The structure of natural numbers

is helpful for proving properties

$\forall n[n \in \mathbb{N}: P(n)]$

The structure of natural numbers

On natural numbers we can define a notion of a successor, a mapping

s: $\mathbb{N} \to \mathbb{N}$

by $s(n) = n+1$

The successor mapping imposes a structure on the set that enables us to count:

1) there is a starting natural number 0
2) for every natural number n, there is a next natural number $s(n) = n+1$.

Cardinality

Cardinals

Def. Two sets A and B have the same cardinality (are equinumerous) if there is a bijection $f: A \to B$. Notation $A \sim B$, or $|A| = |B|$.

$|A| = [A]_\sim$

cardinal numbers are ~ equivalence classes

Prop. The relation ~ is an equivalence relation on sets.

Def. A set A has at most as large cardinality as a set B if there is an injection $f: A \to B$. Notation $|A| \leq |B|$.

Def. A set A has at least as large cardinality as a set B if there is a surjection $f: A \to B$. Notation $|A| \geq |B|$.

Def. A set A has smaller cardinality than a set B if there is an injection $f: A \to B$ and there is no surjection $f: A \to B$. Notation $|A| < |B|$.

Theorem (Cantor)

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Operations on cardinals

$|A| = [A]_{\sim}$

cardinal numbers are ~ equivalence classes

Def. Let A and B be two disjoint sets. Then $|A| + |B| = |A \cup B|$.

Def. Let A and B be two sets. Then $|A| \cdot |B| = |A \times B|$.

Def. Let A and B be two sets. Then $|A|^{|B|} = |A^B|$ where A^B is the set of all functions from B to A, i.e. $A^B = \{f \mid f: B \to A\}$.

Prop. Let A be a set. Then $|\mathcal{P}(A)| = 2^{|A|}$.

Note: $2 = |\{0,1\}|$

Finite sets, finite cardinals

We write $\mathbb{N}_k$ for the set $\{0,1,..., k-1\}$. Then $\mathbb{N}_0 = \varnothing$.

We will also write k for $|\mathbb{N}_k|$.

$|A| = [A]_\sim$

cardinal numbers are ~ equivalence classes

Def. A set A is finite if and only if $|A| = |\mathbb{N}_k|$, for some $k \in \mathbb{N}$. We write then $|A| = k$.

Hence A set A is finite if and only if there is a natural number $k \in \mathbb{N}$ and a bijection $f: A \to \mathbb{N}_k$.

if and only if A has k elements, for some $k \in \mathbb{N}$

E.g. If $|A| = k$ and $|B| = m$ for some $k,m \in \mathbb{N}$ then $|A \times B| = k \cdot m$

The operations on cardinals when restricted to finite cardinals coincide with the operations on natural numbers!
This justifies the notation.

Infinite, countable and uncountable sets

Time for a video!

Hilbert's
infinite hotel :-)

Infinite, countable and uncountable sets

We write $\aleph_0$ for the cardinality of natural numbers.
Hence $\aleph_0 = |\mathbb{N}|$.

$|A| = [A]_\sim$

cardinal numbers are ~ equivalence classes

Def. A set A is countable iff $|A| = \aleph_0$.

Prop. $\mathbb{N}$ is countable.
$\mathbb{Z}$ is countable.
$\mathbb{Q}$ is countable.

Def. A set is infinite iff $|A| \geq \aleph_0$.

Hence, every countable set is infinite

Def. A set is uncountable iff $|A| > \aleph_0$.

Prop. $\mathbb{R}$ is uncountable.

We write c for $|\mathbb{R}|$

Cardinals are unbounded

$|A| = [A]_\sim$

cardinal numbers are ~ equivalence classes

Theorem (Cantor)

For every set A we have $|A| < |\mathcal{P}(A)|$.

Hence, for every cardinal there is a larger one.