

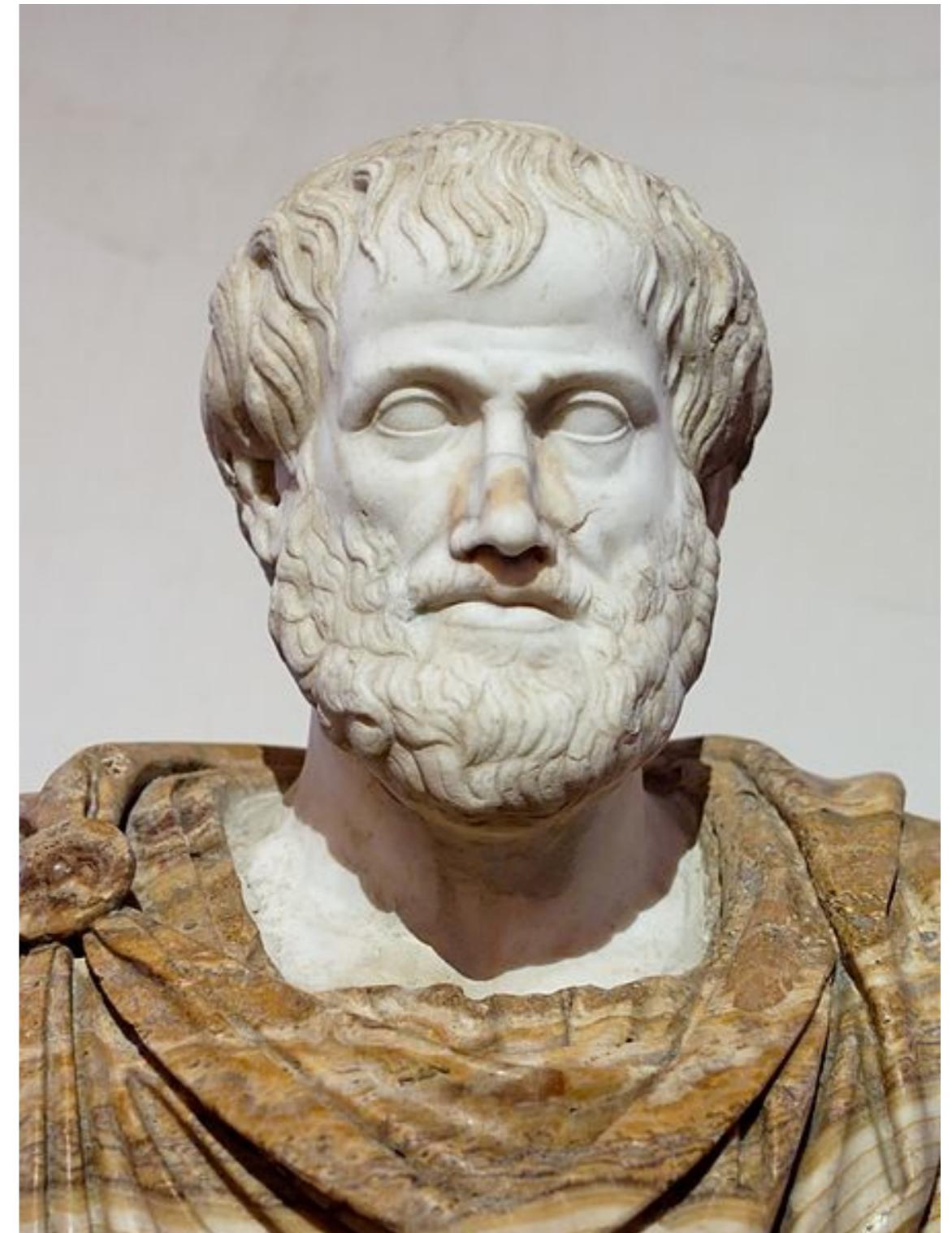
Logic

In the beginning

Aristotle +/- 350 B.C.

Organon

19 syllogisms



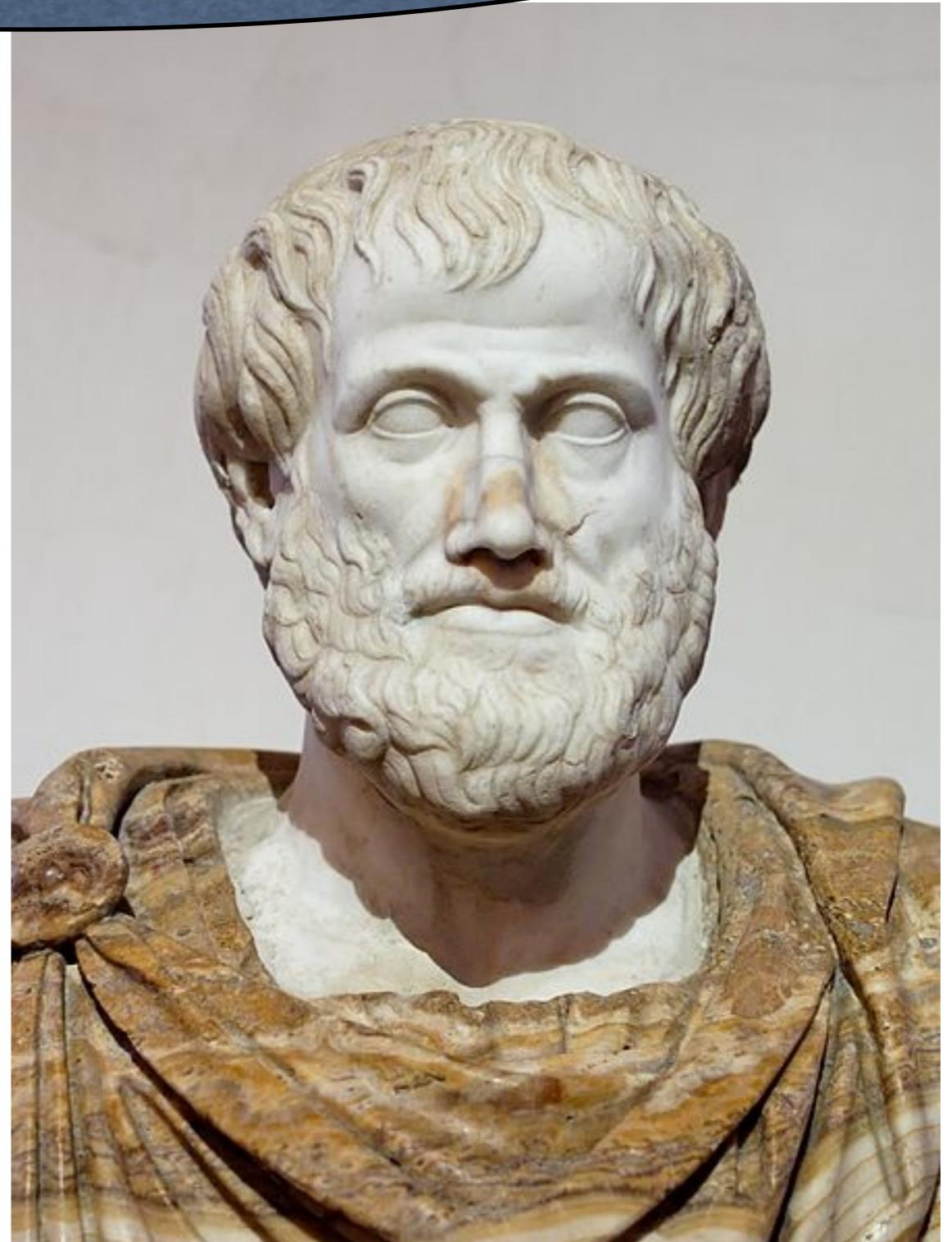
Logic = study of correct reasoning

In the beginning

Aristotle +/- 350 B.C.

Organon

19 syllogisms



Barbara syllogism

All K's are L's
All L's are M's

All K's are M's

Barbara syllogism

only later called so,
in the Middle Ages

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from the two premises

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independent of what the parameters K,L,M are

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independent of what the parameters K,L,M are

Logic (Logos, Greek for word, understanding, reason) deals with general reasoning laws in the form of formulas with parameters.

Propositions

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Def. A proposition (**Aussage**) is a grammatically correct sentence that is either true or false.

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logic deals with patterns!
what matters are not particular propositions but the way in which (abstract) propositions are combined and what follows from them

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Connectives

- ^ for “and”
- ∨ for “or”
- ¬ for “not”
- ⇒ for “if .. then” or “implies”
- ↔ for “if and only if”

logic deals with patterns!
what matters are not particular propositions but the way in which (abstract) propositions are combined and what follows from them

Abstract propositions

Abstract propositions

Definition

Basis Propositional variables are abstract propositions.

Step (Case 1) If P is an abstract proposition, then so is $(\neg P)$.

Step (Case 2) If P and Q are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

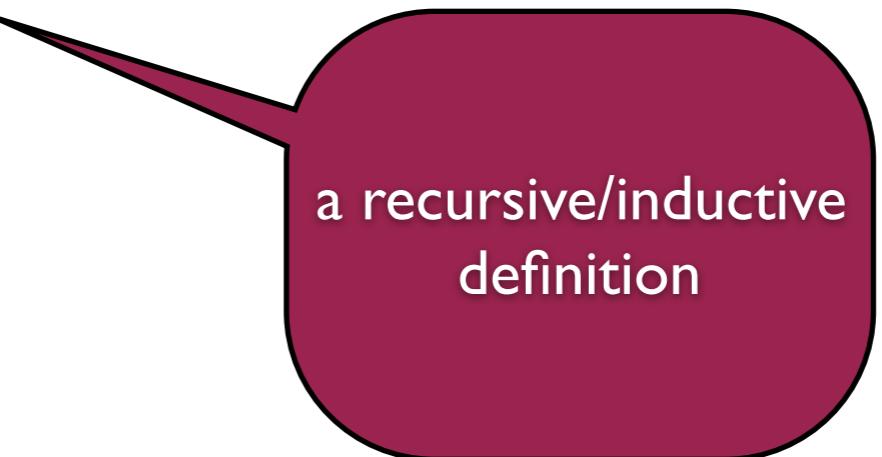
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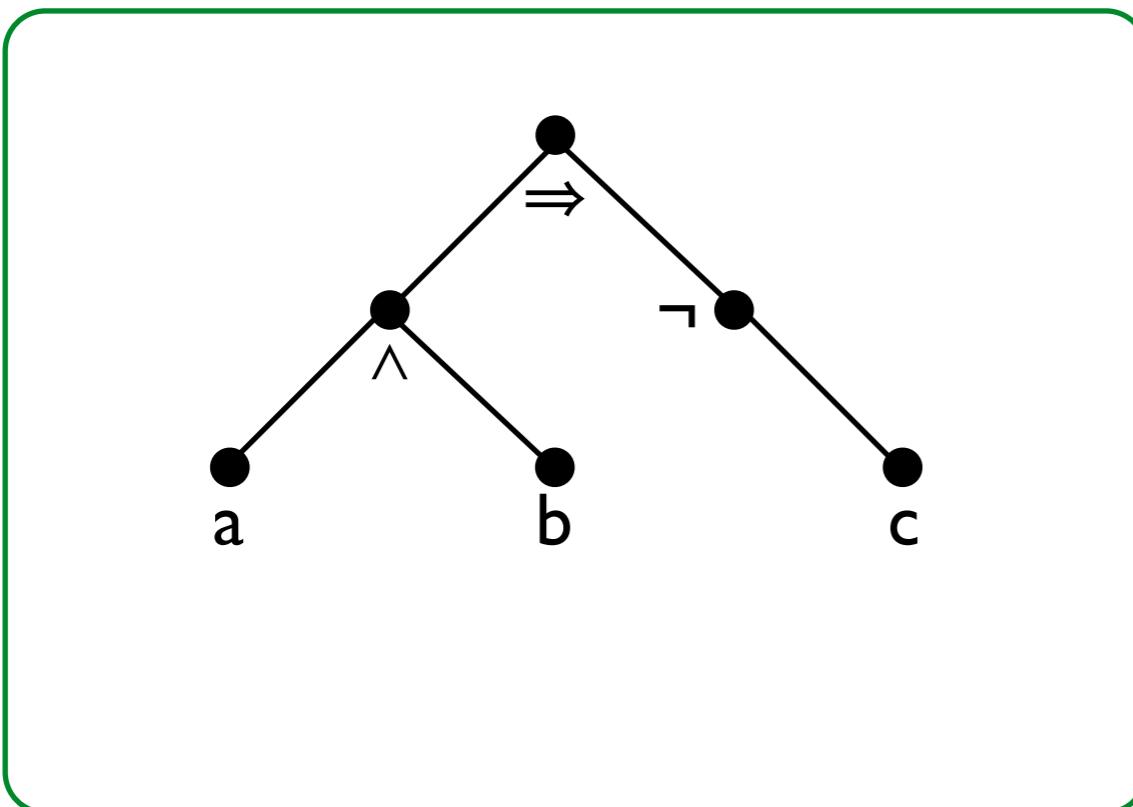
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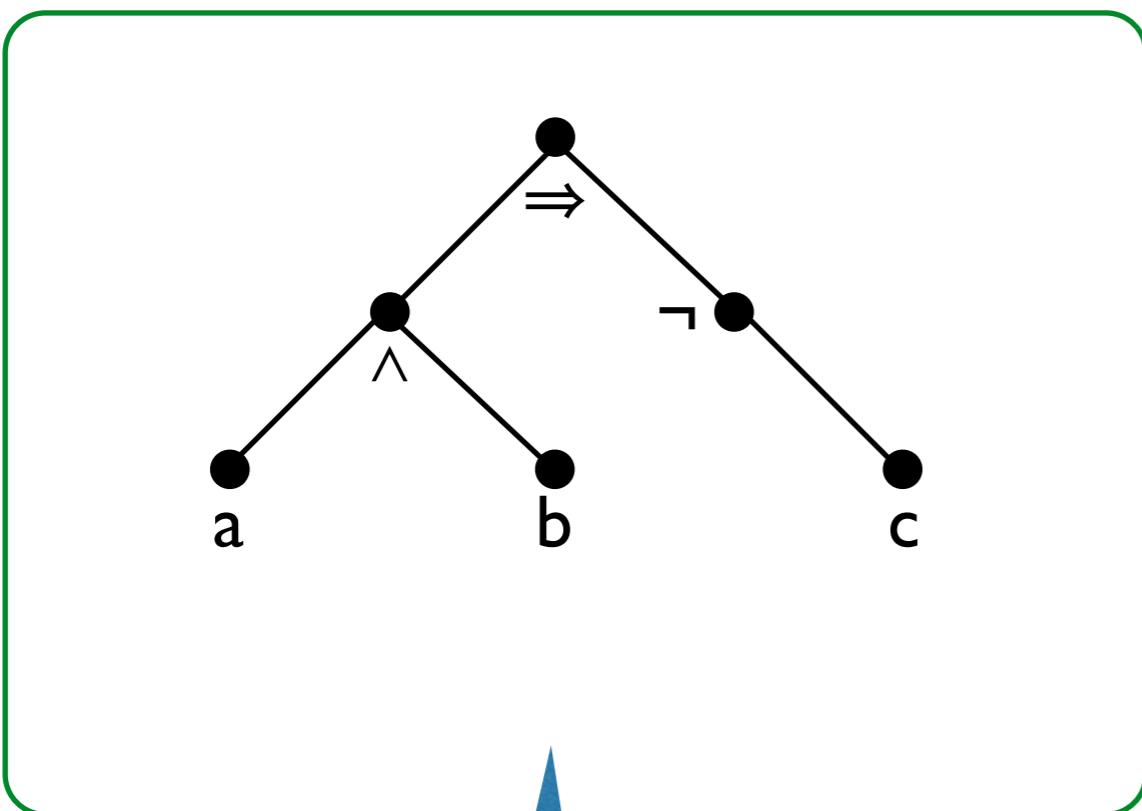


a recursive/inductive definition

...and their structure

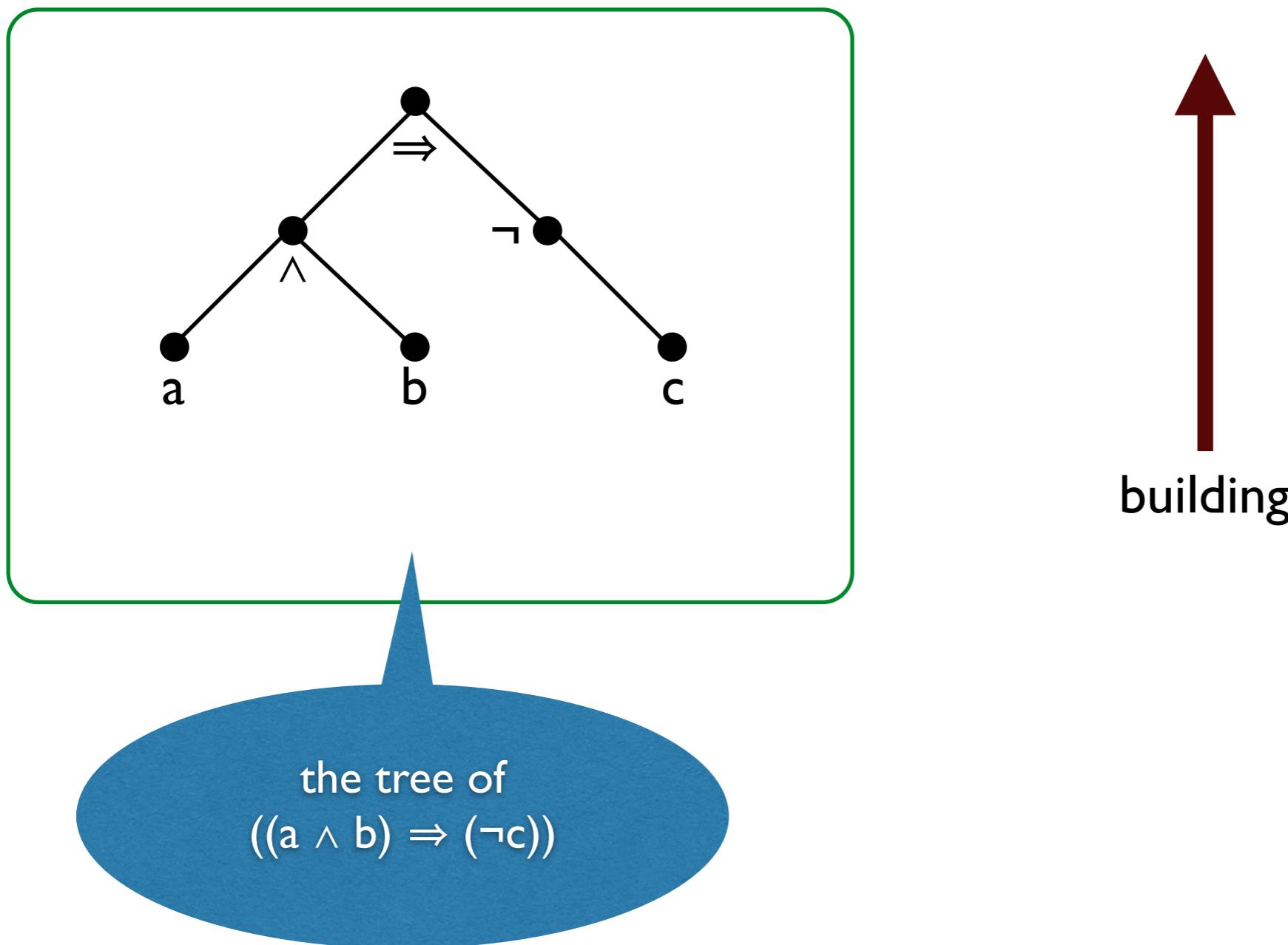


...and their structure

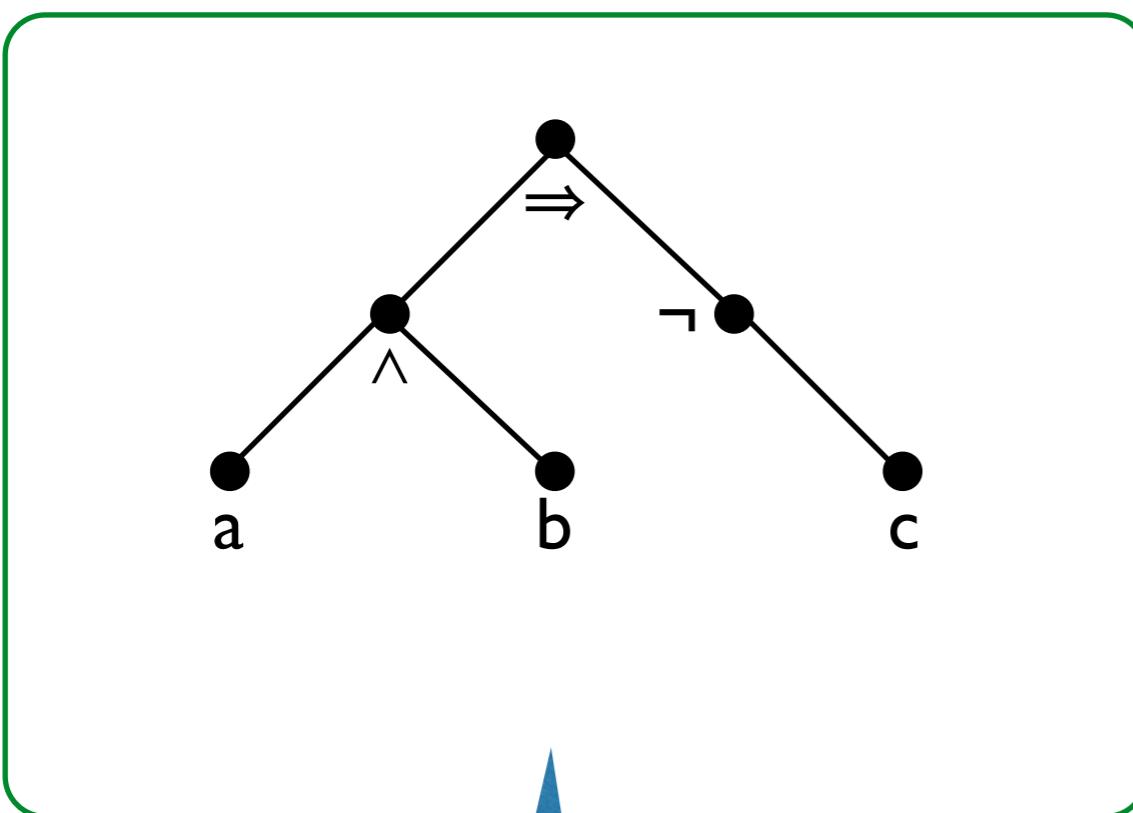


the tree of
 $((a \wedge b) \Rightarrow (\neg c))$

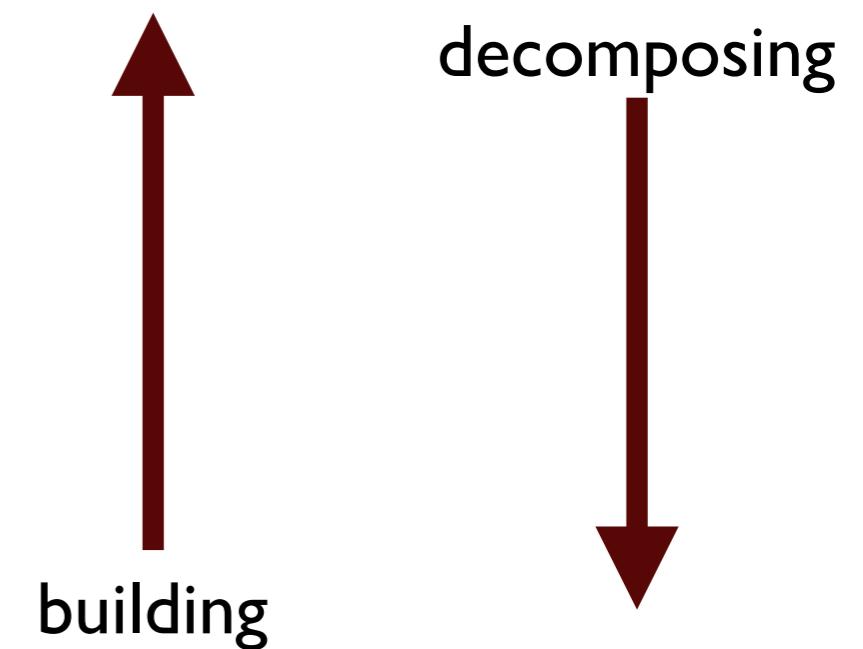
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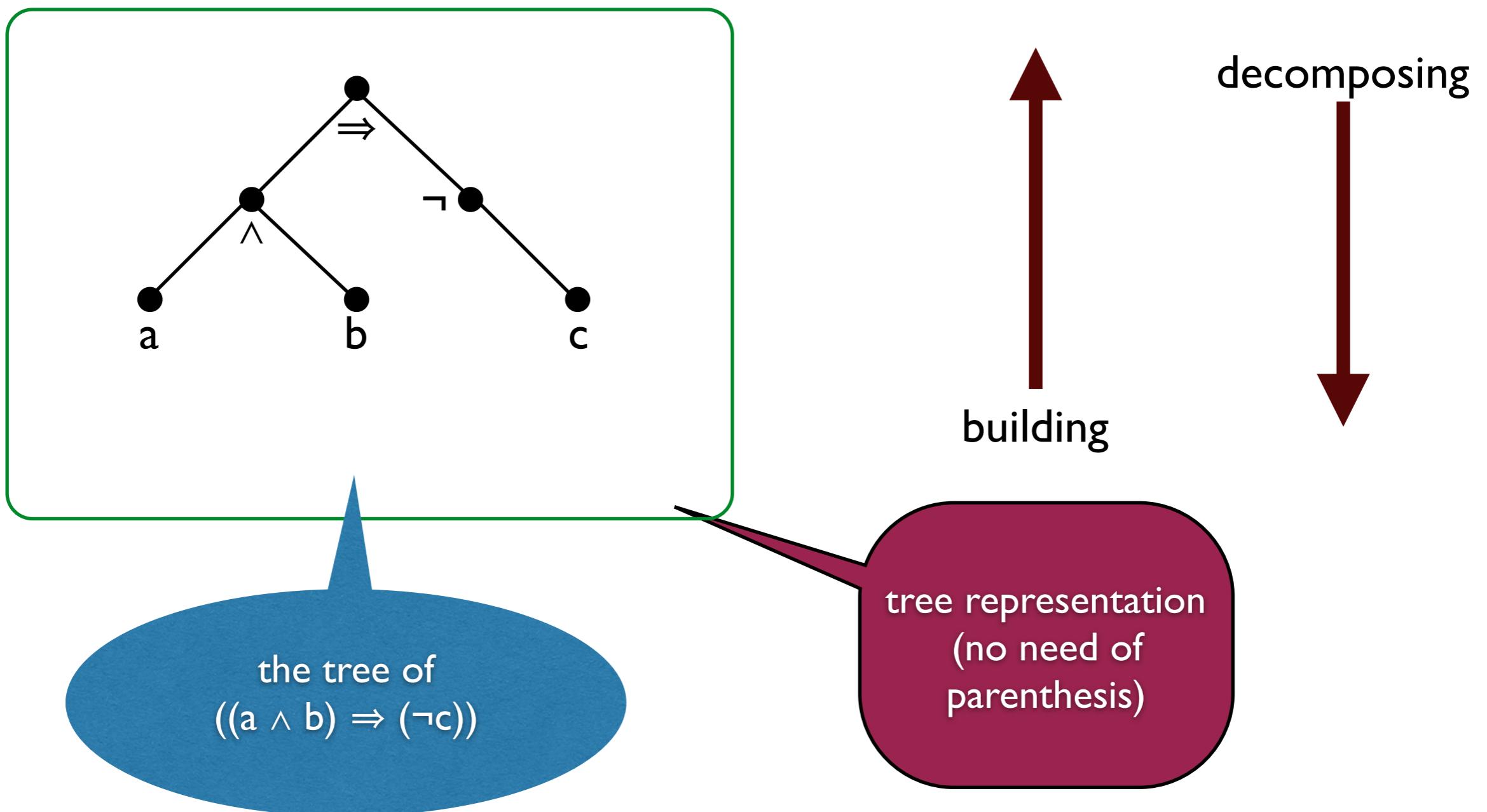
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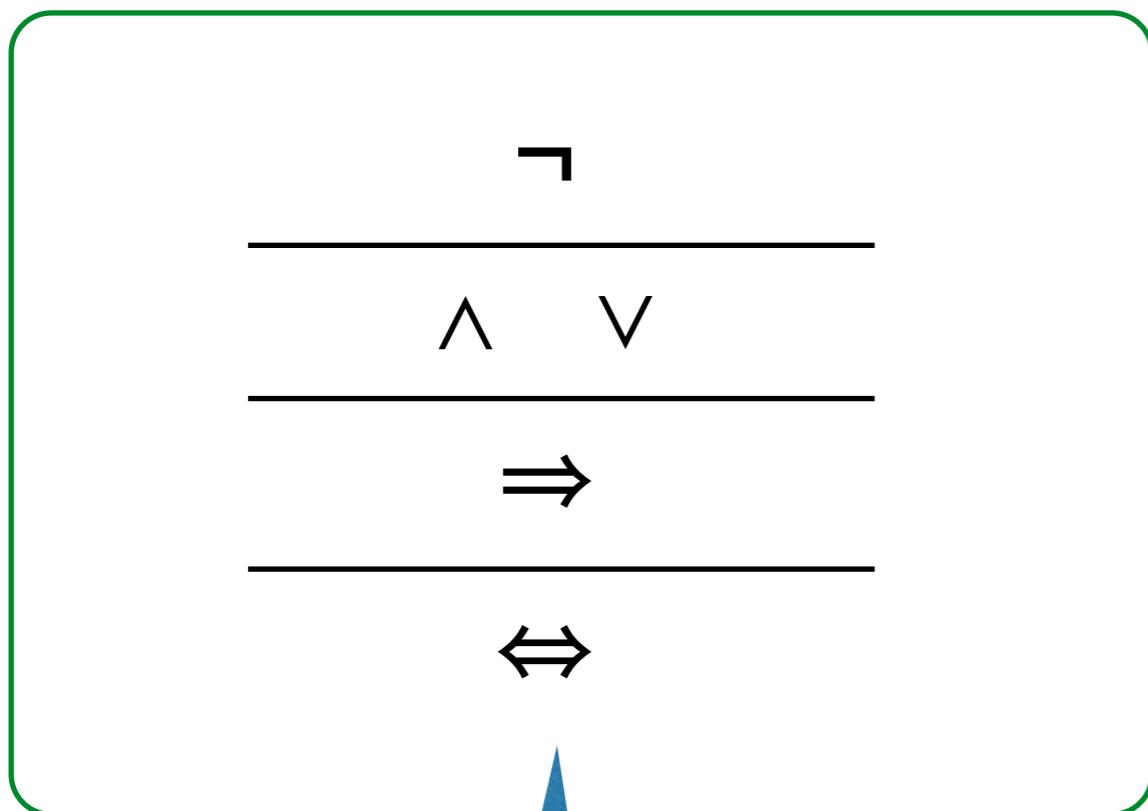
...and their structure



Dropping parenthesis

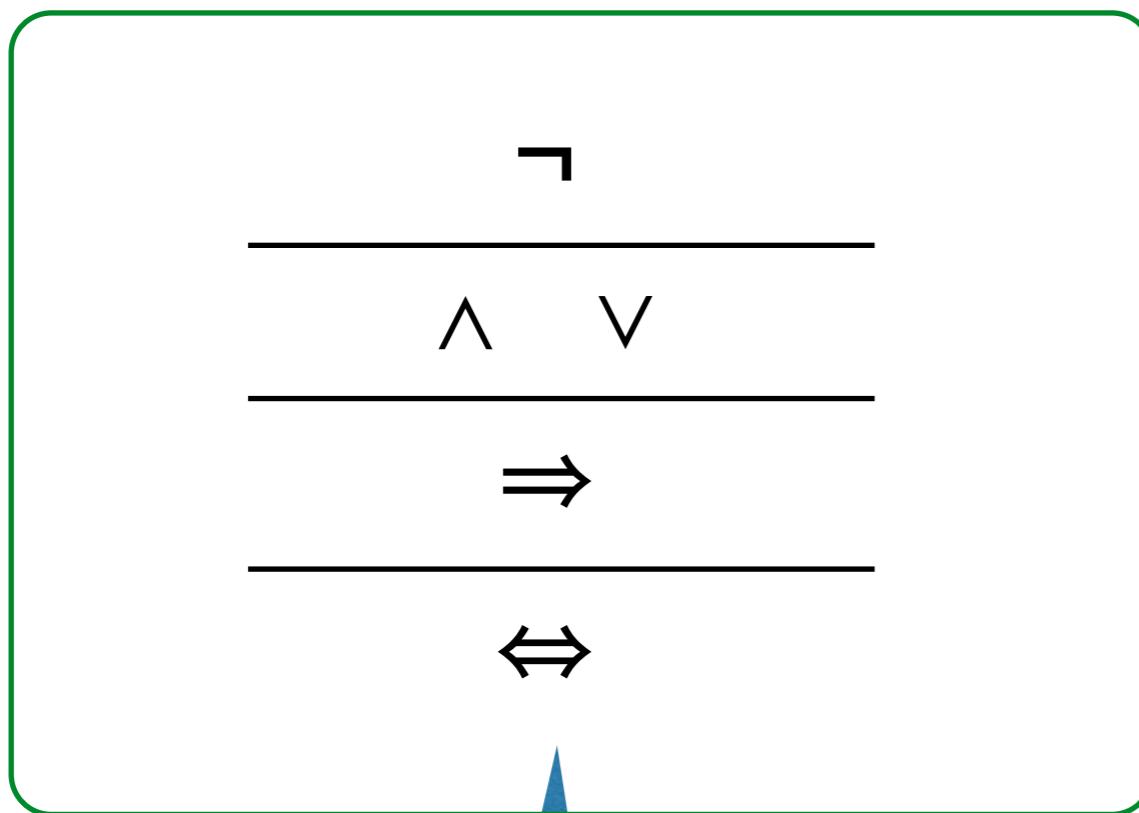
$$\begin{array}{c} \neg \\ \hline \wedge \quad \vee \\ \hline \Rightarrow \\ \hline \Leftrightarrow \end{array}$$

Dropping parenthesis



priority schema
(top binds the most)

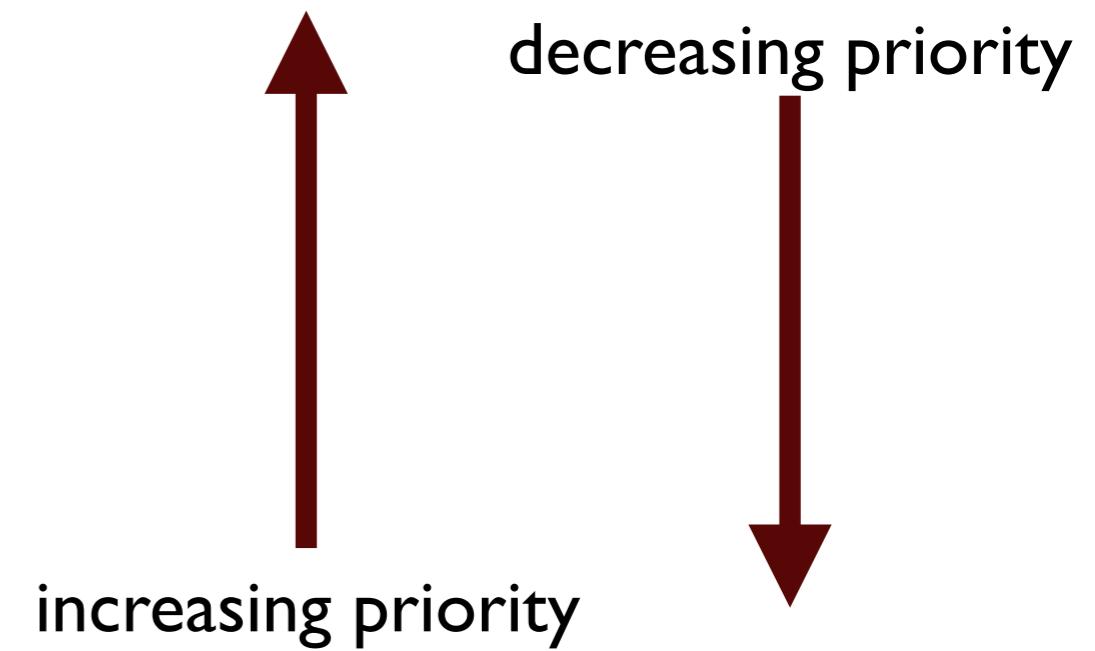
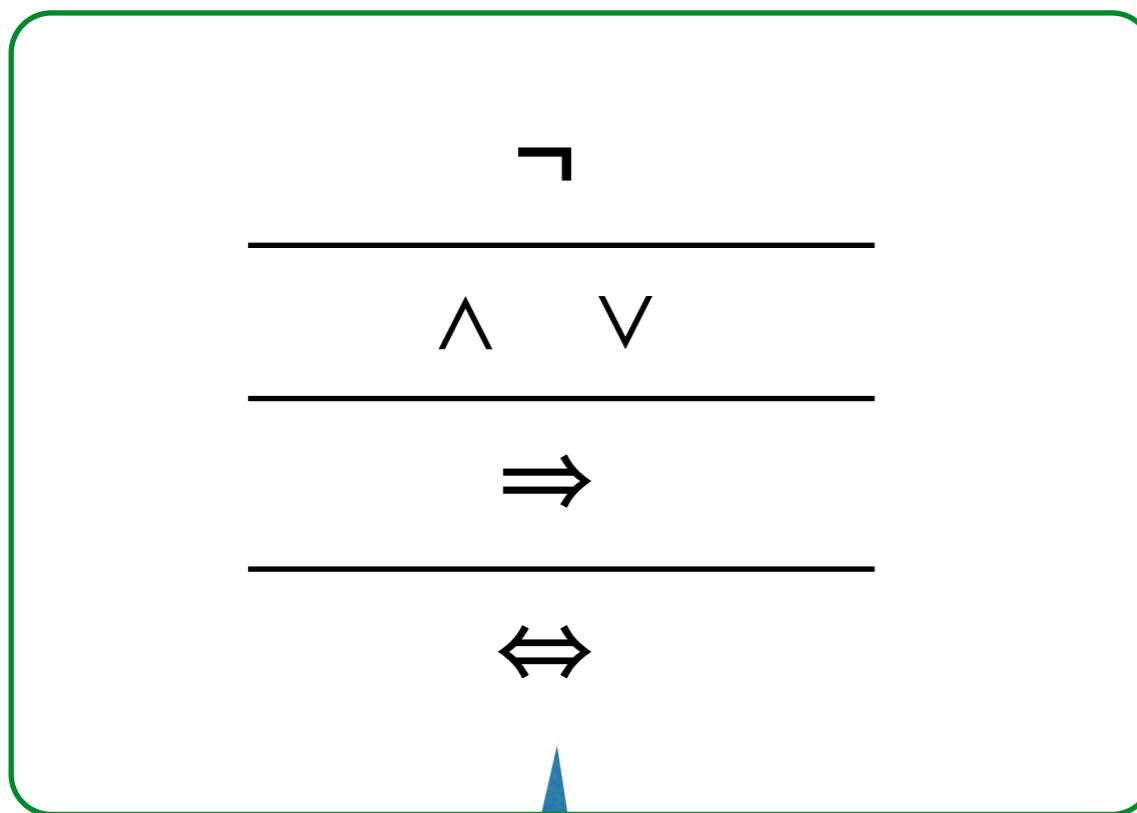
Dropping parenthesis



increasing priority

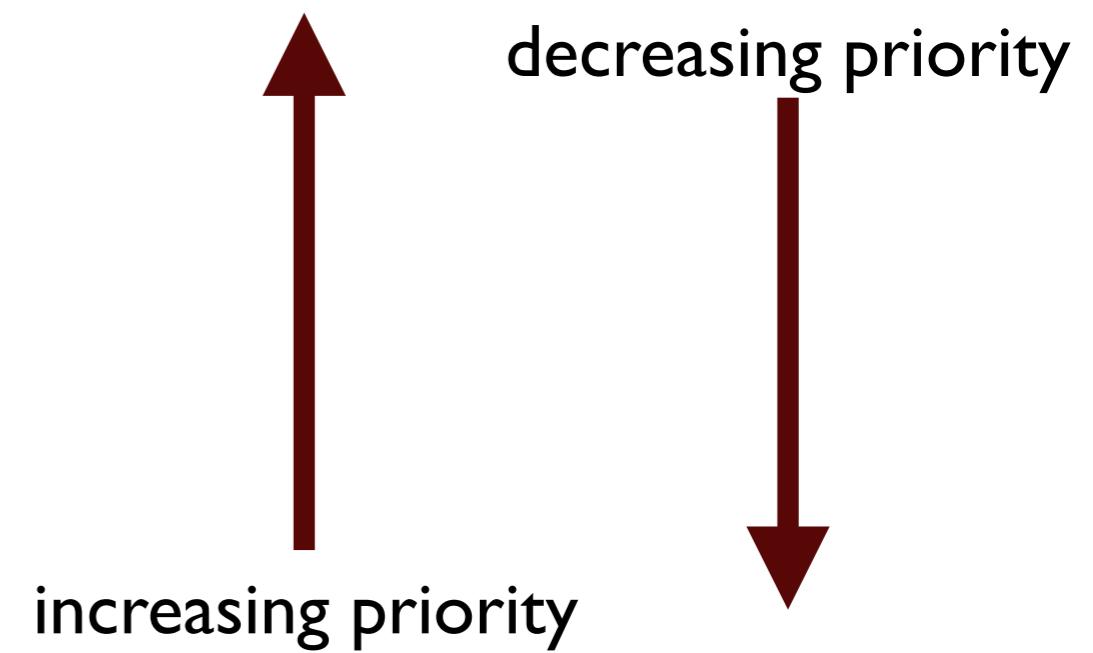
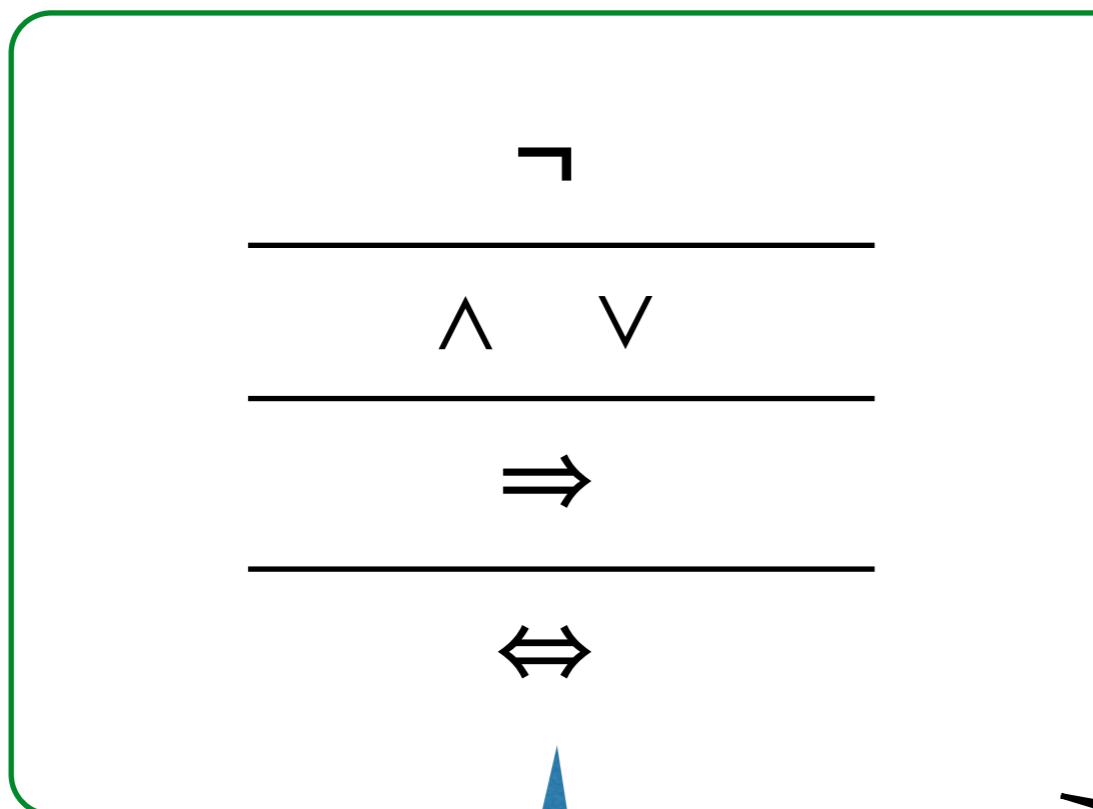
priority schema
(top binds the most)

Dropping parenthesis



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Dropping parenthesis



priority schema
(top binds the most)

Example: $((a \wedge b) \Rightarrow (\neg c))$
becomes
 $a \wedge b \Rightarrow \neg c$

Truth tables

Conjunction

P	Q	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

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P	Q	$P \wedge Q$
0	0	0
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Truth tables

Conjunction

P	Q	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

only true when both
P and Q are true

Truth tables

Disjunction

P	Q	$P \vee Q$
0	0	0
0	1	1
1	0	1
1	1	1

Truth tables

Disjunction

P	Q	$P \vee Q$
0	0	0
0	1	1
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Truth tables

Disjunction

P	Q	$P \vee Q$
0	0	0
0	1	1
1	0	1
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true when either P or Q or both are true

Truth tables

Negation

Truth tables

Negation

unary connective

Truth tables

Negation

unary connective

P	$\neg P$
0	1
1	0

Truth tables

Negation

unary connective

P	$\neg P$
0	1
1	0

Truth tables

Negation

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P	$\neg P$
0	1
1	0

true when P
is false

Truth tables

Implication

Truth tables

Implication

needs more attention

Truth tables

Implication

needs more attention

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

Truth tables

Implication

needs more attention

P	Q	$P \Rightarrow Q$
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Truth tables

Implication

needs more attention

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

only false when P is
true and Q is false

Truth tables

Bi-implication

Truth tables

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$$P \Leftrightarrow Q$$

$$\text{is } (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

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0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

Truth tables

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P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
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Truth tables

Bi-implication

$$P \Leftrightarrow Q$$

$$\text{is } (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
0	0			
0			0	0
	0	0		0

Truth tables

Bi-implication

$$P \Leftrightarrow Q$$

$$\text{is } (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
0	0			
0			0	0
	0	0		0

true when P and Q have the same truth value

Truth-functions

Def. A **truth-function** or **Boolean function** is a function

$$f: \{0, 1\}^n \longrightarrow \{0, 1\}$$

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by its inductive
structure, using the
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Notation in the book...

$$\left\{ \begin{array}{l} a, b \\ (0,0) \mapsto 0 \\ (0,1) \mapsto 1 \\ (1,0) \mapsto 0 \\ (1,1) \mapsto 1 \end{array} \right.$$

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$$P(a,b): (a \wedge b) \vee b$$

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a_1, \dots, a_n are the variables in P (and more) ordered in a sequence

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by its inductive structure, using the truth tables

$$P(a, b): (a \wedge b) \vee b$$

Truth-functions

$a_1, \dots a_n$ are the variables in P (and more) ordered in a sequence

Property: Every abstract proposition $P(a_1, \dots, a_n)$ with ordered and specified variables induces a truth-function.

Note:

The sequence of specified variables matters!

$P(a,b,c): (a \wedge b) \vee b$

induces

a, b, c
(0,0,0) \mapsto 0
(0,0,1) \mapsto 0
(0,1,0) \mapsto 1
(0,1,1) \mapsto 1
(1,0,0) \mapsto 0
(1,0,1) \mapsto 0
(1,1,0) \mapsto 1
(1,1,1) \mapsto 1

Equivalence of propositions

Equivalence of propositions

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{\text{val}}{=} Q$, iff they induce the same truth-function

on any sequence containing their common variables

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Property: The relation $\stackrel{\text{val}}{=}$ is an equivalence on the set of all abstract propositions

i.e., for all abstract propositions P, Q, R ,

- (1) $P \stackrel{\text{val}}{=} P$; (2) if $P \stackrel{\text{val}}{=} Q$, then $Q \stackrel{\text{val}}{=} P$; and
- (3) if $P \stackrel{\text{val}}{=} Q$ and $Q \stackrel{\text{val}}{=} R$, then $P \stackrel{\text{val}}{=} R$

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0				
0	1				
1	0				
1	1				

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	I			
0	I	I			
I	0	0			
I	I	0			

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1		
0	1	1	0		
1	0	0	1		
1	1	0	0		

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0	1	1	0	0	
1	0	0	1	0	
1	1	0	0	0	

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0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

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b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1	0	0
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

Their truth values are the same, so they are equivalent
 $b \wedge \neg b \stackrel{val}{=} c \wedge \neg c$

Tautologies and contradictions

Tautologies and contradictions

Def. An abstract proposition P is a **tautology** iff its truth-function is constant I .

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Def. An abstract proposition P is a **tautology** iff its truth-function is constant 1.

Def. An abstract proposition P is a **contradiction** iff its truth-function is constant 0.

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Def. An abstract proposition P is a **contradiction** iff its truth-function is constant 0.

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Def. An abstract proposition P is a **contingency** iff it is neither a tautology nor a contradiction.

but not all contingencies!

Abstract propositions

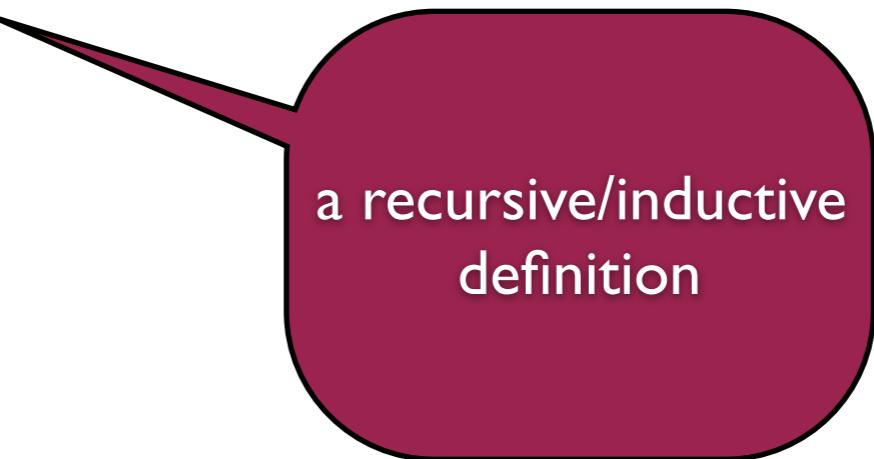
Definition

Basis (Case 1) T and F are abstract propositions.

Basis (Case 2) Propositional variables are abstract propositions.

Step (Case 1) If P is an abstract proposition, then so is $(\neg P)$.

Step (Case 2) If P and Q are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.



a recursive/inductive definition

Propositional Logic Standard Equivalences

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

$$P \Rightarrow Q \stackrel{val}{\neq} Q \Rightarrow P$$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0	1	1	0

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

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$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

Commutativity and Associativity

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$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

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$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	\parallel	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0		

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0	0	1

Idempotence and Double Negation

Idempotence

$$P \wedge P \stackrel{val}{=} P$$

$$P \vee P \stackrel{val}{=} P$$

$$P \Rightarrow P \stackrel{val}{\neq} P$$

$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Idempotence and Double Negation

Idempotence

$$P \wedge P \stackrel{val}{=} P$$

$$P \vee P \stackrel{val}{=} P$$

$$P \Rightarrow P \stackrel{val}{\neq} P$$

$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Double negation

$$\neg\neg P \stackrel{val}{=} P$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$
$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$
$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$
$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$
$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

T/F - elimination

$$P \wedge T \stackrel{val}{=}$$

$$P \wedge F \stackrel{val}{=}$$

$$P \vee T \stackrel{val}{=}$$

$$P \vee F \stackrel{val}{=}$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$
$$\neg F \stackrel{val}{=} T$$

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Contradiction

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T/F - elimination

$$P \wedge T \stackrel{val}{=} P$$

$$P \wedge F \stackrel{val}{=} F$$

$$P \vee T \stackrel{val}{=} T$$

$$P \vee F \stackrel{val}{=} P$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \stackrel{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \stackrel{val}{=} (P \vee Q) \wedge (P \vee R)$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \stackrel{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \stackrel{val}{=} (P \vee Q) \wedge (P \vee R)$$



De Morgan

$$\neg(P \wedge Q) \stackrel{val}{=} \neg P \vee \neg Q$$

$$\neg(P \vee Q) \stackrel{val}{=} \neg P \wedge \neg Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \vee Q$$

$$P \vee Q \stackrel{val}{=} \neg P \Rightarrow Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \vee Q$$

$$P \vee Q \stackrel{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \stackrel{val}{=} \neg Q \Rightarrow \neg P$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \vee Q$$

$$P \vee Q \stackrel{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \stackrel{val}{=} \neg Q \Rightarrow \neg P$$

$$P \Rightarrow Q \stackrel{val}{\neq} \neg P \Rightarrow \neg Q$$

common
mistake!

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \stackrel{val}{=}$$

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \stackrel{val}{=} T$$

Calculating with equivalent propositions

(the use of standard equivalences)

Recall...

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{\text{val}}{=} Q$, iff they induce the same truth-function

on any sequence containing their common variables

Property: The relation $\stackrel{\text{val}}{=}$ is an equivalence on the set of all abstract propositions.

i.e., for all abstract propositions P, Q, R ,

- (1) $P \stackrel{\text{val}}{=} P$; (2) if $P \stackrel{\text{val}}{=} Q$, then $Q \stackrel{\text{val}}{=} P$; and
- (3) if $P \stackrel{\text{val}}{=} Q$ and $Q \stackrel{\text{val}}{=} R$, then $P \stackrel{\text{val}}{=} R$

Substitution

Simple

$$\phi \stackrel{val}{=} \psi$$

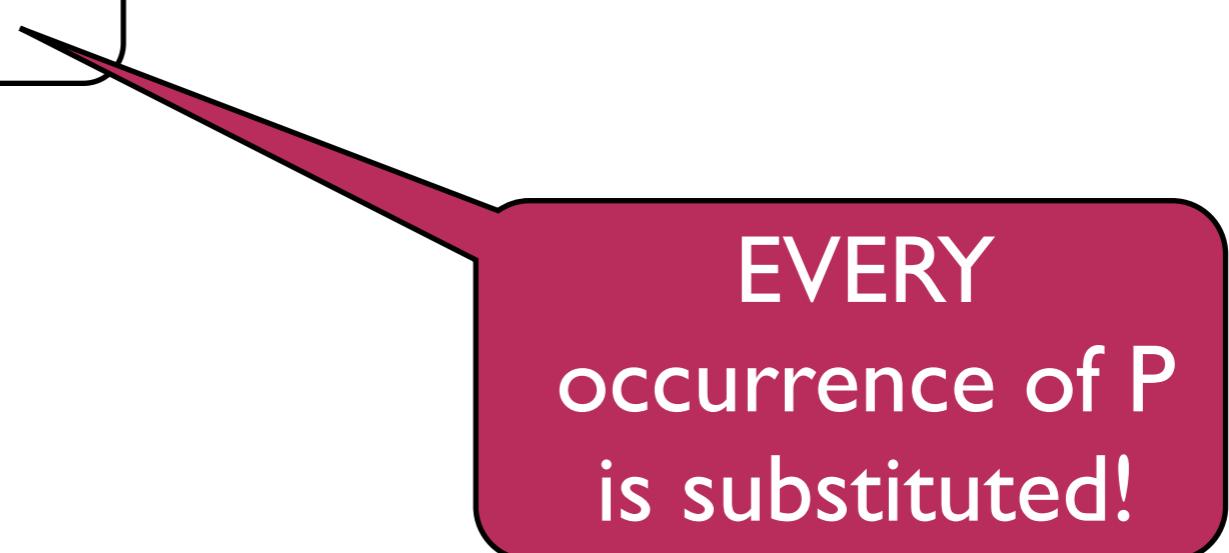
$$\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]$$

Substitution

Simple

$$\phi \stackrel{val}{=} \psi$$

$$\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]$$



EVERY
occurrence of P
is substituted!

Substitution

Simple

$$\phi \stackrel{val}{=} \psi$$

$$\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]$$

Sequential

$$\phi \stackrel{val}{=} \psi$$

$$\phi[\xi/P][\eta/Q] \stackrel{val}{=} \psi[\xi/P][\eta/Q]$$

EVERY
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Substitution

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$$\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]$$

Sequential

$$\phi \stackrel{val}{=} \psi$$

$$\phi[\xi/P][\eta/Q] \stackrel{val}{=} \psi[\xi/P][\eta/Q]$$

Simultaneous

$$\phi \stackrel{val}{=} \psi$$

$$\phi[\xi/P, \eta/Q] \stackrel{val}{=} \psi[\xi/P, \eta/Q]$$

EVERY
occurrence of P
is substituted!

Substitution

meta rule

Simple

$$\phi \stackrel{val}{=} \psi$$

$$\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]$$

Sequential

$$\phi \stackrel{val}{=} \psi$$

$$\phi[\xi/P][\eta/Q] \stackrel{val}{=} \psi[\xi/P][\eta/Q]$$

Simultaneous

$$\phi \stackrel{val}{=} \psi$$

$$\phi[\xi/P, \eta/Q] \stackrel{val}{=} \psi[\xi/P, \eta/Q]$$

EVERY
occurrence of P
is substituted!

The rule of Leibnitz

Leibnitz

$$\phi \stackrel{val}{=} \psi$$

$$C[\phi] \stackrel{val}{=} C[\psi]$$

The rule of Leibnitz

Leibnitz

$$\phi \stackrel{val}{=} \psi$$

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single
occurrence is
replaced!

The rule of Leibnitz

Leibnitz

$$\phi \stackrel{val}{=} \psi$$

$$C[\phi] \stackrel{val}{=} C[\psi]$$

formula that has
 ϕ as a sub formula

single
occurrence is
replaced!

The rule of Leibnitz

Leibnitz

$$\frac{\phi \stackrel{val}{=} \psi}{C[\phi] \stackrel{val}{=} C[\psi]}$$

formula that has
 ϕ as a sub formula

meta rule

single occurrence is replaced!

Strengthening
and
weakening

We had

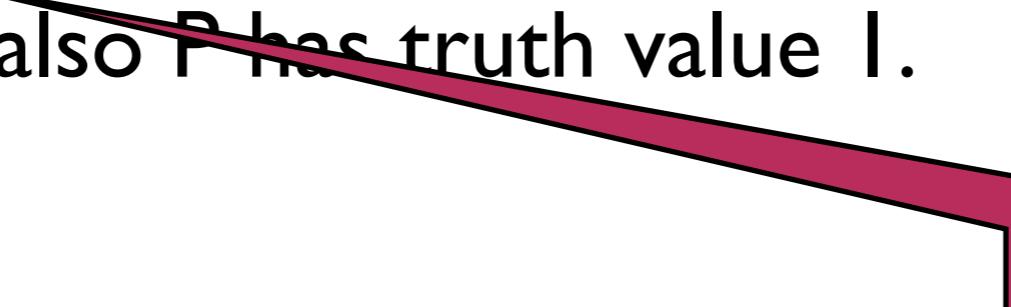
Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{\text{val}}{=} Q$, iff

- (1) Always when P has truth value I,
also Q has truth value I, and
- (2) Always when Q has truth value I,
also P has truth value I.

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Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{\text{val}}{=} Q$, iff

- (1) Always when P has truth value I,
also Q has truth value I, and
- (2) Always when Q has truth value I,
~~also P has truth value I.~~



if we relax this,
we get
strengthening

Strengthening

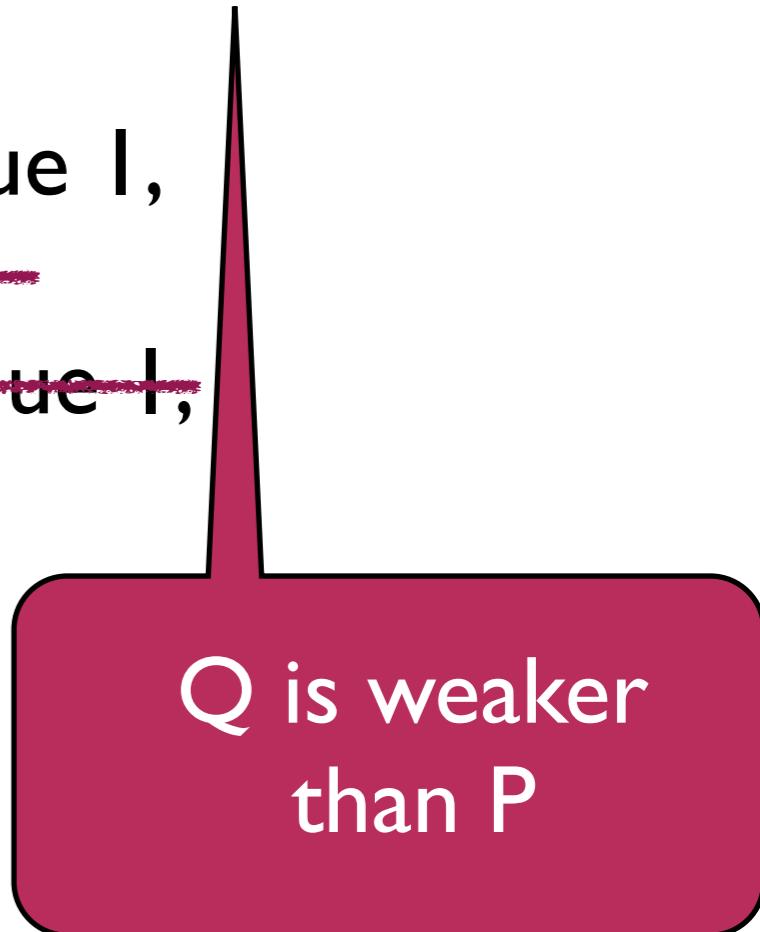
Definition: The abstract proposition P is stronger than Q, notation $P \models^{\text{val}} Q$, iff

- ~~(1) Always when P has truth value I,
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Strengthening

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- ~~(1) Always when P has truth value I,
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Strengthening

Definition: The abstract proposition P is stronger than Q, notation $P \models^{\text{val}} Q$, iff
always when P has truth value I,
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Strengthening

Definition: The abstract proposition P is stronger than Q,
notation $P \models^{\text{val}} Q$, iff

always when P has truth value I,
also Q has truth value I.

always when P is true,
Q is also true

Strengthening

Definition: The abstract proposition P is stronger than Q,

notation $P \models^{\text{val}} Q$, iff

always when P has truth value I,
also Q has truth value I.

always when P is true,
Q is also true

Q is weaker
than P

Properties

Properties

Lemma EI: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

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Properties

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Lemma W2: $P \stackrel{val}{\models} P$

Lemma W3: If $P \stackrel{val}{\models} Q$ and $Q \stackrel{val}{\models} R$ then $P \stackrel{val}{\models} R$

Properties

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Lemma W2: $P \stackrel{val}{\models} P$

Lemma W3: If $P \stackrel{val}{\models} Q$ and $Q \stackrel{val}{\models} R$ then $P \stackrel{val}{\models} R$

Lemma W4: $P \stackrel{val}{\models} Q$ iff $P \Rightarrow Q$ is a tautology.

Standard Weakenings

and-or-weakening

$$P \wedge Q \stackrel{val}{\models} P$$

$$P \stackrel{val}{\models} P \vee Q$$

Extremes

$$\text{F} \stackrel{val}{\models} P$$

$$P \stackrel{val}{\models} \text{T}$$

Calculating with weakenings (the use of standard weakenings)

Substitution

Simple

$$\frac{\begin{array}{c} val \\ \phi \models \psi \end{array}}{\phi\{\xi/P\} \models \psi\{\xi/P\}}$$

Sequential

$$\frac{\begin{array}{c} val \\ \phi \models \psi \end{array}}{\phi[\xi/P][\eta/Q] \models \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\begin{array}{c} val \\ \phi \models \psi \end{array}}{\phi[\xi/P, \eta/Q] \models \psi[\xi/P, \eta/Q]}$$

Substitution

just holds

Simple

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$$\frac{\begin{array}{c} val \\ \phi \models \psi \end{array}}{\phi[\xi/P, \eta/Q] \models \psi[\xi/P, \eta/Q]}$$

Substitution

just holds

Simple

$$\phi \models \psi$$

$$\frac{val}{\phi\{\xi/P\} \models \psi\{\xi/P\}}$$

Sequential

$$\phi \models \psi$$

$$\frac{val}{\phi[\xi/P][\eta/Q] \models \psi[\xi/P][\eta/Q]}$$

Simultaneous

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$$\frac{val}{\phi[\xi/P, \eta/Q] \models \psi[\xi/P, \eta/Q]}$$

EVERY
occurrence of P
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does not hold
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formula that has
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The rule of Leibnitz

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$$\phi \stackrel{val}{\models} \psi$$

$$C[\phi] \stackrel{val}{\models} C[\psi]$$

does not hold
for weakening!

Monotonicity

$$P \stackrel{val}{\models} Q$$

$$P \wedge R \stackrel{val}{\models} Q \wedge R$$

$$P \stackrel{val}{\models} Q$$

$$P \vee R \stackrel{val}{\models} Q \vee R$$