Derivations / Reasoning

Limitations of proofs by calculation

Proofs by calculation are formal and well-structured, but often undirected and not particularly intuitive.

Example

$$P \wedge (P \vee Q) \stackrel{\text{val}}{=} (P \vee F) \wedge (P \vee Q)$$

$$\stackrel{\text{val}}{=} P \vee (F \wedge Q)$$

$$\stackrel{\text{val}}{=} P \vee F$$

$$\stackrel{\text{val}}{=} P$$

we can prove this more intuitively by reasoning

Conclusions

$$P \wedge (P \vee Q) \stackrel{\text{val}}{=} P \quad P \wedge (P \vee Q) \Leftrightarrow P \stackrel{\text{val}}{=} T$$

An example of a mathematical proof

Theorem

If x^2 is even, then x is even $(x \in \mathbb{Z})$.

(sub)goal

Proof

Let $x \in \mathbb{Z}$ be such that x^2 is even.

We need to prove that x is even too.

pure hypothesis

generating hypothesis

Assume that x is odd, towards a contradiction.

conclusion

If x is odd than x = 2y+1 for some $y \in \mathbb{Z}$.

Then $x^2 = (2y+1)^2 = 4y^2 + 4y + 1 = 2(2y^2 + 2y) + 1$ and $2y^2 + 2y \in \mathbb{Z}$.

So, x^2 is odd too, and we have a contradiction.

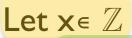
Thanks to Bas Luttik

Exposing logical structure

Theorem

If x^2 is even, then x is even $(x \in \mathbb{Z})$.

Proof



Assume x^2 is even.

Assume that x is odd.

Then x = 2y+1 for some $y \in \mathbb{Z}$.

Then
$$x^2 = (2y+1)^2 = 4y^2 + 4y + 1 = 2(2y^2 + 2y) + 1$$
 and $2y^2 + 2y \in \mathbb{Z}$.

So, x^2 is odd

a contradiction.

So, x is even

(sub)goal

generating hypothesis

pure hypothesis

conclusion

Thanks to Bas Luttik

Single inference rule

Q is a correct conclusion from n premises $P_1, ..., P_n$ iff $(P_1 \land P_2 \land ... \land P_n) \overset{val}{\vDash} Q$

If n=0, then $P_1 \wedge P_2 \wedge ... \wedge P_n \stackrel{\text{val}}{=} T$ Note that $T \models Q$ means that $Q \stackrel{\text{val}}{=} T$

Q holds unconditionally

Derivation

Q is a correct conclusion from n premises $P_1, ..., P_n$ iff $(P_1 \land P_2 \land ... \land P_n) \stackrel{\text{val}}{\models} Q$

a formal system
based on the single
inference rule
for proofs that closely
follow our
intuitive reasoning

Two types of inference rules:

elimination rules

introduction rules

for drawing conclusions out of premises

for simplifying goals

(particularly useful) instances of the single inference rule

and one new special rule!

Conjunction elimination

How do we use a conjunction in a proof?

 $P \wedge Q \stackrel{\text{val}}{\models} P$

 $P \land Q \stackrel{\text{val}}{\models} Q$

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∧-elimination
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 $\parallel \parallel$

(k) $P \wedge Q$

|| ||

 $\{\land$ -elim on $(k)\}$

(m) F

(k) $P \wedge Q$

|| ||

 $\{\land$ -elim on $(k)\}$

(m) Q

(k < m) (k < m)

Implication elimination

How do we use an implication in a proof?

 $P \Rightarrow Q \stackrel{\text{val}}{\models} ???$

 $(P \Rightarrow Q) \land P \stackrel{\text{val}}{\models} Q$

$$\| \|$$

Conjunction introduction

How do we prove a conjunction?

∧-introduction

(k) P

(l) Q

 $\{\land\text{-intro on (k) and (l)}\}\$ (m) $P \land Q$

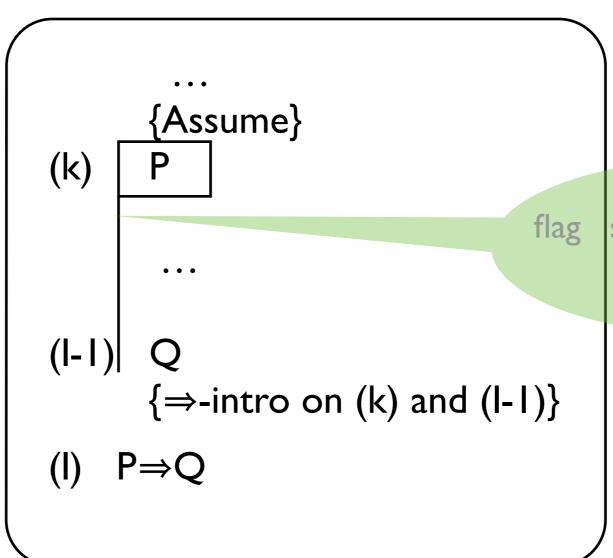
(k < m, l < m)

 $P{\wedge}Q \overset{\mathsf{val}}{\vDash} P{\wedge}Q$

Implication introduction

How do we prove an implication?

⇒-introduction



truly new and necessary for reasoning with hypothesis

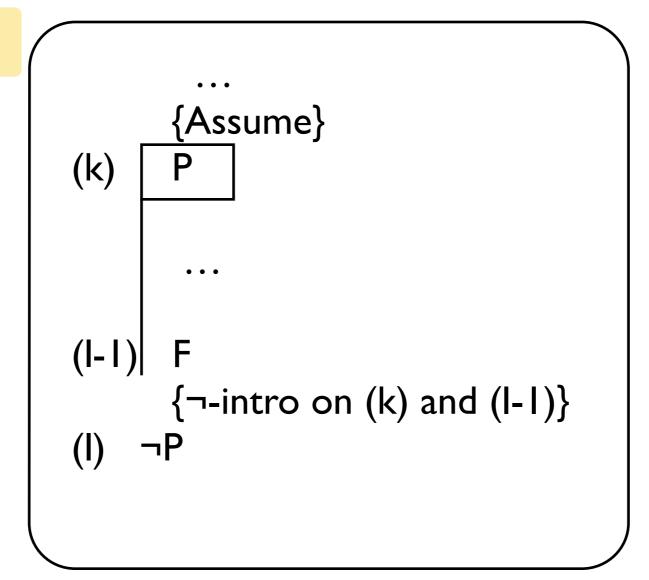
shows the validity of a hypothesis

time for an example!

Negation introduction

How do we prove a negation?

¬-introduction



 $\neg P \stackrel{\text{val}}{=} P \Rightarrow F$

⇒-intro

Negation elimination

How do we use a negation in a proof?

¬-elimination

 $P \wedge \neg P \stackrel{\text{val}}{\models} F$

time for an example!

(k < m, l < m)

F introduction

How do we prove F?

F-introduction

• • •

(k) P

• • •

(I) ¬P

• • •

{F-intro on (k) and (l)}

(m) F

 $P \wedge \neg P \stackrel{\text{val}}{\models} F$

the same as ¬-elim only intended bottom-up

(k < m, l < m)

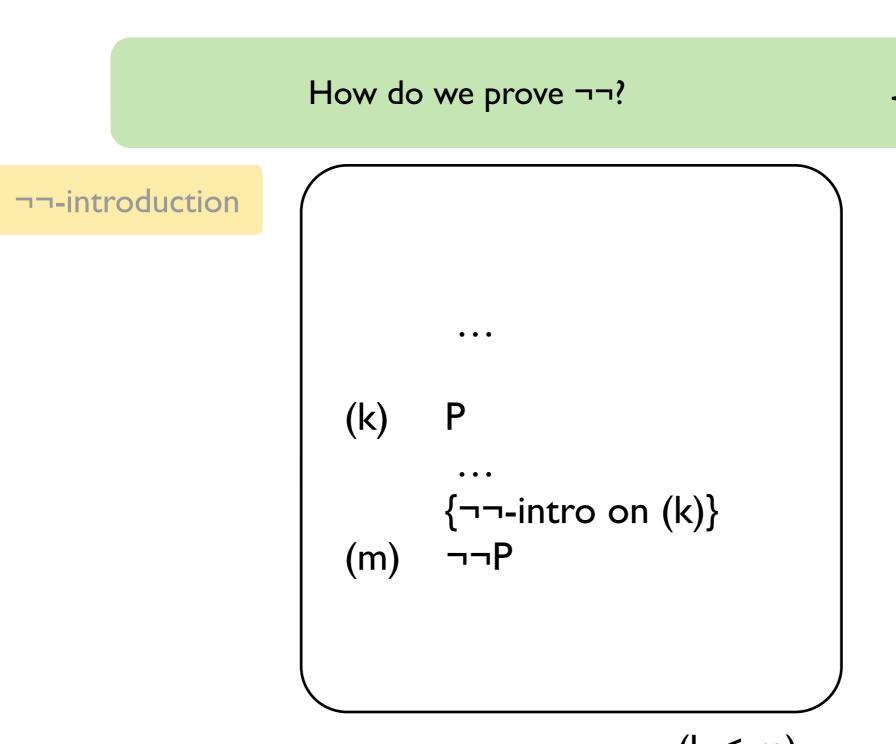
F elimination

How do we use F in a proof? F-elimination (k) $\{F-elim on (k)\}$ (m)

it's very useful!

 $F \stackrel{\text{val}}{\models} P$

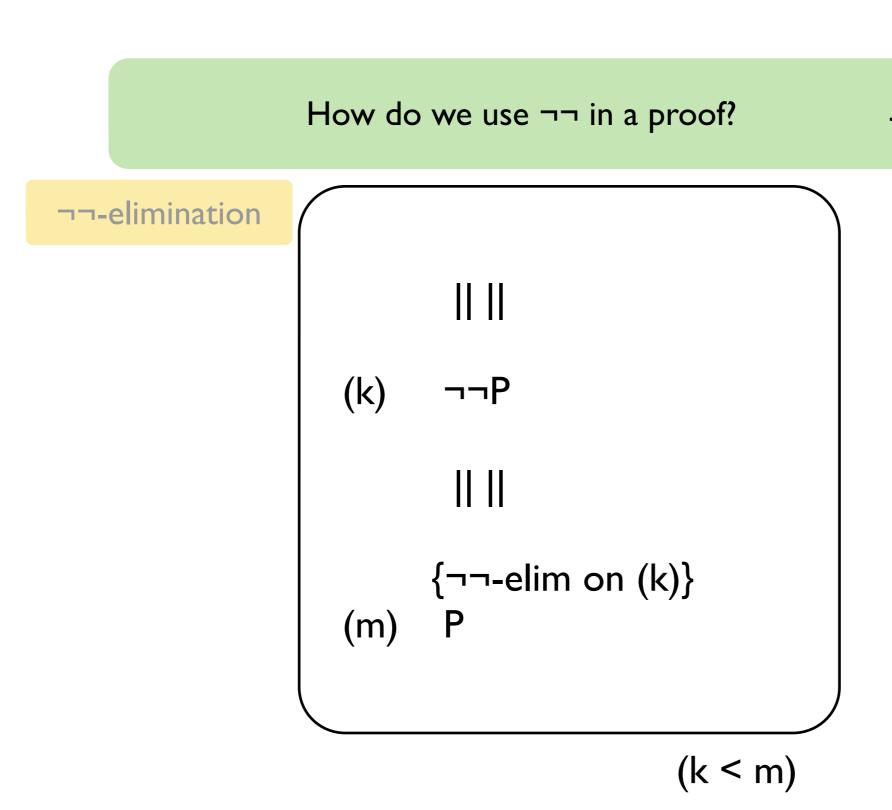
Double negation introduction



P ⊭ ¬¬P

(k < m)

Double negation elimination



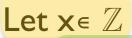
 $\neg \neg P \stackrel{\text{val}}{\models} P$

Proof by contradiction

Theorem

If x^2 is even, then x is even $(x \in \mathbb{Z})$.

Proof



Assume x^2 is even.

Assume that x is odd.

Then x = 2y+1 for some $y \in \mathbb{Z}$.

Then
$$x^2 = (2y+1)^2 = 4y^2 + 4y + 1 = 2(2y^2 + 2y) + 1$$
 and $2y^2 + 2y \in \mathbb{Z}$.

So, x^2 is odd

a contradiction.

So, x is even

(sub)goal

generating hypothesis

pure hypothesis

conclusion

Thanks to Bas Luttik

Proof by contradiction

How do we prove P by a contradiction?

proof by contradiction

```
{Assume}
(k)
         ¬P
        \{\neg-intro on (k) and (l-1)\}
(l)
        \neg \neg P
        \{\neg\neg\text{-elim on (I)}\}\
(|+|)
```

 $\neg P \Rightarrow F \stackrel{\text{val}}{\models} \neg \neg P \stackrel{\text{val}}{\models} P$

¬-intro

¬¬-elim

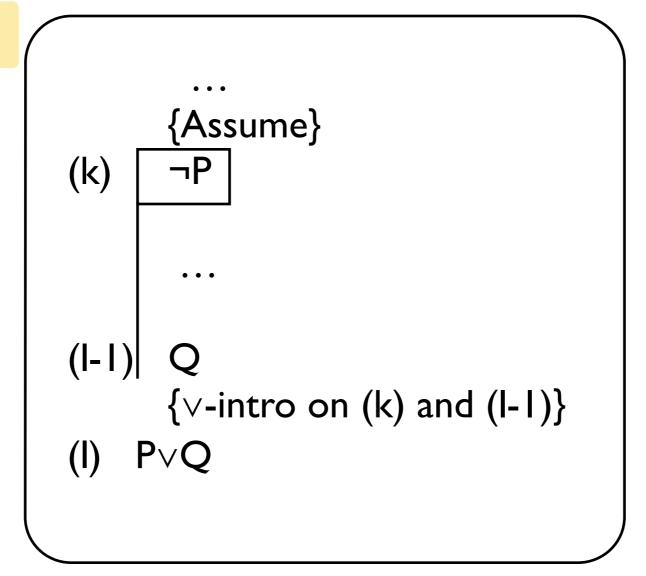
time for an example!

(k < m)

Disjunction introduction

How do we prove a disjunction?

∨-introduction



 $\neg P \Rightarrow Q \stackrel{\text{val}}{\models} P \lor Q$

 $\neg \overline{Q \Rightarrow} P \stackrel{\text{val}}{\vDash} P \lor Q$

⇒-intro

Disjunction introduction

How do we prove a disjunction?

 $\neg P \Rightarrow Q \stackrel{\text{val}}{\models} P \lor Q$

 $\neg \overline{Q \Rightarrow} P \stackrel{\text{val}}{\vDash} P \vee Q$

∨-introduction

⇒-intro

Disjunction elimination

How do we use a disjunction in a proof?

 $P \lor Q \stackrel{\text{val}}{\models} \neg P \Rightarrow Q$

 $P \lor Q \stackrel{\text{val}}{\models} \neg Q \Rightarrow P$

$$(k) \quad P \lor Q$$

$$\{ \lor \text{-elim on (k)} \}$$

(m) $\neg P \Rightarrow Q$

Disjunction elimination

How do we use a disjunction in a proof?

 $P \lor Q \stackrel{\text{val}}{\models} \neg P \Rightarrow Q$

 $P \lor Q \stackrel{\text{val}}{\models} \neg Q \Rightarrow P$

Proof by case distinction

How do we prove R by a case distinction?

proof by case distinction

|| ||

(k) $P\lor Q$

|| ||

I) P⇒R

|| ||

(m) $Q \Rightarrow R$

 $\| \|$

 $\{case-dist on (k), (l), (m)\}$

(n) R

 $(P \lor Q) \land (P \Rightarrow R) \land (Q \Rightarrow R) \stackrel{\text{val}}{\vDash} R$

 $(k \le n, l \le n, m \le n)$

Bi-implication introduction

How do we prove a bi-implication?

 $(P \Rightarrow Q) \land (Q \Rightarrow P) \stackrel{\text{val}}{\vDash} P \Leftrightarrow Q$

⇔-introduction

• • •

(k) P⇒Q

• • •

(I) $Q \Rightarrow P$

• • •

 $\{\Leftrightarrow$ -intro on (k) and (l) $\}$

(m) P⇔Q

(k < m, l < m)

∧-intro

Bi-implication elimination

