

# The structure of natural numbers

is helpful for proving  
properties

$$\forall n[n \in \mathbb{N} : P(n)]$$

# The structure of natural numbers

On natural numbers we can define a notion of a **successor**, a mapping

$$s: \mathbb{N} \rightarrow \mathbb{N}$$

by  $s(n) = n+1$

The successor mapping imposes a structure on the set that enables us to **count**:

- 1) there is a **starting** natural number 0
- 2) for every natural number  $n$ , there is a **next** natural number  $s(n) = n+1$ .

# Cardinality

# Cardinals

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Def.

Two sets  $A$  and  $B$  have the same cardinality (are equinumerous) if there is a bijection  $f:A\rightarrow B$ .  
Notation  $A \sim B$ , or  $|A| = |B|$ .

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A set  $A$  has at most as large cardinality as a set  $B$  if there is an injection  $f:A \rightarrow B$ .  
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A set  $A$  has smaller cardinality than a set  $B$  if there is an injection  $f:A \rightarrow B$  and there is no surjection  $f:A \rightarrow B$ . Notation  $|A| < |B|$ .

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Theorem (Cantor)

If  $|A| \leq |B|$   
and  
 $|B| \leq |A|$ ,  
then  
 $|A| = |B|$ .

# Operations on cardinals

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Let  $A$  and  $B$  be two disjoint sets. Then  
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Let  $A$  and  $B$  be two sets. Then  
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Let  $A$  be a set. Then  $|\mathcal{P}(A)| = 2^{|A|}$ .

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$$\text{Note: } 2 = |\{0, 1\}|$$

# Finite sets, finite cardinals

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# Finite sets, finite cardinals

We write  $\mathbb{N}_k$  for the set  $\{0, 1, \dots, k-1\}$ . Then  $\mathbb{N}_0 = \emptyset$ .

We will also write  $k$  for  $|\mathbb{N}_k|$ .

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cardinal numbers are  $\sim$  equivalence classes

The operations on cardinals when restricted to finite cardinals coincide with the operations on natural numbers!  
This justifies the notation.



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E.g. If  $|A| = k$  and  $|B| = m$  for some  $k, m \in \mathbb{N}$  then  $|A \times B| = k \cdot m$

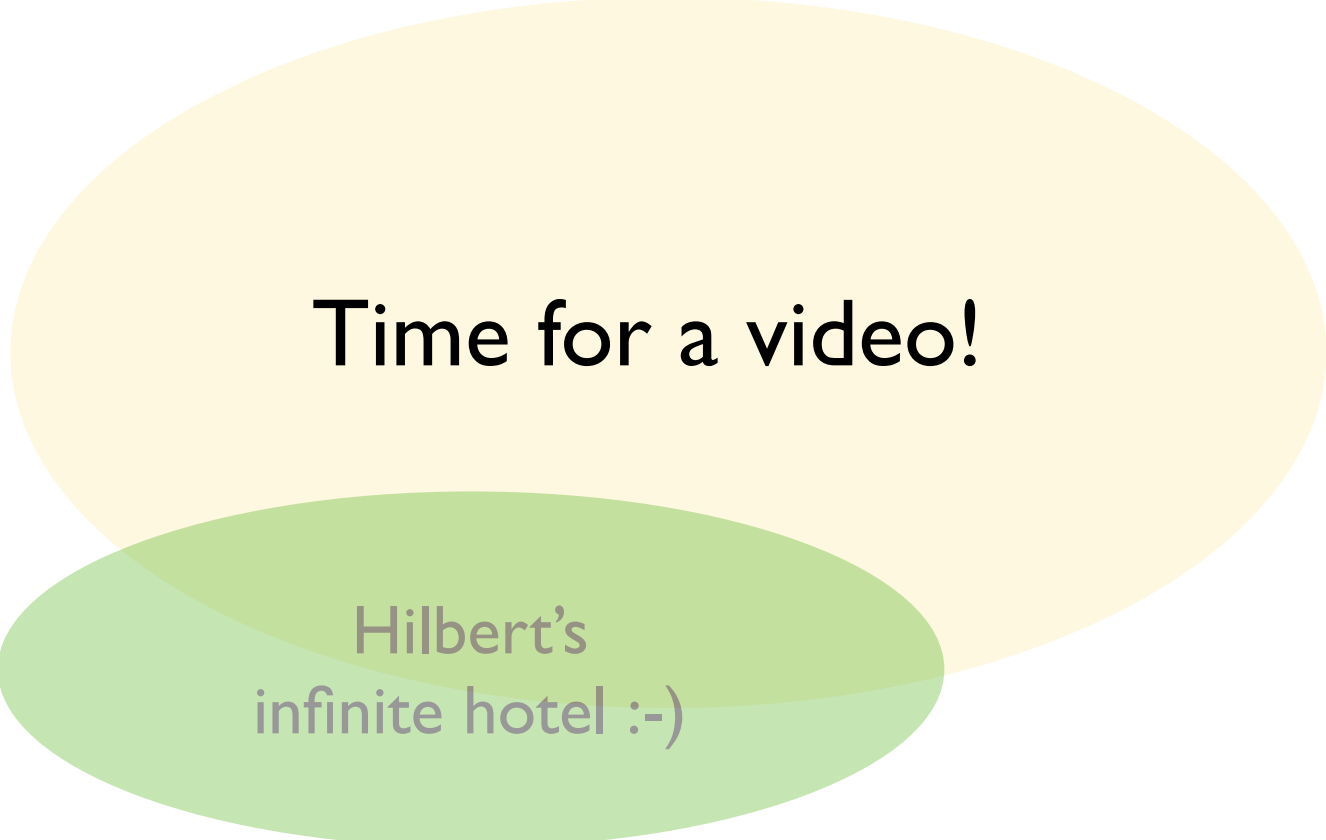
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# Infinite, countable and uncountable sets



Time for a video!

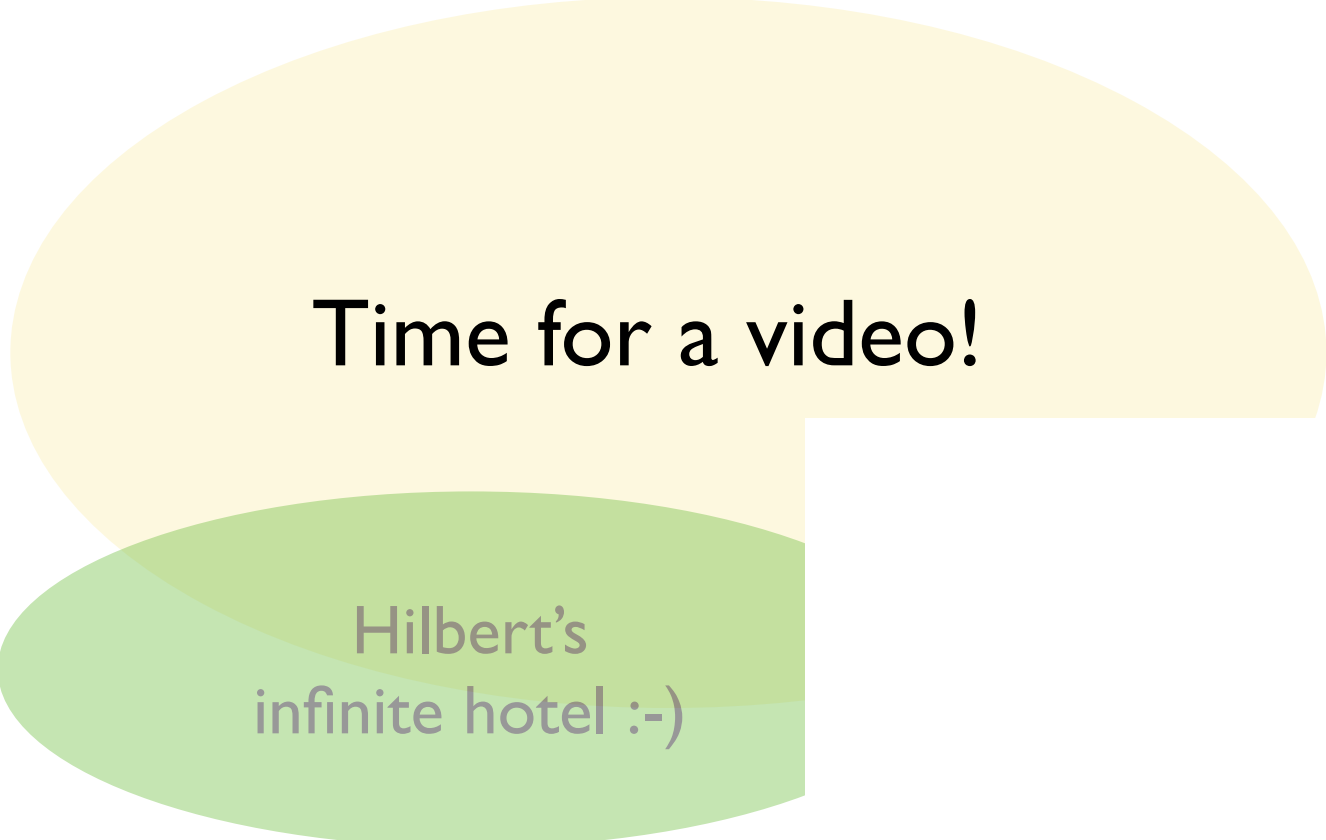
# Infinite, countable and uncountable sets

The image features two overlapping ovals. The larger, upper oval is a pale yellow color and contains the text 'Time for a video!'. The smaller, lower oval is a light green color and overlaps the bottom-left corner of the yellow oval. It contains the text 'Hilbert's infinite hotel :-)' in a smaller, grey font.

Time for a video!

Hilbert's  
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# Infinite, countable and uncountable sets



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We write  $\aleph_0$  for the cardinality of natural numbers.  
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## Theorem (Cantor)

For every set  $A$  we have  $|A| < |\mathcal{P}(A)|$ .

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Hence, for every cardinal there is a larger one.

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