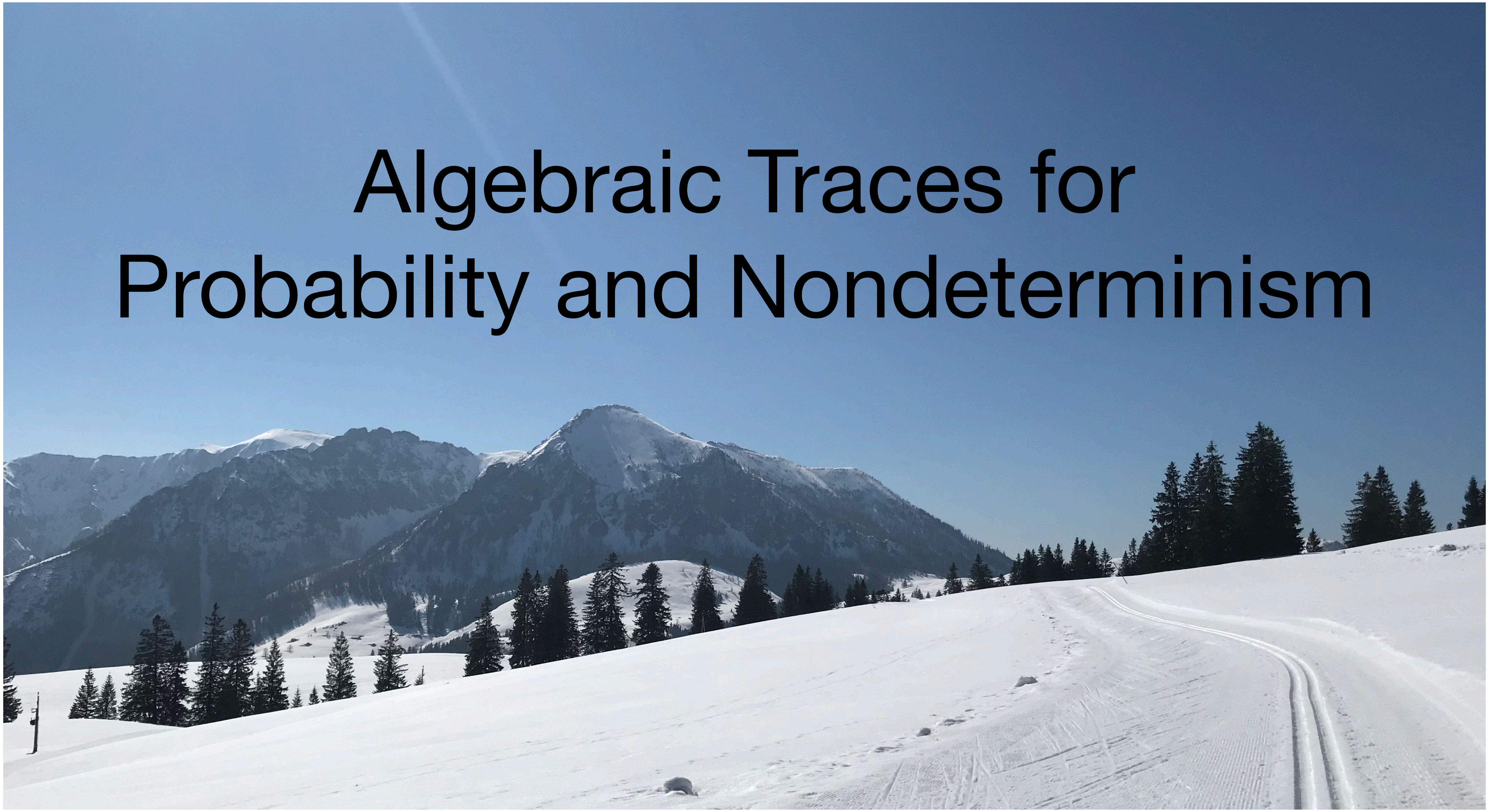
GI Meeting Deduction and Logic - 26.3.21

Algebraic Traces for Probability and Nondeterminism



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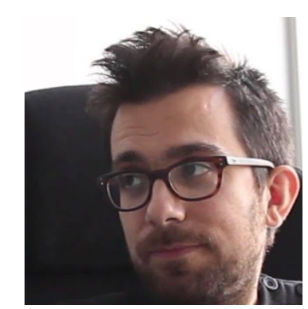

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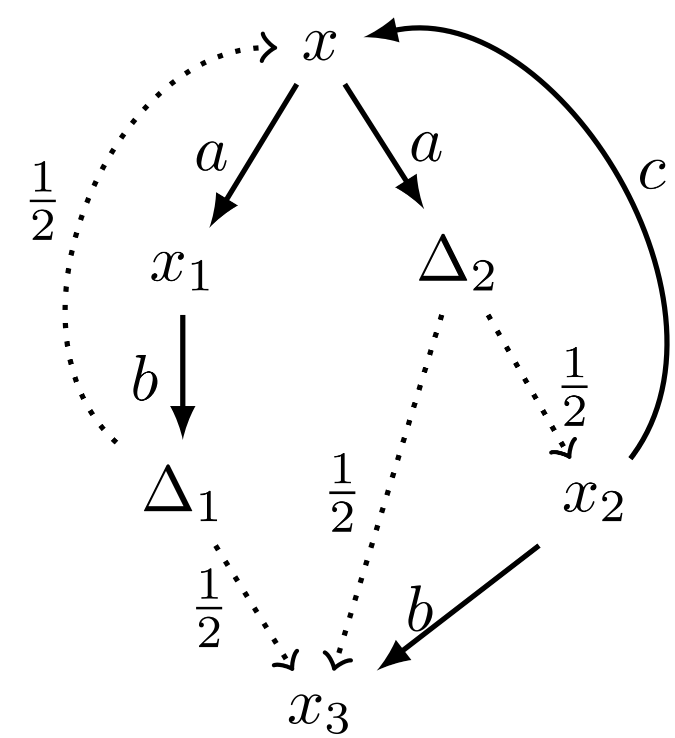
Probabilistic Nondeterministic Labeled Transition Systems

$$t\colon X \to (\mathcal{P}\mathcal{D}X)^A$$

Trace Semantics for these systems is usually defined by means of schedulers and resolutions

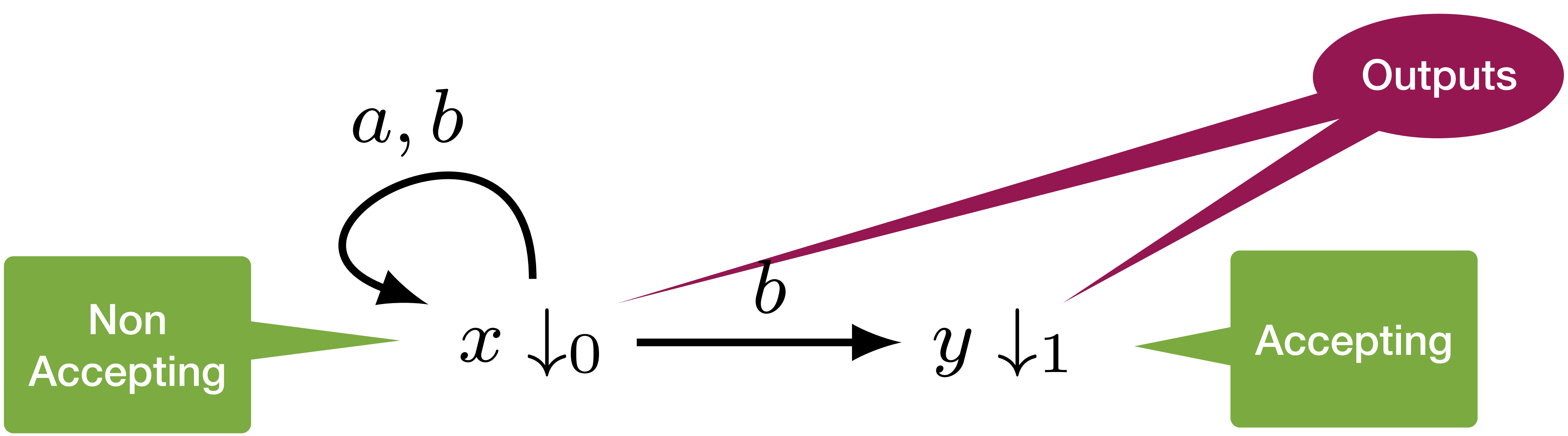


We take a different view: our semantics is based on automata theory, algebra and coalgebra

WARNING: In this talk, we will present our theory in its simplest possible form, hiding all category theory

Nondeterministic Automata

$$\langle o, t \rangle : X \to 2 \times (\mathcal{P}X)^A$$

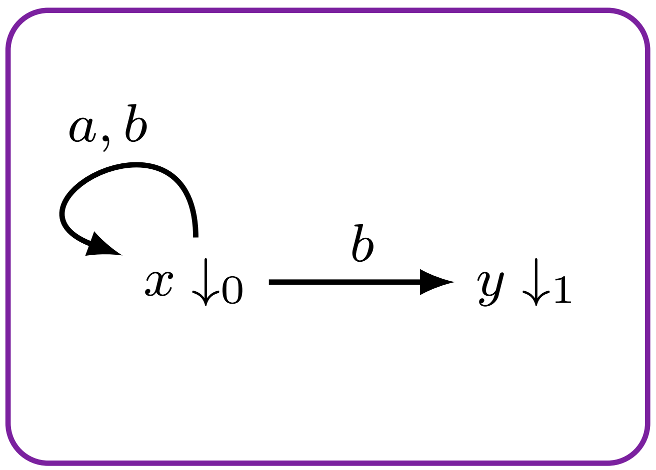


$$X = \{x, y\} \qquad A = \{a, b\}$$

Language Semantics

NFA = LTS + output

$X \to 2 \times (\mathcal{P}X)^A$

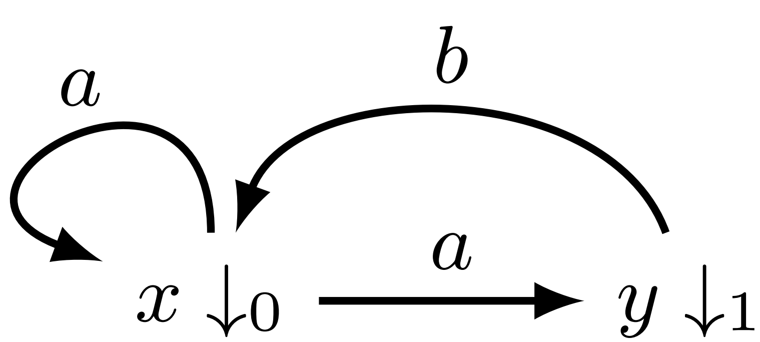


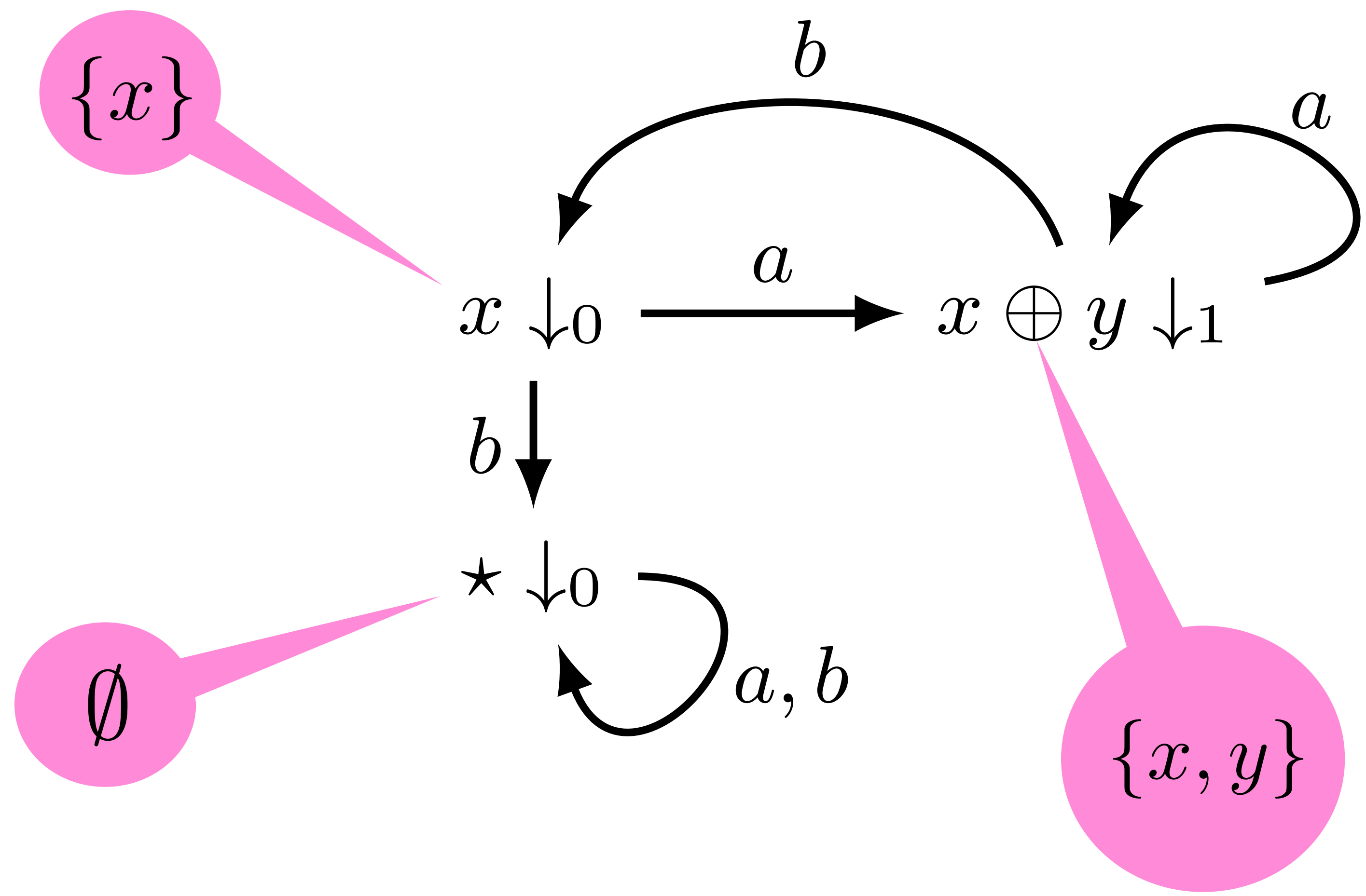
$$[\![\cdot]\!] \colon X \to 2^{A^*}$$

$$[\![x]\!] = (a \cup b)^* b = \{w \in \{a,b\}^* \mid w \text{ ends with a } b\}$$

Determinisation for Nondeterministic Automata

$$\langle o,t\rangle\colon X \to 2\times(\mathcal{P}X)^A \quad\Longrightarrow\quad \langle o^\sharp,t^\sharp\rangle\colon \mathcal{P}X \to 2\times(\mathcal{P}X)^A$$



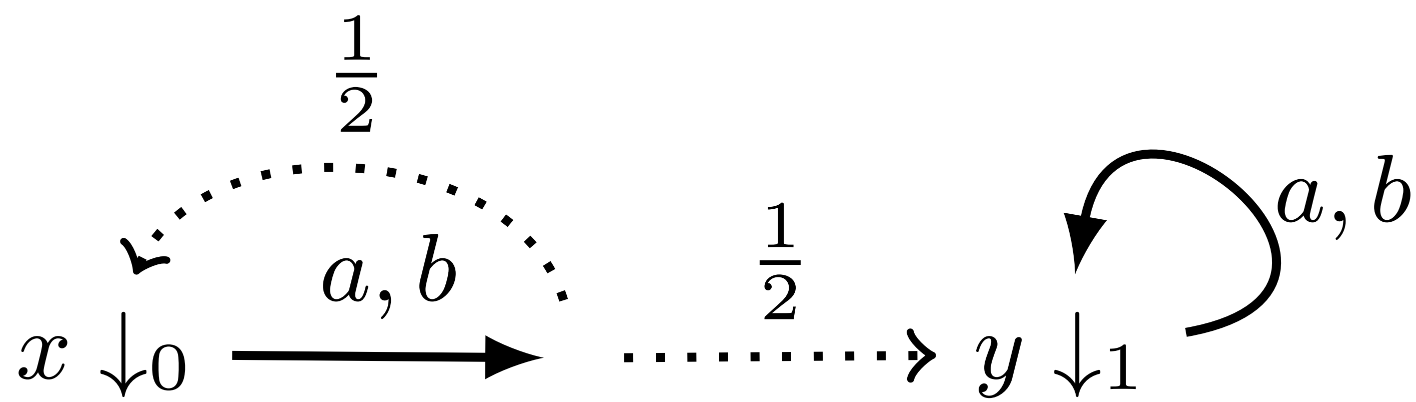


$$[\![\cdot]\!]\colon \mathcal{P}X \to 2^{A^*}$$

$$[\![S]\!](\varepsilon) = o^\sharp(S)$$
$$[\![S]\!](aw) = [\![t^\sharp(S)(a)]\!](w)$$

Probabilistic Automata

$$\langle o, t\rangle \colon X \to [0,1] \times (\mathcal{D}X)^A$$

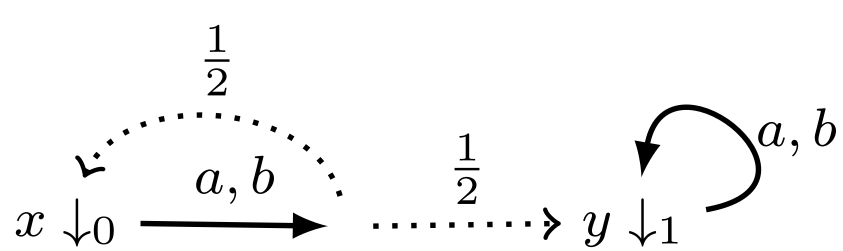


$$X = \{x, y\} \qquad A = \{a, b\}$$

Probabilistic Language Semantics

Rabin PA = PTS + output

$X \to [0,1] \times (\mathcal{D}X)^A$

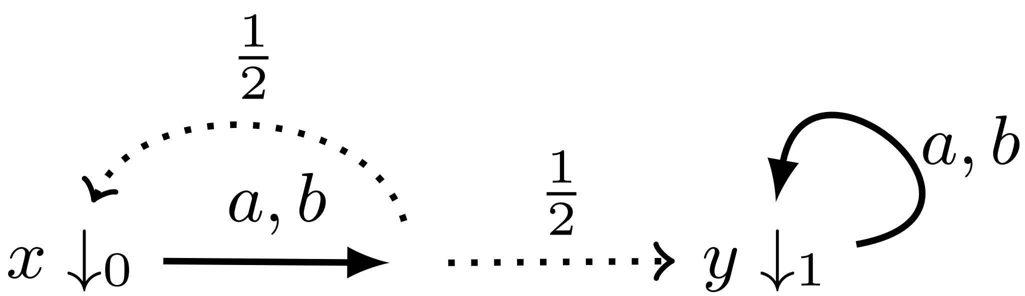


$[\![\cdot]\!]: X \to [0,1]^{A^*}$

$$[\![x]\!] = \left(a \mapsto \frac{1}{2}, aa \mapsto \frac{3}{4}, \ldots\right)$$

Determinisation for Probabilistic Automata

$$\langle o, t\rangle \colon X \to [0,1] \times (\mathcal{D}X)^A \quad \Longrightarrow \quad \langle o^\sharp, t^\sharp\rangle \colon \mathcal{D}X \to [0,1] \times (\mathcal{D}X)^A$$



$$x \downarrow_0$$

$$\Big\downarrow a, b$$

$$x +_{\frac{1}{2}} y \downarrow_{\frac{1}{2}}$$

$$\Big\downarrow a, b$$

$$x +_{\frac{1}{4}} y \downarrow_{\frac{3}{4}}$$

$$\Big\downarrow a, b$$

$$\vdots$$

δ_x

$x \mapsto \frac{1}{4}$
$y \mapsto \frac{3}{4}$

$$[\![\cdot]\!] \colon \mathcal{D}X \to [0,1]^{A^*}$$

$$[\![\Delta]\!](\varepsilon) = o^\sharp(\Delta)$$
$$[\![\Delta]\!](aw) = [\![t^\sharp(\Delta)(a)]\!](w)$$

Toward a GSOS semantics

In the determinisation of **nondeterministic** automata we use terms built of the following syntax

$$s, t \ ::= \ \star, \ s \oplus t, \ x \in X$$

to represent states in $\mathcal{P}X$

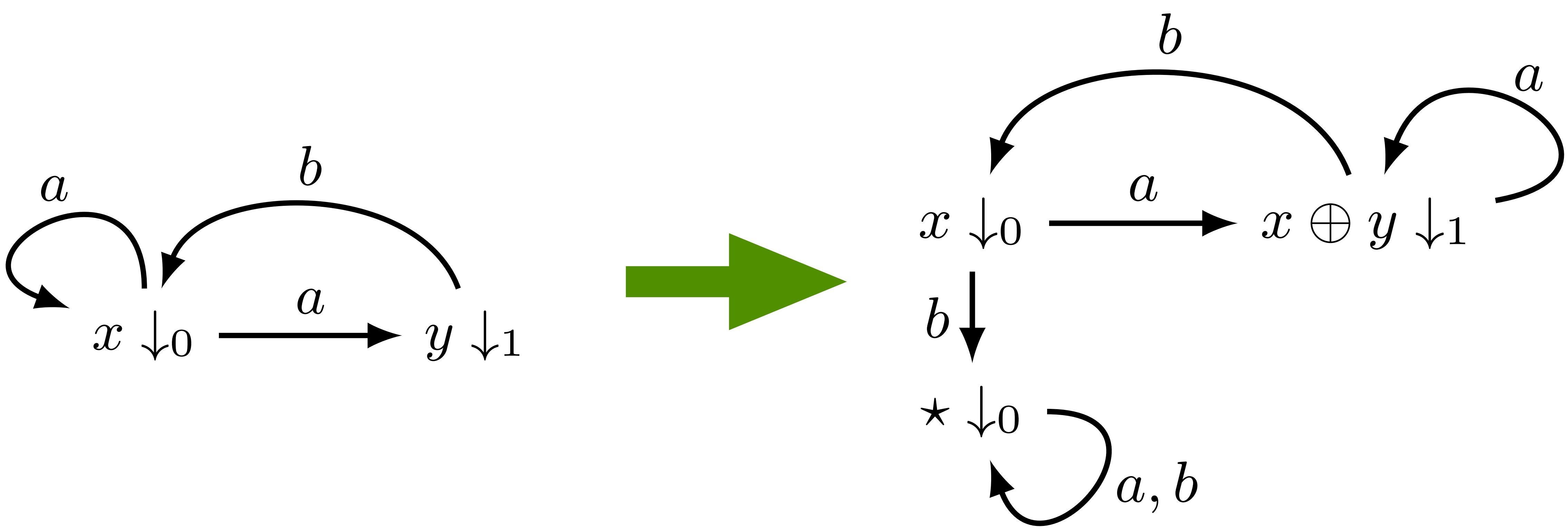
In the determinisation of **probabilistic** automata we use terms built of the following syntax

$$s, t \ ::= \ s +_p t, \ x \in X \quad \text{for all } p \in [0,1]$$

to represent elements of $\mathcal{D}X$

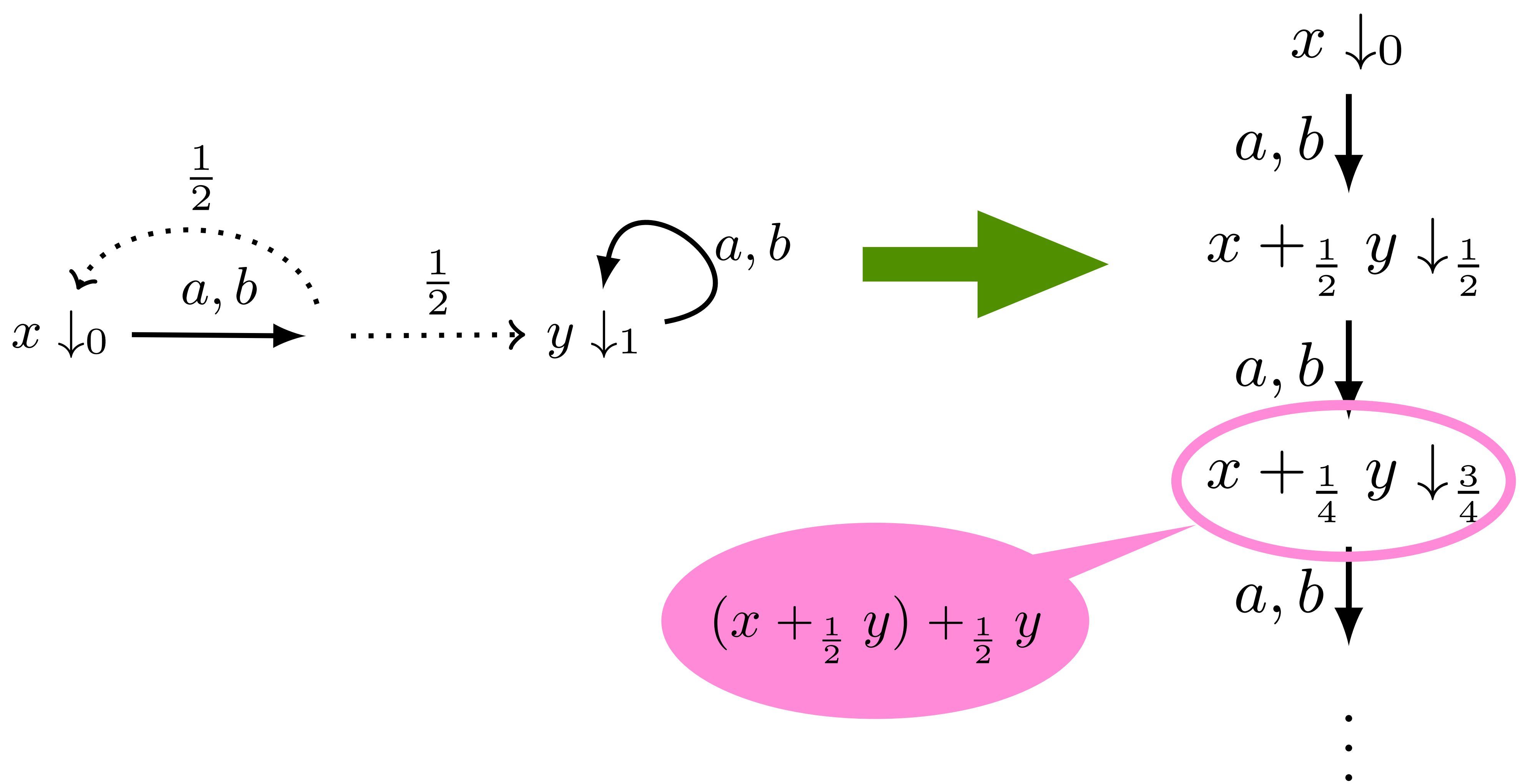
GSOS Semantics for Nondeterministic Automata

$$\frac{-}{\star \xrightarrow{a} \star} \qquad \frac{s \xrightarrow{a} s' \quad t \xrightarrow{a} t'}{s \oplus t \xrightarrow{a} s' \oplus t'} \qquad \frac{-}{\star \downarrow_0} \qquad \frac{s \downarrow_{b_1} \quad t \downarrow_{b_2}}{s \oplus t \downarrow_{b_1 \sqcup b_2}}$$



GSOS Semantics for Probabilistic Automata

$$\frac{s \xrightarrow{a} s' \qquad t \xrightarrow{a} t'}{s +_p t \xrightarrow{a} s' +_p t'} \qquad\qquad \frac{s \downarrow_{q_1} \qquad t \downarrow_{q_2}}{s +_p t \downarrow_{p \cdot q_1 + (1-p) \cdot q_2}}$$



The Algebraic Theory of Semilattices with Bottom

$$s, t \ ::= \ \star, \ s \oplus t, \ x \in X$$

$$\begin{aligned}
(x \oplus y) \oplus z \ &\overset{(A)}{=} \ x \oplus (y \oplus z) \\
x \oplus y \ &\overset{(C)}{=} \ y \oplus x \\
x \oplus x \ &\overset{(I)}{=} \ x \\
x \oplus \star \ &\overset{(B)}{=} \ x
\end{aligned}$$

The set of terms quotiented by these axioms is isomorphic to $\mathcal{P}X$

this theory is a presentation for the powerset monad

The Algebraic Theory of Convex Algebras

$$s, t \ ::= \ s +_p t, \ x \in X \quad \text{for all } p \in [0,1]$$

$$\begin{array}{ccc}
(x +_q y) +_p z & \stackrel{(A_p)}{=} & x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z) \\
x +_p y & \stackrel{(C_p)}{=} & y +_{1-p} x \\
x +_p x & \stackrel{(I_p)}{=} & x
\end{array}$$

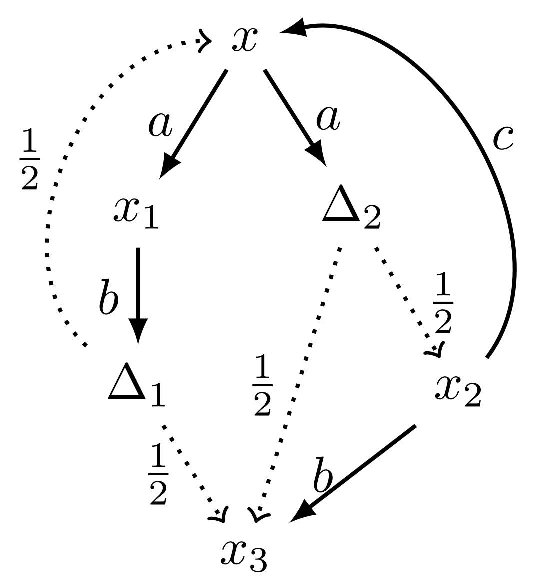
The set of terms quotiented by these axioms is isomorphic to $\mathcal{D}X$

this theory is a presentation for the distribution monad

Probabilistic Nondeterministic Language Semantics ?

NPA

$X \to ? \times (\mathcal{P}\mathcal{D}X)^A$



$$[\![x]\!] = ???$$

$$[\![\cdot]\!] : X \to ?^{A^*}$$

Algebraic Theory for Subsets of Distributions ?

- For our approach it would be convenient to have a theory presenting subsets of distributions

- Monads can be composed by means of distributive laws, but, unfortunately, there is no distributive law between powerset and distributions (Daniele Varacca Ph.D thesis)

convexity is the key !

- Other general approach to compose monads/algebraic theories fail

- Our first step is to decompose the powerset monad...

Three Algebraic Theories

Nondeterminism $\bigoplus$

$$(x \oplus y) \oplus z \overset{(A)}{=} x \oplus (y \oplus z)$$
$$x \oplus y \overset{(C)}{=} y \oplus x$$
$$x \oplus x \overset{(I)}{=} x$$

Monad: $\mathcal{P}_{ne}$

Algebras: **Semilattices**

Probability $+_p$

$$(x +_q y) +_p z \overset{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$$
$$x +_p y \overset{(C_p)}{=} y +_{1-p} x$$
$$x +_p x \overset{(I_p)}{=} x$$

Monad: $\mathcal{D}$

Algebras: **Convex Algebras**

Termination $\star$

no axioms

Monad: $\cdot + 1$

Algebras: **Pointed Sets**

The Algebraic Theory of Convex Semilattices

$$\oplus \quad +_p$$

$$(x \oplus y) \oplus z \stackrel{(A)}{=} x \oplus (y \oplus z) \qquad (x +_q y) +_p z \stackrel{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$$

$$x \oplus y \stackrel{(C)}{=} y \oplus x \qquad x +_p y \stackrel{(C_p)}{=} y +_{1-p} x$$

$$x \oplus x \stackrel{(I)}{=} x \qquad x +_p x \stackrel{(I_p)}{=} x$$

$$(x \oplus y) +_p z \stackrel{(D)}{=} (x +_p z) \oplus (y +_p z)$$

Monad C: non-empty convex subsets of distributions

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convexity comes from the following derived law

$$s \oplus t \stackrel{(C)}{=} s \oplus t \oplus s +_p t$$

Adding Termination

$$\oplus \quad +_p \quad \star$$

$$(x \oplus y) \oplus z \stackrel{(A)}{=} x \oplus (y \oplus z) \qquad (x +_q y) +_p z \stackrel{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$$

$$x \oplus y \stackrel{(C)}{=} y \oplus x \qquad x +_p y \stackrel{(C_p)}{=} y +_{1-p} x$$

$$x \oplus x \stackrel{(I)}{=} x \qquad x +_p x \stackrel{(I_p)}{=} x$$

$$(x \oplus y) +_p z \stackrel{(D)}{=} (x +_p z) \oplus (y +_p z)$$

The Algebraic Theory of Pointed Convex Semilattices

$$x \oplus \star \stackrel{(B)}{=} x$$

The Algebraic Theory of Convex Semilattices with Bottom

$$x \oplus \star \stackrel{(T)}{=} \star$$

The Algebraic Theory of Convex Semilattices with Top

These three algebras are those freely generated by the singleton set 1

They give rise to three different semantics: may, must, and may-must

$$\mathbb{M}_{\mathcal{I}} = (\mathcal{I}, \text{min-max}, +_p^{\mathcal{I}}, [0,0])$$

$$\mathcal{I} = \{[x,y] \mid x, y \in [0,1] \text{ and } x \leq y\}$$

$$\text{min-max}([x_1, y_1], [x_2, y_2]) = [\min(x_1, x_2), \max(y_1, y_2)]$$

$$[x_1, y_1] +_p^{\mathcal{I}} [x_2, y_2] = [x_1 +_p x_2,\ y_1 +_p y_2]$$

The Theory of Pointed Convex Semilattices

$$\mathbb{M}\text{ax} = ([0,1], \max, +_p, 0)$$

The Algebraic Theory of Convex Semilattices with bottom

$$\mathbb{M}\text{in} = ([0,1], \min, +_p, 0)$$

The Algebraic Theory of Convex Semilattices with Top

Syntax and Transitions

For the three semantics, we use the same syntax

$$s, t \ ::= \ \star, \ s \oplus t, \ s +_p t, \ x \in X \quad \text{for all } p \in [0,1]$$

and transitions

$$\frac{-}{\star \xrightarrow{a} \star} \qquad \frac{s \xrightarrow{a} s' \quad t \xrightarrow{a} t'}{s \oplus t \xrightarrow{a} s' \oplus t'} \qquad \frac{s \xrightarrow{a} s' \quad t \xrightarrow{a} t'}{s +_p t \xrightarrow{a} s' +_p t'}$$

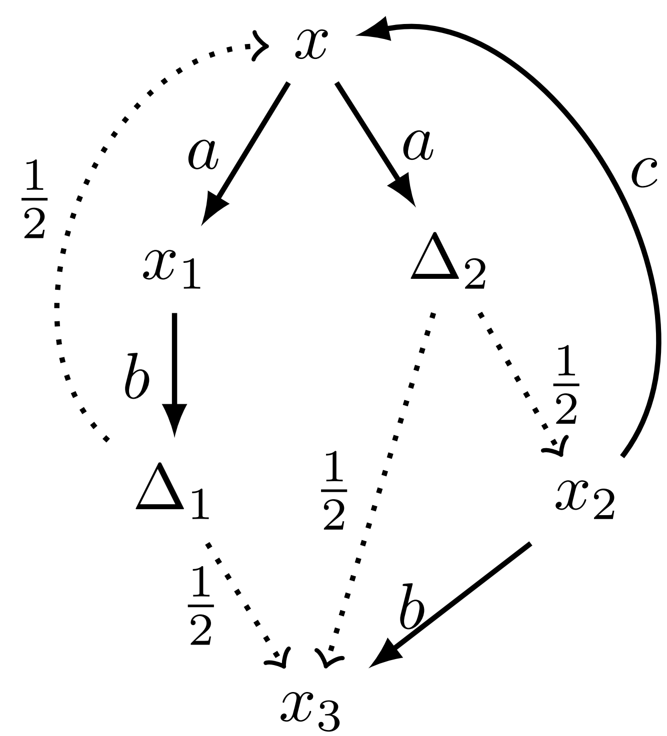
but different output functions...

Example without outputs

$$x \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2)$$

$$x_1 \xrightarrow{b} x +_{\frac{1}{2}} x_3$$

$$x_2 \xrightarrow{b} x_3 \qquad x_2 \xrightarrow{c} x$$



$$x \xrightarrow{b,c} \star$$

$$x_1 \xrightarrow{a,c} \star$$

$$x_2 \xrightarrow{a} \star$$

$$x_3 \xrightarrow{a,b,c} \star$$

$$x \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2) \xrightarrow{b} (x +_{\frac{1}{2}} x_3) \oplus (\star +_{\frac{1}{2}} x_3)$$

Outputs for May

We take as algebra of outputs

$$\mathbb{M}\mathrm{ax} = ([0,1], \max, +_p, 0)$$

that gives rise to the following three rules

$$\frac{-}{\star \downarrow_0} \qquad \frac{s \downarrow_{q_1} \quad t \downarrow_{q_2}}{s \oplus t \downarrow_{\mathtt{max}(q_1,q_2)}} \qquad \frac{s \downarrow_{q_1} \quad t \downarrow_{q_2}}{s +_p t \downarrow_{q_1 +_p q_2}}$$

Outputs for Must

We take as algebra of outputs

$$\mathbb{M}\mathrm{in} = ([0,1], \min, +_p, 0)$$

that gives rise to the following three rules

$$\frac{-}{\star \downarrow_0} \qquad \frac{s \downarrow_{q_1} \quad t \downarrow_{q_2}}{s \oplus t \downarrow_{\mathtt{min}(q_1, q_2)}} \qquad \frac{s \downarrow_{q_1} \quad t \downarrow_{q_2}}{s +_p t \downarrow_{q_1 +_p q_2}}$$

Outputs for May-Must

We take as algebra of outputs

$$\mathbb{M}_{\mathcal{I}} = (\mathcal{I}, \text{min-max}, +_p^{\mathcal{I}}, [0,0])$$

that gives rise to the following three rules

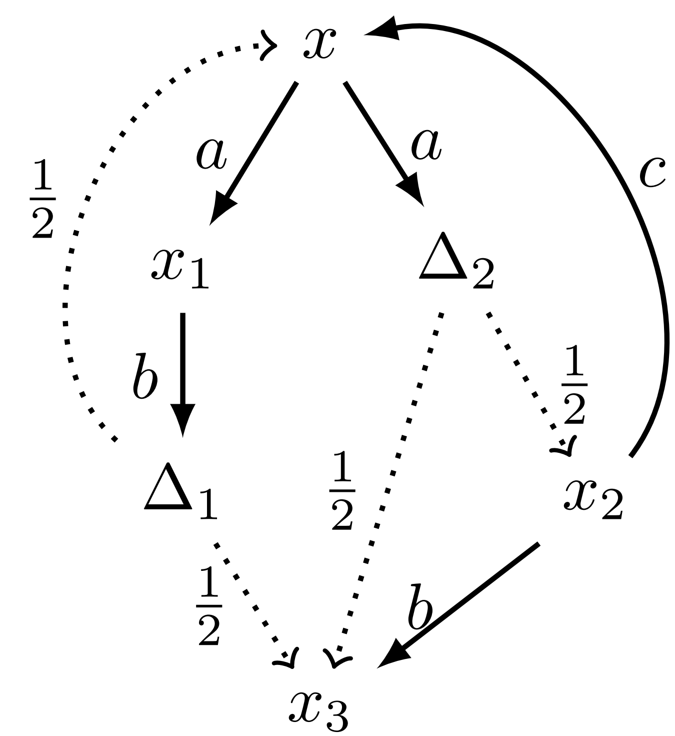
$$\frac{-}{\star \downarrow_{[0,0]}} \qquad \frac{s \downarrow_I \quad t \downarrow_J}{s \oplus t \downarrow_{\texttt{min-max}(I,J)}} \qquad \frac{s \downarrow_I \quad t \downarrow_J}{s +_p t \downarrow_{I +_p^{\mathcal{I}} J}}$$

Example with outputs

$$x \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2)$$

$$x_1 \xrightarrow{b} x +_{\frac{1}{2}} x_3$$

$$x_2 \xrightarrow{b} x_3 \qquad x_2 \xrightarrow{c} x$$



$$x \xrightarrow{b,c} \star$$

$$x_1 \xrightarrow{a,c} \star$$

$$x_2 \xrightarrow{a} \star$$

$$x_3 \xrightarrow{a,b,c} \star$$

All states output 1

$x \downarrow_1$ $\quad$ $x_1 \downarrow_1$ $\quad$ $x_2 \downarrow_1$ $\quad$ $x_3 \downarrow_1$

May $\quad x \downarrow_1 \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2) \downarrow_1 \xrightarrow{b} (x +_{\frac{1}{2}} x_3) \oplus (\star +_{\frac{1}{2}} x_3) \downarrow_1$

Must $\quad x \downarrow_1 \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2) \downarrow_1 \xrightarrow{b} (x +_{\frac{1}{2}} x_3) \oplus (\star +_{\frac{1}{2}} x_3) \downarrow_{\frac{1}{2}}$

Results follow:

- Traces carry a convex semilattice
- The three trace semantics are convex semilattice homomorphisms
- Trace equivalences are congruences w.r.t. convex semilattice operations
- Coinduction up-to these operation is sound

- Both probabilistic and convex bisimilarity implies the three trace equivalences

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- The equivalences are "backward compatible" with standard trace equivalences for nondeterministic and probabilistic systems

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- The may-equivalence coincides with one in Bernardo, De Nicola, Loreti TCS 2014


Thank You