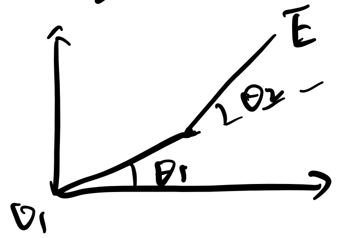
3.16 Lecture 2

Today's topics: $\boxed{\text{Kinematics}}$ Dynamics Control

- Frame, Vectors
- Composition of Rations
- Rotation

Simulation



$\theta_1, \theta_2 \Rightarrow$ position $O_1 . O_2 . E$

$\Rightarrow$ draw the robots

$\Rightarrow$ simulate

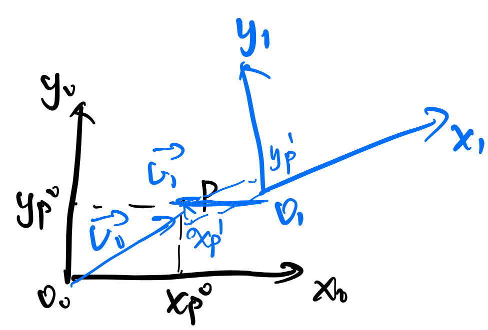
· Frames, Vectors:

2D frames $\rightarrow$ plane (Robotics $\rightarrow$ 3D space)

+ $x_0 O_0 y_0$ is coordinate frame in $\mathbb{R}^2 \Rightarrow$ 2 dimension

$\Rightarrow$ 0 frame (frame $\{0\}$)

+ a point P $\Rightarrow$ has the coordinate $[x_p^0 . y_p^0]$ in frame $\{0\}$



+ point P is associated with the vector $\vec{V_0} = \overrightarrow{O_0 P}$

+ $x_1 O_1 y_1$ is another frame $\{1\}$

P has the coordinate $[x_p^1 . y_p^1]$ in $\{1\}$

P is associated with $\vec{V_1} = \overrightarrow{O_1 P}$

- Points

+ A point corresponds to a particular location in space

+ A point has different representation (coordinates) in different frames

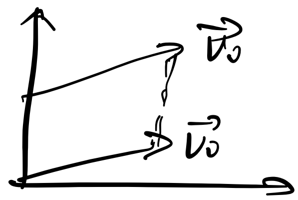
⇒ coordinate of a point → refer to a specific frame

$[x_p^0, y_p^0] \neq [x_p^1, y_p^1]$

- Vectors: P in {0} ⇒ $\vec{OP}$

  - a vector is defined by — direction / — magnitude 方向和大小
  - vector with the same direction & magnitude

    ⇒ same vector



- Rotation

  (1) Rotation of a vector

  (2) Rotation of a frame

(1) + given a fixed frame {0}

+ rotate a vector $\vec{v_0} \to \vec{v_1}$ new vector

+ what is the coordinate of $\vec{v_1}$ in frame {0}

(2) + given a fixed vector $\vec{v}$

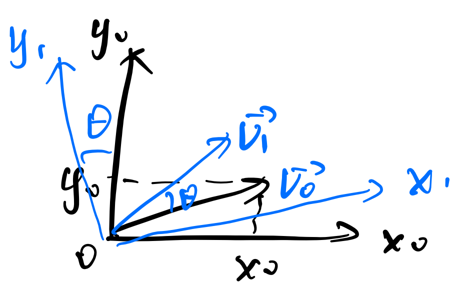
+ rotate a frame $\{0\} \to \{1\}$ new frame

+ what is the coordinate of $\vec{v}$ in frame {1}.

(1) Rotation of a vector.

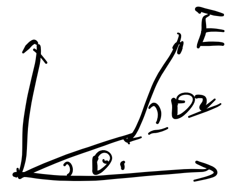
+ given a fixed frame {0}

+ $\vec{v_0} = [x_0, y_0]$ in {0}



+ rotate $\vec{V_0}$ by angle θ → new vector $\vec{V_i}$

⇒ coordinate of $\vec{V_i}$ in {0}



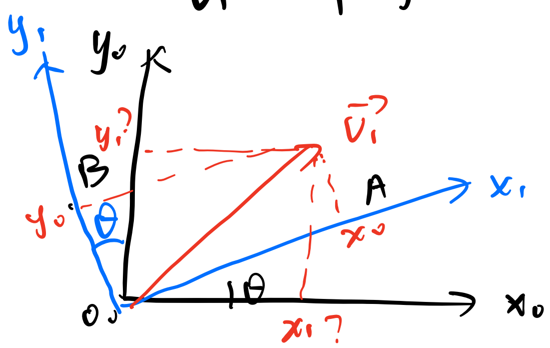
if $\vec{V_i}$ is attached to a moving frame {i}

$\{0\} \xrightarrow[\text{by } \theta]{\text{rotate}} \{1\}$ ⇒ local frame

⇒ local coordinate does not change (during the rotation)

⇒ V_i^{1} = V_0^{0} = $[x_0, y_0]$ (1 → frame {i}; V_i → vector)

↳ V_i^0 → projection



$\vec{V_1} = \vec{O_0A} + \vec{O_0B}$

⇒ projection of

+ $OA \to x_0$

$OB \to y_0$

$$\begin{cases} x_1 = OA\cos\theta - OB\sin\theta \\ y_1 = OA\sin\theta + OB\cos\theta \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = x_0\cos\theta - y_0\sin\theta \\ y_1 = x_0\sin\theta + y_0\cos\theta \end{cases}$$

$$\Rightarrow \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{V_1^0} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{R(\theta)} \underbrace{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{V_0^0}$$

→ Rotation Matrix

⇒ rotation of a vector in a fixed frame {0}

$$\boxed{\vec{V_1} = R(\theta)\vec{V_0}}$$

$\vec{V_0} \xrightarrow{\theta} \vec{V_1}$

$R(\theta)$ = Rotation Matrix

(1) Rotating a 2D frame by an angle θ)

$\{0\} \to \{1\}$: R_1^0

$$\boxed{\vec{v}_1 = R_1^0 \vec{v}_0} \quad (1)$$

+ Rotation matrix in 2D

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

+ properties of a rotation matrix

$$R^{-1}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R^T(\theta)$$

正交矩阵

$\Rightarrow$ $R^{-1}(\theta) = R^T(\theta)$ $\Rightarrow$ $R(\theta)$ is orthogonal

$A \cdot A^T = E$

- $\det(R(\theta)) = \cos^2\theta + \sin^2\theta = 1$

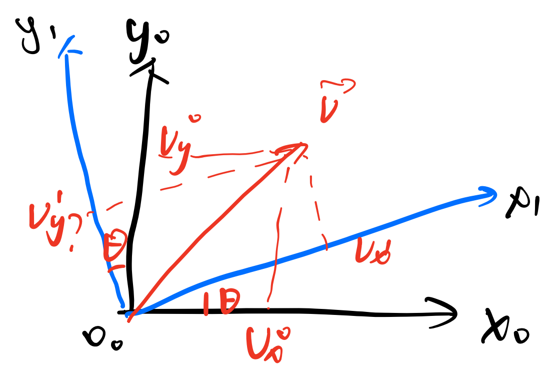
$$\boxed{\det(R(\theta)) = 1}$$

$\Rightarrow$ $R(\theta)$ is special orthogonal

$R(\theta) \in SO(n)$

2D rotation $\Rightarrow$ $n=2$ $\Rightarrow$ $R(\theta) \in SO(2)$

3D $\Rightarrow$ $n=3$ $\Rightarrow$ $R(\theta) \in SO(3)$



(2) Rotation of a frame

+ given a frame $\{0\}$

+ given a fixed vector

$\vec{v} = [v_x^0 \; ; \; v_y^0]$ in $\{0\}$

+ rotate $\{0\} \xrightarrow{\theta}$ new frame $\{1\}$

+ coordinate $\vec{v}$ in $\{1\}$? $\Rightarrow$ $[v_x^1 \; ; \; v_y^1]$

$$\underset{V^0}{\begin{bmatrix} V_x^0 \\ V_y^0 \end{bmatrix}} = \underbrace{R(\theta)}_{R_1^0} \underset{V^1}{\begin{bmatrix} V_x^1 \\ V_y^1 \end{bmatrix}}$$

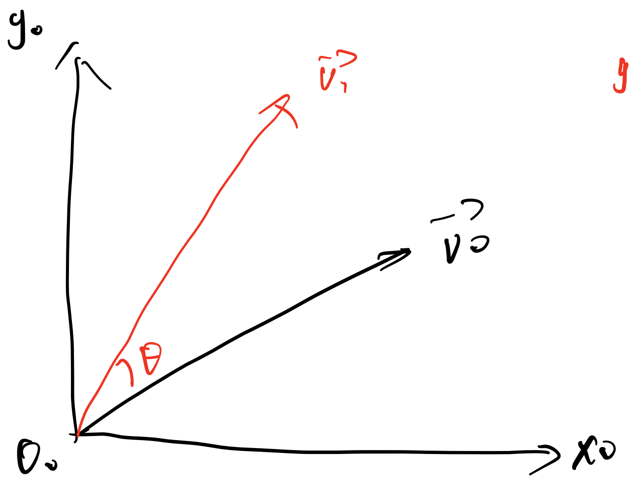
$$\Rightarrow \underset{\text{given}}{V^0} = R_1^0 \underset{\text{find}}{V^1} \quad \Rightarrow V^1 = (R_1^0)^{-1} V^0$$

$$\Rightarrow \boxed{V^1 = (R_1^0)^T V^0}$$

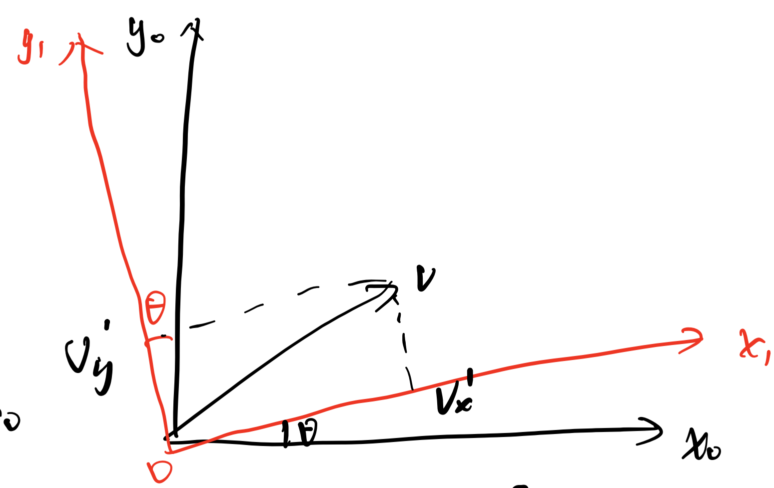
V^0 : coordinate of $\vec{v}$ with respect to $\{0\}$

V^1 : - - - - - - - - $\{1\}$

R_1^0 : Rotation matrix $\{0\} \to \{1\}$



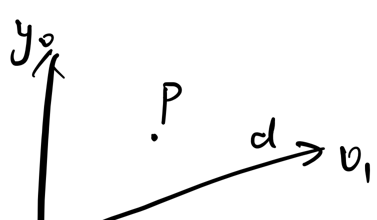
Rotation of vector



Rotation of frame

Rigid motions : Rotation Matrix $\Rightarrow$ rotation

+ any rigid motion = translation + rotation
+ consider a frame $\{0\}$ and point P
+ apply a rigid motion $(d \cdot R)$ to move $\{0\} \to \{1\}$



— $d \in R^3 = \overrightarrow{O_0 O_1}$

= translational distance (pure translation)

$R \in SO_3 = R_1^0$ = rotation matrix (pure rotation)

+ P is rigidly attached to $\{1\}$ ⇒ moving frame

↳ body frame

⇒ $\{0\}$ is the world frame

while $\{1\}$ is moving with respect to $\{0\}$

P is attached to $\{1\}$ ⇒ the local coordinate of P wrt $\{1\}$ (body frame) is unchanged

$P' =$ fixed

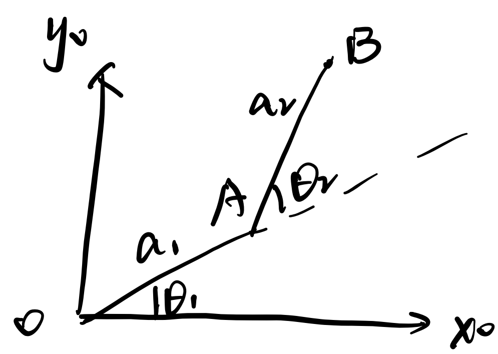
$\{0\} \xrightarrow{(d, R)} \{1\}$

P^0? $P' =$ fixed $\xrightarrow{\text{P will move with } \{1\}}$ new coordinate of P^0?

$P^0 = R_1^0 P' + d^0$

given the location coordinate P. what is the P^0 (wrt world frame) after the rigid motion

Example:

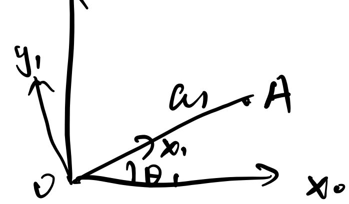


FK of a 2-link robot arm

+ linkage length: a_1, a_2

+ given joint angles: θ_1, θ_2

⇒ simulate / draw the robot?

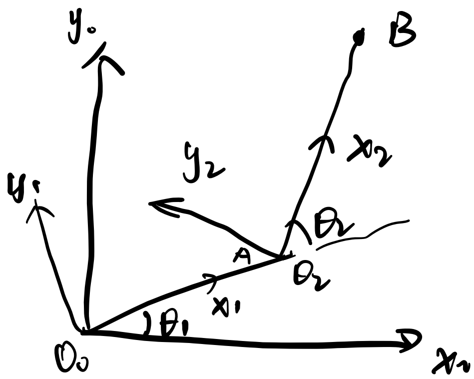


→ 2 links: $\begin{cases} OA = A? \Rightarrow P_A^0 \\ AB = B? \Rightarrow P_B^0 \end{cases}$

+ link 1 is attached to $\{1\}$

$\{0\} \xrightarrow{\theta_1} \underline{\{1\}}$ ⤳ body frame

$O_1 \equiv O_0$



$$P_A^0 = R_1^0 P_A^1 + O_0O_1$$

$$\left.\begin{aligned} R_1^0 &= R(\theta_1) \\ P_A^1 &= \begin{bmatrix} a_1 \\ 0 \end{bmatrix} \\ O_0O_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}\right\} \Rightarrow P_A^0 = R(\theta_1) P_A^1$$

+ link 2 is attached to $\{2\}$

$$P_B^2 = \begin{bmatrix} a_2 \\ 0 \end{bmatrix}$$

$\{0\} \xrightarrow{R(\theta)} \{1\} \xrightarrow{\vec{O_1O_2}\,.\,R(\theta_2)} \{2\}$

$\Downarrow$ rigid motion

$\{1\} \to \{2\} \Rightarrow P_B^1 = \underbrace{P_A^1}_{\vec{O_1O_2} \text{ wrt } \{1\}} + R_2^1 P_B^2$

$$P_B^1 \begin{cases} P_A^1 = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}; \quad R_2^1 = R(\theta_2) \\ P_B^2 = \begin{bmatrix} a_2 \\ 0 \end{bmatrix} \end{cases}$$

$$\left.\begin{aligned} &\Rightarrow P_B^0 \,? \\ &\{0\} \xrightarrow{\theta_1} \{1\} \end{aligned}\right\} \Rightarrow P_B^0 = R_1^0 P_B^1$$

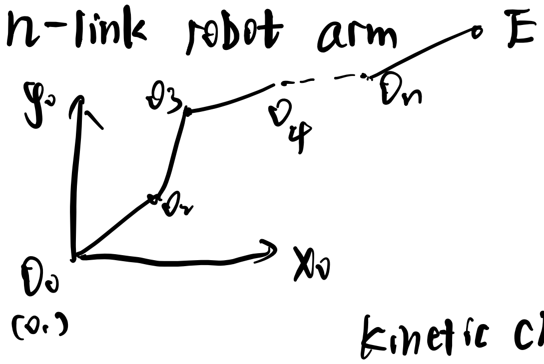
$$\left.\begin{aligned} &\text{link } 1 \to \{1\} \to P_A^1 \\ &\text{link } 2 \to \{2\} \to P_B^2 \end{aligned}\right\} \Rightarrow \text{local coordinates}$$

$\{0\} \to \{1\}$: $P_A^1 \to P_A^0$

$\{1\} \to \{2\}$: $P_B^2 \to P_B^1$

$\{0\} \to \{1\}$ $\quad P_B^1 \to P_B^0$

n-link robot arm



$\{0\} \to \{1\}$: $P_{O_2}^{1} \to P_{O_2}^{0}$

$\{0\} \to \{1\} \to \{2\}$

$P_{O_3}^{0} \leftarrow P_{O_3}^{1} \leftarrow P_{O_3}^{2}$

kinetic change $\Rightarrow$ $\{0\} \to \{1\} \to \cdots \{n\}$

$P_E^{0} \leftarrow P_E^{1} \cdots \leftarrow P_E^{n}$

$\Rightarrow O_1\ O_2 \cdots O_n,\ E$ wrt $\{0\}$

$\Rightarrow$ draw n-link robot arm