The Microcosm Principle and Concurrency in Coalgebras

Ichiro Hasuo Bart Jacobs* Ana Sokolova

Radboud University Nijmegen, the Netherlands
http://www.cs.ru.nl/~{ichiro, bart, anas}

Abstract

Our initial question is: How to specify a parallel composition of transition systems as coalgebras that is guaranteed to be at least associative and compositional? The answer lies in recognizing algebraic structure in the base category and its compatibility with the coalgebra-type functor. Our results apply to both possibilistic and probabilistic systems and to both bisimilarity and trace semantics.

Parallel composition is defined on the level of systems but it also induces parallel composition on states (in a final coalgebra). The two are shown to be related via the compositionality result. This phenomenon is an instance of what Baez and Dolan have called the "microcosm principle": parallel composition is defined inside a category with parallel composition, just like a monoid is defined in a monoidal category. We generalize such nested results in two steps: to algebraic specifications (signatures with equations) and further to 2-categorical treatment of Lawvere theories.

1. Introduction

In concurrency theory people use parallel composition of automata, or states in automata, to describe complex systems consisting of concurrent components. It is desirable that a parallel composition is (1) compositional—the behaviour of the components determines the behaviour of the composite – and (2) it has some algebraic properties like associativity or commutativity. Our question is how to achieve these requirements directly from the specification.

This paper provides answers at multiple levels of abstraction, using category theory in general—and coalgebras in particular—to capture the connections and similarities. It first describes fairly concretely how a tensor $\otimes$ on a base category induces, in presence of a suitable synchronisation map, a tensor on a category of coalgebras. The latter tensor is understood as parallel composition of automata. In presence of a final coalgebra (of behaviours or processes) it is shown that its carrier comes equipped with an induced parallel composition operation $\|$, together with a suitable compositionality result relating $\otimes$ and $\|$.

After this result we switch gears and move to the next level of abstraction. The tensor $\otimes$ is generalised to an arbitrary algebraic signature with equations (Σ, E). We show that if a base category is a model of (Σ, E), then the category of coalgebras of a suitable endofunctor on this basis is also a model of (Σ, E), and that a final coalgebra (if any) is a (Σ, E)-model too—again with a compositionality result. Here we see more clearly how structure "in the large" (on categories) also appears "in the small" (on object in categories), in a connected manner. We borrow from John Baez and James Dolan the phrase "microcosm principle" for this phenomenon, namely what they describe as: "A monoid object is something you define in a monoidal category, but a monoidal category is itself a kind of monoid object!"[1].

At this stage we switch gears once again and move to a 2-categorical level. Here we recognise this microcosm principle more clearly, namely as an $\mathbb{L}$-object appearing in an $\mathbb{L}$-category, for a Lawvere theory $\mathbb{L}$.

Nested structure is thus the underlying theme of this paper. It first appears in our investigations of concurrency—for coalgebras, as systems in general [17]—at separate, but connected, levels. This nested structure is subsequently clarified and axiomatised at different levels of abstraction: via algebraic specifications and via Lawvere theories.

This introduction continues with a more detailed account of the content and contribution of the three parts of the paper, Sections 2, 3 and 4. The reader is assumed to be familiar with the basics of the theory of coalgebras, of algebraic specifications, and also of elementary 2-category theory.

Part I (Section 2) is about abstract concurrency. We identify "concurrency in coalgebras" with an associative tensor structure $\otimes$ on a category $\mathbf{Coalg}_F$ of coalgebras of a functor F. This structure arises from (1) a base category with an associative tensor $\otimes$, and (2) a suitable natural transformation sync connecting F and $\otimes$. Different transformations sync yield different tensor structures (i.e., parallel compo-

*Also part-time at Technical University Eindhoven, the Netherlands.

[1] See http://math.ucr.edu/home/baez/week89.html.

sitions).

In this first part we also motivate the microcosm principle. Once we have parallel composition $\otimes$ of coalgebras, we aim at the following *compositionality* result for final coalgebra semantics, i.e., for the map $\mathrm{beh}(\_)$ to the final coalgebra: for coalgebras $X \xrightarrow{c} FX$ and $Y \xrightarrow{d} FY$,

$$\mathrm{beh}(c \otimes d) = \mathrm{beh}(c) \parallel \mathrm{beh}(d).$$

Here arises a question: what is $\parallel$ on the right hand side? It comes from a parallel composition of processes, that is, *states* of the final coalgebra $Z \xrightarrow{\zeta} FZ$. This is how we first encounter the microcosm principle of Baez and Dolan.

- the category $\mathbf{Coalg}_F$ carries an "outer" binary operation $\mathbf{Coalg}_F \times \mathbf{Coalg}_F \xrightarrow{\otimes} \mathbf{Coalg}_F$;
- an object $Z \xrightarrow{\zeta} FZ$ therein also carries an "inner" binary operation $\zeta \otimes \zeta \xrightarrow{\parallel} \zeta$. Note that $\parallel$ depends on the outer structure $\otimes$.

Part II (Section 3) investigates the microcosm principle for an algebraic specification (Σ, E), instead of just a tensor. It defines the notion of *microcosm models* $X \in \mathbb{C}$, where both a category $\mathbb{C}$ and an object X in $\mathbb{C}$ carry a (Σ, E)-structure. We show that when a (behaviour) functor $F : \mathbb{C} \to \mathbb{C}$ is compatible with (Σ, E) in a suitable sense, then compositionality holds for final coalgebra semantics for F-coalgebras.

In Part III (Section 4) we aim at a fully categorical account of the microcosm principle by employing *Lawvere theories*, that is, presentations of algebraic specification as categories. A microcosm model $X \in \mathbb{C}$ is now identified as a lax cone of the form
$\begin{tikzcd} \mathbb{L} \arrow[r, bend left, "\mathbf{1}"] \arrow[r, bend right, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow X"] & \mathbf{Cat} \end{tikzcd}$. By applying this observation in a coalgebraic setting, we obtain our most general compositionality result for final coalgebra semantics. In this third part we use 2-categorical notions because a 2-category (involving "categories in a category") well accommodates microcosm models (involving "models in a model").

Finally, we point out that there are obvious similarities between our work and the "mathematical operational semantics" of Turi-Plotkin [20], also involving algebraic structure on final coalgebras. We go beyond their work in the following two ways.

- Algebraic structure is carried not only by *states* (as in [20]) but also by *systems*; moreover, this is the *same* algebraic structure, typical for the microcosm principle.
- Equations in an algebraic specification, which are absent in [20], are handled here as integral part of the approach. For instance, parallel composition $\otimes$ of F-coalgebras is associative if F is compatible with associativity of $\otimes$ on the base category.

However, not all process algebraic operations that can be described by GSOS [20] fit in our (present) framework, as we will elaborate later (Remark 3.5).

2. Part I: Concurrency in coalgebra

Assume $\mathbb{C}$ is a category with an associative tensor $\otimes\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. Associativity is given by a natural isomorphism α with components $X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z$, satisfying standard coherence conditions [14]. For convenience we shall call such a $\mathbb{C}$ with associative $\otimes$ a $\otimes$-*category*. If there is also a neutral element $I \in \mathbb{C}$ one speaks of a *monoidal* category. It is called *symmetric* if there are also natural isomorphisms $\gamma\colon X \otimes Y \xrightarrow{\cong} Y \otimes X$

Examples of symmetric $\otimes$-categories (which are in fact monoidal) are $\mathbf{Sets}$ with cartesian product or $\mathbf{Vect}$ with the standard tensor product.

A functor $F\colon \mathbb{C} \to \mathbb{D}$ between two $\otimes$-categories will be called a $\otimes$-*functor* if it comes with a natural transformation sync with components $FX \otimes FY \to F(X \otimes Y)$ that commutes with the associativities α. If the $\otimes$'s are symmetric, F with sync is called symmetric if sync also commutes with the γ's.

Example 2.1. Consider the functor $F\colon \mathbf{Sets} \to \mathbf{Sets}$ given by $F = A \times \_ + 1$ for a constant set A. Assume there is a partial associative operation $\cdot$ on A. Define $\mathrm{sync} : FX \times FY \to F(X \times Y)$ by

$$\mathrm{sync}(u, v) = \begin{cases} \langle c, \langle x, y\rangle\rangle & \text{if } u = \langle a, x\rangle,\ v = \langle b, y\rangle, \\ & \quad \text{and } c = a \cdot b \in A \\ * & \text{otherwise} \end{cases}$$

This F with sync is a $\otimes$-functor. It is also symmetric if the operation $\cdot$ on A is commutative.

Here is our first microcosm phenomenon: A semigroup is defined most generally in a $\otimes$-category $\mathbb{C}$, namely as an object $S \in \mathbb{C}$ with an associative binary operation $m\colon S \otimes S \to S$. Associativity means that $m \circ (\mathrm{id} \otimes m) = m \circ (m \otimes \mathrm{id}) \circ \alpha$. The semigroup is called commutative (in a symmetric $\otimes$-category) if $m \circ \gamma = m$.

Assume now that $\mathbb{C}$ is a (possibly symmetric) $\otimes$-category and that $F\colon \mathbb{C} \to \mathbb{C}$ with sync is a $\otimes$-functor. Then we can define a tensor product on the category $\mathbf{Coalg}_F$ of F-coalgebras via:

$$(X \xrightarrow{c} FX) \otimes (Y \xrightarrow{d} FY) \stackrel{\text{defn}}{=} \Big(X \otimes Y \xrightarrow{c \otimes d} FX \otimes FY \xrightarrow{\mathrm{sync}} F(X \otimes Y) \Big)$$

This makes $\mathbf{Coalg}_F$ into a (symmetric) $\otimes$-category, in such a way that the forgetful functor $U\colon \mathbf{Coalg}_F \to \mathbb{C}$ is a $\otimes$-functor (with identity as sync-map). Note that for coalgebras c, d the notation $c \otimes d$ can mean two things, namely their tensor product in $\mathbf{Coalg}_F$, but also their tensor product in $\mathbb{C}$. Hopefully this will not lead to confusion.

In a next step we assume that F has a final coalgebra $\zeta\colon Z \xrightarrow{\cong} FZ$. The tensor coalgebra $\zeta \otimes \zeta$ yields a unique homomorphism $\|\colon Z \otimes Z \to Z$ to the final coalgebra.

Theorem 2.2. *If F is a $\otimes$-functor with sync on a $\otimes$-category with a final coalgebra, then:*

- *The operation $\|$ on the final coalgebra is a semigroup in $\mathbf{Coalg}_F$. It is commutative if F is symmetric.*
- (Compositionality) *We have*

$$\mathrm{beh}(c \otimes d) = \| \circ (\mathrm{beh}(c) \otimes \mathrm{beh}(d)) \quad , \qquad (1)$$

where $\mathrm{beh}(c)\colon X \to Z$ is the unique homomorphism obtained by finality for a coalgebra $c\colon X \to FX$. $\square$

In the first item above we obtain algebraic structure on a final coalgebra. It may also be understood (apart from the associativity) within the framework of Turi-Plotkin [20], arising from the above sync map as distributive law $\otimes F^2 \Rightarrow F\otimes$.

The compositionality result in the second item can be described more neatly by moving to the slice category $\mathbb{C}/Z$ of arrows with codomain Z and commuting triangles. The semigroup structure yields a $\otimes$-category simply by putting:

$$(X \xrightarrow{f} Z) \otimes (Y \xrightarrow{g} Z) \stackrel{\text{defn}}{=} (\ X \otimes Y \xrightarrow{\| \circ (f \otimes g)} Z\ ).$$

Theorem 2.3. *There is a commuting diagram of (symmetric) $\otimes$-functors*

$$\begin{tikzcd} \mathbf{Coalg}_F \arrow[rr, "\mathrm{beh}"] \arrow[dr, "U"'] & & \mathbb{C}/Z \arrow[dl, "\mathrm{dom}"] \\ & \mathbb{C} & \end{tikzcd}$$

all with identity as sync-map. $\square$

The behaviour functor can be understood as a composition-preserving translation from automata (as coalgebras) to behaviours.

2.1. In $\mathbf{Sets}$: Bisimilarity is a congruence

$\mathbf{Sets}$ is a symmetric $\otimes$-category with cartesian product as a tensor. Let F with sync be a symmetric $\otimes$-functor on $\mathbf{Sets}$ with a final coalgebra $\zeta : Z \to FZ$. When F preserves weak pullbacks, the final F-coalgebra is a fully abstract domain for bisimilarity $\sim$ [17]. That is,

$$x \sim y \iff \mathrm{beh}(c)(x) = \mathrm{beh}(d)(y)$$

for states x in $X \xrightarrow{c} FX$ and y in $Y \xrightarrow{d} FY$.

Let us denote by $x \mid y$ a state $\langle x, y\rangle$ of a composed coalgebra $c \otimes d : X \times Y \to F(X \times Y)$. From (1) we immediately obtain the usual presentation of compositionality:

$$x_1 \sim x_2 \text{ and } y_1 \sim y_2 \quad \Longrightarrow \quad x_1 \mid y_1 \ \sim\ x_2 \mid y_2 \ .$$

Moreover, commutativity and associativity

$$x \mid y \sim y \mid x \qquad x \mid (y \mid z) \sim (x \mid y) \mid z$$

hold because $\|\colon Z \times Z \to Z$ satisfies the corresponding equations. We proceed with some concrete examples in $\mathbf{Sets}$.

Example 2.4 (LTS). Consider the functor $\mathcal{P}_\omega(A \times \_)$ where $\mathcal{P}_\omega$ denotes the finite powerset functor[2]. Its coalgebras are finitely branching LTS. Given $c : X \to \mathcal{P}_\omega(A \times X)$ we write $x \xrightarrow{a} x'$ if $\langle a, x'\rangle \in c(x)$. This functor is a $\otimes$-functor together with sync_{LTS} with components

$$\mathrm{sync}_{LTS} : \mathcal{P}_\omega(A \times X) \times \mathcal{P}_\omega(A \times Y) \to \mathcal{P}_\omega(A \times (X \times Y))$$

defined as

$$\begin{aligned} \mathrm{sync}_{LTS}(U, V) = \{\langle l, \langle u, v\rangle\rangle \mid \langle a, u\rangle \in U, \\ \langle b, v\rangle \in V, l = a \cdot b\}. \end{aligned}$$

It induces commutative and associative composition of finitely branching LTS defined by

$$x \mid y \xrightarrow{l} x' \mid y' \iff x \xrightarrow{a} x',\, y \xrightarrow{b} y',\, l = a \cdot b \ .$$

Bisimilarity is a congruence with respect to the parallel composition.

Remark 2.5. Note that the parallel composition is induced by the associative (or monoidal) structure of F, i.e., it is parametrized by the natural transformation sync. This is to be understood as: the type of synchronisation determines the parallel composition itself. For different types of synchronisation, one would get different parallel composition operators, as is usual in concurrency theory [2, 15, 7]. In the example above, the type of synchronisation is given by the partial commutative semigroup operation $\cdot$ on A. The general style of synchronisation corresponds to synchronous ACP process algebra [2]. If A contains pairs of compatible actions a, $\bar{a}$ and a special silent action τ and if one defines $\bar{a} \cdot a = a \cdot \bar{a} = \tau$ we get the CCS handshaking style of synchronisation [15]. Finally, if $a \cdot a = a$, for $a \in L \subseteq A$, then we get the CSP style of synchronisation [7]. Hence in our generic setting we can obtain any of the used types of communication in classical process calculi. As a result we have that all these possible parallel compositions are commutative and associative, and bisimilarity is a congruence with respect to any of them.

[2] We need the finiteness assumption to ensure existence of a final coalgebra.

Example 2.6 (**Probabilistic systems).** Finitely branching generative probabilistic transition systems [21] are $\mathcal{D}_\omega(A \times \_)$ coalgebras for $\mathcal{D}_\omega$ denoting the finite[3] support subdistribution functor. Its action is: for a set X and a function $f : X \to Y$

$$\mathcal{D}_\omega X = \{\xi : X \to [0,1] \mid \textstyle\sum_{x\in X} \xi(x) \le 1, \\ \hfill |\{x \in X \mid \xi(x) > 0\}| < \infty\}$$

and $(\mathcal{D}_\omega f)(\xi) = \lambda y. \sum_{x\in f^{-1}(\{y\})} \xi(x)$ for $\xi \in \mathcal{D}_\omega X$.

Given a $\mathcal{D}_\omega(A \times \_)$-coalgebra c we write $x \overset{a[p]}{\to} x'$ if $c(x)(\langle a, x'\rangle) = p$. For $\xi \in \mathcal{D}_\omega(A \times X), \psi \in \mathcal{D}_\omega(A \times Y)$, define

$$\mathrm{sync}_{PS}(\xi, \psi)(\langle l, \langle x, y\rangle\rangle) = \sum_{a,b:\, l=a\cdot b} \xi(\langle a, x\rangle) \cdot \psi(\langle b, y\rangle).$$

Then sync_{PS} turns $\mathcal{D}_\omega(A \times \_)$ into a $\otimes$-functor, and therefore it induces a commutative and associative synchronous parallel composition operation on its coalgebras. A composed generative probabilistic system $c \otimes d : X \times Y \to \mathcal{D}_\omega(A \times (X \times Y))$ is defined by $x \mid y \overset{l[p]}{\to} x' \mid y'$ if and only if $p = \sum_{q,r} q \cdot r$ taken over all q, r such that $x \overset{a[q]}{\to} x'$, $y \overset{b[r]}{\to} y'$ and $l = a \cdot b$.

Also here the operation $\cdot$ on A determines the style of synchronisation. We have that (probabilistic) bisimilarity is a congruence with respect to the synchronous parallel composition of generative probabilistic systems (in any synchronisation style), and the parallel composition is commutative and associative.

2.2. In a Kleisli category: Trace semantics

Another semantic equivalence (besides bisimilarity) broadly used in process theory is trace semantics. Since trace semantics is also a final coalgebra semantics, our general framework applies and we obtain compositionality/commutativity/associativity results.

It is shown in [5] (also earlier in [4]) that coinduction in a Kleisli category $\mathcal{K}\ell(T)$ yields trace semantics. This is done in the following way. Given a monad T on $\mathbf{Sets}$ and a functor F, a distributive law $\pi : FT \Rightarrow TF$ lifts F to a functor $\overline{F} : \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$. Then a final $\overline{F}$-coalgebra is yielded by an initial F-algebra. An automaton (or a system) c—identified with a TF-coalgebra, hence with an $\overline{F}$-coalgebra—is provided with a unique coalgebra morphism $\mathrm{tr}(c)$ into the final coalgebra.

$$\begin{array}{ccc} \overline{F}X & \overset{\overline{F}(\mathrm{tr}(c))}{- - - - - \to} & \overline{F}Z \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle \cong \text{final}} \\ X & \underset{\mathrm{tr}(c)}{- - - - - - \to} & Z \end{array} \quad \text{in } \mathcal{K}\ell(T) \qquad (2)$$

[3]The finiteness assumption again ensures existence of a final coalgebra.

The commutation of this diagram (defining $\mathrm{tr}(c)$) is equivalent to the usual recursive definition of trace semantics. Hence the kernel of $\mathrm{tr}(c)$ is trace equivalence $\equiv_{\mathrm{tr}}$, that is, $x \equiv_{\mathrm{tr}} y \iff \mathrm{tr}(c)(x) = \mathrm{tr}(c)(y)$.

The assumption in our general result—algebraic structures in $\mathcal{K}\ell(T)$ and in $\overline{F}$—can be reduced to a condition on T and F. This is done as follows. If T is a $\otimes$-*monad*, that is, if T is a $\otimes$-functor with sync_T and sync_T commutes with the monad unit and multiplication, then $\mathcal{K}\ell(T)$ is a $\otimes$-category (see Proposition 3.6 below). In particular, any commutative monad [11] is a $\otimes$-monad. If moreover F is a $\otimes$-functor and the distributive law $\pi : FT \Rightarrow TF$ is compatible with sync_F and sync_T (as in Theorem 4.12), then the lifted $\overline{F}$ is a $\otimes$-functor on $\mathcal{K}\ell(T)$. Hence the category $\mathcal{K}\ell(T)$ and the functor $\overline{F}$ satisfy the assumption of Theorem 2.2. To the best of our knowledge, this is the first general result on compositionality of trace semantics for coalgebras.

Coalgebras of type TF for a monad T and a functor F are quite common, for example LTS and generative probabilistic systems are such. In a TF-coalgebra, T determines the “branching-type” of the system, while F determines its “transition-type”. We next focus on concrete examples.

Example 2.7 (LTS). Take $T = \mathcal{P}$, the powerset monad, and $F = A\times\_+1$. Then a TF-coalgebra $X \overset{c}{\to} \mathcal{P}(A\times X+1)$ is an LTS with explicit termination[4]. In this setting the results in [5] instantiate as follows. The coinduction diagram (2) in $\mathcal{K}\ell(\mathcal{P})$ yields a function $\mathrm{tr}(c) : X \to \mathcal{P}(A^*)$; the commutation is equivalent to the following recursive definition of $\mathrm{tr}(c)$.

$$\begin{array}{lcl} \varepsilon \in \mathrm{tr}(c)(x) & \iff & x \to \checkmark \\ aw \in \mathrm{tr}(c)(x) & \iff & \exists x'.\ x \overset{a}{\to} x' \text{ and } w \in \mathrm{tr}(c)(x') \end{array}$$

Here ε denotes the empty word.

We obtain compositionality result for this trace semantics. Indeed, $\mathcal{P}$ is a $\otimes$-monad since it is commutative. Moreover the lifted functor $\overline{F}$ is a $\otimes$-functor with $\eta \circ \mathrm{sync}$ as its synchronisation map, where sync is from Example 2.1. The resulting composition $\otimes$ in $\mathbf{Coalg}_{\overline{F}}$ is similar to the one in Example 2.4. By changing sync the composition $\otimes$ can be any of the standard ones (Remark 2.5), and Theorem 2.2 yields the compositionality result of trace semantics, with respect to each of these parallel compositions.

Example 2.8 (**Probabilistic systems).** Generative probabilistic transition systems with explicit termination are TF-coalgebras for F as in Example 2.1, and $T = \mathcal{D}$ denoting the (not necessarily finite support) subdistribution functor, as in Example 2.6. Trace semantics is defined for this monad and this functor and it amounts to trace distribution. For a given generative system with explicit termination

[4]Explicit termination is necessary for defining trace semantics of LTS and of probabilistic systems via coinduction—see [5].

$c : X \to \mathcal{D}(A \times X + 1)$, the trace map $\mathrm{tr}(c) : X \to \mathcal{D}(A^*)$, also called trace distribution, is defined as $\mathrm{tr}(c)(x)(\varepsilon) = c(*)$ and

$$\mathrm{tr}(c)(x)(aw) = \sum_{x' \in X} c(x)(a, x') \cdot \mathrm{tr}(c)(x')(w).$$

Furthermore, $\mathcal{D}$ is a $\otimes$-monad since it is commutative, and F lifts to a functor $\overline{F}$ on $\mathcal{K}\ell(\mathcal{D})$ which is also a $\otimes$-functor. We define a natural transformation $\overline{\mathrm{sync}}$ in the Kleisli category with components $\overline{\mathrm{sync}} : \overline{F}X \times \overline{F}Y \to \mathcal{D}\overline{F}(X \times Y)$ by

$$\overline{\mathrm{sync}}(u, v) = \begin{cases} \mathbf{0} & \text{if } \mathrm{sync}(u, v) = *, \\ \eta \circ \mathrm{sync}(u, v) & \text{otherwise.} \end{cases}$$

Here $\mathbf{0}$ denotes the zero distribution. This natural transformation makes $\overline{F}$ a $\otimes$-functor in $\mathcal{K}\ell(\mathcal{D})$. It induces similar parallel composition of generative systems as the one defined in Example 2.6. Therefore, trace semantics is compositional with respect to parallel composition of generative probabilistic systems.

3. Part II: Microcosm models of algebraic specifications

In this section we generalize the associative tensor to algebraic specifications of operations and equations. All results transfer to the new setting.

The basic scenario is as follows. Let Σ be an algebraic signature. A Σ-*category* is a category $\mathbb{C}$ with Σ-algebraic structure—this is an "outer" model. Then we observe that Σ induces an "inner" functor $\widehat{\Sigma} : \mathbb{C} \to \mathbb{C}$. A Σ-*object* is then $A \in \mathbb{C}$ which carries a $\widehat{\Sigma}$-algebra—an "inner" algebra. This results in nested algebras $A \in \mathbb{C}$ interpreting Σ on different levels. This is a microcosm model of Σ. In the following detailed account we also include a set E of equations—as a generalization of associativity of $\otimes$ in Section 2.

A *signature* Σ is a set of operations $f \in \Sigma$, each with an arity $|f| \in \mathbb{N}$. For a fixed set of variables we consider Σ-terms built from the variables via the operations $f \in \Sigma$. An equation is a pair of terms commonly written as $s = t$. An *algebraic specification* is a pair (Σ, E), where Σ is a signature and E is a set of equations. Terms contain only finitely many variables—since they are finitely generated—so individual equations involve only finitely many variables.

We identify a signature Σ with the endofunctor

$$\Sigma = \coprod_{f \in \Sigma} (\_)^{|f|}$$

that can be defined on any category $\mathbb{C}$ with finite products and arbitrary coproducts. A Σ-*model* in $\mathbb{C}$ is then a Σ-algebra, $\Sigma A \to A$ in $\mathbb{C}$, consisting of separate interpretations $[\![f]\!] \colon A^{|f|} \to A$ of the operations $f \in \Sigma$. The interpretation extends to terms, so that each term t, say with n variables, can be interpreted as a map $[\![t]\!] \colon A^n \to A$ in $\mathbb{C}$. An equation $s = t$ yields a parallel pair of maps $A^n \rightrightarrows A$, which can also be drawn as a diagram. Validity of the equation is most naturally expressed as equations of these maps $A^n \rightrightarrows A$—that is, as commutation of the diagram—but in the following 2-categorical setting one can also consider commutation up-to-isomorphism.

We use the term Σ-*category* for a Σ-model in the 2-category $\mathbf{Cat}$ of categories. Validity of equations in this case is up-to-isomorphism: a (Σ, E)-*category* is defined as a Σ-category such that for each equation $(s = t) \in E$ we have a natural isomorphism α as below[5].

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, bend left, "{[\![s]\!]}"{name=U}] \arrow[r, bend right, "{[\![t]\!]}"'{name=D}] & \mathbb{C} \arrow[from=U, to=D, phantom, "\cong\Downarrow\alpha"] \end{tikzcd} \tag{3}$$

Next we define microcosm models of signatures and equations. Let $\mathbb{C}$ be a Σ-category. By $\widehat{\Sigma}$ we denote the endofunctor on $\mathbb{C}$

$$\widehat{\Sigma} = \coprod_{f \in \Sigma} [\![f]\!] \circ \Delta_{|f|},$$

where $\Delta_n \colon \mathbb{C} \to \mathbb{C}^n$ is the obvious diagonal functor. A Σ-*object* is a $\widehat{\Sigma}$-algebra in $\mathbb{C}$. A *microcosm model* $A \in \mathbb{C}$ of Σ is a Σ-object in a Σ-category $\mathbb{C}$. In a microcosm model $A \in \mathbb{C}$, A is the *inner* model, and $\mathbb{C}$ is the *outer* model. The inner model consists of a collection of maps $[\![f]\!](A, \ldots, A) \to A$ in $\mathbb{C}$, for $f \in \Sigma$. One can then interpret terms t as maps $[\![t]\!] \colon [\![t]\!](A, \ldots, A) \to A$. By this we overload the interpretation notation $[\![\_]\!]$ and let the context make clear which version is meant.

Assume $\mathbb{C}$ is a (Σ, E)-category. If a Σ-object A in $\mathbb{C}$ is such that for each equation $(s = t) \in E$ we have a commuting triangle

$$\begin{tikzcd} {[\![s]\!](A, \ldots, A)} \arrow[rr, "\alpha", "\cong"'] \arrow[dr, "{[\![s]\!]}"'] & & {[\![t]\!](A, \ldots, A)} \arrow[dl, "{[\![t]\!]}"] \\ & A & \end{tikzcd} \tag{4}$$

then we call it a (Σ, E)-*object*.

A *microcosm model* $A \in \mathbb{C}$ of (Σ, E) is a (Σ, E)-object in a (Σ, E)-category $\mathbb{C}$.

Example 3.1. **(1)** Consider the specification of monoids, given by signature $\Sigma = \{\mathsf{m}, \mathsf{e}\}$ of multiplication m with $|\mathsf{m}| = 2$, and unit e with $|\mathsf{e}| = 0$, together with the standard equations $\mathsf{m}(x, \mathsf{m}(y, z)) = \mathsf{m}(\mathsf{m}(x, y), z)$, $\mathsf{m}(x, \mathsf{e}) = x$, and $\mathsf{m}(\mathsf{e}, x) = x$. A microcosm model is then a monoid in a monoidal category. The outer interpretation in the category

[5] Under this 2-categorical interpretation of validity one can formulate coherence as: all possible pairs of 2-cells $[\![s]\!] \overset{\cong}{\Longrightarrow} [\![t]\!]$ that can be built from the basic validity 2-cells are equal. Sometimes the resulting equations are finitely axiomatisable, like for monoidal categories, see [14].

$\mathbb{C}$ is commonly written as $\otimes = [\![\mathsf{m}]\!] : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $I = [\![\mathsf{e}]\!] \in \mathbb{C}$. The equations yield natural isomorphisms:

$$X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z \quad X \otimes I \cong X \quad I \otimes X \cong X$$

The inner model consists of an object A in $\mathbb{C}$ with interpretations $[\![\mathsf{m}]\!] : A \otimes A \to A$ and $[\![\mathsf{e}]\!] : I \to A$ satisfying for example the following instance of (4).

$$\begin{tikzcd}
{[\![\mathsf{m}(_,\mathsf{e})]\!](A) = A \otimes I} \arrow[rr, "\cong"] \arrow[dr, "{[\![\mathsf{m}(_,\mathsf{e})]\!]}"'] & & {A = [\![_]\!](A)} \arrow[dl, "{[\![_]\!] = \mathrm{id}}"] \\
& A &
\end{tikzcd}$$

(2) Theorem 2.2 gives a semigroup microcosm model $\| : \zeta \otimes \zeta \to \zeta$ in a $\otimes$-category of coalgebras.

3.1 Microcosm models in coalgebras

We now focus on the microcosm structure for (Σ, E) in a category $\mathbf{Coalg}_F$. We will observe that a (Σ, E)-structure in the base category lifts to one in $\mathbf{Coalg}_F$, provided that the functor F is "lax-compatible" with (Σ, E).

A *lax* Σ*-functor*[6] is a functor $F : \mathbb{C} \to \mathbb{D}$ between the Σ-categories $\mathbb{C}$ and $\mathbb{D}$ that forms a *lax* algebra homomorphism φ as below.

$$\begin{tikzcd}
\coprod_{f \in \Sigma} \mathbb{C}^{|f|} \arrow[r, "\coprod_{f \in \Sigma} F^{|f|}"] \arrow[d] \arrow[dr, phantom, "\varphi \swarrow"] & \coprod_{f \in \Sigma} \mathbb{D}^{|f|} \arrow[d] \\
\mathbb{C} \arrow[r, "F"'] & \mathbb{D}
\end{tikzcd} \tag{5}$$

Such a natural transformation φ corresponds to a family of natural transformations $\varphi^f : [\![f]\!] F^{|f|} \Rightarrow F[\![f]\!]$, for $f \in \Sigma$. It is not hard to define for each term t (with n variables) a similar natural transformation $\varphi^t : [\![t]\!] F^n \Rightarrow F[\![t]\!]$.

A Σ*-functor* $F : \mathbb{C} \to \mathbb{D}$ is a (proper, non-lax) algebra morphism making (5) commute, that is, we have $\varphi = \mathrm{id}$.[7]

If a lax Σ-functor $F : \mathbb{C} \to \mathbb{D}$ with φ additionally commutes with the equations, that is,

$$F\alpha \circ \varphi^s = \varphi^t \circ \alpha \tag{6}$$

for any equation $(s = t) \in E$ with α the validity isomorphism as in (3), then F is said to be a *lax* (Σ, E)*-functor*.

Proposition 3.2. *Let $\mathbb{C}$ be a (Σ, E)-category and F be a lax (Σ, E)-functor on $\mathbb{C}$. Then the category $\mathbf{Coalg}_F$ is also a (Σ, E)-category via the interpretations*

$$[\![f]\!](\vec{c_i}) \stackrel{\text{defn}}{=} \varphi^f \circ [\![f]\!]_{\mathbb{C}}(\vec{c_i})$$

for a $|f|$-tuple of F-coalgebras $(X_i \xrightarrow{c_i} FX_i)_i$, where $[\![f]\!]_{\mathbb{C}}$ is the interpretation in $\mathbb{C}$.

[6] In this spirit, $\otimes$-functors in Section 2 should have been called "lax $\otimes$-functors".

[7] This corresponds to $\otimes$-functors "with identity as sync-map" in Section 2.

Proof. (Sketch) In order to show that the Σ-category $\mathbf{Coalg}_F$ satisfies equations in E, we first note that the induced interpretation of a term t in $\mathbf{Coalg}_F$ also satisfies $[\![t]\!](\vec{c_i}) = \varphi^t \circ [\![t]\!]_{\mathbb{C}}(\vec{c_i})$. This together with the compatibility of F suffices to show that the validity isomorphisms (3) are coalgebra morphisms. This implies the claim. $\square$

From now on we assume that F has a final coalgebra $Z \xrightarrow{\zeta} FZ$. The next two results generalise Theorem 2.2 and Theorem 2.3.

Theorem 3.3. *Assume that F is a lax (Σ, E)-functor on a (Σ, E) category.*

- *The final coalgebra $\zeta \in \mathbf{Coalg}_F$ is a microcosm model of (Σ, E).*
- (Compositionality) *Using the notation* $\mathrm{beh}(\_)$ *for the unique homomorphism to the final coalgebra, we have*

$$\mathrm{beh}\left([\![f]\!](\vec{c_i})\right) = [\![f]\!]_{\zeta} \circ [\![f]\!]_{\mathbb{C}}(\overrightarrow{\mathrm{beh}(c_i)}) \tag{7}$$

where the first $[\![f]\!]$ is the interpretation in $\mathbf{Coalg}_F$. $\square$

The interpretations $[\![f]\!](\zeta, \ldots, \zeta) \xrightarrow{[\![f]\!]_{\zeta}} \zeta$ that make ζ a (Σ, E)-object are obtained by finality.

Theorem 3.4. *The following diagram of functors between the (Σ, E)-categories commutes.*

$$\begin{tikzcd}
\mathbf{Coalg}_F \arrow[rr, "\mathrm{beh}"] \arrow[dr, "U"'] & & \mathbb{C}/Z \arrow[dl, "\mathrm{dom}"] \\
& \mathbb{C} &
\end{tikzcd}$$

Moreover, the functors are (non-lax) (Σ, E)-functors. $\square$

Under suitable additional assumptions one can construct a left adjoint to the functor $\mathrm{beh} : \mathbf{Coalg}_F \to \mathbb{C}/Z$. Namely by sending an object $X \to Z$ in $\mathbb{C}/Z$ first to its image $X' \rightarrowtail Z$, and then to the least invariant/subcoalgebra (of the final coalgebra) that contains this image.

As mentioned in the introduction, our work is related to the results of Turi-Plotkin on abstract GSOS [20]. We go beyond their work in a couple of ways—see Introduction. Notably, nested algebraic structures are not explicit in [20] because outer models are fixed—algebraic operations are always interpreted by cartesian products. The following remark elaborates on the point where we fall short.

Remark 3.5. In [20], a GSOS rule—which specifies how a behavior-type (F in this paper) and an algebraic signature Σ are compatible—is represented by a distributive law $\Sigma(F \times \mathrm{id}) \Rightarrow F\Sigma^*$. Here Σ^* is a free monad on the functor Σ: having Σ^* instead of Σ makes the format expressive enough to accommodate common examples of GSOS rules.

In the current work, however, compatibility of F with Σ is expressed as "F being a lax Σ-functor"—in the setting of [20] this amounts to a natural transformation $\Sigma F \Rightarrow F\Sigma$. This is less expressive, although it suffices for parallel composition.

We end the section with a brief treatment of Kleisli categories. Algebraic structure on Kleisli categories is a basis for compositionality of trace semantics (Section 2.2).

Assume T is a monad with multiplication μ and unit η on a (Σ, E)-category $\mathbb{C}$. We say that T is a *lax* (Σ, E)-*monad* if it is a lax (Σ, E)-functor via a natural transformation τ like in (5) and additionally the coproduct components $\tau^f : [\![f]\!]T^{|f|} \Rightarrow T[\![f]\!]$, for $f \in \Sigma$, commute properly with the monads unit and multiplication:

$$\begin{aligned} \tau \circ [\![f]\!]\eta^{|f|} &= \eta[\![f]\!] \\ \tau \circ [\![f]\!]\mu^{|f|} &= \mu[\![f]\!] \circ T\tau \circ \tau T^{|f|}. \end{aligned}$$

Proposition 3.6. *If T is a lax (Σ, E)-monad, then its Kleisli category $\mathcal{K}\ell(T)$ is a (Σ, E)-category, and the inclusion $\mathbb{C} \xrightarrow{J} \mathcal{K}\ell(T)$ preserves this structure.* □

In this section we have generalised microcosm models and the related results from associative tensors to algebraic specifications. We have studied models in the same kind of categories as before: categories of coalgebras, slice categories and Kleisli categories. In order to capture models in those categories uniformly we move to a next level of abstraction given by 2-categories.

4. Part III: Microcosm principle for Lawvere theories

In this last part of our exploration we aim at a fully categorical account of the microcosm principle. We identify an algebraic specification (Σ, E) as a Lawvere theory $\mathbb{L}$ which is itself a certain kind of category. Then a microcosm model arises in the following way. An outer model ($\mathbb{L}$-category) is a (pseudo) functor $\mathbb{L} \xrightarrow{\mathbb{C}} \mathbf{Cat}$: this is like the usual *functorial semantics* [13] for a Lawvere theory. We observe that an inner model ($\mathbb{L}$-object) is a lax natural transformation

$$\begin{tikzcd} \mathbb{L} \arrow[r, bend left, "\mathbf{1}"{name=U}] \arrow[r, bend right, "\mathbb{C}"'{name=D}] & \mathbf{Cat} \arrow[from=U, to=D, phantom, "\Downarrow X"] \end{tikzcd} \quad .$$

This observation is applied to formulate and derive the compositionality result for final coalgebra semantics (Theorem 4.11). In this section we heavily rely on 2-categorical notions, on which a detailed account is found in [3].

In the sequel, an *FP-category* is a category with finite products; an *FP-functor* is a functor preserving finite products *on-the-nose* (instead of *up-to-isomorphism*).[8]

[8]To put it more precisely: an FP-category comes with a specific choice of finite products and an FP-functor strictly preserves this choice.

Definition 4.1. By $\mathbf{Nat}$ we denote the category of natural numbers and functions between them. Note that every arrow in $\mathbf{Nat}$ is a (cotuple of) coprojection.

A *Lawvere theory* is a small FP-category $\mathbb{L}$ with an FP-functor $H : \mathbf{Nat}^{\mathrm{op}} \to \mathbb{L}$ which is bijective on objects. The "carrier" object $H1 \in \mathbb{L}$ is denoted by L.

Lawvere theories are introduced in [13][9] as a categorical presentation of algebraic specifications (or algebraic theories). An accessible introduction is given in [12]; see also [8, 16] for recent developments.

The category $\mathbf{Nat}^{\mathrm{op}}$ is there because it is the free FP-category on the trivial category $\mathbf{1}$. A few basic facts on a Lawvere theory $\mathbb{L}$ are mentioned for illustration.

- ($\mathbb{L}$ is "single-sorted") Every object of $\mathbb{L}$ is of the form L^m with $m \in \mathbb{N}$.
- Arrows of $\mathbb{L}$ are classified into 1) projections which are in the image of H; 2) other arrows which are thought of as algebraic "operations". The first group has a privileged status: it happens often that its members require strict preservation, while the members of the second group require only "lax" preservation.
- A Lawvere theory $\mathbb{L}$ is "algebraic" in the sense that each operation has unary output. An arrow $f : L^m \to L^n$—seemingly an operation with n-ary output—can be understood as an n-tuple of operations $L^m \to L$.

A Lawvere theory $\mathbb{L}$ arises from a specification (Σ, E) as its so-called *classifying category* (see e.g. [13, 9]). An arrow $L^m \to L^n$ in $\mathbb{L}$ is an n-tuple $([t_1(\vec{x})], \ldots, [t_n(\vec{x})])$ of terms with m variables $\vec{x}$ modulo equations in E. An equivalent way to describe this construction is via *sketches*: (Σ, E) is identified with a sketch, which in turn induces $\mathbb{L}$ as the free FP-category. See [1] for details.

Example 4.2. The Lawvere theory $\mathbf{Mon}$ for monoids is the classifying category of the specification in Example 3.1. Equivalently, it is the freely generated FP-category by arrows $0 \xrightarrow{\mathsf{e}} 1$ and $2 \xrightarrow{\mathsf{m}} 1$; subject to the commutativity

$$\begin{tikzcd} 1 \arrow[r, "\langle \mathrm{id}, \mathsf{e} \rangle"] \arrow[dr, "\mathrm{id}"'] & 2 \arrow[d, "\mathsf{m}"] & 1 \arrow[l, "\langle \mathsf{e}, \mathrm{id} \rangle"'] \arrow[dl, "\mathrm{id}"] \\ & 1 & \end{tikzcd} \qquad \begin{tikzcd} 3 \arrow[r, "\mathsf{m} \times 1"] \arrow[d, "1 \times \mathsf{m}"'] & 2 \arrow[d, "\mathsf{m}"] \\ 2 \arrow[r, "\mathsf{m}"'] & 1 \end{tikzcd} \quad . \tag{8}$$

These data (arrows and commutative diagrams) form an FP-sketch (see [1]).

In order to state what it means for a category $\mathbb{C}$ to have an $\mathbb{L}$-algebra structure, we exploit the following observation. A category $\mathbb{C}$ can be identified with a functor $\mathbf{1} \to \mathbf{Cat}$, then further with its unique FP-extension $\mathbf{Nat}^{\mathrm{op}} \to \mathbf{Cat}$,

[9]Not, of course, under the name *Lawvere theory*.

since $\mathbf{Nat}^{\mathrm{op}}$ is a free FP-category over $\mathbf{1}$. The resulting FP-functor carries $m \in \mathbf{Nat}$ to $\mathbb{C}^m$ and $m \to n$ to a suitable projection $\mathbb{C}^m \to \mathbb{C}^n$. Similarly, a functor $F : \mathbb{C} \to \mathbb{D}$ can be identified with a natural transformation

$$\begin{tikzcd} \mathbf{Nat}^{\mathrm{op}} \arrow[r, bend left, "\mathbb{C}"] \arrow[r, bend right, "\mathbb{D}"'] \arrow[r, phantom, "\Downarrow F"] & \mathbf{Cat} \end{tikzcd} .$$

Definition 4.3. An $\mathbb{L}$-*category* is a pseudo-functor $\mathbb{L} \overset{[\![\_]\!]}{\to} \mathbf{Cat}$ which is "product preserving", by which we mean that the composite $[\![\_]\!] \circ H$ is an FP-functor. To put it differently: a category $\mathbb{C}$ is said to be an $\mathbb{L}$-*category* if it factors as

$$\mathbf{Nat}^{\mathrm{op}} \xrightarrow{\mathbb{C}} \mathbf{Cat} \quad = \quad \mathbf{Nat}^{\mathrm{op}} \xrightarrow{H} \mathbb{L} \xrightarrow{[\![\_]\!]} \mathbf{Cat}$$

with some pseudo-functor $[\![\_]\!]$. The pseudo-functor $[\![\_]\!]$ will be often denoted by $\mathbb{C}$ as well.

An $\mathbb{L}$-*functor* is a (proper, non-lax) natural transformation $\begin{tikzcd} \mathbb{L} \arrow[r, bend left, "\mathbb{C}"] \arrow[r, bend right, "\mathbb{D}"'] \arrow[r, phantom, "\Downarrow \nu"] & \mathbf{Cat} \end{tikzcd}$. This is understood as a functor $\mathbb{C} \to \mathbb{D}$ which preserves $\mathbb{L}$-algebraic structures. $\mathbb{L}$-categories and $\mathbb{L}$-functors form a category $\mathbb{L}$-$\mathbf{cat}$.

An $\mathbb{L}$-category $\mathbb{C}$ is a category "with an $\mathbb{L}$-algebra structure": each arrow f in $\mathbb{L}$ has its interpretation $\mathbb{C}^m \overset{[\![f]\!]}{\to} \mathbb{C}^n$. We have $[\![\pi_i]\!] = \pi_i$ because $[\![\_]\!] \circ H$ is an FP-functor.

A pseudo-functor (see [3]) is a "functor up-to-isomorphism": coherence conditions have only to be satisfied up to coherent iso 2-cells. This is needed because: in a monoidal category we have $I \otimes X \cong X$ instead of equality.

Example 4.4. The usual notion of monoidal categories coincides with $\mathbb{L}$-categories for $\mathbb{L} = \mathbf{Mon}$. For example, an isomorphism $X \otimes I \cong X$ is derived as follows.

$$\begin{aligned} X \otimes I &= (\mathbb{C} \overset{\otimes}{\leftarrow} \mathbb{C} \times \mathbb{C} \overset{\langle \mathrm{id}, I \rangle}{\leftarrow} \mathbb{C})(X) & \\ &= (\,[\![\mathsf{m}]\!] \circ [\![\langle \mathrm{id}, \mathsf{e} \rangle]\!]\,)(X) & [\![\_]\!] \text{ is product preserving} \\ &\cong (\,[\![\mathsf{m} \circ \langle \mathrm{id}, \mathsf{e} \rangle]\!]\,)(X) & [\![\_]\!] \text{ is a pseudo-functor} \\ &= [\![\mathrm{id}]\!](X) & \text{commutativity of (8)} \\ &\cong \mathrm{id}(X) = X & [\![\_]\!] \text{ is a pseudo-functor} \end{aligned}$$

Let us recall 2-universality of $\mathbf{1} \to \mathbf{Nat}^{\mathrm{op}}$, via which, an object $X \in \mathbb{C}$ is identified with a cone $\begin{tikzcd} \mathbf{Nat}^{\mathrm{op}} \arrow[r, bend left, "\mathbf{1}"] \arrow[r, bend right, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow X"] & \mathbf{Cat} \end{tikzcd}$.

The following definition is the core of our 2-categorical investigation of the microcosm principle.

Definition 4.5. Let $\mathbb{C}$ be an $\mathbb{L}$-category. An object $X \in \mathbb{C}$ is said to be an $\mathbb{L}$-*object* if there exists a lax cone χ—see (9) below—such that $X = \chi \circ H$.

$$\begin{tikzcd} \mathbf{Nat}^{\mathrm{op}} \arrow[r, bend left, "\mathbf{1}"] \arrow[r, bend right, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow X"] & \mathbf{Cat} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbf{Nat}^{\mathrm{op}} \arrow[r, "H"] & \mathbb{L} \arrow[r, bend left, "\mathbf{1}"] \arrow[r, bend right, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow \chi"] & \mathbf{Cat} \end{tikzcd}$$

The lax cone χ will be also denoted by X.

In other words, an $\mathbb{L}$-*object* is a lax cone $\mathbf{1} \overset{\chi}{\Rightarrow} \mathbb{C} : \mathbb{L} \to \mathbf{Cat}$ which is "product preserving", that is, the composition $\chi \circ H$ is (properly, non-lax) natural.

An $\mathbb{L}$-object $X \in \mathbb{C}$ is called a *microcosm model* of $\mathbb{L}$: with $\mathbb{C}$ the *outer model* and X the *inner model*.

Let us illustrate that χ represents an object $X \in \mathbb{C}$ equipped with algebraic operations. The lax cone χ is a family of objects $(\mathbf{1} \overset{\chi_m}{\to} \mathbb{C}^m)_{m \in \mathbb{N}}$ with $\chi_1 = X$. Naturality of $\chi \circ H$ means commutativity of the following diagram, for each $m \in \mathbb{N}$ and $i < m$.

$$\begin{tikzcd} \mathbf{1} \arrow[r, "\chi_m"] \arrow[dr, "\chi_1"'] & \mathbb{C}^m \arrow[d, "{[\![H(\pi_i)]\!] = \pi_i}"] \\ & \mathbb{C} \end{tikzcd}$$

Therefore we have $\chi_m = (X, \ldots, X) \in \mathbb{C}^m$.

Finally, lax naturality of χ says: for each "operation" $f : L^m \to L^n$ in $\mathbb{L}$ we have the following mediating 2-cell χ_f.

$$\begin{tikzcd} \mathbf{1} \arrow[r, "{\chi_m = (X,\ldots,X)}"] \arrow[dr, bend right, "{\chi_n = (X,\ldots,X)}"'] \arrow[r, phantom, shift right=3, "\Downarrow \chi_f"] & \mathbb{C}^m \arrow[d, "{[\![f]\!]}"] \\ & \mathbb{C}^n \end{tikzcd} \qquad (9)$$

This is an arrow $[\![f]\!](X, \ldots, X) \overset{\chi_f}{\to} (X, \ldots, X)$ in $\mathbb{C}$ which interprets f on X. The mediating 2-cells must satisfy obvious coherence conditions: for example $\chi_{g \circ f}$ is a suitable composite of χ_g after χ_f. This interpretation χ_f will be denoted also by $[\![f]\!]$ when no confusion arises. For example, a microcosm model for $\mathbf{Mon}$ is precisely a monoid in a monoidal category (Example 3.1.1).

Proposition 4.6. *An $\mathbb{L}$-functor preserves $\mathbb{L}$-objects.* □

A final object always carries an algebraic structure induced by finality.

Proposition 4.7. *A final object 1 of an $\mathbb{L}$-category $\mathbb{C}$, if it exists, is an $\mathbb{L}$-object.* □

A *morphism of* $\mathbb{L}$-*objects* is a modification (see [3]) $\Xi : \chi \Rrightarrow \chi'$. It is not hard to see that Ξ corresponds to an arrow $\xi : X \to X'$ such that, for each $f : L^m \to L$ in $\mathbb{L}$, the following diagram commutes.

$$\begin{tikzcd} {[\![f]\!](X, \ldots, X)} \arrow[r, "{[\![f]\!](\xi,\ldots,\xi)}"] \arrow[d, "\chi_f"'] & {[\![f]\!](X', \ldots, X')} \arrow[d, "\chi'_f"] \\ X \arrow[r, "\xi"'] & X \end{tikzcd}$$

This suggests that ξ (hence Ξ) is a "morphism of algebras". The category of $\mathbb{L}$-objects in $\mathbb{C}$ and morphisms between them is denoted by $\mathbb{L}$-$\mathbf{obj}_{\mathbb{C}}$.

The next result follows from $\mathbb{C} \cong \mathbf{Cat}(\mathbf{1}, \mathbb{C})$, just like $S \cong \mathbf{Sets}(1, S)$ leads to the presentation of $\lim(\mathbb{J} \to \mathbf{Sets})$ of a diagram in $\mathbf{Sets}$ as the set of "coherent elements" [14, Theorem V.1.1].

Proposition 4.8. *The category $\mathbb{L}\text{-}\mathbf{obj}_{\mathbb{C}}$ is the lax limit of the diagram $\mathbb{C} : \mathbb{L} \to \mathbf{Cat}$.*[10] $\square$

Towards the compositionality result for final coalgebra semantics, we shall show the following: a slice category $\mathbb{C}/X$ is an $\mathbb{L}$-category for a microcosm model $X \in \mathbb{C}$; so is the category $\mathbf{Coalg}_F$ of F-coalgebra when F is compatible with $\mathbb{L}$. In fact, $\mathbb{C}/X$ and $\mathbf{Coalg}_F$ both allow for a characterisation as certain *inserters*: we show that, under certain compatibility assumptions, any inserter is an $\mathbb{L}$-category.

In general, the *inserter* $Ins(F, G)$ for functors $F, G : \mathbb{C} \rightrightarrows \mathbb{D}$ consists of a functor $Ins(F, G) \xrightarrow{R} \mathbb{C}$ and a natural transformation

$$\begin{tikzcd}
 & \mathbb{C} \arrow[dr, "F"] \arrow[dd, phantom, "\swarrow\rho"] & \\
Ins(F,G) \arrow[ur, "R"] \arrow[dr, "R"'] & & \mathbb{D} \\
 & \mathbb{C} \arrow[ur, "G"'] &
\end{tikzcd}$$

which is universal: each (B, β) such that

$$\begin{tikzcd}
 & \mathbb{C} \arrow[dr, "F"] \arrow[dd, phantom, "\swarrow\beta"] & \\
\mathbb{B} \arrow[ur, "B"] \arrow[dr, "B"'] & & \mathbb{D} \\
 & \mathbb{C} \arrow[ur, "G"'] &
\end{tikzcd}$$

induces a unique mediating functor $\overline{B} : \mathbb{B} \to Ins(F, G)$ such that $B = R \circ \overline{B}$ and $\beta = \rho \circ \overline{B}$. Examples include slice categories and categories of coalgebras:

$$\begin{array}{rcl}
\mathbb{C}/X & = & Ins(\,\mathbb{C} \xrightarrow{\mathrm{id}} \mathbb{C}\,,\ \mathbb{C} \xrightarrow{!} \mathbf{1} \xrightarrow{X} \mathbb{C}\,)\ , \\
\mathbf{Coalg}_F & = & Ins(\,\mathbb{C} \xrightarrow{\mathrm{id}} \mathbb{C}\,,\ \mathbb{C} \xrightarrow{F} \mathbb{C}\,)\ .
\end{array}$$

This 2-categorical notion of inserters comes from [10, 19]; see also [6].

Lemma 4.9. *Let $\mathbb{C}$ and $\mathbb{D}$ be $\mathbb{L}$-categories. Assume the following compatibility of $F, G : \mathbb{C} \rightrightarrows \mathbb{D}$ with $\mathbb{L}$.*[11]

- *F is an **oplax** $\mathbb{L}$-**functor**. That is, when identified with a natural transformation $\mathbb{C} \Rightarrow \mathbb{D} : \mathbf{Nat}^{\mathrm{op}} \to \mathbf{Cat}$, there exists an oplax natural transformation φ such that $F = \varphi \circ H$:*

$$\begin{tikzcd}
\mathbf{Nat}^{\mathrm{op}} \arrow[r, bend left, "\mathbb{C}"] \arrow[r, bend right, "\mathbb{D}"'] \arrow[r, phantom, "\Downarrow F"] & \mathbf{Cat}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbf{Nat}^{\mathrm{op}} \arrow[r, "H"] & \mathbb{L} \arrow[r, bend left, "\mathbb{C}"] \arrow[r, bend right, "\mathbb{D}"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathbf{Cat}
\end{tikzcd}
\tag{10}$$

- *G is a **lax** $\mathbb{L}$-**functor**. That is, G factors as $G = \gamma \circ H$ (as above) through a lax natural transformation γ.*

Then the inserter $Ins(F, G)$ is again an $\mathbb{L}$-category. Moreover, the functor $Ins(F, G) \xrightarrow{R} \mathbb{C}$ is a (non-lax) $\mathbb{L}$-functor.

Proof. We must define, for each $L^m \xrightarrow{f} L^n$ in $\mathbb{L}$, its interpretation $Ins(F, G)^m \xrightarrow{[\![f]\!]} Ins(F, G)^n$. This is induced by the following 2-cell via universality of $Ins(F, G)^n \cong Ins(F^n, G^n)$.

$$\begin{tikzcd}
 & \mathbb{C}^m \arrow[r, "[\![f]\!]"] \arrow[dr, "F^m"'] \arrow[dd, phantom, "\swarrow\rho^m"] & \mathbb{C}^n \arrow[dr, "F^n"] \arrow[d, phantom, "\swarrow\varphi_f"] & \\
Ins(F,G)^m \arrow[ur, "R^m"] \arrow[dr, "R^m"'] & & \mathbb{D}^m \arrow[r, "[\![f]\!]"] \arrow[d, phantom, "\searrow\gamma_f"] & \mathbb{D}^n \\
 & \mathbb{C}^m \arrow[r, "[\![f]\!]"'] \arrow[ur, "G^m"] & \mathbb{C}^n \arrow[ur, "G^n"'] &
\end{tikzcd}$$

Here the 2-cell ρ is the one accompanying $Ins(F, G)$; φ_f comes from oplax naturality of φ; γ_f comes from lax naturality of γ. Coherence conditions such as $[\![g \circ f]\!] \cong [\![g]\!] \circ [\![f]\!]$ follow from universality. Naturality of $R : Ins(F, G) \Rightarrow \mathbb{C}$ is because the induced $[\![f]\!]$ is a "mediating" functor. $\square$

Corollary 4.10. 1. *If $\mathbb{C}$ is an $\mathbb{L}$-category and F is a **lax** $\mathbb{L}$-functor, then $\mathbf{Coalg}_F$ is an $\mathbb{L}$-category.*

2. *Given a microcosm model $X \in \mathbb{C}$ for $\mathbb{L}$, the slice category $\mathbb{C}/X$ is an $\mathbb{L}$-category.* $\square$

Concretely, an operation $L^m \xrightarrow{f} L$ in $\mathbb{L}$ is interpreted in the following ways. In $\mathbf{Coalg}_F$, an m-tuple $(X_i \xrightarrow{c_i} FX_i)_i$ of F-coalgebras is carried to the F-coalgebra

$$[\![f]\!](\overrightarrow{X_i}) \xrightarrow{[\![f]\!](\overrightarrow{c_i})} [\![f]\!](\overrightarrow{FX_i}) \xrightarrow{(\varphi_f)_{\overrightarrow{X_i}}} F[\![f]\!](\overrightarrow{X_i}) \tag{11}$$

where $[\![f]\!]$ is the interpretation in the $\mathbb{L}$-category $\mathbb{C}$. In $\mathbb{C}/X$, an m-tuple $(Y_i \xrightarrow{y_i} X)_i$ is carried to

$$[\![f]\!](\overrightarrow{Y_i}) \xrightarrow{[\![f]\!](\overrightarrow{y_i})} [\![f]\!](\overrightarrow{X}) \xrightarrow{X_f} X\ , \tag{12}$$

where X_f is the interpretation on the $\mathbb{L}$-object X.

Theorem 4.11 (Final coalgebra semantics is compositional)**.** *Let $\mathbb{C}$ and F be as in Corollary 4.10.1. Moreover, assume that $\zeta : Z \xrightarrow{\cong} FZ$ is a final F-coalgebra. Then the functor $\mathbf{Coalg}_F \xrightarrow{\mathrm{beh}} \mathbb{C}/Z$ is a (non-lax) $\mathbb{L}$-functor. It makes the following diagram in $\mathbb{L}\text{-}\mathbf{cat}$ commute.*

$$\begin{tikzcd}
\mathbf{Coalg}_F \arrow[rr, "\mathrm{beh}"] \arrow[dr, "U"'] & & \mathbb{C}/Z \arrow[dl, "\mathrm{dom}"] \\
 & \mathbb{C} &
\end{tikzcd}$$

$\square$

The category $\mathbb{C}/Z$ is an $\mathbb{L}$-category because: $\zeta \in \mathbf{Coalg}_F$ is an $\mathbb{L}$-object; so is $Z \in \mathbb{C}$ which is the image under an $\mathbb{L}$-functor $\mathbf{Coalg}_F \to \mathbb{C}$ (Proposition 4.6). The functor beh's preservation of algebraic structures readily instantiates to equations like (1), (7) with the help of (11), (12).

For trace semantics—which is final coalgebra semantics in $\mathcal{K}\ell(T)$ [5]—the assumption on $\mathbb{L}$-structures on $\mathcal{K}\ell(T)$ and $\overline{F}$ can be reduced to those on T and F. Recall that a distributive law $\pi : FT \Rightarrow TF$ lifts F to a functor $\overline{F} : \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$.

[10] To be precise, $\mathbb{L}\text{-}\mathbf{obj}_{\mathbb{C}}$ is the lax "FP-limit": the canonical cone from $\mathbb{L}\text{-}\mathbf{obj}_{\mathbb{C}}$ is the final one among those lax cones $\Delta\mathbb{B} \overset{\beta}{\Rightarrow} \mathbb{C} : \mathbb{L} \to \mathbf{Cat}$ such that $\beta \circ H$ is (properly) natural.

[11] Note that F or G is not required to be an $\mathbb{L}$-functor: only (op)lax compatibility will do.

Theorem 4.12. *1. Assume that a monad T is a lax $\mathbb{L}$-functor—with $T = \tau \circ H$ just like in (10). Assume further that the mediating 2-cells of τ are compatible with the monad structure of T in an obvious way. Then $\mathcal{K}\ell(T)$ is an $\mathbb{L}$-category and $\mathbb{C} \xrightarrow{J} \mathcal{K}\ell(T)$ is an $\mathbb{L}$-functor.*

2. Moreover, suppose that there is a distributive law $\pi : FT \Rightarrow TF$ which makes the following 2-cells equal, for each $f : L^m \to L^n$ in $\mathbb{L}$.

$$
\begin{tikzcd}
\mathbb{C}^m \arrow[rr, "T^m"] \arrow[dr] \arrow[dd, "{[\![f]\!]}"'] & & \mathbb{C}^m \arrow[dr, "F^m"] \arrow[dl, phantom, "\pi^m \swarrow"] & \\
& \mathbb{C}^m \arrow[rr] \arrow[dd] \arrow[dl, phantom, "\varphi_f \swarrow"] & & \mathbb{C}^m \arrow[dd, "{[\![f]\!]}"] \arrow[dll, phantom, "\swarrow \tau_f"] \\
\mathbb{C}^n \arrow[dr, "F^n"'] & & & \\
& \mathbb{C}^n \arrow[rr, "T^n"'] & & \mathbb{C}^n
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C}^m \arrow[rr, "T^m"] \arrow[dd, "{[\![f]\!]}"'] \arrow[ddrr, phantom, "\tau_f \swarrow"] & & \mathbb{C}^m \arrow[dr, "F^m"] \arrow[dd, "\varphi_f"] & \\
& & & \mathbb{C}^m \arrow[dd, "{[\![f]\!]}"] \arrow[dl, phantom, "\swarrow"] \\
\mathbb{C}^n \arrow[rr] \arrow[dr, "F^n"'] & & \mathbb{C}^n \arrow[dr] \arrow[dl, phantom, "\pi^n \swarrow"] & \\
& \mathbb{C}^n \arrow[rr, "T^n"'] & & \mathbb{C}^n
\end{tikzcd}
$$

Then $\overline{F}$ is a lax $\mathbb{L}$-functor and hence $\mathbf{Coalg}_{\overline{F}}$ is an $\mathbb{L}$-category.

Proof. The theorem is proved much like Lemma 4.9. Here we exploit the following 2-universality of $\mathbb{C} \xrightarrow{J} \mathcal{K}\ell(T)$: $(J, \epsilon J)$ is initial among those (B, β)

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "B"] \arrow[d, "T"'] & \mathbb{B} \\
\mathbb{C} \arrow[ur, bend right, "B"'] \arrow[u, phantom, ""] & \arrow[ul, phantom, "\nearrow \beta"]
\end{tikzcd}
$$

which are compatible with the monad structure of T in an obvious sense.[12] □

By applying Theorem 4.11 in this Kleisli setting, we obtain compositionality of trace semantics.

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[12] This is in fact the universality of the unit of the 2-adjunction $\mathcal{K}\ell \dashv \mathit{Ins} : \mathbf{Mnd}(\mathbf{Cat}_*) \to \mathbf{Cat}$. See [18] for details.