Convex Algebras for Probabilistic Systems

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TRENDS ‘17

Convex Algebras

- algebras

- convex (affine) maps

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- algebras $\boxed{(A, \sum_{i=1}^{n} p_i(-)_i)}$ $p_i \in [0,1], \sum_{i=1}^{n} p_i = 1$

- convex (affine) maps

Convex Algebras

infinitely many
finitary operations

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$$\boxed{h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)}$$

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- convex (affine) maps

$$h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)$$

satisfying

- Projection $\quad \sum_{i=1}^{n} p_i a_i = a_k, \quad p_k = 1$
- Barycenter $\quad \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} p_{i,j} a_j\right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} p_i p_{i,j}\right) a_j$

Eilenberg-Moore Algebras

convex algebras abstractly

$\mathcal{EM}(\mathcal{D})$

- objects

$$\begin{array}{c} \mathcal{D}A \\ \downarrow a \\ A \end{array}$$

satisfying

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, equal, "a"'] & \mathcal{D}A \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}\mathcal{D}A \arrow[r, "\mu"] \arrow[d, "\mathcal{D}a"'] & \mathcal{D}A \arrow[d, "a"] \\ \mathcal{D}A \arrow[r, "a"] & A \end{tikzcd}$$

- morphisms

$$\begin{array}{c} \mathcal{D}A \\ \downarrow a \\ A \end{array} \xrightarrow{h} \begin{array}{c} \mathcal{D}B \\ \downarrow b \\ B \end{array}$$

$$\begin{tikzcd} \mathcal{D}A \arrow[r, "\mathcal{D}h"] \arrow[d, "a"'] & \mathcal{D}B \arrow[d, "b"] \\ A \arrow[r, "h"] & B \end{tikzcd}$$

What is $\mathcal{D}$?

Probability distribution functor on $\mathbf{Sets}$

$$\boxed{\mathcal{D}X = \{\mu \colon X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}}$$

Free Convex Algebras

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carried by distributions

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$$\sum p_i \xi_i = \xi \quad \Leftrightarrow \quad \forall x \in X.\ \xi(x) = \sum p_i \xi_i(x)$$

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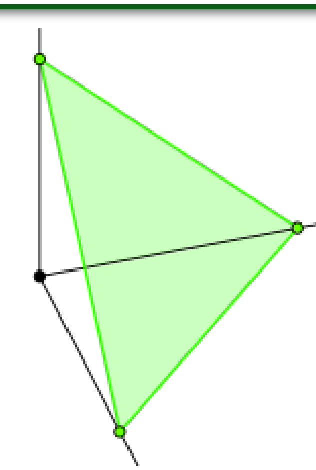
finitely generated free convex algebras are simplexes

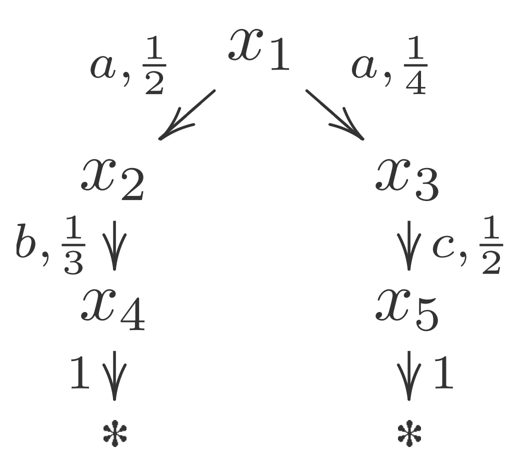


Convexity in Probabilistic Systems Semantics

Traces

Generative PTS

$\mathcal{D}(1 + A \times (-))$



$$\mathrm{tr}(x_1)(ab) = \frac{1}{6} \qquad \mathrm{tr}(x_1)(ac) = \frac{1}{8}$$

$$\mathrm{tr}\colon X \to \mathcal{D}A^*$$

Traces via determinisation

Generative PTS

$\mathcal{D}$ (1 + A x (-))

$$
\begin{tikzcd}
 & x_1 \arrow[dl, "{a, \frac{1}{2}}"'] \arrow[dr, "{a, \frac{1}{4}}"] & \\
x_2 \arrow[d, "{b, \frac{1}{3}}"'] & & x_3 \arrow[d, "{c, \frac{1}{2}}"] \\
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$$\mathrm{tr}\colon X \to \mathcal{D}A^*$$

trace = bisimilarity after determinisation

Happens in convex algebra

Trace axioms for generative PTS

Axioms for bisimilarity

+

$$p \cdot a \cdot (p_1 \mathsf{E}_1 \oplus p_2 \mathsf{E}_2) \quad \equiv \quad p_1 \cdot a \cdot p\mathsf{E}_1 \oplus p_2 \cdot a \cdot p\mathsf{E}_2 \qquad (D)$$

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[Silva, S. MFPS'11]

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soundness and completeness

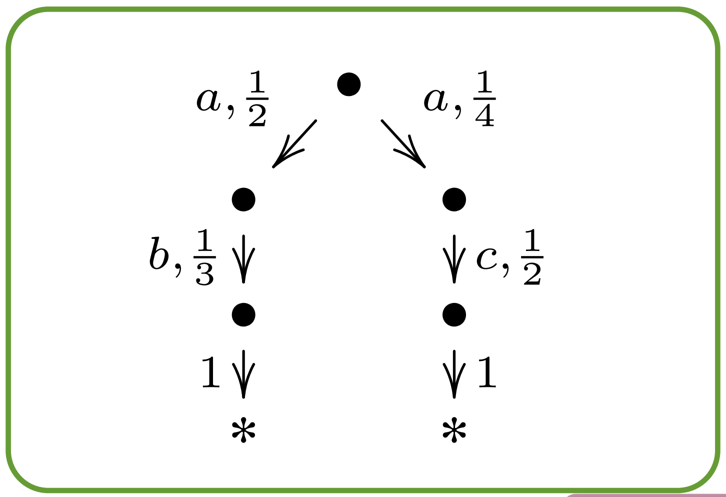
Happens in convex algebra

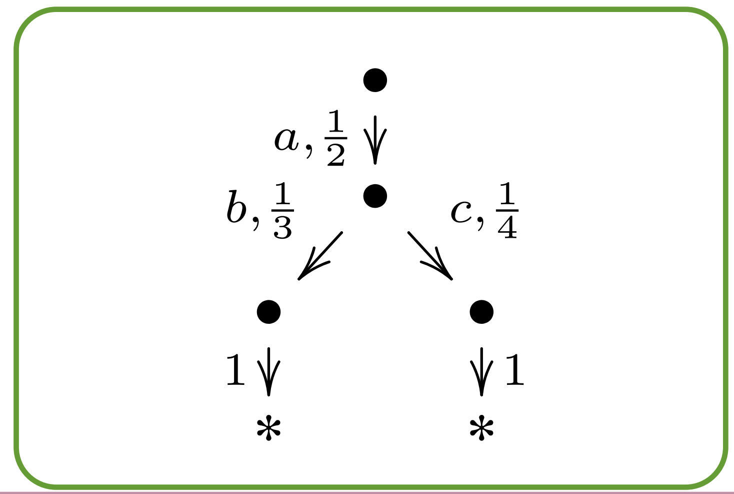
[Silva, S. MFPS'11]

Trace axioms for generative PTS

Generative PTS

$\mathcal{D}\,(1 + A \times (-))$





$$\frac{1}{4}\cdot a\cdot\frac{1}{2}\cdot c\cdot 1\cdot * \overset{(D)}{\equiv} \frac{1}{2}\cdot a\cdot\frac{1}{4}\cdot c\cdot 1\cdot *$$

$$\left(\frac{1}{2}\cdot a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *\right)\oplus\left(\frac{1}{4}\cdot a\cdot\frac{1}{2}\cdot c\cdot 1\cdot *\right) \overset{(Cong)}{\equiv} \left(\frac{1}{2}\cdot a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *\right)\oplus\left(\frac{1}{2}\cdot a\cdot\frac{1}{4}\cdot c\cdot 1\cdot *\right)$$

$$\overset{(D)}{\equiv} \frac{1}{2}\cdot a\cdot\left(\frac{1}{3}\cdot b\cdot 1\cdot *\oplus\frac{1}{4}\cdot c\cdot 1\cdot *\right)$$

The quest for completeness

Inspired lots of new research:

- A. S., H. Woracek Congruences of convex algebras JPAA'15

- S. Milius Proper functors CALCO'17

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[S., Woracek :-)'17]

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Theorem

Every congruence of convex algebras is f.g.
Hence f.p. = f.g.

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[S., Woracek JPAA'15]

Determinisations

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Generative PTS

X → $\mathcal{D}$ (1 + A x X)

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& x_1 \arrow[dl, "{a,\frac{1}{2}}"'] \arrow[dr, "{a,\frac{1}{2}}"] & \\
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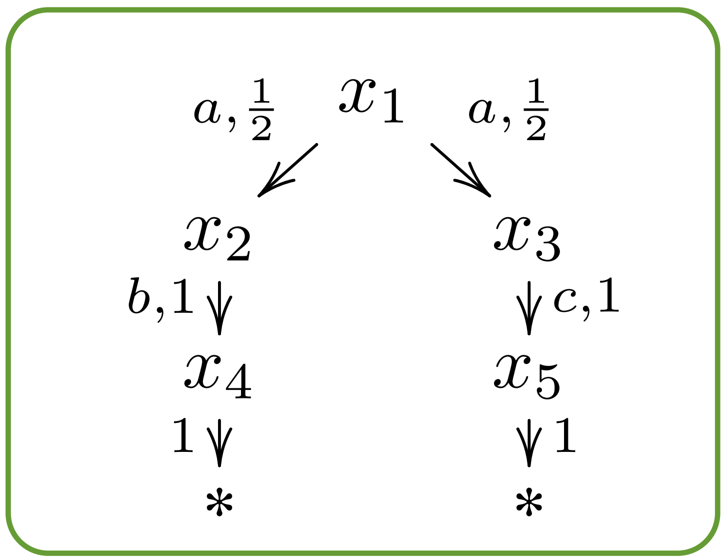

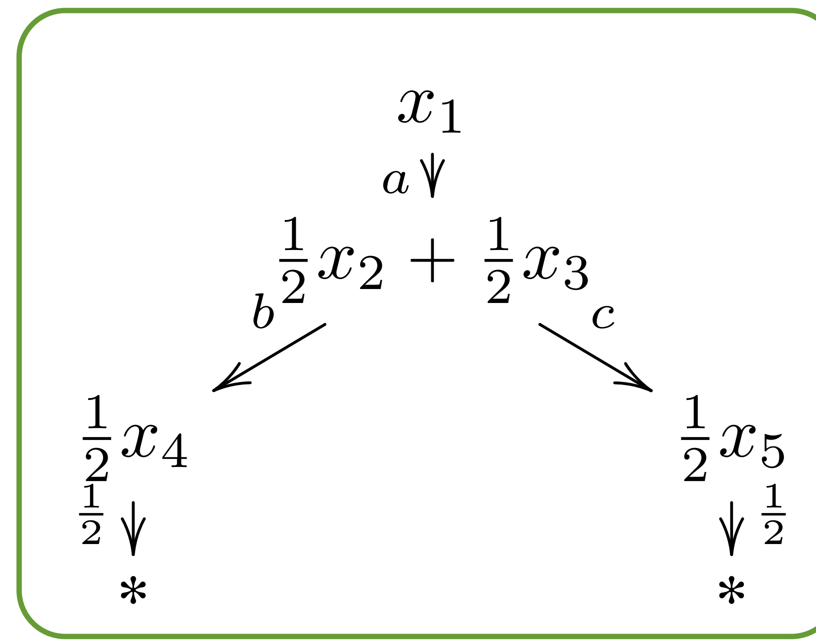

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$$\begin{tikzcd}[column sep=small]
 & x_1 \arrow[d, "a"'] & \\
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* & & *
\end{tikzcd}$$

[Silva, S. MFPS'11]

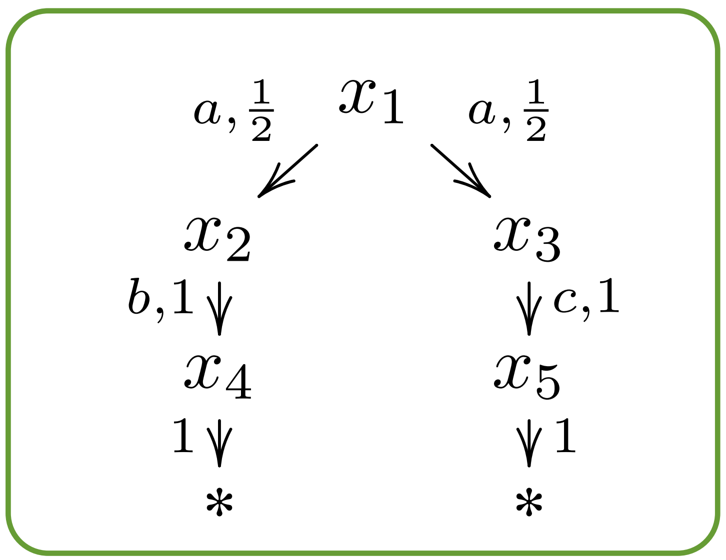
[Jacobs, Silva, S. JCSS'15]

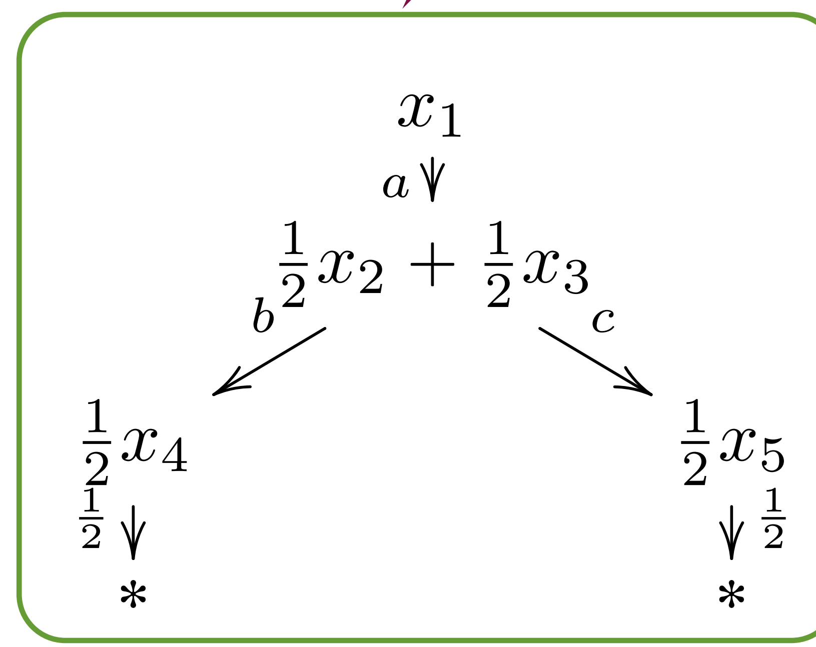
Determinisations

belief-state transformer

Generative PTS

$X \to \mathcal{D}(1 + A \times X)$





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Belief-state transformers

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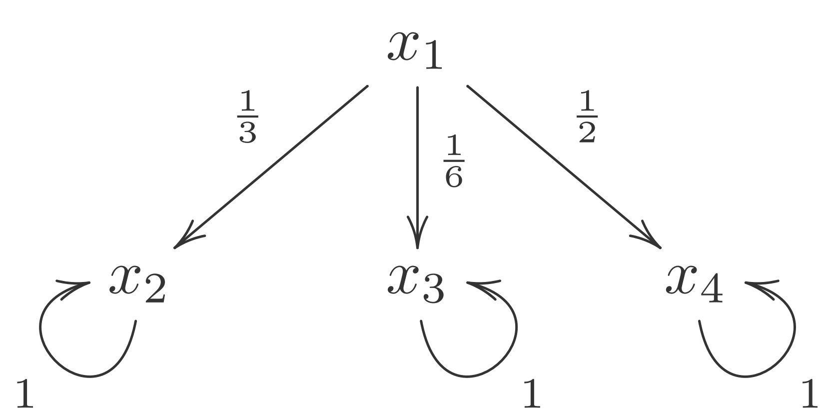
MC

$X \to \mathcal{D}(X)$



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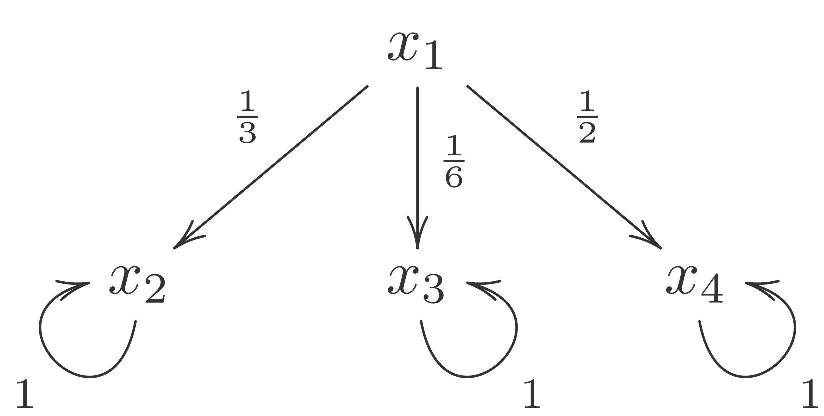

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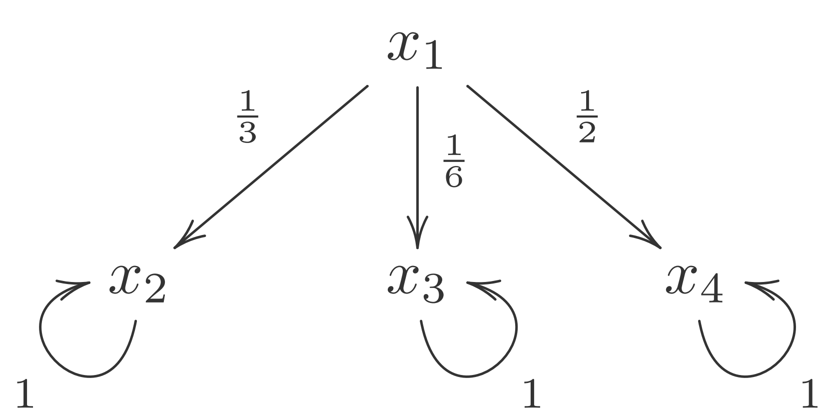

$$\frac{1}{3}x_1 + \frac{2}{3}x_2 \qquad \cdots$$

$$\downarrow$$

$$\frac{7}{9}x_2 + \frac{1}{18}x_3 + \frac{1}{6}x_4$$

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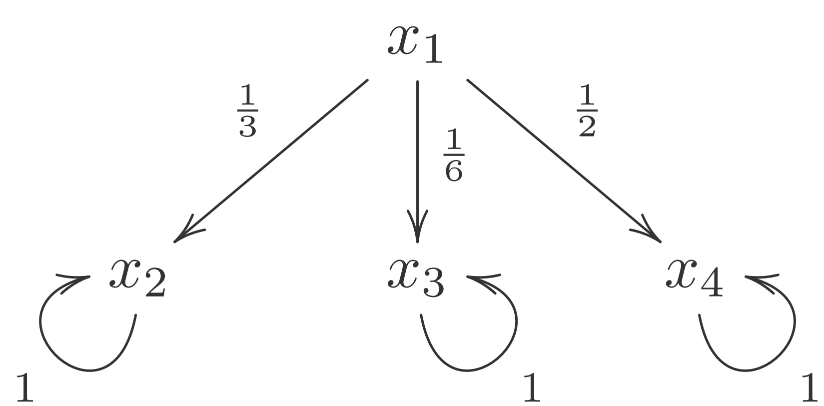

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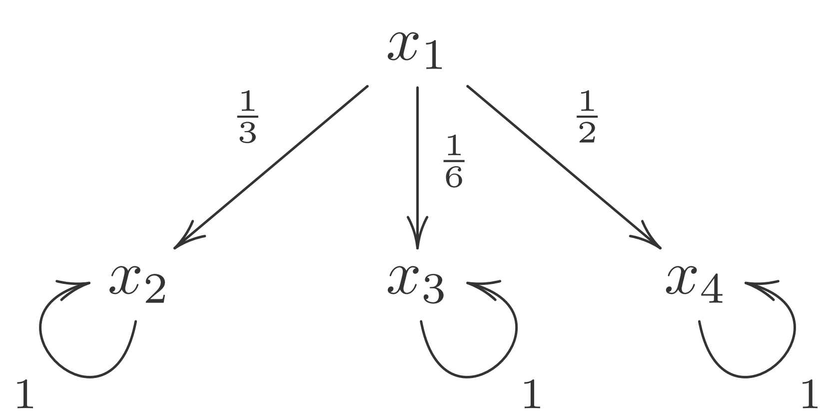

belief-state transformer

belief state

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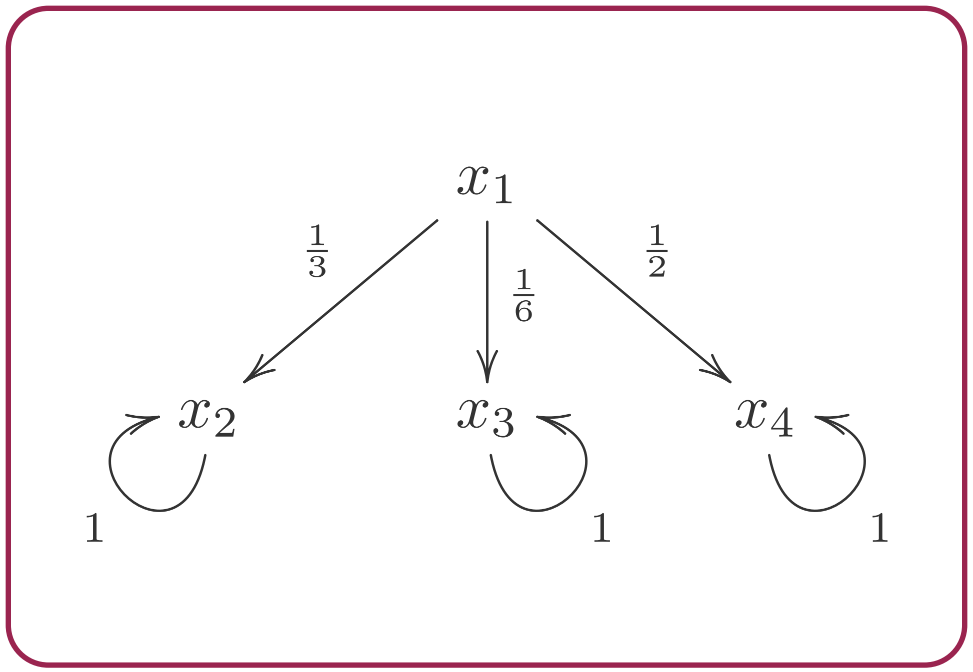

belief-state transformer

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$$\frac{1}{3}x_1 + \frac{2}{3}x_2 \qquad \dots$$

$$\Big\downarrow$$

$$\frac{7}{9}x_2 + \frac{1}{18}x_3 + \frac{1}{6}x_4$$

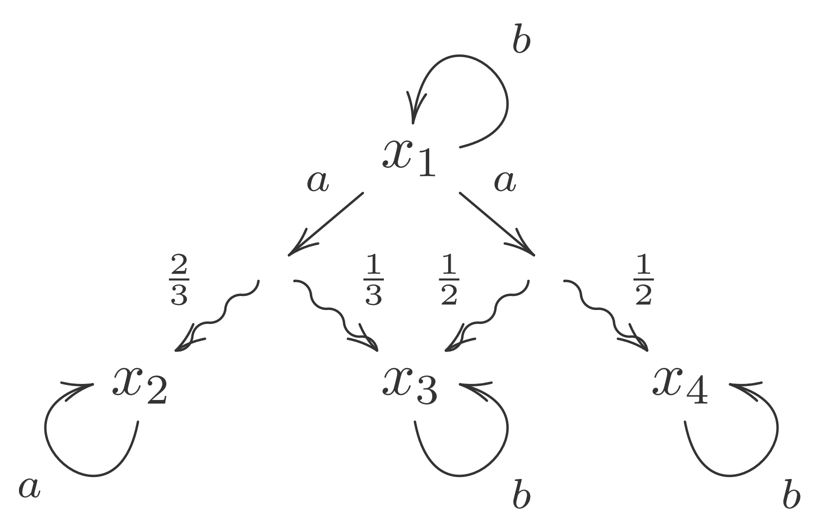
$$\frac{1}{3}\left(\frac{1}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{2}x_4\right) + \frac{2}{3}(1x_2)$$

Belief-state transformers

Belief-state transformers

PA

$X \to (\mathcal{PD}(X))^A$



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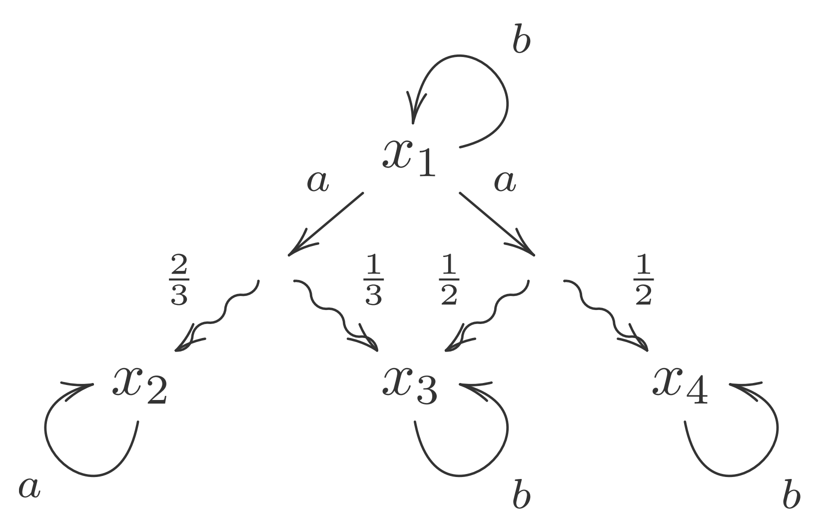

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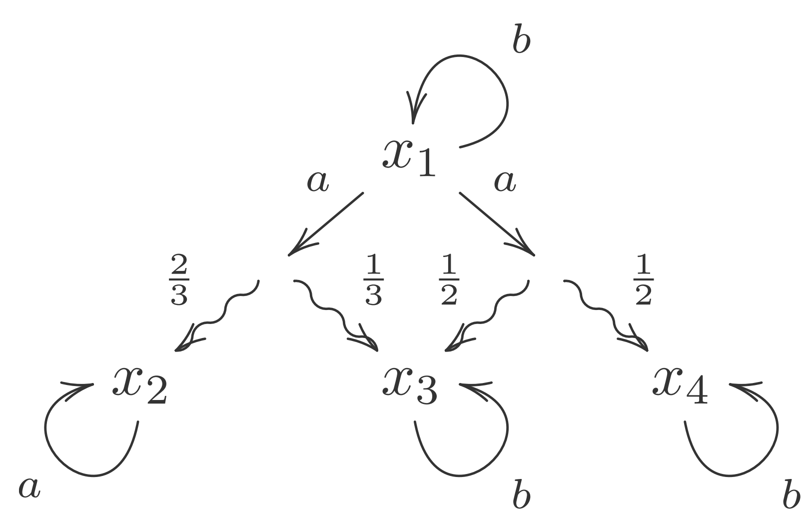

$$\begin{tikzcd}[column sep=small]
 & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & & \dots \\
\frac{8}{9}x_2 + \frac{1}{9}x_3 & \dots & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 &
\end{tikzcd}$$

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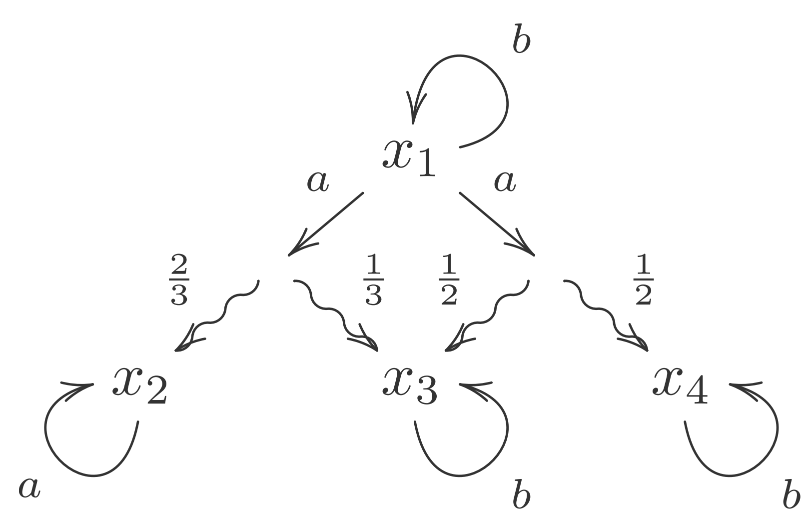
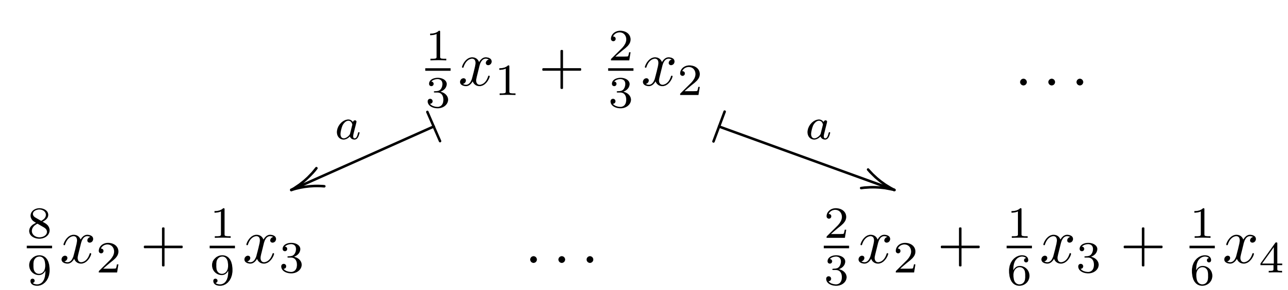


belief-state transformer



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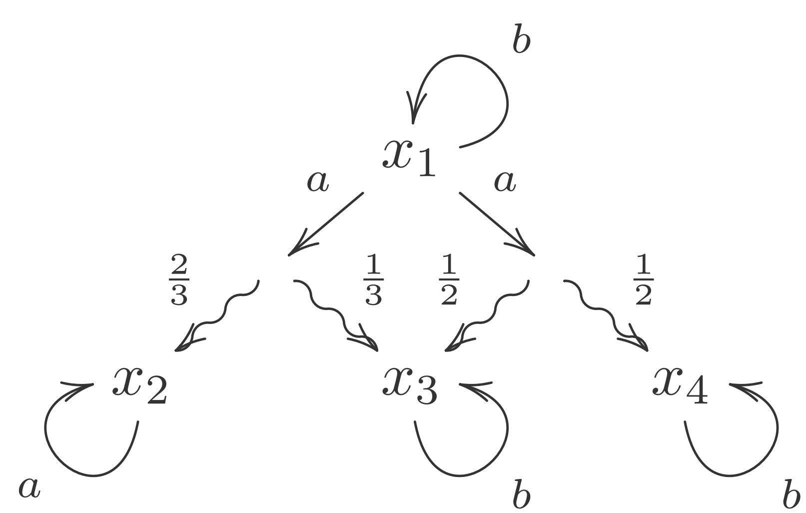
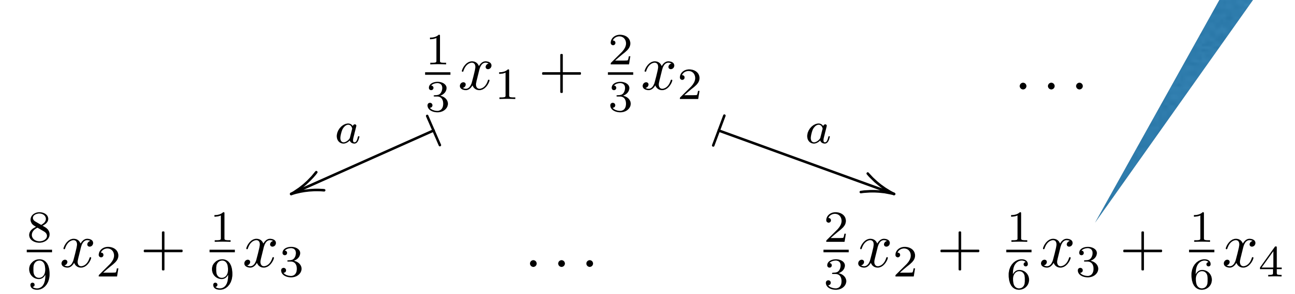


belief-state transformer

belief state



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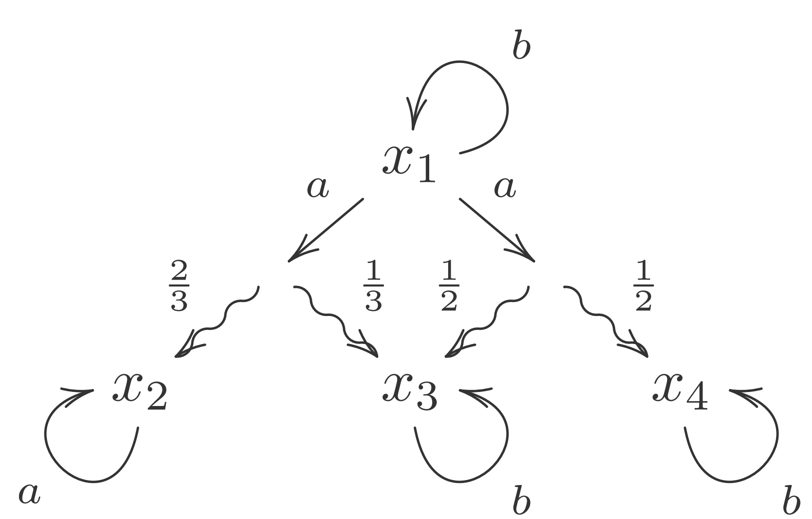

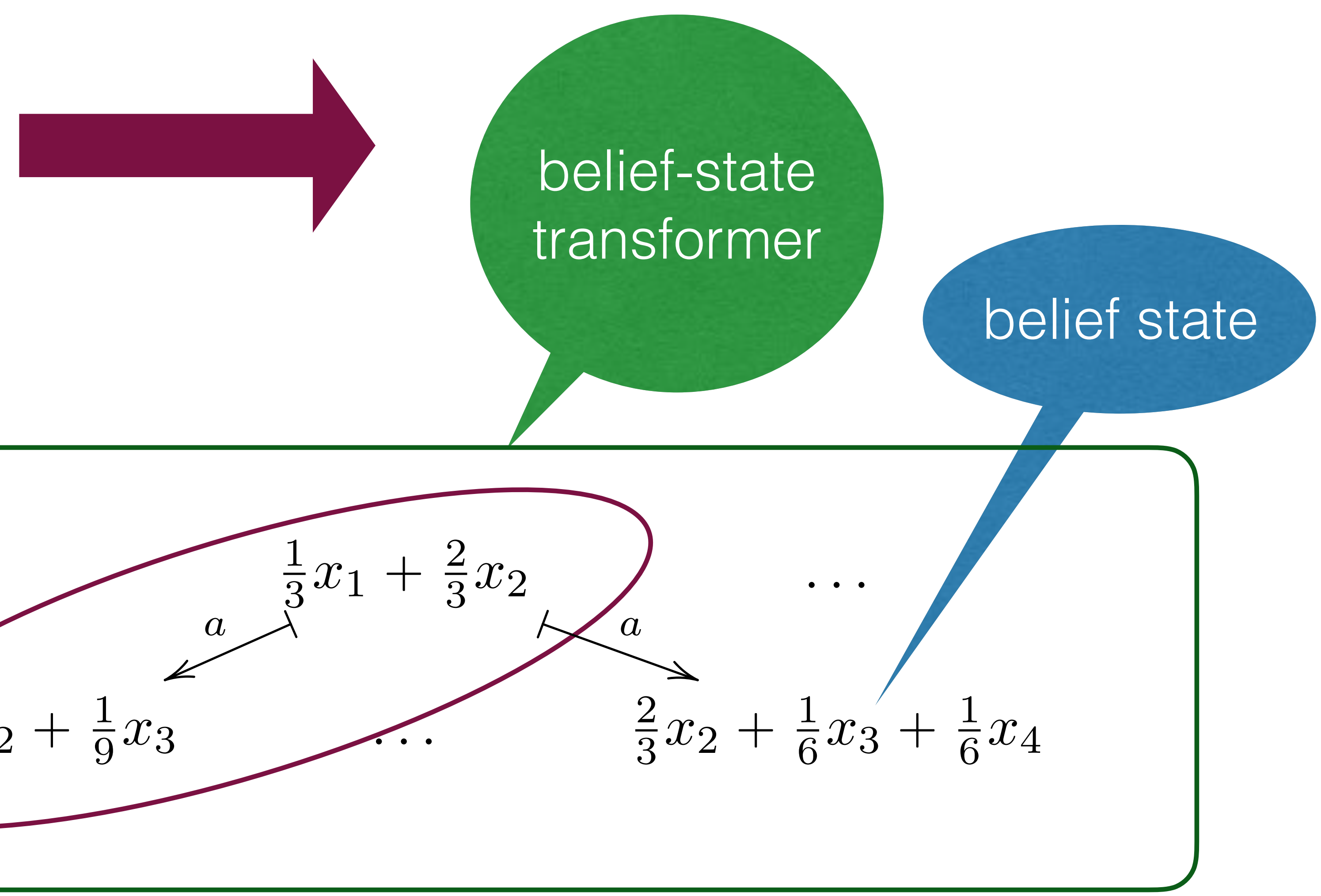

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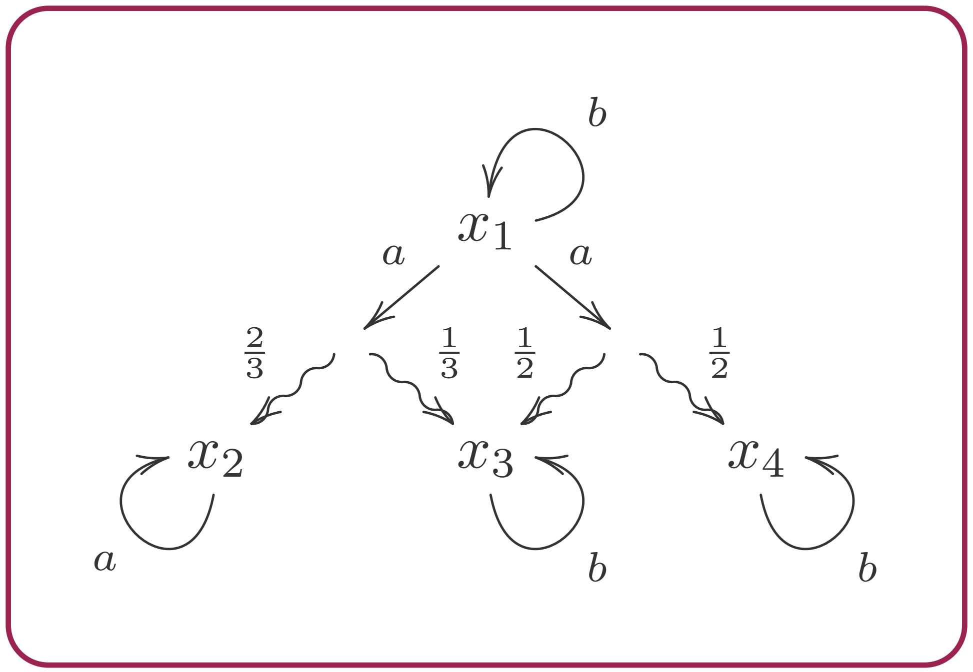

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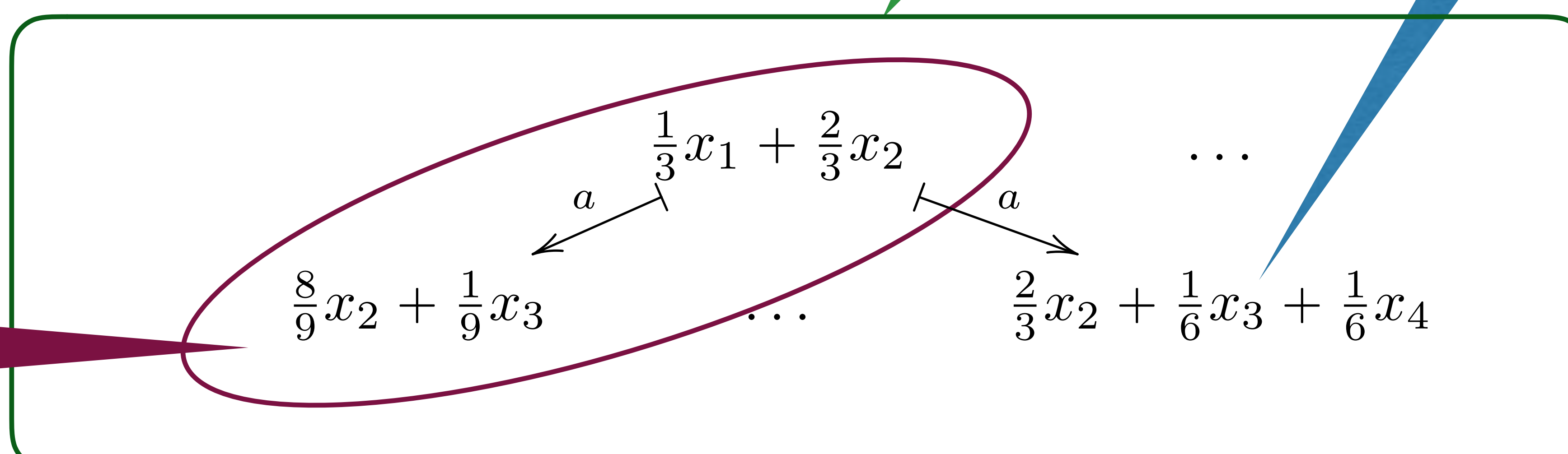


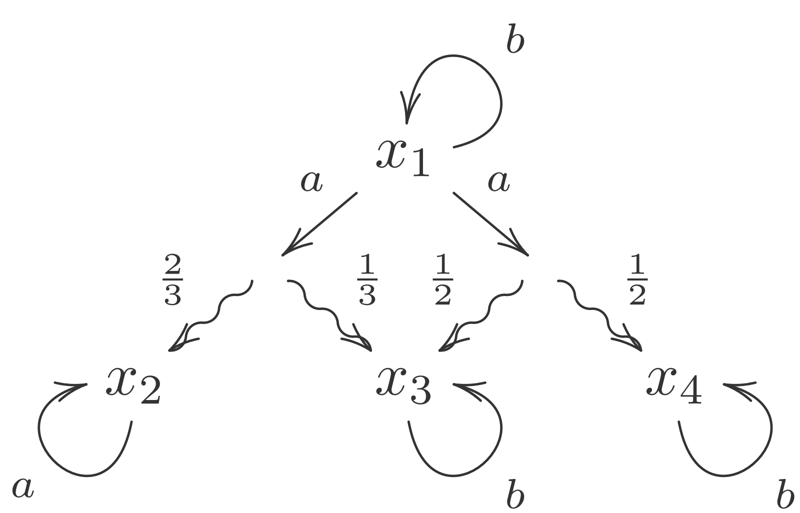
$$\frac{1}{3}\left(\frac{2}{3}x_2 + \frac{1}{3}x_3\right) + \frac{2}{3}(1x_2)$$

Belief-state transformer

foundation ?

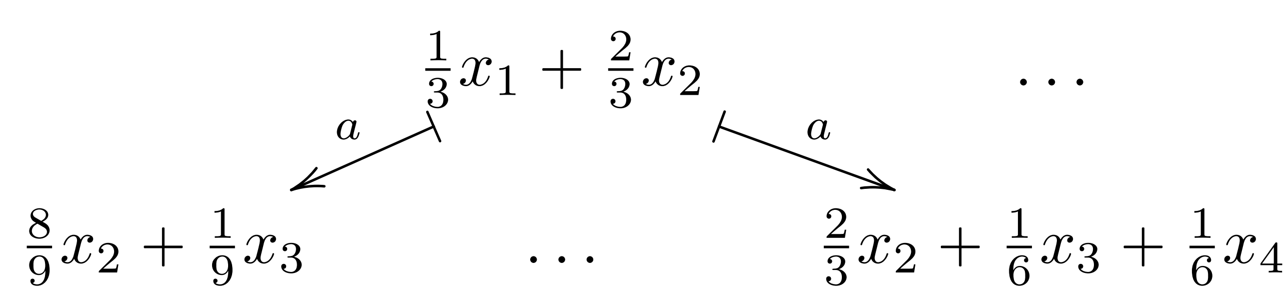
PA

$X \to \mathcal{P}(\mathcal{D}(X))^A$



via a generalised determinisation

LTS on free convex algebra

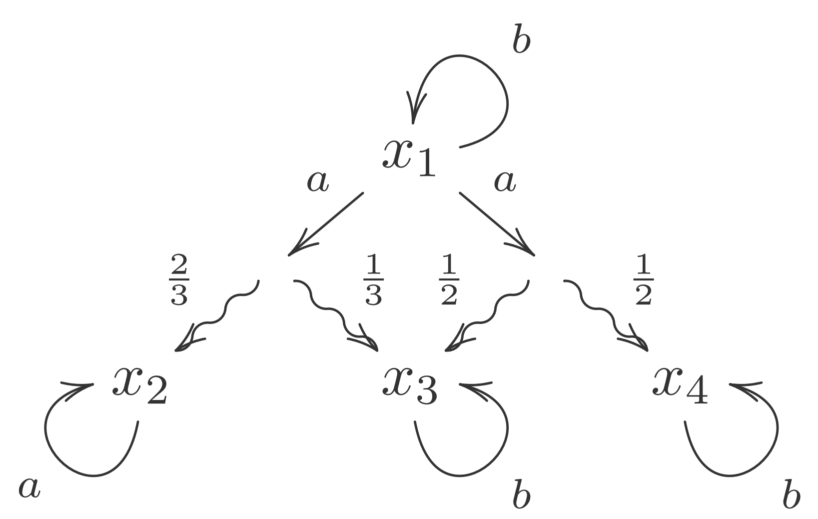


Belief-state transformer

foundation ?

PA

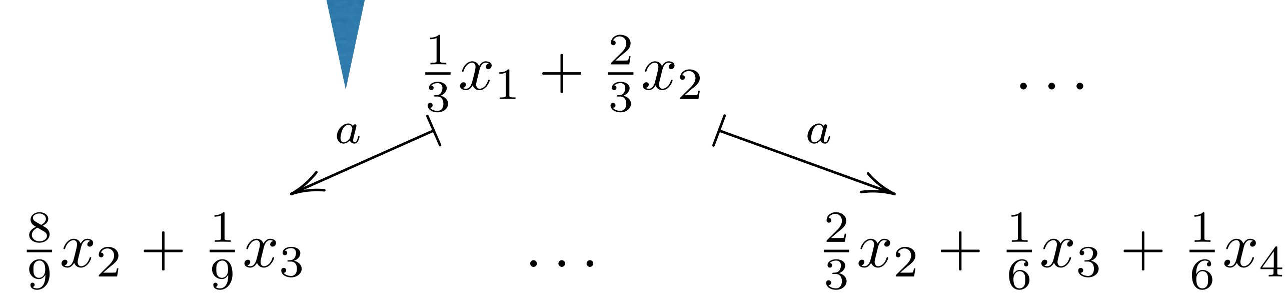
$X \to \mathcal{P}\,(\mathcal{D}(X))^A$



via a generalised determinisation

LTS on free convex algebra

Minkowski sum



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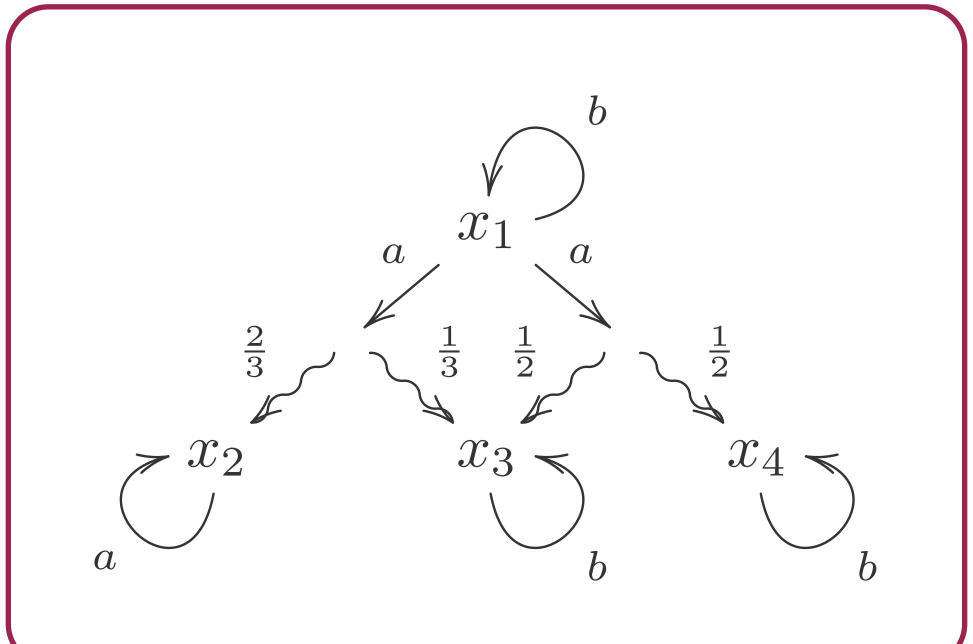

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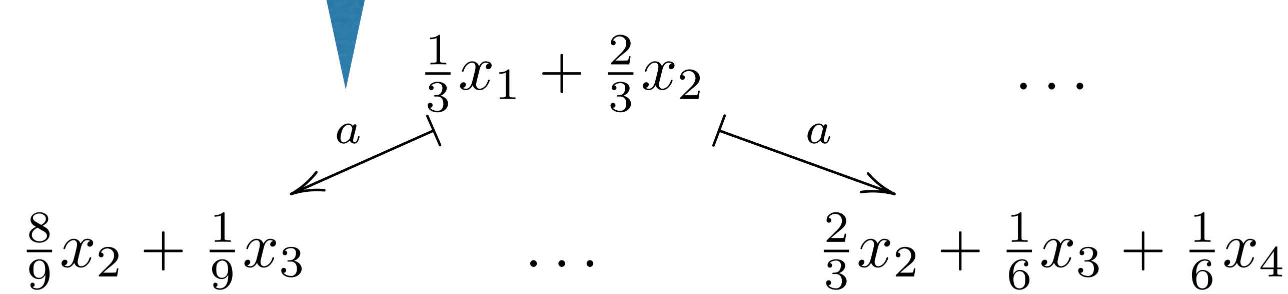

[Bonchi, SIlva, S. CONCUR'17]

Probabilistic Automata

Probabilistic Automata

Can be given different semantics:

1. Bisimilarity
2. Convex bisimilarity
3. Distribution bisimilarity

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probabilistic / combined bisimilarity

Probabilistic Automata

Can be given different semantics:

1. Bisimilarity — strong bisimilarity
2. Convex bisimilarity — probabilistic / combined bisimilarity
3. Distribution bisimilarity — belief-state bisimilarity

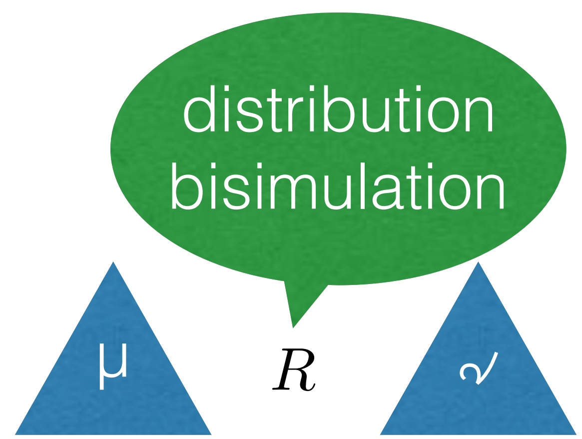
Distribution bisimilarity

Distribution bisimilarity

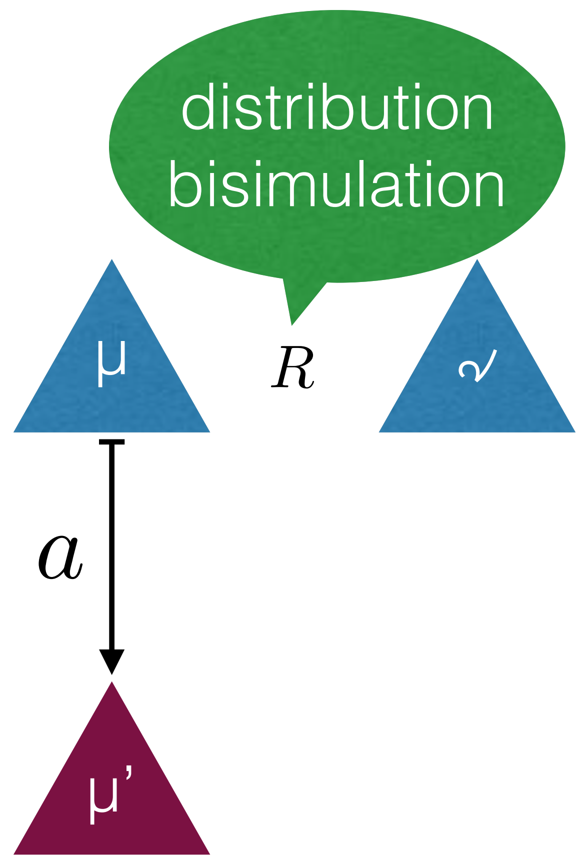
distribution bisimulation

R

Distribution bisimilarity



Distribution bisimilarity



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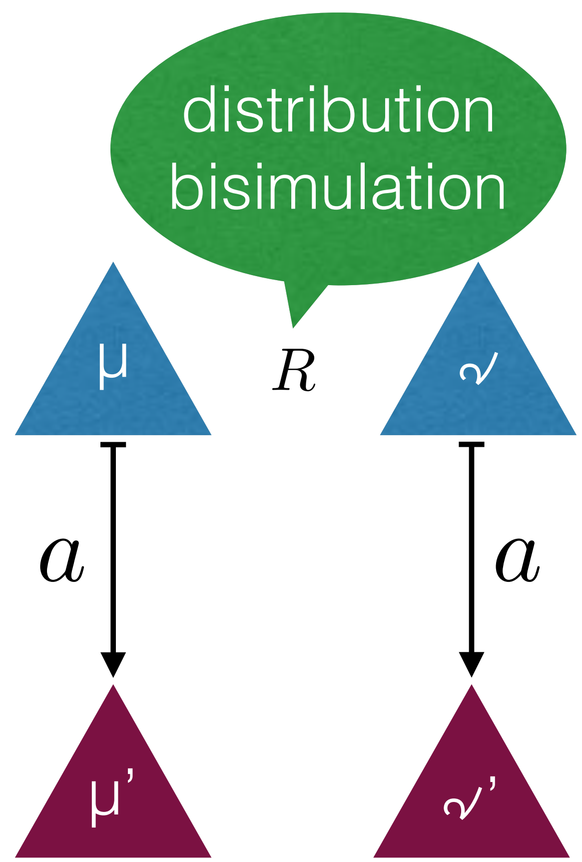

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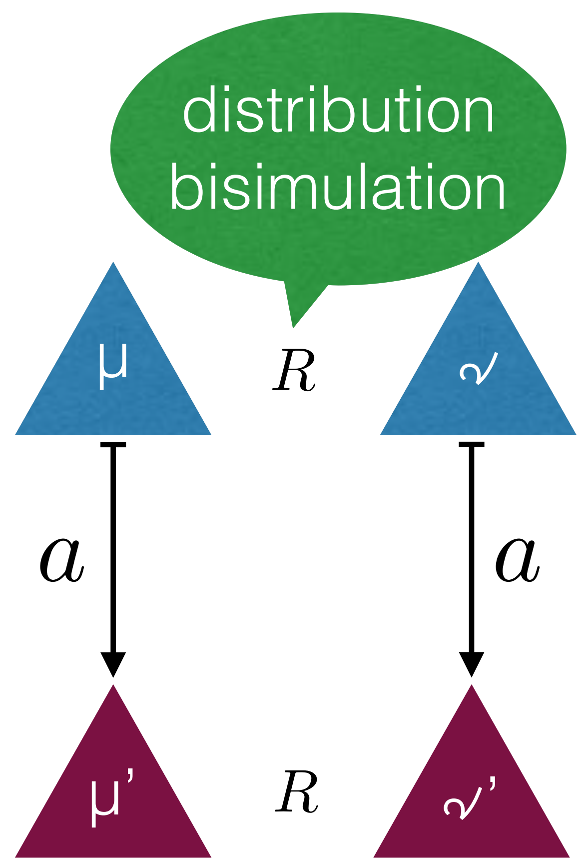

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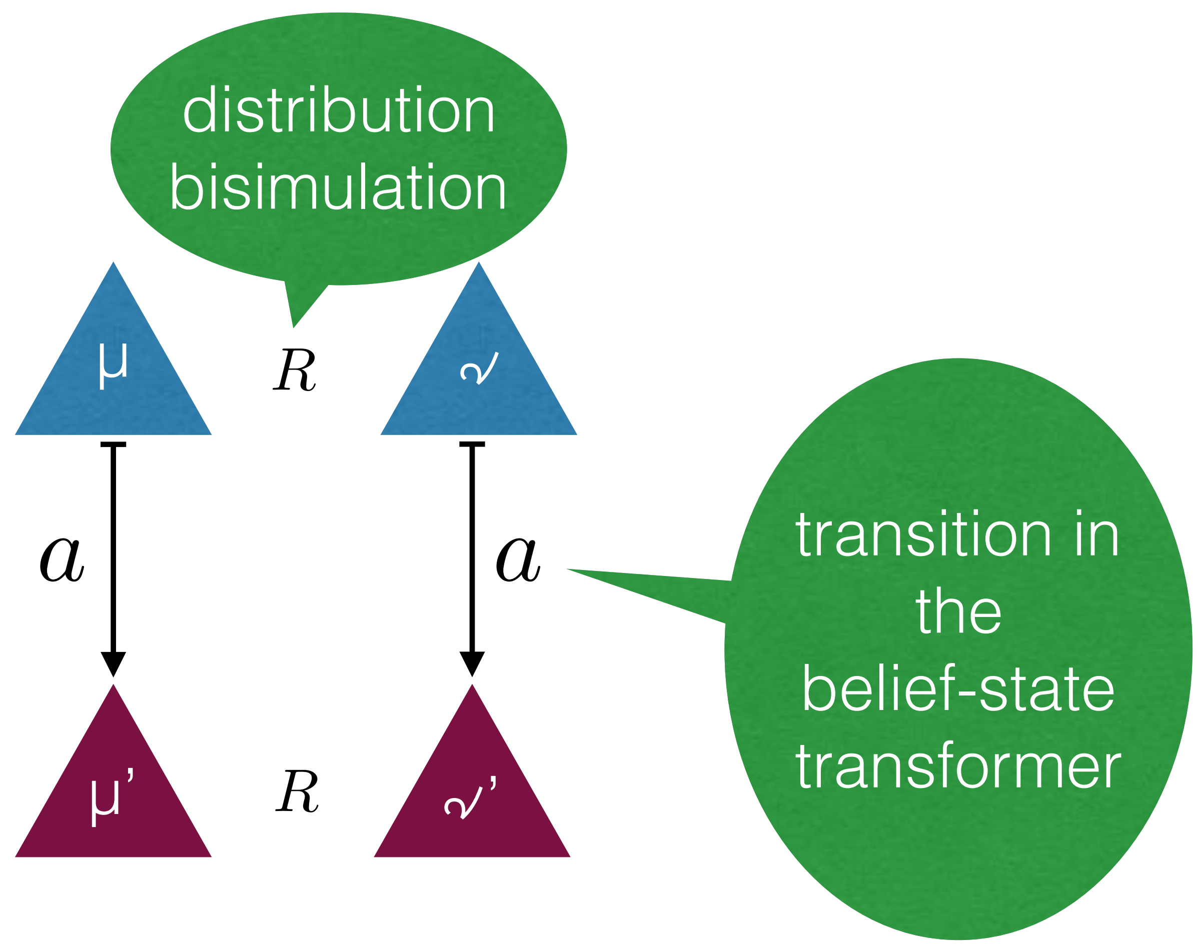
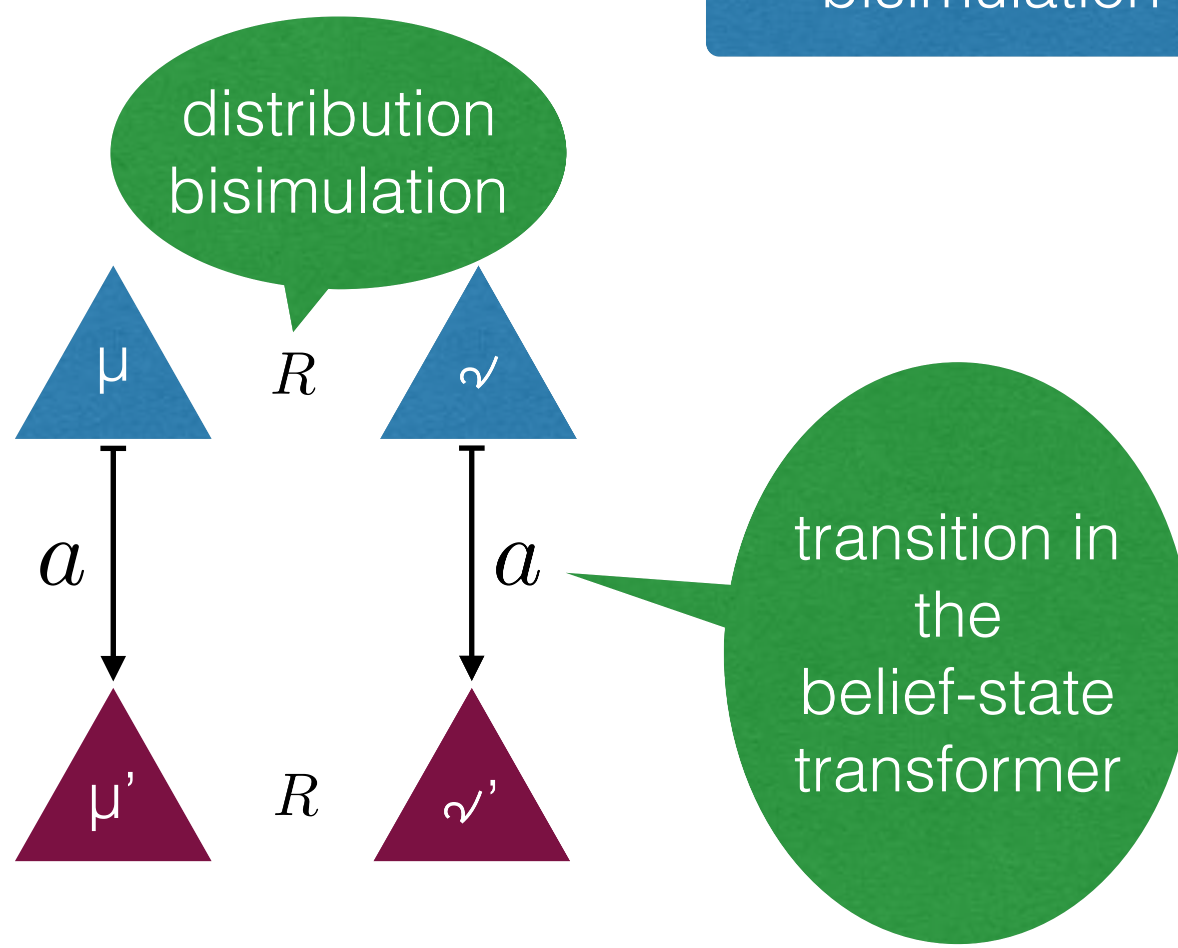


Distribution bisimilarity

$\sim_d$
largest distribution bisimulation



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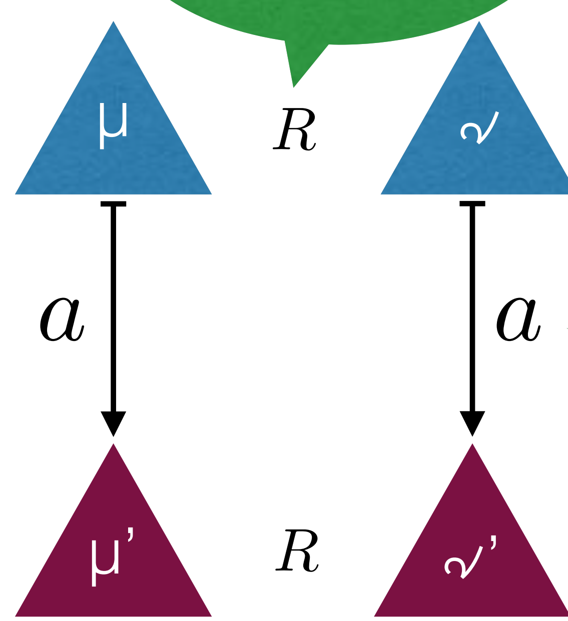

transition in the belief-state transformer

$\sim_d$
is LTS bisimilarity on the belief-state transformer

Coinductive proof method for distribution bisimilarity

[Bonchi, SIlva, S. CONCUR'17]

Coinductive proof method for distribution bisimilarity

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bisimulation up-to convex hull

R

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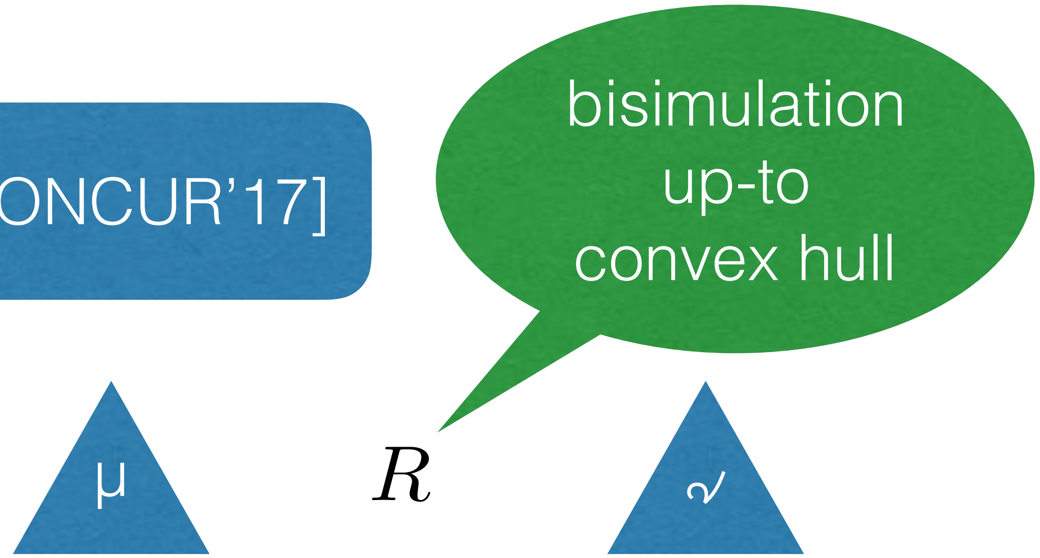
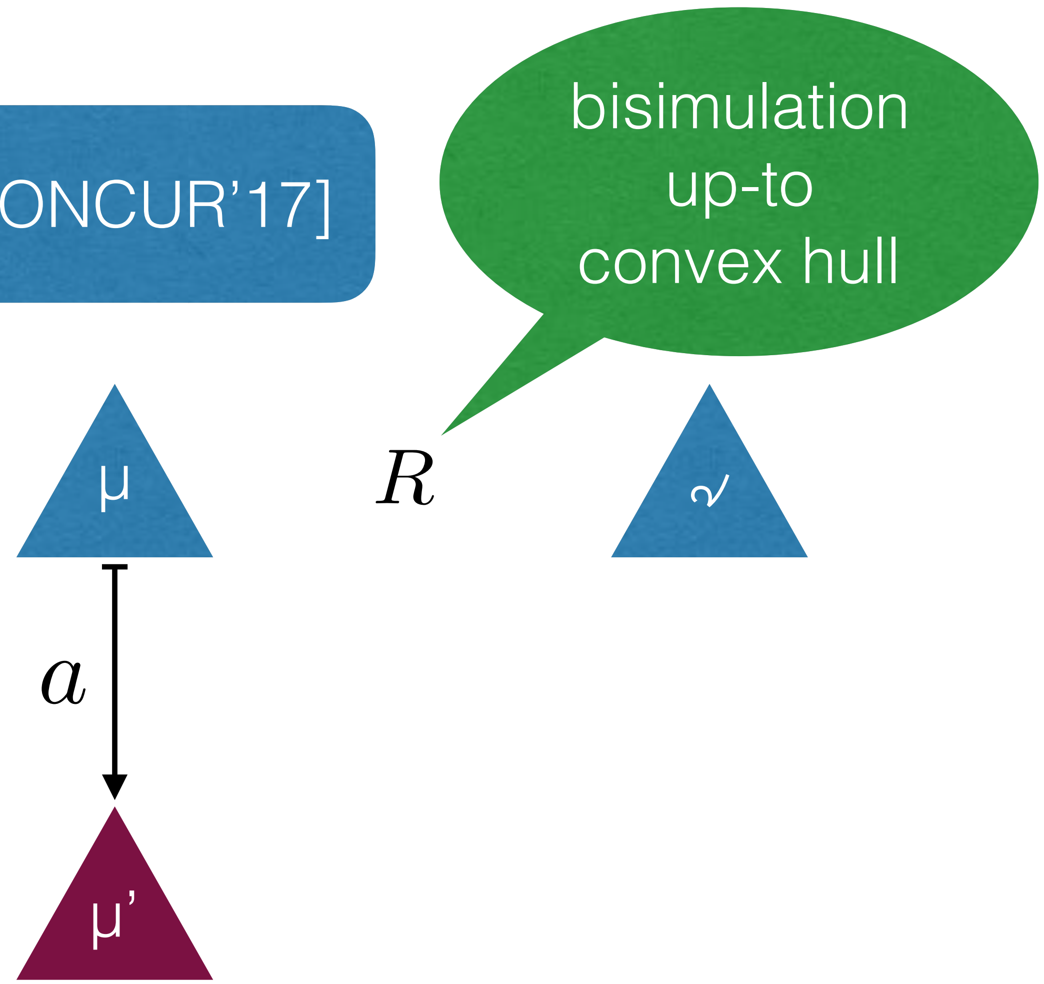


Coinductive proof method for distribution bisimilarity

[Bonchi, SIlva, S. CONCUR'17]



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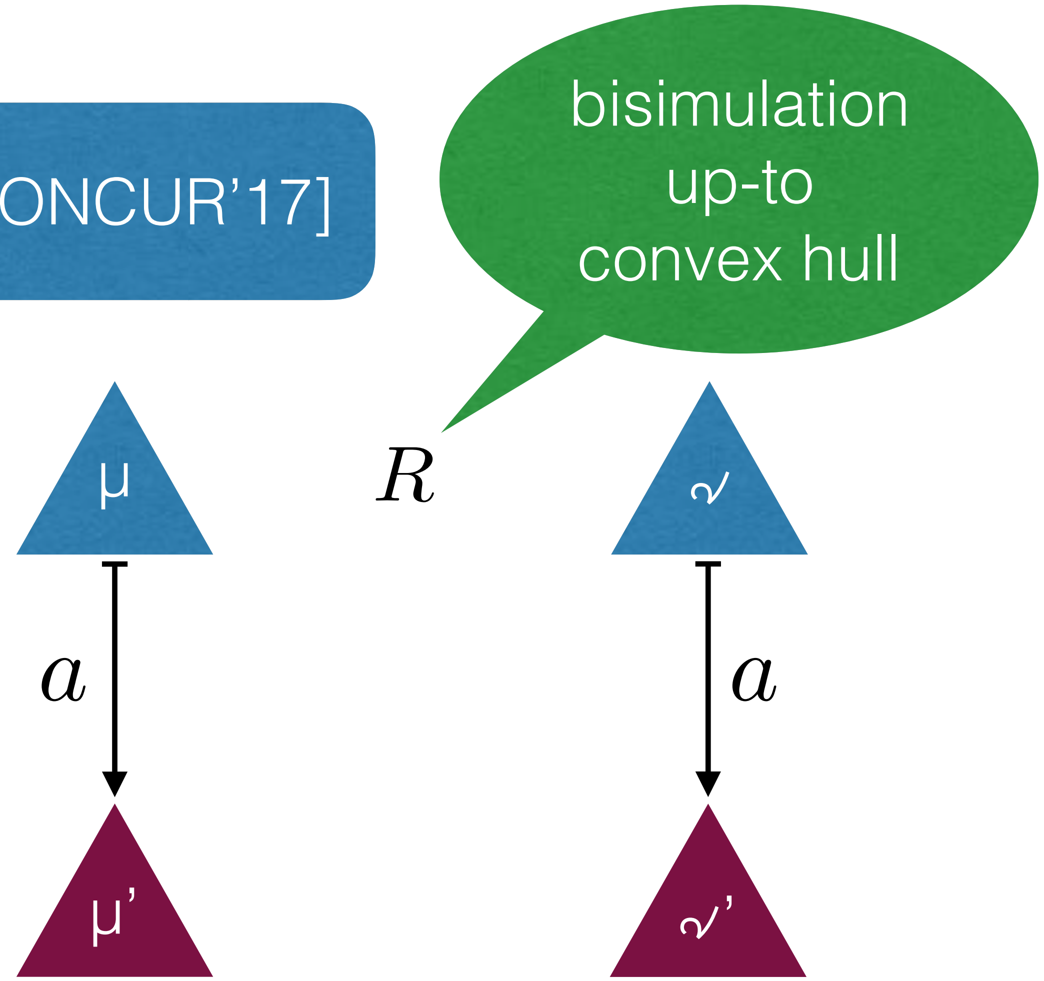

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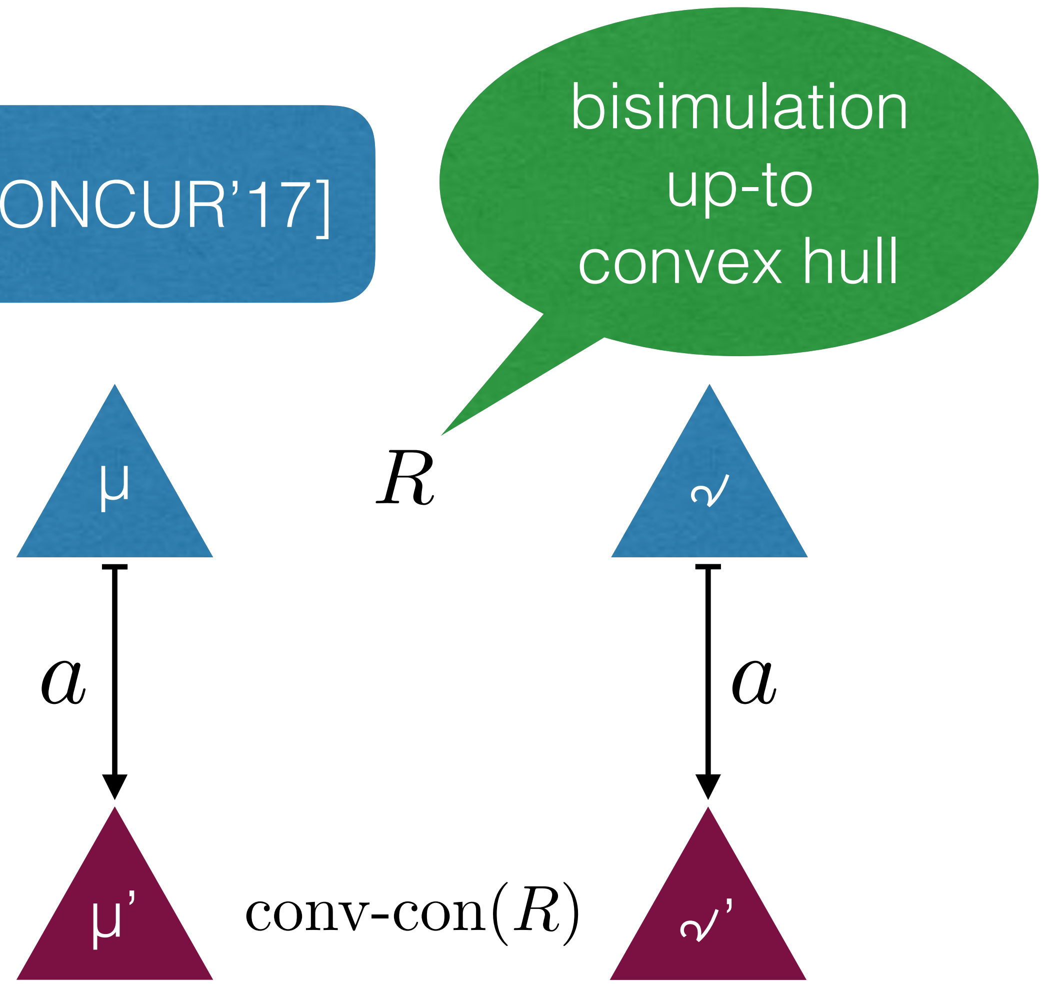

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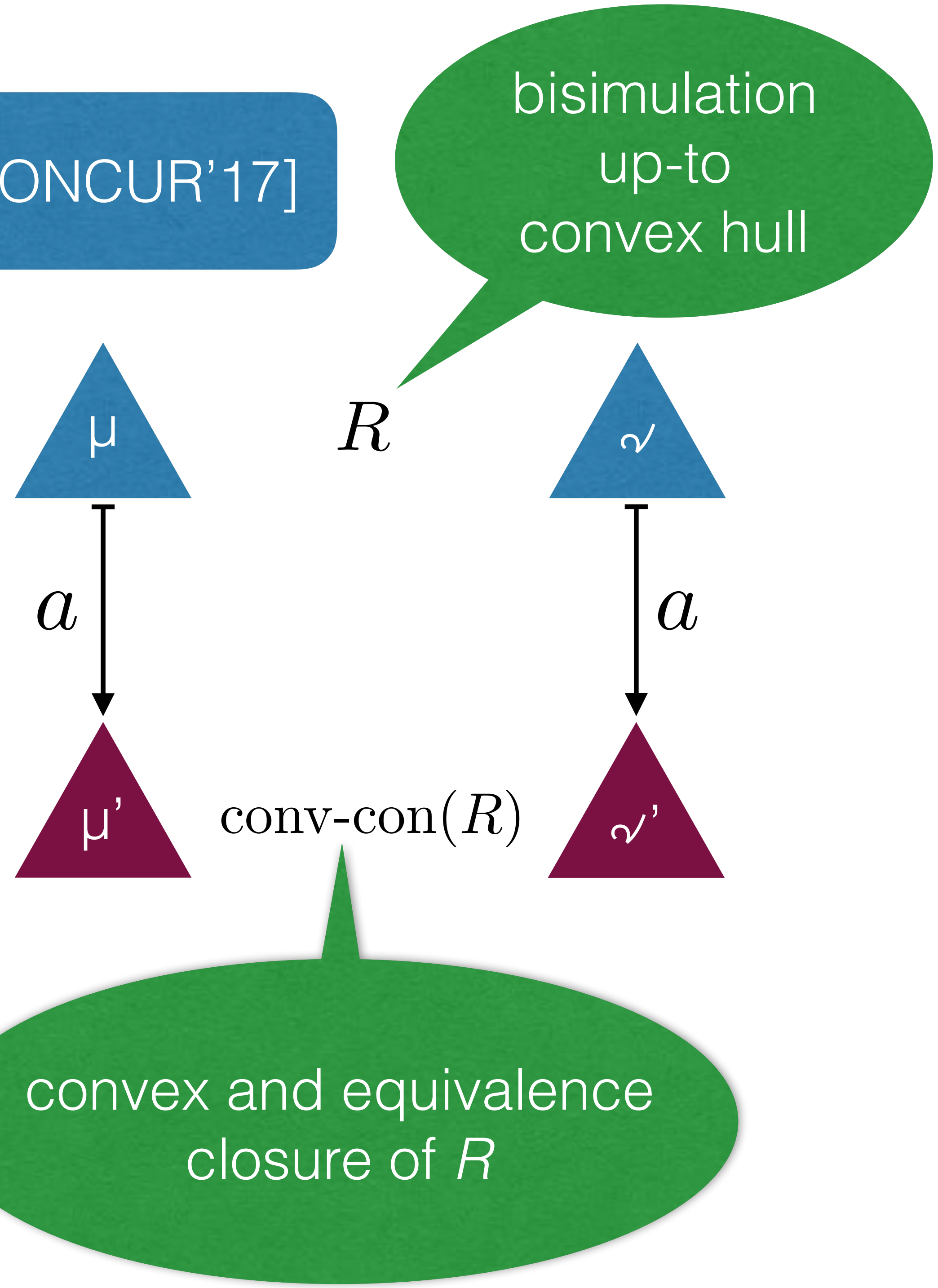
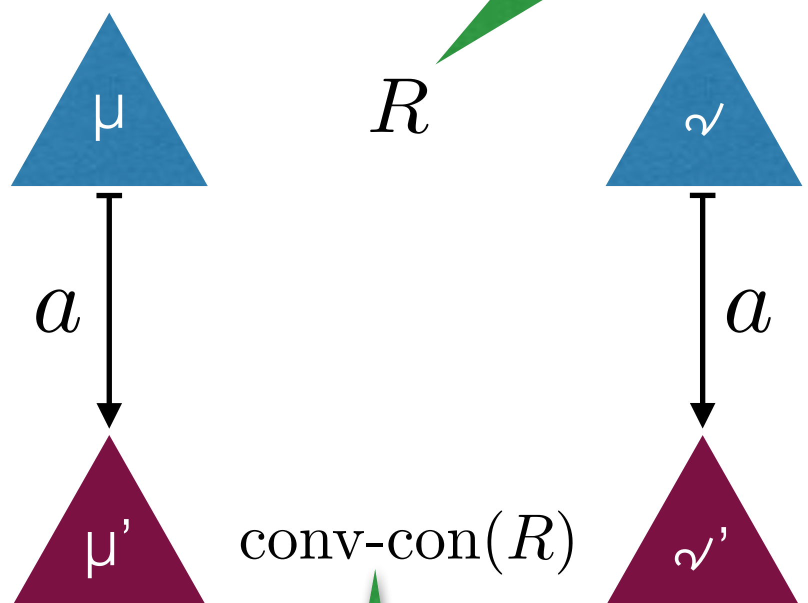


Coinductive proof method for distribution bisimilarity

[Bonchi, SIlva, S. CONCUR'17]

bisimulation up-to convex hull



convex and equivalence closure of *R*

to prove $\mu \sim_d \nu$ it suffices to find a bisimulation up-to convex hull *R* with $\mu \; R \; \nu$

Coinductive proof method for distribution bisimilarity

[Bonchi, SIlva, S. CONCUR'17]

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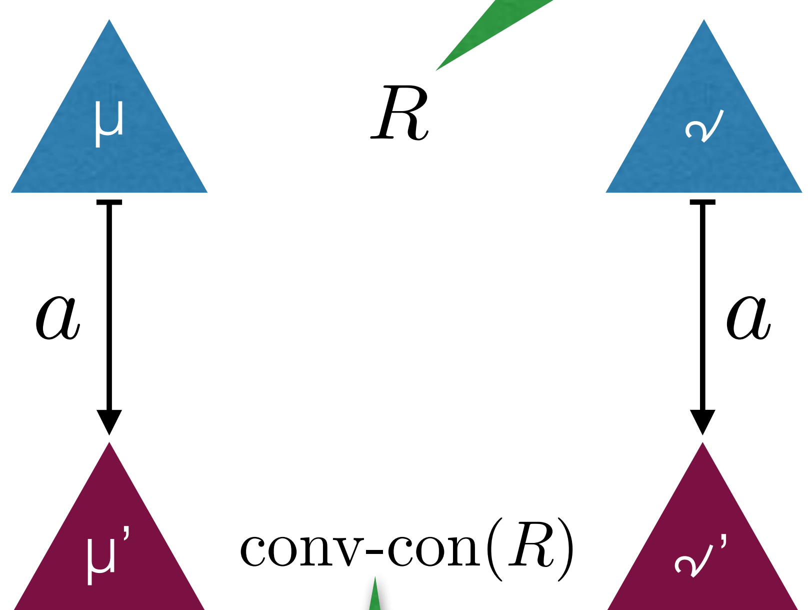

convex and equivalence closure of R

to prove $\mu \sim_d \nu$ it suffices to find a bisimulation up-to convex hull R with $\mu \; R \; \nu$

there always exists a finite one!

Coinductive proof method for distribution bisimilarity

[Bonchi, SIlva, S. CONCUR'17]

bisimulation up-to convex hull

$$\begin{tikzcd} \mu \arrow[d, "a"'] & R & \nu \arrow[d, "a"] \\ \mu' & \text{conv-con}(R) & \nu' \end{tikzcd}$$

convex and equivalence closure of R

to prove μ $\sim_d$ ν
it suffices to find a bisimulation up-to convex hull R
with μ R ν

there always exists a finite one!

by S., Woracek JPAA'15]

Termination?

- We looked at one-point extensions of convex algebras, for termination.

- What are all the possible ways?

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MC and PA belief-state transformers

[S., Woracek CALCO'17]

$\mathbb{X} = \mathbb{D}_X$

Free convex algebras

carrier
$X_* = X + 1 = X \cup \{*\}$

Possible extensions $\mathbb{X}_*$ are:

- the black-hole extension

  $px + (1-p)* = *$

- * imitates a point $w \in X$

  $px + (1-p)* = px + (1-p)w$

- * imitates one of the extremal points $s \in S$ on all other points, and adheres this point

  $px + (1-p)* = px + (1-p)s, \quad x \neq s$

  $ps + (1-p)* = *$

these are all extensions!

It's time to terminate this talk..

convexity is everywhere in probabilistic systems sematics

next: algorithms for computing distribution bisimilarity

Thank You!