \stackrel{c}{\to} \mathcal{D}(X)}$$

Modelling discrete probabilistic systems

Markov chains are $\mathcal{D}$-coalgebras on $\mathbf{Sets}$

$$\boxed{X \stackrel{c}{\to} \mathcal{D}(X)}$$

How about their coalgebraic bisimilarity?

Modelling discrete probabilistic systems

Markov chains are $\mathcal{D}$-coalgebras on $\mathbf{Sets}$

$$\boxed{X \stackrel{c}{\to} \mathcal{D}(X)}$$

$$\begin{tikzcd}
X \arrow[d, "c"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & X \arrow[d, "c"] \\
\mathcal{D}(X) & \mathcal{D}(R) \arrow[l, "\mathcal{D}(\pi_1)"'] \arrow[r, "\mathcal{D}(\pi_2)"] & \mathcal{D}(X)
\end{tikzcd}$$

How about their coalgebraic bisimilarity?

Modelling discrete probabilistic systems

Markov chains are $\mathcal{D}$-coalgebras on $\mathbf{Sets}$

$$\boxed{X \stackrel{c}{\to} \mathcal{D}(X)}$$

$$\begin{tikzcd}
X \arrow[d, "c"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & X \arrow[d, "c"] \\
\mathcal{D}(X) & \mathcal{D}(R) \arrow[l, "\mathcal{D}(\pi_1)"'] \arrow[r, "\mathcal{D}(\pi_2)"] & \mathcal{D}(X)
\end{tikzcd}$$

How about their coalgebraic bisimilarity?

Coincides with Larsen&Skou bisimilarity

de Vink&Rutten '99

Modelling discrete probabilistic systems

Markov chains are $\mathcal{D}$-coalgebras on **Sets**

$$\boxed{X \stackrel{c}{\to} \mathcal{D}(X)}$$

$$\begin{tikzcd} X \arrow[d, "c"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & X \arrow[d, "c"] \\ \mathcal{D}(X) & \mathcal{D}(R) \arrow[l, "\mathcal{D}(\pi_1)"'] \arrow[r, "\mathcal{D}(\pi_2)"] & \mathcal{D}(X) \end{tikzcd}$$

How about their coalgebraic bisimilarity?

$xRy \Rightarrow c(x)(C) = c(y)(C)$

C- equivalence class of R

Coincides with Larsen&Skou bisimilarity

de Vink&Rutten '99

Modelling discrete probabilistic systems

Markov chains are $\mathcal{D}$-coalgebras on $\mathbf{Sets}$

$$X \stackrel{c}{\to} \mathcal{D}(X)$$

$$\begin{tikzcd}
X \arrow[d, "c"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & X \arrow[d, "c"] \\
\mathcal{D}(X) & \mathcal{D}(R) \arrow[l, "\mathcal{D}(\pi_1)"'] \arrow[r, "\mathcal{D}(\pi_2)"] & \mathcal{D}(X)
\end{tikzcd}$$

How about their coalgebraic bisimilarity?

Coincides with Larsen&Skou bisimilarity

$$xRy \Rightarrow c(x)(C) = c(y)(C)$$

C- equivalence class of R

de Vink&Rutten '99

$$xRy \quad \Rightarrow \quad c(x) \equiv_R c(y)$$

Other discrete systems

Other discrete systems

- Markov chains have trivial bisimilarity, the final coalgebra is trivial

Other discrete systems

- Markov chains have trivial bisimilarity, the final coalgebra is trivial
- It gets more interesting with labels, termination, nondeterminism

Other discrete systems

- Markov chains have trivial bisimilarity, the final coalgebra is trivial
- It gets more interesting with labels, termination, nondeterminism
- (Almost) all known probabilistic systems can be modelled as coalgebras of functors built by the grammar

Other discrete systems

- Markov chains have trivial bisimilarity, the final coalgebra is trivial
- It gets more interesting with labels, termination, nondeterminism
- (Almost) all known probabilistic systems can be modelled as coalgebras of functors built by the grammar

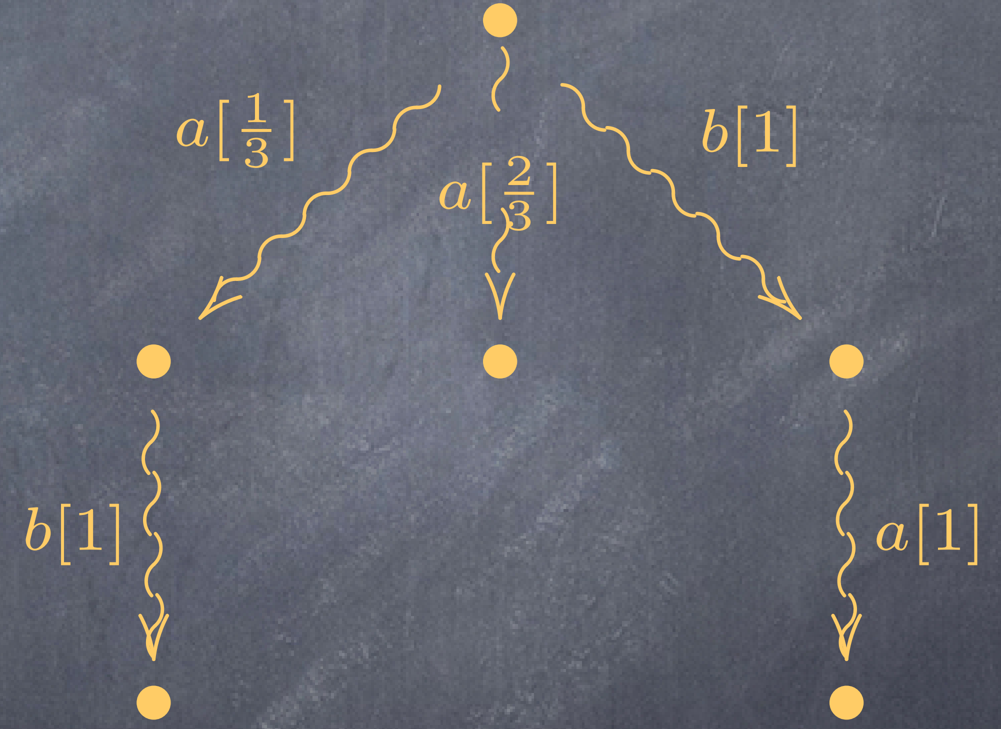
$$F := \_ \mid A \mid \mathcal{D} \mid \mathcal{P} \mid F^A \mid F + F \mid F \times F \mid F \circ F$$

Discrete system types

MC	$\mathcal{D}$
DLTS	$(\_+1)^A$
LTS	$\mathcal{P}(A\times\_) \cong \mathcal{P}^A$
React	$(\mathcal{D}+1)^A$
Gen	$\mathcal{D}(A\times\_)+1$
Str	$\mathcal{D}+(A\times\_)+1$
Alt	$\mathcal{D}+\mathcal{P}(A\times\_)$
Var	$\mathcal{D}(A\times\_)+\mathcal{P}(A\times\_)$
SSeg	$\mathcal{P}(A\times\mathcal{D})$
Seg	$\mathcal{P}\mathcal{D}(A\times\_)$
$\dots$	$\dots$

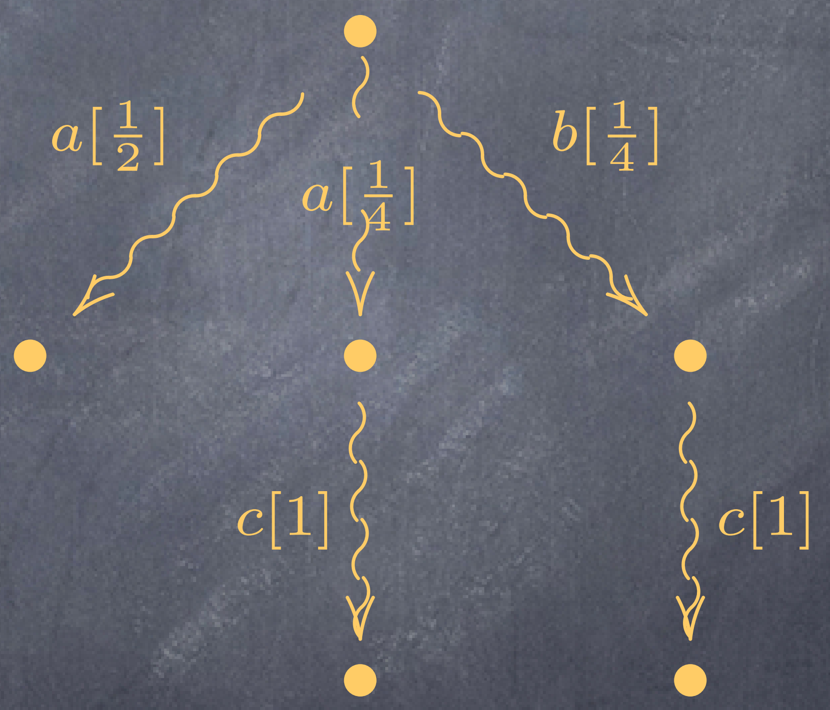
Discrete system types

MC	$\mathcal{D}$
DLTS	$(\_ + 1)^A$
LTS	$\mathcal{P}(A \times \_) \cong \mathcal{P}^A$
React	$(\mathcal{D} + 1)^A$
Gen	$\mathcal{D}(A \times \_) + 1$
Str	$\mathcal{D} + (A \times \_) + 1$
Alt	$\mathcal{D} + \mathcal{P}(A \times \_)$
Var	$\mathcal{D}(A \times \_) + \mathcal{P}(A \times \_)$
SSeg	$\mathcal{P}(A \times \mathcal{D})$
Seg	$\mathcal{PD}(A \times \_)$
$\ldots$	$\ldots$



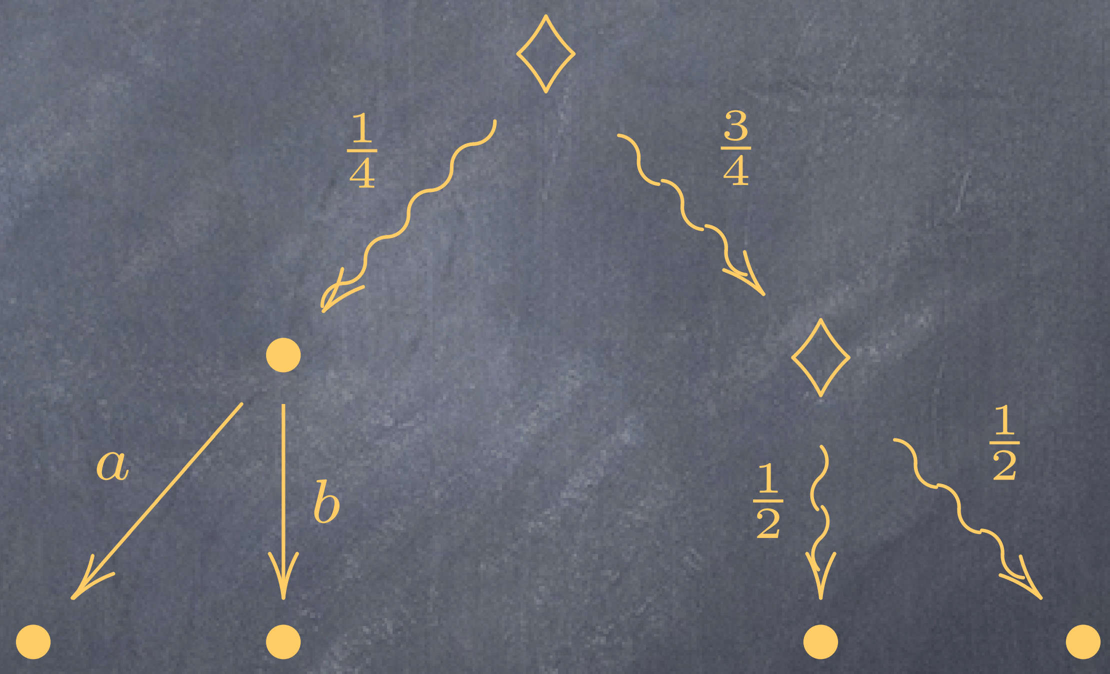
Discrete system types

MC	$\mathcal{D}$
DLTS	$(\_ + 1)^A$
LTS	$\mathcal{P}(A \times \_) \cong \mathcal{P}^A$
React	$(\mathcal{D} + 1)^A$
Gen	$\mathcal{D}(A \times \_) + 1$
Str	$\mathcal{D} + (A \times \_) + 1$
Alt	$\mathcal{D} + \mathcal{P}(A \times \_)$
Var	$\mathcal{D}(A \times \_) + \mathcal{P}(A \times \_)$
SSeg	$\mathcal{P}(A \times \mathcal{D})$
Seg	$\mathcal{P}\mathcal{D}(A \times \_)$
$\ldots$	$\ldots$



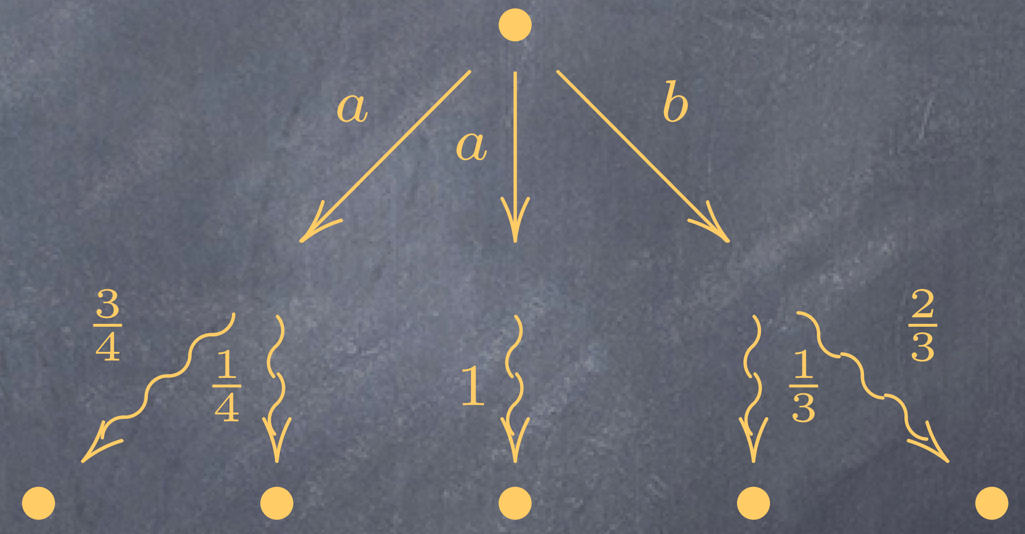
Discrete system types

MC	$\mathcal{D}$
DLTS	$(\_ + 1)^A$
LTS	$\mathcal{P}(A \times \_) \cong \mathcal{P}^A$
React	$(\mathcal{D} + 1)^A$
Gen	$\mathcal{D}(A \times \_) + 1$
Str	$\mathcal{D} + (A \times \_) + 1$
Alt	$\mathcal{D} + \mathcal{P}(A \times \_)$
Var	$\mathcal{D}(A \times \_) + \mathcal{P}(A \times \_)$
SSeg	$\mathcal{P}(A \times \mathcal{D})$
Seg	$\mathcal{P}\mathcal{D}(A \times \_)$
$\dots$	$\dots$



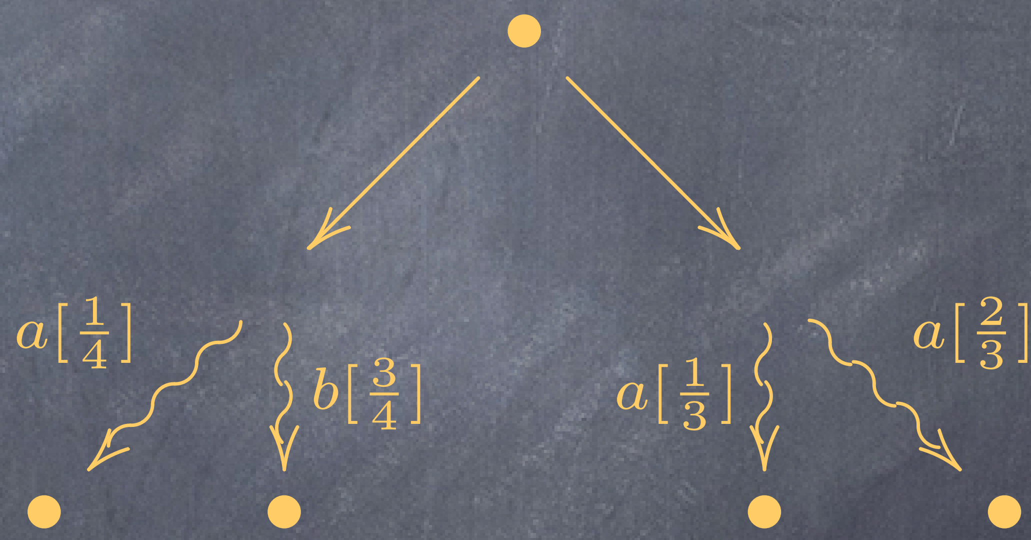
Discrete system types

MC	$\mathcal{D}$
DLTS	$(\_ + 1)^A$
LTS	$\mathcal{P}(A \times \_) \cong \mathcal{P}^A$
React	$(\mathcal{D} + 1)^A$
Gen	$\mathcal{D}(A \times \_) + 1$
Str	$\mathcal{D} + (A \times \_) + 1$
Alt	$\mathcal{D} + \mathcal{P}(A \times \_)$
Var	$\mathcal{D}(A \times \_) + \mathcal{P}(A \times \_)$
SSeg	$\mathcal{P}(A \times \mathcal{D})$
Seg	$\mathcal{P}\mathcal{D}(A \times \_)$
...	...



Discrete system types

MC	$\mathcal{D}$
DLTS	$(\_ + 1)^A$
LTS	$\mathcal{P}(A \times \_) \cong \mathcal{P}^A$
React	$(\mathcal{D} + 1)^A$
Gen	$\mathcal{D}(A \times \_) + 1$
Str	$\mathcal{D} + (A \times \_) + 1$
Alt	$\mathcal{D} + \mathcal{P}(A \times \_)$
Var	$\mathcal{D}(A \times \_) + \mathcal{P}(A \times \_)$
SSeg	$\mathcal{P}(A \times \mathcal{D})$
Seg	$\mathcal{P}\mathcal{D}(A \times \_)$
...	...



Discrete systems

- enter coalgebra, which provides a unifying framework
- become available as examples for generic coalgebra results
- all concrete probabilistic bisimulations (based on Larsen&Skou bisimulation) coincide with coalgebraic bisimulations
  Bartels,S.&deVink '03/'04
  S. '05

Discrete systems

- enter coalgebra, which provides a unifying framework
- become available as examples for generic coalgebra results
- all concrete probabilistic bisimulations (based on Larsen&Skou bisimulation) coincide with coalgebraic bisimulations
  Bartels,S.&deVink '03/'04
  S. '05

original proof: as in de Vink&Rutten

Discrete systems

- enter coalgebra, which provides a unifying framework
- become available as examples for generic coalgebra results
- all concrete probabilistic bisimulations (based on Larsen&Skou bisimulation) coincide with coalgebraic bisimulations
Bartels,S.&deVink '03/'04
S. '05

original proof: as in de Vink&Rutten

next version, simpler

Discrete systems

- enter coalgebra, which provides a unifying framework
- become available as examples for generic coalgebra results
- all concrete probabilistic bisimulations (based on Larsen&Skou bisimulation) coincide with coalgebraic bisimulations
  Bartels,S.&deVink '03/'04
  S. '05

original proof: as in de Vink&Rutten

next version, simpler

relation liftings

Discrete systems

- enter coalgebra, which provides a unifying framework

- become available as examples for generic coalgebra results

- all concrete probabilistic bisimulations (based on Larsen&Skou bisimulation) coincide with coalgebraic bisimulations
  Bartels,S.&deVink '03/'04
  S. '05

original proof: as in de Vink&Rutten

next version, simpler

relation liftings

modular, inductive proof

Bisimilarity for simple Segala automata

An equivalence R on the states of a simple Segala automaton is a bisimulation iff

$$\boxed{X \to \mathcal{P}(A \times \mathcal{D}(X))}$$

Bisimilarity for simple Segala automata

An equivalence R on the states of a simple Segala automaton is a bisimulation iff

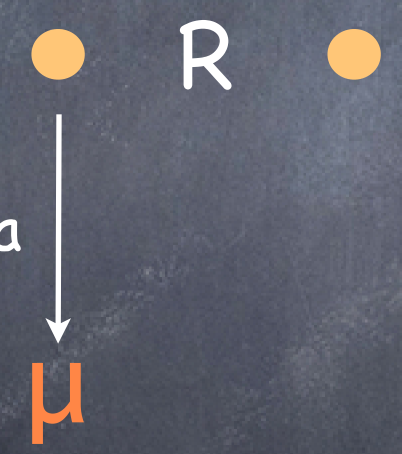
$\boxed{X \to \mathcal{P}(A \times \mathcal{D}(X))}$

- R -

Bisimilarity for simple Segala automata

An equivalence R on the states of a simple Segala automaton is a bisimulation iff

$$X \to \mathcal{P}(A \times \mathcal{D}(X))$$



Bisimilarity for simple Segala automata

An equivalence R on the states of a simple Segala automaton is a bisimulation iff

$$\boxed{X \to \mathcal{P}(A \times \mathcal{D}(X))}$$

$$\begin{tikzcd} \bullet \arrow[d, "a"'] \arrow[r, phantom, "R"] & \bullet \arrow[d, "a"] \\ \mu & \nu \end{tikzcd}$$

Bisimilarity for simple Segala automata

An equivalence R on the states of a simple Segala automaton is a bisimulation iff

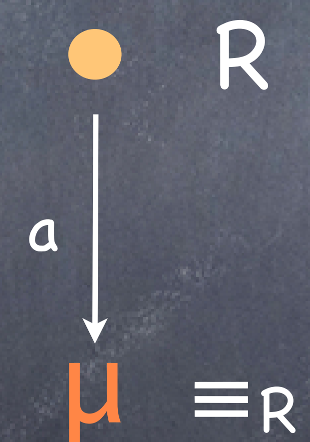
$$X \to \mathcal{P}(A \times \mathcal{D}(X))$$

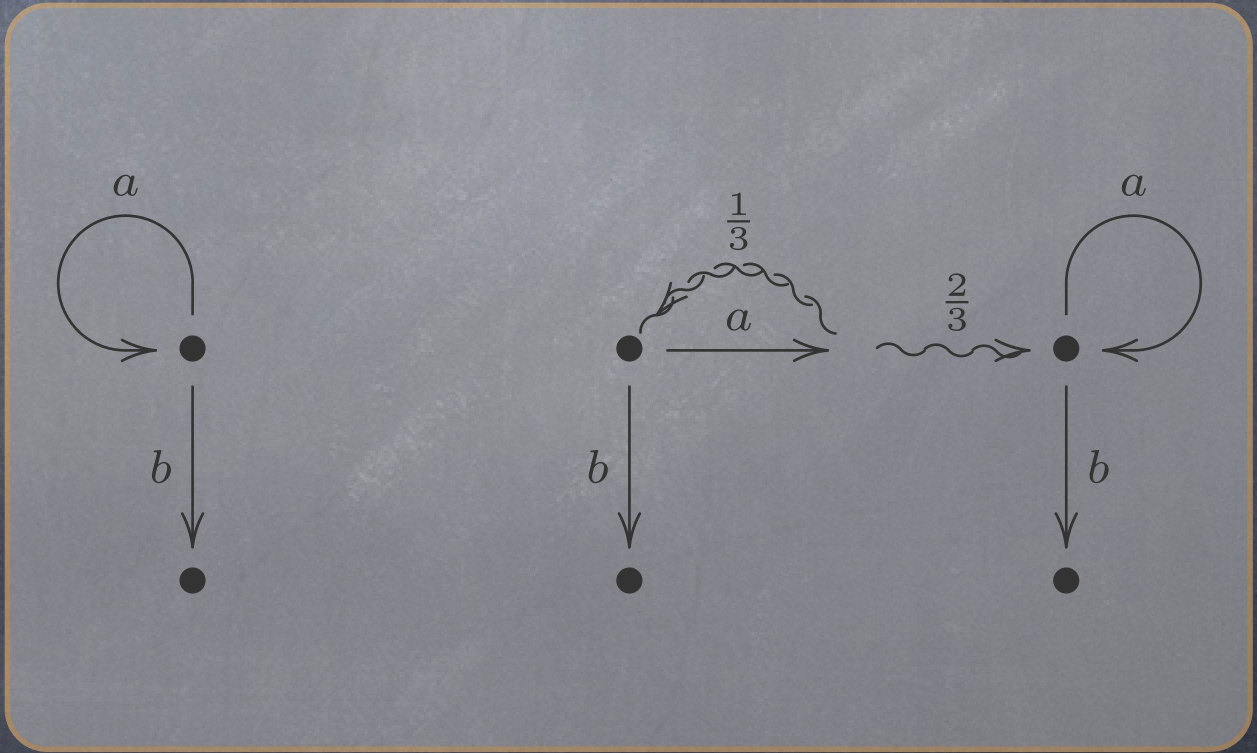
$$\begin{tikzcd} \bullet \arrow[r, phantom, "R"] \arrow[d, "a"'] & \bullet \arrow[d, "a"] \\ \mu \arrow[r, phantom, "\equiv_R"] & \nu \end{tikzcd}$$

Bisimilarity for simple Segala automata

An equivalence R on the states of a simple Segala automaton is a bisimulation iff

$$X \to \mathcal{P}(A \times \mathcal{D}(X))$$

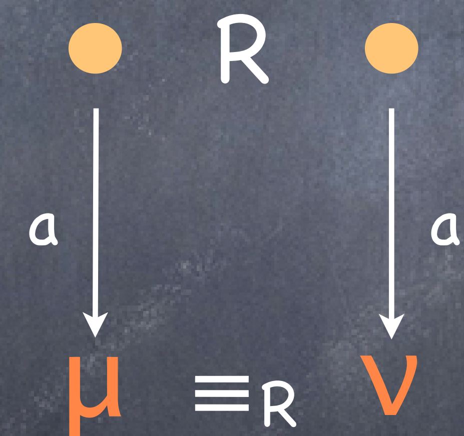


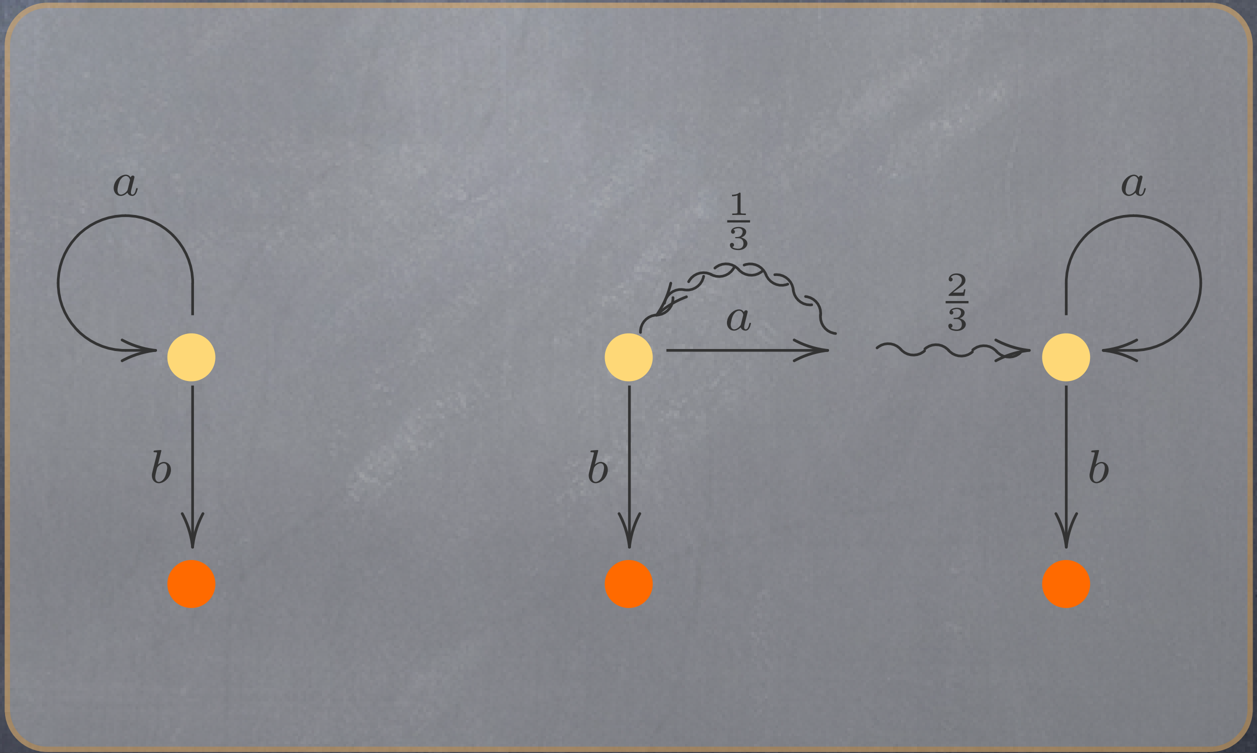


Bisimilarity for simple Segala automata

An equivalence R on the states of a simple Segala automaton is a bisimulation iff

$$X \to \mathcal{P}(A \times \mathcal{D}(X))$$

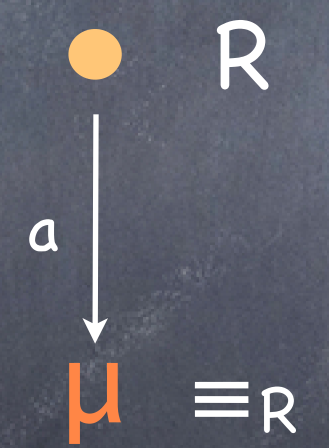




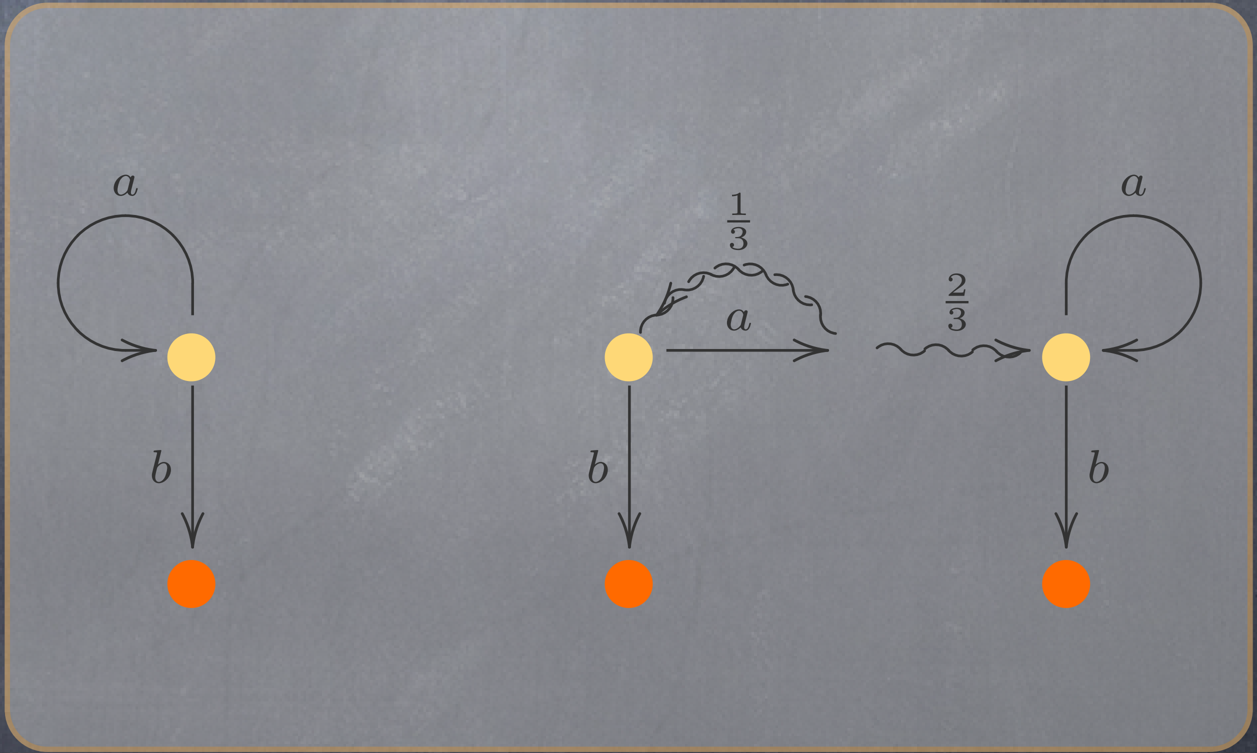
Bisimilarity for simple Segala automata

An equivalence R on the states of a simple Segala automaton is a bisimulation iff

$$X \to \mathcal{P}(A \times \mathcal{D}(X))$$



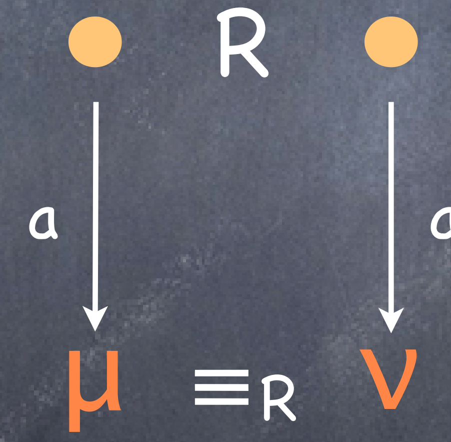
$$\begin{aligned} sRt \quad &\Rightarrow \quad \left(s \xrightarrow{a} \mu \Rightarrow (\exists \nu)\ t \xrightarrow{a} \nu,\ \mu \equiv_R \nu\right) \\ &\Leftrightarrow \quad \langle s, t\rangle \in \mathrm{Rel}(\mathcal{P}(A \times \mathcal{D}))(R) \end{aligned}$$

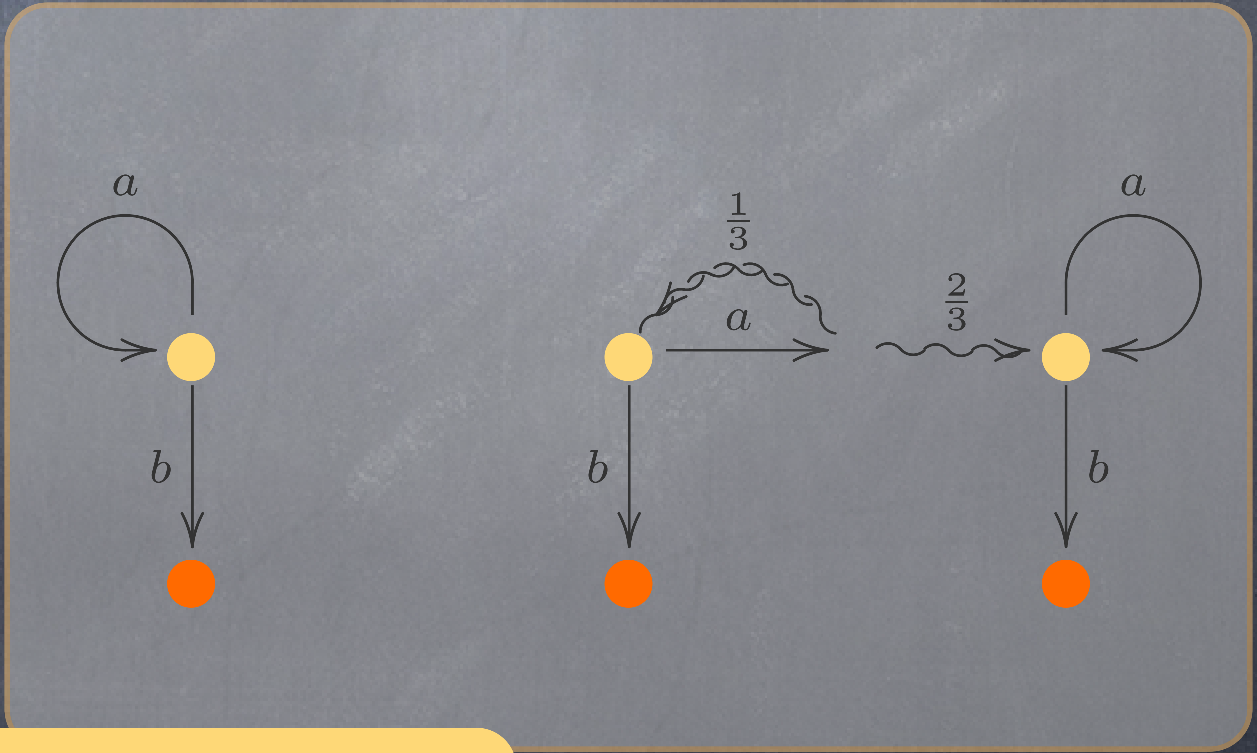


Bisimilarity for simple Segala automata

An equivalence R on the states of a simple Segala automaton is a bisimulation iff

$$X \to \mathcal{P}(A \times \mathcal{D}(X))$$



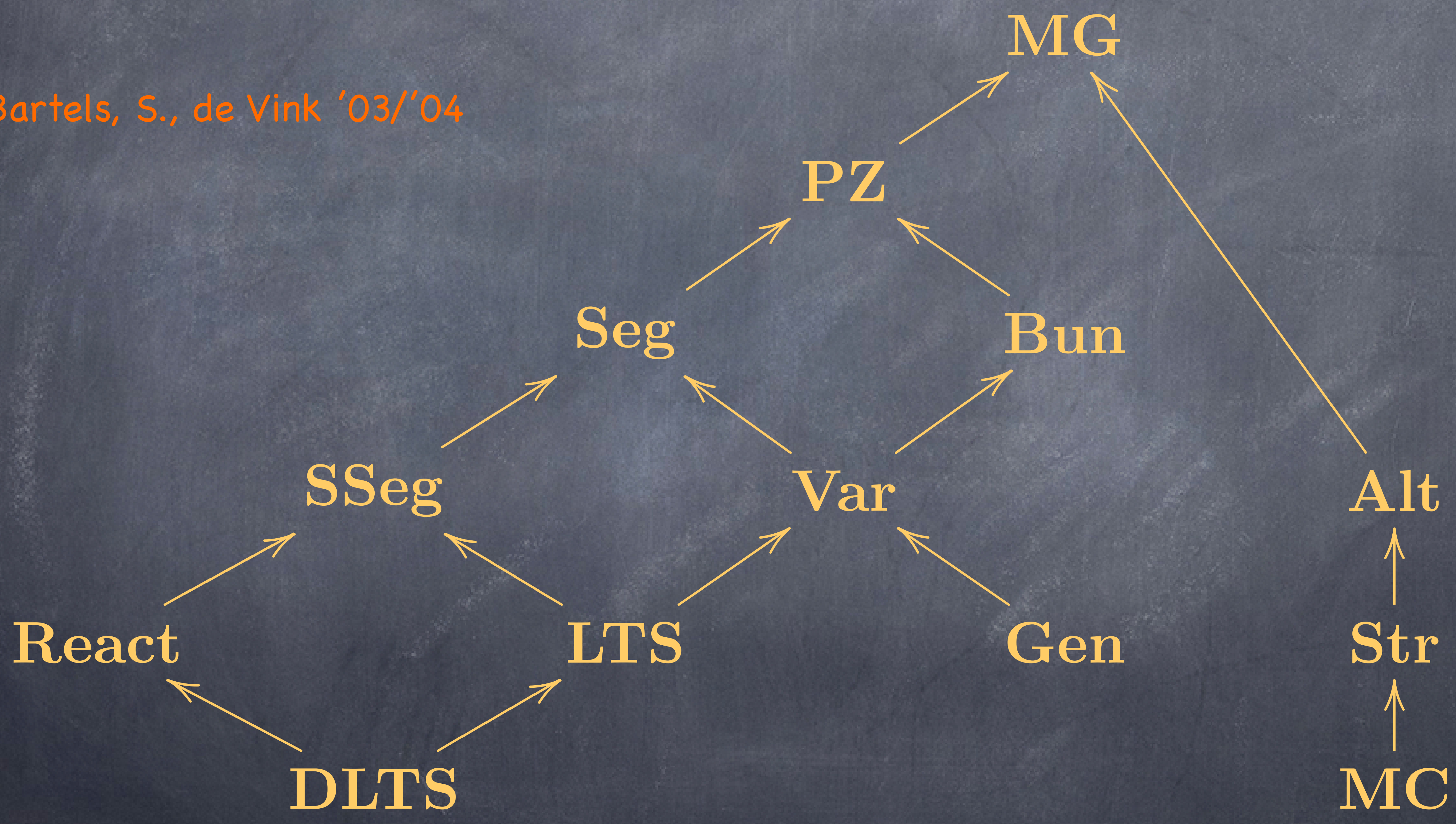


$$\begin{aligned} sRt \quad &\Rightarrow \quad \left(s \xrightarrow{a} \mu \Rightarrow (\exists \nu)\; t \xrightarrow{a} \nu,\ \mu \equiv_R \nu\right) \\ &\Leftrightarrow \quad \langle s, t\rangle \in \mathrm{Rel}(\mathcal{P}(A \times \mathcal{D}))(R) \end{aligned}$$

$$\equiv_R = \mathrm{Rel}(\mathcal{D})(R)$$

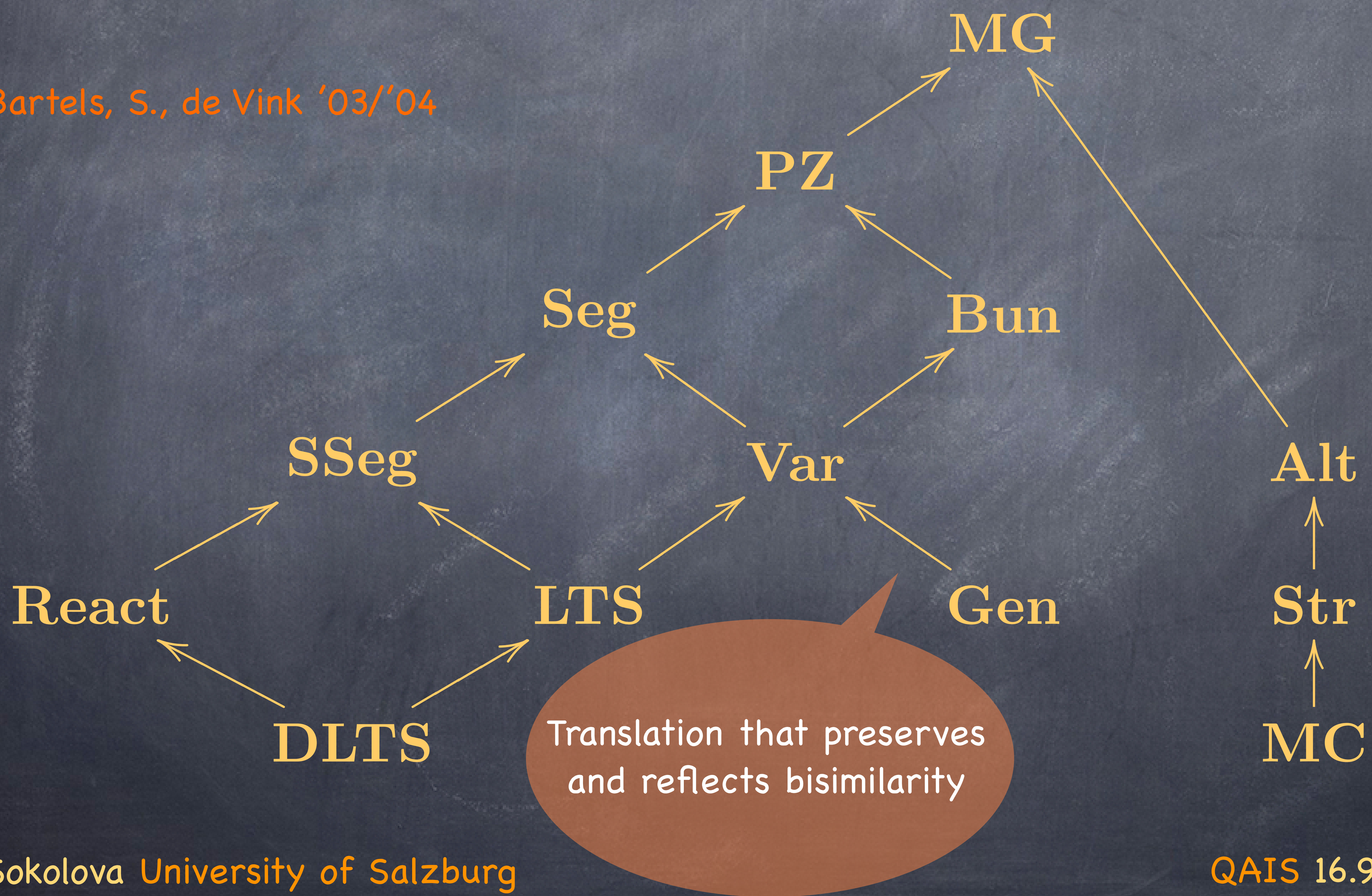
Expressiveness hierarchy

Bartels, S., de Vink '03/'04



Expressiveness hierarchy

Bartels, S., de Vink '03/'04



Expressivity comparison

Theorem If F preserves weak pullbacks and there is an injective natural transformation from F to G, then F-coalgebras $\longrightarrow$ G-coalgebras

Expressivity comparison

Theorem If F preserves weak pullbacks and there is an injective natural transformation from F to G, then F-coalgebras $\longrightarrow$ G-coalgebras

1. If there is an injective natural transformation from F to G, then it induces a translation that preserves and reflects behaviour equivalence

Expressivity comparison

Theorem If F preserves weak pullbacks and there is an injective natural transformation from F to G, then F-coalgebras $\longrightarrow$ G-coalgebras

1. If there is an injective natural t

from F to G, then it induces a transl

$$
\begin{tikzcd}
X \arrow[r, "u_1"] \arrow[d, "c"'] & U \arrow[d, "\gamma"] & X \arrow[l, "u_2"'] \arrow[d, "c"] \\
F(X) \arrow[r, "F(u_1)"] & F(U) & F(X) \arrow[l, "F(u_2)"']
\end{tikzcd}
$$

preserves and reflects behaviour equivalence

Expressivity comparison

Theorem If F preserves weak pullbacks and there is an injective natural transformation from F to G, then F-coalgebras $\longrightarrow$ G-coalgebras

1. If there is an injective natural from F to G, then it induces a transl preserves and reflects behaviour equivalence

$$\begin{tikzcd} X \arrow[r, "u_1"] \arrow[d, "c"'] & U \arrow[d, "\gamma"] & X \arrow[l, "u_2"'] \arrow[d, "c"] \\ F(X) \arrow[r, "F(u_1)"] & F(U) & F(X) \arrow[l, "F(u_2)"'] \end{tikzcd}$$

2. If F preserves weak pullbacks, then behaviour equivalence and bisimilarity coincide

Expressivity comparison

Theorem If F preserves weak pullbacks and there is an injective natural transformation from F to G, then F-coalgebras $\longrightarrow$ G-coalgebras

$$\begin{tikzcd} X \arrow[r, "u_1"] \arrow[d, "c"'] & U \arrow[d, "\gamma"] & X \arrow[l, "u_2"'] \arrow[d, "c"] \\ F(X) \arrow[r, "F(u_1)"] & F(U) & F(X) \arrow[l, "F(u_2)"'] \end{tikzcd}$$

1. If there is an injective natural t from F to G, then it induces a transla preserves and reflects behaviour equivalence

2. If F preserves weak pullbacks, then behaviour equivalence and bisimilarity coincide

bisimilarity always implies behaviour equivalence (pushouts)

Expressivity comparison

Theorem If F preserves weak pullbacks and there is an injective natural transformation from F to G, then F-coalgebras $\longrightarrow$ G-coalgebras

$$\begin{tikzcd} X \arrow[r, "u_1"] \arrow[d, "c"'] & U \arrow[d, "\gamma"] & X \arrow[l, "u_2"'] \arrow[d, "c"] \\ F(X) \arrow[r, "F(u_1)"] & F(U) & F(X) \arrow[l, "F(u_2)"'] \end{tikzcd}$$

1. If there is an injective natural
from F to G, then it induces a transl ion
preserves and reflects behaviour equivalence

2. If F preserves weak pullbacks, then behaviour equivalence and bisimilarity coincide

bisimilarity always implies behaviour equivalence (pushouts)

if not, behaviour equivalence is better

Example embedding

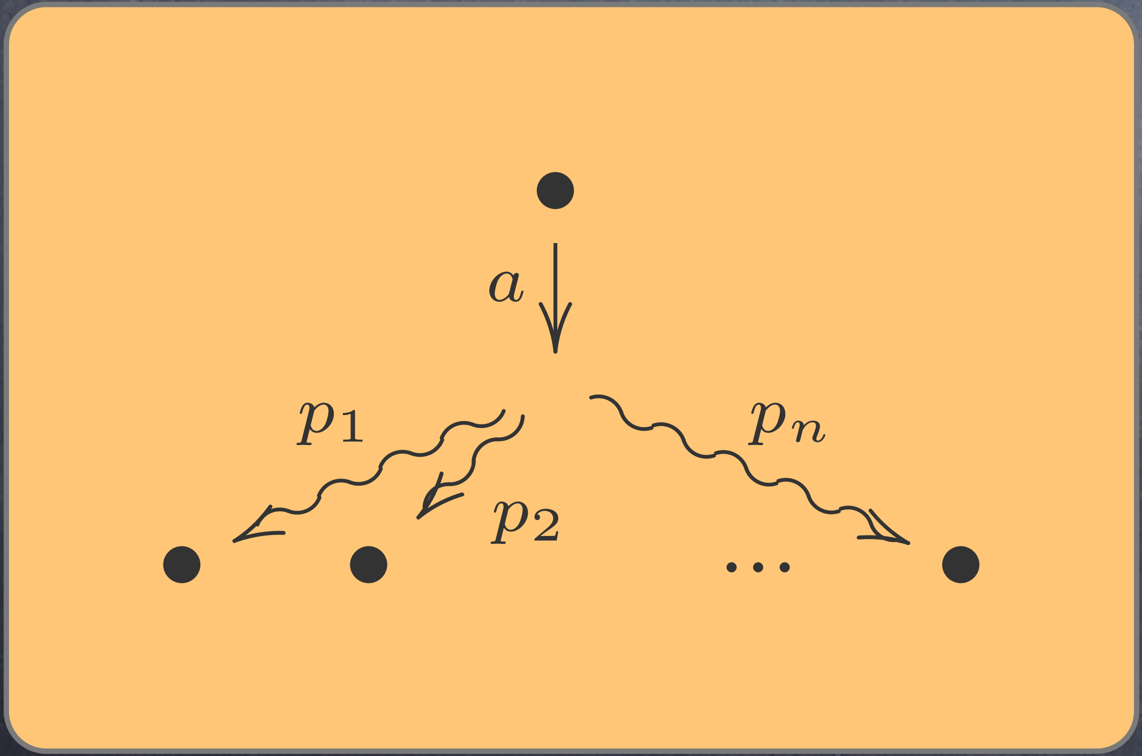
simple Segala system $\longrightarrow$ Segala system

$\mathcal{P}(A \times \mathcal{D})$ $\qquad$ $\mathcal{P}\mathcal{D}(A \times \_)$

Example embedding

simple Segala system $\longrightarrow$ Segala system

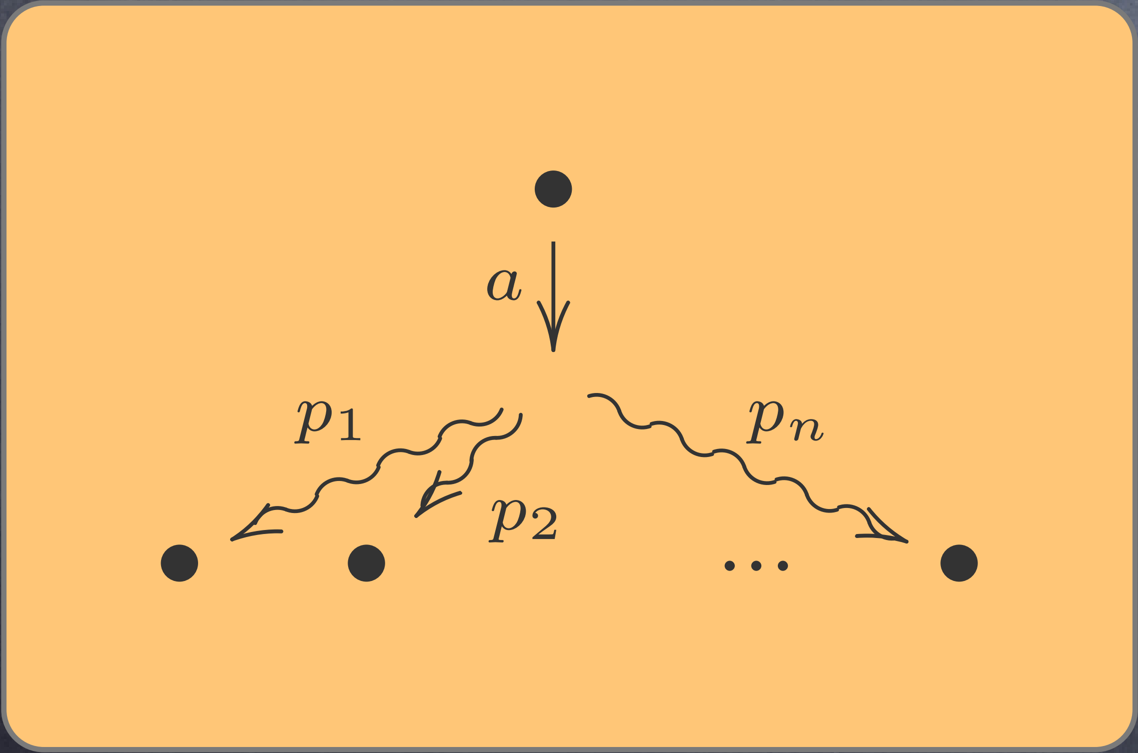
$\mathcal{P}(A \times \mathcal{D})$ $\qquad$ $\mathcal{P}\mathcal{D}(A \times \_)$

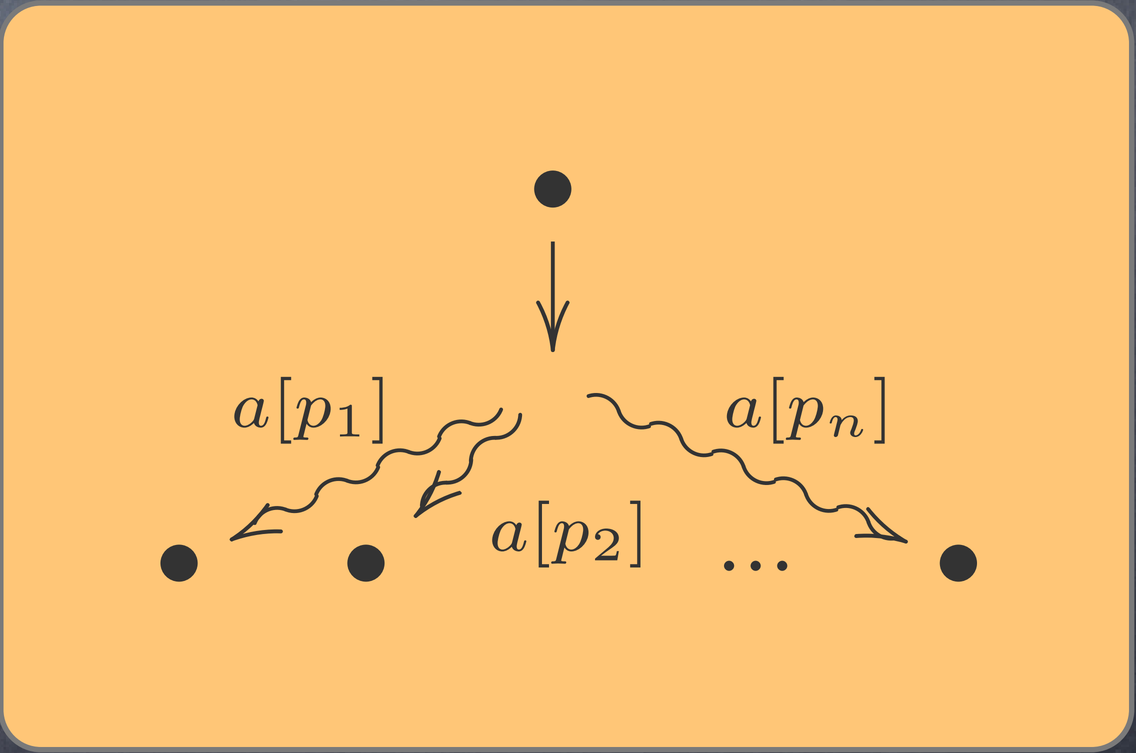


Example embedding

simple Segala system $\longrightarrow$ Segala system

$\mathcal{P}(A \times \mathcal{D})$ $\qquad$ $\mathcal{PD}(A \times \_)$





Basic natural transformations

- $\eta : 1 \Rightarrow \mathcal{P}$ with $\eta_X(*) := \emptyset$,
- $\sigma : \_ \Rightarrow \mathcal{P}$ with $\sigma_X(x) := \{x\}$
- $\delta : \_ \Rightarrow \mathcal{D}$ with $\delta_X(x) := \delta_x$ (Dirac),
- $\iota_l : \mathcal{F} \Rightarrow \mathcal{F} + \mathcal{G}$ and $\iota_r : \mathcal{G} \Rightarrow \mathcal{F} + \mathcal{G}$,
- $\phi + \psi : \mathcal{F} + \mathcal{G} \Rightarrow \mathcal{F}' + \mathcal{G}'$ for $\phi : \mathcal{F} \Rightarrow \mathcal{F}'$ and $\psi : \mathcal{G} \Rightarrow \mathcal{G}'$ (both with i.c.),
- $\kappa : \mathrm{A} \times \mathcal{P} \Rightarrow \mathcal{P}(\mathrm{A} \times \_)$ with $\kappa_X(a, M) := \{\langle a, x\rangle \mid x \in M\}$,
- $\ldots$

Specific coalgebra results

Specific coalgebra results

- Probabilistic GSOS Bartels'02/'04
  Stochastic/weighted GSOS Klin&Sassone'08, Klin'09

Specific coalgebra results

- Probabilistic GSOS Bartels'02/'04
  Stochastic/weighted GSOS Klin&Sassone'08, Klin'09
- Monad for probability and nondeterminism (a vicious combination) Varacca'02, Varacca&Winskel'06

Specific coalgebra results

- Probabilistic GSOS Bartels'02/'04
  Stochastic/weighted GSOS Klin&
  Klin'09

- Monad for probability and nondeter
  (a vicious combination) Varacca'02, Varacca&Winskel'06

$\mathcal{P}$,$\mathcal{D}$ are monads, but $\mathcal{PD}$ is not

Specific coalgebra results

- Probabilistic GSOS Bartels'02/'04
  Stochastic/weighted GSOS Klin&
  Klin'09

$\mathcal{P}$,$\mathcal{D}$ are monads,
but $\mathcal{PD}$ is not

- Monad for probability and
  (a vicious combination) Varacca'02, Varacca&Winskel'06

  used for traces of systems with probability and nondeterminism Jacobs'08

Specific coalgebra results

- Probabilistic GSOS Bartels'02/'04
  Stochastic/weighted GSOS Klin&S
  Klin'09
- Monad for probability and nondeterm
  (a vicious combination) Varacca'02, Varacca&Winskel'06

  used for traces of systems with probability and nondeterminism Jacobs'08
- Probabilistic anonymity Hasuo&Kawabe'07

$\mathcal{P}$,$\mathcal{D}$ are monads, but $\mathcal{PD}$ is not

Generic results

Generic results

- Modal logics, also via dual adjunctions
  Cirstea, Kupke, Kurz, Pattinson, Schroeder, Venema,..

Generic results

- Modal logics, also via dual adjunctions
  Cirstea, Kupke, Kurz, Pattinson, Schroeder, Venema,..
- Weak bisimulation (?) Rothe/S.,deVink,Woracek/Brengos'12

Generic results

- Modal logics, also via dual adjunctions
  Cirstea, Kupke, Kurz, Pattinson, Schroeder, Venema,..
- Weak bisimulation (?) Rothe/S.,deVink,Woracek/Brengos'12
- Generic trace theory ($\mathcal{D}$ is a monad) Hasuo,Jacobs,S.'06/'07

Generic results

- Modal logics, also via dual adjunctions
  Cirstea, Kupke, Kurz, Pattinson, Schroeder, Venema,..
- Weak bisimulation (?) Rothe/S.,deVink,Woracek/Brengos'12
- Generic trace theory ($\mathcal{D}$ is a monad) Hasuo,Jacobs,S.'06/'07
- Forward and backward simulations Hasuo

Generic results

- Modal logics, also via dual adjunctions
  Cirstea, Kupke, Kurz, Pattinson, Schroeder, Venema,..
- Weak bisimulation (?) Rothe/S.,deVink,Woracek/Brengos'12
- Generic trace theory ($\mathcal{D}$ is a monad) Hasuo,Jacobs,S.'06/'07
- Forward and backward simulations Hasuo
- Syntax and axioms for quantitative behaviors Kleene coalgebras
  Bonchi, Bonsaque,Rutten,Silva

Generic results

- Modal logics, also via dual adjunctions
  Cirstea, Kupke, Kurz, Pattinson, Schroeder, Venema,..
- Weak bisimulation (?) Rothe/S.,deVink,Woracek/Brengos'12
- Generic trace theory ($\mathcal{D}$ is a monad) Hasuo,Jacobs,S.'06/'07
- Forward and backward simulations Hasuo
- Syntax and axioms for quantitative behaviors Kleene coalgebras
  Bonchi, Bonsaque,Rutten,Silva
- Semantics via determinizations
  Bonchi, Bonsaque,Rutten,Silva/Jacobs,Silva,S.

Generic results

- Modal logics, also via dual adjunctions
  Cirstea, Kupke, Kurz, Pattinson, Schroeder, Venema,..

- Weak bisimulation (?) Rothe/S.,deVink,Woracek/Brengos'12

- Generic trace theory ($\mathcal{D}$ is a monad) Hasuo,Jacobs,S.'06/'07

- Forward and backward simulations Hasuo

- Syntax and axioms for quantitative behaviors Kleene coalgebras
  Bonchi, Bonsaque,Rutten,Silva

- Semantics via determinizations
  Bonchi, Bonsaque,Rutten,Silva/Jacobs,Silva,S.

Generic results

- Modal logics, also via dual adjunctions
  Cirstea, Kupke, Kurz, Pattinson, Schroeder, Venema,..
- Weak bisimulation (?) Rothe/S.,deVink,Woracek/Brengos'12
- Generic trace theory ($\mathcal{D}$ is a monad) Hasuo,Jacobs,S.'06/'07
- Forward and backward simulations Hasuo
- Syntax and axioms for quantitative behaviors Kleene coalgebras
  Bonchi, Bonsaque,Rutten,Silva
- Semantics via determinizations
  Bonchi, Bonsaque,Rutten,Silva/Jacobs,Silva,S.

Generic results

- Modal logics, also via dual adjunctions
  Cirstea, Kupke, Kurz, Pattinson, Schroeder, Venema,..
- Weak bisimulation (?) Rothe/S.,deVink,Woracek/Brengos'12
- Generic trace theory ($\mathcal{D}$ is a monad) Hasuo,Jacobs,S.'06/'07
- Forward and backward simulations Hasuo
- Syntax and axioms for quantitative behaviors Kleene coalgebras
  Bonchi, Bonsaque,Rutten,Silva
- Semantics via determinizations
  Bonchi, Bonsaque,Rutten,Silva/Jacobs,Silva,S.

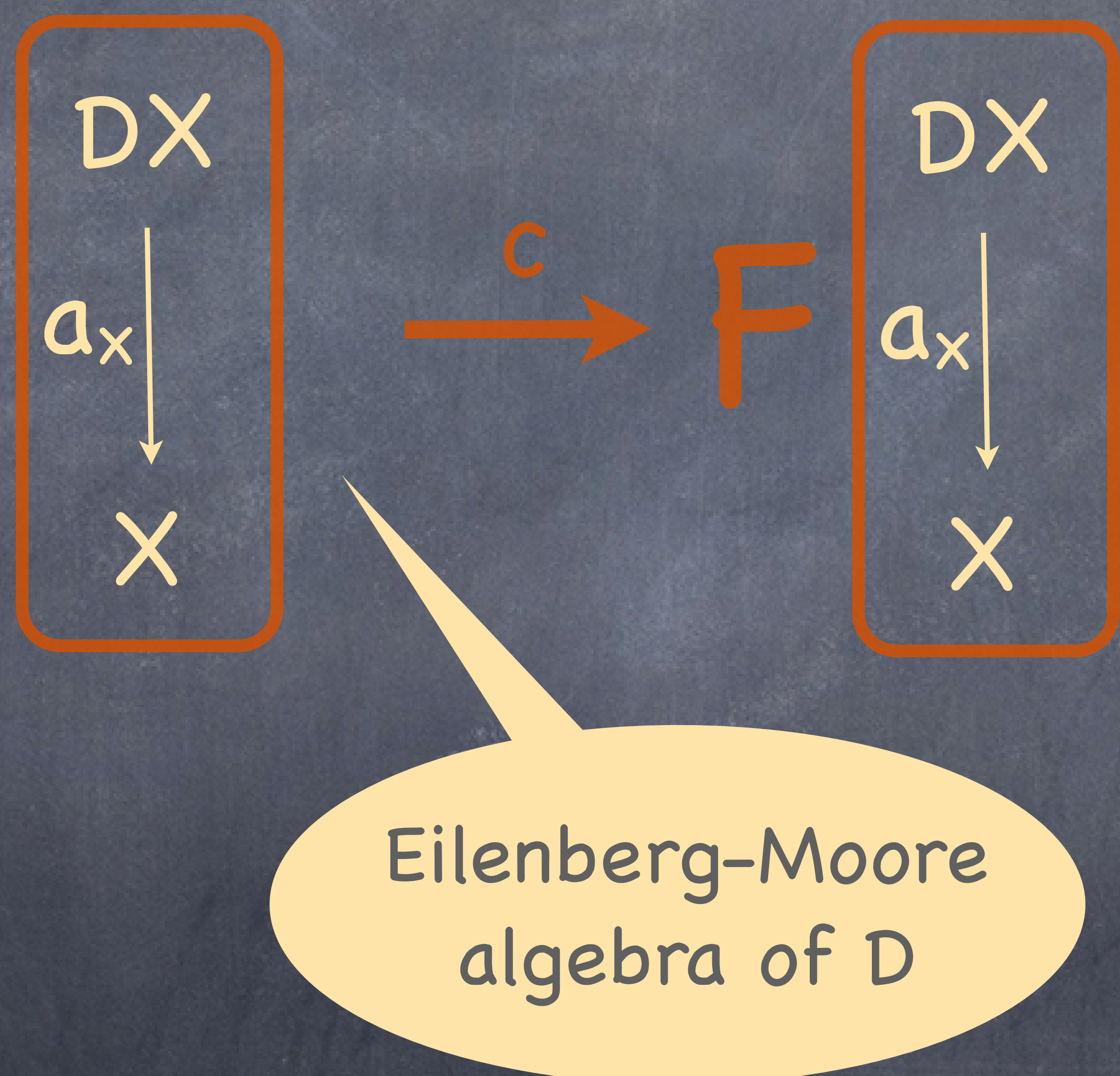
Generic results

- Modal logics, also via dual adjunctions
  Cirstea, Kupke, Kurz, Pattinson, Schroeder, Venema,..
- Weak bisimulation (?) Rothe/S.,deVink,Woracek/Brengos'12
- Generic trace theory ($\mathcal{D}$ is a monad) Hasuo,Jacobs,S.'06/'07
- Forward and backward simulations Hasuo
- Syntax and axioms for quantitative behaviors Kleene coalgebras
  Bonchi, Bonsaque,Rutten,Silva
- Semantics via determinizations
  Bonchi, Bonsaque,Rutten,Silva/Jacobs,Silva,S.

Coalgebras over algebras

$$\boxed{\begin{array}{c} DX \\ {\scriptstyle a_X}\big\downarrow \\ X \end{array}} \xrightarrow{\;c\;} F\,\boxed{\begin{array}{c} DX \\ {\scriptstyle a_X}\big\downarrow \\ X \end{array}}$$

Coalgebras over algebras



Coalgebras over algebras

$$\begin{pmatrix} DX \\ \downarrow a_X \\ X \end{pmatrix} \xrightarrow{\;c\;} F \begin{pmatrix} DX \\ \downarrow a_X \\ X \end{pmatrix}$$

Eilenberg-Moore algebra of D

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, equal] & TX \arrow[d, "a"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "T(a)"] \arrow[d, "\mu"'] & TX \arrow[d, "a"] \\ TX \arrow[r, "a"'] & X \end{tikzcd}$$

Eilenberg-Moore algebras of D

CA = convex algebras

- Variety of algebras of type

$$\mathcal{T}_{\rm ca} := \left\{ (p_i)_{i=1}^n \in \mathbb{R}^n \mid \; n \in \mathbb{N}^+, \, p_1, \dots, p_n \geq 0, \, \sum_{i=1}^n p_i = 1 \right\}.$$

- two axioms

$$f_{(\delta_{ij})_{i=1}^n}(x_1, \dots, x_n) = x_j, \quad n \in \mathbb{N}^+, \; j = 1, \dots, n,$$

$$f_{(p_i)_{i=1}^n}\left(f_{(p_{1j})_{j=1}^m}(x_1, \dots, x_m), \dots, f_{(p_{nj})_{j=1}^m}(x_1, \dots, x_m)\right) = \\ = f_{(\sum_{i=1}^n p_i p_{ij})_{j=1}^m}(x_1, \dots, x_m)$$

CA = convex algebras

Eilenberg-Moore algebras of D

PCA - $\leq$

- Variety of algebras of type

$$\mathcal{T}_{\mathrm{ca}} := \left\{ (p_i)_{i=1}^n \in \mathbb{R}^n \mid \ n \in \mathbb{N}^+, p_1, \ldots, p_n \geq 0, \sum_{i=1}^n p_i = 1 \right\}.$$

- two axioms

$$f_{(\delta_{ij})_{i=1}^n}(x_1, \ldots, x_n) = x_j, \quad n \in \mathbb{N}^+, \ j = 1, \ldots, n,$$

$$f_{(p_i)_{i=1}^n}\left(f_{(p_{1j})_{j=1}^m}(x_1, \ldots, x_m), \ldots, f_{(p_{nj})_{j=1}^m}(x_1, \ldots, x_m)\right) = \\ = f_{(\sum_{i=1}^n p_i p_{ij})_{j=1}^m}(x_1, \ldots, x_m)$$

Eilenberg-Moore algebras of D

CA = convex algebras

PCA - ≤

- Variety of algebras of type

$$\mathcal{T}_{\mathrm{ca}} := \left\{(p_i)_{i=1}^n \in \mathbb{R}^n \mid n \in \mathbb{N}^+, p_1, \dots, p_n \geq 0, \sum_{i=1}^n p_i = 1\right\}$$

TCA

- two axioms

$$f_{(\delta_{ij})_{i=1}^n}(x_1, \dots, x_n) = x_j, \quad n \in \mathbb{N}^+, \ j = 1, \dots, n,$$

$$f_{(p_i)_{i=1}^n}\left(f_{(p_{1j})_{j=1}^m}(x_1, \dots, x_m), \dots, f_{(p_{nj})_{j=1}^m}(x_1, \dots, x_m)\right) = \\ = f_{(\sum_{i=1}^n p_i p_{ij})_{j=1}^m}(x_1, \dots, x_m)$$

Eilenberg-Moore algebras of D

CA = convex algebras

PCA - $\leq$

- Variety of algebras of type

$$\mathcal{T}_{\mathrm{ca}} := \left\{ (p_i)_{i=1}^n \in \mathbb{R}^n \mid n \in \mathbb{N}^+, p_1, \ldots, p_n \geq 0, \sum_{i=1}^n |p_i| = 1 \right\}$$

TCA

- two axioms

$$f_{(\delta_{ij})_{i=1}^n}(x_1, \ldots, x_n) = x_j, \quad n \in \mathbb{N}^+, \ j = 1, \ldots, n,$$

$$f_{(p_i)_{i=1}^n}\left(f_{(p_{1j})_{j=1}^m}(x_1, \ldots, x_m), \ldots, f_{(p_{nj})_{j=1}^m}(x_1, \ldots, x_m)\right) = $$
$$= f_{(\sum_{i=1}^n p_i p_{ij})_{j=1}^m}(x_1, \ldots,$$

have nice properties

Probabilities are not that special...

- subsets, multisets and distributions have something in common: they are instances of the same functor

Probabilities are not that special...

- subsets, multisets and distributions have something in common: they are instances of the same functor
- given a monoid $(M, +, 0)$ and a subset $S \subseteq M$

$$V_S(X) = \{\varphi : X \to M \mid \mathsf{supp}(\varphi) \text{ is finite}, \sum_{x \in X} \varphi(x) \in S\}$$

$$V_S(f)(\varphi)(y) = \sum_{x \in f^{-1}(\{y\})} \varphi(x)$$

Probabilities are not that special...

- subsets, multisets and distributions have something in common: they are instances of the same functor

$\mathcal{P}_f = V_S$
$M = (\{0,1\}, \vee, 0)$
$S = M$

- given a monoid $(M, +, 0)$ and a subset $S \subseteq M$

$$V_S(X) = \{\varphi : X \to M \mid \mathsf{supp}(\varphi) \text{ is finite}, \sum_{x \in X} \varphi(x) \in S\}$$

$$V_S(f)(\varphi)(y) = \sum_{x \in f^{-1}(\{y\})} \varphi(x)$$

Probabilities are not that special...

$\mathcal{P}_f = V_S$
$M = (\{0,1\}, \vee, 0)$
$S = M$

$\mathcal{M}_f = V_S$
$M = (\mathbb{N}, +, 0)$
$S = M$

- subsets, multisets and distributions have something in common: they are instances of the same functor

- given a monoid $(M, +, 0)$ and a subset $S \subseteq M$

$$V_S(X) = \{\varphi : X \to M \mid \mathsf{supp}(\varphi) \text{ is finite}, \sum_{x \in X} \varphi(x) \in S\}$$

$$V_S(f)(\varphi)(y) = \sum_{x \in f^{-1}(\{y\})} \varphi(x)$$

Probabilities are not that special...

- subsets, multisets and distributions have something in common: they are instances of the same functor

$\mathcal{P}_f = V_S$
$M = (\{0,1\}, \vee, 0)$
$S = M$

$\mathcal{M}_f = V_S$
$M = (\mathbb{N}, +, 0)$
$S = M$

$\mathcal{D}_f = V_S$
$M = (\mathbb{R}^{\geq 0}, +, 0)$
$S = [0,1]$

- given a monoid $(M, +, 0)$ and a subset $S \subseteq M$

$$V_S(X) = \{\varphi : X \to M \mid \mathsf{supp}(\varphi) \text{ is finite}, \sum_{x \in X} \varphi(x) \in S\}$$

$$V_S(f)(\varphi)(y) = \sum_{x \in f^{-1}(\{y\})} \varphi(x)$$

Probabilities are not that special...

$\mathcal{P}_f = V_S$
$M = (\{0,1\}, \vee, 0)$
$S = M$

$\mathcal{M}_f = V_S$
$M = (\mathbb{N}, +, 0)$
$S = M$

$\mathcal{D}_f = V_S$
$M = (\mathbb{R}^{\geq 0}, +, 0)$
$S = [0,1]$

- subsets, multisets and distributions have something in common: they are instances of the same functor

- given a monoid $(M, +, 0)$ and a subset $S \subseteq M$

$$V_S(X) = \{\varphi : X \to M \mid \mathsf{supp}(\varphi) \text{ is finite}, \sum_{x \in X} \varphi(x) \in S\}$$

additional structure on M
adds structure to the functor
(monad...)

Part 2

Continuous probabilistic systems

Live beyond sets

in $\mathbf{Meas}$

Live beyond sets

in $\mathbf{Meas}$

the category of measure spaces
and measurable maps

Live beyond sets

in $\mathbf{Meas}$

the category of measure spaces and measurable maps

objects: measure spaces (X, S_X)

Live beyond sets

in $\mathbf{Meas}$

the category of measure spaces
and measurable maps

objects: measure spaces (X, S_X)

σ-algebra

Live beyond sets

in $\mathbf{Meas}$

the category of measure spaces and measurable maps

objects: measure spaces (X, S_X)

σ-algebra

closed w.r.t.∅, complements, countable unions

Live beyond sets

in $\mathbf{Meas}$

the category of measure spaces
and measurable maps

closed w.r.t.$\emptyset$,
complements,
countable unions

objects: measure spaces (X, S_X)

σ-algebra

arrows: measurable maps

$f : X \to Y$ with $f^{-1}(S_Y) \subseteq S_X$

Markov and Giry

- Markov processes are coalgebras of the Giry monad on $\mathbf{Meas}$

Markov and Giry

- Markov processes are coalgebras of the Giry monad on $\mathbf{Meas}$
- The Giry functor (monad) acts on objects and arrows as

Markov and Giry

- Markov processes are coalgebras of the Giry monad on $\mathbf{Meas}$
- The Giry functor (monad) acts on objects and arrows as

$$\mathcal{G}(X, S_X) = (\mathcal{G}X, S_{\mathcal{G}X})$$

Markov and Giry

- Markov processes are coalgebras of the Giry monad on $\mathbf{Meas}$
- The Giry functor (monad) acts on objects and arrows as

$$\mathcal{G}(X, S_X) = (\mathcal{G}X, S_{\mathcal{G}X})$$

all subprobability measures

Markov and Giry

- Markov processes are coalgebras of the Giry monad on $\mathbf{Meas}$
- The Giry functor (monad) acts on objects and arrows as

$$\mathcal{G}(X, S_X) = (\mathcal{G}X, S_{\mathcal{G}X})$$

all subprobability measures

$$\mathcal{G}X = \{\varphi : S_X \to [0,1] \mid \varphi(\emptyset) = 0, \varphi(\uplus_i M_i) = \sum_i \varphi(M_i)\}$$

Markov and Giry

- Markov processes are coalgebras of the Giry monad on $\mathbf{Meas}$
- The Giry functor (monad) acts on objects and arrows as

$$\mathcal{G}(X, S_X) = (\mathcal{G}X, S_{\mathcal{G}X})$$

all subprobability measures

the smallest σ-algebra making the evaluation maps measurable

$$ev_M : \mathcal{G}X \to [0,1]$$

$$ev_M(\varphi) = \varphi(M)$$

$$\mathcal{G}X = \{\varphi : S_X \to [0,1] \mid \varphi(\emptyset) = 0, \varphi(\uplus_i M_i) = \sum_i \varphi(M_i)\}$$

Markov and Giry

- Markov processes are coalgebras of the Giry monad on $\mathbf{Meas}$
- The Giry functor (monad) acts on objects and arrows as

$$\mathcal{G}(X, S_X) = (\mathcal{G}X, S_{\mathcal{G}X})$$

all subprobability measures

the smallest σ-algebra making the evaluation maps measurable

$$ev_M : \mathcal{G}X \to [0,1]$$

$$ev_M(\varphi) = \varphi(M)$$

Markov and Giry

- Markov processes are coalgebras of the Giry monad on $\mathbf{Meas}$

- The Giry functor (monad) acts on objects and arrows as

$$\mathcal{G}(X, S_X) = (\mathcal{G}X, S_{\mathcal{G}X})$$

all subprobability measures

the smallest σ-algebra making the evaluation maps measurable

$X \to [0,1]$

$= \varphi(M)$

$$\mathcal{G}(f)\left(S_X \stackrel{\varphi}{\to} [0,1]\right) \quad = \quad \left(S_Y \stackrel{f^{-1}}{\to} S_X \stackrel{\varphi}{\to} [0,1]\right)$$

Properties, other spaces

- $\mathbf{Meas}$ may not have weak pullbacks
- Analytic spaces have semi-pullbacks
  Edalat '99
- It has a final coalgebra
  Moss&Viglizzo'06

Properties, oth

bisimilarity is difficult to handle

- $\mathbf{Meas}$ may not have weak pullbacks
- Analytic spaces have semi-pullbacks
  Edalat '99
- It has a final coalgebra
  Moss&Viglizzo'06

Properties, oth

bisimilarity is difficult to handle

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

- $\mathbf{Meas}$ may not have weak pullbacks
- Analytic spaces have semi-pullbacks
  Edalat '99
- It has a final coalgebra
  Moss&Viglizzo'06

Properties, oth

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

bisimilarity is difficult to handle

- $\mathbf{Meas}$ may not have weak pullbacks

so people turned to analytic or Polish spaces

- Analytic spaces have semi-pullbacks
  Edalat '99

- It has a final coalgebra
  Moss&Viglizzo'06

Properties, oth

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

bisimilarity is difficult to handle

- $\mathbf{Meas}$ may not have weak pullbacks

so people turned to analytic or Polish spaces

- Analytic spaces have semi-pullbacks
  Edalat '99

- It has a final coal
  Moss&Viglizzo'06

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

Properties, oth

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

bisimilarity is difficult to handle

- Meas may not have weak pullbacks

so people turned to analytic or Polish spaces

- Analytic spaces have semi-pullbacks
  Edalat '99

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

- It has a final coal
  Moss&Viglizzo'06

Properties, oth

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

bisimilarity is difficult to handle

- Meas may not have weak pullbacks

so people turned to analytic or Polish spaces

- Analytic spaces have semi-pullbacks
Edalat '99

$$\begin{tikzcd} & & S \arrow[dl] \arrow[dr] & & \\ & X \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & \\ U & & Z & & V \end{tikzcd}$$

- It has a final coal
Moss&Viglizzo'06

Properties, oth

bisimilarity is difficult to handle

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

- Meas may not have weak pullbacks

so people turned to analytic or Polish spaces

- Analytic spaces have semi-pullbacks Edalat '99
- It has a final coal
Moss&Viglizzo'06

bisimilarity is an equivalence

$$\begin{tikzcd} & & S \arrow[dl] \arrow[dr] & & \\ & X \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & \\ U & & Z & & V \end{tikzcd}$$

Main research in continuous systems

- Labelled Markov processes
  bisimulation as before, logical characterization of bisimulation
  Desharnais,Panangaden,..(book '09)
- Stochastic relations
  generalization of Markov processes
  Doberkat (book'07)

Main research in continuous systems

reactive labels

- Labelled Markov processes
  bisimulation as before, logical characterization of bisimulation
  Desharnais,Panangaden,..(book '09)
- Stochastic relations
  generalization of Markov processes
  Doberkat (book'07)

Main research in continuous systems

reactive labels

analytic spaces

- Labelled Markov processes
  bisimulation as before, logical characterization of bisimulation
  Desharnais,Panangaden,..(book '09)

- Stochastic relations
  generalization of Markov processes
  Doberkat (book'07)

Main research in continuous systems

reactive labels

analytic spaces

- Labelled Markov processes
  bisimulation as before, logical characterization of bisimulation
  Desharnais,Panangaden,..(book '09)

finite conjunctions
negation free logic

- Stochastic relations
  generalization of Markov processes
  Doberkat (book'07)

Main research in continuous systems

reactive labels

analytic spaces

- Labelled Markov processes
  bisimulation as before, logical characterization of bisimulation
  Desharnais,Panangaden,..(b

finite conjunctions
negation free logic

Kleisli morphisms in the Kleisli category of the Giry monad

- Stochastic relations
  generalization of Markov processes
  Doberkat (book'07)

Main research in continuous systems

reactive labels

analytic spaces

- Labelled Markov processes
  bisimulation as before, logical characterization of bisimulation
  Desharnais,Panangaden,..(b

finite conjunctions
negation free logic

Kleisli morphisms in the Kleisli category of the Giry monad

- Stochastic relations
  generalization of Markov processes
  Doberkat (book'07)

$X \to \mathcal{G}Y$

Main research in continuous systems

reactive labels

analytic spaces

- Labelled Markov processes
  bisimulation as before, logical characterization of bisimulation
  Desharnais,Panangaden,..(b

finite conjunctions
negation free logic

Kleisli morphisms in the
Kleisli category of the
Giry monad

Polish and
analytic spaces

- Stochastic relations
  generalization of Markov processes
  Doberkat (book'07)

$$X \to \mathcal{G}Y$$

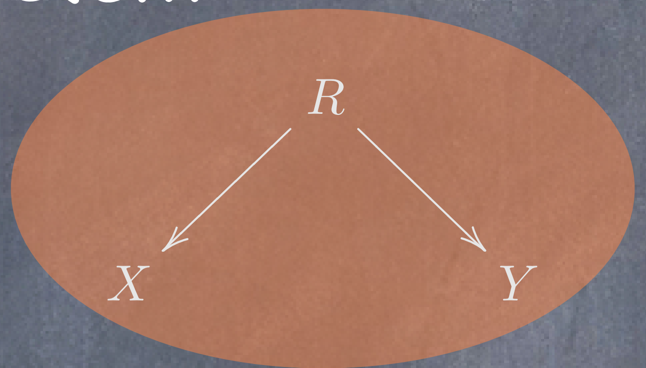
Measure spaces are fine

Measure spaces are fine

- bisimilarity is the problem

Measure spaces are fine

- bisimilarity is the problem



Measure spaces are fine

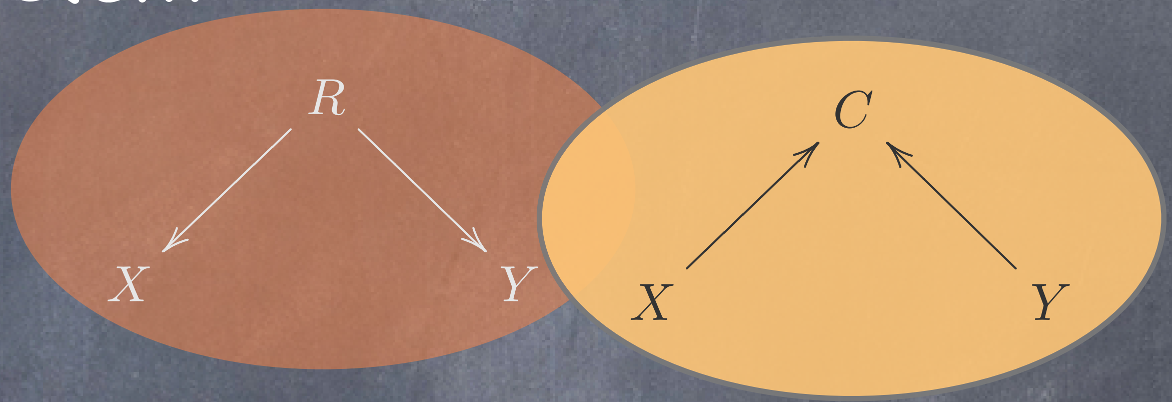
- bisimilarity is the problem

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

- behaviour equivalence is the solution
  Danos, Desharnais, Laviolette,Panangaden '04/'06

Measure spaces are fine

- bisimilarity is the problem



- behaviour equivalence is the solution
  Danos, Desharnais, Laviolette,Panangaden '04/'06

Measure spaces are fine

- bisimilarity is the problem

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & C & \\ X \arrow[ur] & & Y \arrow[ul] \end{tikzcd}$$

- behaviour equivalence is the solution
  Danos, Desharnais, Laviolette,Panangaden '04/'06

same logical characterization

all works smoothly

Measure spaces are fine

- bisimilarity is the problem

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & C & \\ X \arrow[ur] & & Y \arrow[ul] \end{tikzcd}$$

- behaviour equivalence is the solution
  Danos, Desharnais, Laviolette,Panangaden '04/'06

same logical characterization

No need of Polish/ analytic spaces (?)

all works smoothly

Discrete to continuous

$$\begin{tikzcd} \mathbf{Meas} \arrow[loop left, "\mathcal{G}"] \arrow[r, bend left, "U"] & \mathbf{Sets} \arrow[l, bend left, "D"] \arrow[loop right, "\mathcal{D}"] \end{tikzcd}$$

with $\quad D \dashv U$

Discrete to continuous

forgetful

$$\begin{tikzcd} \mathbf{Meas} \arrow[loop left, "\mathcal{G}"] \arrow[r, bend left, "U"] & \mathbf{Sets} \arrow[loop right, "\mathcal{D}"] \arrow[l, bend left, "D"] \end{tikzcd}$$

with $D \dashv U$

Discrete to continuous

discrete

forgetful

$$\begin{tikzcd} \mathbf{Meas} \arrow[loop left, "\mathcal{G}"] \arrow[r, bend left, "U"] & \mathbf{Sets} \arrow[loop right, "\mathcal{D}"] \arrow[l, bend left, "D"] \end{tikzcd} \qquad \text{with} \quad D \dashv U$$

Discrete to continuous

discrete

forgetful

$$
\begin{tikzcd}
\mathbf{Meas} \arrow[loop left, "\mathcal{G}"] \arrow[r, bend left, "U"] & \mathbf{Sets} \arrow[l, bend left, "D"] \arrow[loop right, "\mathcal{D}"]
\end{tikzcd}
$$

with $D \dashv U$

obvious natural transformation

$$\rho : \mathcal{D}U \Rightarrow U\mathcal{G}$$

Discrete to continuous

discrete

forgetful

$$\begin{tikzcd} \mathbf{Meas} \arrow[loop left, "\mathcal{G}"] \arrow[r, bend left, "U"] & \mathbf{Sets} \arrow[l, bend left, "D"] \arrow[loop right, "\mathcal{D}"] \end{tikzcd} \quad \text{with} \quad D \dashv U$$

obvious natural transformation

$$\rho : \mathcal{D}U \Rightarrow U\mathcal{G}$$

We can translate chains into processes:

Discrete to continuous

discrete

forgetful

$$\begin{tikzcd} \mathbf{Meas} \arrow[loop left, "\mathcal{G}"] \arrow[r, bend left, "U"] & \mathbf{Sets} \arrow[l, bend left, "D"] \arrow[loop right, "\mathcal{D}"] \end{tikzcd}$$

with $D \dashv U$

obvious natural transformation

$$\rho : \mathcal{D}U \Rightarrow U\mathcal{G}$$

We can translate chains into processes:

$$\begin{tikzcd} \mathsf{Coalg}_{\mathcal{D}} \arrow[r, "\rho D \circ -"'] \arrow[dr] \arrow[rr, bend left, "\mathcal{T}"] & \mathsf{Coalg}_{U\mathcal{G}D} \arrow[r, "(*)"'] \arrow[d] & \mathsf{Coalg}_{\mathcal{G}} \arrow[d] \\ & \mathbf{Sets} \arrow[r, "D"] & \mathbf{Meas} \end{tikzcd}$$

Discrete to continuous

discrete

forgetful

$$\begin{tikzcd} \mathbf{Meas} \arrow[loop left, "\mathcal{G}"] \arrow[r, bend left, "U"] & \mathbf{Sets} \arrow[loop right, "\mathcal{D}"] \arrow[l, bend left, "D"] \end{tikzcd} \qquad \text{with} \quad D \dashv U$$

We can translate chains into processes:

$$\left(X \xrightarrow{c} \mathcal{D}(X) = \mathcal{D}UD(X)\right) \longmapsto \left(X \xrightarrow{c} \mathcal{D}UD(X) \xrightarrow{\rho_{DX}} U\mathcal{G}D(X)\right)$$

$$\begin{tikzcd} \mathsf{Coalg}_{\mathcal{D}} \arrow[r, "\rho D\circ -"'] \arrow[rr, bend left, "\mathcal{T}"] \arrow[dr] & \mathsf{Coalg}_{U\mathcal{G}D} \arrow[r, "(*)"'] \arrow[d] & \mathsf{Coalg}_{\mathcal{G}} \arrow[d] \\ & \mathbf{Sets} \arrow[r, "D"] & \mathbf{Meas} \end{tikzcd}$$

Discrete to continuous

discrete

forgetful

$$\begin{tikzcd} \mathbf{Meas} \arrow[loop left, "\mathcal{G}"] \arrow[r, bend left, "U"] & \mathbf{Sets} \arrow[loop right, "\mathcal{D}"] \arrow[l, bend left, "D"] \end{tikzcd}$$

with $D \dashv U$

We can translate chains into processes:

$$\Big(X \xrightarrow{c} \mathcal{D}(X) = \mathcal{D}UD(X)\Big) \longmapsto \Big(X \xrightarrow{c} \mathcal{D}UD(X) \xrightarrow{\rho_{DX}} U\mathcal{G}D(X)\Big)$$

$$\begin{tikzcd} \mathsf{Coalg}_{\mathcal{D}} \arrow[r, "\rho D \circ -"'] \arrow[rr, bend left, "\mathcal{T}"] \arrow[dr] & \mathsf{Coalg}_{U\mathcal{G}D} \arrow[r, "(*)"'] \arrow[d] & \mathsf{Coalg}_{\mathcal{G}} \arrow[d] \\ & \mathbf{Sets} \arrow[r, "D"] & \mathbf{Meas} \end{tikzcd}$$

$$\frac{X \longrightarrow U\mathcal{G}D(X) \quad \text{in } \mathbf{Sets}}{D(X) \longrightarrow \mathcal{G}D(X) \quad \text{in } \mathbf{Meas}}$$

Discrete to continuous

$$\begin{tikzcd} \mathbf{Meas} \arrow[loop left, "\mathcal{G}"] \arrow[r, bend left, "U"] & \mathbf{Sets} \arrow[l, bend left, "D"] \arrow[loop right, "\mathcal{D}"] \end{tikzcd}$$

discrete

forgetful

with $D \dashv U$

We can translate chains into processes:

$$\big(X \xrightarrow{c} \mathcal{D}(X)\big) \mapsto \big(X \xrightarrow{c} \mathcal{D}UD(X) \xrightarrow{\rho_{DX}} U\mathcal{G}D(X)\big)$$

preserves and reflects behaviour equivalence

$$\begin{tikzcd} \mathrm{Coalg}_{\mathcal{D}} \arrow[r, "\rho D \circ -"'] \arrow[rr, bend left, "\mathcal{T}"] \arrow[dr] & \mathrm{Coalg}_{U\mathcal{G}D} \arrow[r, "(*)"'] \arrow[d] & \mathrm{Coalg}_{\mathcal{G}} \arrow[d] \\ & \mathbf{Sets} \arrow[r, "D"] & \mathbf{Meas} \end{tikzcd}$$

$$\frac{X \longrightarrow U\mathcal{G}D(X) \quad \text{in } \mathbf{Sets}}{D(X) \longrightarrow \mathcal{G}D(X) \quad \text{in } \mathbf{Meas}}$$

Final message

Teaser for Prakash :-)

Final message

- Both discrete and continuous probabilistic systems are coalgebras

Teaser for Prakash :-)

Final message

often just nice examples

- Both discrete and continuous probabilistic systems are coalgebras

Teaser for Prakash :-)

Final message

often just nice examples

also some very interesting results

- Both discrete and continuous probabilistic systems are coalgebras

Teaser for Prakash :-)

Final message

often just
nice examples

also some very interesting results

need advertising?

- Both discrete and continuous probabilistic systems are coalgebras

Teaser for Prakash :-)

Final message

often just nice examples

also some very interesting results

need advertising?

- Both discrete and continuous probabilistic systems are coalgebras
- Observation: behaviour equivalence (cospan) is more suitable than bisimilarity (span)

Teaser for Prakash :-)

Final message

often just nice examples

also some very interesting results

need advertising?

- Both discrete and continuous probabilistic systems are coalgebras

- Observation: behaviour equivalence (cospan) is more suitable than bisimilarity (span)

Teaser for Prakash :-)

Final message

often just nice examples

also some very interesting results

need advertising?

- Both discrete and continuous probabilistic systems are coalgebras
- Observation: behaviour equivalence (cospan) is more suitable than bisimilarity (span)
- Measure spaces are enough, one can forget about Polish or analytic ones (unless one loves them)