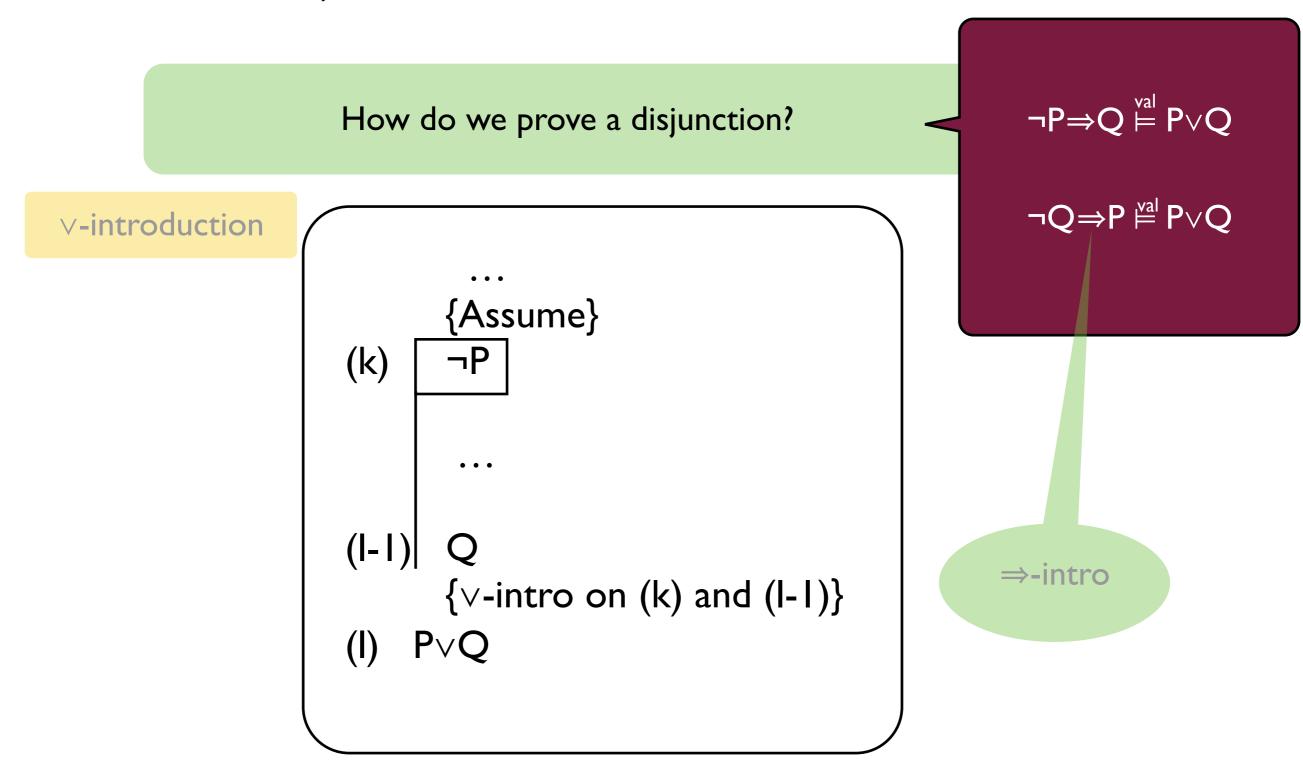
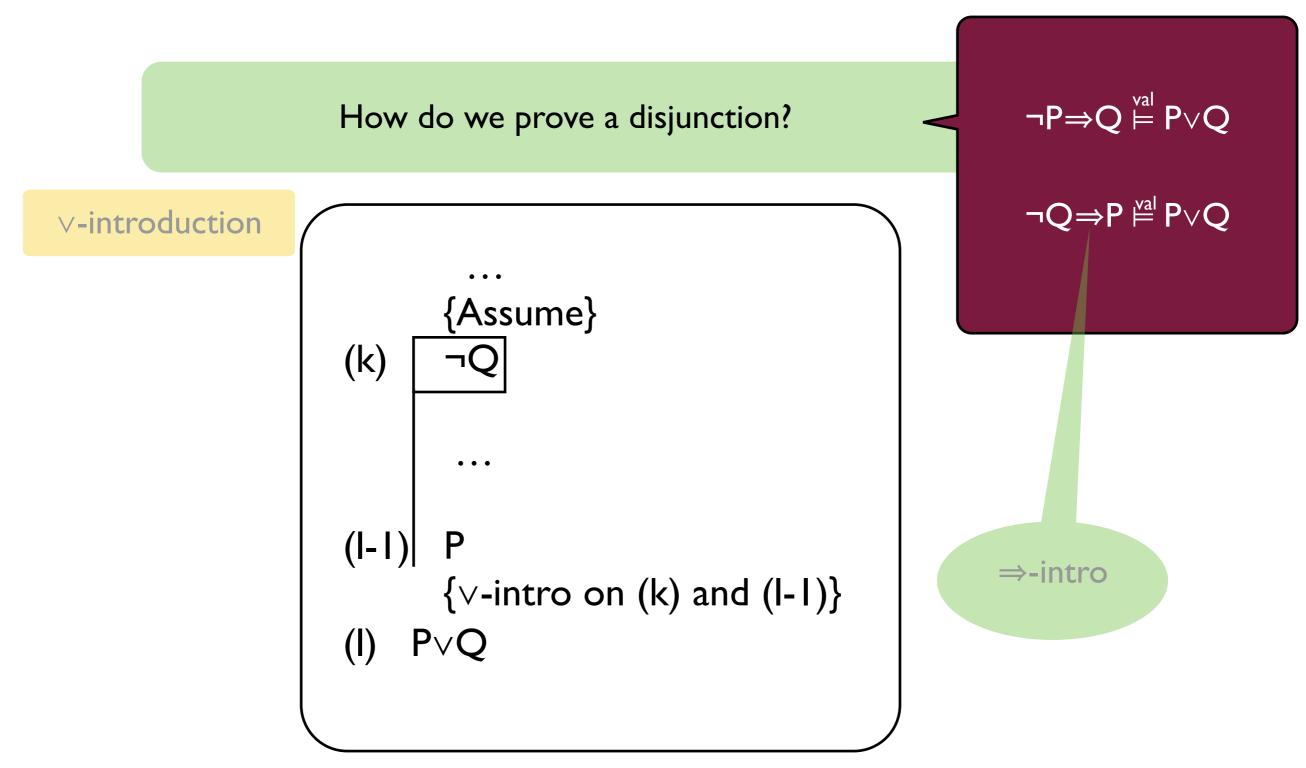
Disjunction introduction



Disjunction introduction



Disjunction elimination

How do we use a disjunction in a proof?

 $P \lor Q \stackrel{\text{val}}{\models} \neg P \Rightarrow Q$

 $P \lor Q \stackrel{\text{val}}{\models} \neg Q \Rightarrow P$

$$(k)$$
 $P \lor Q$

Disjunction elimination

How do we use a disjunction in a proof?

 $P \lor Q \stackrel{\text{val}}{\models} \neg P \Rightarrow Q$

 $P \lor Q \stackrel{\text{Yal}}{\models} \neg Q \Rightarrow P$

$$\parallel \parallel$$

$$(k)$$
 $P \lor Q$

Proof by case distinction

How do we prove R by a case distinction?

proof by case distinction

 $\| \|$

(k) $P\lor Q$

|| ||

l) P⇒R

|| ||

(m) $Q \Rightarrow R$

 $\| \|$

{case-dist on (k), (l), (m)}

(n) R

 $(P \lor Q) \land (P \Rightarrow R) \land (Q \Rightarrow R) \stackrel{\text{val}}{\vDash} R$

 $(k \le n, l \le n, m \le n)$

Bi-implication introduction

How do we prove a bi-implication?

 $(P \Rightarrow Q) \land (Q \Rightarrow P) \stackrel{\text{val}}{\models} P \Leftrightarrow Q$

⇔-introduction

• • •

(k) P⇒Q

• • •

(I) $Q \Rightarrow P$

• • •

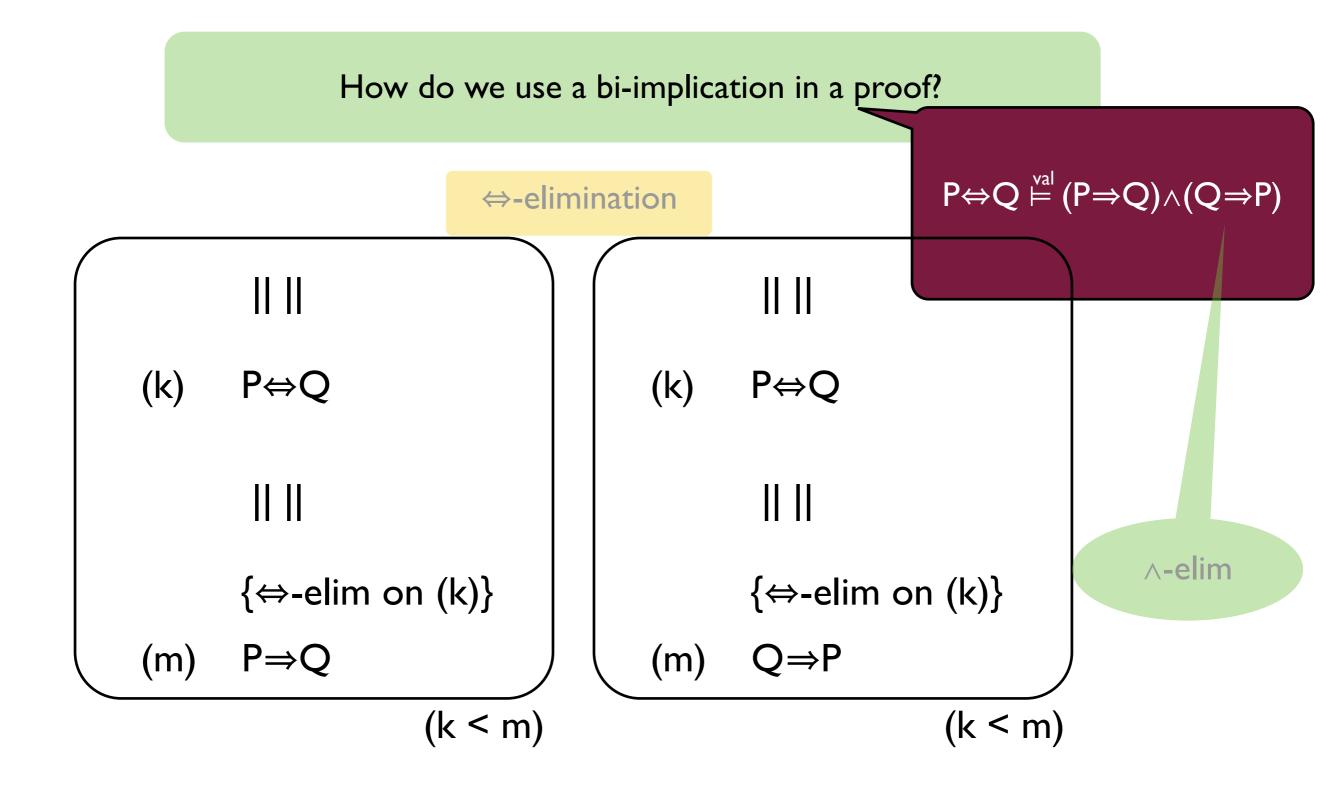
 $\{\Leftrightarrow$ -intro on (k) and (l) $\}$

(m) P⇔Q

 $(k \le m, l \le m)$

∧-intro

Bi-implication elimination



Derivations / Reasoning with quantifiers

Proving a universal quantification

To prove

$$\forall x[x \in \mathbb{Z} \land x \ge 2 : x^2 - 2x \ge 0]$$

Proof

Let $x \in \mathbb{Z}$ be arbitrary and assume that $x \ge 2$.

Then, for this particular x, it holds that $x^2 - 2x = x(x-2) \ge 0$ (Why?)

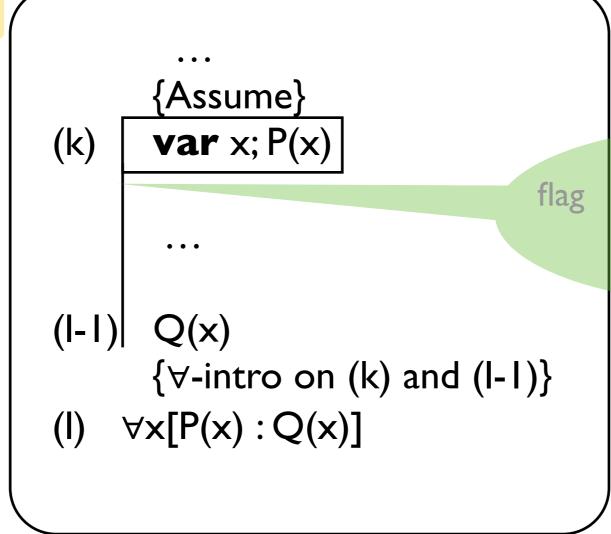
Conclusion: $\forall x[x \in \mathbb{Z} \land x \ge 2 : x^2 - 2x \ge 0].$

V introduction

How do we prove a universal quantification?

10 W do We prove a diliversal qualitatication.

∀-introduction



similar to
⇒-intro
with
generating
hypothesis

shows the validity of a hypothesis

Using a universal quantification

We know

 $\forall x[x \in \mathbb{Z} \land x \ge 2 : x^2 - 2x \ge 0]$

Whenever we encounter an $a \in \mathbb{Z}$ such that $a \ge 2$, we can conclude that $a^2 - 2a \ge 0$.

For example, $(52387^2 - 2 \cdot 52387) \ge 0$ since $52387 \in \mathbb{Z}$ and $52387 \ge 2$.

∀ elimination

How do we use a universal quantification in a proof?

∀-elimination

(k) $\forall x[P(x):Q(x)]$

(I) P(a)

II II {∀-elim on (k) and (l)}

(m) Q(a)

similar to implication but we need a witness

a is
an object
(variable, number,..)
which is "known" in line
(I)

the same "a" from line (I)

time for an example!

 $(k \le m, l \le m)$

3 introduction

How do we prove an existential quantification?

3-introduction

```
{Assume}
(k)
        \forall x[P(x): \neg Q(x)]
(I-I)
        \{\exists-intro on (k) and (I-I)\}
(I) \exists x [P(x) : Q(x)]
```

 $\neg \ \forall x [P(x):\neg Q(x)] \stackrel{\text{val}}{\vDash}$ $\exists x \ [P(x):Q(x)]$

and ¬-intro

3 elimination

How do we use an existential quantification in a proof?

3-elimination

(k) $\exists x [P(x) : Q(x)]$

 $\parallel \parallel$

(I) $\forall x[P(x): \neg Q(x)]$

(m) F

(k < m, l < m)

 $\exists x [P(x) : Q(x)] \stackrel{\text{val}}{\vDash} \\ \neg \ \forall x [P(x) : \neg Q(x)]$

and ¬- elimination

time for an example!

Proofs with 3-introduction and 3-elimination are unnecessarily long and cumbersome...

There are alternatives!

Proving an existential quantification

To prove

$$\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \ge 0]$$

Proof

It suffices to find a witness, i.e., an $x \in \mathbb{Z}$ satisfying $x^3 - 2x - 8 \ge 0$.

x = 3 is a witness, since $3 \in \mathbb{Z}$ and $3^3 - 2 \cdot 3 - 8 = 13 \ge 0$

Conclusion: $\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \ge 0]$.

also x = 5 is a witness...

Alternative 3 introduction

How do we prove an existential quantification?

3*-introduction

by finding a witness

(k) P(a)

• • •

(I) Q(a)

• • •

 $\{\exists *-intro on (k) and (l)\}$

(m) $\exists x [P(x) : Q(x)]$

strategy: wait until a witness object appears

does not always work

(k < m, l < m)

Using an existential quantification

We know

$$\exists x[x \in \mathbb{R} : a - x < 0 < b - x]$$

We can declare an $x \in \mathbb{Z}$ (a witness) such that

$$a - x < 0 < b - x$$

and use it further in the proof. For example:

From a - x < 0, we get a < x.

From b - x > 0, we get x < b.

Hence, a < b.

Alternative 3 elimination

How do we use an existential quantification in a proof?

∃*-elimination

 $\| \|$

(k) $\exists x [P(x) : Q(x)]$

 $\{\exists *-elim on (k)\}$

(m) Pick x with P(x) and Q(x)

we pick a witness

x must be new!

time for an example!

(k < m)