Coalgebraic behaviour via coinduction

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Joint work with: Ichiro Hasuo RIMS, KU, JP, and Bart Jacobs, RUN, NL

Outline

- introduction - formal methods, models and semantics
- from LTS to coalgebras
- Bisimilarity can't be traced, BUT
  * bisimilarity via coinduction in $\mathbf{Sets}$
  * trace semantics also via coinduction...

Formal methods

are mathematically based techniques for

- specification
- development
- verification

of software and hardware systems

Formal methods

In general:

- models - transition systems, automata, terms,...
  with a clear semantics
- analysis - model checking
  process algebra
  theorem proving...

Formal methods

Here:

- models - transition systems, coalgebras
- analysis - via behavior semantics

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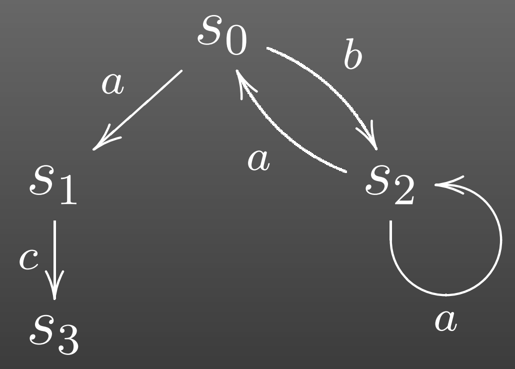
Aim: One framework for many models and semantics !

Standard model - LTS

labelled transition systems A - labels



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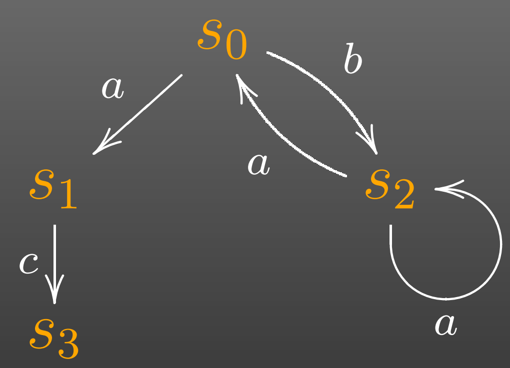

states $+$ transitions $\alpha : S \to \mathcal{P}(A \times S)$

$$\alpha(s_0) = \{\langle a, s_1\rangle, \langle b, s_2\rangle\},\ \alpha(s_1) = \{\langle c, s_3\rangle\},\ \ldots$$

Behavior semantics

are used for verification

- behavior equivalence ($\equiv$) identifies states with same behavior
- behavior preorder ($\sqsubseteq$) orders states according to behavior

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there are many of them: bisimilarity, trace, ...

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verification amounts to:

- given
  * Sys - model of the system, LTS
  * Spec - model of the specification, LTS

Behavior semantics

verification amounts to:

- given
  - * Sys - model of the system, LTS
  - * Spec - model of the specification, LTS
- verify if

  Sys $\equiv$ Spec or Sys $\sqsubseteq$ Spec

Bisimulation - LTS

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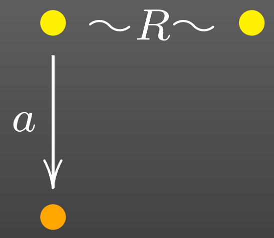

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Transfer condition: $\langle s, t \rangle \in R \quad \Longrightarrow$

$$s \xrightarrow{a} s' \Rightarrow (\exists t')\ t \xrightarrow{a} t',\ \langle s', t' \rangle \in R,$$
$$t \xrightarrow{a} t' \Rightarrow (\exists s')\ s \xrightarrow{a} s',\ \langle s', t' \rangle \in R$$

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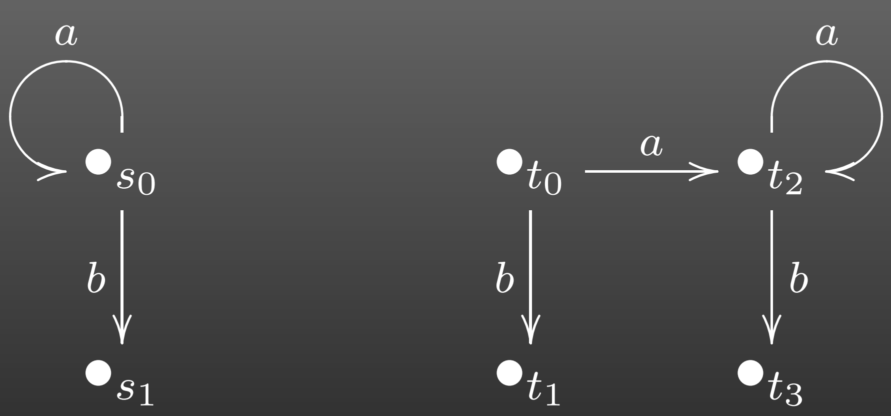
... two states are bisimilar if they are related by some bisimulation

Bisimulation - LTS

Example: Consider the LTS



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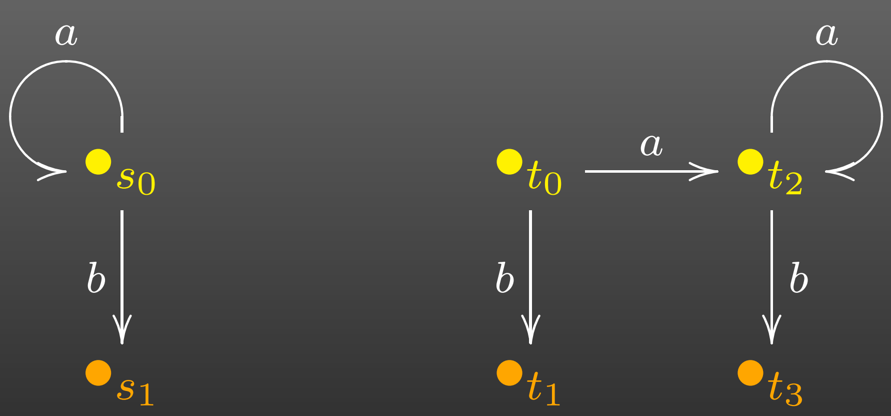


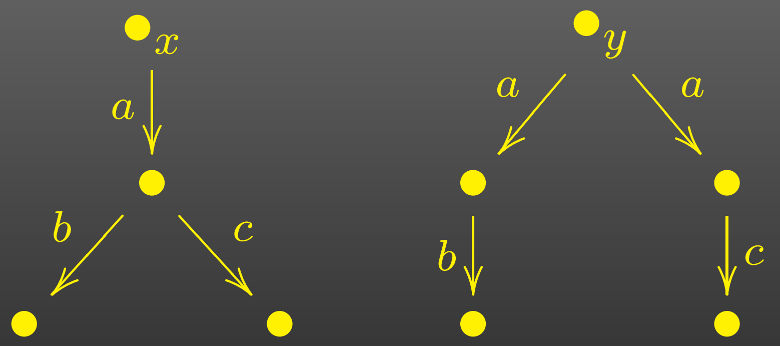
the coloring is a bisimulation, so s_0 and t_0 are bisimilar

LT/BT spectrum

Bisimilarity is not the only semantics...

LT/BT spectrum

Are these non-deterministic systems equal ?



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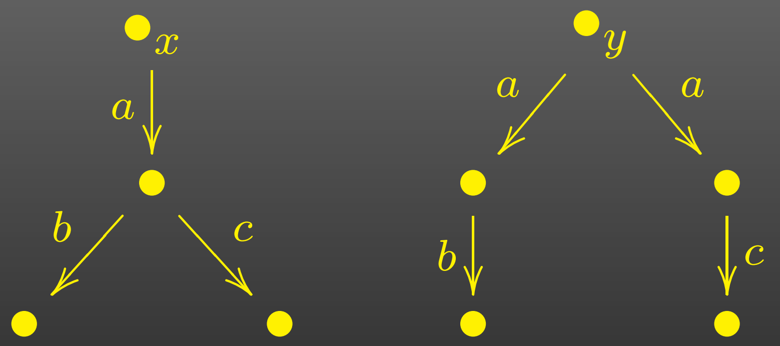

x and y are:

- different wrt. bisimilarity

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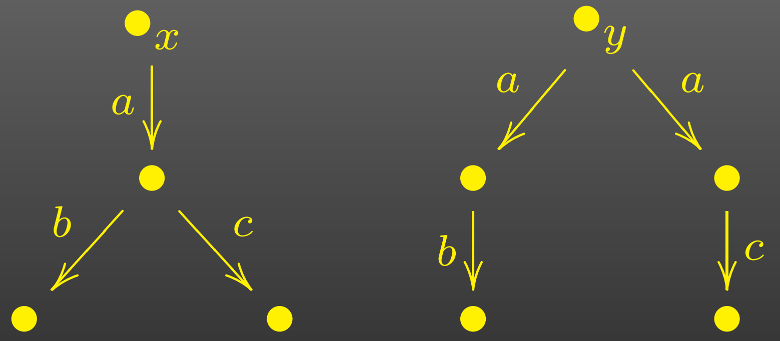

x and y are:

- different wrt. bisimilarity, but
- equivalent wrt. trace semantics
  $\mathrm{tr}(x) = \mathrm{tr}(y) = \{ab, ac\}$

Traces - LTS

For LTS with explicit termination (NA)

trace = the set of all possible linear behaviors

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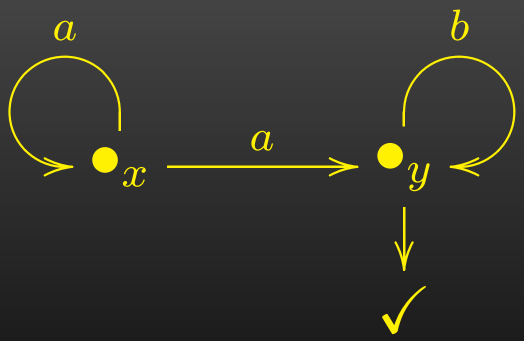

$$\mathrm{tr}(y) \;=\; b^*, \qquad \mathrm{tr}(x) \;=\; a^+ \cdot \mathrm{tr}(y) \;=\; a^+ \cdot b^*$$

Other models

deterministic systems

$$\begin{tikzcd} s_0 \arrow[r] & s_1 \arrow[r] & s_2 \arrow[r] & s_3 \arrow[loop right] \end{tikzcd}$$

Other models

deterministic systems

$$s_0 \longrightarrow s_1 \longrightarrow s_2 \longrightarrow s_3 \circlearrowleft$$

states $+$ transitions $\alpha : S \to S$

$$\alpha(s_0) = s_1,\ \alpha(s_1) = s_2,\ \ldots$$

Other models

labelled deterministic systems A - labels

$$\begin{tikzcd} s_0 \arrow[r, "a"] & s_1 \arrow[r, "b"] & s_2 \arrow[r, "b"] & s_3 \arrow[loop below, "a"] \end{tikzcd}$$

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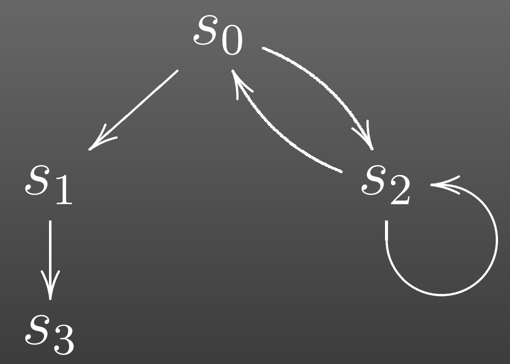
states $+$ transitions $\alpha : S \to A \times S$

$$\alpha(s_0) = \langle a, s_1 \rangle,\ \alpha(s_1) = \langle b, s_2 \rangle,\ \ldots$$

Other models

transition systems



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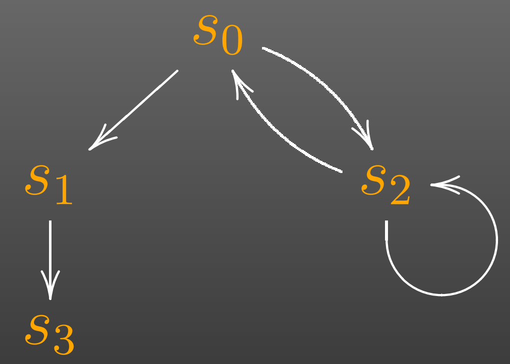
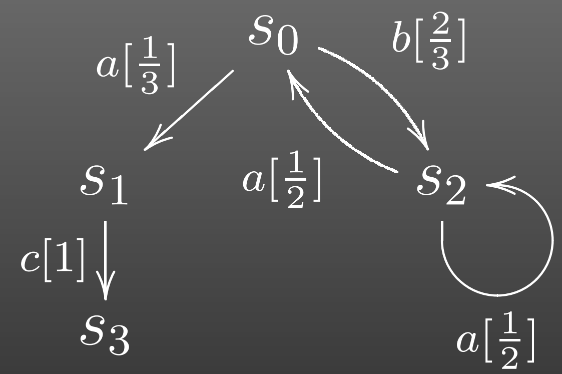


states $+$ transitions $\alpha : S \to \mathcal{P}(S)$

$$\alpha(s_0) = \{s_1, s_2\},\ \alpha(s_1) = \{s_3\},\ \ldots$$

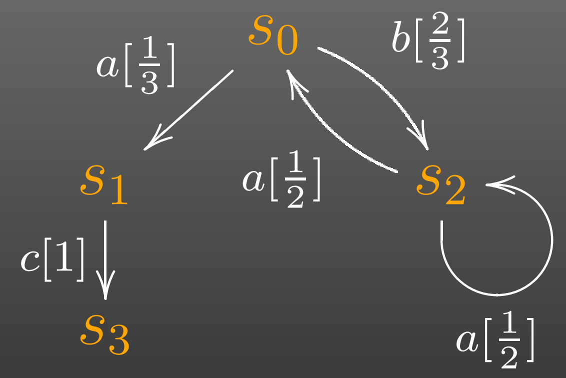
Other models

generative probabilistic systems A - labels



Other models

generative probabilistic systems $\quad A$ - labels



states $+$ transitions $\alpha : S \to \mathcal{D}(A \times S) + 1$

$$\alpha(s_0) = \left(\langle a, s_1\rangle \mapsto \tfrac{1}{3}, \langle b, s_2\rangle \mapsto \tfrac{2}{3}\right),$$
$$\alpha(s_1) = \left(\langle c, s_3\rangle \mapsto 1\right), \ \ldots$$

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$$\begin{tikzcd} \bullet \arrow[r, phantom, "\sim R \sim"] \arrow[d] & \bullet \arrow[d] \\ \mu \arrow[r, phantom, "\equiv_{R,A}"] & \nu \end{tikzcd}$$

$\equiv_{R,A}$ relates distributions that assign the same probability to each label and each R-class

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Transfer condition: $\langle s, t\rangle \in R \quad \Longrightarrow$

$$s \rightarrow \mu \Rightarrow (\exists \nu)\ t \rightarrow \nu,\ \ \mu \equiv_{R,A} \nu$$

Bisimulation - generative

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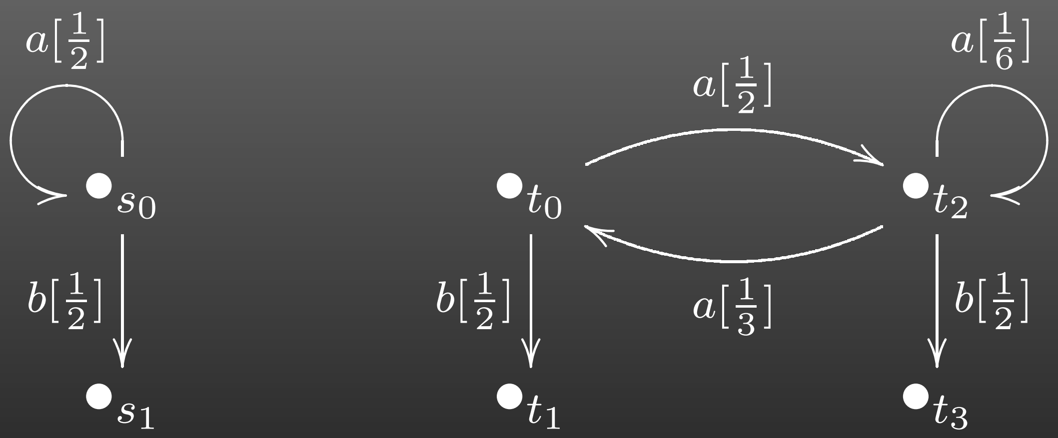
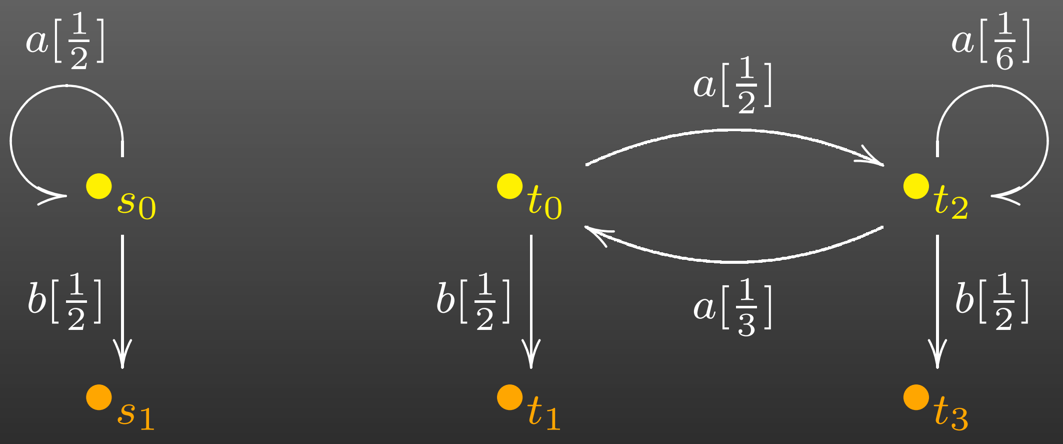
Bisimulation - generative

Consider the generative systems



Bisimulation - generative

Example: Consider the generative systems



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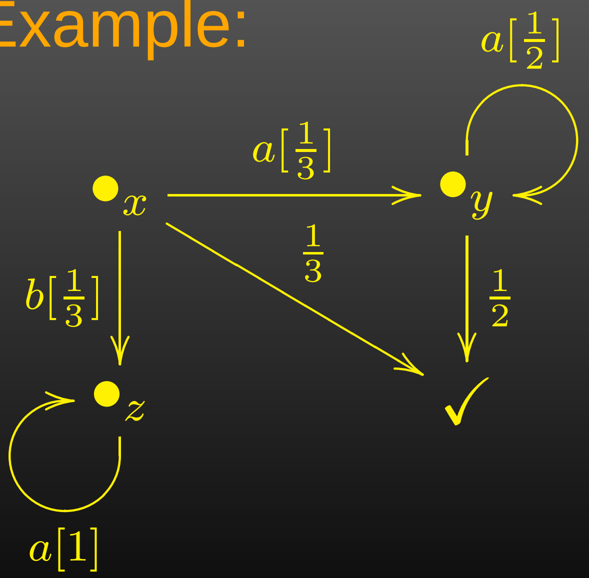

$$\mathrm{tr}(x): \quad \langle\rangle \mapsto \frac{1}{3}$$

$$a \mapsto \frac{1}{3}\cdot\frac{1}{2}$$

$$a^2 \mapsto \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$\cdots$$

Coalgebras

are an elegant generalization of transition systems with

states + transitions

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$$\langle S, \alpha : S \to \mathcal{F}S \rangle, \text{ for } \mathcal{F} \text{ a functor}$$

- rich mathematical structure
- a uniform way for treating transition systems
- general notions and results, generic notion of bisimulation

Coalgebraic bisimulation

A bisimulation on

$$\langle S, \alpha : S \longrightarrow \mathcal{F}S \rangle$$

is $R \subseteq S \times S$ such that

Coalgebraic bisimulation

A bisimulation on

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is $R \subseteq S \times S$ such that γ exists:

$$\begin{tikzcd} S & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & S \arrow[d, "\alpha"] \\ \mathcal{F}S \arrow[from=u, "\alpha"'] & \mathcal{F}R \arrow[l, "\mathcal{F}\pi_1"] \arrow[r, "\mathcal{F}\pi_2"'] & \mathcal{F}S \end{tikzcd}$$

Coalgebraic bisimulation

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$$\bullet_s \rightsquigarrow R \rightsquigarrow \bullet_t$$

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$$\begin{tikzcd} \bullet_s \arrow[r, squiggly, dash, "R"] \arrow[d] & \bullet_t \\ \alpha(s) & \end{tikzcd}$$

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is $R \subseteq S \times S$ such that

$$\begin{array}{ccc} \bullet_s & \sim\sim\sim R \sim\sim\sim & \bullet_t \\ \downarrow & & \downarrow \\ \alpha(s) & \mathrm{Rel}(\mathcal{F})(R) & \alpha(t) \end{array}$$

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\alpha(s) \arrow[r, phantom, "\mathrm{Rel}(\mathcal{F})(R)"] & \alpha(t)
\end{tikzcd}$$

Transfer condition: $\langle s, t \rangle \in R \quad \Longrightarrow$

$$\langle \alpha(s), \beta(t) \rangle \in \mathrm{Rel}(\mathcal{F})(R)$$

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Theorem: Coalgebraic and concrete bisimilarity coincide !

Trace of a coalgebra ?

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- Power&Turi '99 - $\mathcal{P}(1 + \Sigma \times \_)$
- Jacobs '04 - $\mathcal{P}\mathcal{F}$
- Hasuo&Jacobs CALCO '05, CALCO Jnr '05 - $\mathcal{P}\mathcal{F}$, $\mathcal{D}\mathcal{F}$
- Hasuo&Jacobs&Sokolova CMCS'06, LMCS 3(4:11)'07
  Generic Trace Semantics via Coinduction
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main idea: coinduction in a Kleisli category

Coinduction

$$\begin{tikzcd} \mathcal{F}X \arrow[rr, dashed, "\mathcal{F}(\mathrm{beh})"] & & \mathcal{F}Z \\ X \arrow[u, "\alpha"] \arrow[rr, dashed, "\mathrm{beh}"'] & & Z \arrow[u, "\cong"'] \end{tikzcd}$$

system final coalgebra

Coinduction

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system final coalgebra

- finality = $\exists!$(morphism for any $\mathcal{F}$- coalgebra)
- beh gives the behavior of the system
- this yields final coalgebra semantics

Coinduction

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, dashed, "\mathcal{F}(\mathrm{beh})"] & \mathcal{F}Z \\
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\end{tikzcd}$$

system final coalgebra

- f.c.s. in $\mathbf{Sets}$ = bisimilarity
- f.c.s. in a Kleisli category = trace semantics

Types of systems

For trace semantics systems are suitably modelled as coalgebras in $\mathbf{Sets}$

$$X \xrightarrow{c} \mathcal{T}\mathcal{F}X$$

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needed: distributive law $\mathcal{F}\mathcal{T} \Rightarrow \mathcal{T}\mathcal{F}$

Distributive law

is needed since branching is irrelevant:

LTS with ✓ - $\mathcal{PF} = \mathcal{P}(1 + \Sigma \times \_\,)$

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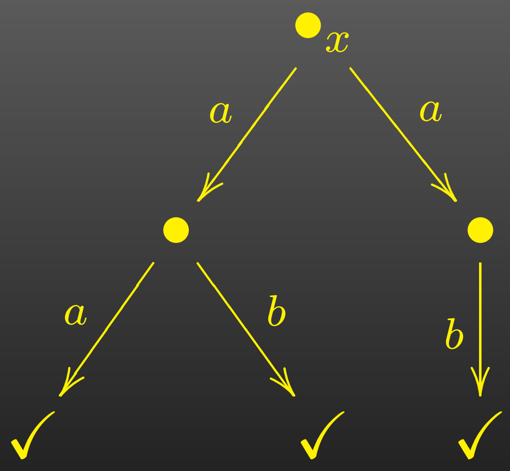

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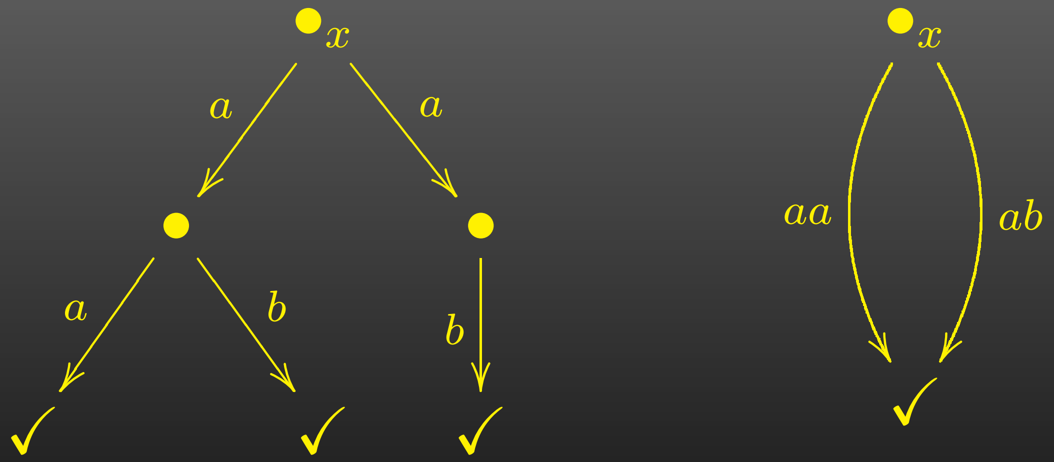

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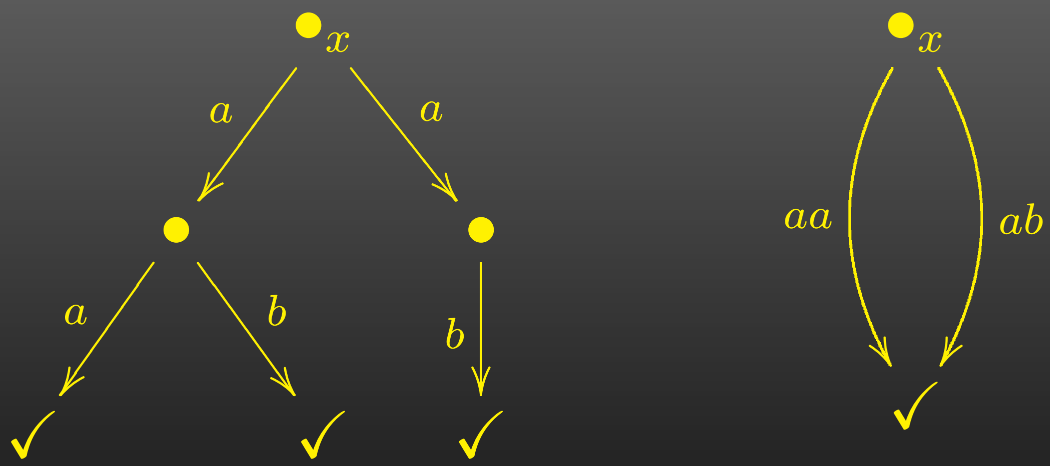

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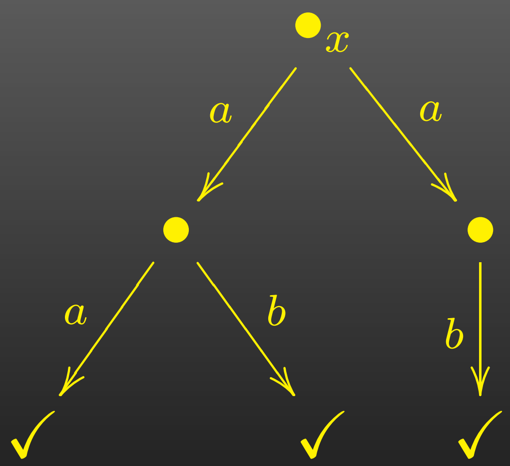

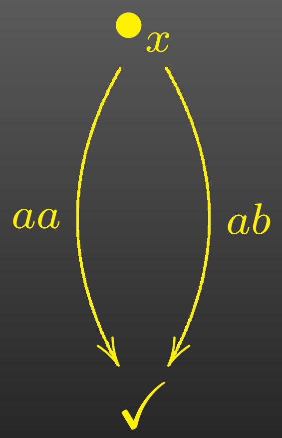

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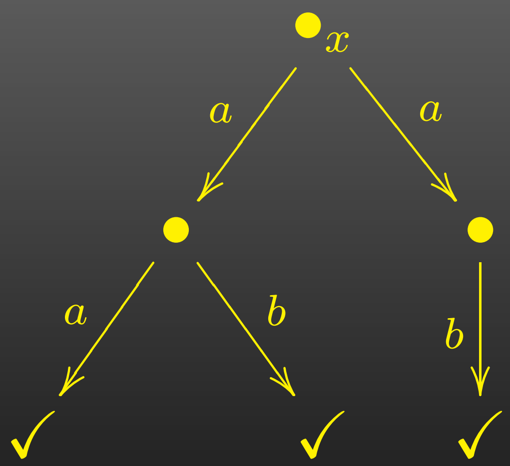

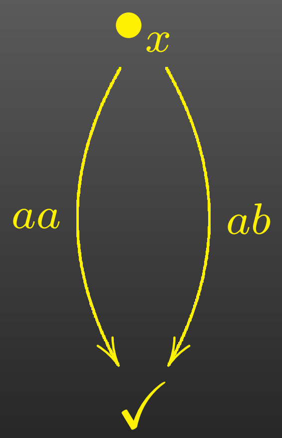

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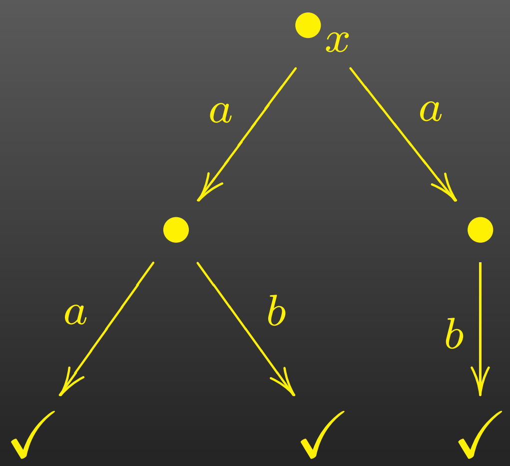

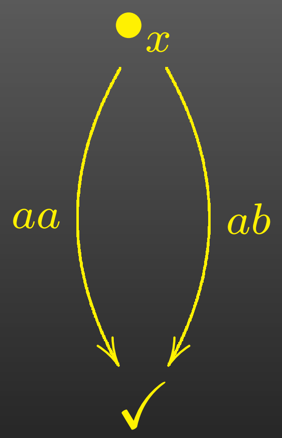

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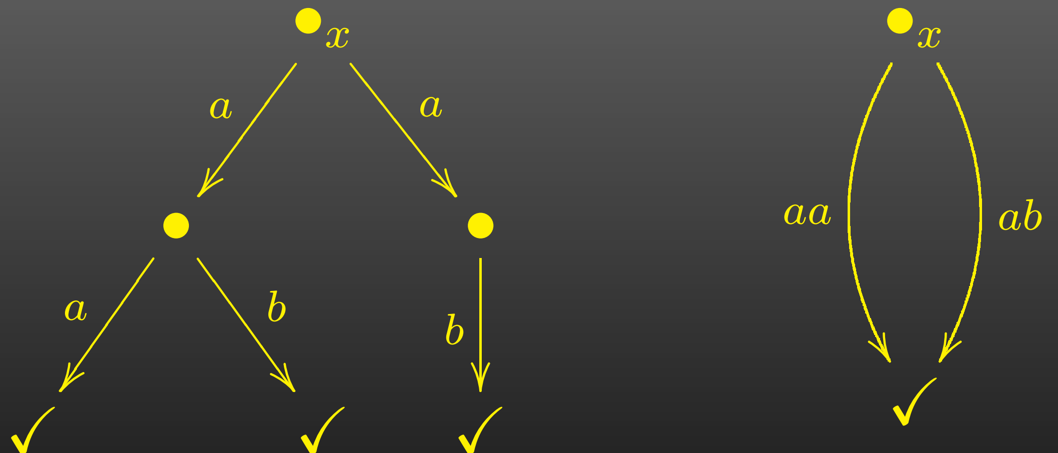

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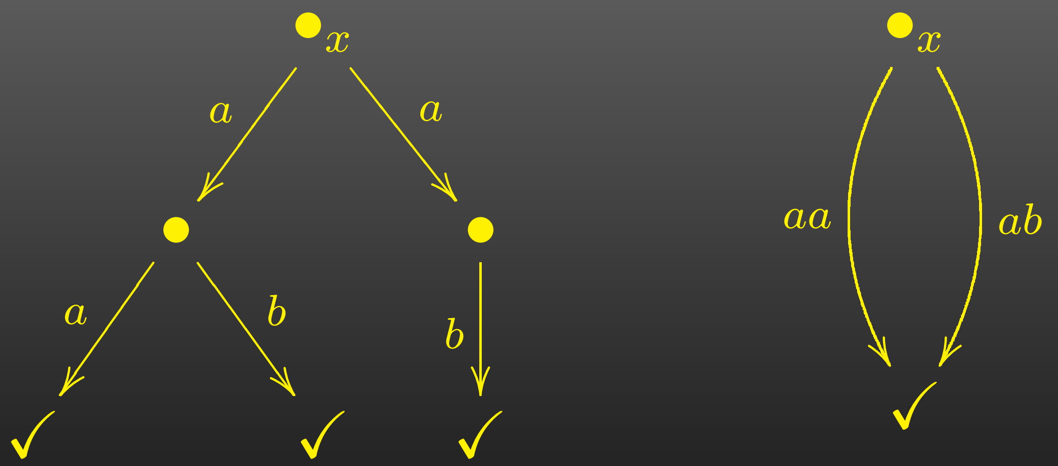

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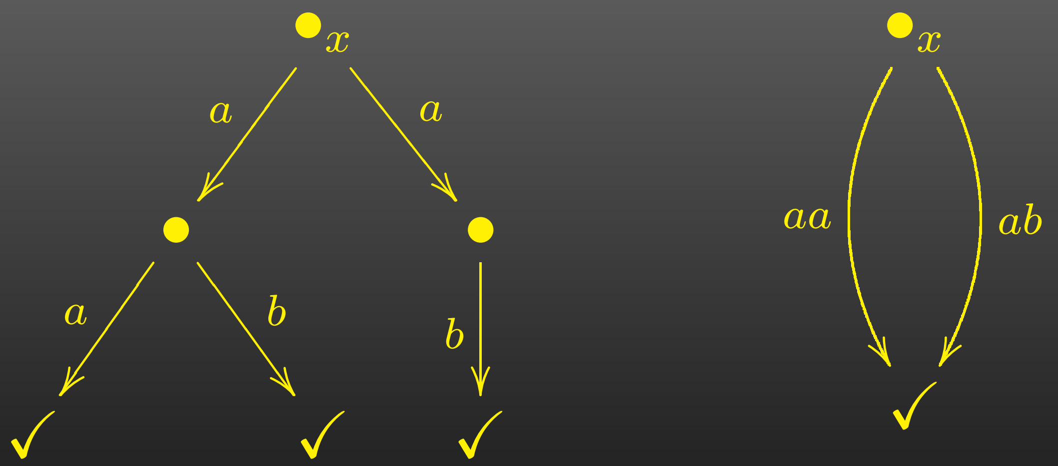

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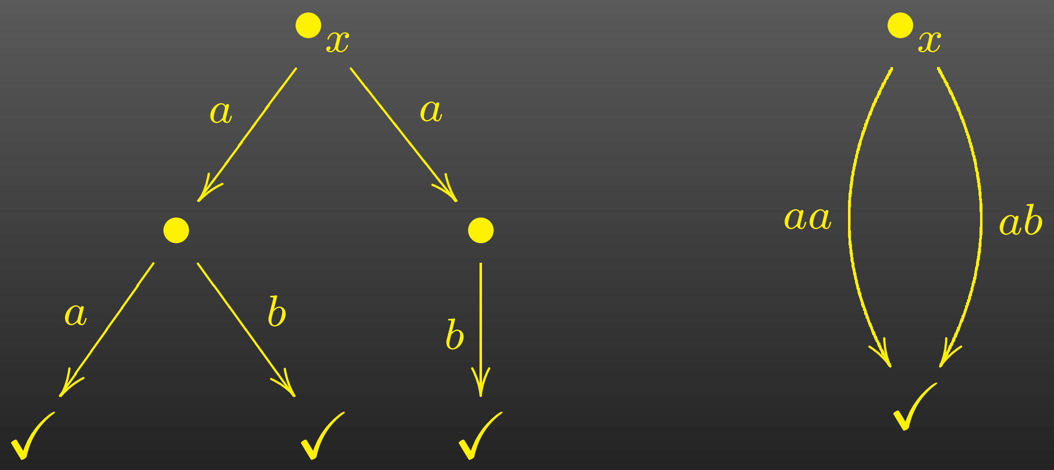

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- objects - sets
- arrows - $X \xrightarrow{f} Y$ are functions $f : X \to \mathcal{T}Y$

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Main Theorem

If ♣, then

$$\begin{tikzcd} \mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}A \arrow[d, "\eta_A \circ \alpha"', "\cong"] \\ A \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}A \\ A \arrow[u, "\eta_{\mathcal{F}A} \circ \alpha^{-1}", "\cong"'] \end{tikzcd}$$

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proof: via limit-colimit coincidence Smyth&Plotkin ’82

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- $\mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}$ should be locally monotone

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$$\begin{tikzcd}
\mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}X \arrow[r, dashed, "\mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}(\mathrm{tr}_c)"] & \mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}A \\
X \arrow[u, "c"] \arrow[r, dashed, "\mathrm{tr}_c"'] & A \arrow[u, "\cong"']
\end{tikzcd}$$

It works for...

- branching types:
  - * lift monad $1 + \_$

    systems with non-termination, exception
  - * powerset monad $\mathcal{P}$

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    $\mathcal{D}X = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) \leq 1\}$

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  all with pointwise order !

together with...

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$$\mathcal{F} = \mathit{id} \mid \Sigma \mid F \times F \mid \coprod_i F_i$$

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$$\mathcal{F} = \mathit{id} \mid \Sigma \mid F \times F \mid \coprod_i F_i$$

  * modular distributive law between commutative monads and shapely functors

  * our monads are commutative

Hence, it works...

- for LTS with explicit termination

$$\mathcal{P}(1 + \Sigma \times \_\ )$$

Hence, it works...

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Note: Initial $1 + \Sigma \times \_$ - algebra is

$$\Sigma^* \xrightarrow[\cong]{[\mathrm{nil},\mathrm{cons}]} 1 + \Sigma \times \Sigma^*$$

Finite traces - LTS with $\checkmark$

the finality diagram in $\mathcal{K}\ell(\mathcal{P})$

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\mathcal{F}_{\mathcal{K}\ell(\mathcal{P})} X \arrow[r, dashed, "\mathcal{F}_{\mathcal{K}\ell(\mathcal{P})}(\mathrm{tr}_c)"] & \mathcal{F}_{\mathcal{K}\ell(\mathcal{P})} \Sigma^* \\
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amounts to

- $\langle\rangle \in \mathrm{tr}_c(x) \iff \checkmark \in c(x)$
- $a\cdot w \in \mathrm{tr}_c(x) \iff (\exists x')\langle a, x'\rangle \in c(x),\ w \in \mathrm{tr}_c(x')$

Finite traces - generative ✓

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amounts to $\mathrm{tr}_c(x)$:

- $\langle\rangle \mapsto c(x)(\checkmark)$
- $a \cdot w \mapsto \sum_{y\in X} c(x)(a,y)\cdot c(y)(w)$

Conclusions

- Systems as coalgebras
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- Main technical result: initial algebra = final coalgebra