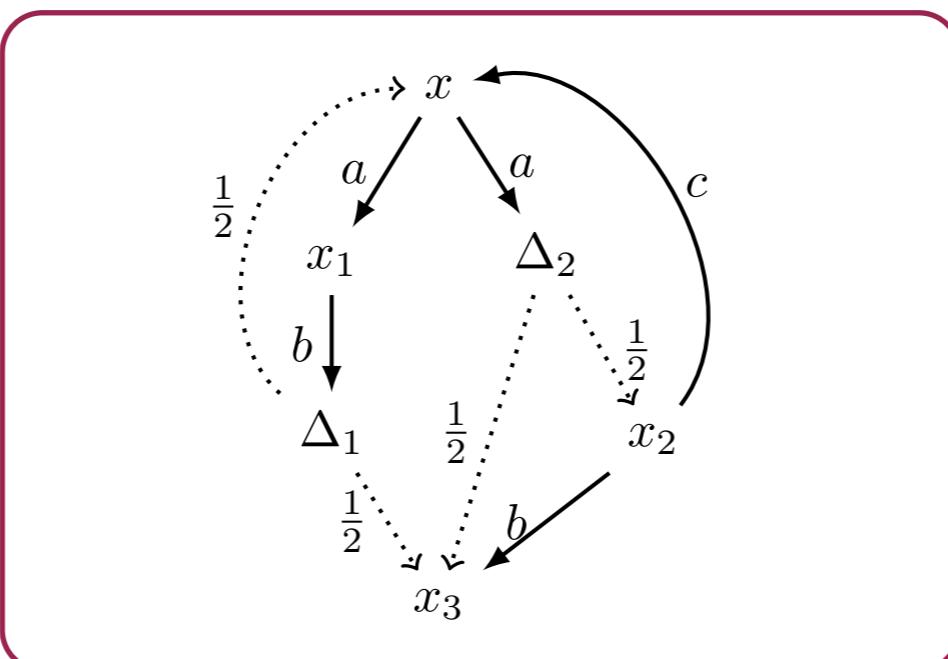


The Theory of Traces for Nondeterminism and Probability

Ana Sokolova  UNIVERSITY
of SALZBURG



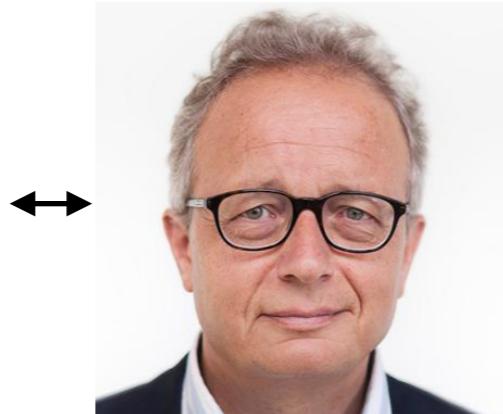
It's all about leaving
a trace...



Joint work with



Ichiro Hasuo
NII
National Institute of Informatics



Bart Jacobs
Radboud University



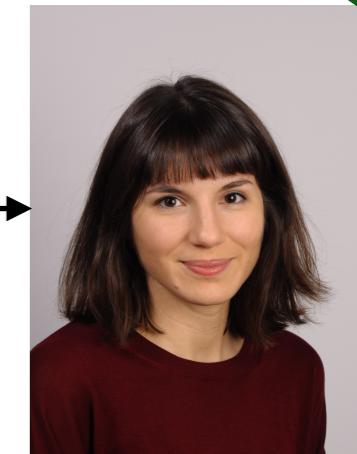
Alexandra Silva



Harald Woracek
TU
WIEN



Filippo Bonchi



Valeria Vignudelli
ENS
ENS DE LYON

I will talk about:

- 1.** The absolute basics of coalgebra
- 2.** Trace semantics via determinisation
- 3.** ...enabled by algebraic structure

I will talk about:

Mathematical framework
based on category theory
for state-based
systems semantics

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Mathematical framework
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for
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Mathematical framework
based on category theory
for state-based
systems semantics

for
nondeterministic/
probabilistic
systems

systems with
algebraic effects



Coalgebras

Uniform framework for dynamic transition systems, based on category theory.



Coalgebras

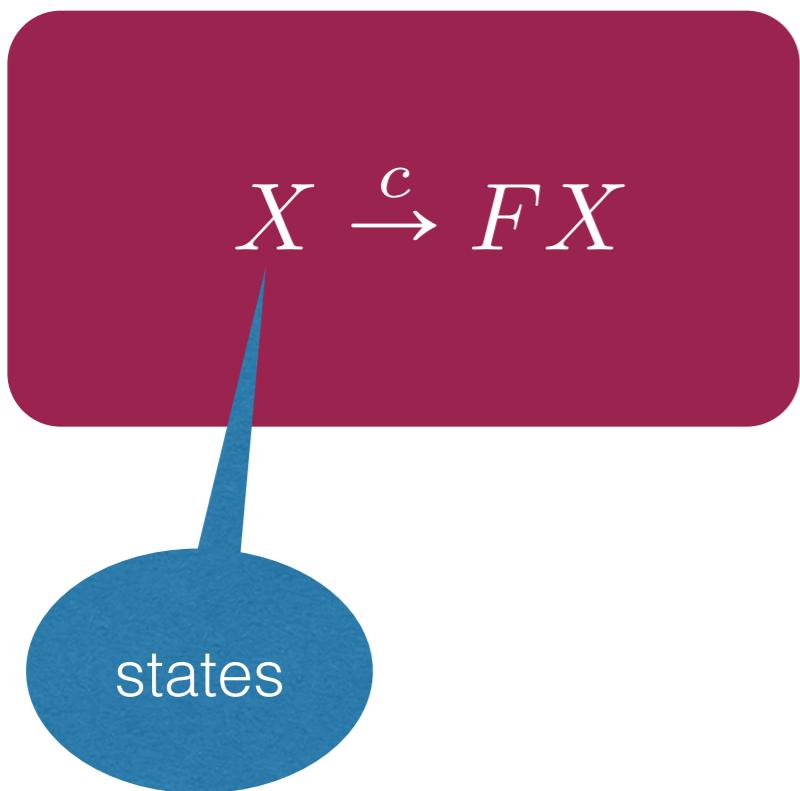
Uniform framework for dynamic transition systems, based on category theory.

$$X \xrightarrow{c} FX$$



Coalgebras

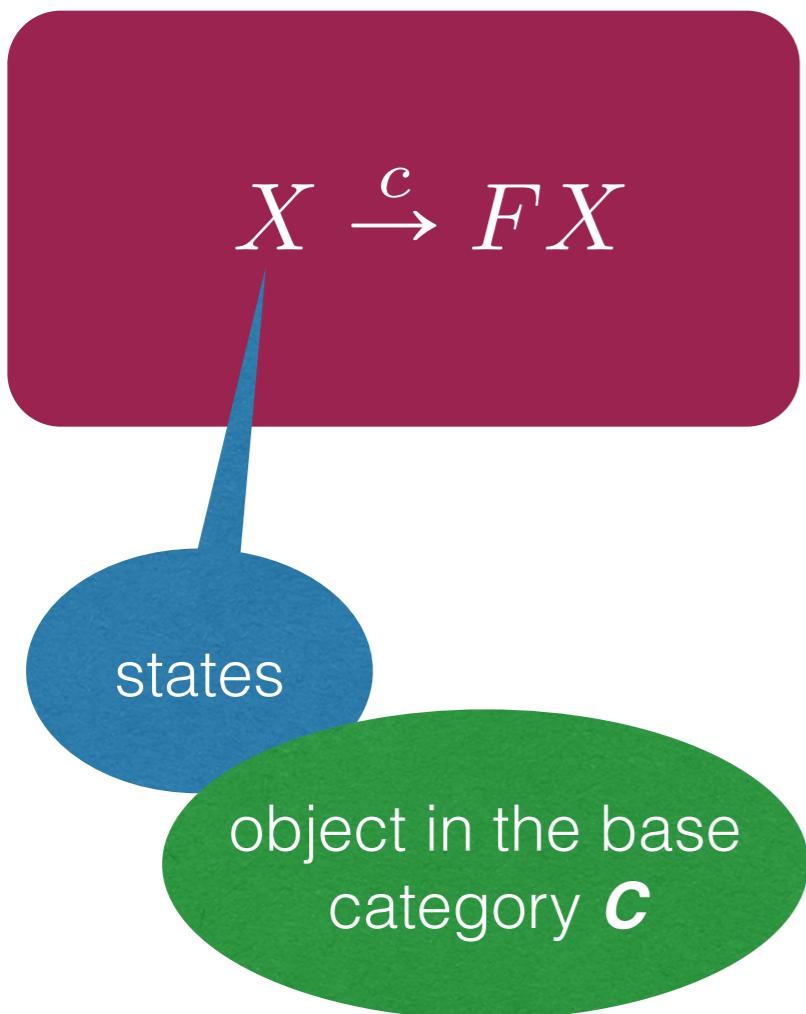
Uniform framework for dynamic transition systems, based on category theory.





Coalgebras

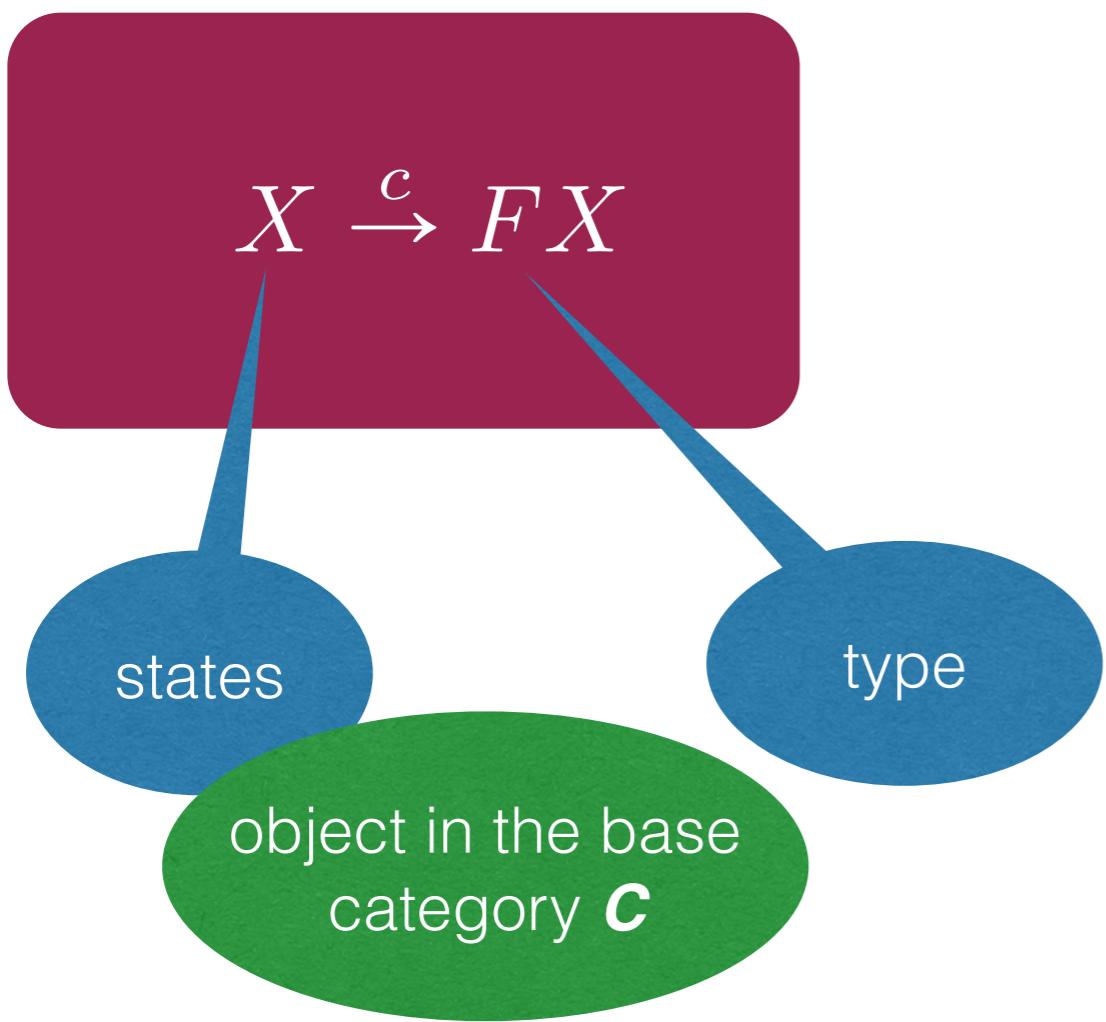
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Coalgebras

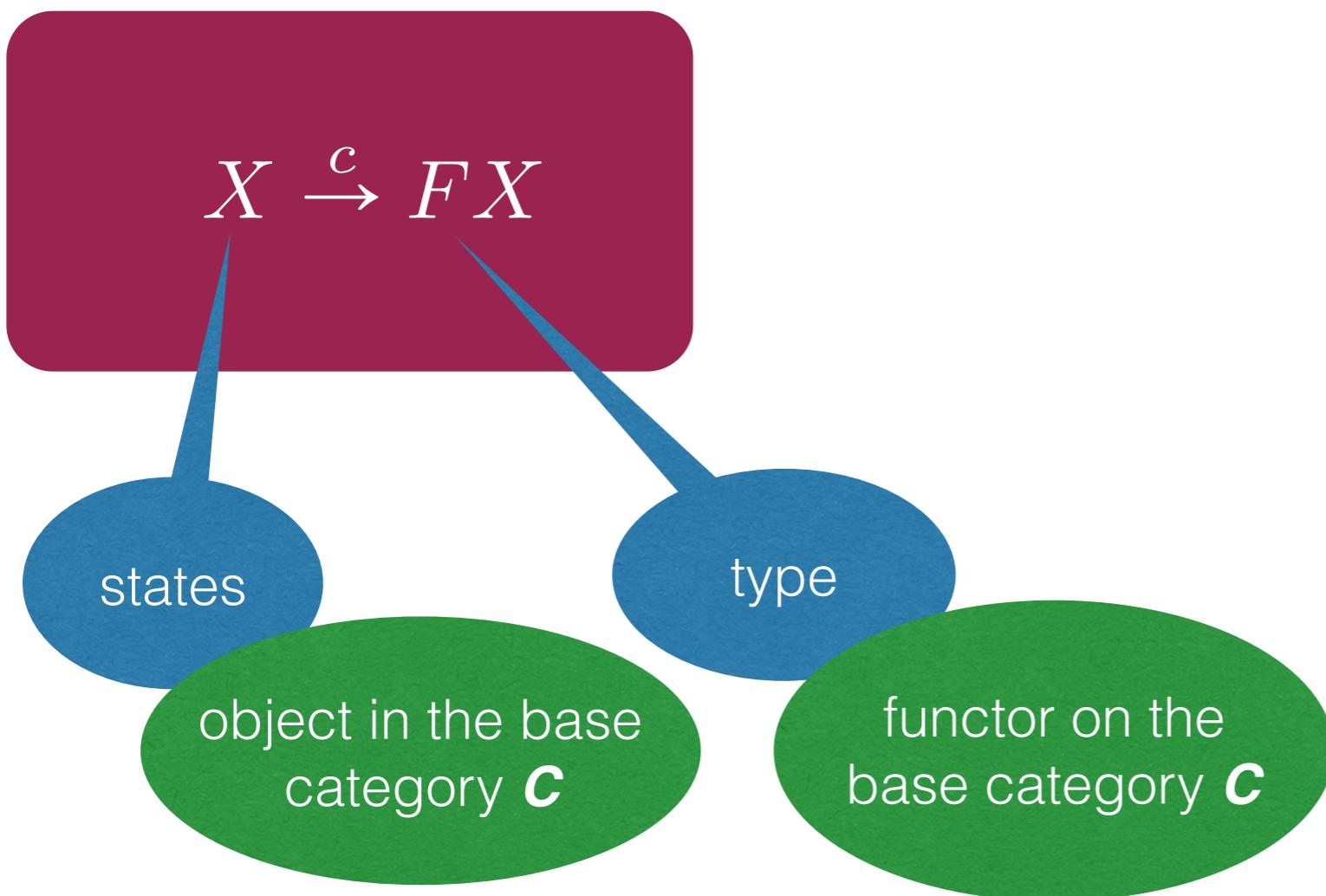
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Coalgebras

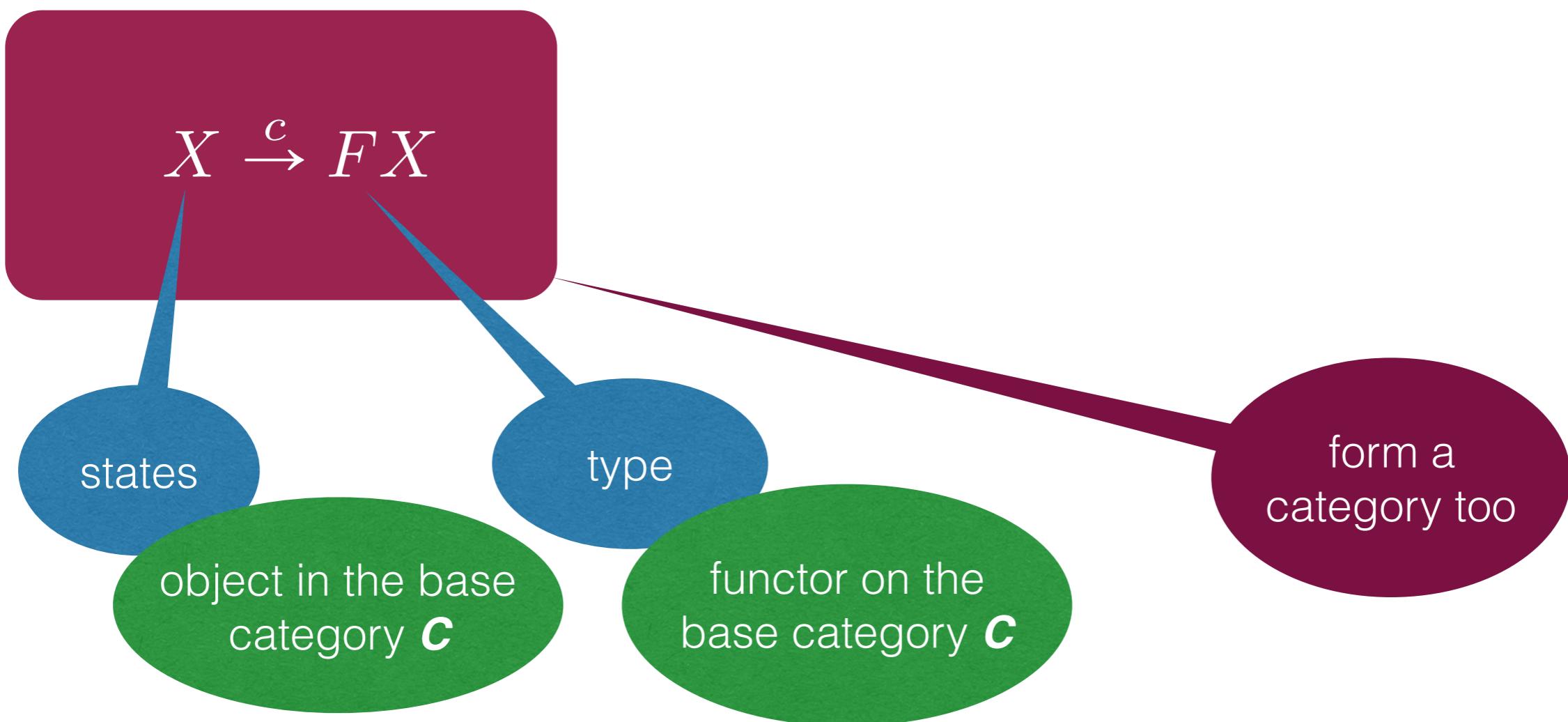
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Coalgebras

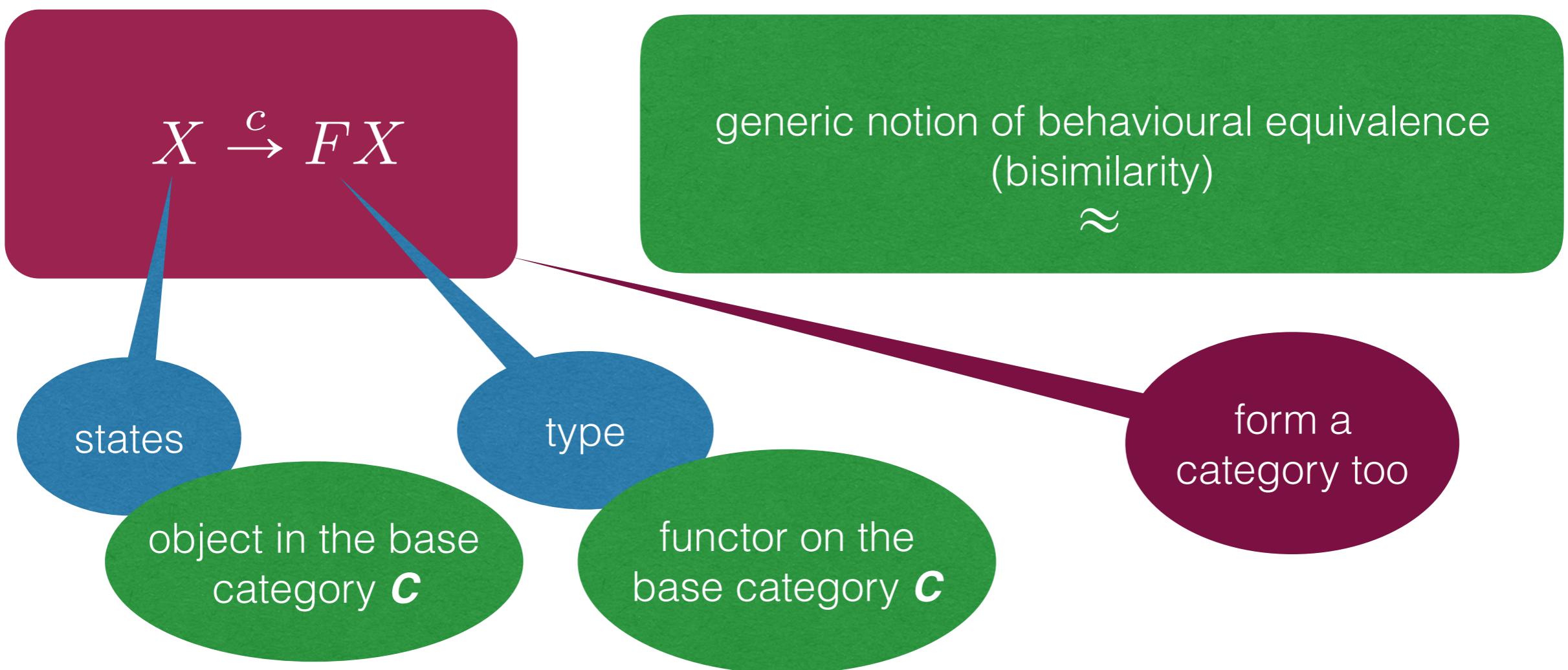
Uniform framework for dynamic transition systems, based on category theory.





Coalgebras

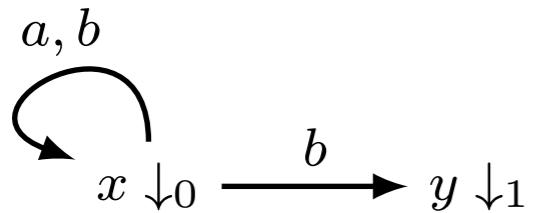
Uniform framework for dynamic transition systems, based on category theory.



Examples

NFA

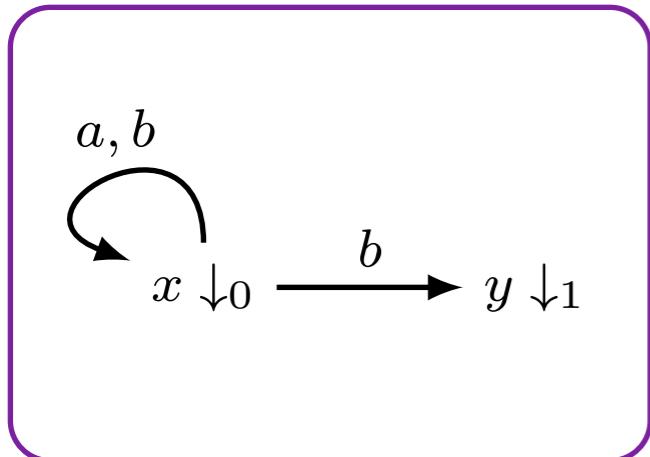
$$X \rightarrow 2 \times (\mathcal{P}X)^A$$



Examples

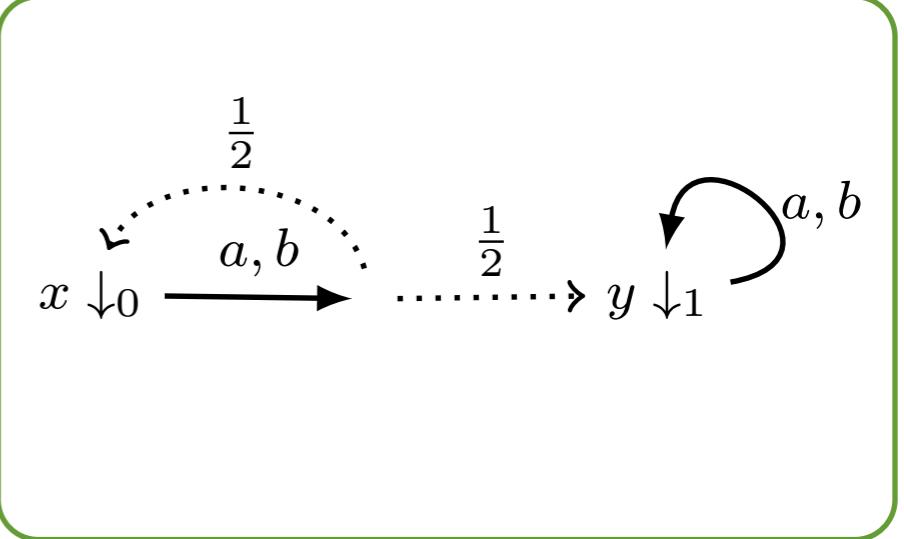
NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$



Rabin PA

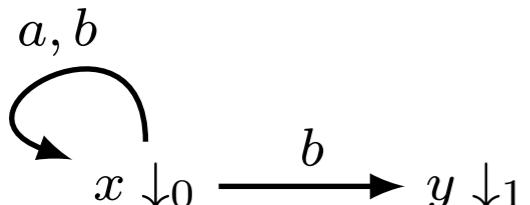
$$X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1} X)^A$$



Examples

NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$

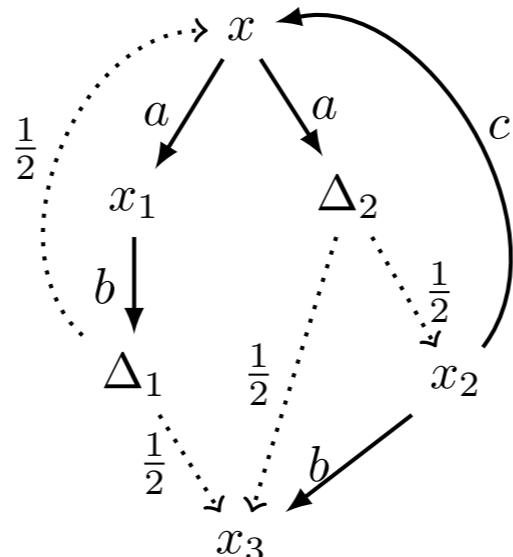
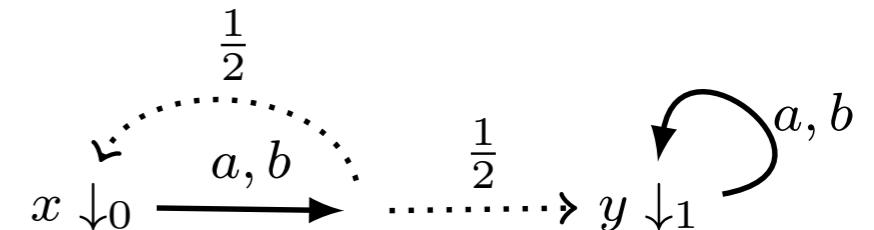


Rabin PA

$$X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1} X)^A$$

Simple NPA

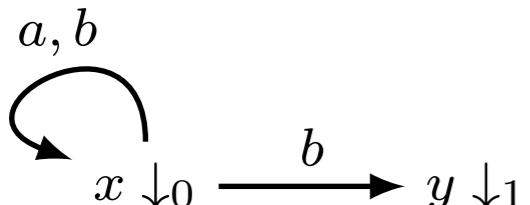
$$X \rightarrow ? \times (\mathcal{PDX})^A$$



Examples

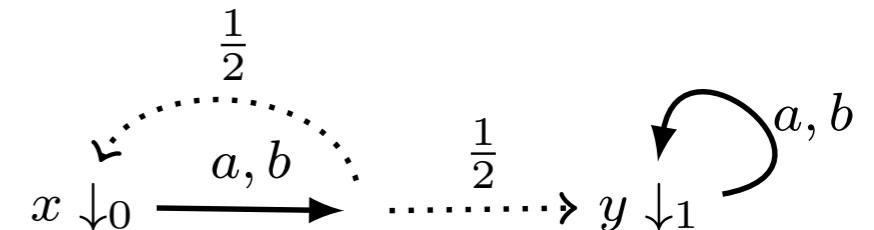
NFA

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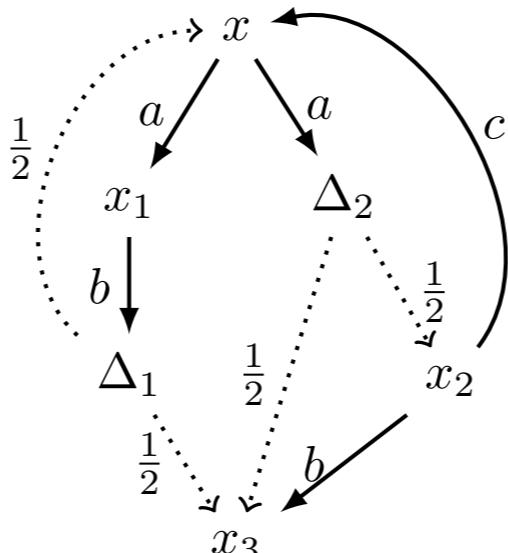
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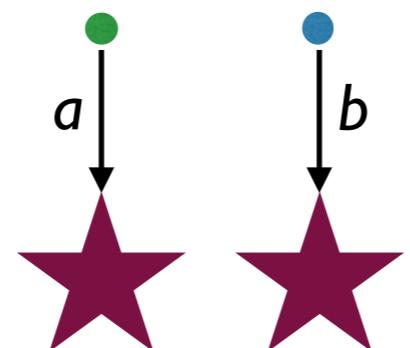
systems with
nondeterminism
and
probability

In general

In general

Automata

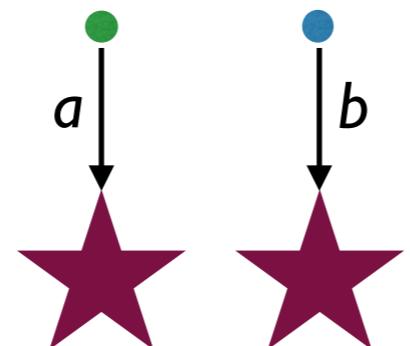
$$X \rightarrow O \times (MX)^A$$



In general

Automata

$$X \rightarrow O \times (MX)^A$$

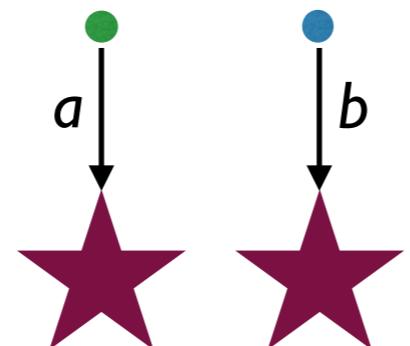


with
observations
in O

In general

Automata

$$X \rightarrow O \times (MX)^A$$



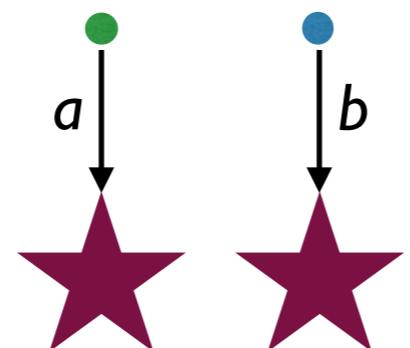
with
observations
in O

and M-effects

In general

Automata

$$X \rightarrow O \times (MX)^A$$



with
observations
in O

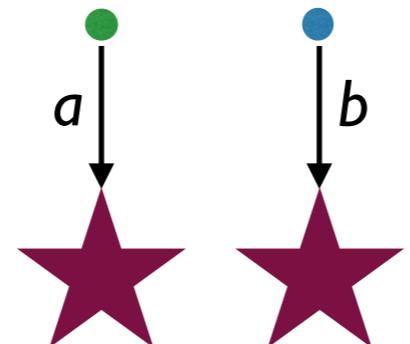
and M-effects

for a monad M

In general

Automata

$$X \rightarrow O \times (MX)^A$$



with
observations
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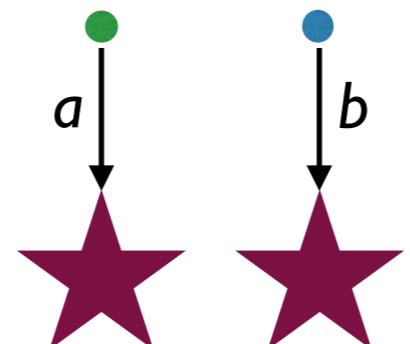
for a monad M

$$\begin{aligned}\mu: MM &\Rightarrow M \\ \eta: Id &\Rightarrow M\end{aligned}$$

In general

Automata

$$X \rightarrow O \times (MX)^A$$



with
observations
in O

and M-effects

for a monad M

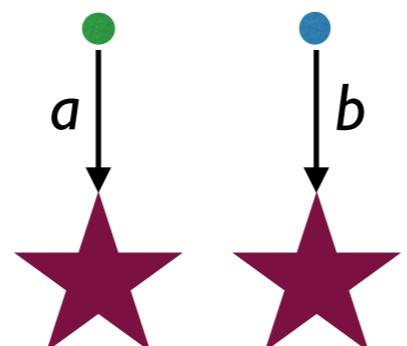
$$\begin{aligned}\mu: MM &\Rightarrow M \\ \eta: Id &\Rightarrow M\end{aligned}$$

providing
algebraic
effects

In general

Automata

$$X \rightarrow O \times (MX)^A$$



with
observations
in O

and M-effects

we write

$$x \downarrow o, \quad x \xrightarrow{a} t_x$$

$$\begin{aligned}\mu: MM &\Rightarrow M \\ \eta: Id &\Rightarrow M\end{aligned}$$

providing
algebraic
effects

In our examples

NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$

Rabin PA

$$X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1} X)^A$$

Simple PA

$$X \rightarrow ? \times (\mathcal{PDX})^A$$

In our examples

NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$

$M = \mathcal{P}$
for nondeterminism

Rabin PA

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Powerset, subsets

Rabin PA

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In our examples

NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$

$M = \mathcal{P}$
for nondeterminism

Powerset, subsets

Rabin PA

$$X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1} X)^A$$

$M = \mathcal{D}_{\leq 1}$
for probability

Simple PA

$$X \rightarrow ? \times (\mathcal{PDX})^A$$

In our examples

NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$

$M = \mathcal{P}$
for nondeterminism

Powerset, subsets

Rabin PA

$$X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1} X)^A$$

$M = \mathcal{D}_{\leq 1}$
for probability

Subdistributions

Simple PA

$$X \rightarrow ? \times (\mathcal{PDX})^A$$

In our examples

NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$

$M = \mathcal{P}$
for nondeterminism

Powerset, subsets

Rabin PA

$$X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1} X)^A$$

$M = \mathcal{D}_{\leq 1}$
for probability

Subdistributions

Simple PA

$$X \rightarrow ? \times (\mathcal{PDX})^A$$

$M = \mathcal{PD} ???$
for nondeterminism
and probability

In our examples

NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$

$M = \mathcal{P}$
for nondeterminism

Powerset, subsets

Rabin PA

$$X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1} X)^A$$

$M = \mathcal{D}_{\leq 1}$
for probability

Subdistributions

Simple PA

$$X \rightarrow ? \times (\mathcal{C}X)^A$$

$M = \mathcal{C}$
for nondeterminism
and probability !

In our examples

NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$

$M = \mathcal{P}$
for nondeterminism

Powerset, subsets

Rabin PA

$$X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1} X)^A$$

$M = \mathcal{D}_{\leq 1}$
for probability

Subdistributions

Simple PA

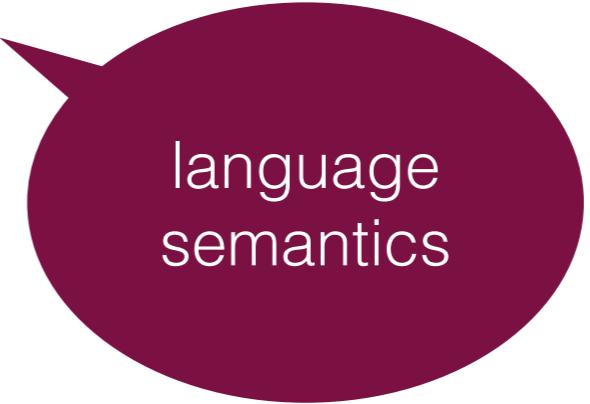
$$X \rightarrow ? \times (\mathcal{C}X)^A$$

$M = \mathcal{C}$
for nondeterminism
and probability !

Nonempty f.g. convex
subsets of
distributions

Trace Semantics

Trace Semantics

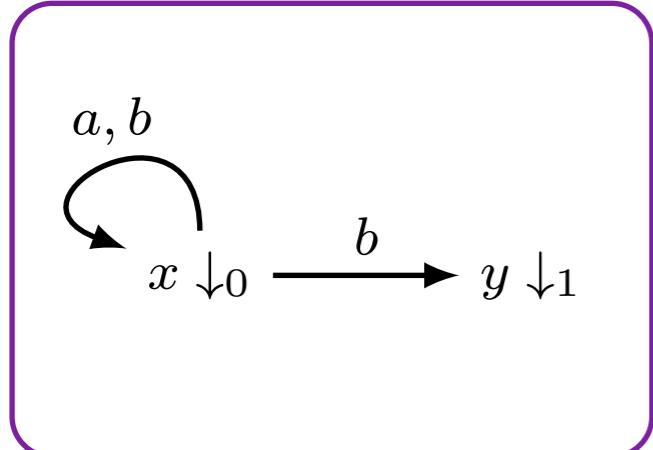


language
semantics

Trace Semantics

NFA = LTS + termination

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$

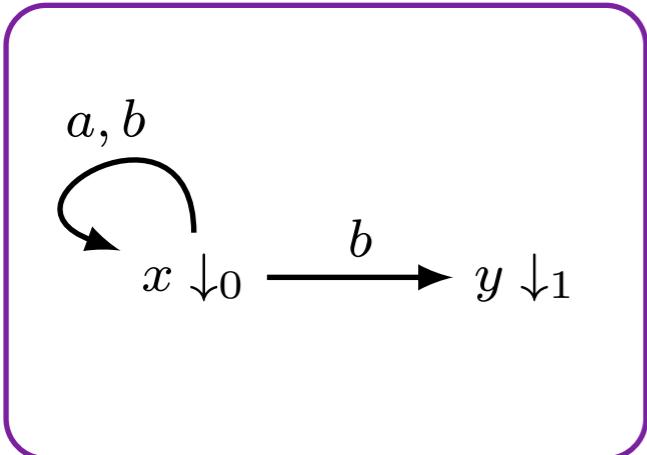


language
semantics

Trace Semantics

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$$X \rightarrow 2 \times (\mathcal{P}X)^A$$



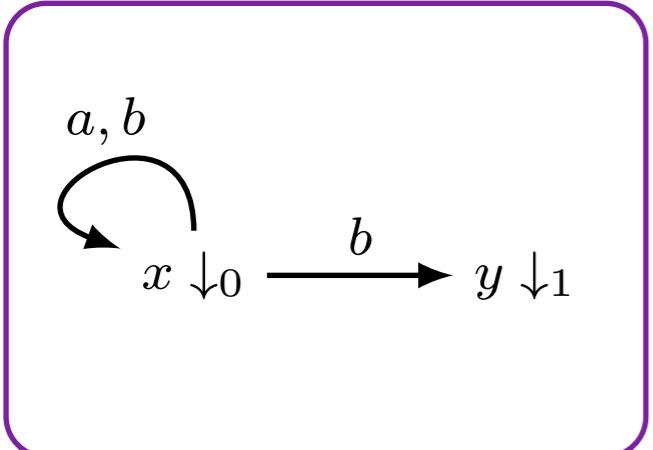
language
semantics

$$\text{tr}: X \rightarrow 2^{A^*}$$

Trace Semantics

NFA = LTS + termination

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$



language
semantics

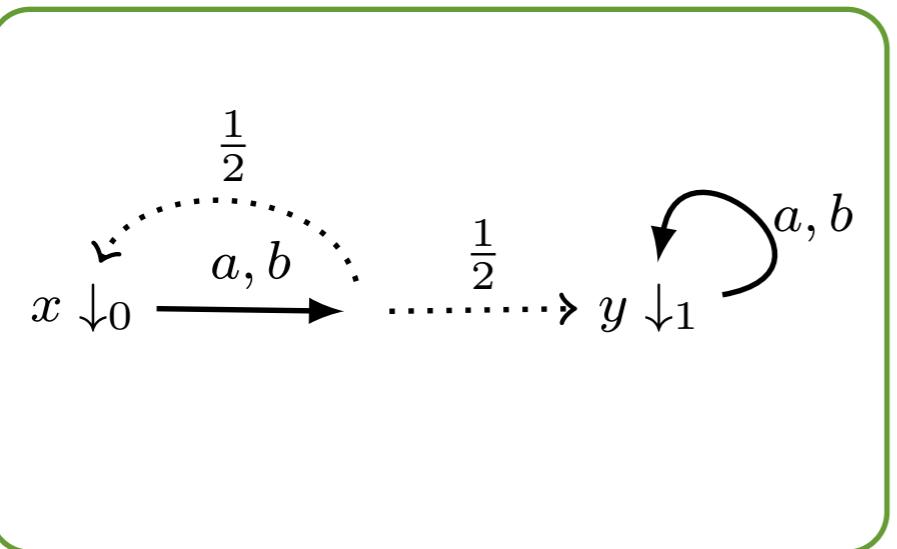
$$\text{tr}: X \rightarrow 2^{A^*}$$

$$\text{tr}(x) = (a \cup b)^* b = \{w \in \{a, b\}^* \mid w \text{ ends with a } b\}$$

Trace Semantics

Rabin PA = RPTS + termination

$$X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1} X)^A$$



$$\text{tr}(x) = \left(a \mapsto \frac{1}{2}, aa \mapsto \frac{3}{4}, \dots \right)$$

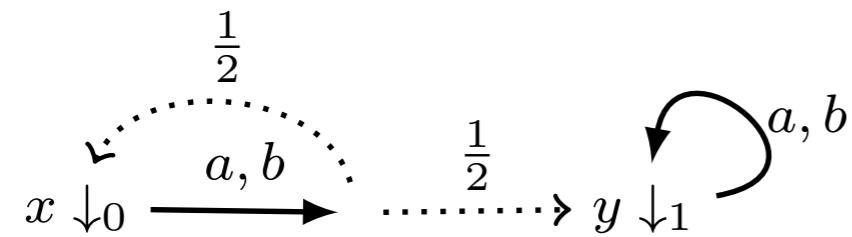
$$\text{tr}: X \rightarrow [0, 1]^{A^*}$$

Trace Semantics

Rabin PA = RPTS + termination

$$X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1} X)^A$$

probabilistic
language
semantics



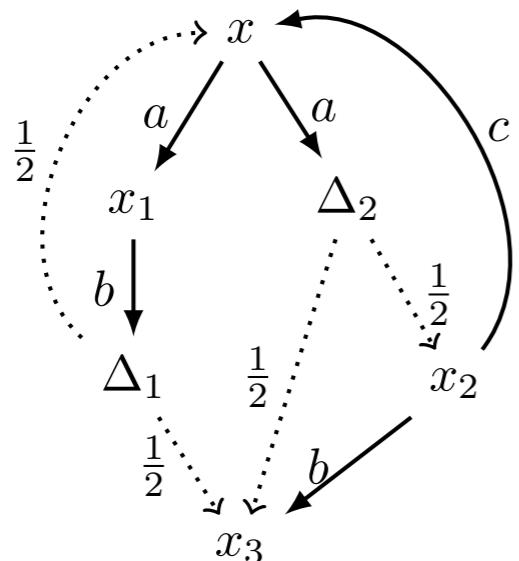
$$\text{tr}(x) = \left(a \mapsto \frac{1}{2}, aa \mapsto \frac{3}{4}, \dots \right)$$

$$\text{tr}: X \rightarrow [0, 1]^{A^*}$$

Trace Semantics

Simple NPA

$$X \rightarrow ? \times (\mathcal{PDX})^A$$



$$\text{tr}(x) = ???$$

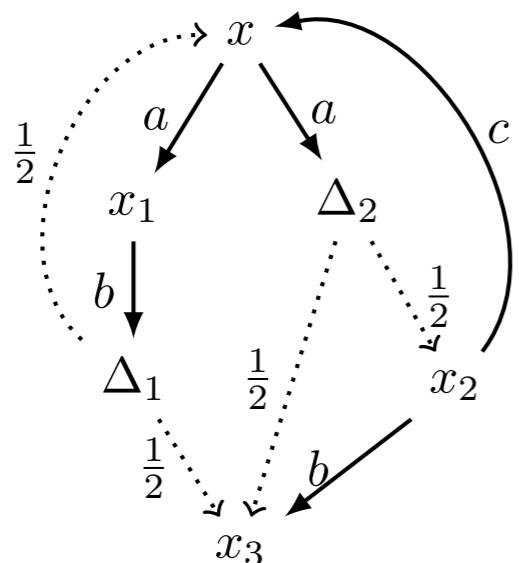
$$\text{tr}: X \rightarrow ?^{A^*}$$

Trace Semantics

Simple NPA

$$X \rightarrow ? \times (\mathcal{PDX})^A$$

nondet.
probabilistic
language
semantics ?



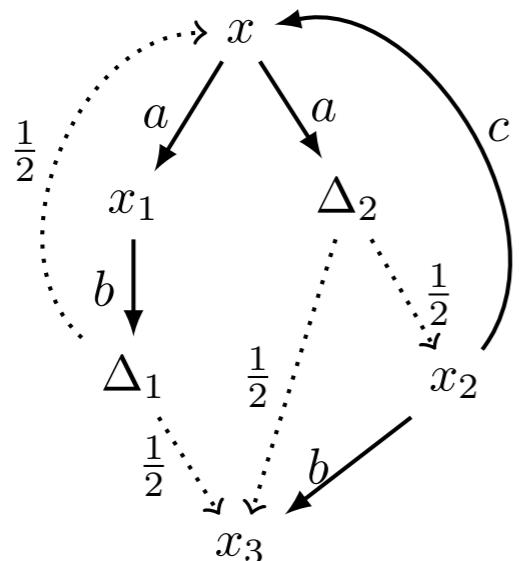
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$$\text{tr}: X \rightarrow ?^{A^*}$$

Trace Semantics

Simple NPA

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nondet.
probabilistic
language
semantics ?

Existing definitions
are “local”
given in terms of
schedulers

$$\text{tr}(x) = ???$$

$$\text{tr}: X \rightarrow ?^{A^*}$$

Trace semantics coalgebraically?

NFA / LTS

Two ideas:

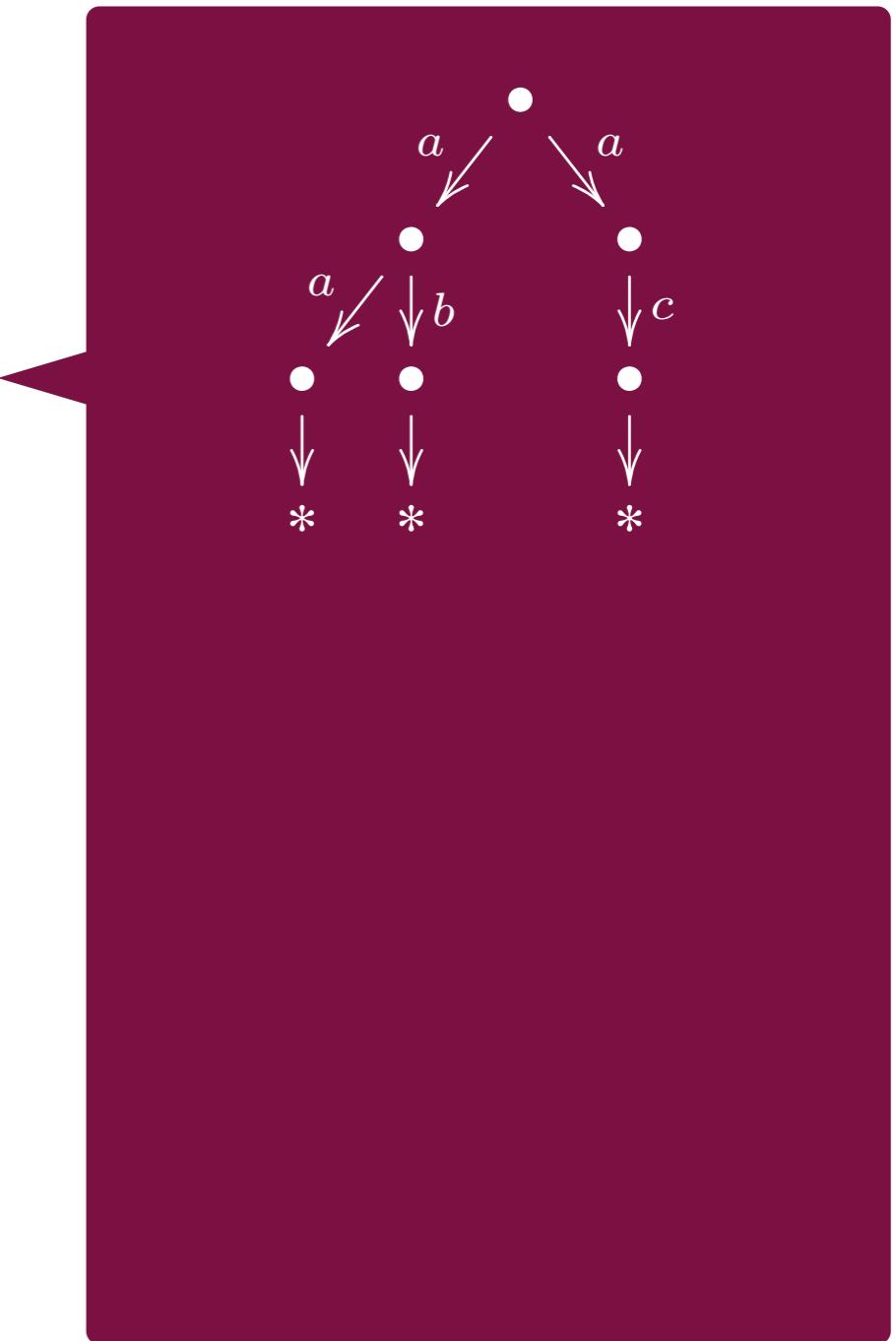
- (1) unfold branching + transitions on words
- (2) trace = bisimilarity after determinisation

Trace semantics coalgebraically?

NFA / LTS

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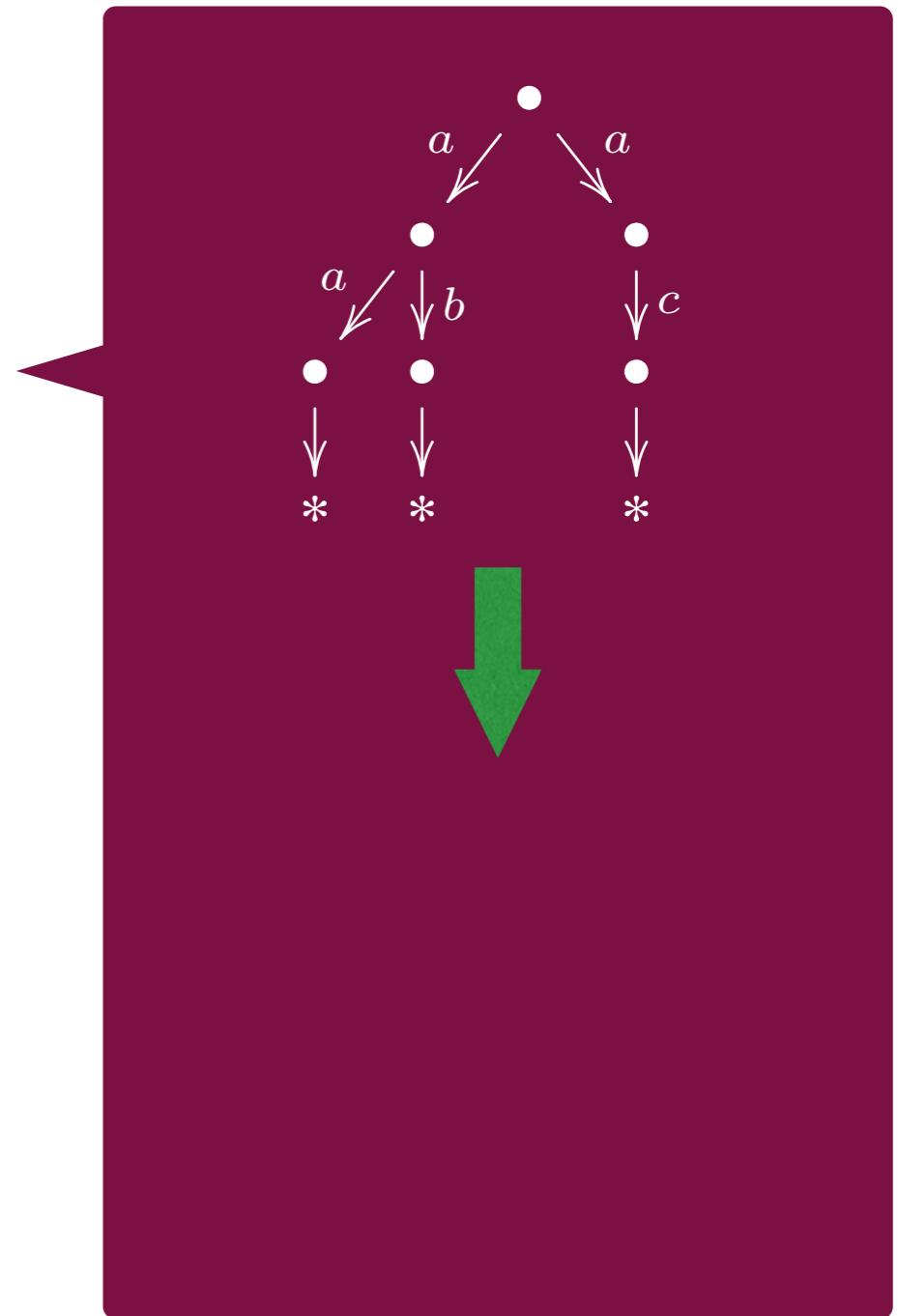


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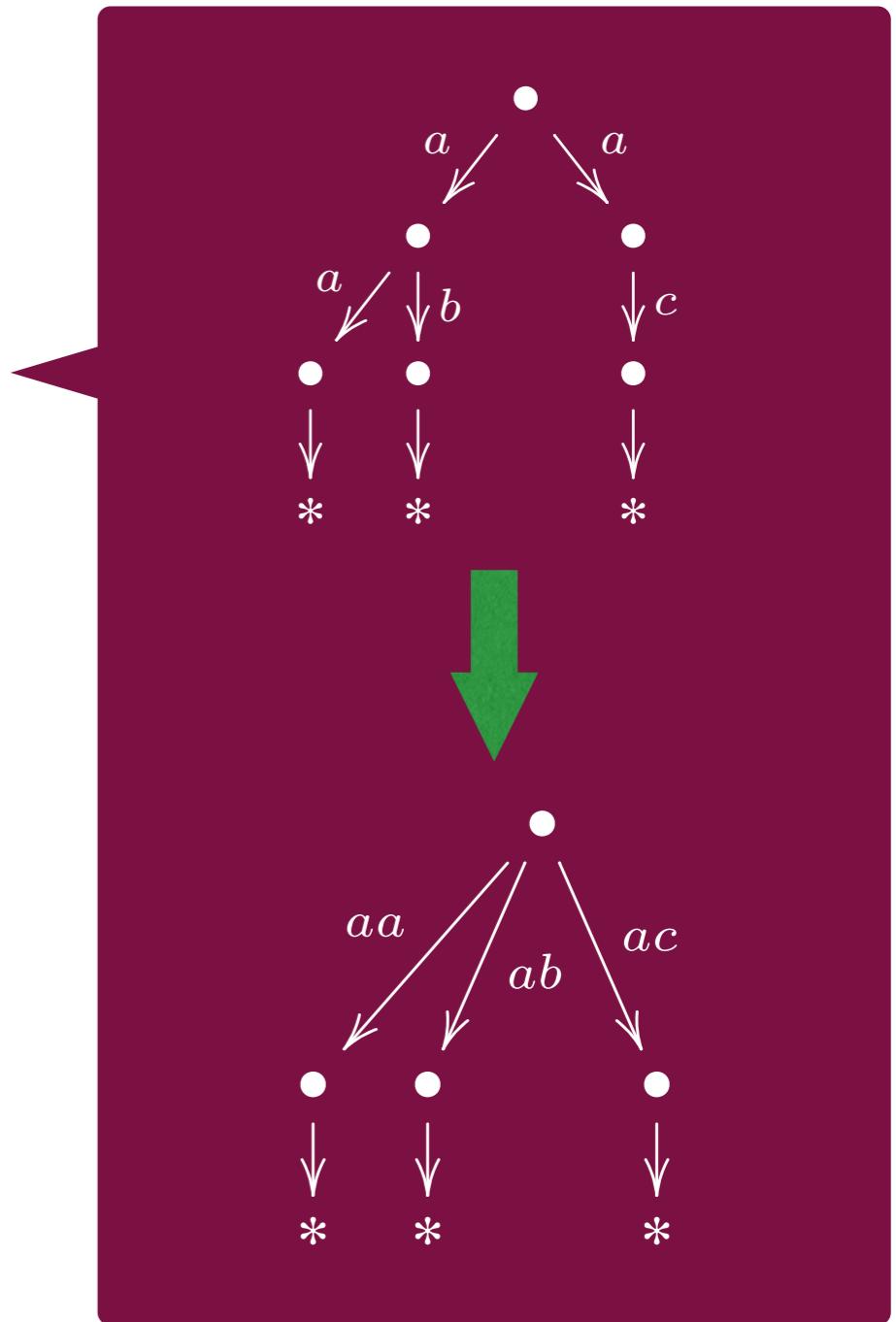


Trace semantics coalgebraically?

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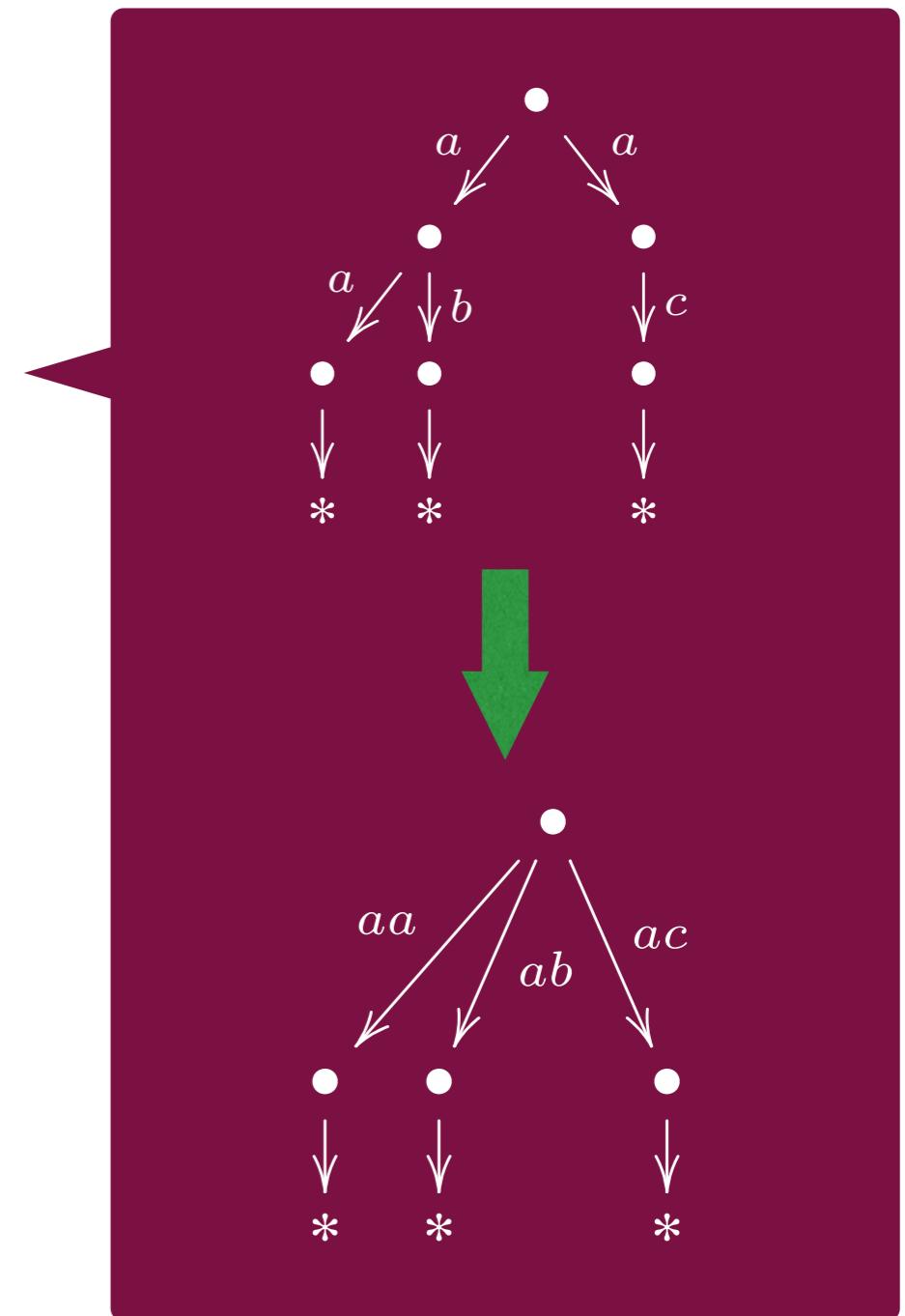
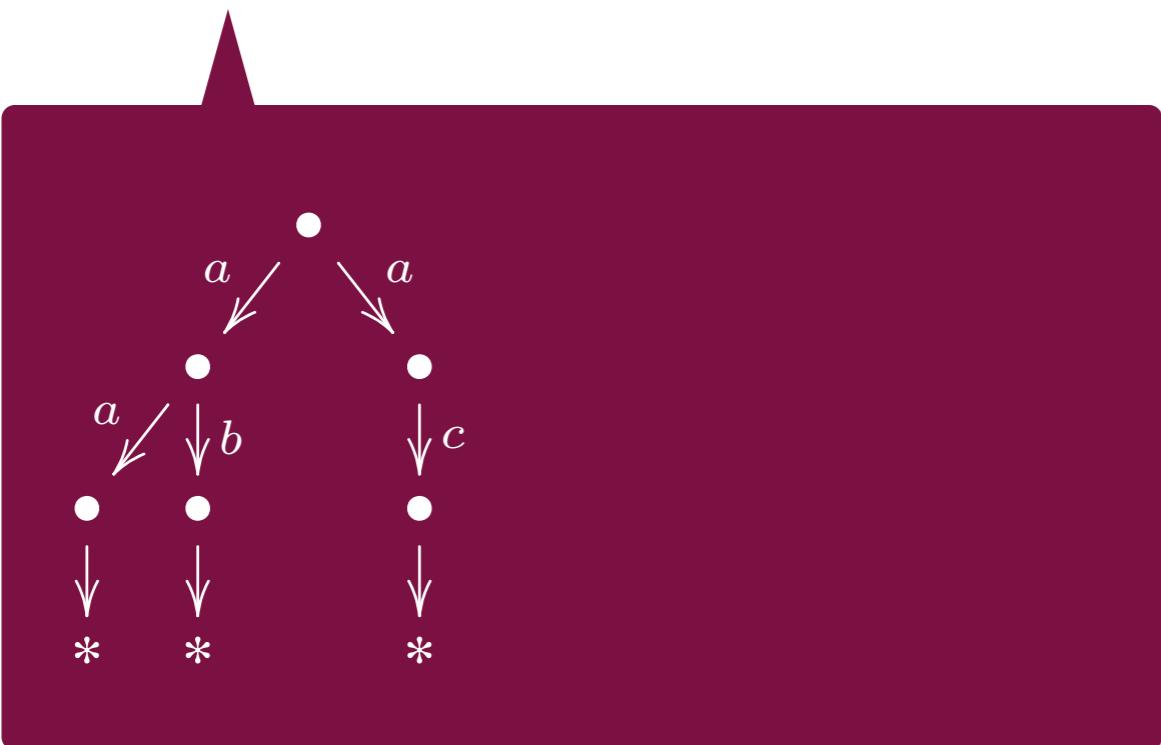


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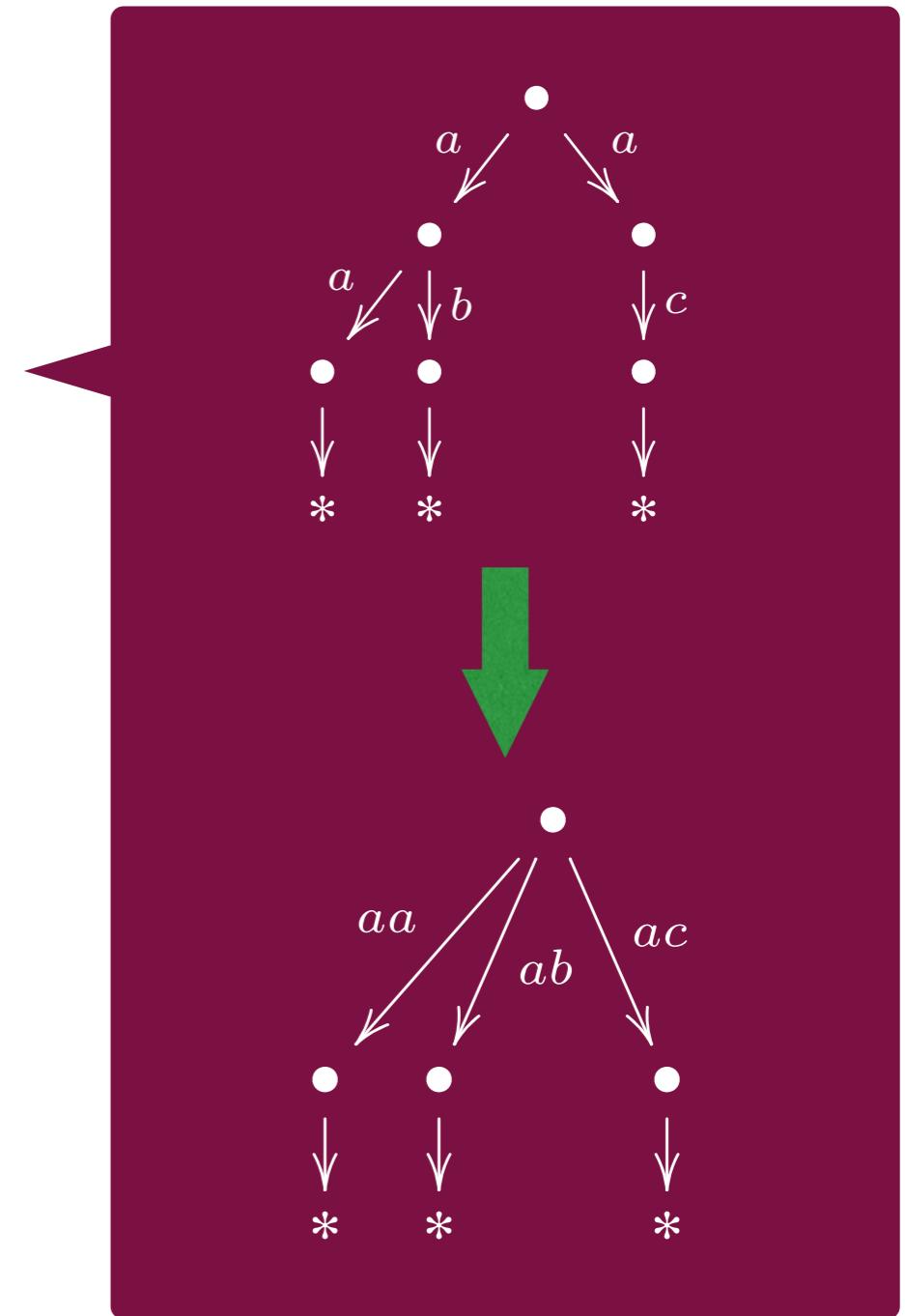
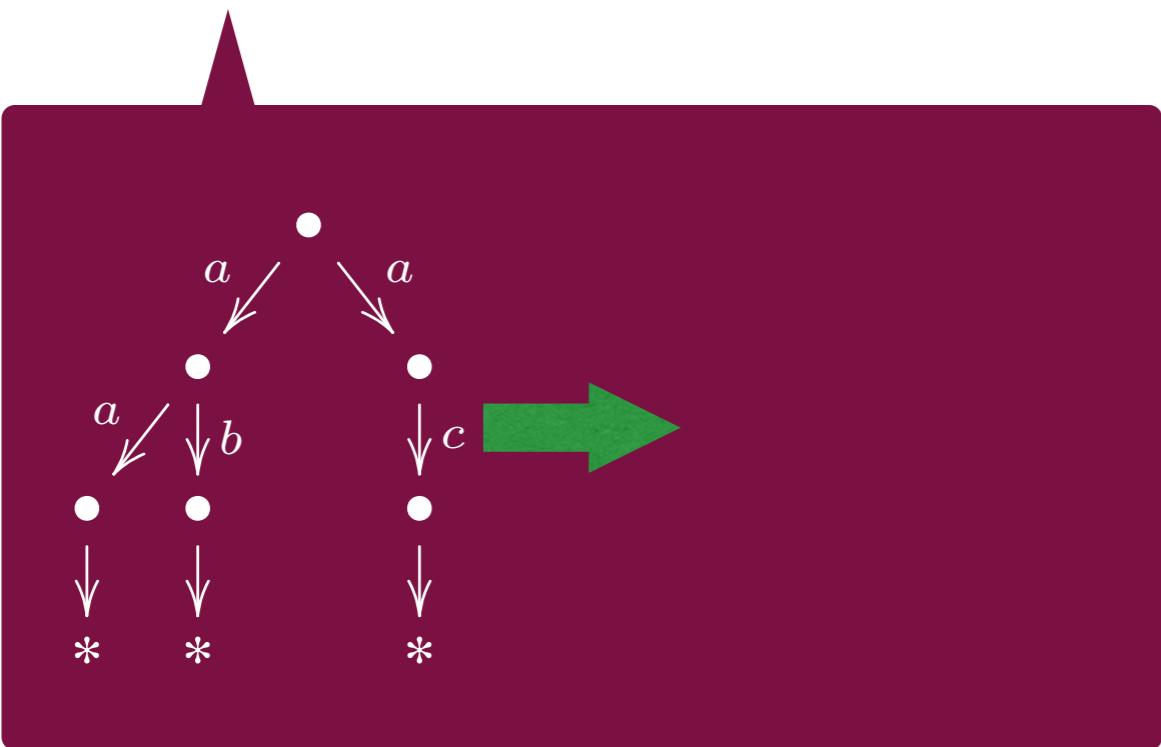


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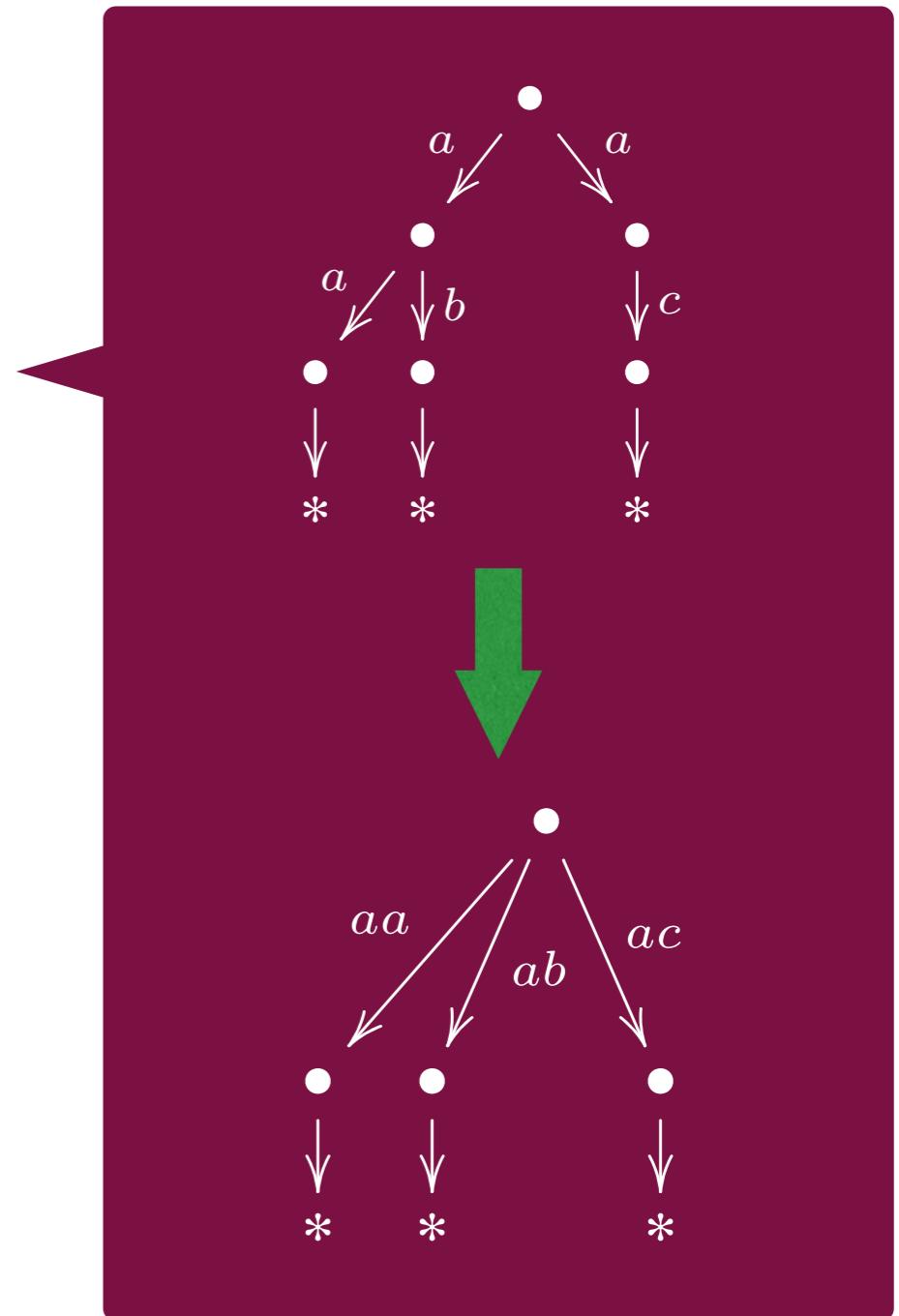
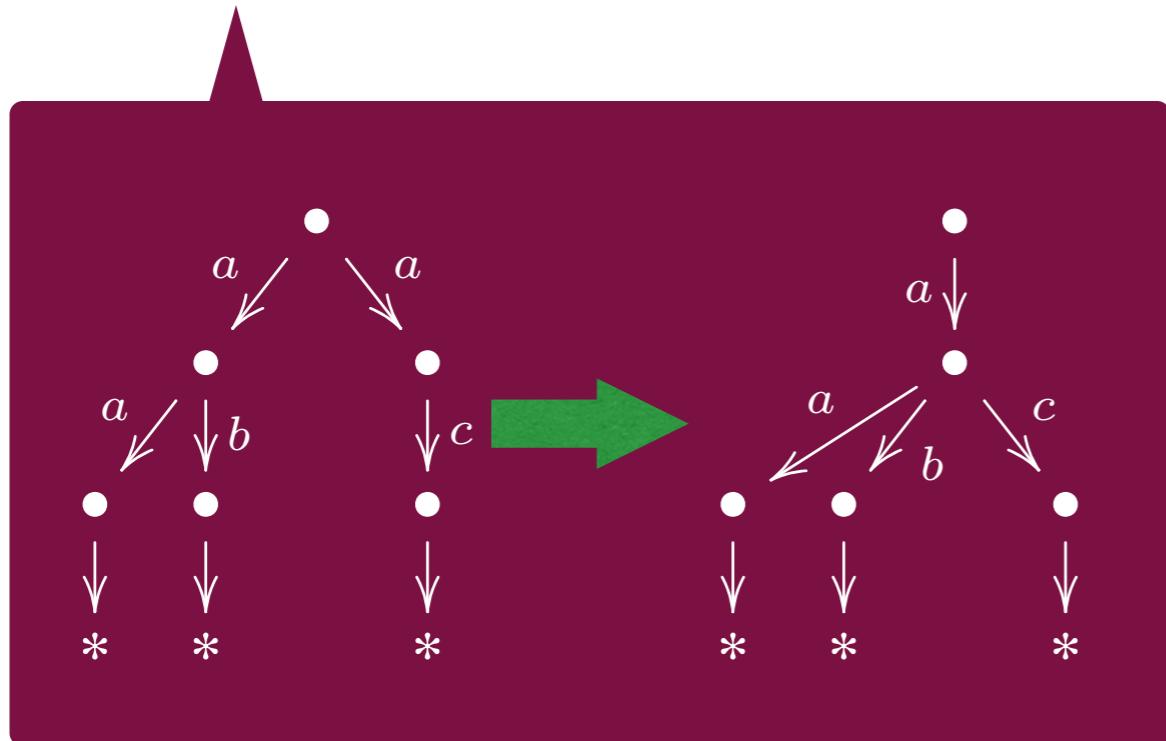


Trace semantics coalgebraically?

NFA / LTS

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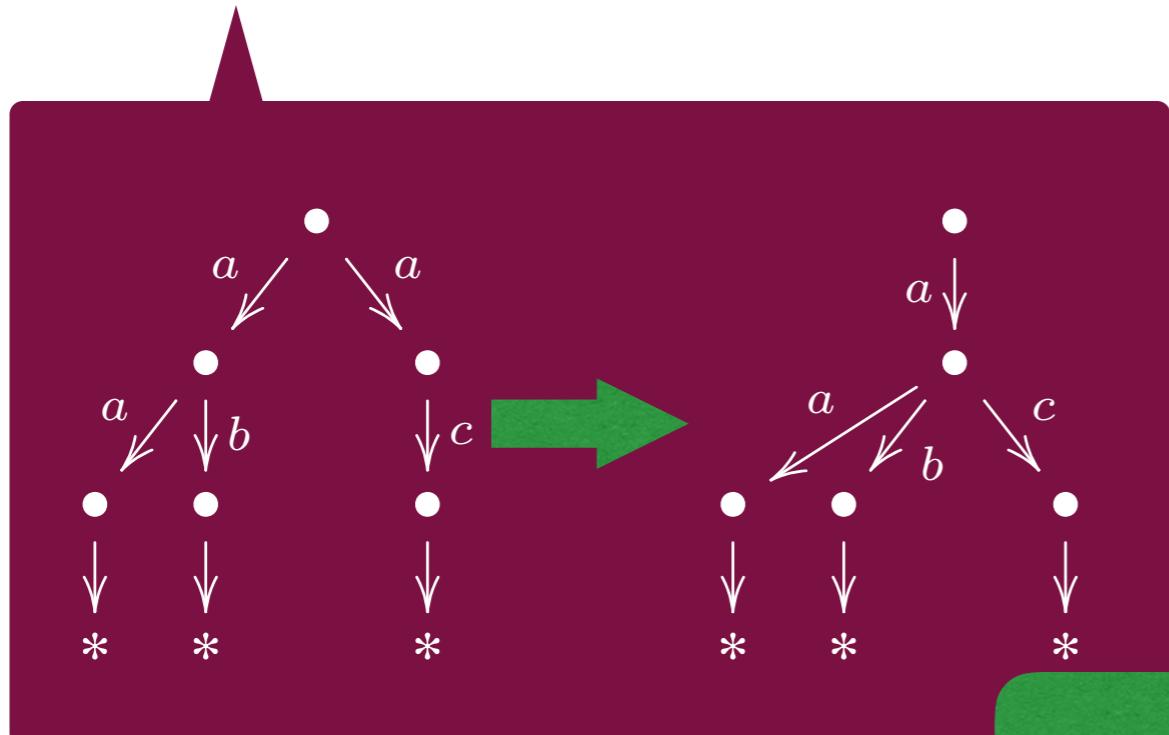
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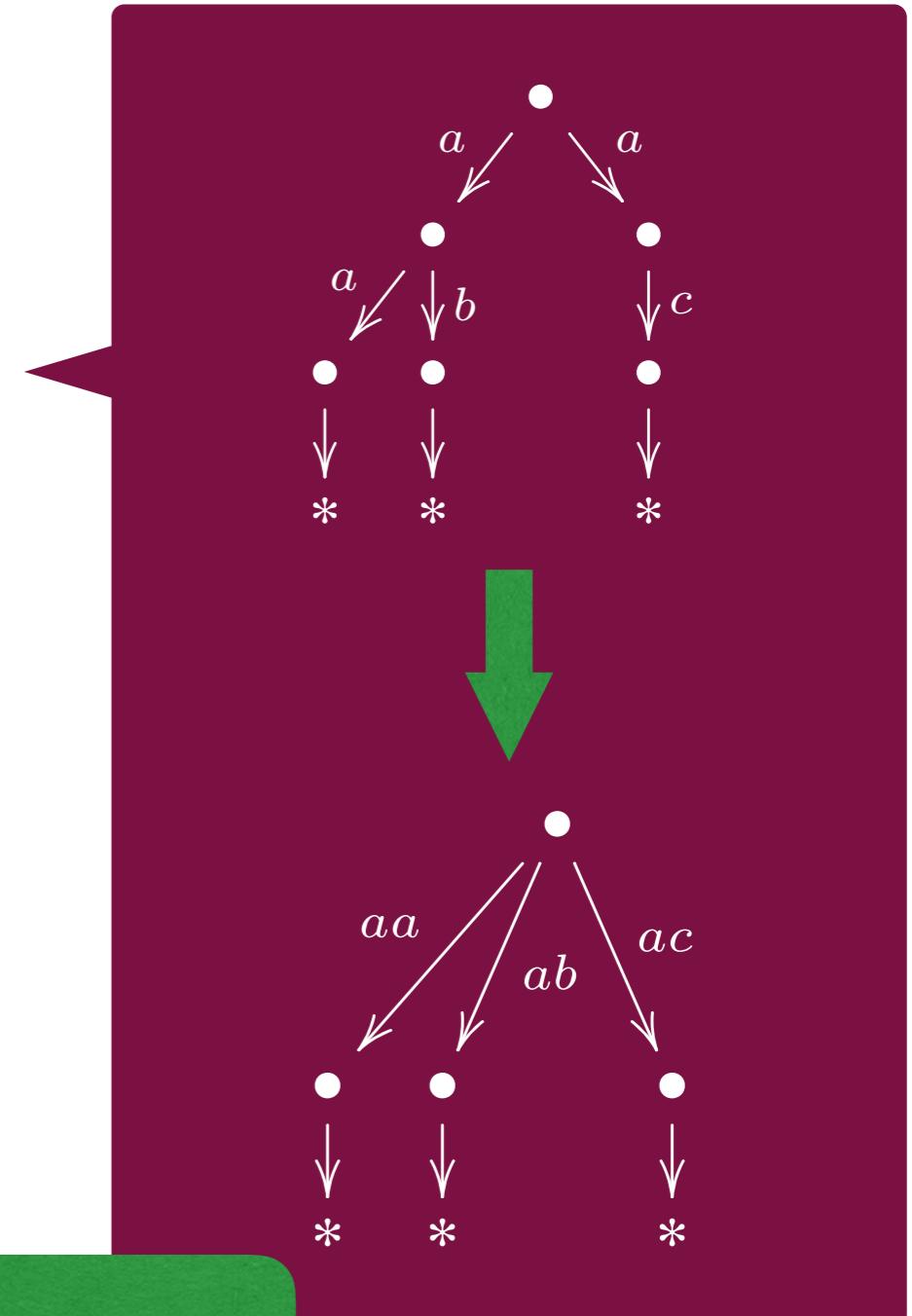
Trace semantics coalgebraically?

Two ideas:

- (1) unfold branching + transitions on words
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monads !



Trace semantics coalgebraically

Trace semantics coalgebraically

Two approaches:

- (1) modelling in a Kleisli category
- (2) modelling in an Eilenberg-Moore category

Trace semantics coalgebraically

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algebras of a monad M

Trace semantics coalgebraically

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Hasuo,
Jacobs, S.
LMCS '07

algebras of a monad M

Trace semantics coalgebraically

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Bonsangue, Rutten
FSTTCS'10

algebras of a monad M

Trace semantics coalgebraically

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algebras of a monad M

(1) and (2) are related

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Trace semantics coalgebraically

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algebras of a monad M

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Jacobs, Silva, S.
JCSS'15

Traces via determinisation

Traces via determinisation

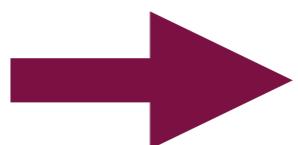
Automaton with M-effects

$$X \rightarrow O \times (MX)^A$$

Traces via determinisation

Automaton with M-effects

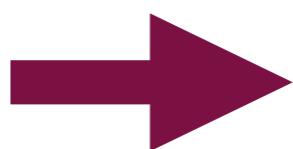
$$X \rightarrow O \times (MX)^A$$



Traces via determinisation

Automaton with M-effects

$$X \rightarrow O \times (MX)^A$$



Determinisation

$$MX \rightarrow O \times (MX)^A$$

Traces via determinisation

Automaton with M-effects

$$X \rightarrow O \times (MX)^A$$

Determinisation

$$MX \rightarrow O \times (MX)^A$$

trace = bisimilarity after
determinisation

Traces via determinisation

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Algebras for M

Traces via determinisation

Automaton with M-effects

$$X \rightarrow O \times (MX)^A$$

Determinisation

$$MX \rightarrow O \times (MX)^A$$

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Algebras for M

ideally
we have a
presentation

Traces via determinisation

Automaton with M-effects

$$X \rightarrow O \times (MX)^A$$

O has to
be an
M-algebra !

trace = bisimilarity after
determinisation

Determinisation

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Algebras for M

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presentation

Traces via determinisation

Automaton with M-effects

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trace = bisimilarity after
determinisation

Determinisation

$$MX \rightarrow O \times (MX)^A$$

MX carries
the free M-
algebra

Algebras for M

ideally
we have a
presentation

Traces via determinisation

Automaton with M-effects

$$X \rightarrow O \times (MX)^A$$

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determinisation

Determinisation

$$MX \rightarrow O \times (MX)^A$$

MX carries
the free M-
algebra

Algebras for M

ideally
we have a
presentation

Eilenberg-Moore algebras

Traces via determinisation

Automaton with M-effects

$$X \rightarrow O \times (MX)^A$$

O has to
be an
M-algebra !

trace = bisimilarity after
determinisation

$$\text{tr}: X \rightarrow O^{A^*}$$

Determinisation

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$$\text{tr}: X \rightarrow O^{A^*}$$

$$\text{tr}(x)(a_1 a_2 \dots a_n) = o \iff x \xrightarrow{a_1} t_1 \xrightarrow{a_2} t_2 \dots t_{n-1} \xrightarrow{a_n} t_n \wedge t_n \downarrow o$$

Determinisation

$$MX \rightarrow O \times (MX)^A$$

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ideally
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presentation

Eilenberg-Moore algebras



Eilenberg-Moore Algebras

abstractly

$\mathcal{EM}(M)$

- objects

$$\boxed{\begin{array}{c} MA \\ \downarrow a \\ A \end{array}}$$

satisfying

$$A \xrightarrow{\eta} MA \quad \begin{array}{c} \cong \\ a \end{array} \quad \downarrow a \quad A$$

$$MMA \xrightarrow{\mu} MA \quad Ma \downarrow \quad \downarrow a \\ MA \xrightarrow{a} A$$

- morphisms

$$\boxed{\begin{array}{c} MA \\ \downarrow a \\ A \end{array}} \xrightarrow{h} \boxed{\begin{array}{c} MB \\ \downarrow b \\ B \end{array}}$$

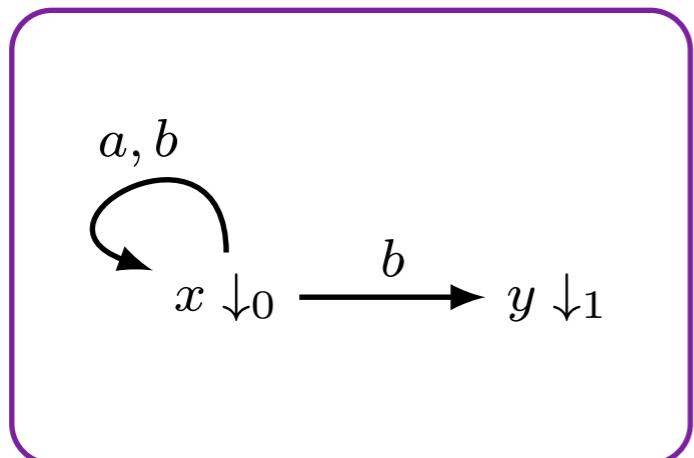
$$\boxed{\begin{array}{ccc} MA & \xrightarrow{Mh} & MB \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{h} & B \end{array}}$$

Traces via determinisation

Traces via determinisation

NFA

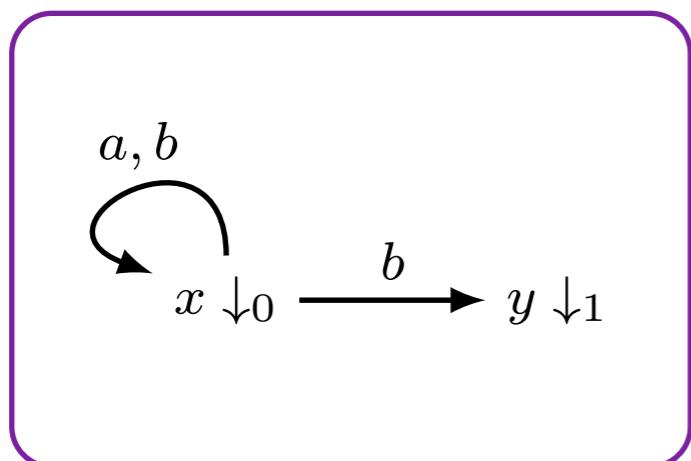
$$X \rightarrow 2 \times (\mathcal{P}X)^A$$



Traces via determinisation

NFA

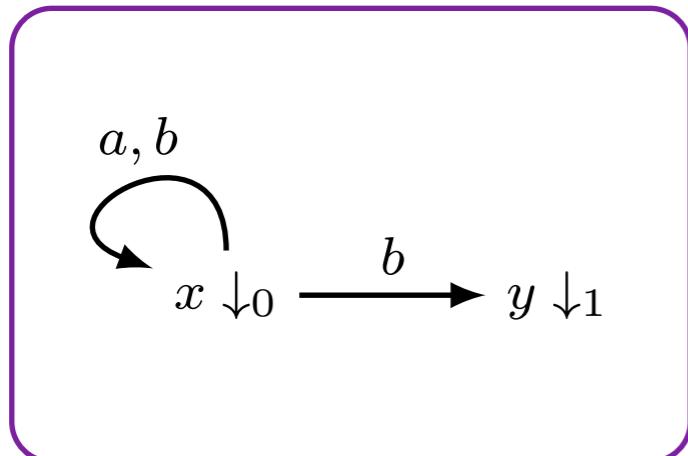
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Traces via determinisation

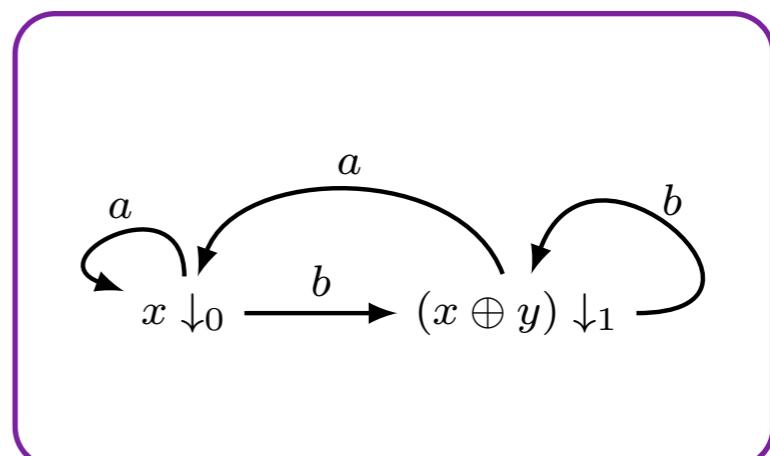
NFA

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DFA

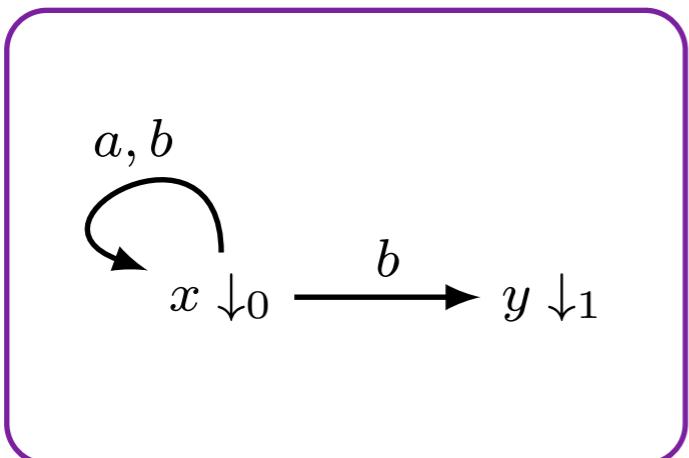
$$\mathcal{P}X \rightarrow 2 \times (\mathcal{P}X)^A$$



Traces via determinisation

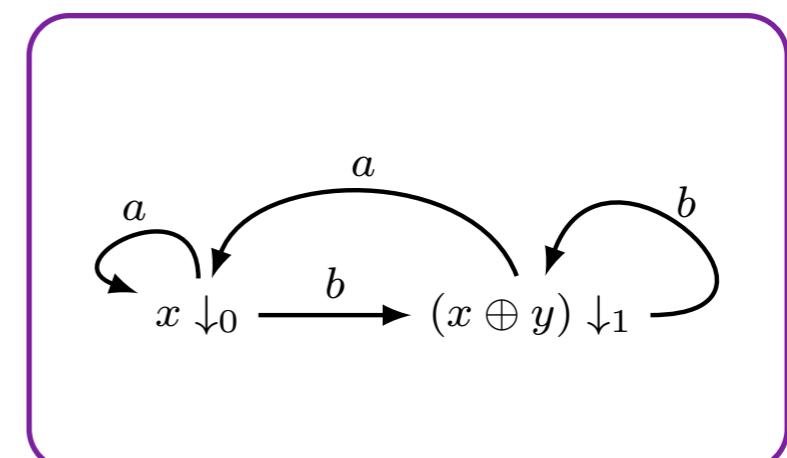
NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$



DFA

$$\mathcal{P}X \rightarrow 2 \times (\mathcal{P}X)^A$$



$$x \xrightarrow{a} t_x, y \xrightarrow{a} t_y$$

$$x \oplus y \xrightarrow{a} t_x \oplus t_y$$

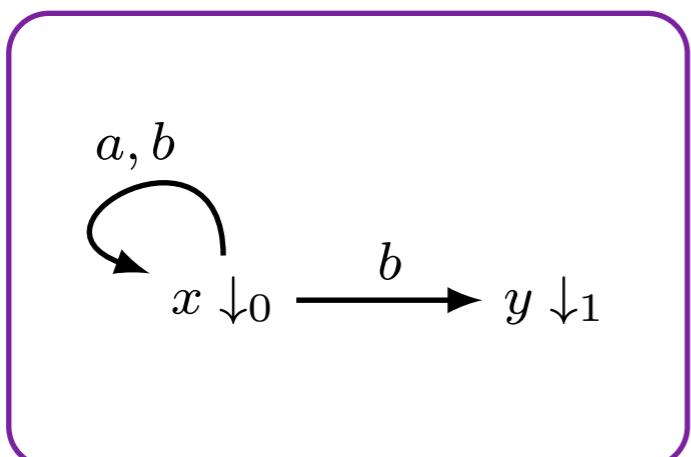
$$x \downarrow o_x, y \downarrow o_y$$

$$x \oplus y \downarrow o_x \oplus o_y$$

Traces via determinisation

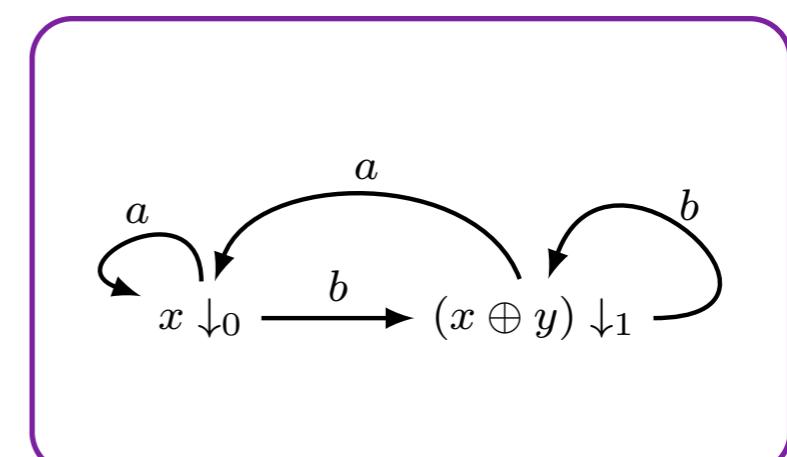
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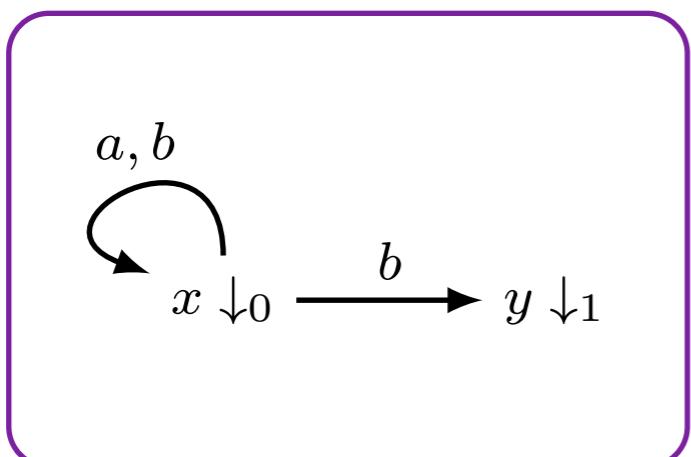
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Algebras for \mathcal{P}

Traces via determinisation

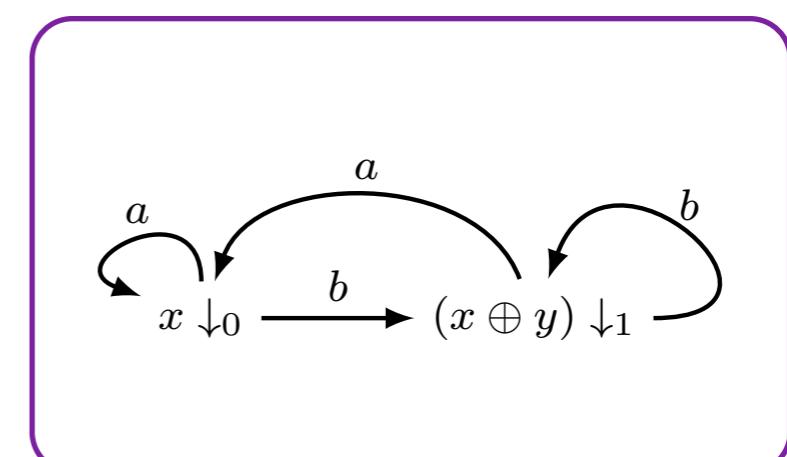
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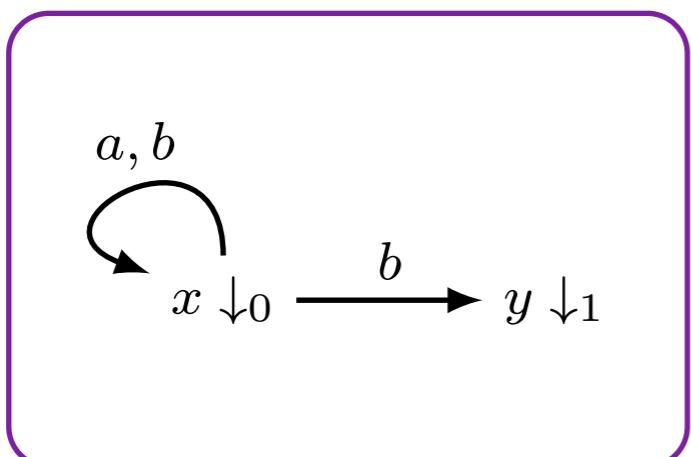
Algebras for \mathcal{P}

join
semilattices
with bottom

Traces via determinisation

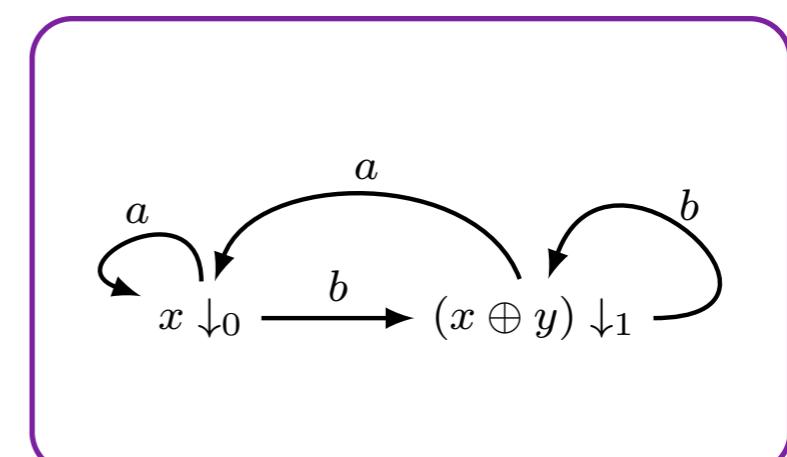
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finite powerset !

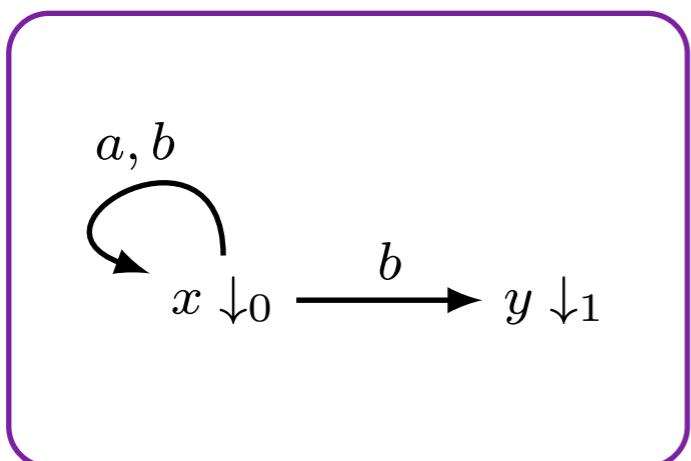
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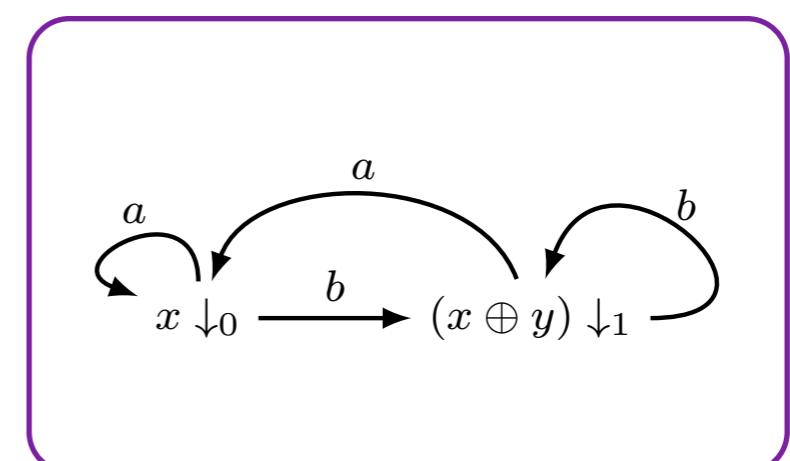
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Algebras for \mathcal{P}

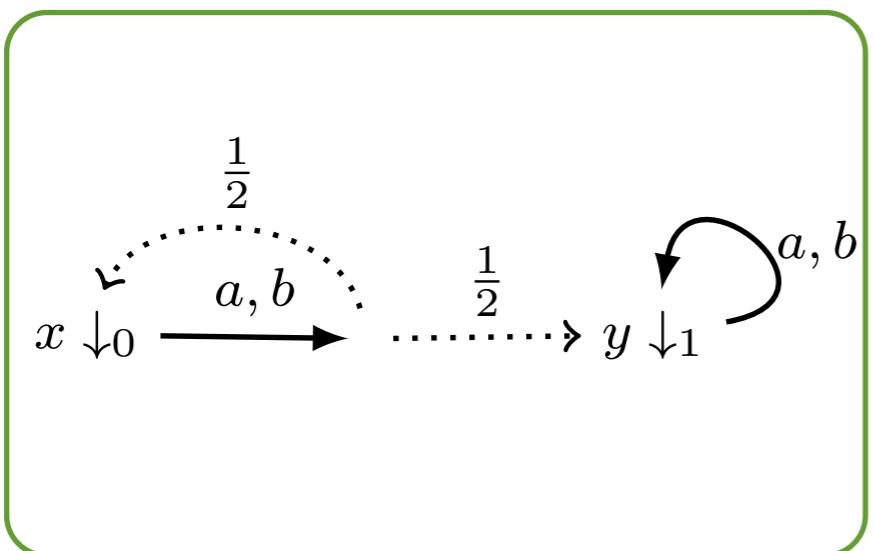
join
semilattices
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$2 = \mathcal{P}1$

Traces via determinisation

Rabin PA

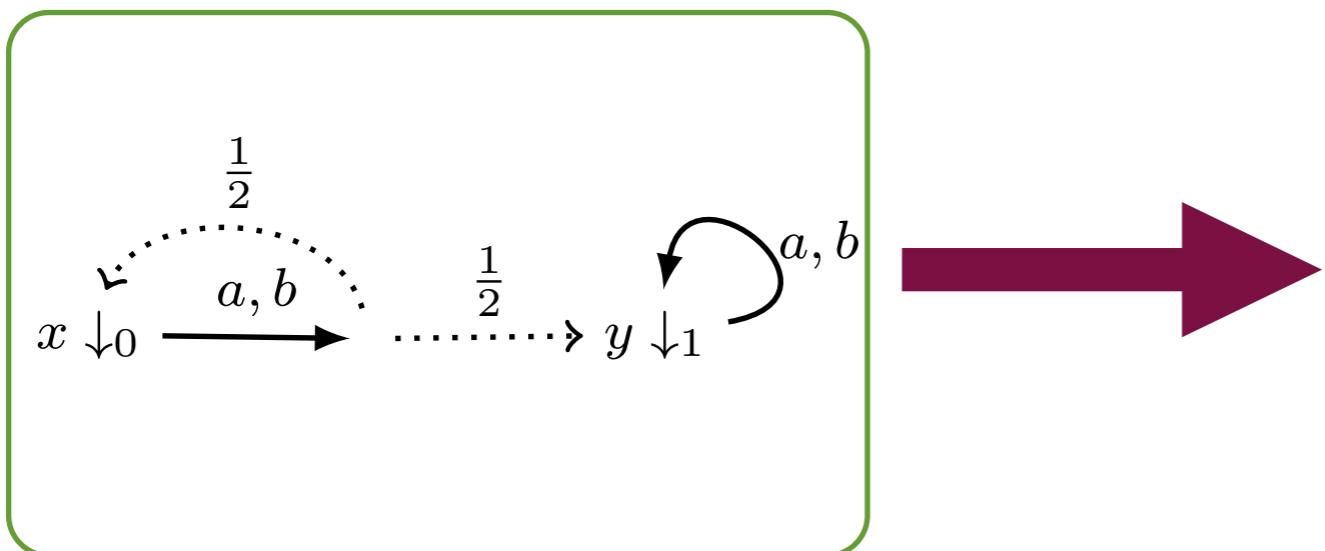
$$X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1} X)^A$$



Traces via determinisation

Rabin PA

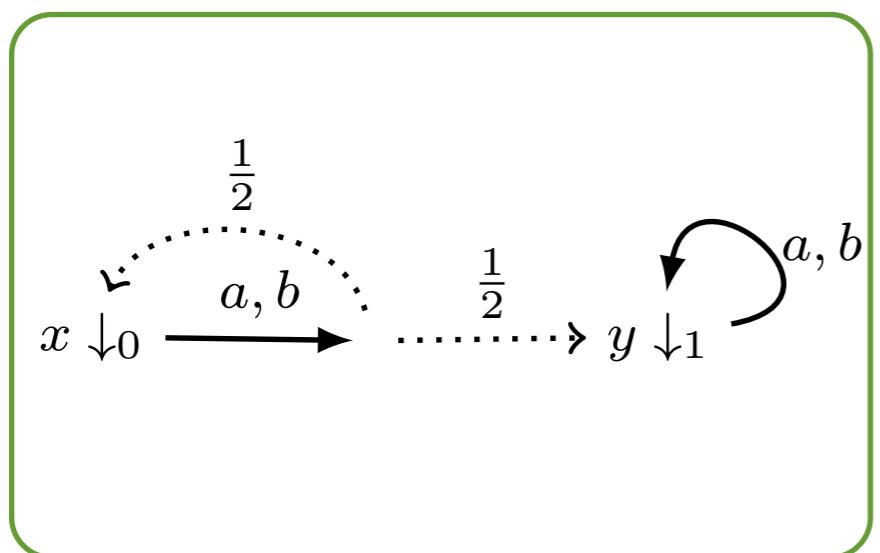
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Traces via determinisation

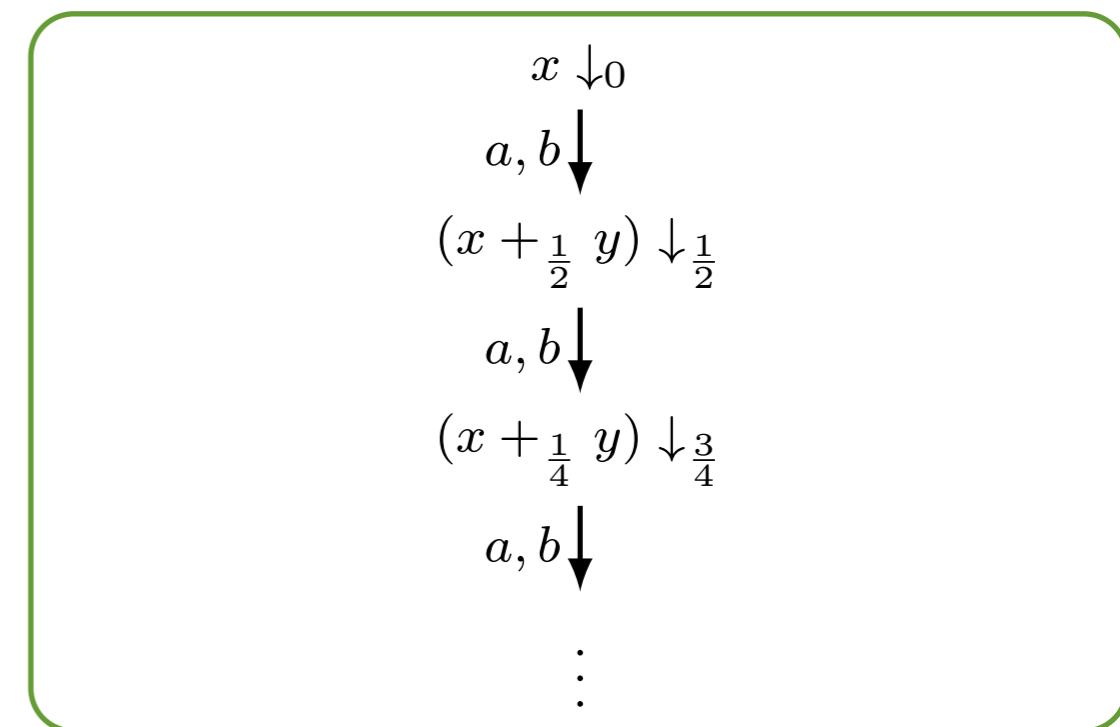
Rabin PA

$$X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1} X)^A$$



DPA

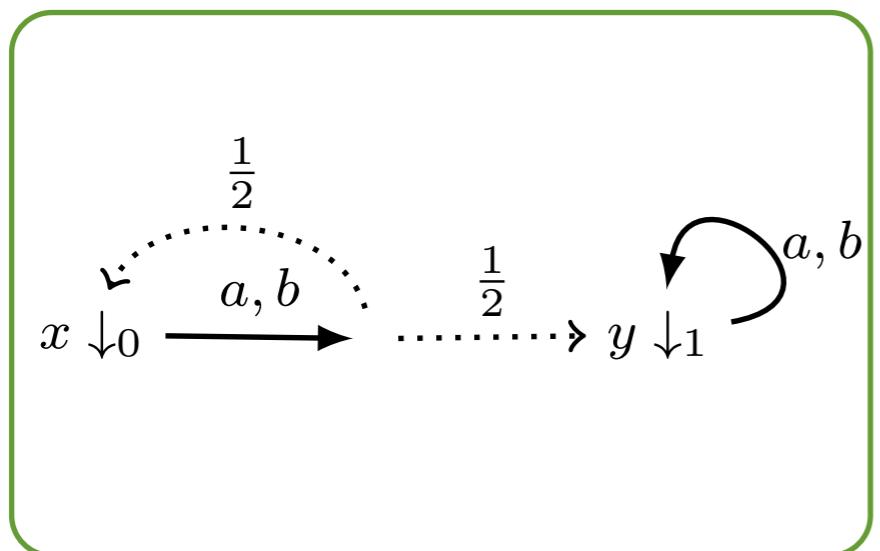
$$\mathcal{D}_{\leq 1} X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1} X)^A$$



Traces via determinisation

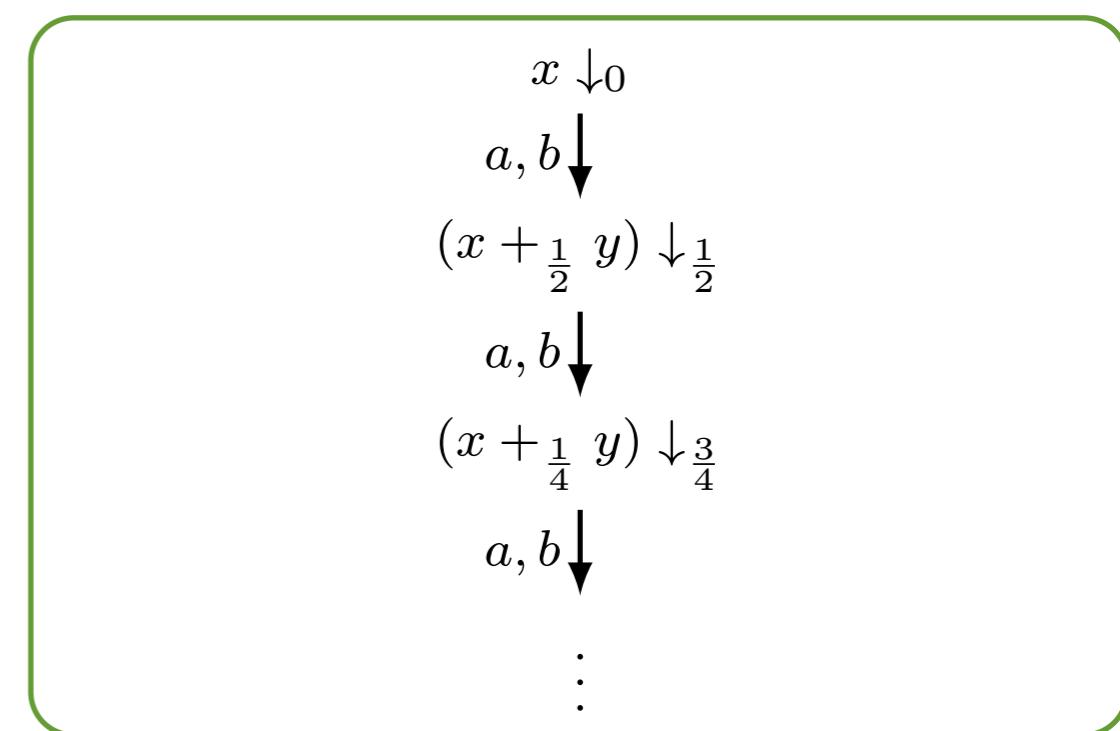
Rabin PA

$$X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1} X)^A$$



DPA

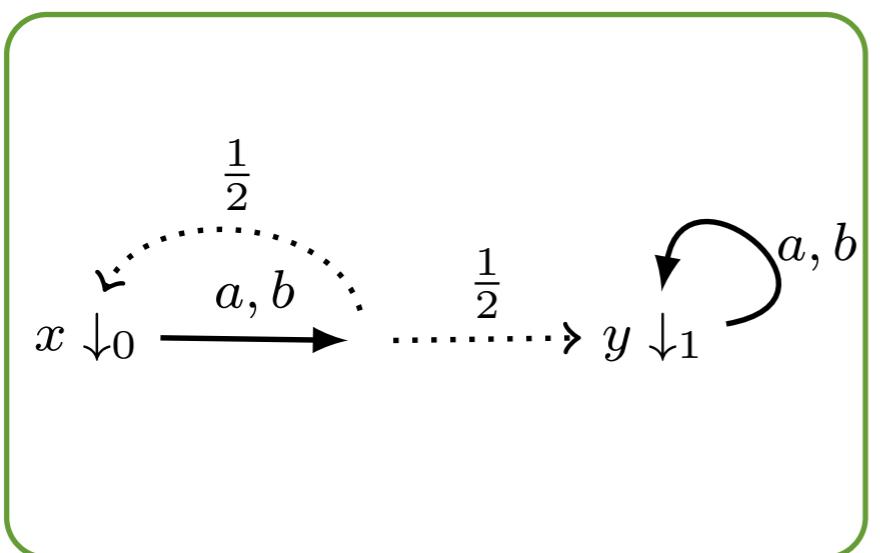
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Traces via determinisation

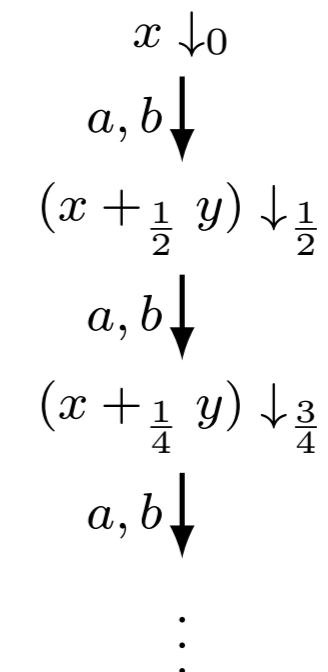
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DPA

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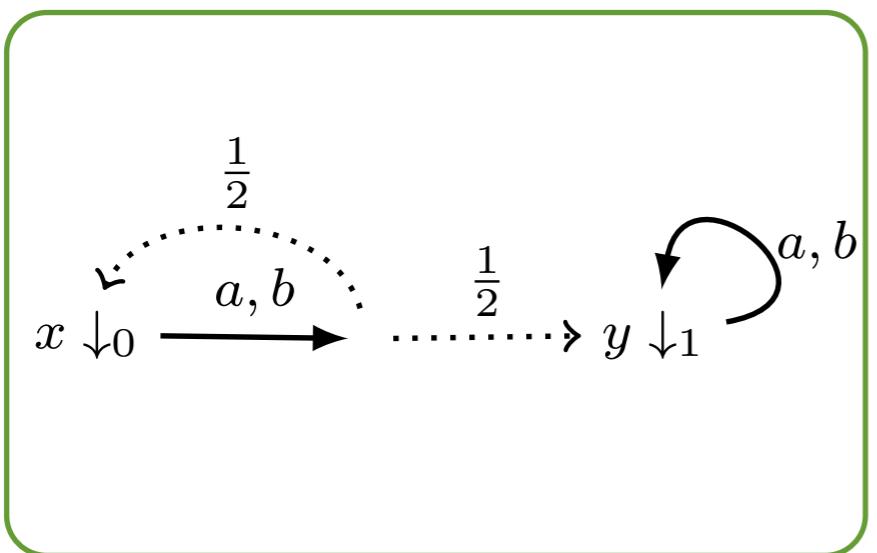
Algebras for $\mathcal{D}_{\leq 1}$

positive
convex
algebras

Traces via determinisation

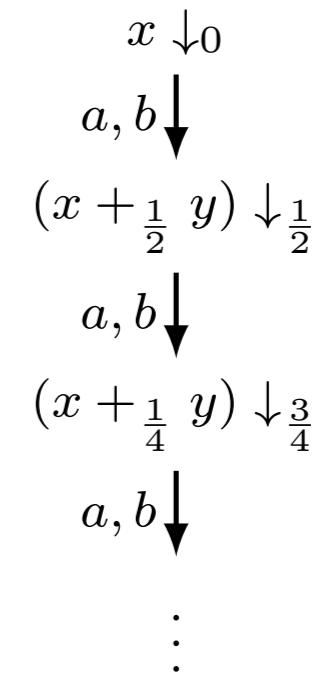
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Algebras for $\mathcal{D}_{\leq 1}$

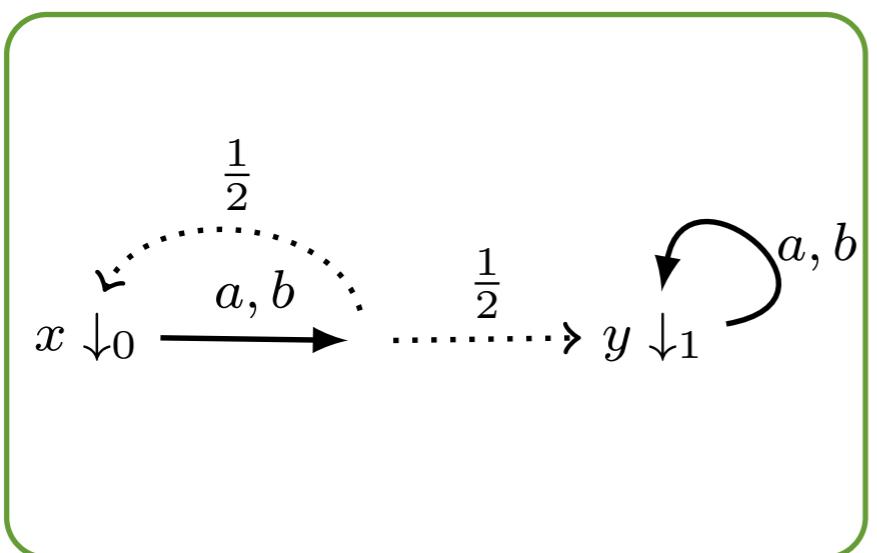
positive
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algebras

finitely supported
subdistributions!

Traces via determinisation

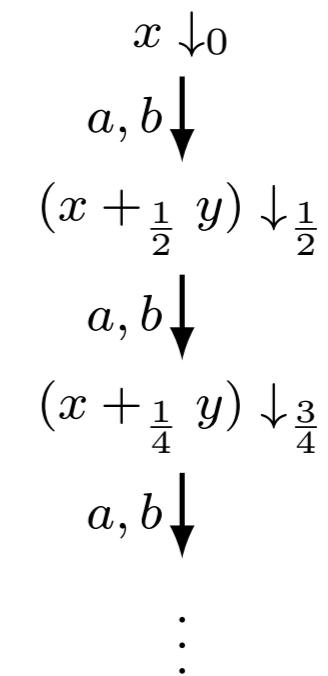
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Algebras for $\mathcal{D}_{\leq 1}$

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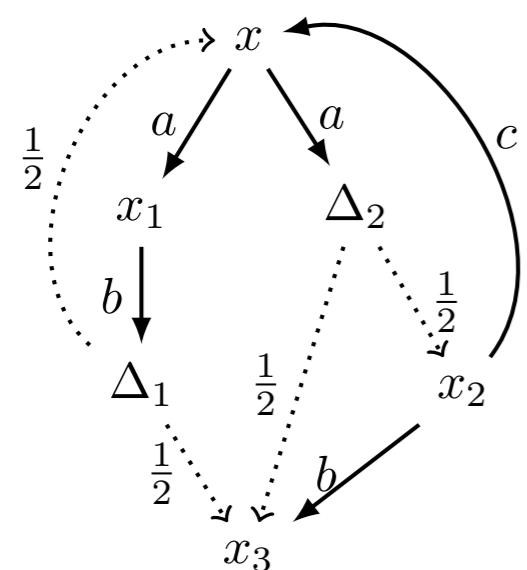
finitely supported
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$[0, 1] = \mathcal{D}_{\leq 1} 1$

Traces via determinisation

Simple NPA

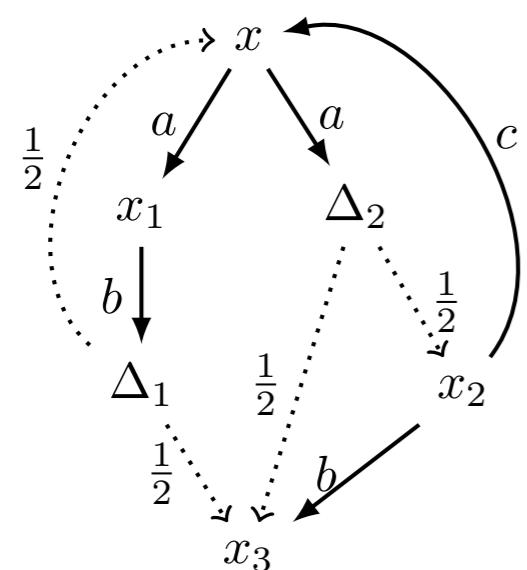
$$X \rightarrow ? \times (\mathcal{C}X)^A$$



Traces via determinisation

Simple NPA

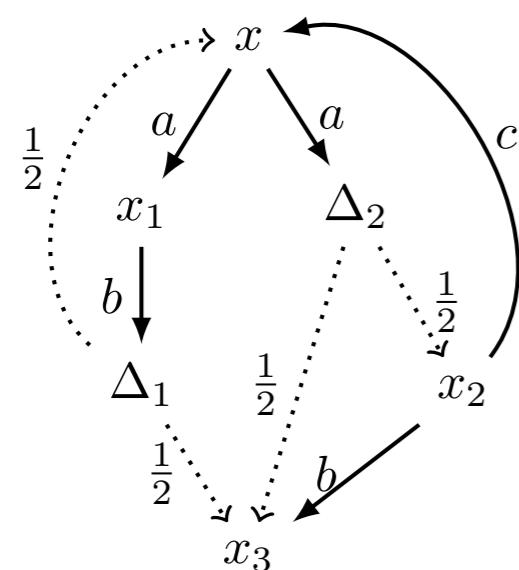
$$X \rightarrow ? \times (\mathcal{C}X)^A$$



Traces via determinisation

Simple NPA

$$X \rightarrow ? \times (\mathcal{C}X)^A$$



DNPA

$$\mathcal{C}X \rightarrow ? \times (\mathcal{C}X)^A$$

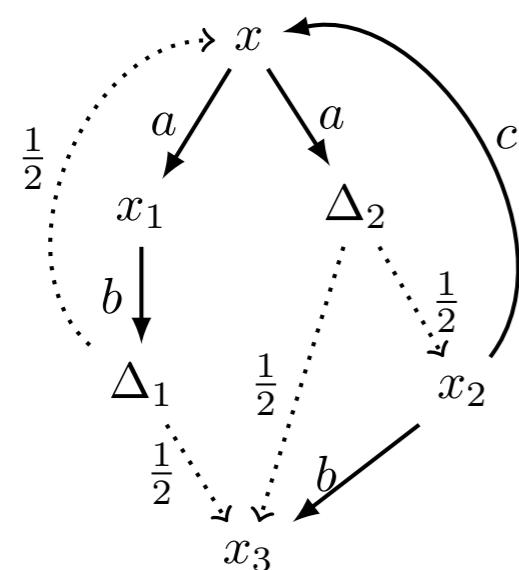
$$\begin{array}{c} x_1 \\ \downarrow a \\ x_1 \oplus (x_3 + \frac{1}{2} x_2) \end{array}$$



Traces via determinisation

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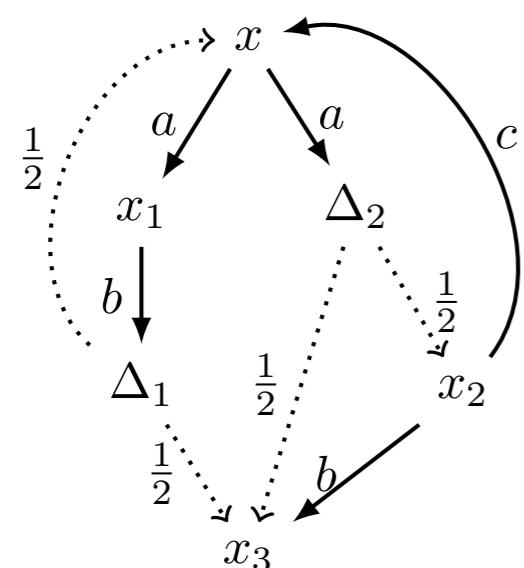
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Algebras for C

Traces via determinisation

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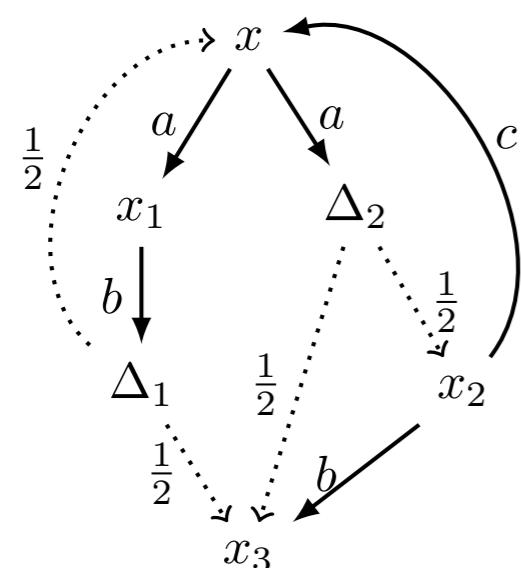
Algebras for C

convex
semilattices

Traces via determinisation

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DNPA

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Algebras for C

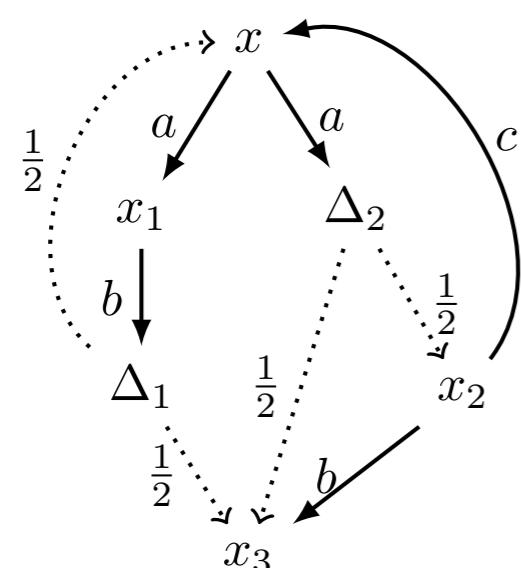
convex
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finitely generated
convex sets of distr...

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Algebras for C

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Presentation for ℓ

Algebras for ℓ

finitely generated
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convex
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Presentation for ℓ

Algebras for ℓ

finitely generated
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convex
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Vignudelli '19

Presentation for ℓ

Algebras for ℓ

finitely generated
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Bonchi, S.,
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$$\mathbb{A} = (A, \oplus, +_p)$$

Presentation for ℓ

Algebras for ℓ

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$$p \in (0, 1)$$

Presentation for ℓ

Algebras for ℓ

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$$\begin{aligned}(x \oplus y) \oplus z &\stackrel{(A)}{=} x \oplus (y \oplus z) \\ x \oplus y &\stackrel{(C)}{=} y \oplus x \\ x \oplus x &\stackrel{(I)}{=} x\end{aligned}$$

$$\begin{aligned}(x +_q y) +_p z &\stackrel{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z) \\ x +_p y &\stackrel{(C_p)}{=} y +_{1-p} x \\ x +_p x &\stackrel{(I_p)}{=} x\end{aligned}$$

$$(x \oplus y) +_p z \stackrel{(D)}{=} (x +_p z) \oplus (y +_p z)$$

Bonchi, S.,
Vignudelli '19

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convex
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Bonchi, S.,
Vignudelli '19

semilattice

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Bonchi, S.,
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semilattice

convex
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Bonchi, S.,
Vignudelli '19

semilattice

S., Woracek
'15, '17, '18

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Bonchi, S.,
Vignudelli '19

semilattice

S., Woracek
'15, '17, '18

convex
algebra

distributivity

Three variants for “*e*”

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Three variants for “ e ”

Bonchi, S.,
Vignudelli ‘19

We explore the whole space
and
prove coincidence with “local”
trace semantics

Three variants for “ \mathcal{C} ”

Algebras for “ \mathcal{C} ”

nonempty f.g.
convex subsets of
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Bonchi, S.,
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Three variants for “ \mathcal{C} ”

Algebras for “ \mathcal{C} ”

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l.pointed
convex
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Intervals in $[0,1]$
with
min-max
Minkowski
 $[0,0] = “\mathcal{C}”^1$

Bonchi, S.,
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II.
with bottom

Bonchi, S.,
Vignudelli ‘19

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$[0,1]$
with
max, $+_p$
 $0 = “\mathcal{C}”^1$

Three variants for “ ℓ ”

Algebras for “ ℓ ”

nonempty f.g.
convex subsets of
subdistr...

I. pointed
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semilattices

Intervals in $[0,1]$
with
min-max
Minkowski
 $[0,0] = \ell^1$

II.
with bottom

$[0,1]$
with
max, $+_p$
 $0 = \ell^1$

III.
with top

$[0,1]$
with
min, $+_p$
 $0 = \ell^1$

Bonchi, S.,
Vignudelli '19

We explore the whole space
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prove coincidence with “local”
trace semantics

Three things to take home:

- 1.** Semantics via determinisation
is easy for automata with M-effects
- 2.** Having a presentation for M gives us syntax
- 3.** Having the syntax makes determinisation natural !

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follow
also a sound
up-to context
proof technique

combining
nondeterminism
and probability
becomes easy

Thank You !