Week 6, Task 6

Consider the following two predicates on $\mathbb{Z}^2$:

$$A(m,n) = m < n$$

$$B(m,n) = \exists x \, [x \in D \colon m < x < n]$$

(a) Show that $A(m,n) \overset{\text{val}}{=} B(m,n)$ if $D = \mathbb{R}$.

Let $D = \mathbb{R}$ and assume that $A(m,n)$ is true for $(m,n) \in \mathbb{Z}^2$.

This means that $m < n$.

Let $x = \frac{m+n}{2} \in \mathbb{R} = D$

Then $x - m = \frac{m+n}{2} - m = \frac{n-m}{2} > 0$ since $n - m > 0$ since $n > m$ by assumption.

Hence $x > m$.

Also $n - x = n - \frac{m+n}{2} = \frac{n-m}{2} > 0$, since $n > m$ (as before),

and therefore $n > x$.

So we have found a real number x for which it holds that $m < x < n$, and therefore

$B(m,n)$ is true for this particular pair $(m,n) \in \mathbb{Z}^2$.

This shows that whenever $A(m,n)$ is true, also $B(m,n)$ is true. (*)

For the opposite, assume $(m,n) \in \mathbb{Z}^2$ are such that

$B(m,n)$ is true.

This means that there is an element $x \in \mathbb{R} = D$ for which $m < x < n$. But then, from the transitivity of "<" on real numbers, we get that $m < n$ holds, i.e., $A(m,n)$ is true.

This shows that whenever

$B(m,n)$ is true, also $A(m,n)$ is true. (**)

From (*) and (**) and the definition of $\overset{val}{=}$ we get that $A(m,n) \overset{val}{=} B(m,n)$.

(b) Show that $A(m,n) \overset{val}{\neq} B(m,n)$ if $D = \mathbb{Z}$.

Here it suffices to give a counter example, i.e., concrete values for $m, n \in \mathbb{Z}$ for which $A(m,n)$ and $B(m,n)$ have different truth value.

Consider $m = 0$ and $n = 1$.

Then we have that $A(m,n) = A(0,1)$ is the proposition "$0 < 1$" which is true, but

$B(m,n)$ is the proposition

$$\exists x [x \in \mathbb{Z} : 0 < x < 1]$$

which is not true, since there is no integer number between 0 and 1.

This shows (b).

REMARK: Notice that $B(m,n) \overset{val}{\models} A(m,n)$

since whenever $B(m,n)$ is true, also $A(m,n)$ is true

(this is proven in the second part of (a))

Hence, it wouldn't be possible to find an example where $B(m,n)$ is true, but $A(m,n)$ is false.