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THEORY AND TIME SERIES ESTIMATION OF THE QUADRATIC EXPENDITURE SYSTEM

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INTRODUCTION

IN THIS PAPER WE OBTAIN a closed-form characterization of the class of complete systems of theoretically plausible demand functions which are quadratic in total expenditure (Section 1) and estimate one system belonging to this class using U.S. per capita time series data (Section 2). The particular quadratic expenditure system (QES) we estimate is a generalization of the familiar linear expenditure system (LES). We estimate both the LES and the QES under two alternative assumptions about the dynamics of tastes (constant tastes and linear habit formation) and two alternative assumptions about stochastic structure (disturbances independent over time and first order serial correlation). Unfortunately, the data do not permit sharp distinctions among various hypotheses about serial correlation, habit formation, and the functional form of the demand system.

1. THEORY

From the perspective of empirical demand analysis, demand theory serves two distinct functions. First, it provides a set of restrictions on individual behavior which can be tested against particular bodies of data; for example, an individual's ordinary demand functions must satisfy the strong axiom of revealed preference.[2] Second, it is a rich source of functional forms of demand systems which serve as maintained hypotheses for estimation; examples include the LES and the translog. Virtually all estimates of complete demand systems—that is, of demand systems which describe the allocation of expenditure, μ, among some exhaustive set of consumption categories—utilize per capita time series data.[3] Since the

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[2] Households rather than individuals are often taken to be the basic units of analysis and assumed to have well-behaved preferences. See Samuelson [**36**] for the classic statement of the problem of aggregating the preferences of the individuals who comprise a household into a single preference ordering.

[3] We use the term "expenditure" to mean "total expenditure on the enumerated consumption categories;" we avoid using "income" in this sense, but we retain "income" in the phrase "income-consumption curve."

We use the phrase "time series data" to refer to per capita consumption series constructed from national income and product accounts data. Panel data—data on the consumption patterns of particular individuals or households in successive time periods—present very different analytical problems and fall outside the scope of "time series data." Pollak and Wales [**32**] use U.K. household budget data for two periods to estimate a QES, even though such data exhibit relatively little price variation.

restrictions of the theory apply to the behavior of individuals but need not hold for per capita demand functions, time series data cannot be used to test or refute the theory itself.[4] Nevertheless, demand theory is an important source of specifications for empirical demand analysis.

Most work on the specification of functional forms has focused on either "flexible functional forms" (e.g., the translog, the generalized Cobb-Douglas, and the generalized Leontief) or on special forms related to separability (e.g., direct separability, indirect separability, generalized separability, implicit separability).[5] Relatively little attention has been devoted to functional forms which are defined by specifying the way expenditure enters the demand equations.[6]

It is well known that if a demand system generated by a well behaved utility function exhibits expenditure proportionality,

$$h^i(P, \mu) = B^i(P)\mu \qquad (i = 1, \ldots, n), \tag{1a}$$

then the demand functions must be of the form

$$h^i(P, \mu) = \frac{g_i(P)}{g(P)}\mu, \tag{1b}$$

where $g(P)$ is a function homogeneous of degree one and g_i denotes its partial derivative. We shall call a system of demand functions generated by a well behaved preference ordering "theoretically plausible."[7] The demand functions (1b) are generated by the indirect utility function

$$\Psi(P, \mu) = \frac{\mu}{g(P)}. \tag{2}$$

What if the demand functions are linear in expenditure:

$$h^i(P, \mu) = B^i(P)\mu + C^i(P)? \tag{3a}$$

W. M. Gorman [**13**] has shown that any theoretically plausible demand system linear in expenditure must be of the form

$$h^i(P, \mu) = \frac{g_i(P)}{g(P)}\mu + f_i(P) - \frac{g_i(P)}{g(P)}f(P) \tag{3b}$$

[4] That aggregate or per capita demand functions cannot necessarily be rationalized by a utility function even when each individual's demand behavior can be has long been recognized and discussed in terms of the non-existence of the "representative consumer." See Chipman [**7**, pp. 689–798] and Muellbauer [**27**] for detailed discussions of the representative consumer.

A referee argues that it is possible to test the theory if supplementary assumptions are made (e.g., that Engel curves are linear and parallel). But it is not clear why this should be viewed as a test of the theory rather than of the supplementary assumptions.

[5] For flexible functional forms, see Lau and Mitchell [**22**], Diewert [**9**], Diewert [**10**, Section 2], and Christensen, Jorgenson and Lau [**8**]. For separability, see Houthakker [**17**], Pollak [**31**], Blackorby, Primont, and Russell [**4**], and Hanoch [**14**].

[6] Muellbauer [**26**] and Carlevaro [**6**] are interesting exceptions.

[7] A utility function is "well behaved" if it is increasing in its arguments and strictly quasi-concave. Under certain regularity conditions this is equivalent to the requirement that the demand functions imply a Slutsky matrix which is symmetric and negative semi-definite. See Hurwicz and Uzawa [**19**] for a precise statement.

where $f(P)$ and $g(P)$ are functions homogeneous of degree one.[8] Gorman showed that these demand functions are generated by an indirect utility function of the form

$$\text{(4a)} \qquad \Psi(P, \mu) = \frac{\mu - f(P)}{g(P)}.$$

To facilitate later comparisons, we subject Gorman's indirect utility function (4a) to the increasing monotonic transformation $T(z) = -z^{-1}$ to obtain the equivalent indirect utility function

$$\text{(4b)} \qquad \Psi(P, \mu) = -\frac{g(P)}{\mu - f(P)}.$$

The LES

$$h^i(P, \mu) = \frac{a_i}{p_i}\mu + b_i - \frac{a_i}{p_i}\sum p_k b_k$$

(Klein and Rubin [**21**], Samuelson [**35**], Geary [**12**], Stone [**37**]) is a special case of (3).[9]

Systems of demand functions quadratic in expenditure

$$\text{(5)} \qquad h^i(P, \mu) = A^i(P)\mu^2 + B^i(P)\mu + C^i(P)$$

have hardly been mentioned in the literature.[10] We will show that if a system of demand functions is quadratic in expenditure and is theoretically plausible, then it is of the form

$$\text{(6a)} \qquad \begin{aligned} h^i(P, \mu) &= \frac{1}{g^2}\left(\alpha_i - \frac{g_i}{g}\alpha\right)\mu^2 \\ &\quad + \left[\frac{g_i}{g} - \frac{2f}{g^2}\left(\alpha_i - \frac{g_i}{g}\alpha\right)\right]\mu + \frac{f^2}{g^2}\left(\alpha_i - \frac{g_i}{g}\alpha\right) - \frac{g_i}{g}f + f_i \end{aligned}$$

[8] If we require linearity to hold for all price-expenditure situations in which P and μ are strictly positive, then we are back to expenditure proportionality: the budget constraint implies $\sum p_k C^k(P) = 0$ and non-negativity of consumption near zero expenditure implies $C^i(P) \geq 0$. Hence, $C^i(P) = 0$ for all i. But we may still require linearity in a region of the price expenditure space ("local linearity").

[9] Pollak [**30**] obtained a closed form characterization of the subclass of (3) which is generated by additive direct utility functions, a subclass which includes the LES.

[10] Houthakker [**16**, p. 5] conjectures in a footnote that quadratic Engel curves cannot be derived from any explicit utility function. But his principal reservation about polynomial forms is that they ignore "the essential non-negativity of consumption" and, that when non-negativity is allowed for, the resulting Engel curves have kinks where goods enter or leave the optimal consumption pattern. Houthakker's argument rules out demand systems whose Engel curves are quadratic for all price-expenditure configurations, but they may be quadratic in a region of the price-expenditure space. If we confine ourselves to such a region, there is no objection in principle to demand systems "locally quadratic in expenditure."

Diewert [**10**, pp. 129–130] gives an example of a theoretically plausible QES with zero intercepts. Muellbauer [**26**] obtains the indirect utility function corresponding to the class of theoretically plausible demand systems of the form $h^i(P, \mu) = A^i(P)\gamma(P, \mu) + B^i(P)\mu$, a class which includes the QES with zero intercepts. Carlevaro [**6**] obtains the indirect utility function corresponding to the class of theoretically plausible demand systems of the form $h^i(P, \mu) = b_i + \gamma^i[(\mu - \sum p_k b_k)/T(P)](\mu - \sum p_k b_k)/p_i$ where $T(P)$ is a "price index" function and the γ's are functions of a single variable; this class includes some demand systems quadratic in expenditure.

where f, g, and α are functions homogeneous of degree one, or, equivalently

$$h^i(P, \mu) = \frac{1}{g^2}\left(\alpha_i - \frac{g_i}{g}\alpha\right)(\mu - f)^2 + \frac{g_i}{g}(\mu - f) + f_i. \tag{6b}$$

These demand functions are generated by the indirect utility function

$$\Psi(P, \mu) = -\frac{g(P)}{\mu - f(P)} - \frac{\alpha(P)}{g(P)} \tag{7}$$

whose relation to the transformed Gorman form (4b) is evident.

Using Roy's identity, it is easy to verify that the indirect utility function (7) generates a QES of the form (6). The hard part is to show that every theoretically plausible QES is of the form (6). We shall prove that any QES which satisfies what Hurwicz [**18, 19**] calls the "mathematical integrability conditions" (i.e., the Slutsky symmetry conditions) is of the form (6).

We begin with a lemma summarizing the implications of the fact that any system of theoretically plausible demand functions satisfies the budget constraint and the Slutsky symmetry conditions.

LEMMA: *Let*

$$h^i(P, \mu) = A^i(P)\mu^2 + B^i(P)\mu + C^i(P) \tag{5}$$

be a theoretically plausible QES. Then

$$\sum p_k A^k = 0, \tag{8a}$$

$$\sum p_k B^k = 1, \tag{8b}$$

$$\sum p_k C^k = 0, \tag{8c}$$

$$A^i_j + A^i B^j = A^j_i + A^j B^i, \tag{9a}$$

$$B^i_j + 2A^i C^j = B^j_i + 2A^j C^i, \tag{9b}$$

$$C^i_j + C^j B^i = C^j_i + C^i B^j. \tag{9c}$$

These equalities hold as identities in P.

PROOF: The budget constraint

$$\mu = \mu^2 \sum p_k A^k + \mu \sum p_k B^k + \sum p_k C^k$$

holds as an identity, so terms in like powers of μ must be equal. This implies (8). The Slutsky symmetry conditions imply that

$$\frac{\partial h^i}{\partial p_j} + h^j \frac{\partial h^i}{\partial \mu}$$

is symmetric in i and j (i.e., is equal to the corresponding expression with i and j interchanged) and these equalities hold identically in (P, μ). Calculating this expression from (5) we find

$$\frac{\partial h^i}{\partial p_j} + h^j \frac{\partial h^i}{\partial \mu} = 2A^iA^j\mu^3 + [A^i_j + A^jB^i + 2A^iB^j]\mu^2 + [B^i_j + B^jB^i + 2A^iC^j]\mu + [C^i_j + C^jB^i].$$

Since this holds as an identity in μ, terms in like powers of μ are equal to the corresponding terms with i and j interchanged. Equating terms in like powers of μ and cancelling where possible yields (9).

THEOREM: *Any theoretically plausible QES,*

$$h^i(P, \mu) = A^i(P)\mu^2 + B^i(P)\mu + C^i(P), \tag{5}$$

can be written in the form

$$h^i(P, \mu) = \frac{1}{g^2}\left(\alpha_i - \frac{g_i}{g}\alpha\right)\mu^2 + \left[\frac{g_i}{g} - \frac{2f}{g^2}\left(\alpha_i - \frac{g_i}{g}\alpha\right)\right]\mu + \frac{f^2}{g^2}\left(\alpha_i - \frac{g_i}{g}\alpha\right) - \frac{g_i}{g}f + f_i, \tag{6a}$$

where f, g, and α are functions homogeneous of degree one, or equivalently

$$h^i(P, \mu) = \frac{1}{g^2}\left(\alpha_i - \frac{g_i}{g}\alpha\right)(\mu - f)^2 + \frac{g_i}{g}(\mu - f) + f_i.\text{[11]} \tag{6b}$$

The demand functions (6) *are generated by the indirect utility function*

$$\Psi(P, \mu) = -\frac{g(P)}{\mu - f(P)} - \frac{\alpha(P)}{g(P)}. \tag{7}$$

PROOF: The second assertion is easily verified using Roy's identity. The proof of the first is broken down into three steps: (i) we first show that there exists a function $f(P)$, homogeneous of degree one, and n functions $\{\gamma^i(P)\}$ such that (5) can be written as

$$h^i(P, \mu) = A^i\mu^2 + (\gamma^i - 2A^if)\mu + (A^if^2 - \gamma^if + f_i) \tag{10a}$$

or, equivalently,

$$h^i(P, \mu) = A^i(\mu - f)^2 + \gamma^i(\mu - f) + f_i. \tag{10b}$$

(ii) We then show that there exists a function $g(P)$, homogeneous of degree one, such that $g_i/g = \gamma^i$. (iii) Finally, we show that there exists a function $\alpha(P)$, homogeneous of degree one, such that

$$A^i = \frac{1}{g^2}\left(\alpha_i - \frac{g_i}{g}\alpha\right).$$

[11] We assume throughout that the demand functions are differentiable enough to support the calculus arguments we employ. Strictly speaking, we establish our results only for this subclass of theoretically plausible QES.

In each part of the proof we appeal to a mathematical theorem on the existence of local solutions to systems of partial differential equations. The theorem guarantees the existence of a function $z=\omega(P)$ which satisfies a system of partial differential equations

$$\frac{\partial z}{\partial p_i}=\phi^i(P,z)$$

provided the symmetry conditions

$$\phi^i_j(P,z)+\phi^i_z(P,z)\phi^j(P,z)=\phi^j_i(P,z)+\phi^j_z(P,z)\phi^i(P,z)$$

hold (see Hurwicz and Uzawa [**19**, Appendix]).

(i) We first show the existence of γ's and f which satisfy

(11a) $\quad B^i=\gamma^i-2A^if$

and

(11b) $\quad C^i=A^if^2-\gamma^if+f_i$

or, equivalently,

(12a) $\quad \gamma^i=B^i+2A^if$

and

(12b) $\quad f_i=C^i+B^if+A^if^2.$

We define the functions $\phi^i(P,z)$ by

$$\phi^i(P,z)=C^i+B^iz+A^iz^2.$$

By direct calculation

$$\begin{aligned}\phi^i_j+\phi^i_z\phi^j=C^i_j+B^iC^j+[B^i_j+2A^iC^j]z+B^iB^jz+[A^i_j+A^iB^j]z^2\\ +(A^jB^i+A^iB^j)z^2+2A^iA^jz^3.\end{aligned}$$

The term in z^3 is clearly symmetric, as are the second terms in z and z^2. The symmetry of each of the remaining terms follows directly from (9). This establishes the existence of the function $f(P)$.

To show that $f(P)$ is homogeneous of degree one, we multiply (12b) by p_i and sum over all goods to obtain

$$\sum p_kf_k=\sum p_kC^k+f\sum p_kB^k+f^2\sum p_kA^k.$$

Making use of (8), we find

$$\sum p_kf_k=f.$$

The converse of Euler's theorem implies that f is homogeneous of degree one.

We define the γ's by (12a).

(ii) To show the existence of a function $g(P)$ such that $g_i/g=\gamma^i$, we define the function $\phi^i(P,z)$ by $\phi^i(P,z)=\gamma^i(P)z$. Substituting for γ^i from (12a), we calculate

$\phi_j^i+\phi_z^i\phi^j$. Substituting for f_j from (12b) we find

$$\phi_j^i+\phi_z^i\phi^j=(B_j^i+2A^iC^j)z+2(A_j^i+A^iB^j)fz+2A^iA^jf^2z$$
$$+(B^i+2A^if)(B^j+2A^jf)z.$$

The last two terms are clearly symmetric in i and j; the symmetry of the first two terms follows directly from (9). Hence, there exists a function g such that $g_i/g=\gamma^i$.

To show that $g(P)$ is homogeneous of degree one, we substitute g_i/g for γ^i in (12a), multiply by p_ig, and sum over all goods to obtain

$$\sum p_kg_k=g\sum p_kB^k+2fg\sum p_kA^k.$$

Making use of (8) we find

$$\sum p_kg_k=g.$$

The converse of Euler's theorem implies that g is homogeneous of degree one.

(iii) To show that there exists a function $\alpha(P)$ such that

$$A^i=\frac{1}{g^2}\left(\alpha_i-\frac{g_i}{g}\alpha\right),$$

we solve for α_i and obtain

$$\alpha_i=g^2A^i+\frac{g_i}{g}\alpha. \tag{13}$$

We define the function $\phi^i(P,z)$ by

$$\phi^i(P,z)=g^2A^i+\frac{g_i}{g}z.$$

Calculating $\phi_j^i+\phi_z^i\phi^j$ and substituting $g\gamma^j$ for g_j, we obtain

$$\phi_j^i+\phi_z^i\phi^j=g^2A_j^i+2g^2\gamma^jA^i+g^2\gamma^iA^j+\gamma^i\gamma^jz+\gamma_j^iz.$$

Using (12a) to substitute for γ^j and γ^i in the second and third terms yields

$$\phi_j^i+\phi_z^i\phi^j=g^2(A_j^i+A^iB^j)+g^2(A^iB^j+A^jB^i)+6g^2A^iA^jf+\gamma^i\gamma^jz+\gamma_j^iz.$$

The symmetry of the first term follows from (9). The next three terms are clearly symmetric. Since $\gamma^i=g^i/g$, $\gamma_j^i=\gamma_i^j$, the last term is also symmetric.

To show that $\alpha(P)$ is homogeneous of degree one, we multiply (13) by p_i and sum over all goods to obtain

$$\sum p_k\alpha_k=g^2\sum p_kA^k+\frac{\alpha}{g}\sum p_kg_k.$$

By (8), the first term on the right hand side is zero. Since g is homogeneous of degree one, Euler's theorem implies

$$\sum p_k\alpha_k=\alpha.$$

The converse of Euler's theorem implies that α is homogeneous of degree one.

Q.E.D.

Estimation of a QES requires that we specify the form of the homogeneous functions $f(P)$, $g(P)$, and $\alpha(P)$ which appear in (6). There are no universally accepted guidelines for selecting a particular specification, but two desiderata are likely to command general agreement. First, for certain parameter values, the QES specification should reduce to the LES, since this will permit testing of a frequently estimated demand system against a more general specification. Second, the QES specification should add a relatively small number of additional parameters to the $2n-1$ parameters of the LES. The following two specifications satisfy both of these disiderata, each requiring only one additional parameter for each of the n goods, so that $3n-1$ independent parameters must be estimated. The first specification, estimated by Pollak and Wales [**34**], is a QES of the form

$$\begin{aligned}(14)\qquad p_i h^i(P,\mu) &= p_i b_i + a_i(\mu - \sum p_k b_k) \\ &\quad + (c_i - a_i)\lambda \prod p_k^{-c_k}(\mu - \sum p_k b_k)^2, \qquad \sum a_k = 1, \qquad \sum c_k = 1.\end{aligned}$$

It is a special case of (6) in which

$$(15a)\qquad g(P) = \prod p_k^{a_k}, \qquad \sum a_k = 1,$$

$$(15b)\qquad f(P) = \sum p_k b_k,$$

$$(15c)\qquad \alpha(P) = -\lambda \prod p_k^{2a_k} / \prod p_k^{c_k} = -\lambda g(P)^2 / \prod p_k^{c_k}, \qquad \sum c_k = 1.$$

The second specification, estimated in this paper, is a QES of the form

$$\begin{aligned}(16)\qquad p_i h^i(P,\mu) &= p_i b_i + a_i(\mu - \sum p_k b_k) \\ &\quad + (c_i p_i - a_i \sum p_k c_k) \prod p_k^{-2a_k}(\mu - \sum p_k b_k)^2, \qquad \sum a_k = 1,\end{aligned}$$

and corresponds to the specification

$$(17a)\qquad g(P) = \prod p_k^{a_k}, \qquad \sum a_k = 1,$$

$$(17b)\qquad f(P) = \sum p_k b_k,$$

$$(17c)\qquad \alpha(P) = \sum p_k c_k.$$

Hereafter, we shall use the term QES to refer to this particular demand system rather than the entire class, (6).[12,13]

[12] Pollak and Wales [**34**] use the term QES to refer to (14) rather than (16).

[13] We have not established conditions on the functions $f(P)$, $g(P)$, and $\alpha(P)$ which guarantee that the implied preferences are well-behaved. But for both (14) and (16), a straightforward continuity argument shows that these forms include well behaved non-trivial generalizations of the LES. If the a's and b's are such that the associated LES is "well behaved," then (14) is well behaved for values of the c's sufficiently close to the corresponding a's, and (16) is well behaved for values of the c's sufficiently close to zero.

Except for very special cases, neither of these QES forms corresponds to preferences exhibiting direct additivity or even generalized separability (Pollak [**30**]). It would be interesting to investigate the class of QES exhibiting these properties.

2. ESTIMATION FROM TIME SERIES DATA

In this section we present estimates of the LES and the QES based on per capita U.S. consumption data for 1929–1975, excluding the war years, 1942–1946.[14] To estimate the LES and the QES, we must specify both the dynamic and the stochastic structure of these demand systems; we consider two dynamic and two stochastic specifications of each, so we estimate a total of eight demand models.

It is convenient to write the QES in "share" form (i.e., to divide (16) by μ) and to attach time subscripts to the b's and an additive disturbance term to the share equations:[15]

$$(18) \qquad w_i = \frac{p_i b_{it}}{\mu} + a_i\left(1 - \sum \frac{p_k}{\mu} b_{kt}\right) + \left(\frac{p_i}{\mu} c_i - a_i \sum \frac{p_k}{\mu} c_k\right) \prod \left(\frac{p_k}{\mu}\right)^{-2a_k} \left(1 - \sum \frac{p_k}{\mu} b_{kt}\right)^2 + u_{it}, \qquad \sum a_k = 1.$$

The two dynamic specifications we consider are constant tastes and linear habit formation.[16] With linear habit formation the b's are assumed to depend linearly on per capita consumption lagged one period:

$$(19) \qquad b_{it} = b_i^* + \beta_i x_{it-1}.$$

In the constant tastes model, the β's are constrained to equal zero, so $b_{it} = b_i^*$.

We estimate two stochastic specifications, one in which disturbances are independent over time and the other exhibiting first order serial correlation. With first order serial correlation

$$(20) \qquad u_{it} = \rho u_{it-1} + e_{it} \qquad (i = 1, \ldots, 4;\ t = 2, \ldots, T)$$

where for each i the $e_{i2}, \ldots, e_{iT}$ are independently normally distributed random variables with mean zero and for each t the $e_{1t}, e_{2t}, e_{3t}, e_{4t}$ have a contemporaneous covariance matrix Ω.[17] When ρ is set equal to zero, $u_{it} = e_{it}$ and (20) reduces to the specification in which disturbances are independent over time.

[14] Estimation of complete systems of theoretically plausible demand functions is relatively recent. Stone [**37**] was the first to estimate the LES. Maximum likelihood estimates of the LES are presented in Parks [**28**] and Pollak and Wales [**33**]. Many of the complete systems which have been estimated are generalizations of the LES; see, for example, Wales [**38**], Brown and Heien [**5**], Phlips [**29**], and Lluch [**23**]. Estimates based on flexible functional forms include Diewert [**11**], Christensen, Jorgenson, and Lau [**8**], and Berndt, Darrough, and Diewert [**3**]. For a survey of recent work on complete demand systems, see Barten [**1**].

[15] We have also estimated the share form of the QES given by (14) for all models except the QES with habit formation, for which we encountered convergence problems. The results in terms of price elasticities, marginal budget shares, and regularity conditions for these models are very similar to those presented in this paper and, hence, are not reported here.

[16] The specification which we describe as "linear habit formation" is equally consistent with the linear model of "interdependent preferences" described in Pollak [**32**].

[17] Berndt and Savin [**2**] show that a single correlation coefficient ρ must be used for all goods when the covariance matrix of the u's is assumed to be singular and serial correlation takes this simple form (i.e., u_{it} related to u_{it-1}, but not to u_{jt-1}).

Each of the eight demand models we estimate is a special case of the QES in share form, (18), with linear habit formation, (19), and first order serial correlation, (20). The other seven models are obtained by setting certain parameters of the most general model equal to zero. To obtain maximum likelihood estimates of the parameters which are not constrained to equal zero in a particular model, we drop the share equation for any good (since the covariance matrix is singular) and maximize the concentrated likelihood function.[18]

Our estimates are based on four broad consumption categories constructed from the national income and product accounts: "food," "clothing," "shelter," and "miscellaneous." We describe the construction of these categories in an appendix. We view demand theory as a model of the allocation of total expenditure on consumption among an exhaustive set of consumption categories. Since our basic theoretical framework is a one period allocation model, these categories should include the flow of consumption services provided by consumer durables, but not purchases of durables. In principle, our shelter category does reflect the theoretically appropriate flow of services concept. Our miscellaneous category is defined to exclude purchases of durables, and thus consists of services and various non-durable goods purchases. Expenditure, μ, refers to the sum of current dollar expenditure on these four consumption categories.

In Table I we report the values of the logarithms of the likelihood functions (a common multiplicative factor is omitted from each) and the number of sample price-expenditure situations for which the regularity conditions of demand theory are satisfied in each of the eight models we have estimated. To check the regularity conditions, we evaluated whether the Slutsky matrix implied by our parameter

TABLE I

LIKELIHOOD VALUES AND REGULARITY CONDITIONS

	LES		QES	
	$b_{it}=b_i^*$	$b_{it}=b_i^*+\beta_i x_{it-1}$	$b_{it}=b_i^*$	$b_{it}=b_i^*+\beta_i x_{it-1}$
		$\rho=0$		
Likelihood	563.2	672.8	612.1	673.2
Regularity Conditions	28	10	22	0
Number of Parameters	7	11	11	15
		ρ Unconstrained		
Estimated Value of ρ	.95	−.30	.94	.95
Likelihood	667.6	677.5	675.5	679.6
Regularity Conditions	40	30	40	40
Number of Parameters	8	12	12	16

NOTE: The regularity conditions were tested at 40 of the 42 data points. Both 1929 and 1947 were excluded in order to include lagged consumption values in the habit formation models.

[18] For models in which ρ is not constrained to zero we vary ρ from minus one to plus one by steps of .1 and maximize the likelihood function conditional on each such value of ρ. This provides a starting point (or points) for use in maximizing the unconditional likelihood.

estimates was negative semi-definite at each of the 40 price-expenditure situations corresponding to our data.[19]

A number of interesting conclusions can be drawn from Table I. Since the likelihood ratio test shows that the first order serial correlation coefficient is significant at the 1 per cent level for both dynamic specifications of both the LES and the QES, we begin by focusing on the four models which allow first order serial correlation.[20] Within this framework, habit formation is significant in both the LES and the QES. For the LES, it is significant at the 1 per cent level and for the QES at the 10 per cent level.[21] The inclusion of quadratic terms in the constant tastes model is significant at the 1 per cent level.[22] But the inclusion of quadratic terms in the linear habit formation model is not significant even at the 25 per cent level.[23] Finally, in three of the four models with serial correlation, the regularity conditions of the theory of household behavior are satisfied at all of the price-expenditure situations corresponding to our data points, while in the remaining model (the LES with linear habit formation) they are satisfied at 30 of 40 price-expenditure situations.

When no allowance is made for serial correlation, the likelihood ratio test yields somewhat different results. Habit formation is now highly significant in both the LES and the QES: twice the change in the log likelihood is 219.2 for the LES and 122.2 for the QES, while the 1 per cent Chi-square critical level is 13.3. It is tempting to conclude that some of the apparent significance of habit formation might better be interpreted as serial correlation; we return to this question below. The quadratic terms are significant at the 1 per cent level in the constant tastes model, but clearly insignificant in the model with habit formation.[24] Regularity conditions are satisfied at between one-half and three-quarters of the sample points for the constant tastes specification and at one-quarter of the sample points or less in the habit formation specification.[25]

Perhaps the most interesting difference between the QES and LES estimates (see Table II) involves the relationship between habit formation and serial correlation. For the QES, the β's are all close to zero and none is individually significant (although as noted above the β's are jointly significant at the 10 per cent level); for the LES, the β's are all much larger and individually significant. For the QES, ρ is close to unity and significant, while for the LES it is significantly negative. Thus, for the QES it appears that the dynamic and stochastic aspects of

[19] The specification of the model guarantees that the calculated Slutsky matrix will be symmetric. Our approach to regularity conditions is that used in Pollak and Wales [**34**]. Jorgenson and Lau [**20**] consider only whether regularity conditions are satisfied at a single price-expenditure situation, but they test the significance of the ability of their estimated parameters to satisfy the regularity conditions.

[20] The 1 per cent Chi-square critical level is 6.6 for 1 degree of freedom.

[21] For the LES, twice the change in the log likelihood is 19.8, and for the QES it is 8.2. The 10 per cent Chi-square critical level for 4 degrees of freedom is 7.8; the 1 per cent critical level is 13.3.

[22] Twice the change in the log likelihood is 15.8.

[23] Twice the change in the log likelihood is 4.2.

[24] With constant tastes, twice the change in the log likelihood is 97.8; with linear habit formation it is .8.

[25] It is interesting to note that Lluch and Williams [**24**] also find that the LES does much better at satisfying the regularity conditions when ρ is not constrained to zero.

TABLE II
ESTIMATED EQUATIONS

Parameter	LES	QES
a_F	.38 (.022)	.37 (.053)
a_C	.24 (.015)	.17 (.030)
a_S	.17 (.020)	.31 (.078)
a_M	.21	.15
b_F^*	51.8 (20.8)	345.2 (64.9)
b_C^*	13.3 (14.9)	76.1 (34.3)
b_S^*	−3.5 (4.4)	347.3 (48.9)
b_M^*	1.0 (13.7)	147.4 (24.7)
β_F	.91 (.025)	.11 (.128)
β_C	.94 (.052)	−.12 (.110)
β_S	1.03 (.007)	−.19 (.110)
β_M	1.00 (.024)	.02 (.099)
c_F		.0005 (.471)
c_C		.23 (.243)
c_S		1.70 (.499)
c_M		.70 (.207)
ρ	−.30 (.082)	.95 (.015)

NOTES:
1. The food, clothing, shelter, and miscellaneous categories are denoted by F, C, S, and M, respectively.
2. The normalization rule is $\Sigma_1^4 a_k = 1$.
3. The numbers in parentheses are asymptotic standard errors.
4. The b_i^* are measured in 1972 dollars per capita.
5. Since the LES has two β's greater than or equal to one, the argument used in McCarthy [**25**] implies that the dynamic system is unstable. We are grateful to Michael D. McCarthy for pointing this out to us.

demand are "best modeled" almost entirely by positive serial correlation with very little habit formation, while for the LES they are "best modeled" by strong habit formation, offset to some extent by negative serial correlation.[26] However, a careful examination of the likelihood functions reveals, for both the LES and the QES, a local maximum for ρ positive and close to .95 and another for ρ negative and close to $-.3$.[27] For the LES the likelihood is slightly higher with ρ negative and for the QES it is higher with ρ positive.[28] However, since the differences in the

[26] By "best modeled" we mean only that the sample likelihood is highest, not that the model is preferable on other grounds. Although high positive serial correlation is not uncommon in estimates based on aggregate time series data, we would prefer to specify the (non-stochastic) dynamic structure of the QES in a way which would reduce or eliminate it. We have estimated several alternative specifications of habit formation (not reported here) but all yield estimates of high positive serial correlation. Thus, although we are sympathetic to the interpretation of high positive serial correlation as evidence of model misspecification, it is not easily exorcised.

[27] The possibility of a bimodal likelihood function in the context of a dynamic model with autoregressive errors has been discussed recently in Hendry and Serba [**15**]. They conclude that "it is important to graph . . . to check for multiple optima . . . and, if such occur, the estimates must be treated with caution even when they correspond to the sample global minimum" [**15**, p. 989].

[28] For the LES the local maximum with ρ positive yields β's close to zero, while for the QES the local maximum with ρ negative yields β's close to one.

likelihood values are very small in both these cases, any strong conclusion about the relative importance of first order serial correlation and habit formation does not appear to be warranted.[29]

In Table III we present estimates of the marginal budget shares and own price elasticities in selected years for the QES and LES models with linear habit formation and first order serial correlation. Estimates of the parameters of the demand equations from which these marginal budget shares and elasticities are calculated were given in Table II. In the LES, marginal budget shares are necessarily independent of prices, expenditure, and past consumption; our estimates for food, clothing, shelter, and miscellaneous are .38, .24, .17, and .21. For the QES, marginal budget shares vary with prices, expenditure, and past consumption. Over the period 1930–1975, the marginal budget shares for food and clothing have fallen substantially (from .34 to .07 and from .16 to .10) while those for shelter and miscellaneous have increased (from .35 to .58 and from .15 to .25). Examining the estimated own price elasticities for the four consumption

TABLE III

MARGINAL BUDGET SHARES AND OWN PRICE ELASTICITIES FOR THE LES AND QES WITH HABIT FORMATION AND FIRST ORDER SERIAL CORRELATION

Marginal Budget Shares

	F	*C*	*S*	*M*
LES	.38	.24	.17	.21
QES				
1930	.34	.16	.35	.15
1940	.30	.15	.39	.16
1950	.25	.14	.44	.17
1960	.19	.13	.49	.19
1970	.10	.11	.56	.23
1975	.07	.10	.58	.25

LES Own Price Elasticities

	F	*C*	*S*	*M*
1930	−.31	−.14	−.13	−.10
1940	−.39	−.26	−.18	−.23
1950	−.39	−.26	−.18	−.23
1960	−.38	−.25	−.17	−.21
1970	−.39	−.26	−.18	−.22
1975	−.38	−.26	−.17	−.22

QES Own Price Elasticities

	F	*C*	*S*	*M*
1930	−.41	−.42	−.40	−.28
1940	−.44	−.55	−.50	−.38
1950	−.44	−.64	−.64	−.48
1960	−.41	−.65	−.70	−.50
1970	−.31	−.62	−.74	−.50
1975	−.26	−.61	−.74	−.50

NOTES:
1. For the LES, the marginal budget share estimates are independent of prices, expenditure, and past consumption.
2. The observed average budget shares for the four goods in 1975 were .32, .12, .36, and .20.

[29] This is particularly true in view of the fact that with serial correlation and habit formation the QES is not a significant improvement over the LES as judged by the likelihood ratio test.

categories in the QES and the LES, we see that in both systems all goods are inelastic, and that the LES elasticities are almost all closer to zero than the corresponding QES values. Furthermore, the LES elasticities are remarkably stable over time, although the elasticities calculated for 1930 do differ from the rest, all being substantially closer to zero. The QES own price elasticities are not as stable as those from the LES, but the elasticities for clothing, shelter, and miscellaneous change very little from 1960 to 1970 to 1975; food, however, exhibits considerably more variation over the period (from $-.41$ to $-.26$). With the exception of food, the QES elasticities for the early years in the sample period are closer to zero than those in later years.

3. CONCLUSION

We have obtained a closed form characterization of the class of complete systems of theoretically plausible demand functions which are quadratic in expenditure, and presented estimates based on U.S. per capita time series data of a particular QES belonging to this class. The QES we have estimated includes the LES as a special case, and we have estimated both of these demand systems under two dynamic specifications (constant tastes and linear habit formation) and two stochastic specifications (disturbances independent over time and first order serial correlation).

The estimates we obtained for these eight closely related demand specifications are interesting but somewhat inconclusive. The likelihood ratio test indicates that first order serial correlation is significant in both the LES and the QES for both dynamic specifications; habit formation is significant in both the LES and the QES for both stochastic specifications (i.e., both with and without serial correlation). With serial correlation and linear habit formation, the estimated LES and QES imply strikingly different consumption responses to changes in prices and total expenditure, but the inclusion of quadratic terms is not significant even at the 25 per cent level when we allow for both serial correlation and linear habit formation. The lack of significance of the quadratic terms lends some empirical support to one of the least appealing features of the LES, namely, the linearity of the income-consumption curves. In contrast, Pollak and Wales [**34**] find the quadratic terms highly significant in estimates of the QES based on U.K. household budget data.

We are somewhat skeptical about these results. For the QES, the serial correlation coefficient is close to unity (.95) and the habit formation coefficients are all close to zero, while for the LES, the serial correlation coefficient is significantly negative ($-.30$) and the habit formation parameters are highly significant. For both the QES and the LES, however, the likelihood functions have two local maxima, one at which the serial correlation coefficient is positive and close to .95 and another at which it is negative and close to $-.3$. Furthermore, the LES and the QES predict fairly similar responses to changes in prices and expenditure when these responses are calculated using parameter estimates corresponding to similar values of the serial correlation coefficient. Because the differences in the values of the likelihood functions at the two maxima are small, sharp distinctions among various hypotheses about serial correlation, habit formation, and the functional form of the demand system appear unwarranted.

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APPENDIX

Data

Constant (1972) dollar expenditures on the various categories of goods were obtained from Table 2.7 of *The National Income and Product Accounts* for 1929–1972 and from Table 2.7 of *The Survey of Current Business*, July 1976, for 1973–1975. Corresponding current dollar expenditures were obtained from Tables 2.6 of the same sources. In terms of the categories reported in Table 2.7, we defined our four broad categories of goods as follows (numbers in parentheses correspond to those in Table 2.7):

I. Food
  1. Food (17)

II. Clothing
  1. Clothing and Shoes (23)
  2. Shoe cleaning and repair (61)
  3. Cleaning, laundering, dyeing, etc. (62)

III. Shelter
  1. Housing (38)
  2. Semidurable house furnishings (32)
  3. Cleaning and polishing preparations, and miscellaneous household supplies and paper products (33)
  4. Fuel oil and coal (28)
  5. Household operation, excluding domestic service (43 excluding 48)

IV. Miscellaneous
  1. Toilet articles and preparations (31)
  2. Tobacco products (30)
  3. Drug preparations and sundries (34)
  4. Nondurable toys and sports supplies (35)
  5. Domestic service (48)
  6. Barbershops, beauty parlors and baths (63)
  7. Medical care services (64)
  8. Admissions to specified spectator amusements (69)

Per capita consumption of each good was calculated by dividing annual expenditure in 1972 dollars by population. The population figures are "resident population" in the U.S. and are taken from Table 2 of *The Statistical Abstract of the U.S.*, 1976. Price indices were determined by dividing current dollar expenditure by constant dollar expenditure for each of the four categories.

Although not much purpose would be served by presenting these data here, a few comments about general trends seem appropriate in view of our serial correlation findings. The price indices for all goods were lower in 1941 than 1929, although the decline over the period was not monotonic. In the postwar period, on the other hand, all price indices rose, monotonically for the shelter and miscellaneous categories, and almost monotonically for the food and clothing indices. As far as the dependent variable is concerned, the clothing and miscellaneous shares remained roughly constant in the prewar years, while the food share increased and the shelter share decreased. In the postwar period the shares of food and clothing fell, while those for shelter and miscellaneous rose, the changes being monotonic with but a few exceptions. Thus in the postwar period, and to a lesser extent in the prewar period, the variables tended to trend either up or down. This lends some support to the interpretation of high positive serial correlation as a model misspecification involving an omitted variable that is itself trending over time.

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