Today's topics: – dynamics (simulation)

– linear control

Euler-Lagrange dynamics

$\Rightarrow$ dynamics $\ddot{q}(q, \dot{q}, \tau)$

$\Rightarrow$ Equation of motion (EOM)

$$\boxed{\underbrace{D(q)}_{\text{Inertia Matrix}}\ddot{q} + \underbrace{C(q,\dot{q})}_{\text{Coriolis Matrix}}\dot{q} + \underbrace{g(q)}_{\text{gravity vector}} = \tau} \rightarrow \text{System dynamics } \dot{x} = f(x,u)$$

$\Rightarrow$ Simulation:

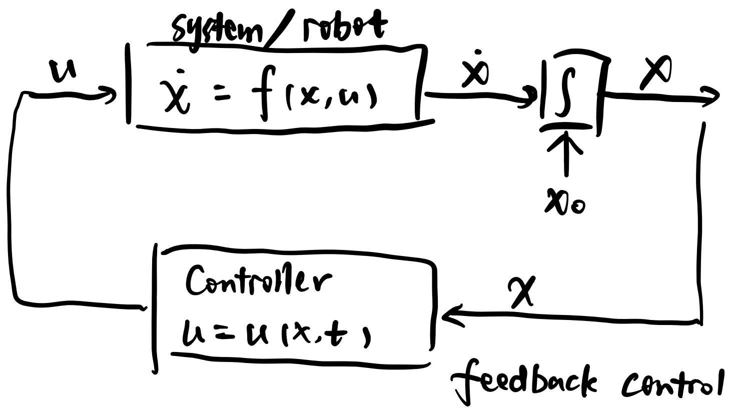
kinematics: $q \Rightarrow$ draw the robot

dynamics: $q(t) \Rightarrow$ simulate the robot motion

$q(0)$

System dynamics: $\dot{x} = f(x,u)$

- x = system state = $\begin{bmatrix} q \\ \dot{q} \end{bmatrix}$
- u = control input = τ



$\Rightarrow$ close-loop system (闭环控制系统)

EOM: $D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$

$\Rightarrow D(q)\ddot{q} = \tau - C(q,\dot{q}) - g(q)$

$\Rightarrow \ddot{q} = D(q)^{-1}(\tau - C(q,\dot{q})\dot{q} - g(q))$

$= f_q(q, \dot{q}, \tau)$

system state : $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \Rightarrow$ measured by sensor

? if we apply $u \Rightarrow$ future state $x(t)$

$\Rightarrow \dot{x}$?

$$\Rightarrow \dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} x(:) \\ f_q(x,u) \end{bmatrix} = f(x,u)$$

$x(2)$

Example: $q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \Rightarrow \dot{q} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \in \mathbb{R}^4$

$x(3)$

$x(4)$

$$\dot{x} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} x(3) \\ x(4) \\ f_q \end{bmatrix}$$

EOM: $\underline{\dot{x} = f(x,u)} \Rightarrow$ simulate the system dynamics

dynamic equation

+ given $u(x,t)$

+ given the initial condition : $x(t=0) = x_0$

$\Rightarrow$ solve $x(t)$? $= \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} \Rightarrow q(t) \overset{FK}{\Rightarrow}$ draw the robot

$\Rightarrow$ robot simulation

$$\dot{x} = f(x,u) = f(x, u(x,t)) = f_{cl}(x,t)$$

$\boxed{\dot{x} = f_{cl}(x,t)} \Rightarrow$ ode equation

close-loop

$\Downarrow x_0$

ode solver

$\Downarrow$

$x(t)$

Example :

① $\dot{x} = 2$ (constant velocity)

$$\frac{dx}{dt} = 2 \Rightarrow dx = 2dt \Rightarrow \int dx = \int 2\,dt$$

$$\Rightarrow x - x_0 = 2t \Rightarrow x(t) = x_0 + 2t$$

② $\dot{x} = x = \frac{dx}{dt}$

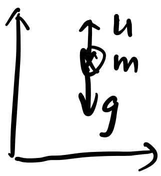
$\Rightarrow \int \frac{dx}{x} = \int dt \quad \Rightarrow x = x_0 \cdot e^t$

+ More complex equation

$\dot{x} = fcn(u, t)$ → ode solver

MATLAB ode

eg.



$m = 0.5$

$g = 9.81$

$m\ddot{y} = u - mg$

$\Rightarrow \ddot{y} = \frac{u}{m} - g$

$\Rightarrow y(t)$?

$X = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \quad X_0 = \begin{bmatrix} y_0 \\ \dot{y}_0 \end{bmatrix}$ → initial position → initial velocity

tspan → simulation time

Mode:

1) main function

① define parameters of the system

② X_0, tspan

③ $[t, x] = ode45(@sys\_dynamics, tspan, x_0)$

④ animation

2) dynamics function

$\dot{x} = f(x, u) \quad u = u(x, t)$

$\dot{x} = sys\_dynamics(t, x)$

$\dot{x} = dX = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} X(2) \\ u/m - g \end{bmatrix}$

3) controller function

$u = controller(t, x)$

$\Rightarrow \dot{x} = fcn(t, x)$

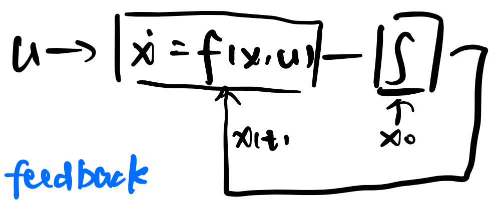
EOM: $\dot{x} = f(x, u) \;;\; u = u(t, x)$

$\Rightarrow \dot{x} = \text{sys-dynamics}(t, x)$

MATLAB: $\underbrace{[t, x]}_{x(t)} = \underbrace{\text{ode45}}_{\text{ode solver}}(@\underbrace{\text{sys-dynamics}}_{\text{dynamics}}, \underbrace{\text{tspan}}_{\text{simulation time}}, \underbrace{x_0}_{\text{initial condition}})$

Linear Control:

Simulation



Controller $u = u(x, x_d)$

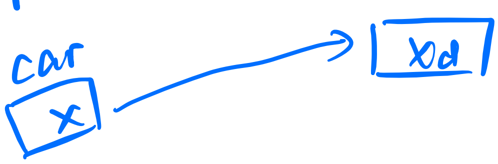
state feedback

feedback control

desired state (reference input)

goal: $x \to x_d$

Example:

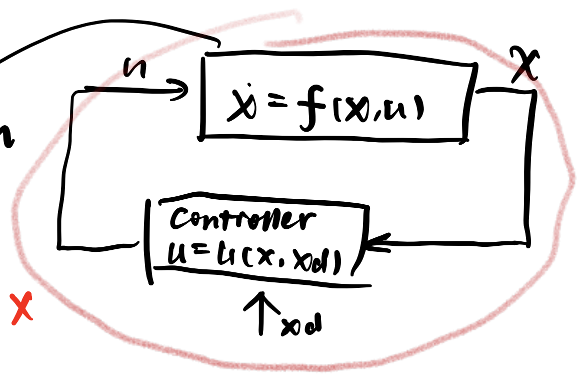


goal: $x \to x_d$

Simulation

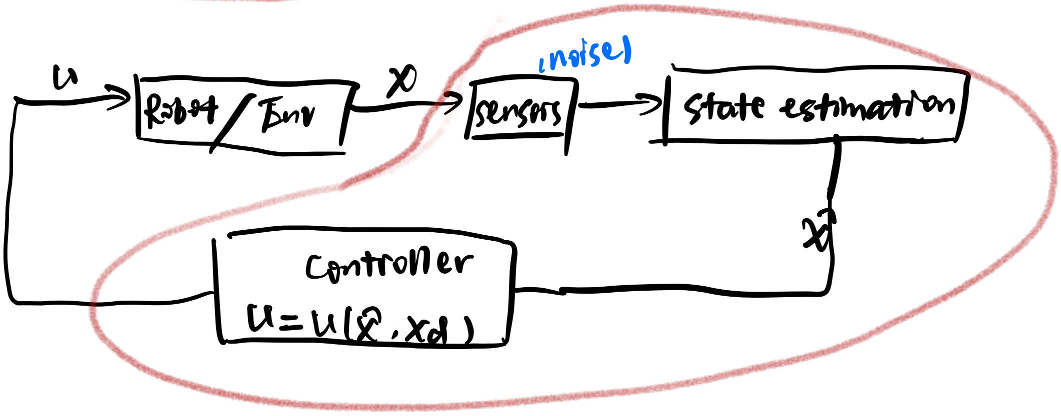
(No sensor)

prediction of X



$\Rightarrow$ feedback control system

Robot:



Linear Control

dynamics: Nonlinear system $\Rightarrow$ linear system

$$\dot{x} = f(x,u) \xrightarrow[\text{linearization}]{} \dot{x} = Ax + Bu$$

linearization $\to$ approximation

A, B $\to$ constant matrix (around $x_d(t)$, $u_d(t)$)

Taylor Series Approximation (1st order)

$$f(x) \simeq f(a) + \frac{\partial f}{\partial x}\Big|_{x=a}(x-a)$$

(around $x=a$)

$$\dot{x} = f(x,u)\Big|_{x=x_d,\, u=u_d}$$

$$\simeq \underbrace{f(x_d, u_d)}_{\dot{x}_d} + \overbrace{\frac{\partial f}{\partial x}(x_d, u_d)}^{A}(x - x_d) + \overbrace{\frac{\partial f}{\partial u}(x_d, u_d)}^{B}(u - u_d)$$

$\dot{x}_d = f(x_d, u_d)$ $\to$ forward dynamics

$$\Rightarrow \underbrace{\dot{x} - \dot{x}_d}_{\delta \dot{x}} = A\overbrace{(x - x_d)}^{\delta x} + B\overbrace{(u - u_d)}^{\delta u}$$

$$\Rightarrow \delta \dot{x} = A\,\delta x + B\,\delta u$$

Example: 1-link pendulum

$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$; $u = \tau$

$\Rightarrow I\ddot{\theta} = \tau - mgl\sin\theta$ $\quad [I = ml^2]$

$\Rightarrow \ddot{\theta} = \frac{\tau}{ml^2} - \frac{g}{l}\sin\theta$ $\to$ nonlinear

$= f_\theta(\theta, \dot{\theta}, \tau)$

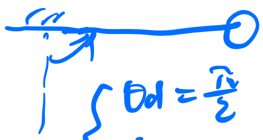
$\Rightarrow$ system dynamics: $\dot{x} = f(x,u)$

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x(2) \\ -\frac{g}{l}\sin\theta + \frac{\tau}{ml^2} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} x(2) \\ -\frac{g}{l}\sin x(1) + \frac{u}{ml^2} \end{bmatrix} = f(x,u)$$

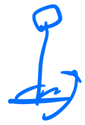
goal : design a controller (??) ⇒ drive $\theta \to \theta_d$

$\dot{\theta}_d = 0$



$$\begin{cases} \theta_d = \frac{\pi}{2} \\ \dot{\theta}_d = 0 \end{cases}$$

$$\Rightarrow x_d = \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}$$



$$\begin{cases} \theta_d = \pi \\ \dot{\theta}_d = 0 \end{cases}$$

$$x_d = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

u_d ? system dynamics : $\dot{x} = f(x,u)$

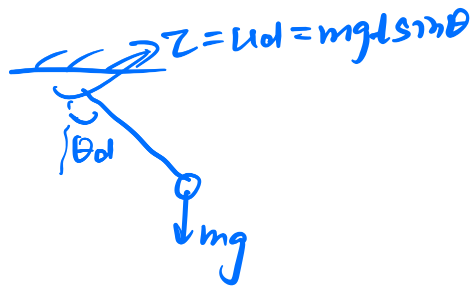
$$\Rightarrow \dot{x}_d = f(x_d, u_d)$$

$$\Rightarrow \dot{x}_d = \begin{bmatrix} \dot{\theta}_d \\ \ddot{\theta}_d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow f(x_d, u_d) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u_d ?$$

$$\ddot{\theta}_d = -\frac{g}{l}\sin\theta + \frac{1}{ml^2}u_d = 0$$

$$\Rightarrow u_d = mgl\sin\theta_d$$



$$\Rightarrow \delta x = x - x_d \qquad \delta u = u - u_d$$

Control goal: $\delta x \to 0$, $\delta u \to 0$ $\Rightarrow \delta\dot{x} = A\delta x + B\delta u \to 0$

$\Rightarrow \delta x$ stay at 0

$\Rightarrow x$ stay at x_d

linearization :

$$\delta\dot{x} = A\delta x + B\delta u$$

$$f(x,u) = \begin{bmatrix} f(1) \\ f(2) \end{bmatrix}$$

$$A = \frac{\partial f}{\partial x}(x_d, u_d)$$

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x(2) \\ \frac{g}{l}\sin x(1) + \frac{1}{ml^2}u \end{bmatrix}$$

$$X = \begin{bmatrix} x(1) \\ x(2) \end{bmatrix}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \qquad \frac{\partial f_1}{\partial x_1} = 0, \ \frac{\partial f_1}{\partial x_2} = 1, \ \frac{\partial f_2}{\partial x_1} = -\frac{g}{l}\cos x(1), \ \frac{\partial f_2}{\partial x_2} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & 0 \end{bmatrix}$$

$$A = \frac{\partial f}{\partial x}(x_d, u_d) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_d & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}(x_d, u_d) = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}(x_d, u_d) \qquad \frac{\partial f_1}{\partial u} = 0 \quad \frac{\partial f_2}{\partial u} = \frac{1}{ml^2}$$

$$\Rightarrow B = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

$$\Rightarrow \delta\dot{x} = A\,\delta x + B\,\delta u$$

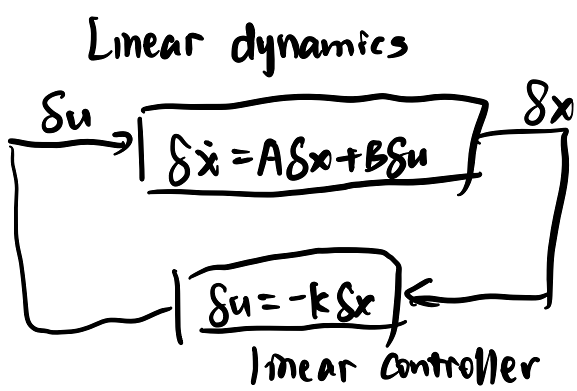
$$\delta x = x - x_d = \begin{bmatrix} \theta - \theta_d \\ \dot{\theta} - \dot{\theta}_d \end{bmatrix} = \begin{bmatrix} \theta - \theta_d \\ \dot{\theta} \end{bmatrix}$$

$$\delta u = \tau - mgl\sin\theta_d$$

Linear Control:

$$\delta u = -K\,\delta x$$

control goal: $\delta x \to 0 \Rightarrow x - x_d$

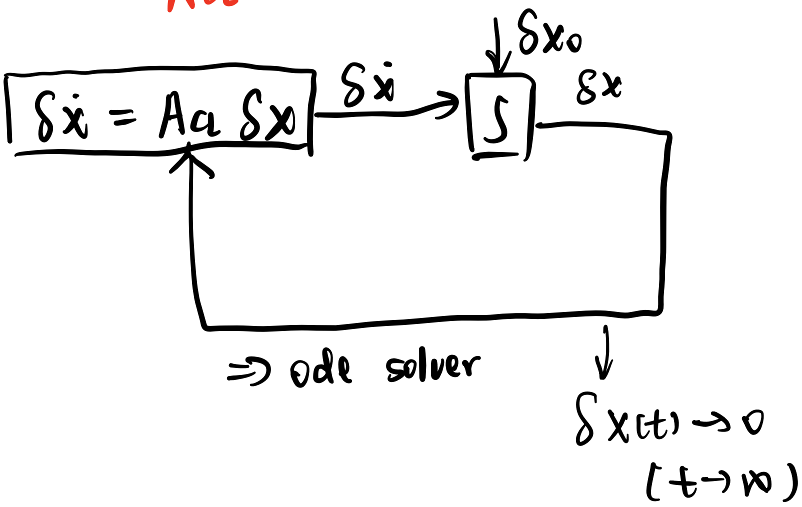


$\Rightarrow$ close-loop system

$$\delta\dot{x} = A\delta x + B\delta u$$
$$= A\delta x + B(-K\delta x)$$
$$= (A - BK)\,\delta x$$

$$\Rightarrow \delta\dot{x} = A_{cl}\,\delta x$$
$$\Downarrow$$
$$\delta x(t)$$

$= \underbrace{(A - BK)}_{A_a} \delta x$



Example: $x \in \mathbb{R}$

$\dot{x} = ax \quad \Rightarrow x(t)\,?$

close-loop dynamics

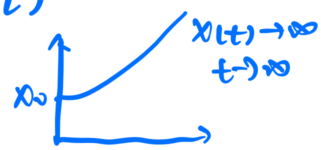
$$\frac{dx}{dt} = ax$$

$$\Rightarrow \int_0^t \frac{dx}{x} = \int_0^t a\,dt$$

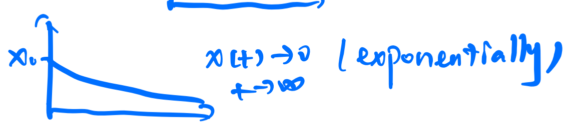
$$\Rightarrow x(t) = x_0 e^{at}$$

① $a=0 \Rightarrow x(t) = x_0 \quad (\forall t)$

$a>0$ ② $a=1 \Rightarrow x(t) = x_0 e^{t}$



$a<0$ ③ $a=-1 \Rightarrow x(t) = x_0 e^{-t}$



$\Rightarrow$ exponential stability

($x \to 0$ exponentially)