Sets*

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We start by defining the most important notions of (naive) set theory.

A *set* is a collection of different objects, the *elements* of the set. Examples of sets are $A = \{1, 3, 7, 18\}$, $\mathbb{N}$, or $B = \{n \mid n \in \mathbb{N} \wedge n \leq 10\}$. We already used different ways to specify a set: we specified the finite set A by listing its elements, and we specified the finite set B by specifying a property that its elements must satisfy. In general, sets are specified via properties, as in $\{x \mid P(x)\}$ where $P(x)$ is a property, a predicate over x, that is either true or false. The x's for which $P(x)$ is true belong to the specified set $\{x \mid P(x)\}$, and only those. We will learn about predicates in much more detail later in the semester.

We write $x \in S$ for "x is an element of the set S". We also read $x \in S$ as "x is in S" or "x belongs to S". All elements of a set are distinct, and the elements are not ordered. We always use curly brackets { and } to denote sets. The same set (we discuss set equality precisely below) can be described in multiple different ways. For example

$$\begin{aligned} B &= \{n \mid n \in \mathbb{N} \wedge n \leq 10\} \\ &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\ &= \{1, 3, 5, 0, 2, 4, 6, 7, 10, 9, 8\}. \end{aligned}$$

Sets can be finite, like the sets A and B above, or infinite like $\mathbb{N}$. Part of set theory is the study of cardinals or representations of the size of sets. For a *finite* set S, its cardinality denoted by $|S|$ is the number of elements in S. The set with no elements is called the *empty set* and we denote it by $\emptyset$. The empty set is finite, and $|\emptyset| = 0$.

Definition 1 (Subset, $\subseteq$). *Let X and Y be sets. The set X is a* subset *(D. Teilmenge) of the set Y if and only if (D. genau dann wenn) every element of X is an element of Y.*

In the previous definition we used the phrase "if and only if" (and we will do so very often throughout the course). We will also just write "iff" (D. "gdw") as an abbreviation of "if and only if". The meaning of a statement "a iff b" is that both (1) and (2) hold, where

(1) if a is true, then b is as well; and
(2) if b is true, then a is too.

* Notes from the lectures Formale Systeme on naive set theory. I write these notes as the things that we cover this week on this topic are not to be found (in this form) in the book. I also write them so that you—the students—see the difference between lecture notes (written text) and slides (overview presentation), and so that you do not need to rely on hand-written notes.

Definition 2 (Set equality, $=$). *Two sets X and Y are equal iff both $X \subseteq Y$ and $Y \subseteq X$.*

Definition 3 (Proper subset, $\subset$). *Let X and Y be two sets. The set X is a proper subset of the set Y, notation $X \subset Y$, iff $X \subseteq Y$ and $X \neq Y$.*

Next we define several operations on sets, that you have probably already encountered at school. For what follows let X and Y be two sets.

Definition 4 (Union, $\cup$). *The union (D. Vereinigung) of X and Y, denoted by $X \cup Y$ is the set*

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}.$$

Definition 5 (Intersection, $\cap$). *The intersection (D. Durchschnitt) of X and Y, denoted by $X \cap Y$ is the set*

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}.$$

Definition 6 (Set difference, $\setminus$). *The set difference (D. Differenz) of X and Y, denoted by $X \setminus Y$ is the set*

$$X \setminus Y = \{x \mid x \in X \text{ and } x \notin Y\}.$$

The definition of intersection enables the definition of disjoint sets.

Definition 7 (Disjoint sets). *The sets X and Y are* disjoint *if $X \cap Y = \emptyset$.*

Definition 8 (Complement, $-^c$). *The complement (D. Komplement) of X in a universal set U, denoted by X^c is the set*

$$X^c = \{x \mid x \in U \text{ and } x \notin X\}.$$

From the definitions of complement and set difference, we immediately get that with a universal set U, $X^c = U \setminus X$.

The need for a universal set U stems from Russell's paradox: Let C be the set of sets that are not an element of itself, i.e.,

$$C = \{x \mid \; x \notin x\}.$$

For such a set C we get: $C \in C$ iff $C \notin C$ which is a contradiction. This shows that such a set C can not exist, and hence we assume the existence of a universal set U and whenever we write $\{x \mid P(x)\}$ we actually mean the set $\{x \mid x \in U \text{ and } P(x)\}$.

Russell wrote about this paradox to Frege in 1903, pointing an inconsistency in Frege's theory. The discovery of this and similar paradoxes was the driving force for the significant development of logic and the foundations of mathematics in the first half of the 20th century. The importance of these development is nicely stated by the Oxford Companion p.500 according to the Wikipedia article on the history of logic: "The development of the modern "symbolic" or "mathematical" logic during this period by the likes of Boole, Frege, Russell, and Peano is the most significant in the two-thousand-year history of logic, and is arguably one of the most important and remarkable events in human intellectual history."

Definition 9 (Direct product, $\times$). *The direct product, or just product (D. Kartesisches Produkt) of X and Y, denoted by $X \times Y$ is the set*

$$X \times Y = \{(x, y) \mid x \in X \textit{ and } y \in Y\}.$$

In the definition of a product, and from now on, by (x, y) we denote the *ordered pair* of the elements x and y. Hence, the direct product has a different structure of elements than its arguments. This is in contrast to all other set operations we have defined previously. Two ordered pairs (x_1, y_1) and (x_2, y_2) in $X \times Y$ are equal iff $x_1 = x_2$ and $y_1 = y_2$. Note that we have not defined $X \times Y \times Z$, nor we have defined $X \cup Y \cup Z$. Both $\times$ and $\cup$ are *binary* operations, i.e., operations with two arguments, and hence $X \times Y \times Z$ makes no sense. We could define it to be equal to $(X \times Y) \times Z$ or to $X \times (Y \times Z)$ but

$$(X \times Y) \times Z \neq X \times (Y \times Z)$$

which makes either choice unnatural. We will learn later how to define the product of more than two sets.

Definition 10 (Powerset, $\mathcal{P}$). *The powerset (D. Potenzmenge) of X, denoted by $\mathcal{P}(X)$ or 2^X is the set of all subsets of X. Hence,*

$$\mathcal{P}(X) = \{S \mid S \subseteq X\}.$$

Clearly, also the elements of the powerset of a given set X have a different structure than the elements of X – namely they are sets of elements of X.

Example 1. Let $X = \{0, 1\}$ and $Y = \{a, b, c\}$. Then

$$\begin{aligned}
X \cup Y &= \{0, 1, a, b, c\}; \\
X \cap Y &= \emptyset, \quad \text{hence } X \text{ and } Y \text{ are disjoint;} \\
X \setminus Y &= \{0, 1\} = X; \\
X \times Y &= \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c)\}; \\
\mathcal{P}(X) &= \{\emptyset, \{0\}, \{1\}, X\}; \\
\mathcal{P}(Y) &= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, Y\};
\end{aligned}$$

Both X and Y here are finite sets, with $|X| = 2$ and $|Y| = 3$.

Much later in the semester we will prove that $|\mathcal{P}(X)| = 2^{|X|}$ which justifies the alternative notation 2^X for the powerset of X.

So far we have defined several binary operations on sets: union, intersection, set difference and product, as well as two *unary* operations (i.e. operations with a single argument) $-^c$ and $\mathcal{P}$.

Set properties

Next we prove some properties of sets. A *proof* is a sequence of statements each of which is either an assumption or a logical consequence of the previous statements.

We next list 32 properties of sets, some very standard and simple ones, and some less so but maybe more interesting, and present the proofs of some of them. I encourage you to try to prove all the others yourself.

Here, we will write textual mathematical proofs, as it is the case in most of mathematics and computer science. However, in order to understand better the structure of proofs, in this course we will also learn some formal ways to write proofs.

The proofs that we next present are based on the given definitions and nothing else but some basic logic laws, e.g. if a is true and b is true, then a and b is true.

In the following list we present the 32 set properties (also mentioned on the slides in the same order) that we like to note at this point. Each property is stated as a proposition. There is no particular reason why we have numbered and ordered the properties in this way, except that sometimes the properties with a smaller number (stated before) are useful in the proofs of properties with higher number (stated after). Not all properties have names, we only list the names that are commonly known. Also, not all properties are equally "important". The properties with names are the more important properties, the others are here to either give us a bit more feeling about the operations that we just defined, or to provide us with examples of what all one could state and prove about sets. We could write infinitely many properties of the kind, these 32 are to be seen as chosen examples that serve the purpose of introducing basic properties of sets.

For the rest of this section, let X, Y, Z and their decorated versions like X' and X'', be arbitrary sets.

Proposition 1. $\emptyset \subseteq X$.

Proposition 2. *If* $X \subseteq Y$ *and* $Y \subseteq Z$*, then* $X \subseteq Z$.

Proposition 3. $X \cap Y \subseteq X$ *and* $X \cap Y \subseteq Y$.

Proposition 4. $X \subseteq X \cup Y$ *and* $Y \subseteq X \cup Y$.

Proposition 5. *If* $X' \subseteq Y'$ *and* $X'' \subseteq Y''$*, then* $X' \cap Y' \subseteq X'' \cap Y''$.

Proposition 6. *If* $X' \subseteq Y'$ *and* $X'' \subseteq Y''$*, then* $X' \cup Y' \subseteq X'' \cup Y''$.

Proposition 7. $X \cap Y = X$ *iff* $X \subseteq Y$.

Proof. This is an "iff" property, hence we need to prove both directions, which means that we need to prove that
(1) If $X \cap Y = X$, then $X \subseteq Y$; and
(2) If $X \subseteq Y$, then $X \cap Y = X$.

To prove (1), we assume that $X \cap Y = X$. The task is now to show that $X \subseteq Y$. Due to Definition 1, this means that we must show that every element of X is an element of Y. Let x be an arbitrary element in X. From our assumption that $X = X \cap Y$, and Definition 2, we get that $x \in X \cap Y$. From the definition of $X \cap Y$, Definition 5, this means that $x \in X$ and $x \in Y$. In particular, $x \in Y$ which was our goal. This proves (1).

To prove (2), we assume that $X \subseteq Y$. The task now is to show that $X \cap Y = X$. Due to Definition 2, this means we have two new goals:

(2.1) $X \cap Y \subseteq X$; and
(2.2) $X \subseteq X \cap Y$.

The property (2.1) is stated in Proposition 3, and it is easy to prove and independent of the assumption that $X \subseteq Y$. Namely, take an arbitrary $x \in X \cap Y$. Then from the definition of intersection, Definition 5, $x \in X$ and $x \in Y$. In particular, $x \in X$. This proves (2.1). Note that, we could have just referred to Proposition 3, as it is stated before and we can assume we prove the properties in the order of stating them.

In order to prove (2.2), we must again unfold the definition of a subset. Let $x \in X$. The goal is to prove that $x \in X \cap Y$. (Note that this does not hold in general, i.e., we will need our assumption somewhere.) Then, from the assumption that $X \subseteq Y$ and the definition of subset, we get that $x \in Y$. Hence x is in both X and Y, which means that $x \in X \cap Y$ and completes the proof of (2.2), the proof of (2), and the whole proof. □

Note the different level of detail in the proof. While we try to give justification for every single statement in a proof, we sometimes spell out this justification in full detail, with a full reference to all definitions used and needed, and sometimes just say a brief argument e.g. "from the assumption and the definition of a subset".

A proof can have a varying level of detail, depending on the expertise of the intended reader. However, independent of what is communicated, every step (every new statement) in a proof must have a justification from the definitions, and the previous statements.

Proposition 8 (Idempotence of $\cap$). $X \cap X = X$.

Proposition 9 (Idempotence of $\cup$). $X \cup X = X$.

Proposition 10. $X \cap \emptyset = \emptyset$.

Proposition 11. $X \cup \emptyset = X$.

Proposition 12 (Commutativity of $\cap$). $X \cap Y = Y \cap X$.

Proposition 13 (Commutativity of $\cup$). $X \cup Y = Y \cup X$.

Proposition 14 (Associativity of $\cap$). $X \cap (Y \cap Z) = (X \cap Y) \cap Z$.

Proposition 15 (Associativity of $\cup$). $X \cup (Y \cup Z) = (X \cup Y) \cup Z$.

Proposition 16 (Absorbtion). $X \cap (X \cup Y) = X$.

Proposition 17 (Absorbtion). $X \cup (X \cap Y) = X$.

Proof. We will provide a briefer proof, and use previous propositions assuming they were already proven. We need to prove the equality of two sets, which means that by Definition 2 we must prove:
(1) $X \cup (X \cap Y) \subseteq X$; and
(2) $X \subseteq X \cup (X \cap Y)$.

For (1), we proceed as follows. From the definition of a subset, it is easy to see that $X \subseteq X$. We denote this property by (3). From Proposition 3, we know that $X \cap$

$Y \subseteq X$. We denote this property by (4). From (3) and (4), and Proposition 6, we get $X \cup (X \cap Y) \subseteq X \cup X$. We denote this property by (5). Further, from idempotence of union, Proposition 8, we know that $X \cup X = X$. This property we denote by (6). From (5) and (6), we get $X \cup (X \cap Y) \subseteq X$ which shows (1).

For (2), we use Proposition 4: It tells us that $X \subseteq X \cup Y$ for any sets X and Y. Replacing Y here with $X \cap Y$, we obtain (2) which completes the proof. $\square$

Proposition 18 (Distributivity of $\cap$ over $\cup$). $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

Proposition 19 (Distributivity of $\cup$ over $\cap$). $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

Proposition 20. $X \setminus Y \subseteq X$.

Proposition 21. $(X \setminus Y) \cap Y = \emptyset$.

Proposition 22. $X \cup Y = X \cup (Y \setminus X)$.

Proposition 23. $X \setminus X = \emptyset$.

Proposition 24. $X \setminus \emptyset = X$.

Proposition 25. $\emptyset \setminus X = \emptyset$.

Proposition 26. *If $X \subseteq Y$, then $X \setminus Y = \emptyset$.*

Proof. Assume that $X \subseteq Y$. Assume further that $x \in X \setminus Y$, i.e., $X \setminus Y \neq \emptyset$. Then $x \in X$ and $x \notin Y$, which contradicts the assumption that $X \subseteq Y$. Hence, our last assumption is wrong and such an x can not exist, showing that $X \setminus Y = \emptyset$. $\square$

Proposition 27 (Double complement). $(X^c)^c = X$.

Proposition 28 (De Morgan for $\cap$). $(X \cap Y)^c = X^c \cup Y^c$.

Proposition 29 (De Morgan for $\cup$). $(X \cup Y)^c = X^c \cap Y^c$.

Proof. We will do one proof for both directions. Note that we have $S = T$ for two sets S and T iff for every element x, $x \in S$ iff $x \in T$.

Now let x be an arbitrary element. We have

$$\begin{array}{lll}
x \in (X \cup Y)^c & \text{iff, by Def. 8,} & x \notin X \cup Y \\
 & \text{iff} & \text{it is not true that } x \in X \cup Y \\
 & \text{iff !} & x \notin X \textbf{ and } x \notin Y \\
 & \text{iff, by Def. 8,} & x \in X^c \text{ and } x \in Y^c \\
 & \text{iff} & x \in X^c \cap Y^c.
\end{array}$$

Note that the step marked by "!" is the crucial point of this proof and it does not follow from our definitions or previous properties about sets. It is a logical law, also knows as the law of De Morgan, and I hope that after some reflecting you understand that it is true – and also understand that we need logic to understand and give proofs. $\square$

Proposition 30. $X \times \emptyset = \emptyset$.

Proposition 31. $\emptyset \times X = \emptyset$.

Proposition 32. *If* $X \subseteq Y$, *then* $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$.

Proof. Assume $X \subseteq Y$, and denote this statement by (1). We need to show that $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$. Let $S \in \mathcal{P}(X)$, and denote this assumption by (2). The task is now to show that $S \in \mathcal{P}(Y)$.

From (2), and the definition of a powerset, Definition 10, we know that $S \subseteq X$. We denote this statement by (3). From (3), (1), and Proposition 2, we immediately get $S \subseteq Y$, which by the definition of powerset, shows that $S \in \mathcal{P}(Y)$ and completes the proof. $\square$