

The structure of natural numbers

is helpful for proving
properties

$$\forall n[n \in \mathbb{N} : P(n)]$$

The structure of natural numbers

On natural numbers we can define a notion of a **successor**, a mapping

$$s: \mathbb{N} \rightarrow \mathbb{N}$$

by $s(n) = n+1$

The successor mapping imposes a structure on the set that enables us to **count**:

- 1) there is a **starting** natural number 0
- 2) for every natural number n , there is a **next** natural number $s(n) = n+1$.

Cardinality

Cardinals

Def.

Two sets A and B have the same cardinality (are equinumerous) if there is a bijection $f:A \rightarrow B$.
Notation $A \sim B$, or $|A| = |B|$.

Prop.

The relation \sim is an equivalence relation on sets.

Def.

A set A has at most as large cardinality as a set B if there is an injection $f:A \rightarrow B$.
Notation $|A| \leq |B|$.

Def.

A set A has at least as large cardinality as a set B if there is a surjection $f:A \rightarrow B$.
Notation $|A| \geq |B|$.

Def.

A set A has smaller cardinality than a set B if there is an injection $f:A \rightarrow B$ and there is no surjection $f:A \rightarrow B$. Notation $|A| < |B|$.

$$|A| = [A]_{\sim}$$

cardinal
numbers are
 \sim equivalence
classes

Theorem (Cantor)

If $|A| \leq |B|$
and
 $|B| \leq |A|$,
then
 $|A| = |B|$.

Operations on cardinals

Def.

Let A and B be two disjoint sets. Then $|A| + |B| = |A \cup B|$.

Def.

Let A and B be two sets. Then $|A| \cdot |B| = |A \times B|$.

Def.

Let A and B be two sets. Then $|A|^{|B|} = |A^B|$ where A^B is the set of all functions from B to A , i.e. $A^B = \{f \mid f: B \rightarrow A\}$.

Prop.

Let A be a set. Then $|\mathcal{P}(A)| = 2^{|A|}$.

$$|A| = [A]_{\sim}$$

cardinal
numbers are
 \sim equivalence
classes

$$\text{Note: } 2 = |\{0, 1\}|$$

Finite sets, finite cardinals

We write \mathbb{N}_k for the set $\{0, 1, \dots, k-1\}$. Then $\mathbb{N}_0 = \emptyset$.

We will also write k for $|\mathbb{N}_k|$.

Def.

A set A is finite if and only if $|A| = |\mathbb{N}_k|$, for some $k \in \mathbb{N}$. We write then $|A| = k$.

Hence

A set A is finite if and only if there is a natural number $k \in \mathbb{N}$ and a bijection $f: A \rightarrow \mathbb{N}_k$.

$$|A| = [A]_{\sim}$$

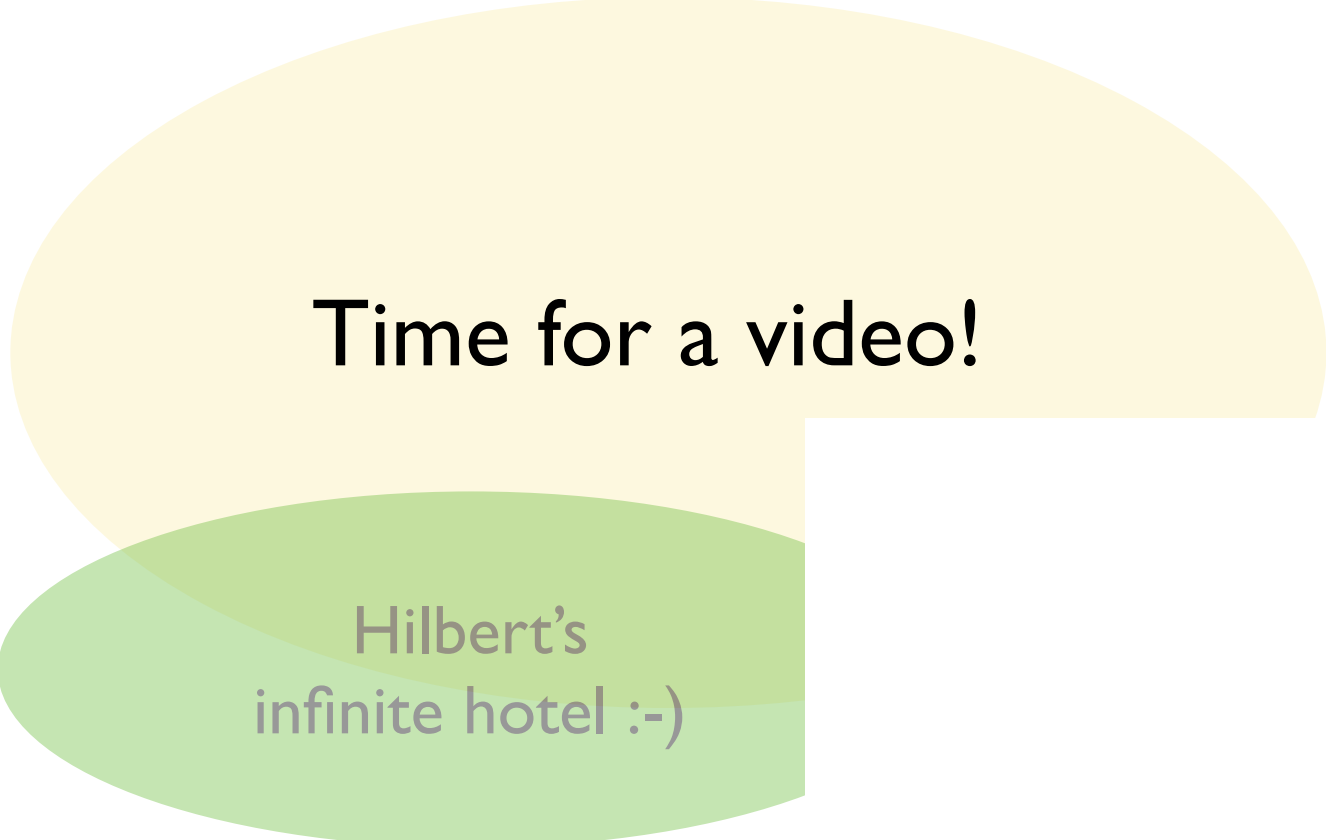
cardinal numbers are \sim equivalence classes

if and only if A has k elements, for some $k \in \mathbb{N}$

E.g. If $|A| = k$ and $|B| = m$ for some $k, m \in \mathbb{N}$ then $|A \times B| = k \cdot m$

The operations on cardinals when restricted to finite cardinals coincide with the operations on natural numbers!
This justifies the notation.

Infinite, countable and uncountable sets



Time for a video!

Hilbert's
infinite hotel :-)

Infinite, countable and uncountable sets

We write \aleph_0 for the cardinality of natural numbers.
Hence $\aleph_0 = |\mathbb{N}|$.

Def.

A set A is countable iff $|A| = \aleph_0$.

Prop.

\mathbb{N} is countable.
 \mathbb{Z} is countable.
 \mathbb{Q} is countable.

Def.

A set is infinite iff $|A| \geq \aleph_0$.

Def.

A set is uncountable iff $|A| > \aleph_0$.

Prop.

\mathbb{R} is uncountable.

$$|A| = [A]_{\sim}$$

cardinal
numbers are
 \sim equivalence
classes

Hence, every countable set
is infinite

We write c for $|\mathbb{R}|$

Cardinals are unbounded

Theorem (Cantor)

For every set A we have $|A| < |\mathcal{P}(A)|$.

Hence, for every cardinal
there is a larger one.

$$|A| = [A]_{\sim}$$

cardinal
numbers are
 \sim equivalence
classes