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Regular flexibility of nested CES functions ☆

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Abstract

In this paper we define the class of *regular-flexible* functional forms as the class of cost functions which are globally well-behaved and can provide a local approximation to any globally well-behaved cost function. We then offer a constructive proof of flexibility for the Nonseparable N-stage Constant-Elasticity-of-Substitution functional form (NNCES), by describing a technique through which the NNCES form can be parameterized so as to locally capture any regular configuration of input demands and second-order curvature conditions.

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1. Introduction

In the last 25 years, economists have devised a number of flexible functional forms that can provide a second-order local approximation to an arbitrary cost

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function. Although second-order local approximation is now considered to be a rather restrictive criterion for econometric applications, flexible functional forms still represent an attractive tool for applied general equilibrium modelling exercises, where it is common practice to incorporate local information on elasticities obtained from other studies, without directly engaging in econometric estimation. [3] To be suited to this type of applications, a cost function must be *globally regular*, i.e., nondecreasing and concave in prices everywhere in price space. This is because the computational algorithms employed for finding equilibria typically involve a search path in price space which can contain points lying arbitrarily far from an equilibrium point. Lack of global regularity can thus cause an algorithm to fail even when concavity is maintained at the equilibrium point. Unfortunately, traditional flexible forms do not guarantee global regularity, and may therefore be unsuitable for use in applied equilibrium models.

In this paper we argue that, when used in equilibrium modelling applications, traditional flexible functional forms suffer from an excess of flexibility, for they can locally represent arbitrary cost functions whereas, when optimizing behaviour is assumed, only globally regular functions (a subclass) need be represented. We thus propose a restricted notion of flexibility, which we term *regular flexibility*. Functional forms with this property provide a second-order local approximation to any cost function if and only if the function to be approximated is globally regular. We examine the N-stage Nonseparable Constant-Elasticity-of-Substitution (NNCES) functional form as a candidate for regular flexibility and offer a constructive proof of regular flexibility for the single-output, homogeneous case. [4] The calibration procedure we describe in our proof provides modellers with a flexible and robust tool for incorporating exogenous elasticity estimates in their models.

2. Flexibility and regular flexibility

The following discussion will focus on the dual representation of production technologies with N inputs, one output and constant returns to scale, [5] by means of a real-valued, continuous, twice differentiable unit cost function, $C(\pi)$, where π denotes the input price vector. A unit cost function is said to be well-behaved at a point π if its first derivatives, C_i $(i = 1, \dots, N)$, which represent the conditional

[3] For an example of this approach, see Despotakis (1980).

[4] Although Pollak and Wales (1987) suggest that the Two-stage Nonseparable CES functional form might be flexible, and in spite of the long standing and widespread use of nested CES functions, ours is the first proof of flexibility for this class of functions.

[5] Our analysis applies, without changes, to the representation of homothetic preferences. Extensions to joint production and nonhomothetic preferences are warranted but beyond the scope of the present paper.

input demands, are nonnegative and homogeneous of degree zero in prices, and if the matrix of partial second derivatives, $[C_{ij}]$, is negative semidefinite. [6] A cost function is globally regular if it is well-behaved everywhere in price space.

Flexibility of the second order means that, with an appropriate choice of parameters, at a given point in price space the derivatives of the cost function up to the second order can assume any combination of values compatible with homogeneity. The local curvature properties of a cost function up to the second order can be summarized by a unit cost, C, input value shares, $\theta_i = \pi_i C_i / C$, $\forall i$, $(\Sigma_i \theta_i = 1)$, and Allen–Uzawa elasticities of substitution (AUES), $\sigma_{ij} = C_{ij} C / (C_i C_j)$, $\forall i, j$. [7] Using the Cournot restriction $\Sigma_j \pi_j C_{ij} = 0$, $\forall i$, diagonal elements of the AUES matrix can be expressed as a function of the off-diagonal elements, i.e., $\sigma_{ii} = (\Sigma_{j \neq i} \theta_j \sigma_{ij}) / \theta_i$, $\forall i$. Homogeneity and symmetry of the AUES matrix imply that only $H = N(N-1)/2$ of its elements are independent. Taking into account the adding-up condition for value shares, we obtain a total number of independent derivatives up to the second order equal to $T = N + H = N(N+1)/2$. In order to provide flexibility, traditional flexible functional forms feature a number of independent parameters equal to T so that the mapping is unique.

It could be argued that the above definition of flexibility is too broad, since only globally regular functional forms should be considered as candidates for the representation of the 'true' technology. Hence we propose an alternative, restricted notion of flexibility.

Definition. A cost function is *regular-flexible* of the second order if it is globally regular and is able to locally represent any regular configuration of compensated demands and second-order curvature conditions.

It should be noted that global *concavity* can be imposed on several of the traditional flexible functional forms, but such restrictions can limit the set of local curvature conditions that can be represented. For example, consider a homogeneous version of the Generalized Leontief (GL) form (Diewert, 1971), which can be written as $C(\pi) = 2\Sigma_i \Sigma_j b_{ij} (\pi_i \pi_j)^{\frac{1}{2}}$, where $[b_{ij}]$ is a symmetric matrix of share parameters. The off-diagonal elements of the Hessian have the form $C_{ij} = b_{ij} (\pi_i \pi_j)^{-\frac{1}{2}}$, $i \neq j$. If we impose the restriction $b_{ij} \geqslant 0$, $\forall i, j$, the GL is globally concave. It does, however, lose flexibility, for under such restriction all off-diagonal elements of the Hessian $[C_{ij}]$ must be nonnegative, implying that all inputs

[6] A configuration of nonnegative input demands together with a negative semidefinite matrix of partial second derivatives is said to be *regular*.

[7] Although a AUES matrix provides a complete local characterization of the second-order curvature conditions, it has persuasively been argued (Blackorby and Russell, 1989) that it is not very informative. As we will be discussing nested CES functions, however, it is convenient to cast our analysis in terms of the AUES. Such choice does not entail any loss of generality.

must be substitutes. Diewert and Wales (1987) have shown that it is possible to impose restrictions on the parameters of the Normalized Quadratic (NQ) cost function so as to guarantee global concavity without destroying flexibility. Notice, however, that such restrictions do not ensure global *regularity*, since conditional input demands may become negative. [8] Hence, the resulting NQ representation is not regular-flexible.

3. A regular-flexible functional form

In this section the NNCES functional form is examined as a candidate for regular flexibility. We define the NNCES cost function as a nested CES aggregation of the N input prices, where the maximum nesting depth is equal to N and all the share parameters are nonnegative. The NNCES functional form can be formally described as follows. A given *nest* may be either a *terminal nest* or an *intermediate nest*. An intermediate nest, $C^{\ell}(\pi)$, of level $\ell = 0, \ldots, N-1$ is a CES aggregation of a maximum of N subnests of level $\ell + 1$, $C_k^{\ell+1}(\pi)$ $(k = 1, \ldots, N)$:

$$C^{\ell}(\pi) = \left[\sum_k C_k^{\ell+1}(\pi)^{1-\gamma^{\ell}}\right]^{1/(1-\gamma^{\ell})}, \tag{1}$$

where $\gamma^{\ell} \geqslant 0$. [9] A terminal nest at level $\ell = 1, \ldots, N$, is a CES aggregation of input prices:

$$C^{\ell}(\pi) = \left(\sum_i \alpha_i^{\ell} \pi_i^{1-\gamma^{\ell}}\right)^{1/(1-\gamma^{\ell})}, \tag{2}$$

where $\alpha_i^{\ell} \geqslant 0$, $\forall i$. Let $s_{ik}^{\ell+1}$ denote the fraction of total input of good i in a nest of level ℓ entering the kth subnest at level $\ell + 1$:

$$s_{ik}^{\ell+1} = \frac{\left(C_k^{\ell+1}\right)^{-\gamma^{\ell}} C_{ki}^{\ell+1}}{\sum_h \left[\left(C_h^{\ell+1}\right)^{-\gamma^{\ell}} C_{hi}^{\ell+1}\right]}, \quad \forall i, k, \tag{3}$$

where $C_{ki}^{\ell+1}$ is the partial derivative of $C_k^{\ell+1}$ with respect to π_i. Then, the elasticity of substitution between two inputs within a nest of level ℓ can be expressed as a function of the elasticity of substitution of the nest, γ^{ℓ}, the

[8] Loss of monotonicity can also occur under the restricted GL representation discussed above.

[9] For $\gamma^{\ell} = 1$ the right-hand side of (1) is undefined, but its limit for γ^{ℓ} approaching unity is a Cobb–Douglas cost function.

elasticities of substitution between these two inputs within each of the subnests of level $\ell + 1$, $\sigma_{kij}^{\ell+1}$, the input value shares in the nest, θ_i^{ℓ}, and the fractions $s_{ik}^{\ell+1}$:

$$\sigma_{ij}^{\ell} = \gamma^{\ell} + \sum_k \left(\sigma_{kij}^{\ell+1} - \gamma^{\ell} \right) \frac{s_{ik}^{\ell+1} s_{jk}^{\ell+1}}{\sum_r \theta_r^{\ell} s_{rk}^{\ell+1}}, \quad i \neq j. \tag{4}$$

In terms of the above definitions, the NNCES cost function can be described simply as a nest of level 0.

The NNCES functional form has a maximum number of nonnegative parameters equal to $\sum_{i=0}^{N} N^i$, instead of the $T = (N^2 + N)/2$ free parameters of traditional flexible forms. Since the number of parameters exceeds the number of independent derivatives, calibration of the NNCES function involves a point-to-set mapping, instead of the point-to-point mapping of strongly flexible functional forms. [10]

We shall show that regular flexibility of the NNCES can be proven constructively. The proof takes the form of a procedure through which the NNCES form can be parameterized to capture any regular configuration of unit cost, C^0, value shares, θ_i^0, $\forall i$, and AUESs, σ_{ij}^0, $\forall i, j$. We will focus on a restricted version of the NNCES form, which will be referred to as 'Lower-Triangular Leontief' NNCES (LTL-NNCES). In this form, each intermediate nest of level $\ell < N$ contains only two subnests, a Left subnest and a Right subnest. The Right subnest is either an intermediate nest of level $\ell + 1$ when $\ell + 1 < N$ or an empty nest when $\ell + 1 = N$. The Left subnest is a terminal nest of level $\ell + 1$ which aggregates prices for $N - \ell + 1$ inputs with an elasticity of substitution equal to zero. The nested structure of the LTL-NNCES form is shown in Fig. 1.

Theorem. *The NNCES functional form is regular-flexible.*

Proof. We will prove flexibility for the LTL-NNCES mapping. By proving regular flexibility for LTL-NNCES form, which is a restricted version of the NNCES form, we will also prove that the unrestricted NNCES is regular-flexible.

Under the LTL-NNCES restriction, Eq. (4) has the form

$$\sigma_{ij}^{\ell} = \gamma^{\ell} - \gamma^{\ell} \frac{s_i^{\ell+1} s_j^{\ell+1}}{\sum_r \theta_r^{\ell} s_r^{\ell+1}} + \left(\sigma_{ij}^{\ell+1} - \gamma^{\ell} \right) \frac{\left(1 - s_i^{\ell+1}\right)\left(1 - s_j^{\ell+1}\right)}{1 - \sum_r \theta_r^{\ell} s_r^{\ell+1}}, \quad i \neq j. \tag{5}$$

[10] The lack of parsimony in parameters and the nonnegativity conditions can make these functions difficult to estimate, but not impossibly so (Pollak and Wales, 1987).

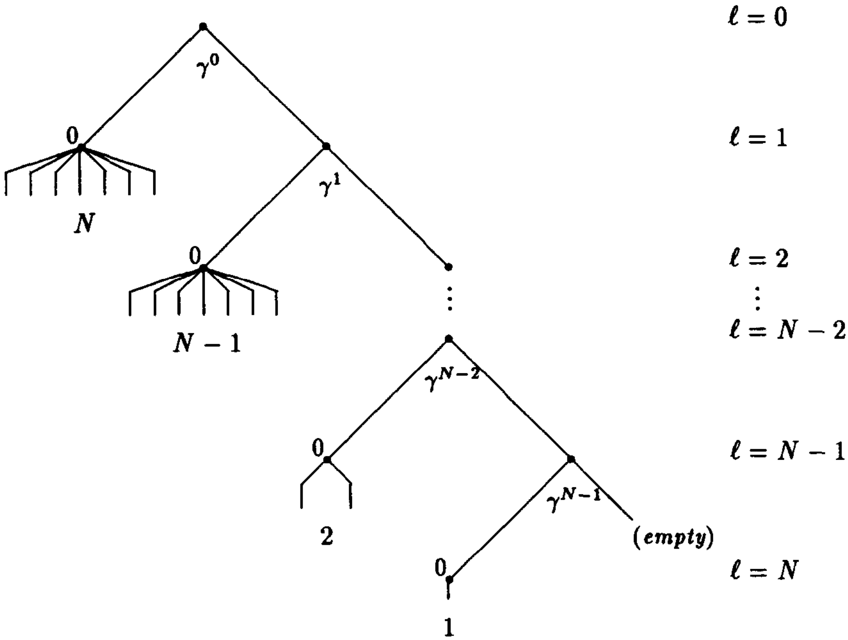

Fig. 1. The nested structure of the LTL-NNCES form.

(The subscript k has been dropped since there is only one terminal nest at each level ℓ.) A benchmark calibration is a matrix of nonnegative fractions, s_i^ℓ, $\forall i$, ℓ, with unitary row sums, and a vector of elasticity parameters γ^ℓ, $\forall \ell$, satisfying Eqs. (5) for given values σ_{ij}^0, $i \neq j$.

To obtain these parameters, we perform a sequence of steps moving down the tree structure of the LTL-NNCES cost function. A step from level ℓ to level $\ell + 1$ assigns all inputs of one good and fractions of the remaining goods to the Left nest of level $\ell + 1$, reducing the number of goods to be allocated. A running account of unallocated inputs and substitution parameters is maintained for the subsequent steps to be performed at a lower dimension. Proof of flexibility is based on an inductive argument: if at each step at least one commodity can be eliminated from the system, and if the lower dimensional problem generated by each step is equivalent to the previous problem, the calibration procedure guarantees a parameterization.

Let us focus on the first step (i.e., $\ell = 0$) and assume benchmark prices to be unity. By Lemma 1 in the appendix, negative semidefiniteness of the AUES matrix implies that there is a good p such that

$$\sigma_{pp}^\ell = \min_i \left(\sigma_{ip}^\ell \right) \leqslant 0. \tag{6}$$

We can then choose good p as the 'pivot' good, and assign all inputs of good p to the Left subnest of level $\ell+1$, letting $s_p^{\ell+1}=1$. Fractions of the remaining goods are allocated to the same subnest in order to satisfy the pairwise AUES conditions for good p represented by (5), which reduce to the following conditions:

$$\sigma_{ip}^{\ell}=\gamma^{\ell}\left(1-\frac{s_i^{\ell+1}}{\sum_r \theta_r^{\ell} s_r^{\ell+1}}\right), \quad i \neq p. \tag{7}$$

A solution to this system of equations is given by

$$s_i^{\ell+1}=\frac{\gamma^{\ell}-\sigma_{ip}^{\ell}}{\gamma^{\ell}-\sigma_{pp}^{\ell}}, \quad \forall i. \tag{8}$$

The step is feasible if all resulting fractions are nonnegative and if they are less than or equal to unity, i.e., if

$$0 \leqslant s_i^{\ell+1} \leqslant 1, \quad \forall i. \tag{9}$$

Let us select γ^{ℓ} to be equal to the maximum off-diagonal AUES value, i.e.,

$$\gamma^{\ell}=\max_i\left(\sigma_{ip}^{\ell}\right). \tag{10}$$

Then (6) and (10) imply that the right-hand side of (8) satisfies condition (9), ensuring that the step is feasible.

The formation of the Left subnest at level $\ell+1$ effectively eliminates good p from the system and reduces the inputs of the goods which remain to be allocated. Total cost is updated as follows:

$$C^{\ell+1}=\left(1-\sum_i \theta_i^{\ell} s_i^{\ell+1}\right)C^{\ell}=\frac{-\sigma_{pp}^{\ell}}{\gamma^{\ell}-\sigma_{pp}^{\ell}}C^{\ell}. \tag{11}$$

Input value shares are also adjusted:

$$\theta_i^{\ell+1}=\frac{\left(1-s_i^{\ell+1}\right)\theta_i^{\ell}C^{\ell}}{C^{\ell+1}}=\frac{\sigma_{ip}^{\ell}-\sigma_{pp}^{\ell}}{-\sigma_{pp}^{\ell}}\theta_i^{\ell}, \quad \forall i. \tag{12}$$

Combining (5) and (8) and solving for the updated cross AUESs, $\sigma_{ij}^{\ell+1}$, we obtain

$$\sigma_{ij}^{\ell+1}=\left(\gamma^{\ell}-\sigma_{pp}^{\ell}\right)\frac{\sigma_{ip}^{\ell}\sigma_{pj}^{\ell}-\sigma_{ij}^{\ell}\sigma_{pp}^{\ell}}{\left(\sigma_{ip}^{\ell}-\sigma_{pp}^{\ell}\right)\left(\sigma_{pj}^{\ell}-\sigma_{pp}^{\ell}\right)}, \quad i \neq j, \tag{13}$$

which, after some manipulation, yields

$$C_{ij}^{\ell+1}=C_{ij}^{\ell}-\frac{C_{ip}^{\ell}C_{pj}^{\ell}}{C_{pp}^{\ell}}, \quad i \neq j. \tag{14}$$

From (14), by Lemma 2 in the appendix, we conclude that the updated Hessian (and thereby the updated AUES matrix) is negative semidefinite.

Since the updated AUES matrix has the same properties of the original matrix, a new step can be performed at a lower dimension. We may therefore conclude that in at most N steps the described procedure will always yield a solution to the benchmark conditions (5). To prove regular flexibility of the LTL-NNCES, it remains to be shown that only *regular* benchmark conditions can be represented by this functional form. This trivially follows from the fact that the LTL-NNCES form is a composite of regular functions, hence it must itself be regular. Thus the LTL-NNCES form (and consequently the unrestricted NNCES form) is regular-flexible. □

Once the fractions s_i^{ℓ} have been obtained, we can compute conditional demands for all inputs in each nest simply as $s_i^{\ell}\theta_i^0 C^0$, $\forall i$, ℓ. The share parameters α_i^{ℓ} can then be recovered using the standard calibration procedure of CES functions.[11] The algorithm described in our proof thus provides modellers with a practical and robust technique for finding and parameterizing a globally regular nested structure that locally approximates any given regular configuration of input demands and second-order curvature conditions.

Appendix

Lemma 1. A symmetric negative semidefinite matrix, A, has at least one dominant diagonal entry, i.e., there is one column p for which $a_{pp} = \min_i(a_{ip})$.

Proof. Assume the contrary, i.e., no dominant diagonal exists, and without loss of generality suppose that $a_{11} > a_{21} = a_{12} = \min_i(a_{i1})$. We may then conclude that $a_{22} < a_{12}$, otherwise the vector $x = (1, -1, 0, \dots, 0)$ would produce:

$$x^T A x = a_{11} - a_{12} - a_{12} + a_{22} > 0,$$

contradicting negative semidefiniteness of A.

We can apply this reasoning to the first $N-1$ diagonal entries (p, p), at each step swapping a pair of rows together with the corresponding columns so that the minimal element in the pth column is located immediately below the diagonal. Then, because of symmetry, we obtain $a_{NN} < a_{iN}$, $\forall i$, a contradiction. □

Lemma 2. Suppose that a symmetric, negative semidefinite matrix, A, can be partitioned as follows:

$$A = \begin{bmatrix} \beta & v^T \\ v & B \end{bmatrix},$$

[11] See, for example, Shoven and Whalley (1992).

where β is a scalar and v is an $(N-1)$ dimensional vector; then the matrix $A' = B - vv^T / \beta$ is also negative semidefinite.

Proof. Consider the following matrix, E, and its inverse, E^{-1}:

$$E = \begin{bmatrix} 1 & 0 \\ -v/\beta & I \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & 0 \\ v\beta & I \end{bmatrix},$$

where I is an identity matrix of dimension $N-1$. Now consider the matrix $D = EAE^{-1}$; one can verify that A' appears in the lower right-hand corner of D. Since E is nonsingular, the matrix D inherits negative semidefiniteness from A: this in turn implies that all the minors of D must be negative semidefinite, including A'. □

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