Predicate logic

Limitations of propositional logic

Propositional logic only allows us to reason about completed statements about things, not about the things themselves.

Example

Some chicken cannot fly
All chicken are birds

Some birds cannot fly

this reasoning can not be expressed in propositional logic

Example

Every player except the winner looses a match

Unary predicate (example)

Consider the statement $2m>3$.

Whether this statement is true or false depends on the value of m (and on the domain of values).

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on $\mathbb{Z}$

… -3 -2 -1 0 1 2 3 …

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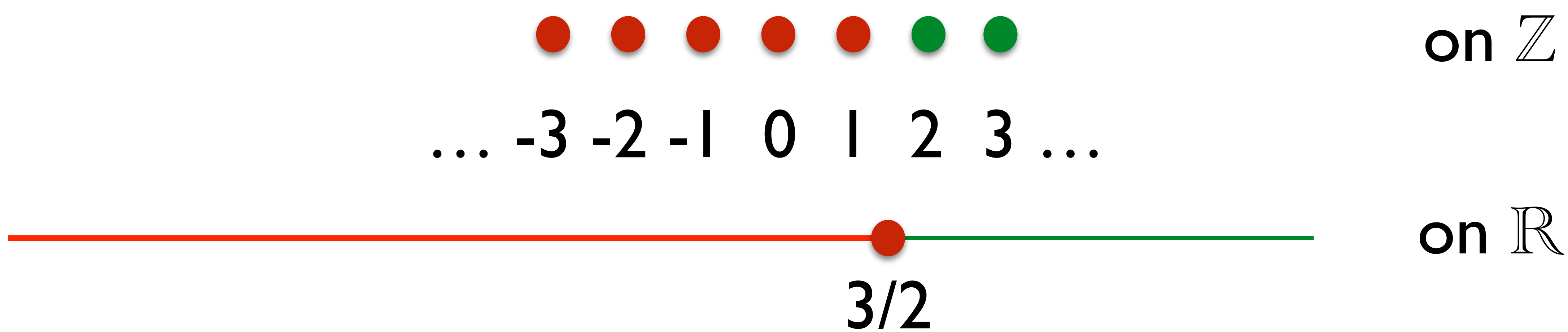

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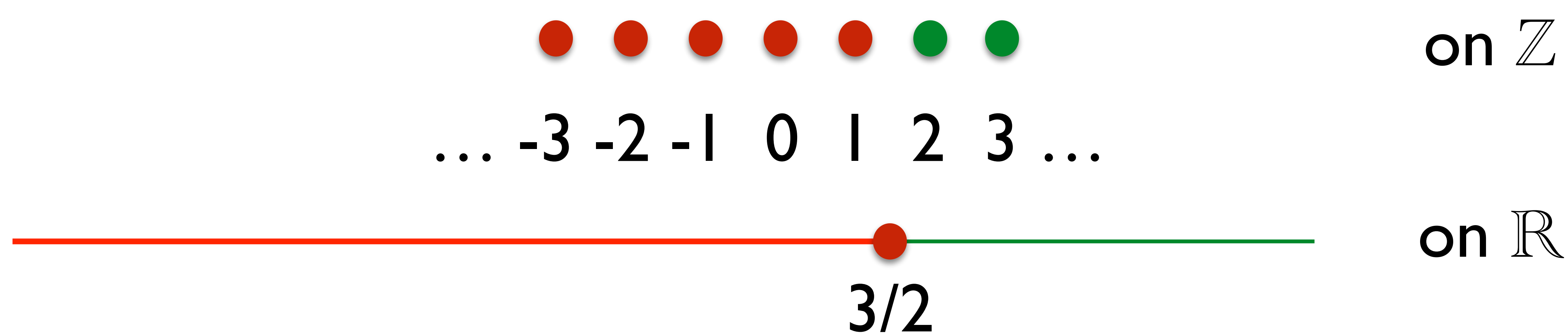

Note: $2m > 3 \overset{\text{val}}{=} m > 3/2$ on $\mathbb{Z}$ and $\mathbb{R}$

$2m > 3 \overset{\text{val}}{=} m \geq 2$ on $\mathbb{Z}$ but not on $\mathbb{R}$

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a unary relation

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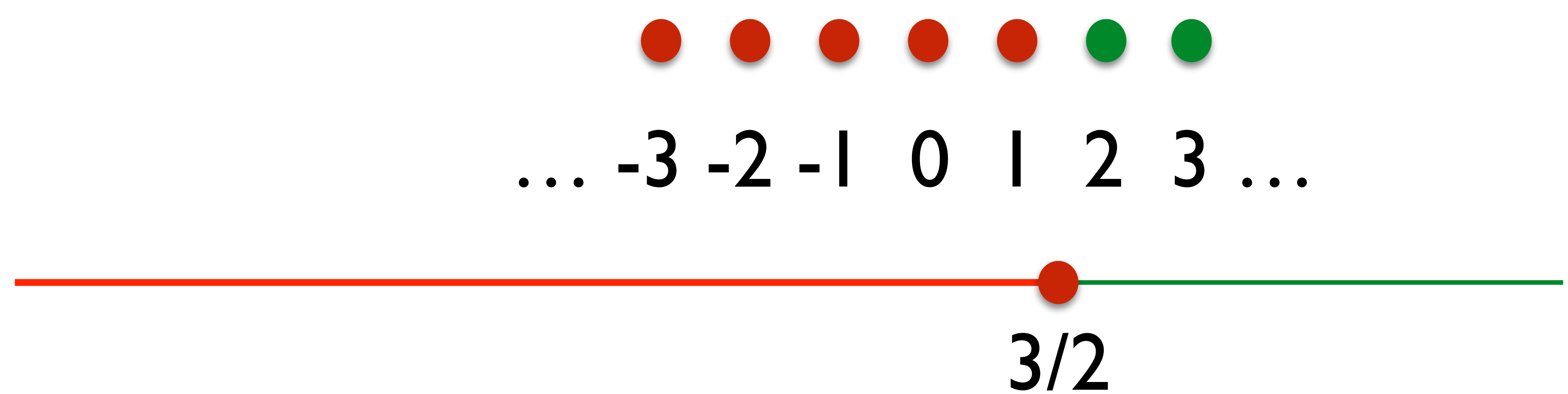

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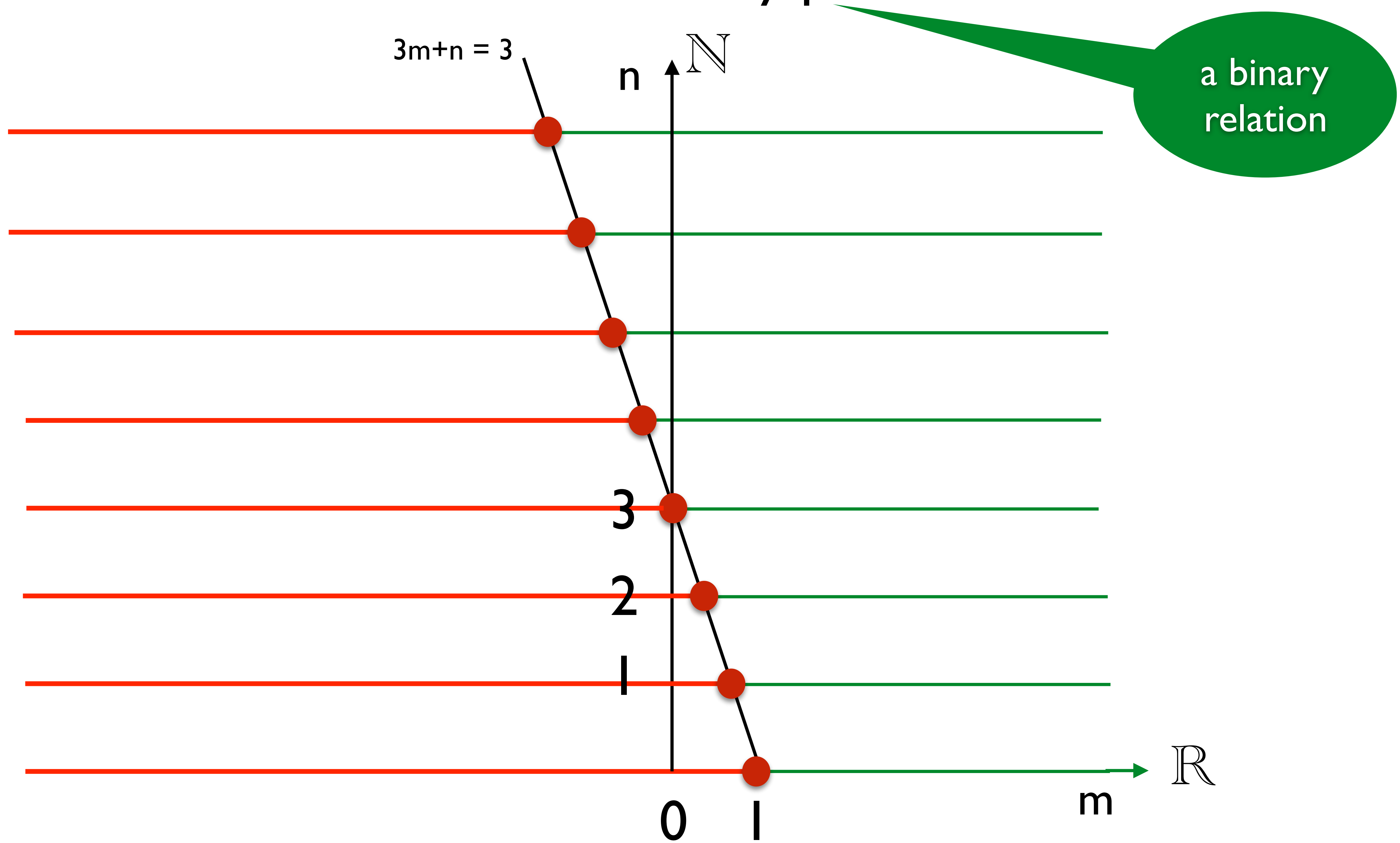
Binary predicate (example)

The statement $3m+n > 3$ is a binary predicate on $\mathbb{R} \times \mathbb{N}$.



Predicates

In general, an n-ary predicate is an n-ary relation.

If it is on a domain D, then it's a relation $P(x_1, .., x_n) \subseteq D^n$ or equivalently a function $P: D^n \rightarrow \{0,1\}$.

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for m=2

$2 \cdot 2 > 3$

is a true proposition

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unary predicate

binary predicate

proposition,
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other standard (!) notation:

$\exists m\ (m\in \mathbb{R} \wedge \forall n\ (n \in \mathbb{N} \Rightarrow 3m+n>3))$

Additional Notation Rules

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We write $\forall_x\ [P]$ for $\forall_x\ [T : P]$

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also for ∃

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also for $\exists$

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We also write $\exists_{m,} \forall_n [(m,n) \in \mathbb{R} \times \mathbb{N} : 3m + n > 3]$

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And even $\exists_{m,\,n}\ [(m,n) \in \mathbb{R} \times \mathbb{N} : 3m + n > 3]$

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but only for the same quantifier!

Quantification - task

Let P be the set of all tennis players.
Let $w \in P$ be the winner.

For $p, q \in P$, write $p \neq q$ for "p and q are different players" .

Let M be the set of all matches.
For $p \in P$ and $m \in M$, write $L(p,m)$ for
"player p loses match m".

Write the following sentence as a formula with predicates and quantifiers:

Every player except the winner loses a match.

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Thanks to Bas Luttik

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Equivalences with quantifiers

Renaming bound variables

Bound variables

$$\forall_x[P{:}Q] \overset{val}{=} \forall_y[P[y/x]{:}Q[y/x]]$$

$$\exists_x[P{:}Q] \overset{val}{=} \exists_y[P[y/x]{:}Q[y/x]]$$

if y does not occur in
P or Q (not even in ∀y, ∃y)

Domain splitting

Examples:

$$\begin{aligned} &\forall_x[x \leqslant 1 \vee x \geqslant 5\colon x^2 - 6x + 5 \geqslant 0] \\ \stackrel{val}{=}\ &\forall_x[x \leqslant 1\colon x^2 - 6x + 5 \geqslant 0] \wedge \forall_x[x \geqslant 5\colon x^2 - 6x + 5 \geqslant 0] \end{aligned}$$

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$$\exists_k[0 \leqslant k \leqslant n : k^2 \leqslant 10]$$
$$\overset{val}{=} \exists_k[0 \leqslant k \leqslant n-1 \vee k = n : k^2 \leqslant 10]$$
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Domain splitting

$$\forall_x[P \vee Q : R] \stackrel{val}{=} \forall_x[P : R] \wedge \forall_x[Q : R]$$
$$\exists_x[P \vee Q : R] \stackrel{val}{=} \exists_x[P : R] \vee \exists_x[Q : R]$$

Equivalences with quantifiers

One-element domain

$$\forall_x[x = n\colon Q] \stackrel{val}{=} Q[n/x]$$

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Empty domain

$$\forall_x[F : Q] \stackrel{val}{=} T$$

$$\exists_x[F : Q] \stackrel{val}{=} F$$

Equivalences with quantifiers

One-element domain

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$$\forall_x[F : Q] \stackrel{val}{=} T$$

$$\exists_x[F : Q] \stackrel{val}{=} F$$

"All Marsians are green"

Domain weakening

Intuition: The following are equivalent

$$\forall_x[x \in D : A(x)] \quad \text{and} \quad \forall_x[x \in D \Rightarrow A(x)]$$
$$\exists_x[x \in D : A(x)] \quad \text{and} \quad \exists_x[x \in D \wedge A(x)]$$

The same can be done to parts of the domain

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Domain weakening

$$\forall_x [P \wedge Q : R] \stackrel{val}{=} \forall_x [P : Q \Rightarrow R]$$
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$$P \wedge Q \stackrel{val}{\models} P$$

De Morgan with quantifiers

De Morgan

$$\neg\forall_x[P:Q] \stackrel{val}{=} \exists_x[P:\neg Q]$$
$$\neg\exists_x[P:Q] \stackrel{val}{=} \forall_x[P:\neg Q]$$

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not for all = at least for one not

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Hence: $\neg \forall = \exists \neg$ and $\neg \exists = \forall \neg$

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Hence: $\neg\forall = \exists\neg$ and $\neg\exists = \forall\neg$

It holds further that:

$$\neg\forall_x\neg = \exists_x\neg\neg = \exists_x$$
$$\neg\exists_x\neg = \forall_x\neg\neg = \forall_x$$

Substitution

meta rule

Simple

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P] \overset{val}{=} \psi[\xi/P]}$$

EVERY occurrence of P is substituted!

Sequential

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{=} \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{=} \psi[\xi/P, \eta/Q]}$$

Substitution

holds also for quantified formulas!

meta rule

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The rule of Leibniz

meta rule

Leibniz

$$\frac{\phi \overset{val}{=} \psi}{C[\phi] \overset{val}{=} C[\psi]}$$

formula that has ϕ as a sub formula

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The rule of Leibniz

meta rule

Leibniz

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Other equivalences with quantifiers

Exchange trick

$$\forall_x[P{:}Q] \overset{val}{=} \forall_x[\neg Q{:}\neg P]$$

$$\exists_x[P{:}Q] \overset{val}{=} \exists_x[Q{:}P]$$

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No wonder as

$$\forall_x[P{:}Q] \stackrel{val}{=} \forall_x[P \Rightarrow Q]$$

$$\exists_x[P{:}Q] \stackrel{val}{=} \exists_x[P \wedge Q]$$

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Term splitting

$$\forall_x[P{:}Q \wedge R] \stackrel{val}{=} \forall_x[P{:}Q] \wedge \forall_x[P{:}R]$$

$$\exists_x[P{:}Q \vee R] \stackrel{val}{=} \exists_x[P{:}Q] \vee \exists_x[P{:}R]$$

Other equivalences with quantifiers

Monotonicity of quantifiers

$$\forall_x[P{:}Q \Rightarrow R] \Rightarrow (\forall_x[P{:}Q] \Rightarrow \forall_x[P{:}R]) \stackrel{val}{=} T$$

$$\forall_x[P{:}Q \Rightarrow R] \Rightarrow (\exists_x[P{:}Q] \Rightarrow \exists_x[P{:}R]) \stackrel{val}{=} T$$

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tautologies

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tautologies

Lemma E1: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Other equivalences with quantifiers

Monotonicity of quantifiers

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$$\forall_x[P{:}Q \Rightarrow R] \Rightarrow (\exists_x[P{:}Q] \Rightarrow \exists_x[P{:}R]) \overset{val}{=} T$$

tautologies

Lemma E1: $P \overset{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

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Lemma W5: If $Q \overset{val}{\models} R$ then $\forall_x[P{:}Q] \overset{val}{\models} \forall_x[P{:}R]$.