Coalgebra Course

14.6.2012 + 21.6.2012

(lecture notes)

Properties and Comparison of Behavior Semantics

So far, we have dealt with the following behavior semantics.

Let $c : S \to F(S)$,
$d : T \to F(T)$
be two coalgebras,
and let $s \in S$, $t \in T$

I. Final Coalgebra Semantics

The states s and t are equivalent by final coalgebra semantics, notation $s \Leftrightarrow t$ if there exists a final F-coalgebra $\zeta : Z \to F(Z)$ and $beh_c(s) = beh_d(t)$ where beh_c (beh_d) is the unique homomorphism from c (d) to the final ζ.

II. Bisimilarity

The states s and t are bisimilar, notation $s \sim t$ if there exists a bisimulation relation R that relates them (we also say that witnesses that $s \sim t$), i.e. a relation $R \subseteq S \times T$ with a coalgebra structure $r : R \to F(R)$ making $\pi_1 : R \to S$, $\pi_2 : R \to T$ coalgebra homomorphisms, i.e., making the following diagram commute

$$\begin{tikzcd} S \arrow[d, "c"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "r"] & T \arrow[d, "d"] \\ F(S) & F(R) \arrow[l, "F\pi_1"'] \arrow[r, "F\pi_2"] & F(T) \end{tikzcd}$$

III. Behavioral Equivalence

The states s and t are behaviorally equivalent, notation $s \approx t$ if there exists a cocongruence (U, u_1, u_2) that identifies them (we also say that witnesses that $s \approx t$), i.e. a cospan (U, u_1, u_2) with a coalgebra structure $u : U \to F(U)$ making $u_1 : S \to U$, $u_2 : T \to U$ coalgebra homomorphisms, i.e., making the following diagram commute

$$\begin{tikzcd}
S \arrow[r, "u_1"] \arrow[d, "c"'] & U \arrow[d, "u"] & T \arrow[l, "u_2"'] \arrow[d, "d"] \\
F(S) \arrow[r, "Fu_1"] & F(U) & F(T) \arrow[l, "Fu_2"']
\end{tikzcd}$$

Such that $u_1(s) = u_2(t)$.
↳ "identifying part"

So far (before 14.6.2012) we had only given the definition of behavioral equivalence.

Now, the following holds. (if F is wpp and has a final)

Theorem: On the states of a single coalgebra $c : S \to F(S)$ all three behavior semantics $\underline{\leftrightarrow}$, $\sim$, and $\approx$ are equivalences.

Proof: For $\underline{\leftrightarrow}$ it is simple, since on a single coalgebra $c : S \to F(S)$, $\underline{\leftrightarrow} = \ker(beh_c)$ where the kernel is the equivalence

$$\ker(f) = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$$

$f : X \to Y$ any map.

Since it is defined using equality, it is very simple to show that it's always an equivalence.

For bisimilarity, for wpp functors, we have shown this in class before 14.6.2012.

For behavior equivalence - it was your HW week 9. ■

[Actually the result reads as follows:

- Behavioral equivalence is always an equivalence.
- Bisimilarity is an equivalence if the functor is wpp.
- Final coalgebra semantics is an equivalence if final coalgebra exists]

Next we see that for behavioral equivalence on a single coalgebra one does not even need a cospan, just a homomorphism is sufficient.

Proposition 1: We have $s \approx t$ in $c: S \to F(S)$ iff there exists a coalgebra $u: U \to F(U)$ and a (surjective*) coalgebra homomorphism $h: S \to U$ from c to u, s.t.

$$h(s) = h(t).$$

Before we prove this proposition, let's learn about coequilizers - we need them for the one direction of (party) the proof.

But as an illustration, we'll learn all that's important about them.

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* recall that in the original def. of a cocongruence it is required that u_1, u_2 are jointly surjective. If that's the case h is surjective and ~~vice~~ vice-versa. I see no reason for requiering that for us.

You can read about coequilizers at many places, but for example in "Universal Coalgebra : a theory of systems" Section 4 (in particular 4.2).

Coequilizers in Sets

A coequilizer is a special colimit. In particular it is the colimit of two parallel arrows, i.e., $X \underset{g}{\overset{f}{\rightrightarrows}} Y$.

More concretely, a coequilizer is an object (set) C together with an arrow $h : Y \to C$ making

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} C$$

commute that is

$$h \circ f = h \circ g$$

and (as every colimit) it has the universal property: Given any other candidate coequilizer C' with h' in a situation i.e. $h' \circ f = h' \circ g$, $h' : Y \to C'$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] \arrow[dr, "h'"'] & C \\ & & C' \end{tikzcd}$$

there is a unique mediating arrow $m : C \to C'$ making the triangle commute

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] \arrow[dr, "h'"'] & C \arrow[d, dotted, "m"] \\ & & C' \end{tikzcd}$$

Coequilizers (like all colimits) exist in <u>Sets</u> and they are concretely constructed as follows.

$C = Y/\Theta$

where Θ is the smallest equivalence on Y containing the pairs $\{(f(x), g(x)) \mid x \in X\}$

(Θ is the equivalence generated by these pairs).

Coequalizers in $CoAlg(F)$

Coequalizers also exist in $CoAlg(F)$ and they are constructed "on top" of the Sets ones.

Here is how. Let $c: X \to F(X)$ and $d: Y \to F(Y)$ be two coalgebras and $f: X \to Y$, $g: X \to Y$ two (parallel) coalgebra homomorphisms from c to d, i.e.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "c"'] & Y \arrow[r, "h"] \arrow[d, "d"] & C \\ F(X) \arrow[r, shift left, "Ff"] \arrow[r, shift right, "Fg"'] & F(Y) \arrow[r, "Fh"] & F(C) \end{tikzcd}$$

Moreover, let h and C be the Sets coequalizer of f and g.

$$\begin{aligned} \text{Now,}\quad Fh \circ d \circ f &\underset{f\text{-hom.}}{=} Fh \circ Ff \circ c \underset{F\text{-functor}}{=} F(h \circ f) \circ c \\ &\underset{\text{Sets' coeq.}}{=} F(h \circ g) \circ c \\ &= Fh \circ Fg \circ c \\ &\underset{g\text{-hom.}}{=} Fh \circ d \circ g \end{aligned}$$

Hence, $F(C)$ together with the map $Fh \circ d$ is a candidate coequalizer in Sets of f and g.

Therefore there exists a unique $m: C \to F(C)$ such that $m \circ h = Fh \circ d$, i.e., turning ~~[illegible]~~ C into a coalgebra and h into a coalgebra homomorphism.

as in the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "c"'] & Y \arrow[r, "h"] \arrow[d, "d"] & C \arrow[d, dashed, "m"] \\ F(X) \arrow[r, shift left, "Ff"] \arrow[r, shift right, "Fg"'] & F(Y) \arrow[r, "Fh"'] & F(C) \end{tikzcd}$$

One still needs to check that $m: C \to F(C)$ together with h has the universal property in $CoAlg(F)$, that is if $m': C' \to F(C')$ with $h': Y \to C'$ (a hom. from d to m')

is a candidate coequilizer, i.e.
$h' \circ f = h' \circ g$ then there is a unique map

$u: C \to C'$ that is a coalgebra hom. from m to m'

and $\underline{m' = u \circ m}$

Now, such a unique map u certainly exists since C with h is a coequilizer in sets but C' with h' is a candidate. It remains to check that it is indeed a coalgebra hom. i.e., $\underline{Fu \circ m = m' \circ u}$ $\to$ This was your <u>Hw week 10</u>.

<u>Here is a hint</u>:

First let's learn about epis and monos.

An arrow h is <u>epi</u> if $f \circ h = g \circ h \Rightarrow f = g$
(for any compatible arrows f and g)

An arrow h is <u>mono</u> if $h \circ f = h \circ g \Rightarrow f = g$
(for any compatible arrows f and g)

$\boxed{*}$ Prove first that any coequilizer map in <u>sets</u> is necessarily epi.

[In <u>sets</u> epis are surjections, monos are injections]

$\boxed{**}$ Then prove that

$$Fu \circ m \circ h = m' \circ u \circ h$$

where h is the coequilizer map.

With this you are done.

Proof of Proposition 1:

The one direction $(\Leftarrow)$ is clear. If such an h exists, then (U, h, h) is a cocongruence witnessing $s \approx t$.

For the other direction $(\Rightarrow)$ let $s \approx t$ witnessed by a cocongruence (U, u_1, u_2) with $u: U \to F(U)$. $(u_1, u_2 : S \to U)$

Consider the sets coequilizer $h: U \to C$ of u_1, u_2.

As in the construction of a coequilizer in $CoAlg(F)$, there exists a (unique) coalgebra structure $m: C \to F(C)$ such that h is a coalgebra homomorphism, so we have the situation

$$\begin{tikzcd} S \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_2"'] \arrow[d, "c"'] & U \arrow[r, "h"] \arrow[d, "u"] & C \arrow[d, dashed, "m"] \\ F(S) \arrow[r, shift left, "Fu_1"] \arrow[r, shift right, "Fu_2"'] & F(U) \arrow[r, "Fh"'] & F(C) \end{tikzcd}$$

and $h \circ u_1 = h \circ u_2$.

Let $e = h \circ u_1 = h \circ u_2$. This is obviously a homomorphism from $c: S \to F(S)$ to $m: C \to F(C)$, and we have

$$e(s) = h(u_1(s)) = h(u_2(t)) = e(t)$$

since $u_1(s) = u_2(t)$

[Moreover, if u_1 and u_2 are jointly surjective, then h is surjective]

∎

(This wraps-up what we slowly did on 14.6.)

Now how do the semantics relate.

<u>Proposition 2</u>: In the presence of a final F-coalgebra

$\xi: Z \to F(Z)$

behavioral equivalence and final coalgebra semantics coincide.

<u>Proof</u>: We consider a single coalgebra $c: S \to F(S)$ and two states $s, t \in S$.

Let $s \Leftrightarrow t$. Then by definition $beh_c(s) = beh_c(t)$ where beh_c is the unique homomorphism from c to ξ, so by Prop. 1, $s \approx t$.

Let $s \approx t$. Then there is a (by Prop. 1) coalgebra $u: U \to F(U)$ and a coalgebra homomorphism h from c to u s.t. $h(s) = h(t)$.

But then we have the following situation

$$\begin{tikzcd} S \arrow[r, "h"] \arrow[d, "c"'] \arrow[rr, bend left, "beh_c"] & U \arrow[r, "beh_u"] \arrow[d, "u"] & Z \arrow[d, "\xi", "\cong"'] \\ F(S) \arrow[r, "Fh"] \arrow[rr, bend right, "F beh_c"'] & F(U) \arrow[r, "F beh_u"'] & F(Z) \end{tikzcd}$$

So, by finality (since $beh_u \circ h$ is a homomorphism from c to ξ)

we get $\underline{beh_u \circ h = beh_c}$ and hence

$$beh_c(s) = beh_u \circ h(s) \underset{h(s)=h(t)}{=} beh_u \circ h(t) = beh_c(t)$$

showing that $s \Leftrightarrow t$. ∎

Next we focus on the relationship between bisimulations and cocongruences.

Proposition 3: for coalgebras over sets

this one you can find as Lemma 4.3.4 in my PhD thesis

Let $c: S \to F(S)$ and $d: T \to F(T)$ be two coalgebras.

(1) If $R \subseteq S \times T$ is a bisimulation between c and d then the pushout (P, p_1, p_2) of (R, π_1, π_2)

$$\begin{tikzcd} & R \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S \arrow[dr, "p_1"'] & & T \arrow[dl, "p_2"] \\ & P & \end{tikzcd}$$

is a cocongruence between c and d.

(2) If F is wpp and (U, u_1, u_2) is a cocongruence between c and d then the pullback

$$Q = \{(s,t) \in S \times T \mid u_1(s) = u_2(t)\}$$ of (U, u_1, u_2)

$$\begin{tikzcd} & Q \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S \arrow[dr, "u_1"'] & & T \arrow[dl, "u_2"] \\ & U & \end{tikzcd}$$

is a bisimulation between c and d.

__Proof__: (1). Let $r: R \to F(R)$ be a ~~bisimulation~~ coalgebra structure witnessing the bisimulation property. Applying F to the pushout square we get

$$Fp_1 \circ F\pi_1 = Fp_2 \circ F\pi_2$$

So since R is a bisimulation, the outer ~~square~~ hexagon in the diagram

$$\begin{tikzcd}
 & & R \arrow[dll, "\pi_1"'] \arrow[d, "r"] \arrow[drr, "\pi_2"] & & \\
S \arrow[dd, "c"'] \arrow[drr, "p_1"] & & F(R) \arrow[ddll, "F\pi_1"'] \arrow[ddrr, "F\pi_2"] & & T \arrow[dd, "d"] \arrow[dll, "p_2"'] \\
 & & P & & \\
F(S) \arrow[drr, "Fp_1"'] & & & & F(T) \arrow[dll, "Fp_2"] \\
 & & F(P) & &
\end{tikzcd}$$

Commutes : $Fp_1 \circ c \circ \pi_1 \underset{\text{bisim.}}{=} Fp_1 \circ F\pi_1 \circ r \underset{\text{pushout}}{=} Fp_2 \circ F\pi_2 \circ r \underset{\text{bisim.}}{=} Fp_2 \circ d \circ \pi_2$.

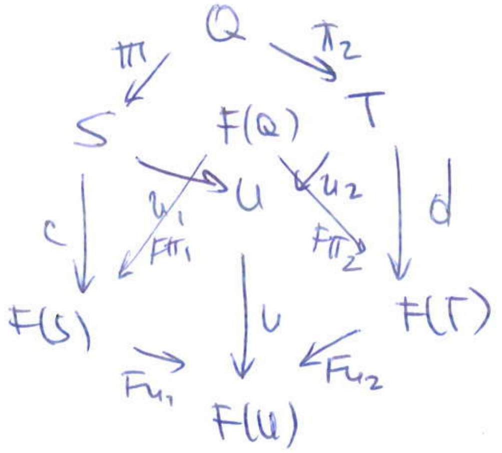
So $(F(P), Fp_1 \circ c, Fp_2 \circ d)$ is a candidate pushout for (R, π_1, π_2) and hence there is a unique arrow $u : P \to F(P)$ such that $Fp_1 \circ c = u \circ p_1$, $Fp_2 \circ d = u \circ p_2$

i.e.

$$\begin{tikzcd}
 & & R \arrow[dll, "\pi_1"'] \arrow[d, "r"] \arrow[drr, "\pi_2"] & & \\
S \arrow[dd, "c"'] \arrow[drr, "p_1"] & & F(R) \arrow[ddll, "F\pi_1"'] \arrow[ddrr, "F\pi_2"] & & T \arrow[dd, "d"] \arrow[dll, "p_2"'] \\
 & & P \arrow[dd, dashed, "u"] & & \\
F(S) \arrow[drr, "Fp_1"'] & & & & F(T) \arrow[dll, "Fp_2"] \\
 & & F(P) & &
\end{tikzcd}$$

(P, p_1, p_2) is a cocongruence with $u: P \to F(P)$ between c and d.

(2) Applying the functor F to the pullback square we get



$Fu_1 \circ F\pi_1 = Fu_2 \circ F\pi_2$

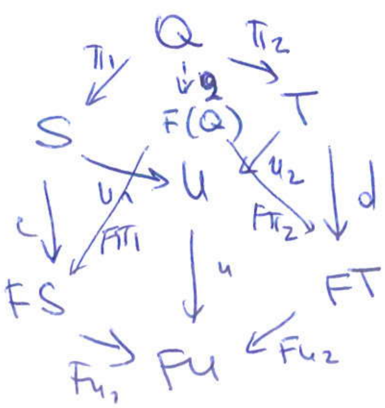
and ~~since (U, u_1, u_2) is a cocongruence~~

$(F(Q), F\pi_1, F\pi_2)$ is a weak pullback for $(F(U), Fu_1, Fu_2)$

Now since (U, u_1, u_2) is a cocongruence, also the outer hexagon commutes:

$$Fu_1 \circ c \circ \pi_1 \underset{\text{cocong.}}{=} u \circ u_1 \circ \pi_1 \underset{\text{pullback}}{=} u \circ u_2 \circ \pi_2 = Fu_2 \circ d \circ \pi_2$$

which means that $(Q, c \circ \pi_1, d \circ \pi_2)$ is a candidate pullback for $(F(U), Fu_1, Fu_2)$ and so there is a (not necessarily unique) mediating arrow $g : Q \to F(Q)$ such that $F\pi_1 \circ g = c \circ \pi_1$ and $F\pi_2 \circ u = d \circ \pi_2$ in the picture



witnessing that Q is a bisimulation between c and d. ∎

We mentioned recently how pushouts in <u>Sets</u> look like: The pushout of a span (S, s_1, s_2) is $[s_1 : S \to X,\ s_2 : S \to Y]$

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ X \arrow[dr, "p_X"'] & \wedge & Y \arrow[dl, "p_Y"] \\ & (X+Y)/\Theta & \end{tikzcd}$$

$(X+Y)/\Theta$ where

Θ is the equivalence generated by

$$\{(k_1(s_1(s)), k_2(s_2(s))) \mid s \in S\}$$

and p_X, p_Y are the canonical maps

$$p_X(x) = [k_1(x)]_\Theta \quad , \quad p_Y(y) = [k_2(y)]_\Theta$$

Now we just need the following consequence of it.

<u>Corollary $\otimes$</u>: The pushout of a relation $R \subseteq S \times T$ (i.e., of the span (R, π_1, π_2)) identifies all elements related by R.

<u>Proof</u>: (direct).

Let $(s,t) \in R$ (so R ~~is~~ relates s and t).

Then $(k_1(\pi_1(s,t)), k_2(\pi_2(s,t))) \in \Theta$

$$\begin{gathered} (k_1(\pi_1(s,t)), k_2(\pi_2(s,t))) \\ \| \\ (k_1(s), k_2(t)) \end{gathered}$$

Hence, $p_S(s) = [k_1(s)]_\Theta = [k_2(t)]_\Theta = p_T(t)$

so the pushout $((S+T)/\Theta, p_S, p_T)$ indeed identifies s and t.

$\blacksquare$

Theorem 4: Let $c: S \to F(S)$, $d: T \to F(T)$ be two coalgebras, and let $s \in S$, $t \in T$.

(1) If $s \sim t$, then $s \approx t$, i.e., bisimilarity implies behavioral equivalence.

(2) If F is wpp then also $s \approx t$ implies $s \sim t$, i.e., bisimilarity and behavioral equivalence coincide.

Proof: (Consequence of Corollary ⊛ and Prop. 3)

(1) If $s \sim t$, then there is a bisimulation $R \subseteq S \times T$ with $(s,t) \in R$. From Proposition 3(1), the pushout of R is a cocongruence and from Corollary ⊛ it identifies s and t, so $s \approx t$.

(2) Let $s \approx t$. So there exists a cocongruence (u, u_1, u_2) identifying s and t. From Prop. 3(2), the set of all pairs identified by this cocongruence is a bisimulation, so $s \sim t$. ∎

Corollary 5: If F is wpp and the final F-coalgebra exists, then bisimilarity and final coalgebra semantics coincide.

Proof: ~~Proof~~ Direct consequence of ~~Corollary~~ Proposition 2 and Theorem 4. ∎

[To prove it directly or find a direct proof was your last HW assignment]

Finally let's remark that all we did in these notes is about coalgebras on <u>sets</u>.

For a more general treatment see the book, Chapter 4.

Some of the results still hold in general, but one needs more assumptions.

For example, Thm. 4 (1) holds but one needs that the base category has pushouts.

At the very end of this topic we still mention

<u>Bisimilarity & Coinduction</u>

<u>Thm.</u> [Rutten & Turi '93]
Final coalgebras satisfy the coinduction proof principle: for any bisimulation R on $\zeta: Z \to F(Z)$ it holds

$$\underline{R \subseteq \Delta_Z}$$

on the final.

<u>Proof-easy</u>: Let R be a bisimulation on the final.

Then

$$\begin{array}{ccccc} Z & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & Z \\ \zeta\downarrow & & \downarrow r & & \downarrow\zeta \\ F(Z) & \xleftarrow{F\pi_1} & F(R) & \xrightarrow{F\pi_2} & F(Z) \end{array}$$

So both π_1 and π_2 are homom. from $r: R \to F(R)$ to the final. Hence $\pi_1 = \pi_2$, so

$$\begin{aligned} (s,t) \in R &\iff (\pi_1(s,t), \pi_2(s,t)) \in R \\ &\implies s = \pi_1(s,t) = \pi_2(s,t) = t \\ &\iff (s,t) \in \Delta_Z. \end{aligned}$$

∎

Now, this coinduction proof principle is more commonly used in proofs by coinduction than what we did in the beginning.

For examples, check Jacobs & Rutten – A Tutorial on Coalgebra and Coinduction

of the PhD thesis of Falk Bartels.

Due to lack of time we will not cover relation liftings and bisimulations via relation liftings in class.

(Chapter 3 of the book or shorter version with nice modular examples in my PhD thesis)

Please read this yourself (directions on the webpage)

it is a nice topic that sheds more light on bisimulations.