- Linear Control

linear system: $\dot{x} = Ax + Bu$ (constant: A, B)

nonlinear system: $\dot{x} = f(x,u)$

$\Rightarrow$ linearization: ① 应用 linear control ② 判别稳定性

$\dot{x} = f(x,u)$ 在 x_d, u_d 展开

$$\dot{x} \approx \underbrace{f(x_d, u_d)}_{\dot{x}_d} + \underbrace{\frac{\partial f}{\partial x}(x_d,u_d)}_{A}(x-x_d) + \underbrace{\frac{\partial f}{\partial u}(x_d,u_d)}_{B}(u-u_d)$$

approximation

x close to x_d

u close to u_d

$x \to x_d$ desire state

$u \to u_d$

$X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$

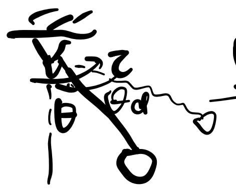
$x_d = 1$: $\dot{x}_d = 0$ $\ddot{x}_d = 0$

$$\Rightarrow \dot{x} - \dot{x}_d = A(x-x_d) + B(u-u_d)$$

$$\Rightarrow \delta\dot{x} = A\,\delta x + B\cdot\delta u$$

control goal: $\delta x \to 0$ $\delta u \to 0$

[例]



$\theta \to \theta_d$

x_d

u_d:

$\theta \to \theta_d$ $\dot{\theta}_d = 0$ $\ddot{\theta}_d = 0$.

$u_d = ?$

$\ddot{\theta} = \frac{\tau}{mL^2} - \frac{g}{L}\sin\theta = 0$

$\Rightarrow \tau = mgL\sin\theta$

$$I\ddot{\theta} = \tau - mgL\sin\theta \quad I = mL^2$$

$$\Rightarrow \ddot{\theta} = \frac{\tau}{mL^2} - \frac{g}{L}\sin\theta$$ nonlinear

$\dot{x} = f(x,u)$ $X = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$ $\dot{x} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix}$

$X = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

- $$\dot{x} = \begin{bmatrix} x(2) \\ \frac{\tau}{mL^2} - \frac{g}{L}\sin\theta \end{bmatrix} = f(x,u)$$ 在 x_d u_d 处展开

$f = \begin{pmatrix} x(2) \\ \frac{\tau}{mL^2} - \frac{g}{L}\sin\theta \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

$u = \tau$

$$\Rightarrow \delta\dot{x} = A\cdot\delta x + B\delta u$$

$$A = \frac{\partial f}{\partial x}(x_d, u_d) \qquad B = \frac{\partial f}{\partial u}(x_d, u_d)$$

☆ $$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}(x_d, u_d) \qquad B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}(x_d, u_d)$$

$$\frac{\partial f_1}{\partial x_1} = 0 \quad \frac{\partial f_1}{\partial x_2} = 1 \quad \frac{\partial f_2}{\partial x_1} = -\frac{g}{L}\cos\theta \quad \frac{\partial f_2}{\partial x_2} = 0$$

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_d & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix}$$

$$\Rightarrow \delta\dot{x} = A\delta x + B\delta u$$

- linear control : $\delta u = -K\delta x$

$$\delta\dot{x} = A\delta x + B\delta u$$

$$= A\delta x + B(-K)\cdot\delta x$$

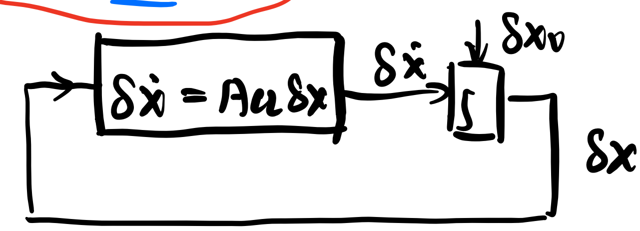
$$= \underbrace{(A - BK)}_{A_{CL}}\cdot\delta x$$

$$\Rightarrow \delta\dot{x} = A_{CL}\,\delta x$$

一阶微分方程

control goal : $\delta x \to 0$

CL : close-loop

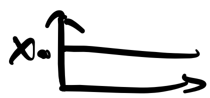


$$\dot{x} = -ax \quad \Rightarrow \frac{dx}{dt} = -ax$$

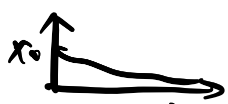
$$\Rightarrow \frac{1}{x}dx = -a\,dt \quad \Rightarrow \int_0^t \frac{1}{x}dx = \int_0^t -a\,dt$$

$$\Rightarrow x(t) = x_0 \cdot e^{-at}$$

① $a = 0$ $\quad x(t) = x_0$



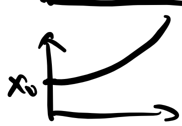
② $a > 0$ $\quad x(t) = x_0 e^{-at}$



$\Rightarrow$ exponential stability.

$\lim_{t\to\infty} x(t) = 0$

③ $a < 0$ $\quad x(t) = x_0 \cdot e^{-at}$



$\lim_{t\to\infty} x(t) \to +\infty$

$\delta x \to 0$: 设计 controller. $\to$ 设计 K，使 $\delta x \to 0$.

求 A_{CL} $\Rightarrow$ 判断 $x(t)$ 是否 exponentially stable

$\Rightarrow$ 2种判据 :

判据1：特征值法.

定理：A_{CL} 的所有特征值为负 ⇒ 稳定

eigenvalues

$$A_{n\times n}\alpha_{n\times 1} = \underset{1\times 1}{\lambda}\,\alpha_{n\times 1}$$ λ：A的特征值.

$$(\underset{n\times n}{A}-\lambda E_{n\times n})\alpha = 0$$

$$|A-\lambda E| = 0$$ ⇒ 求解λ

【例1】：$A=\begin{bmatrix}-6 & 3\\ 4 & 5\end{bmatrix}$ 求λ

$$|A-\lambda E| = 0 \qquad A-\lambda E = \begin{pmatrix}-6 & 3\\ 4 & 5\end{pmatrix} - \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} = \begin{pmatrix}-6-\lambda & 3\\ 4 & 5-\lambda\end{pmatrix}$$

$$(\lambda-5)(\lambda+6)-12=0 \Rightarrow \lambda^2-\lambda-42=0 \Rightarrow \lambda_1=6 \quad \lambda_2=-7$$

$\lambda_1=6$ ↓ 不稳定

判据2：Lyapunov 稳定性分析

$\delta x \to 0$

System dynamics：$\delta\dot{x} = A_{CL}\,\delta x$ $\dot{x} = A_{CL}\,x$

定义：Lyapunov 函数：$V(x) = x^T P x$

① 二次型 $P=\begin{pmatrix}p_1 & 0\\ 0 & p_2\end{pmatrix}$ $x=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ ⇒ $V(x)=p_1x_1^2+p_2x_2^2$

② 正定：$V(x)\geq 0$ 当且仅当 $V(0)=0$

负定：$-V(x)$ 为正定. $V(x)$ 负定

③ $\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$

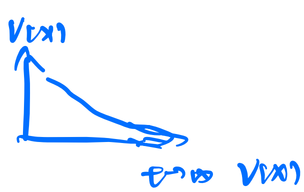
定理：对系统 $\dot{x} = f(x,u)$，$t\geq 0$，其中 $f(0)=0$. 若存在矩阵P，使得：

① $V(x)$ 正定 ② $\dot{V}(x)$ 负定 ③ $x\to\infty$，$V(x)\to\infty$

则系统稳定

$t\to\infty$，$V(x)\to 0$.

$V(x)\downarrow$



【例】设系统状态方程：

$$\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2+x_2^2) \\ \dot{x}_2 = -x_1 - x_2(x_1^2+x_2^2) \end{cases}$$

判断系统是否稳定

解：令 $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 正定

① $V(x) = x^T P x = x_1^2 + x_2^2 \geq 0$，当且仅当 $x_1 = x_2 = 0$ $V(x) = 0$ 正定

② $\dot{V}(x) = 2x_1\cdot\dot{x}_1 + 2x_2\cdot\dot{x}_2 = -2(x_1^2+x_2^2)^2 \leq 0$ 当且 … $\Rightarrow$ 负定

③ $x_1 \to \infty$，$x_2 \to \infty$，$V(x) \to \infty$

$\Rightarrow$ 稳定

判据2推广：只对线性系统

$u = -Kx$

$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$ $\quad \dot{x} = Ax + Bu = A_{cL}x$

$\Rightarrow \dot{V}(x) = x^T \underbrace{(A_{cL}^T P + P\cdot A_{cL})}_{-Q} x$

$\dot{V}(x) = -x^T Q \cdot x$

定理①：给定一个正定矩阵P，若存在正定矩阵Q

Lyapunov equation $A_{cL}^T \cdot P + P\cdot A_{cL} = -Q$ 成立

$\Rightarrow$ 系统稳定

$Q = E$ （特征值全为正）

定理②：线性系统中，对于任意给定的正定矩阵Q，有唯一的正定对称矩阵P，

使：$A_{cL}^T P + P\cdot A_{cL} = -Q$ 成立，

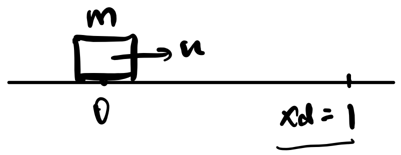
$\Rightarrow$ 则系统稳定。

$\Downarrow$

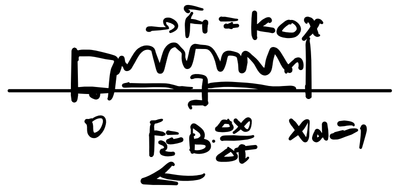
P

- PD control :

质量 弹簧 阻尼.

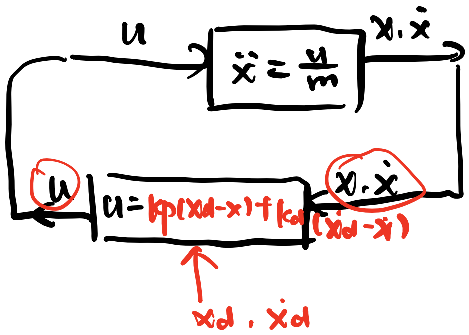


$$u = ma = m\ddot{x} \Rightarrow \ddot{x} = \frac{u}{m}$$



$$u = F_1 + F_2 = k \cdot \Delta x - B\frac{\Delta x}{\Delta t}$$

$$u = k_p(x_d - x) + k_d(\dot{x}_d - \dot{x})$$



close-loop system.