Exemplaric Expressivity of Modal Logics

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Abstract. This paper investigates expressivity of modal logics for transition systems, Markov chains, and Markov processes, as coalgebras of the powerset, finitely supported distribution, and measure functor, respectively. Expressivity means that logically indistinguishable states (satisfying the same formulas) are behaviourally indistinguishable. The investigation is based on the framework of dual adjunctions (between spaces and logics) and focuses on a crucial injectivity property. The main contribution concerns this property for Markov chains *i.e.* finitely supported probabilistic systems, showing that a logic with suitable standard modal operators and (only) finite conjunctions is expressive. A direct proof is presented, and also an indirect one via a simplicification of an earlier proof of this result for Markov processes in the current framework of dual adjunctions.

1 Introduction

During the last decade, coalgebra has become accepted as an abstract framework for describing state-based dynamical systems. Fairly quickly it was recognised, first in [23], that modal logic is the natural logic for coalgebras—and also that coalgebras provide obvious models for modal logics. Intuitively there is indeed a connection, because modal operators can be interpreted in terms of next (or previous) states, with respect to some transition system (or, more abstractly, coalgebra). The last few years have shown a rapid development in this (combined) area [20,14,24,19,21,3,26,27,25,5,16]. One of the more interesting aspects is the use of dualities (or *dual adjunctions*).

Here is a brief "historical" account of how the emergence of dual adjunctions in logical settings can be understood. For reasoning about functors the idea of *predicate lifting* was used already early in [12,13]. This involves the extension of a predicate (formula) $P \subseteq X$ to a *lifted* predicate $\overline{P} \subseteq TX$, for an endofunctor $T\colon \mathbf{Sets} \to \mathbf{Sets}$ whose coalgebras $X \to TX$ we wish to study. The notion of *invariant* arises via such predicate

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liftings. Liftings can be described as a function $\mathcal{P}X \to \mathcal{P}TX$, or actually as a natural transformation $\mathcal{P} \Rightarrow \mathcal{P}T$, for the contravariant powerset functor $\mathcal{P}\colon \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$, *c.f.* [24]. With the introduction of polyadic modal operators [26,16] this natural transformation morphed into maps of the form $\coprod_{n\in\mathbb{N}}(\mathcal{P}X)^n \to \mathcal{P}TX$, or more abstractly via a functor L into a natural transformation $L\mathcal{P} \Rightarrow \mathcal{P}T$. All this takes place in a situation:

$$\begin{tikzcd} T \arrow[loop left] \mathbf{Sets}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{P}"] & \mathbf{Sets} \arrow[l, bend left, "\mathcal{P}"] \arrow[loop right] L \end{tikzcd} \qquad \text{with} \quad L\mathcal{P} \stackrel{\sigma}{\Longrightarrow} \mathcal{P}T \tag{1}$$

where we have a dual adjunction $\mathcal{P} \dashv \mathcal{P}$. The category $\mathbf{Sets}$ on the left describes the spaces on which we have coalgebra structures (of the functor T). $\mathbf{Sets}$ on the right describes the logical universe, on which there is a functor L for modal operators.

The above dual adjunction thus provides the raw setting for considering coalgebras and their (modal) logics. For specific kinds of coalgebras (given by particular functors T) there may be more structure around. In this paper we shall study examples with the categories $\mathbf{Sets}$ of sets and **Meas** of measurable spaces on the left in (1), and the categories **BA** of Boolean algebras and **MSL** of meet semilattices on the right. The latter capture Boolean logic and logic with only finite conjunctions, respectively. Section 2 will describe the adjunctions involved.

Section 3 will enrich these dual adjunctions with endofunctors like T and L in the above diagram (1). It also contains two "folklore" results about the natural transformation involved (the σ in (1)). The most important one is Theorem 1 that relates a certain injectivity condition to the fundamental property of *expressivity* of the logic—which means that logically indistinguishable states are also behaviourally indistinguishable. This theorem is known for some time already in the community and has appeared in print in various places [16,17,5], in one form or another. We present a convenient formulation (and proof) that is useful in our setting, but we do not claim it as our contribution.

What we do claim as contribution appears in Section 4. There we use Theorem 1 to prove expressivity for three concrete examples. In all these cases we describe appropriate modalities, and prove expressivity by adding:

- Boolean logic for image-finite transition systems, as coalgebras of the finite powerset functor on $\mathbf{Sets}$;
- finite conjunctions for Markov chains, as coalgebras of the finitely supported discrete distribution functor on $\mathbf{Sets}$;
- finite conjunctions for Markov processes, as coalgebras of the Giry functor on the category of measure spaces.

The first point goes back to [11]. Here we cast it in the framework of dual adjunctions, with an explicit description of the "modality" endofunctor L on the category **BA** of Boolean algebras and its relevant properties. The second point is our main technical contribution. There is already an expressivity result for Markov chains with the standard modalities and Boolean logic (including negation), *c.f.* [5,22]. Here, we first give a direct proof showing that finite conjunctions suffice, just as they do for non-discrete probabilistic systems [8,6]. In the third point we reformulate the expressivity result of [8,6] within our uniform setting of dual adjunctions. Additionally we elaborate on the relation between the discrete and non-discrete Markov chains/processes and show precisely how expressivity for Markov processes yields expressivity for Markov chains—yielding a (non-trivial) indirect proof for the second point.

2 Dual adjunctions

In this section we shall be interested in adjoint situation of the form:

$$\begin{tikzcd} \mathbb{C}^{\mathrm{op}} \arrow[r, bend left, "P"] & \mathbb{A} \arrow[l, bend left, "F"] \end{tikzcd} \qquad \text{with} \quad F \dashv P \tag{2}$$

We informally call this a *dual* adjunction because one of the categories involved occurs naturally in opposite form. In the next section we shall extend such situations with endofunctors, on $\mathbb{C}$ for systems as coalgebras and on $\mathbb{A}$ for logics, but at this preparatory stage we only look at the adjunctions themselves.

Such situations (2) are familiar in duality-like settings, for instance with $\mathbb{C} = \mathbb{A} = \mathbf{Sets}$, and $P = F =$ "contravariant powerset"; with $\mathbb{C} = \mathbf{Sets}$, $\mathbb{A} = \mathbf{PreOrd}$, $P =$ "contravariant powerset", $F =$ "upsets"; or with $\mathbb{C} =$ "topological spaces" and $\mathbb{A} =$ "frames". Such situations are studied systematically in [15], and more recently also in the context of coalgebras and modal logic [18,17,3,4,16]. Typically the functor P describes predicates on spaces and the functor F theories on logical models.

In this situation it is important to keep track of the direction of arrows. To be explicit, the (components of the) unit and counit of the adjunction $F \dashv P$ are maps $\eta_A: A \to PFA$ in $\mathbb{A}$ and $\varepsilon_X: FPX \to X$ in $\mathbb{C}^{\mathrm{op}}$, *i.e.* $\varepsilon_X: X \to FPX$ in $\mathbb{C}$. The familiar triangular identities are $P\varepsilon \circ \eta P = \mathrm{id}$ in $\mathbb{A}$ and $\varepsilon F \circ F\eta = \mathrm{id}$ in $\mathbb{C}^{\mathrm{op}}$, *i.e.* $F\eta \circ \varepsilon F = \mathrm{id}$ in $\mathbb{C}$.

2.1 Examples

The following three instances of the dual adjunction (2) will be used throughout the paper.

Sets versus Boolean algebras

The first dual adjunction is between sets and Boolean algebras:

$$\begin{tikzcd} \mathbf{Sets}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{P}"] & \mathbf{BA} \arrow[l, bend left, "\mathcal{F}_u"] \end{tikzcd} \tag{3}$$

Here, **BA** is the category of Boolean algebras. The functor $\mathcal{P}$ is (contravariant) powerset and $\mathcal{F}_u$ sends a Boolean algebra A to the set of its ultrafilters. These ultrafilters are filters (see below) $\alpha \subseteq A$ such that for each $a \in A$, either $a \in \alpha$ or $\neg a \in \alpha$, but not both. The unit $\eta_A: A \to \mathcal{P}\mathcal{F}_u(A)$ for this adjunction is given by $\eta(a) = \{\alpha \in \mathcal{F}_u(A) \mid a \in \alpha\}$. The adjunction (3) amounts to the standard correspondence:

$$\frac{X \longrightarrow \mathcal{F}_u(A) \quad \text{in } \mathbf{Sets}}{A \longrightarrow \mathcal{P}(X) \quad \text{in } \mathbf{BA}}$$

Sets versus meet semilattices

The second example uses the category **MSL** of meet semilattices, in a situation:

$$\begin{tikzcd} \mathbf{Sets}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{P}"] & \mathbf{MSL} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd} \tag{4}$$

The functor $\mathcal{F}$ sends a meet semilattice A to the set $\mathcal{F}(A)$ of filters, *i.e.* to the upsets $\alpha \subseteq A$ which are closed under finite meets: $\top \in \alpha$ and $x, y \in \alpha \Rightarrow x \wedge y \in \alpha$. Here the unit is as before, $\eta_A(a) = \{\alpha \in \mathcal{F}(A) \mid a \in \alpha\}$.

Measure spaces versus meet semilattices

Our third example is less standard. It uses the category **Meas** of measurable spaces, instead of $\mathbf{Sets}$. An object of **Meas** is a pair $(X, \mathcal{S}(X))$ of a set X together with a σ-algebra $\mathcal{S}(X) \subseteq \mathcal{P}(X)$. The latter is a collection of "measurable" subsets closed under $\emptyset$, complements (negation), and countable unions. We shall use that it is closed, in particular, under finite conjunctions. A morphism $X \to Y$ in **Meas** is any measurable function $f: X \to Y$, *i.e.* a function satisfying $f^{-1}(N) \in \mathcal{S}(X)$ for each measurable set $N \in \mathcal{S}(Y)$.

Interestingly, in this case we also have an adjunction with meet semilattices:

$$\begin{tikzcd} \mathbf{Meas}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{S}"] & \mathbf{MSL} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd} \tag{5}$$

The functor $\mathcal{F}$ is the filter functor from (4) that maps a meet semilattice to the set of its filters, with a σ-algebra generated by the subsets $\eta(a) \subseteq \mathcal{F}(A)$, for $a \in A$. The bijective correspondence,

$$\frac{X \xrightarrow{f} \mathcal{F}(A) \quad \text{in } \mathbf{Meas}}{A \xrightarrow[g]{} \mathcal{S}(X) \quad \text{in } \mathbf{MSL}}$$

is given as follows.

- Given $f: X \to \mathcal{F}(A)$ in **Meas** we obtain $\widehat{f}: A \to \mathcal{S}(X)$ as:

  $$\widehat{f}(a) = f^{-1}(\eta(a)) = \{x \in X \mid f(x) \in \eta(a)\} = \{x \in X \mid a \in f(x)\}.$$

  This $\widehat{f}$ is well-defined because f is a measurable function—so that $f^{-1}(\eta(a)) \in \mathcal{S}(X)$—and preserves finite meets $\top, \wedge$ because η and f^{-1} do.

- Conversely, given $g\colon A \to \mathcal{S}(X)$ in **MSL** one defines $\widehat{g}\colon X \to \mathcal{F}(A)$ as $\widehat{g}(x) = \{a \in A \mid x \in g(a)\}$. This yields a filter because g preserves finite meets.

It is obvious that $\widehat{\widehat{f}} = f$ and $\widehat{\widehat{g}} = g$. The unit $\eta_A\colon A \to \mathcal{SF}(A)$ of this adjunction is as before: $\eta_A(a) = \{\alpha \in \mathcal{F}(A) \mid a \in \alpha\}$.

Remark 1. In the end we notice that the adjunction $\mathbf{Sets}^{\mathrm{op}} \leftrightarrows$ **MSL** can be obtained from the adjunction $\mathbf{Meas}^{\mathrm{op}} \leftrightarrows$ **MSL** in the following manner. The forgetful functor $U\colon \mathbf{Meas} \to \mathbf{Sets}$ has a left adjoint D which equips a set X with the discrete σ-algebra $\mathcal{P}(X)$ in which all subsets are measurable. Then, when we switch to opposite categories, this forgetful functor $U\colon \mathbf{Meas}^{\mathrm{op}} \to \mathbf{Sets}^{\mathrm{op}}$ is left adjoint to D. Hence the adjunction $\mathcal{F} \dashv P$ between sets and meet semilattices can be obtained as $U\mathcal{F} \dashv \mathcal{S}D$ by composition of adjoints in:

$$\begin{tikzcd}
\mathbf{Meas}^{\mathrm{op}} \arrow[rr, "\mathcal{S}"] \arrow[d, bend left, "U"] & & \mathbf{MSL} \arrow[ll, bend left, "\mathcal{F}"] \arrow[dll, bend left, "\mathcal{F}"] \\
\mathbf{Sets}^{\mathrm{op}} \arrow[u, "D"] \arrow[urr, "\mathcal{P}"] & &
\end{tikzcd}$$

3 Logical set-up

We now extend the adjunction (2) with endofunctors T and L as in:

$$\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \arrow[loop left, "T"] \arrow[r, bend left, "P"] & \mathbb{A} \arrow[l, bend left, "F"] \arrow[loop right, "L"]
\end{tikzcd} \qquad \text{with} \quad F \dashv P \tag{6}$$

We shall be interested in coalgebras of the functor T—describing dynamical systems—and in algebras of L—capturing logical models. These models typically contain certain logical connectives, as incorporated in categories of Boolean algebras or of meet semilattices; the functor L adds modal operators. Via a suitable relation between T and L one can capture logics for dynamical systems in this set-up. But before doing so we recall the following standard result.

Proposition 1. *In the situation of the previous diagram we have bijective correspondence between natural transformations:*

$$\frac{LP \overset{\sigma}{\Longrightarrow} PT}{TF \underset{\tau}{\Longrightarrow} FL} \qquad \textit{i.e.} \qquad \frac{\begin{tikzcd} \mathbb{C}^{op} \arrow[r, bend left, "LP"{name=U}] \arrow[r, bend right, "PT"'{name=D}] & \mathbb{A} \arrow[from=U, to=D, phantom, "\Downarrow\sigma"] \end{tikzcd}}{\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "FL"{name=U}] \arrow[r, bend right, "TF"'{name=D}] & \mathbb{C}^{op} \arrow[from=U, to=D, phantom, "\Downarrow\tau"] \end{tikzcd}}$$

Proof. The correspondence is obtained as follows.

- For $\sigma\colon LP \Rightarrow PT$ one puts:

$$\overline{\sigma} \stackrel{\text{def}}{=} \Big(TF \xrightarrow{\varepsilon TF} FPTF \xrightarrow{F\sigma F} FLPF \xrightarrow{FL\eta} FL\Big)$$

- Conversely, for $\tau\colon TF \Rightarrow FL$ one similarly takes:

$$\overline{\tau} \stackrel{\text{def}}{=} \Big(LP \xrightarrow{\eta LP} PFLP \xrightarrow{P\tau P} PTFP \xrightarrow{PT\varepsilon} PT\Big). \qquad \Diamond$$

Assumptions 1 *In the situation (6) we shall assume the following.*

- *There is an "interpretation" natural transformation $\sigma\colon LP \Rightarrow PT$.*
- *The functor $L\colon \mathbb{A} \to \mathbb{A}$ has an initial algebra of "formulas". We shall write it as $L(Form) \stackrel{\cong}{\Rightarrow} Form$.*
- *There is a factorization system $(\mathcal{M}, \mathcal{E})$ on the category $\mathbb{C}$ with $\mathcal{M} \subseteq$ Monos and $\mathcal{E} \subseteq$ Epis. We shall write the maps in $\mathcal{M}$ as $\rightarrowtail$ and those in $\mathcal{E}$ as $\twoheadrightarrow$. The factorization system satisfies the "diagonal fill-in" property: If the outer square*

$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "h"] \\ A \arrow[r, tail, "m"'] & B \end{tikzcd}$$

  commutes, then there exists a diagonal map making the two triangles commute.
- *The functor T preserves the maps in $\mathcal{M}$, i.e. $m \in \mathcal{M} \Rightarrow T(m) \in \mathcal{M}$.*

For an arbitrary coalgebra $X \xrightarrow{c} TX$ by initiality of *Form* one obtains an interpretation of formulas as predicates on the state space X as in:

$$\begin{tikzcd} L(\mathit{Form}) \arrow[r, dashed, "L\llbracket - \rrbracket"] \arrow[dd, "\cong"'] & LPX \arrow[d, "\sigma_X"] \\ & PTX \arrow[d, "P(c)"] \\ \mathit{Form} \arrow[r, dashed, "\llbracket - \rrbracket"'] & PX \end{tikzcd}$$

The adjunction $F \dashv P$ yields a *theory* map *th*$\colon X \to F(Form)$ corresponding to the interpretation $\llbracket - \rrbracket\colon Form \to PX$. Intuitively, for a state $x \in X$ we have a theory *th*$(x) \in F(Form)$ of formulas that hold in x. Two states $x, y \in X$ will be called *logically indistinguishable*, written as $\equiv y$, if their theories are the same: *th*$(x) =$ *th*(y). In general, logical equivalence is the subobject of $X \times X$ which is the following equaliser in $\mathbb{C}$,

$$\begin{tikzcd} \equiv \arrow[r, tail] & X \times X \arrow[r, shift left, "th \circ \pi_1"] \arrow[r, shift right, "th \circ \pi_2"'] & \mathcal{F}(\mathit{Form}) \end{tikzcd}$$

We are interested in comparing logical indistinguishability to behavioural equivalence. Recall that behavioural equivalence involves "cospans": two states $x, y \in X$ of a coalgebra are *behaviourally equivalent* if there exist coalgebra homomorphisms f and g with $f(x) = g(y)$. Behavioural equivalence coincides with bisimilarity in case the functor involved preserves weak pullbacks. In the context of expressivity of modal logics behavioural equivalence works better, as commonly accepted in the community, and noted explicitly for probabilistic systems in [6].

It is known and not difficult to show that behavioural equivalence implies logical indistinguishability. For the converse we now present our version of a "folklore" result (see also [16]).

Theorem 1. *In the context of Assumptions 1, if the transpose $\overline{\sigma}: TF \Rightarrow FL$ of $\sigma: LP \Rightarrow PT$, according to Proposition 1, is componentwise mono, then logically indistinguishable elements are behaviourally equivalent.*

Proof. One factors the theory map $th: X \to F(\mathit{Form})$ as $X \overset{e}{\twoheadrightarrow} X' \overset{m}{\rightarrowtail} F(\mathit{Form})$. Then: $x \equiv y$ iff $e(x) = e(y)$. The main point is to obtain a (quotient) coalgebra on X' via the diagonal fill-in property of the factorisation in:

$$\begin{tikzcd}
TX \arrow[r, "T(e)"] & TX' \arrow[r, tail, "T(m)"] & TF(\mathit{Form}) \arrow[r, tail, "\overline{\sigma}"] & FL(\mathit{Form}) \\
X \arrow[u, "c"] \arrow[r, two heads, "e"'] & X' \arrow[u, dashed] \arrow[rr, tail, "m"'] & & F(\mathit{Form}) \arrow[u, "\cong"']
\end{tikzcd}$$

Logically indistinguishable elements are then equated by a coalgebra homomorphism (namely e), and are thus behaviourly equivalent. $\Diamond$

The main technical part of applying Theorem 1 is showing that the natural transformation $\overline{\sigma}$ is mono. That will be the topic of the next section.

3.1 Examples

In the remainder of this section we extend the three adjunctions in the examples from Section 2 with suitable coalgebra functors—the T in (6)—that we are interested in. Moreover, we discuss that the needed assumptions hold in each case.

Transition systems

We shall write $\mathcal{P}_f: \mathbf{Sets} \to \mathbf{Sets}$ for the finite powerset functor: $\mathcal{P}_f(X) = \{S \subseteq X \mid S \text{ is finite}\}$. A coalgebra $X \to \mathcal{P}_f(X)$ is an image-finite unlabelled transition system.

The category $\mathbf{Sets}$ has a standard factorisation system given by monos (injections) and epis (surjections), with a diagonal fill-in property. The functor $\mathcal{P}_f$ preserves injections.

Markov chains

The second endofunctor on $\mathbf{Sets}$ is the finitely supported discrete subdistribution functor $\mathcal{D}_f \colon \mathbf{Sets} \to \mathbf{Sets}$. It is described as follows.

$$\mathcal{D}_f(X) = \{\varphi \colon X \to [0,1] \mid \text{supp}(\varphi) \text{ is finite and } \textstyle\sum_{x \in X} \varphi(x) \leq 1\}.$$

The support set of a distribution φ is defined as supp$(\varphi) = \{x \mid \varphi(x) > 0\}$. A function $f \colon X \to Y$ yields a mapping $\mathcal{D}_f(f) \colon \mathcal{D}_f(X) \to \mathcal{D}_f(Y)$ by

$$\mathcal{D}_f(f)(\varphi) = \lambda y \in Y. \sum_{x \in f^{-1}(y)} \varphi(x).$$

A coalgebra $X \to \mathcal{D}_f(X)$ is a Markov chain [28,2]. In this context subdistributions are more common than distributions (with sum $= 1$ instead of ≤ 1, like above), but the difference does not matter here. The functor $\mathcal{D}_f$ preserves injections.

Markov processes

On the category **Meas** we consider the Giry functor (or monad) from [10]. It maps a measurable space $(X, \mathcal{S}(X)$ to the space $(\mathcal{G}(X), \mathcal{S}(\mathcal{G}(X)))$ of subprobability measures $\varphi \colon \mathcal{S}(X) \to [0,1]$, satisfying $\varphi(\emptyset) = 0$ and $\varphi(\bigcup_i M_i) = \sum_i \varphi(M_i)$ for countable unions of pairwise disjoint subsets $M_i \in \mathcal{S}(X)$. For each $M \in \mathcal{S}(X)$ there is an evaluation function $ev_M \colon \mathcal{G}(X) \to [0,1]$ given by $\varphi \mapsto \varphi(M)$. The set $\mathcal{G}(X)$ is equipped with the smallest σ-algebra $\mathcal{S}(\mathcal{G}(X))$ making all these maps ev_M measurable. It is generated by the collection:

$$\big(\Box_r(M)\big)_{r \in [0,1], M \in \mathcal{S}(X)} \quad \text{where} \quad \begin{aligned} \Box_r(M) &= \{\varphi \in \mathcal{G}(X) \mid \varphi(M) \geq r\} \\ &= ev_M^{-1}([r,1]). \end{aligned}$$

These $\Box_r$'s will be used later as modalities, see (13).

On a measurable function $f \colon X \to Y$ one defines $\mathcal{G}(f) \colon \mathcal{G}(X) \to \mathcal{G}(Y)$ by:

$$\mathcal{G}(f)\Big(\mathcal{S}(X) \xrightarrow{\varphi} [0,1]\Big) = \Big(\mathcal{S}(Y) \xrightarrow{f^{-1}} \mathcal{S}(X) \xrightarrow{\varphi} [0,1]\Big).$$

This $\mathcal{G}(f)$ is a measurable function since for $N \in \mathcal{S}(Y)$ one has $\mathcal{G}(f)^{-1}(\Box_r(N)) = \Box_r(f^{-1}(N))$, where $f^{-1}(N) \in \mathcal{S}(X)$.

A coalgebra $X \to \mathcal{G}(X)$ is a Markov process, see [8].

As factorisation system on **Meas** we take as "abstract monos" the collection $\mathcal{M}$ given by morphisms $f \colon X \to Y$ for which: f, as function $X \to Y$, is injective, and f^{-1}, as function $\mathcal{S}(Y) \to \mathcal{S}(X)$, is surjective.

As "abstract epis" $\mathcal{E}$ we take those morphisms $f \colon X \to Y$ for which f, as function $X \to Y$, is surjective.

Every morphism $f \colon X \to Y$ in **Meas** factors as $X \overset{e}{\twoheadrightarrow} f(X) \overset{m}{\rightarrowtail} Y$ where the image $m \colon f(X) \hookrightarrow Y$ is given the σ-algebra $\mathcal{S}(f(X)) = \{m^{-1}(M) \mid M \in \mathcal{S}(Y)\}$. Clearly,

$m^{-1}\colon \mathcal{S}(Y) \to \mathcal{S}(f(X))$ is surjective. It is not hard to see that this factorisation system satisfies the diagonal fill-in property.

Finally we check that $\mathcal{G}$ preserves the maps in $\mathcal{M}$. Given $f\colon X \to Y$ in $\mathcal{M}$, so that f is injective and $f^{-1}\colon \mathcal{S}(Y) \to \mathcal{S}(X)$ is surjective, we claim that also $\mathcal{G}(f) \in \mathcal{M}$.

– The map $\mathcal{G}(f)$ is injective: Assume $\varphi, \psi \in \mathcal{G}(X)$ with $\mathcal{G}(f)(\varphi) = \mathcal{G}(f)(\psi)$, *i.e.* $\varphi \circ f^{-1} = \psi \circ f^{-1}$. For $M \in \mathcal{S}(X)$ we can write $M = f^{-1}(N)$ for some $N \in \mathcal{S}(Y)$ since f^{-1} is surjective. But then: $\varphi(M) = \varphi(f^{-1}(N)) = \psi(f^{-1}(N)) = \psi(M)$.
– The map $\mathcal{G}(f)^{-1}\colon \mathcal{S}(\mathcal{G}(Y)) \to \mathcal{S}(\mathcal{G}(X))$ is surjective: Let $\Box_r(M) \in \mathcal{S}(\mathcal{G}(X))$ be a generator, where $M \in \mathcal{S}(X)$ and $r \in [0,1]$. Since f^{-1} is surjective we can find a measurable subset $N \in \mathcal{S}(Y)$ with $M = f^{-1}(N)$. But then $\mathcal{G}(f)^{-1}(\Box_r(N)) = \Box_r(f^{-1}(N)) = \Box_r(M)$. This is enough to conclude surjectivity of $\mathcal{G}(f)^{-1}$ since all set operations are preserved by inverse images.

Remark 2. In Remark 1 we have used the (discrete) adjunction between sets and meaurable spaces in order to relate the adjunctions with meet semilattices. Here we shall also relate the distribution functor $\mathcal{D}_f$ and the Giry functor $\mathcal{G}$, in the situation:

$$\begin{tikzcd}
\mathbf{Meas}^{\mathrm{op}} \arrow[loop left, "\mathcal{G}"] \arrow[rr, "\mathcal{S}"] \arrow[d, bend left, "U"] & & \mathbf{MSL} \arrow[ll, bend left, "\mathcal{F}"'] \arrow[dll, bend left, "\mathcal{F}"] \\
\mathbf{Sets}^{\mathrm{op}} \arrow[loop left, "\mathcal{D}_f"] \arrow[u, "D"] \arrow[urr, "\mathcal{P}"] & &
\end{tikzcd}$$

There is an obvious natural transformation $\rho\colon \mathcal{D}_f U \Rightarrow U\mathcal{G}$ with component $\rho_X\colon \mathcal{D}_f(X) \to \mathcal{G}(X)$ on $X \in \mathbf{Meas}$, with σ-algebra $\mathcal{S}(X)$, given by:

$$\left[\begin{array}{c}\varphi \in \mathcal{D}_f(X) \text{ i.e. } \varphi\colon X \to [0,1] \text{ with} \\ \mathrm{supp}(\varphi) \text{ finite and } \sum_{x\in X}\varphi(x) \le 1\end{array}\right] \longmapsto \left[\begin{array}{l}\text{the measure } \mathcal{S}(X) \to [0,1] \text{ given by} \\ M \mapsto \varphi(M) = \sum_{x\in M}\varphi(x)\end{array}\right]$$

This ρ captures the standard way in which a discrete measure (on points) forms a proper measure (on events/subsets).

4 Expressivity results

The aim of this section is to show that the functors $\mathcal{P}_f$, $\mathcal{D}_f$ and $\mathcal{G}$ given in the examples of Section 3 have expressive logics via Theorem 1. Thus we shall define, for each of them, an associated modality functor with *interpretation* natural transformation σ whose transpose $\overline{\sigma}$ is (componentwise) mono. We shall first describe the finite powerset $\mathcal{P}_f$ case because it is already well-studied, see *e.g.* [11,18,5]. This will set the scene for the probabilistic examples. Their expressivity is the main topic of this paper.

4.1 Boolean logic for image-finite transition systems

Expressivity of modal logic for image-finite transition systems has originally been proved in [11]. Such transition systems can be captured as coalgebras of the finite powerset functor $\mathcal{P}_f$. Coalgebraic generalisations of this expressivity result have been studied in *e.g.* [26,5].

Here we shall reproduce this expressivity result for $\mathcal{P}_f$ in the context of dual adjunctions, following [5]. We do so not only in order to prepare for the more complicated probabilistic examples $\mathcal{D}_f$ and $\mathcal{G}$ later on, but also to indicate where the negations of Booleans algebras are used. The main point of the latter examples $\mathcal{D}_f$ and $\mathcal{G}$ is that they do not involve negation (nor disjunctions).

For a transition system $c: X \to \mathcal{P}_f(X)$ as coalgebra, the familiar modal operator $\Box(c)$ is described as:

$$\begin{aligned}\Box(c)(S) &= \{x \in X \mid \forall y.\, x \to y \Rightarrow y \in S\} \\ &= \{x \in X \mid c(x) \subseteq S\} \\ &= c^{-1}(\Box S),\end{aligned}$$

where $\Box: \mathcal{P}(X) \to \mathcal{P}(\mathcal{P}_f(X))$ is defined independently of the coalgebra c as:

$$\Box(S) = \{u \in \mathcal{P}_f(X) \mid u \subseteq S\}. \tag{7}$$

It is not hard to see that $\Box$ preserves finite meets (intersections) and is thus a morphism in the category **MSL**. This leads to the following more abstract description.

Proposition 2. *There is an endofunctor L:* ***BA*** $\to$ ***BA*** *in a situation:*

$$\begin{tikzcd} \mathbf{Sets}^{op} \arrow[loop left, "\mathcal{P}_f"] \arrow[r, bend left, "\mathcal{P}"] & \boldsymbol{BA} \arrow[l, bend left, "\mathcal{F}_u"] \arrow[loop right, "L"] \end{tikzcd}$$

which is an instance of (6), such that the definition of $\Box$ in (7) corresponds to an interpretation natural transformation $\boxtimes: L\mathcal{P} \Rightarrow \mathcal{P}\mathcal{P}_f$.

The Boolean algebras LA can be understood as models of Boolean logic with a finite meet preserving modal operator $\blacksquare$.

This shows that the familiar $\Box$ operator fits into the current framework of dual adjunctions, so that we can use Theorem 1. The details of the functor L are not so relevant, but are given for reasons of completeness.

Proof. The construction of the functor L (and $\boxtimes$) follows [18]. There is an obvious forgetful functor V: **BA** $\to$ **MSL**, which has a left adjoint G. We write $L = GV$: **BA** $\to$ **BA** for the resulting functor (actually comonad). By θ and ξ we denote the unit and the counit of this adjunction, respectively. Notice that morphisms $LA \to B$ in **BA** correspond, via the adjunction, to functions $VA \to VB$ that preserve finite meets. The map $\Box: \mathcal{P}(X) \to \mathcal{P}\mathcal{P}_f(X)$ from (7) is formally such a map $V\mathcal{P}(X) \to V\mathcal{P}\mathcal{P}_f(X)$ in the category **MSL**. Hence it corresponds to a map $\boxtimes: L\mathcal{P}(X) \to \mathcal{P}\mathcal{P}_f(X)$ as claimed.

We then put $\blacksquare = \theta_{VA} \circ V(\xi_A) : VLA \to VLA$. The unit $\theta_{VA} : VA \to VLA$ of the adjunction $G \dashv V$ embeds elements (formulas) of a Boolean algebra A into its extension LA with $\blacksquare$. This finite meet preserving endofunction (or modal operator) satisfies:

1. idempotency: $\blacksquare \circ \blacksquare = \blacksquare$;
2. naturality: $VLf \circ \blacksquare = \blacksquare \circ VLf$, for $f : A \to B$ in **BA**;
3. $V\widehat{f} \circ \blacksquare = f \circ V(\xi_A)$ for $f : VA \to VB$ finite meet preserving, with $\widehat{f} : LA \to B$ as corresponding transpose. $\lozenge$

This last point yields a relation between the three boxes, as expressed by the following commuting diagram.

$$
\begin{tikzcd}
VL\mathcal{P}X \arrow[r, "\blacksquare"] \arrow[d, "V(\xi)"'] & VL\mathcal{P}X \arrow[d, "V(\boxtimes)"] \\
V\mathcal{P}X \arrow[r, "\Box"] & V\mathcal{P}\mathcal{P}_f X
\end{tikzcd}
\tag{8}
$$

The next result now gives a semantical reformulation, following [5], of the expressivity result of [11]. It uses Theorem 1.

Theorem 2. *The transpose $\overline{\boxtimes} : \mathcal{P}_f\mathcal{F}_u \Rightarrow \mathcal{F}_u L$ of $\boxtimes$ in Proposition 2, according to Proposition 1, is componentwise mono. Hence Boolean modal logic with $\Box$ is expressive for image-finite transition systems.*

Proof. By unraveling the definition of $\overline{\boxtimes}$ given in the proof of Proposition 1 we see, for finite $S \subseteq \mathcal{F}_u A$,

$$\overline{\boxtimes}(S) = \Big\{\varphi \in LA \mid S \in \boxtimes\Big(L(\eta)(\varphi)\Big)\Big\},$$

where $\eta = \lambda a. \{\alpha \mid a \in \alpha\}$ is the unit $A \to \mathcal{P}\mathcal{F}_u A$ of the adjunction $\mathcal{F}_u \dashv \mathcal{P}$.

In order to prove injectivity of $\overline{\boxtimes}$, assume $S, M \in \mathcal{P}_f\mathcal{F}_u(A)$, $S \neq M$, say $\alpha \in S$, $\alpha \notin M = \{\beta_1, \ldots, \beta_n\}$ for ultrafilters $\alpha, \beta_i \in \mathcal{F}_u(A)$. Then $\alpha \neq \beta_i$, so there are elements $b_i \in \beta_i$ with $b_i \notin \alpha$. These b_i exist because we work in a Boolean algebra, with negation $\neg$, and each ultrafilter γ satisfies either $a \in \gamma$ of $\neg a \in \gamma$, for each $a \in A$.

We can put these b_i into LA as $\theta(b_i)$, take their join and write $a = \blacksquare(\bigvee_i \theta(b_i)) \in LA$. Then, for any set $W \in \mathcal{P}_f\mathcal{F}_u(A)$,

$$
\begin{array}{rll}
a \in \overline{\boxtimes}(W) & \Longleftrightarrow W \in \boxtimes\Big(L(\eta)(\blacksquare \bigvee_i \theta(b_i))\Big) & \\
& \Longleftrightarrow W \in \boxtimes\Big(\blacksquare L(\eta)(\bigvee_i \theta(b_i))\Big) & \text{by 2. in the proof of Proposition 2} \\
& \Longleftrightarrow W \in \boxtimes\Big(\blacksquare \bigvee_i L(\eta)(\theta(b_i))\Big) & \\
& \Longleftrightarrow W \in \boxtimes\Big(\blacksquare \bigvee_i \theta(\eta(b_i))\Big) & \text{by naturality} \\
& \Longleftrightarrow W \in \Box\Big(V(\xi) \bigvee_i \theta(\eta(b_i))\Big) & \text{via (8)} \\
& \Longleftrightarrow W \in \Box\Big(\bigcup_i V(\xi)\theta(\eta(b_i))\Big) & \text{since } \xi \text{ is a map in } \mathbf{BA} \\
& \Longleftrightarrow W \in \Box\Big(\bigcup_i \eta(b_i)\Big) & \text{by the triangular equations} \\
& \Longleftrightarrow W \subseteq \bigcup_i \eta(b_i) & \\
& \Longleftrightarrow \forall \alpha \in W. \exists i. b_i \in \alpha. &
\end{array}
$$

Then $a \in \overline{\boxtimes}(S)$, but $a \notin \overline{\boxtimes}(M)$, proving that $\overline{\boxtimes}$ is mono. $\Diamond$

Remark 3. As is well-known, the finite powerset $\mathcal{P}_f$ is a monad on $\mathbf{Sets}$—and hence a comonad on $\mathbf{Sets}^{\mathrm{op}}$. The functor $L\colon \mathbf{BA} \to \mathbf{BA}$ in Proposition 2 is a comonad by construction. We do not use these comonad structures in this paper but we do like to point out that they are related in the following sense: the functor $\mathcal{P}\colon \mathbf{Sets}^{\mathrm{op}} \to \mathbf{BA}$ with natural transformation $\boxtimes\colon L\mathcal{P} \Rightarrow \mathcal{P}\mathcal{P}_f$ is a morphism of comonads. Hence there is a close connection between the two sides of the dual adjunction in Proposition 2.

The underlying reason is that the natural transformation $\Box\colon \mathcal{P} \Rightarrow \mathcal{P}\mathcal{P}_f$ from (7) commutes with the monad operations $\iota = \{-\}, \mu = \bigcup$ of the monad $\mathcal{P}_f$, in the sense that the following diagrams commute (in **MSL**).

$$\begin{tikzcd}
& V\mathcal{P} \arrow[r, "\Box"] \arrow[dl, equal] \arrow[d, "\Box"'] & V\mathcal{P}\mathcal{P}_f \arrow[d, "\Box\mathcal{P}_f"] \\
V\mathcal{P} & V\mathcal{P}\mathcal{P}_f \arrow[l, "V\mathcal{P}\iota"] \arrow[r, "V\mathcal{P}\mu"'] & V\mathcal{P}\mathcal{P}_f^2
\end{tikzcd}$$

As a consequence we get, for example, the result:

$$\begin{tikzcd}
& L\mathcal{P} \arrow[dl, "\xi"'] \arrow[d, "\boxtimes"] \\
\mathcal{P} & \mathcal{P}\mathcal{P}_f \arrow[l, "\mathcal{P}\iota"]
\end{tikzcd}$$

since:

$$\mathcal{P}\iota \circ \boxtimes = \mathcal{P}\iota \circ \xi \circ G(\Box) = \xi \circ GV\mathcal{P}\iota \circ G(\Box) = \xi.$$

Similarly for commutation of $\boxtimes$ with (co)multiplications.

4.2 Finite-conjunctions logic for Markov chains

The situation for the finitely supported distribution functor $\mathcal{D}_f$ is similar, but a bit more complicated. We obtain an expressivity result for finitely supported probabilistic systems and logic with the standard modalities and only finite conjunctions.

It has been shown in [5] that Boolean logic together with probabilistic modalities is expressive for Markov chains. That result also fits in the framework of dual adjunctions, reasoning similarly as in the case of transition systems. Here we provide an expressivity result for a weaker logic, namely with finite conjunctions only.

We start from the "probabilistic modalities" $\Box_r\colon \mathcal{P}(X) \to \mathcal{P}\mathcal{D}_f(X)$ which can be defined independent of the coalgebra under consideration as:

$$\Box_r(S) = \{\varphi \in \mathcal{D}_f(X) \mid \textstyle\sum_{s \in S} \varphi(x) \geq r\}, \tag{9}$$

for any $r \in \mathbb{Q} \cap [0, 1]$. It is not hard to see that $\Box_r$ is a monotone function, and thus a map in the category **PoSets** of posets and monotone functions.

From these $\Box_r$ we get a modality functor like in Proposition 2, bringing us in the framework of dual adjunctions.

Proposition 3. *There is an endofunctor $K\colon \boldsymbol{MSL} \to \boldsymbol{MSL}$ in a situation:*

$$\begin{tikzcd} \mathbf{Sets}^{op} \arrow[loop left, "\mathcal{D}_f"] \arrow[r, bend left, "\mathcal{P}"] & \boldsymbol{MSL} \arrow[l, bend left, "\mathcal{F}"] \arrow[loop right, "K"] \end{tikzcd}$$

which is an instance of (6), such that all $\Box_r$ in (9) correspond to an interpretation natural transformation $\boxtimes\colon K\mathcal{P} \Rightarrow \mathcal{P}\mathcal{D}_f$.

Proof. In order to construct K, we now consider the forgetful functor $V\colon \mathbf{MSL} \to \mathbf{PoSets}$ with its left adjoint H. As before, we denote the unit and the counit of this adjunction by θ and ξ, respectively. We then define the functor $K\colon \mathbf{MSL} \to \mathbf{MSL}$ as:

$$K(A) = \textstyle\coprod_{q\in\mathbb{Q}\cap[0,1]} HVA$$

Here we use that the category **MSL** has arbitrary coproducts—which follows for instance from Linton's Theorem (see [1]), using that **MSL** is algebraic over $\mathbf{Sets}$ (and thus cocomplete). A map $KA \to B$ in **MSL** now corresponds to a $(\mathbb{Q} \cap [0,1])$-indexed family of (monotone) functions $VA \to VB$ in **PoSets**. The family of maps $\Box_r\colon \mathcal{P}(X) \to \mathcal{P}\mathcal{D}_f(X)$ from (9) is formally a $(\mathbb{Q}\cap[0,1])$-indexed family of functions $V\mathcal{P}(X) \to V\mathcal{P}\mathcal{D}_f(X)$ in **PoSets**. Hence, this family corresponds to a (natural) map $\boxtimes\colon K\mathcal{P}(X) \to \mathcal{P}\mathcal{D}_f(X)$. $\Diamond$

As before, the meet semilattices KA can be seen as models of logic with only finite conjunctions, with a family of order preserving modal operators $\blacksquare_r\colon VKA \to VKA$, for $r \in \mathbb{Q}\cap[0,1]$, defined as composite:

$$V\Big(\textstyle\coprod_q HVA\Big) \xrightarrow{V(\nabla)} VHVA \xrightarrow{V(\xi_A)} VA \xrightarrow{\theta_{VA}} VHVA \xrightarrow{V(\kappa_r)} V\Big(\textstyle\coprod_q HVA\Big)$$

where κ_r is a coprojection and $\nabla = [\mathrm{id}]_q$ is the cotuple of identities. These monotone modal operators satisfy idempotency and naturality, and commute appropriately with $\Box$ and $\boxtimes$. We do not elaborate on these $\blacksquare_r$'s because we do not need them (explicitly) in the expressivity proof below.

Recall that the unit of the filter-powerset adjunction $\eta\colon A \to \mathcal{P}\mathcal{F}(A)$, given by $\eta(a) = \{\alpha \in \mathcal{F}A \mid a \in \alpha\}$, preserves finite meets. We define for an arbitrary subset $\alpha \subseteq A$ the set of filters $\uparrow\alpha = \{\beta \in \mathcal{F}A \mid \alpha \subseteq \beta\}$ that contain α. This map $\uparrow\colon \mathcal{P}A \to \mathcal{P}\mathcal{F}(A)$ can be seen as free extension of η from A to the complete lattice $(\mathcal{P}\mathcal{F}(A), \supseteq)$, since:

$$\uparrow\alpha = \{\beta \in \mathcal{F}A \mid \forall a \in \alpha.\, a \in \beta\} = \bigcap_{a\in\alpha} \eta(a).$$

As a result $\alpha \subseteq \alpha'$ implies $\uparrow\alpha \supseteq \uparrow\alpha'$. We further note that:

$$\begin{aligned} \uparrow\{a_1,\ldots,a_n\} &= \{\beta \in \mathcal{F}A \mid a_1,\ldots,a_n \in \beta\} \\ &= \{\beta \in \mathcal{F}A \mid a_1 \wedge \cdots \wedge a_n \in \beta\} \\ &= \eta(a_1 \wedge \cdots \wedge a_n). \end{aligned} \tag{10}$$

The next result holds in greater generality than stated—since the requirement that A is a semilattice is not used—but we formulate it in the way that we shall use it.

Lemma 1. *Let A be arbitrary meet semilattice and $S \subseteq \mathcal{F}(A)$ be finite. Then for each $\alpha \subseteq A$ there is a finite $\alpha_f \subseteq \alpha$ with $S \cap {\uparrow}\alpha = S \cap {\uparrow}\alpha_f$.*

Proof. Write $S - {\uparrow}\alpha = \{\beta_1, \ldots, \beta_n\}$, where by construction $\alpha \not\subseteq \beta_i$, say via elements $a_i \in \alpha, a_i \notin \beta_i$. Take $\alpha_f = \{a_1, \ldots, a_n\} \subseteq \alpha$. Then ${\uparrow}\alpha_f \supseteq {\uparrow}\alpha$ and so $S \cap {\uparrow}\alpha_f \supseteq S \cap {\uparrow}\alpha$. For the converse, assume $\beta \in S \cap {\uparrow}\alpha_f$. If $\beta \notin {\uparrow}\alpha$, there must be an i with $\beta = \beta_i$ and thus $a_i \notin \beta$. This contradicts $\beta \in {\uparrow}\alpha_f$. $\Diamond$

We can now state and prove the expressivity result for logic with finite conjunctions and finitely supported probabilistic systems. It uses Theorem 1.

Theorem 3. *The transpose $\overline{\boxtimes}\colon \mathcal{D}_f\mathcal{F} \Rightarrow \mathcal{F}K$ of $\boxtimes$ in Proposition 3, according to Proposition 1, is componentwise mono. Hence modal logic with finite conjunctions and probabilistic modalities is expressive for finitely supported probabilistic systems.*

Proof. The transpose $\overline{\boxtimes}\colon \mathcal{D}_f\mathcal{F}(A) \to \mathcal{F}K(A)$ is given on a distribution $\Phi\colon \mathcal{F}(A) \to [0,1]$ on filters of A as:

$$\overline{\boxtimes}(\Phi) = \left\{\varphi \in KA \mid \Phi \in \boxtimes_{\mathcal{F}(A)}\left(K(\eta)(\varphi)\right)\right\}.$$

We first note that for an element $a \in VA$ we have $\theta(a) \in VHVA$ and thus $a' = V(\kappa_r)(\theta(a)) \in VKA$. For such elements a' we reason as follows, writing forgetful functors $V\colon \mathbf{MSL} \to \mathbf{PoSets}$ explicitly in order to justify all manipulations.

$$\begin{aligned}
& a' = V(\kappa_r)(\theta(a)) \in V(\overline{\boxtimes})(\Phi) \\
& \Longleftrightarrow \Phi \in \big(V(\boxtimes \circ K(\eta)) \circ V(\kappa_r) \circ \theta\big)(a) \\
& \Longleftrightarrow \Phi \in \big(V([\xi \circ H(\Box_q)]_q \circ [\kappa_q \circ HV(\eta)]_q \circ \kappa_r) \circ \theta\big)(a) \\
& \qquad\qquad \text{by definition of } \boxtimes \text{ and } K \\
& \Longleftrightarrow \Phi \in \big(V([\xi \circ H(\Box_q)]_q \circ \kappa_r \circ VHV(\eta) \circ \theta\big)(a) \\
& \Longleftrightarrow \Phi \in \big(V(\xi) \circ VH(\Box_r) \circ \theta \circ V(\eta)\big)(a) \\
& \Longleftrightarrow \Phi \in \big(V(\xi) \circ \theta \circ \Box_r \circ V(\eta)\big)(a) \\
& \Longleftrightarrow \Phi \in \big(\Box_r \circ V(\eta)\big)(a) \\
& \Longleftrightarrow \textstyle\sum_{\alpha \in \eta(a)} \Phi(\alpha) \geq r \\
& \Longleftrightarrow \Phi(\eta(a)) \geq r.
\end{aligned}$$

Towards injectivity of $\overline{\boxtimes}$ assume $\overline{\boxtimes}(\Phi) = \overline{\boxtimes}(\Psi)$. By the reasoning above we then get $\Phi(\eta(a)) \geq r \;\Leftrightarrow\; \Psi(\eta(a)) \geq r$ for all $r \in \mathbb{Q} \cap [0,1]$ and all $a \in A$. This yields:

$$\Phi(\eta(a)) = \Psi(\eta(a)). \tag{11}$$

Let $S = \{\alpha \in \mathcal{F}A \mid \Phi(\alpha) > 0\} \cup \{\alpha \in \mathcal{F}A \mid \Psi(\alpha) > 0\}$ be the joined support of Φ and Ψ. Since Φ and Ψ have a finite support, S is also finite. We now reach our second conclusion: for all filters $\alpha \in \mathcal{F}A$,

$$
\begin{aligned}
\Phi(\uparrow \alpha) &= \Phi(S \cap \uparrow \alpha) && \text{since values outside } S \text{ do not contribute to the sum} \\
&= \Phi(S \cap \uparrow \alpha_f) && \text{with } \alpha_f \subseteq \alpha \text{ finite, as in Lemma 1} \\
&= \Phi(\uparrow \alpha_f) \\
&= \Phi(\eta(\textstyle\bigwedge \alpha_f)) && \text{by (10), since } \alpha_f \text{ is finite} \\
&= \Psi(\eta(\textstyle\bigwedge \alpha_f)) && \text{by (11)} \\
&= \dots && \text{as before} \\
&= \Psi(\uparrow \alpha).
\end{aligned}
\tag{12}
$$

If $\Phi \neq \Psi$ we can now construct a contradiction: let α be a maximal filter with $\Phi(\alpha) \neq \Psi(\alpha)$. Such a maximal filter exists since Φ and Ψ have finite support. Thus $\Phi(\beta) = \Psi(\beta)$ for $\beta \supsetneq \alpha$. But then:

$$
\begin{aligned}
\Phi(\uparrow \alpha) &= \Phi(\alpha) + \textstyle\sum_{\beta \supsetneq \alpha} \Phi(\beta) \\
&= \Phi(\alpha) + \textstyle\sum_{\beta \supsetneq \alpha} \Psi(\beta) \\
&\neq \Psi(\alpha) + \textstyle\sum_{\beta \supsetneq \alpha} \Psi(\beta) \\
&= \Psi(\uparrow \alpha),
\end{aligned}
$$

contradicting (12). $\Diamond$

The line of reasoning leading to expressivity of modal logic with finite conjunctions for $\mathcal{D}_f$ also applies to the (finite) multiset functor

$$\mathcal{M}_f(X) = \{\varphi: X \to \mathbb{N} \mid \text{supp}(\varphi) = \{x \in X \mid \varphi(x) > 0\} \text{ is finite}\}$$

with modal operators $\Box_k: \mathcal{P}(X) \to \mathcal{P}\mathcal{M}_f(X)$, for $k \in \mathbb{N}$ given by:

$$\Box_k(S) = \{\varphi \in \mathcal{M}_f(X) \mid \textstyle\sum_{x \in S} \varphi(x) \geq k\}.$$

This leads to expressivity for so-called graded modal logic, see [26].

4.3 Finite-conjunctions logic for Markov processes

We now present an expressivity result for general, non-discrete, probabilistic systems and logic with the standard modalities and only finite conjunctions. This expressivity result was first shown in [7,8] for Markov processes over analytic spaces, and recently for general Markov processes over any measurable space [6]. It is common in the categorial treatment of non-discrete probabilistic systems (c.f. [9,7,8]) to make the detour through analytic or Polish spaces. The main reason is that bisimilarity (in terms of spans) can not be described in general measure spaces, due to non-existence of pullbacks. However, as we already noted before, behavioural equivalence can, which is also one of the main points of [6] where an explicit characterization of behavioural equivalence under the name *event bisimulation* is given. Hence, we consider general measure spaces.

We start from the "probabilistic modalities" $\Box_r : \mathcal{S}(X) \to \mathcal{SG}(X)$ which can be defined independently of the coalgebra under consideration as:

$$\Box_r(M) = \{\varphi \in \mathcal{G}(X) \mid \varphi(M) \geq r\}, \tag{13}$$

for any $r \in \mathbb{Q} \cap [0,1]$. Note that they are well-defined *i.e.* $\Box_r(M) \in \mathcal{SG}(X)$ by definition.

These modalities are obviously monotone. Hence we can use the functor $K : \mathbf{MSL} \to \mathbf{MSL}$ from Proposition 3 to transform these $\Box_r : V\mathcal{S}(X) \to V\mathcal{SG}(X)$ in **PoSets** into a natural transformation $\boxtimes : K\mathcal{S} \Rightarrow \mathcal{SG}$ in the situation:

$$\begin{tikzcd} \mathbf{Meas}^{\mathrm{op}} \arrow[loop left, "\mathcal{G}"] \arrow[r, bend left, "\mathcal{S}"] & \mathbf{MSL} \arrow[loop right, "K"] \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

We can now present the expressivity result for logic with finite conjunctions and finitely supported probabilistic systems (from [8,6]), using Theorem 1.

Theorem 4. *The transpose $\overline{\boxtimes} : \mathcal{GF} \Rightarrow \mathcal{F}K$ of $\boxtimes$ described above, according to Proposition 1, is componentwise mono. Hence modal logic with finite conjunctions and probabilistic modalities is expressive for Markov processes.*

Proof. The transpose $\overline{\boxtimes} : \mathcal{GF}(A) \to \mathcal{F}K(A)$ is given, on a measure $\Phi : \mathcal{F}(A) \to [0,1]$ on filters, of A as:

$$\overline{\boxtimes}(\Phi) = \left\{\varphi \in KA \mid \Phi \in \boxtimes_{\mathcal{F}(A)}\left(K(\eta)(\varphi)\right)\right\}.$$

As before, $\overline{\boxtimes}(\Phi) = \overline{\boxtimes}(\Psi)$ implies $\Phi(\eta(a)) = \Psi(\eta(a))$ for all $a \in A$. Now we use the same measure-theoretic result that is used in [8]: Let φ, ψ be finite measures on a measure space (X, Σ) with generated σ-algebra. If the whole space X is a countable union of generators, and if φ and ψ coincide on the generators, then $\varphi = \psi$.

In our case Φ, Ψ are finite measures on $\mathcal{F}(A)$ with its σ-algebra $\mathcal{SF}(A)$ generated by the sets $\eta(a)$, for $a \in A$. The whole space $\mathcal{F}(A)$ is $\eta(\top)$ since $\top$ is a top element in A which is contained in every filter. Moreover, Φ and Ψ coincide on the generators. Hence, $\Phi = \Psi$ which completes the proof. $\Diamond$

4.4 Relating Markov chains and Markov processes

In Remarks 1 and 2 we have already seen how the categories and functors for Markov chains and for Markov processes are related. Here we complete the picture by showing how these relations can be used to obtain expressivity for chains from expressivity for processes. This depends on auxiliary results which are of interest on their own.

In order to disambiguate the two modal operators $\Box_r$ for chains (9) and processes (13) we shall now write them with additional superscripts, namely:

$$\begin{array}{rcl} \mathcal{P}(X) \xrightarrow{\Box_r^{\mathcal{D}_f}} \mathcal{PD}_f(X) & \text{is} & S \longmapsto \{\varphi \in \mathcal{D}_f(X) \mid \sum_{x \in S} \varphi(x) \geq r\} \\ \mathcal{S}(X) \xrightarrow{\Box_r^{\mathcal{G}}} \mathcal{SG}(X) & \text{is} & M \longmapsto \{\varphi \in \mathcal{G}(X) \mid \varphi(M) \geq r\}. \end{array}$$

They give rise to natural transformations:

$$K\mathcal{P} \xrightarrow{\boxtimes^{\mathcal{D}_f}} \mathcal{P}\mathcal{D}_f \quad \text{and} \quad K\mathcal{S} \xrightarrow{\boxtimes^{\mathcal{G}}} \mathcal{S}\mathcal{G}$$

with transposes:

$$\mathcal{D}_f\mathcal{F} \xrightarrow{\overline{\boxtimes}^{\mathcal{D}_f}} \mathcal{F}K \quad \text{and} \quad \mathcal{G}\mathcal{F} \xrightarrow{\overline{\boxtimes}^{\mathcal{G}}} \mathcal{F}K$$

We next show how they are related via the natural transformation $\rho\colon \mathcal{D}_f U \Rightarrow U\mathcal{G}$ given by $\rho(\varphi) = \lambda M.\, \sum_{x\in M} \varphi(x)$, where U is the forgetful functor **Meas** $\to$ $\mathbf{Sets}$.

Lemma 2. *For a set X, a distribution $\varphi \in \mathcal{D}_f(X)$ and a subset $S \in \mathcal{P}(X) = \mathcal{S}(DX)$, where $D(X)$ is the set X with the discrete σ-algebra $\mathcal{P}(X)$, one has:*

$$\varphi \in \Box_r^{\mathcal{D}_f}(S) \Longleftrightarrow \rho(\varphi) \in \Box_r^{\mathcal{G}}(S).$$

In a diagram:

$$\begin{tikzcd}
V\mathcal{P}(X) \arrow[r, "\Box_r^{\mathcal{D}_f}"] \arrow[d, equal] & V\mathcal{P}\mathcal{D}_f(X) \arrow[r, equal] & V\mathcal{P}\mathcal{D}_f UD(X) \\
V\mathcal{S}D(X) \arrow[r, "\Box_r^{\mathcal{G}}"] & V\mathcal{S}\mathcal{G}D(X) \arrow[r, hook] & V\mathcal{P}U\mathcal{G}D(X) \arrow[u, "V\mathcal{P}(\rho_{DX})"']
\end{tikzcd}$$

*where V is the forgetful functor **MSL** $\to$ **PoSets**. As a result:*

$$\boxtimes^{\mathcal{D}_f} = \mathcal{P}(\rho) \circ \boxtimes^{\mathcal{G}} \qquad \textit{and} \qquad \overline{\boxtimes}^{\mathcal{D}_f} = U(\overline{\boxtimes}^{\mathcal{G}}) \circ \rho\mathcal{F},$$

where the latter equation involves the diagram:

$$\begin{tikzcd}
\mathcal{D}_f\mathcal{F} \arrow[rr, "\overline{\boxtimes}^{\mathcal{D}_f}"] \arrow[d, equal] & & \mathcal{F}K \arrow[d, equal] \\
\mathcal{D}_f U\mathcal{F} \arrow[r, "\rho\mathcal{F}"] & U\mathcal{G}\mathcal{F} \arrow[r, "U(\overline{\boxtimes}^{\mathcal{G}})"] & U\mathcal{F}K
\end{tikzcd}$$

Proof. By unravelling the definitions we get:

$$\begin{aligned}
\varphi \in \Box_r^{\mathcal{D}_f}(S) &\Longleftrightarrow \rho(\varphi)(S) = \textstyle\sum_{x\in S} \varphi(x) \geq r \\
&\Longleftrightarrow \rho(\varphi) \in \Box_r^{\mathcal{G}}(S).
\end{aligned}$$

Then:

$$\begin{aligned}
\boxtimes^{\mathcal{D}_f} &= [\xi \circ H(\Box_r^{\mathcal{D}_f})]_r \\
&= [\xi \circ H(V\mathcal{P}(\rho) \circ \Box_r^{\mathcal{G}})]_r \\
&= [\mathcal{P}(\rho) \circ \xi \circ H(\Box_r^{\mathcal{G}})]_r \\
&= \mathcal{P}(\rho) \circ [\xi \circ H(\Box_r^{\mathcal{G}})]_r \\
&= \mathcal{P}(\rho) \circ \boxtimes^{\mathcal{G}}.
\end{aligned}$$

Finally, for $\varPhi \in \mathcal{D}_f\mathcal{F}(A)$,

$$\begin{aligned}
\big(U(\overline{\boxtimes}^{\mathcal{G}}) \circ \rho_{\mathcal{F}(A)}\big)(\varPhi) &= \{\psi \in KA \mid \rho(\varPhi) \in \boxtimes^{\mathcal{G}}(K(\eta)(\psi))\} \\
&= \{\psi \in KA \mid \varPhi \in \rho^{-1} \boxtimes^{\mathcal{G}} (K(\eta)(\psi))\} \\
&= \{\psi \in KA \mid \varPhi \in \big(\mathcal{P}(\rho) \circ \boxtimes^{\mathcal{G}}(K(\eta)(\psi))\big)\} \\
&= \{\psi \in KA \mid \varPhi \in \boxtimes^{\mathcal{D}_f}(K(\eta)(\psi))\} \\
&= \overline{\boxtimes}^{\mathcal{D}_f}(\varPhi).
\end{aligned}$$

$\Diamond$

Lemma 3. *The natural transformation $\rho\mathcal{F}\colon \mathcal{D}_f U\mathcal{F} \Rightarrow U\mathcal{G}\mathcal{F}$ is componentwise mono.*

Proof. For a meet semilattice A we need to show that the mapping $\rho_{\mathcal{F}(A)}\colon \mathcal{D}_f\mathcal{F}(A) \to \mathcal{G}\mathcal{F}(A)$, given by $\rho(\varPhi) = \lambda M \in \mathcal{S}\mathcal{F}(A).\, \varPhi(M)$ is injective—where, as before, $\varPhi(M) = \sum_{\alpha\in M} \varPhi(\alpha)$. Assume therefore $\varPhi, \varPsi \in \mathcal{D}_f\mathcal{F}(A)$ satisfy $\rho(\varPhi) = \rho(\varPsi)$. In order to show $\varPhi = \varPsi$ we assume an arbitrary $\alpha \in \mathcal{F}(A)$ and wish to show $\varPhi(\alpha) = \varPsi(\alpha)$.

Let $S = \mathrm{supp}(\varPhi) \cup \mathrm{supp}(\varPsi)$ be the join of the two (finite) supports. We may assume $\alpha \in S$, because otherwise $\varPhi(\alpha) = 0 = \varPsi(\alpha)$ and we are done. We form two subsets $B, C \subseteq A$ in the following way. For each $\beta \in S - \{\alpha\}$ we have $\beta \neq \alpha$, so that either $\exists b \in \alpha.\, b \notin \beta$ or $\exists c \in \beta.\, c \notin \alpha$. In the first case we choose such a $b \in \alpha - \beta$ and put it in B, and in the second case we take a $c \in \beta - \alpha$ and put it in C. Since S is finite both B and C are finite (and obtained in finitely many steps). We now define $M \subseteq \mathcal{F}(A)$ as:

$$\begin{aligned}
M &= \{\gamma \in \mathcal{F}(A) \mid B \subseteq \gamma \text{ and } C \cap \gamma = \emptyset\} \\
&= \left(\bigcap_{b\in B} \eta(b)\right) \cap \left(\bigcap_{c\in C} \neg\eta(c)\right).
\end{aligned}$$

This second line describes M as countable (actually finite) intersection of measurable subsets. Hence $M \in \mathcal{S}\mathcal{F}(A)$ so that $\varPhi(M) = \varPsi(M)$ since $\rho(\varPhi) = \rho(\varPsi)$.

Next we claim:

$$M \cap S = \{\alpha\}.$$

The inclusion $(\supseteq)$ is obvious by construction of B, C. For $(\subseteq)$ assume $\gamma \in M \cap S$, but $\gamma \neq \alpha$. Then we have constructed either:

– a $b \in B$ with $b \in \alpha - \gamma$; this is impossible since $B \subseteq \gamma$.
– a $c \in C$ with $c \in \gamma - \alpha$. But since $C \cap \gamma = \emptyset$ this is also impossible.

Hence $\gamma = \alpha$.

We now have $\varPhi(\alpha) = \varPhi(\{\alpha\}) = \varPhi(M \cap S) = \varPhi(M) = \varPsi(M) = \varPsi(\alpha)$, as required. Thus $\rho_{\mathcal{F}(A)}$ is injective. $\Diamond$

Corollary 1. *Expressivity of modal logic for Markov chains follows from expressivity for Markov processes, in the sense that $\overline{\boxtimes}^{\mathcal{D}_f}$ is componentwise mono because $\overline{\boxtimes}^{\mathcal{G}}$ is.*

Proof. By Theorem 4 we know that $\overline{\boxtimes}^{\mathcal{G}}$ is componentwise mono. Hence so is $U(\overline{\boxtimes}^{\mathcal{G}})$, because the forgetful functor $U\colon \mathbf{Meas} \to \mathbf{Sets}$ is a right adjoint (and thus preserves monos). By Lemma 2 we know that $\overline{\boxtimes}^{\mathcal{D}_f} = U(\overline{\boxtimes}^{\mathcal{G}}) \circ \rho\mathcal{F}$. Lemma 3 shows that $\rho\mathcal{F}$ is componentwise mono. This completes the proof. $\Diamond$

5 Conclusions

We have analysed the semantics and logic of three examples of possibilistic and probabilistic state-based systems in a uniform categorical framework and proved expressivity in each of these cases.

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