Predicate logic

Limitations of propositional logic

Propositional logic only allows us to reason about completed statements about things, not about the things themselves.

Example

Some chicken cannot fly
All chicken are birds

Some birds cannot fly

this reasoning can not be expressed in propositional logic

Example

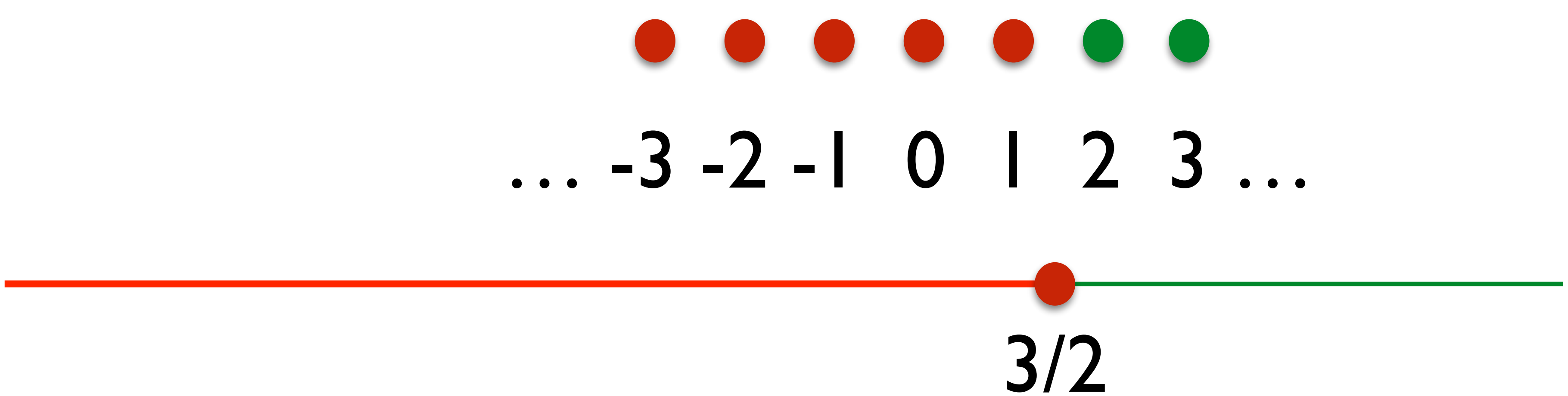
Every player except the winner looses a match

Unary predicate (example)

Consider the statement 2m>3.

a unary relation

Whether this statement is true or false depends on the value of m (and on the domain of values).



on $\mathbb{Z}$

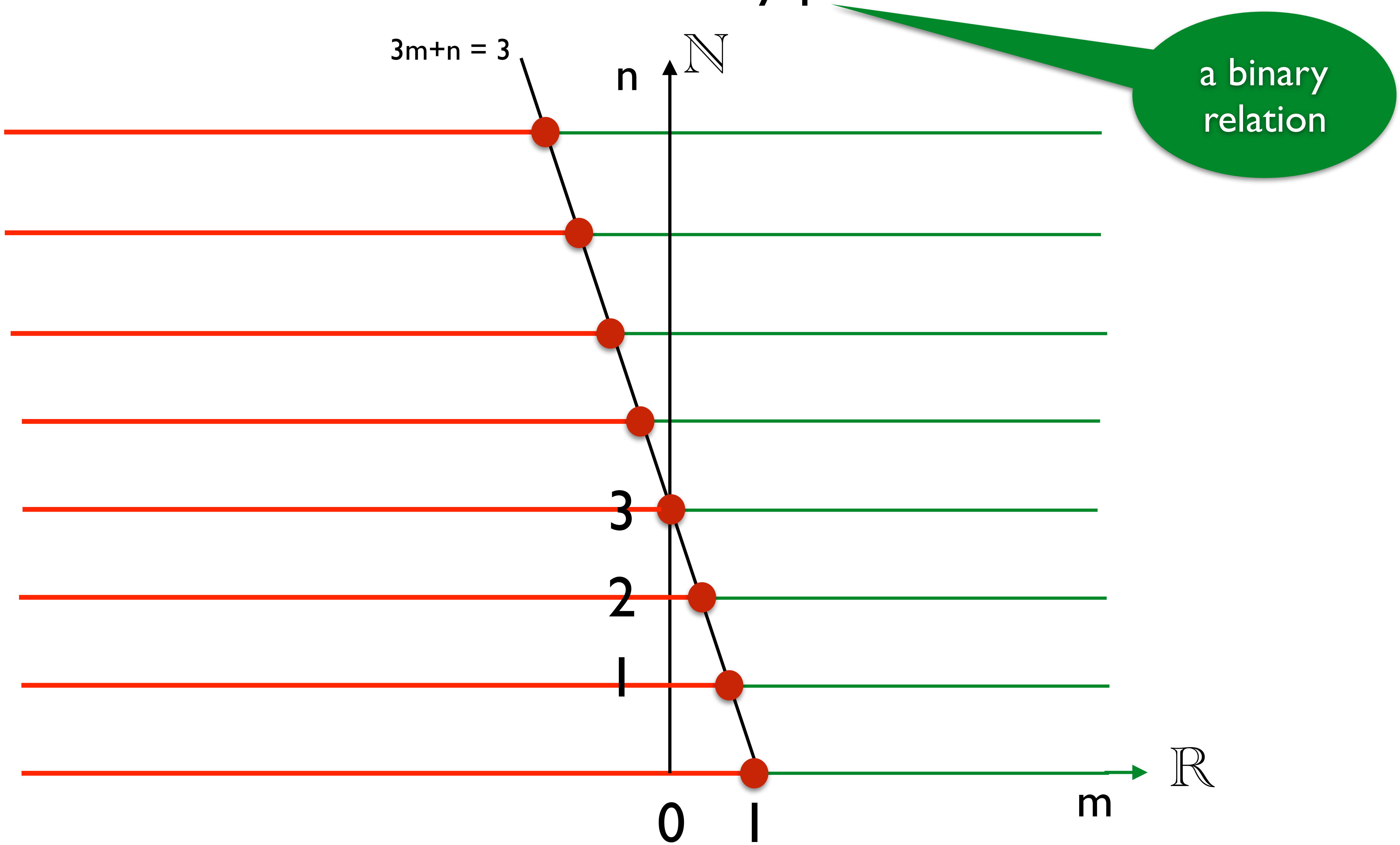
on $\mathbb{R}$

Note: $2m > 3 \overset{\text{val}}{=} m > 3/2$ on $\mathbb{Z}$ and $\mathbb{R}$

$2m > 3 \overset{\text{val}}{=} m \geq 2$ on $\mathbb{Z}$ but not on $\mathbb{R}$

Binary predicate (example)

The statement $3m+n > 3$ is a binary predicate on $\mathbb{R} \times \mathbb{N}$.



Predicates

In general, an n-ary predicate is an n-ary relation.

If it is on a domain D, then it's a relation $P(x_1, .., x_n) \subseteq D^n$ or equivalently a function $P: D^n \rightarrow \{0,1\}$.

true for certain values of the variables

$2m>3$

We can turn a predicate, into a proposition in three ways:

1. By assigning values to the variables.
2. By universal quantification.
3. By existential quantification.

for $m=2$

$2 \cdot 2 > 3$

is a true proposition

Universal quantification

The unary predicate $2m > 3$ on $\mathbb{Z}$ can be turned into a proposition by universal quantification:

false, e.g. for $m = 1$

For all m in $\mathbb{Z}$, $2m > 3$

universal quantifier

standard (!) notation:

$\forall x\ (P(x) \Rightarrow Q(x))$

$\forall x.\ P(x) \Rightarrow Q(x)$

Notation: $\forall_m\ [\underbrace{m \in \mathbb{Z}}_{\text{domain (predicate)}} : \underbrace{2m > 3}_{\text{predicate}}]$

In general: $\forall_x\ [P(x) : Q(x)]$ for "all x satisfying P satisfy Q"

Existential quantification

The unary predicate $2m > 3$ on $\mathbb{Z}$ can also be turned into a proposition by existential quantification:

true, e.g. $m = 2$

There exists m in $\mathbb{Z}$, $2m > 3$

existential quantifier

standard (!) notation:

$\exists x \, (P(x) \wedge Q(x))$

$\exists x. \, P(x) \wedge Q(x)$

Notation: $\exists_m \, [\underbrace{m \in \mathbb{Z}}_{\text{domain (predicate)}} : \underbrace{2m > 3}_{\text{predicate}}]$

In general: $\exists_x \, [P(x) : Q(x)]$ for "there exists x satisfying P that satisfies Q"

Quantification

The binary predicate $3m+n > 3$ on $\mathbb{R} \times \mathbb{N}$ can also be turned into a proposition by quantification:

in 8 possible ways

One way is: $\exists_m [m \in \mathbb{R} : \forall_n [n \in \mathbb{N} : 3m + n > 3]]$

unary predicate

binary predicate

proposition, nullary predicate

standard (!) notation:

$$\exists m\ (m \in \mathbb{R} \land \forall n\ (n \in \mathbb{N} \Rightarrow 3m+n>3))$$

Notation

also for $\exists$

We write $\forall_x$ [P] for $\forall_x$ [T : P]

We also write $\exists_{m,} \forall_n$ [(m,n)$\in \mathbb{R} \times \mathbb{N}$: 3m + n >3]

for $\exists_m$ [m$\in \mathbb{R}$: $\forall_n$ [n$\in \mathbb{N}$: 3m + n >3]]

And even $\exists_{m,\,n}$ [(m,n)$\in \mathbb{R} \times \mathbb{N}$: 3m + n >3]

for $\exists_m$ [m$\in \mathbb{R}$: $\exists_n$ [n$\in \mathbb{N}$: 3m + n >3]]

but only for the same quantifier!

Quantification - task

Thanks to Bas Luttik

Let P be the set of all tennis players.
Let $w \in P$ be the winner.

For $p, q \in P$, write $p \neq q$ for "p and q are different players" .

Let M be the set of all matches.
For $p \in P$ and $m \in M$, write $L(p,m)$ for
"player p loses match m".

Write the following sentence as a formula with predicates and quantifiers:

Every player except the winner loses a match.

Equivalences with quantifiers

Renaming bound variables

Bound variables

$$\forall_x[P{:}Q] \overset{val}{=} \forall_y[P[y/x]{:}Q[y/x]]$$

$$\exists_x[P{:}Q] \overset{val}{=} \exists_y[P[y/x]{:}Q[y/x]]$$

if y does not occur in
P or Q (not even in ∀y, ∃y)

Domain splitting

Examples:

$$
\begin{aligned}
&\forall_x[x \leqslant 1 \vee x \geqslant 5 \colon x^2 - 6x + 5 \geqslant 0] \\
\overset{val}{=}\ &\forall_x[x \leqslant 1 \colon x^2 - 6x + 5 \geqslant 0] \wedge \forall_x[x \geqslant 5 \colon x^2 - 6x + 5 \geqslant 0]
\end{aligned}
$$

$$
\begin{aligned}
&\exists_k[0 \leqslant k \leqslant n : k^2 \leqslant 10] \\
\overset{val}{=}\ &\exists_k[0 \leqslant k \leqslant n-1 \vee k = n : k^2 \leqslant 10] \\
\overset{val}{=}\ &\exists_k[0 \leqslant k \leqslant n-1 : k^2 \leqslant 10] \vee \exists_k[k = n : k^2 \leqslant 10]
\end{aligned}
$$

Domain splitting

$$\forall_x[P \vee Q : R] \overset{val}{=} \forall_x[P : R] \wedge \forall_x[Q : R]$$

$$\exists_x[P \vee Q : R] \overset{val}{=} \exists_x[P : R] \vee \exists_x[Q : R]$$

Equivalences with quantifiers

Renaming bound variables

Bound variables

$$\forall_x[P{:}Q] \overset{val}{=} \forall_y[P[y/x]{:}Q[y/x]]$$

$$\exists_x[P{:}Q] \overset{val}{=} \exists_y[P[y/x]{:}Q[y/x]]$$

if y does not occur in
P or Q (not even in ∀y, ∃y)

Domain splitting

Examples:

$$\begin{aligned} & \forall_x[x \leqslant 1 \vee x \geqslant 5 : x^2 - 6x + 5 \geqslant 0] \\ \stackrel{val}{=} \; & \forall_x[x \leqslant 1 : x^2 - 6x + 5 \geqslant 0] \wedge \forall_x[x \geqslant 5 : x^2 - 6x + 5 \geqslant 0] \end{aligned}$$

$$\begin{aligned} & \exists_k[0 \leqslant k \leqslant n : k^2 \leqslant 10] \\ \stackrel{val}{=} \; & \exists_k[0 \leqslant k \leqslant n - 1 \vee k = n : k^2 \leqslant 10] \\ \stackrel{val}{=} \; & \exists_k[0 \leqslant k \leqslant n - 1 : k^2 \leqslant 10] \vee \exists_k[k = n : k^2 \leqslant 10] \end{aligned}$$

Domain splitting

$$\forall_x[P \vee Q : R] \stackrel{val}{=} \forall_x[P : R] \wedge \forall_x[Q : R]$$

$$\exists_x[P \vee Q : R] \stackrel{val}{=} \exists_x[P : R] \vee \exists_x[Q : R]$$

Equivalences with quantifiers

One-element domain

$$\forall_x[x = n\colon Q] \overset{val}{=} Q[n/x]$$

$$\exists_x[x = n\colon Q] \overset{val}{=} Q[n/x]$$

Example:

$$\forall_x[x = 3 : 2 \cdot x \geqslant 1] \overset{val}{=} 2 \cdot 3 \geqslant 1$$

Empty domain

$$\forall_x[F : Q] \overset{val}{=} T$$

$$\exists_x[F : Q] \overset{val}{=} F$$

"All Marsians are green"

Domain weakening

Intuition: The following are equivalent

$$\forall_x [x \in D : A(x)] \quad \text{and} \quad \forall_x [x \in D \Rightarrow A(x)]$$
$$\exists_x [x \in D : A(x)] \quad \text{and} \quad \exists_x [x \in D \wedge A(x)]$$

The same can be done to parts of the domain

Domain weakening

$$\forall_x [P \wedge Q : R] \stackrel{val}{=} \forall_x [P : Q \Rightarrow R]$$
$$\exists_x [P \wedge Q : R] \stackrel{val}{=} \exists_x [P : Q \wedge R]$$

$$P \wedge Q \stackrel{val}{\models} P$$

De Morgan with quantifiers

De Morgan

$$\neg\forall_x[P:Q] \overset{val}{=} \exists_x[P:\neg Q]$$
$$\neg\exists_x[P:Q] \overset{val}{=} \forall_x[P:\neg Q]$$

not for all = at least for one not

not exists = for all not

Hence: $\neg\forall = \exists\neg$ and $\neg\exists = \forall\neg$

It holds further that:

$$\neg\forall_x\neg = \exists_x\neg\neg = \exists_x$$
$$\neg\exists_x\neg = \forall_x\neg\neg = \forall_x$$

Substitution

holds also for quantified formulas!

meta rule

Simple

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P] \overset{val}{=} \psi[\xi/P]}$$

EVERY occurrence of P is substituted!

Sequential

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{=} \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{=} \psi[\xi/P, \eta/Q]}$$

The rule of Leibniz

holds also for quantified formulas!

meta rule

Leibniz

$$\frac{\phi \stackrel{val}{=} \psi}{C[\phi] \stackrel{val}{=} C[\psi]}$$

formula that has ϕ as a sub formula

single occurrence is replaced!

Other equivalences with quantifiers

Exchange trick

$$\forall_x[P{:}Q] \overset{val}{=} \forall_x[\neg Q{:}\neg P]$$

$$\exists_x[P{:}Q] \overset{val}{=} \exists_x[Q{:}P]$$

No wonder as

$$\forall_x[P{:}Q] \overset{val}{=} \forall_x[P \Rightarrow Q]$$

$$\exists_x[P{:}Q] \overset{val}{=} \exists_x[P \wedge Q]$$

Term splitting

$$\forall_x[P{:}Q \wedge R] \overset{val}{=} \forall_x[P{:}Q] \wedge \forall_x[P{:}R]$$

$$\exists_x[P{:}Q \vee R] \overset{val}{=} \exists_x[P{:}Q] \vee \exists_x[P{:}R]$$

Other equivalences with quantifiers

Monotonicity of quantifiers

$$\forall_x[P{:}Q \Rightarrow R] \Rightarrow (\forall_x[P{:}Q] \Rightarrow \forall_x[P{:}R]) \stackrel{val}{=} T$$

$$\forall_x[P{:}Q \Rightarrow R] \Rightarrow (\exists_x[P{:}Q] \Rightarrow \exists_x[P{:}R]) \stackrel{val}{=} T$$

tautologies

Lemma E1: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Lemma W4: $P \stackrel{val}{\models} Q$ iff $P \Rightarrow Q$ is a tautology.

Lemma W5: If $Q \stackrel{val}{\models} R$ then $\forall_x[P{:}Q] \stackrel{val}{\models} \forall_x[P{:}R]$.

still hold (in predicate logic)