Exemplaric Expressivity of Modal Logics

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Outline

- Expressivity:

  logical equivalence = behavioral equivalence

- For three examples:
  1. Transition systems
  2. Markov chains
  3. Markov processes

Boolean modal logic

Finite conjunctions probabilistic modal logic

Via dual adjunctions

Dual

Predicates on spaces

$$\begin{tikzcd} T \arrow[loop left] & \mathbb{C}^{op} \arrow[rr, bend left, "P"] & & \mathbb{A} \arrow[ll, bend left, "F"] & L \arrow[loop right] \end{tikzcd}$$

$$\begin{tikzcd} PA \arrow[r, "\eta P"] \arrow[dr, equal] & PFPA \arrow[d, "P\varepsilon"] \\ & PA \end{tikzcd}$$

$$\begin{tikzcd} FA \arrow[r, "\varepsilon F"] \arrow[dr, equal] & FPFA \arrow[d, "F\eta"] \\ & FA \end{tikzcd}$$

Behaviour (coalgebras)

Theories on models

Logics (algebras)

$F \dashv P, \quad \eta_A : A \to PFA$ in $\mathbb{A}$, $\quad \varepsilon_X : X \to FPX$ in $\mathbb{C}$

Logical set-up

$$\begin{tikzcd} T \circlearrowright \mathbb{C}^{op} \arrow[r, bend left, "P"] & \mathbb{A} \circlearrowleft L \arrow[l, bend left, "F"] \end{tikzcd}$$

- If L has an initial algebra of formulas

$$L : \mathit{Form} \xrightarrow{\cong} \mathit{Form}$$

- A natural transformation

$$\sigma : LP \Rightarrow PT$$

gives interpretations

for arbitrary coalgebra
$$\begin{tikzcd} TX \\ X \arrow[u, "c"'] \end{tikzcd}$$

$$\begin{tikzcd} L(\mathit{Form}) \arrow[r, dashed, "L[\![-]\!]"] \arrow[dd, "\cong"'] & LPX \arrow[d, "\sigma_X"] \\ & PTX \arrow[d, "Pc"] \\ \mathit{Form} \arrow[r, dashed, "{[\![-]\!]}"'] & PX \end{tikzcd}$$

Logical equivalence behavioural equivalence

- The interpretation map yields a theory map

$$\frac{[\![-]\!] \colon \mathit{Form} \to PX}{th \colon X \to F(\mathit{Form})}$$

Aim: expressivity

- which defines logical equivalence

$$x \equiv y \quad \Leftrightarrow \quad th(x) = th(y)$$

- behavioural equivalence is given by

$$x \sim y \quad \Leftrightarrow \quad h_1(x) = h_2(y)$$

for some coalgebra homomorphisms h_1 and h_2

$$\begin{tikzcd} \mathbb{C}^{op} \arrow[loop left, "T"] \arrow[r, bend left, "P"] & \mathbb{A} \arrow[loop right, "L"] \arrow[l, bend left, "F"] \end{tikzcd}$$

Expressivity

- Bijective correspondence between

$$\begin{tikzcd} \mathbb{C}^{op} \arrow[r, bend left, "LP"] \arrow[r, bend right, "PT"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbb{A} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "FL"] \arrow[r, bend right, "TF"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathbb{C}^{op} \end{tikzcd}$$

If and the transpose of the interpretation is componentwise mono, then expressivity.

Factorisation system on $\mathbb{C}$

$(\mathcal{M}, \mathcal{E}), \ \mathcal{M} \subseteq \text{Monos}, \ \mathcal{E} \subseteq \text{Epis}$

with diagonal fill-in

T preserves $\mathcal{M}$

Sets vs. Boolean algebras

unit $\eta : A \to \mathcal{P}\mathcal{F}_u(A)$

$\eta(a) = \{\alpha \in \mathcal{F}_u(A) \mid a \in \alpha\}$

contravariant powerset

$$\begin{tikzcd} \mathbf{Sets}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{P}"] & \mathbf{BA} \arrow[l, bend left, "\mathcal{F}_u"] \end{tikzcd}$$

Boolean algebras

ultrafilters

upsets $\alpha \subseteq A$, $\top \in \alpha$

$a, b \in \alpha \Rightarrow a \wedge b \in \alpha$

$\forall a \in A. a \in \alpha$ xor $\neg a \in \alpha$

standard correspondence

$$\frac{f : X \longrightarrow \mathcal{F}_u(A) \quad \text{in Sets}}{g : A \longrightarrow \mathcal{P}(X) \quad \text{in } \mathbf{BA}}$$

via

$$\frac{a \in f(x)}{x \in g(a)}$$

Sets vs. meet semilattices

unit $\eta : A \to \mathcal{P}\mathcal{F}(A)$
$\eta(a) = \{\alpha \in \mathcal{F}(A) \mid a \in \alpha\}$

contravariant powerset

$$\begin{tikzcd} \mathbf{Sets}^{\mathrm{op}} \arrow[rr, bend left, "\mathcal{P}"] & & \textsf{MSL} \arrow[ll, bend left, "\mathcal{F}"] \end{tikzcd}$$

meet semilattices

filters

upsets $\alpha \subseteq A$, $\top \in \alpha$
$a, b \in \alpha \Rightarrow a \wedge b \in \alpha$

"the same" correspondence

$$\dfrac{f : X \longrightarrow \mathcal{F}(A) \quad \text{in } \mathbf{Sets}}{g : A \longrightarrow \mathcal{P}(X) \quad \text{in } \textsf{MSL}}$$

via

$$\dfrac{a \in f(x)}{x \in g(a)}$$

Measure spaces vs. meet semilattices

unit $\eta : A \to \mathcal{SF}(A)$
$\eta(a) = \{\alpha \in \mathcal{F}(A) \mid a \in \alpha\}$

maps a measure space to its σ-algebra

$$\begin{tikzcd} \mathbf{Meas}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{S}"] & \mathbf{MSL} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

σ-algebra: "measurable" subsets closed under empty, complement, countable union

"the same" correspondence

$$\frac{f : X \longrightarrow \mathcal{F}(A) \quad \text{in } \mathbf{Meas}}{g : A \longrightarrow \mathcal{S}(X) \quad \text{in } \mathbf{MSL}}$$

via

$$\frac{a \in f(x)}{x \in g(a)}$$

by

measure spaces

objects: pairs $(X, \mathcal{S}(X))$
arrows: measurable functions

$\to Y$ with $f^{-1}(\mathcal{S}(Y)) \subseteq \mathcal{S}(X)$

Behaviour via coalgebras

$$\begin{tikzcd} T \circlearrowright \mathbb{C}^{op} \arrow[r, bend left, "P"] & \mathbb{A} \circlearrowleft L \arrow[l, bend left, "F"] \end{tikzcd}$$

- Transition systems

$\mathcal{P}_f$-coalgebras in **Sets**

$$\begin{tikzcd} \mathcal{P}_f \circlearrowright \mathbf{Sets}^{op} \arrow[r, bend left, "\mathcal{P}"] & \textbf{BA} \arrow[l, bend left, "\mathcal{F}_u"] \end{tikzcd}$$

- Markov chains

$\mathcal{D}_f$-coalgebras in **Sets**

$$\begin{tikzcd} \mathcal{D}_f \circlearrowright \mathbf{Sets}^{op} \arrow[r, bend left, "\mathcal{P}"] & \textbf{MSL} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

Giry monad

- Markov processes

$\mathcal{G}$-coalgebras in **Meas**

$$\begin{tikzcd} \mathcal{G} \circlearrowright \textbf{Meas}^{op} \arrow[r, bend left, "\mathcal{S}"] & \textbf{MSL} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

The Giry monad

$$(X, \mathcal{S}X) \mapsto (\mathcal{G}X, \mathcal{S}\mathcal{G}X)$$

$$\mathcal{G}X = \{\varphi : \mathcal{S}X \to [0,1] \mid \varphi(\quad = 0, \varphi(\cup_i M_i) = \textstyle\sum_i \varphi(M_i)\}$$

countable
union of
pairwise
disjoint

the smallest making

$$ev_M : \mathcal{G}X \to [0,1]$$
$$\varphi \mapsto \varphi(M)$$

measurable

generated by

$$\{\Box_r(M) \mid r \in \mathbb{Q} \cap [0,1]\}$$

$$\Box_r(M) = \{\varphi \in \mathcal{G}X \mid \varphi(M) \geq r\}$$

subprobability
measures

Logic for transition systems

- Modal operator

$$\Box(S) = \{u \in \mathcal{P}_f(X) \mid u \subseteq S\}$$

$\Box$ corresponds to $\boxtimes$: $L\mathcal{P} \Rightarrow \mathcal{P}\mathcal{P}_f$

models of boolean logic with fin.meet preserving modal operators

$$\begin{tikzcd} \mathcal{P}_f \circlearrowright \mathbf{Sets}^{op} \arrow[r, bend left, "\mathcal{P}"] & \mathbf{BA} \circlearrowleft L \arrow[l, bend left, "\mathcal{F}_u"] \end{tikzcd}$$

: GV

rgetful

componentwise mono tr

$: \mathcal{P}_f\mathcal{F}_u \Rightarrow \mathcal{F}_u L$

expressivity

Logic for Markov chains

- Probabilistic modalities

$$\Box_r(S) = \{\varphi \in \mathcal{D}_f(X) \mid \sum_{x\in S}\varphi(x) \geq r\}$$

models of logic with fin.conj. and monotone modal operators

$\Box$ corresponds to $\boxtimes$: $K\mathcal{P} \Rightarrow \mathcal{P}\mathcal{D}_f$

$$\begin{tikzcd} \mathcal{D}_f \circlearrowright \mathbf{Sets}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{P}"] & \mathbf{MSL} \circlearrowleft K \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

$\sqcup_r HV$

getful

componentwise mono tr : $\mathcal{D}_f\mathcal{F} \Rightarrow \mathcal{F}K$

expressivity

Logic for Markov processes

- General probabilistic modalities

$$\Box_r(M) = \{\varphi \in \mathcal{G}(X) \mid \varphi(M) \geq r\}$$

$\Box$ corresponds to $\boxtimes$: $K\mathcal{S} \Rightarrow \mathcal{S}\mathcal{G}$

models of logic with fin.conj. and monotone modal operators

$$\begin{tikzcd} \mathcal{G} \circlearrowright \textbf{Meas}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{S}"] & \textbf{MSL} \circlearrowleft K \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

expressivity

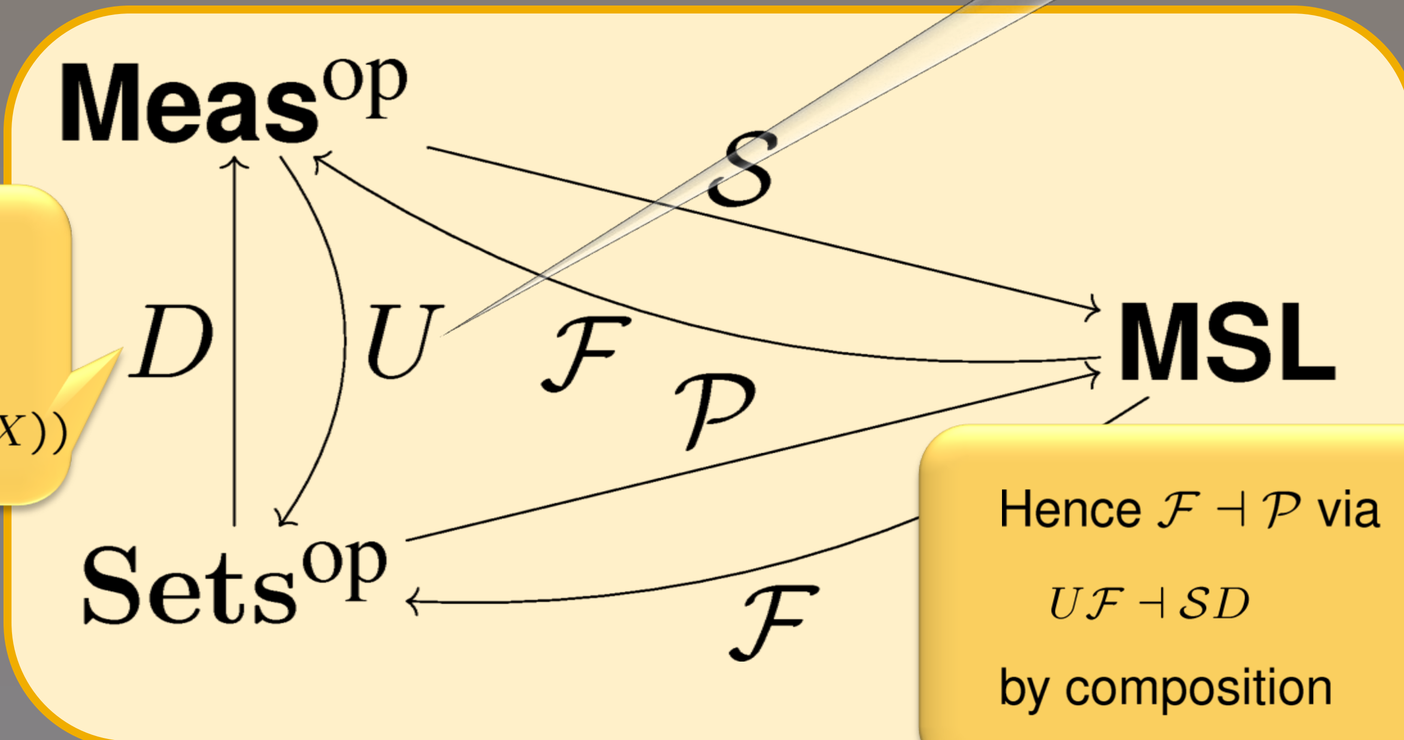
componentwise mono tr : $\mathcal{G}\mathcal{F} \Rightarrow \mathcal{F}K$

ame K

Discrete to indescrete

- The adjunctions are related:

forgetful functor



discrete measure space

$D(X) = (X, \mathcal{P}(X))$

Hence $\mathcal{F} \dashv \mathcal{P}$ via

$U\mathcal{F} \dashv \mathcal{S}D$

by composition

Discrete to indiscrete

- Markov chains as Markov processes

$$
\begin{tikzcd}
\mathcal{G} \circlearrowright \mathbf{Meas}^{\mathrm{op}} \arrow[dd, bend left, "U"] & & \\
& & \mathbf{MSL} \arrow[ull, "\mathcal{S}"'] \arrow[ull, bend right, "\mathcal{F}"'] \arrow[dll, bend left, "\mathcal{F}"] \\
\mathcal{D}_f \circlearrowright \mathbf{Sets}^{\mathrm{op}} \arrow[uu, "D"] \arrow[urr, "\mathcal{P}"] & &
\end{tikzcd}
$$

$$
\begin{array}{c} \mathcal{D}_f X \\ \uparrow c \\ X \end{array} \quad \longmapsto \quad \begin{array}{c} \mathcal{G}(X, \mathcal{P}X) \\ \uparrow \overline{\rho} \circ Dc \\ (X, \mathcal{P}X) \end{array}
$$

via an embedding natural transformation $\rho : \mathcal{D}_f U \Rightarrow U\mathcal{G}$

$$\rho(\varphi) = \left[M \mapsto \sum_{x \in M} \varphi(x)\right]$$

Discrete to indiscrete

$$\varphi \in \Box_r^{\mathcal{D}_f}(S) \Leftrightarrow \rho(\varphi) \in \Box_r^{\mathcal{G}}(S)$$

$\varphi \in \mathcal{D}_f(X)$

$S \in \mathcal{P}(X) = \mathcal{S}(DX)$

$\boxtimes^{\mathcal{D}_f} = \mathcal{P}(\rho) \circ \boxtimes^{\mathcal{G}}$ and $\overline{\boxtimes}^{\mathcal{D}_f} = U(\overline{\boxtimes}^{\mathcal{G}}) \circ \rho\mathcal{F}$

$\rho\mathcal{F}\colon \mathcal{D}_f U\mathcal{F} \Rightarrow U\mathcal{G}\mathcal{F}$ is componentwise mono

Expressivity for Markov chains follows from expressivity for Markov processes !

Conclusions

- Expressivity
- For three examples:
  1. Transition systems
  2. Markov chains
  3. Markov processes

Boolean modal logic

Finite conjunctions probabilistic modal logic

in the setting of dual adjunctions !