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A Comparison of the Performance of Flexible Functional Forms for Use in Applied General Equilibrium Modelling ⋆

CARLO PERRONI[1 ⋆⋆] and THOMAS F. RUTHERFORD[2 ‡]
[1]*University of Warwick, Coventry CV4 7AL, UK*
[2]*University of Colorado, Boulder, Colorado 80309-0256, USA*

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Abstract. This paper describes a procedure for testing the global properties of functional forms which recognizes their specific role in economic equilibrium modelling. This procedure is employed to investigate the global regularity and third-order curvature properties of three widely used flexible functional forms, the Translog, the Generalized Leontieff and the Normalized Quadratic functional forms. We contrast the properties of these functions with a globally regular flexible form, the Nonseparable Nested Constant-Elasticity-of-Substitution functional form. Our results indicate that inherently regular representations are best suited for equilibrium analysis.

Key words: applied general equilibrium, flexible functional forms.

I. Introduction

Flexible functional forms (FFFs) have been widely used in applied econometric work, but there have been relatively few instances in which they have been employed to model production and consumption choices in applied general equilibrium models (Hudson and Jorgenson, 1984; Jorgenson and Slesnick, 1985; Reister and Edmonds, 1981). Few modellers have adopted FFFs for the reason that, in spite of their superior local approximation, they generally exhibit poor global properties.

While in econometric modelling functional forms are used to estimate the local characteristics of technologies or preference orderings from a given set of observations, in applied general equilibrium analysis they are used as a global representation of technologies and preferences. In these applications, the information available to the modeller for the specification of technologies and preferences is typically local, i.e., limited to a small region of production or consumption sets. This local information is extrapolated to the full domain of the modelling exercise by specifying production or utility functions that are locally consistent with such

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information, an approach which is often referred to as ‘calibration’ (Shoven and Whalley, 1992). The global properties of functional forms thus become important in general equilibrium modelling. Lack of global regularity, which may not be crucial for econometric estimation, can cause numerical solution methods to fail even when functions are well behaved at the equilibrium point. Furthermore, when analyzing discrete policy changes, third-order curvature properties, which are of little consequence for the purposes of econometric estimation, can crucially affect estimates of welfare impacts, both in total and at the margin.

This paper explores the global properties of four different functional forms in order to assess their comparative performance and suitability for use in applied equilibrium modelling. For this purpose, we develop a testing methodology which investigates both the global regularity properties and the third-order curvature properties of functional forms, reflecting the specific role of functional forms in applied general equilibrium modelling *vis-à-vis* econometric applications. Summary measures of global regularity are obtained by computing the area of the region in price space over which a cost function is well behaved. We also obtain summary measures of global third-order curvature behaviour by computing the area of the region over which the function remains close, in a sense to be defined by the modeller, to a given local specification of curvature conditions.

We employ this procedure to test the behaviour of the Translog (TL; Christensen, Jorgenson and Lau, 1971), Generalized Leontieff (GL; Diewert, 1971), Normalized Quadratic (NQ; Diewert and Wales, 1987), and Nonseparable Nested CES (NNCES; Perroni and Rutherford, 1995) forms. We choose to focus on the three-input case both because it represents a good compromise between simplicity and generality and because of its practical relevance (e.g., in the modelling of substitution possibilities among labour, capital and energy inputs).

Our tests uncover fundamental differences in the global behaviour of these functional forms. We find that the Translog, Generalized Leontieff and Normalized Quadratic forms are all prone to loss of regularity, particularly when they are calibrated to large cross-elasticity values. Globally regular functions, like the NNCES, are also found to be better at preserving the initial calibration information over the domain of modelling exercises. These results lead us to conclude that inherently regular representations are better suited for equilibrium modelling than traditional FFFs.

The paper is structured as follows. Section II illustrates the implications of the global properties of functional forms by means of a simple numerical example. Section III describes the criteria we adopt to evaluate the global performance of FFFs. Section IV describes the testing procedure and Section V presents test results. Section VI concludes.

II. Functional Forms and Global Behaviour: Implications for Numerical Modelling

To motivate our discussion, we will use a simple numerical example which illustrates the implications of the global properties of functional forms for equilibrium modelling. Consider a cost-minimizing firm producing a certain good or service through constant-returns-to-scale technologies, using labour (L), capital (K), and energy (E) as inputs. Suppose that we observe that, for input prices all equal to unity, the firm employs respectively 0.5 units of labour, 0.25 units of capital and 0.25 units of energy per unit of output. Suppose that we also know how, starting from this initial situation, unit input requirements for each input i ($i = L, K, E$), respond to changes in input prices for all inputs j ($j = L, K, E$), and that this information can be summarized by the following matrix of compensated input demand elasticities $[\sigma_{ij}^C]$:

$$[\sigma_{ij}^C] = \begin{bmatrix} -0.292 & 0.167 & 0.125 \\ 0.333 & -0.250 & -0.083 \\ 0.250 & -0.083 & -0.167 \end{bmatrix}.$$

In this scenario, labour and capital are substitutes ($\sigma_{LK}^C = 0.167 > 0$), labour and energy are substitutes ($\sigma_{LE}^C = 0.125 > 0$), but capital and energy are complements ($\sigma_{KE}^C = -0.083 < 0$). Functional forms such as the CES form are unable to capture such a pattern of substitution possibilities; in this case we need to resort to a flexible functional form – a functional form that is able to provide a local second-order approximation to any cost function.

For this example we will employ two alternative FFFs, the TL and the NNCES (for definitions and for calibration formulae see the Appendix). When calibrated to the above configuration of compensated demands and elasticity values, the TL and the NNCES are locally equivalent, i.e., they provide an equivalent local approximation of behavioural responses. For discrete changes from the initial calibration point, however, they behave differently, as the following experiment shows.

Consider the introduction of an *ad valorem* tax on energy at rate t, which raises the gross-of-tax price of energy inputs to $(1 + t)$. For a given t, we can compute the change in conditional demands and unit cost relative to the calibration point, as well as the *marginal effective tax rate*, i.e., the impact on unit cost of a marginal increase in t. These are summarized in Table I, for different values of t, for both the TL (upper panel) and the NNCES (lower panel) specifications. For low tax rates the behaviour of the TL and NNCES is similar, but for higher taxes differences in behaviour become apparent. For example, for $t = 100\%$ the reduction in energy demand is 6.8% under the TL specification and 5.8% under the NNCES. Impacts on unit cost, in total and at the margin, are also different: for $t = 1000\%$ the marginal

Table I. Impacts of energy taxes under alternative functional specifications – % changes.

TRANSLOG

	Tax rate 10%	100%	1000%
Change in compensated labour demand	1.3	12.5	94.0
Change in compensated capital demand	−0.4	−2.6	−22.2
Change in compensated energy demand	−0.8	−6.8	−19.8
Change in unit cost	2.5	23.2	177.0
Marginal effective tax rate	24.4	21.5	15.1

NNCES

	Tax rate 10%	100%	1000%
Change in compensated labour demand	1.2	10.5	50.3
Change in compensated capital demand	−0.4	−2.9	−7.7
Change in compensated energy demand	−0.8	−5.3	−13.0
Change in unit cost	2.5	23.5	199.5
Marginal effective tax rate	24.6	22.3	18.5

effective tax rate under the TL specification is more than three percentage points higher than under the NNCES.

Even though these differences appear to be small in absolute terms, in relative terms they are significant. Consider, for example, the last line in Table I, which reports marginal effective tax rate estimates. If we were to ignore the information we have on benchmark elasticities and replace both the NNCES and the TL with a fixed coefficient specification – for which all cross elasticities are zero – we would obtain a constant marginal effective tax rate equal to 25.0%. This estimate would be 0.4 percentage points (25.0%–24.6%) in excess of the marginal effective tax rate estimate obtained under a NNCES specification for a 10% energy tax; in comparison, the difference between the corresponding TL and NNCES estimates for a 10% tax rate is 0.2 percentage points (24.6%–24.4%). In other words, the estimate error from using the ‘wrong’ functional form can be as large as one half of the error from using the ‘wrong’ elasticities in calibration, even for a relatively small tax.

For even larger values of t (not shown), the TL form yields negative input demands, and thus ceases to be an economically meaningful representation of technologies. Even when such extreme price changes can be ruled out as being outside the domain of a given modelling exercise, they can still occur during intermediate function evaluation along the computation path to an equilibrium; such occurrences would typically cause computation to abort or to fail to converge. The NNCES specification is more robust in this respect, as it always generates positive input demands.

In summary, the global properties of functional forms have important implications for applied general equilibrium modelling. Functional forms that are not globally well behaved can jeopardize convergence in computation. Furthermore, different functional forms can exhibit significant differences in behavioural responses, which can involve significant variations in induced general equilibrium effects.

III. Evaluating the Global Properties of Functional Forms

Although modellers have long recognized that the choice of functional forms is an important determinant of model behaviour, research on the performance of FFFs in equilibrium models has been rather scant. Caves and Christensen (1980) built a number of examples to compare the regions over which the TL and GL cost functions are well behaved (i.e., non-negative, monotonic, and concave in prices). Performance was found to depend on the initial specification of second-order curvature conditions, with the TL being preferable when the cross Allen–Uzawa elasticities of substitution (AUES; Allen, 1938; Uzawa, 1962) are close to unity, and the GL being a good choice for cross AUESs close to zero. Reister and Edmonds (1981) analyzed the effects of replacing a Constant-Elasticity-of-Substitution (CES) specification (Uzawa, 1962) with a TL form in a simple general equilibrium model and found substantial differences in their global behaviour. Comparing the global properties of the TL and GL functions, Despotakis (1986) concluded that the law of change of the AUES is different for these two functional forms, and he noted that these differences can have important consequences for equilibrium analysis.

None of the studies mentioned above, however, attempts a systematic comparison of functional forms. Here, we propose a methodology for testing functional forms which enables us to assess their comparative performance and suitability for use in applied modelling exercises. In this section, we will first propose criteria for evaluating the global properties of FFFs, and then, in the next section, we will describe a systematic testing methodology which builds on these criteria.

Our discussion will be focused on production technologies with N inputs, one output and constant returns-to-scale (for a discussion of how a single-output, constant-returns representation of technologies can be used to approximate homothetic and non-homothetic preferences, see Perroni, 1992). A sufficient and convenient way of summarizing such technologies is given by a continuous unit cost function, $C(p,1)$, where p denotes the input price vector and 1 is the production level; this will be hereafter referred to simply as $C(p)$ or C. The first derivatives of the cost function, denoted with C_i $(i = 1, \ldots, N)$, represent conditional input demands (Shephard's Lemma); they are homogeneous of degree zero in prices and, by Euler's

Theorem, satisfy the adding-up condition $\sum_i p_i C_i = C$. The matrix of partial second derivatives $[C_{ij}]$ (the Hessian) is homogeneous of degree -1 in prices, and satisfies the Cournot aggregation condition $\sum_j p_j C_{ij} = 0$ $(i = 1, \ldots, N)$.

We will normalize prices so that price vectors lie in the unit simplex, $S^N \equiv \{p \mid p_i \geqslant 0;\ \sum_i p_i = 1\}$. We will also express conditional demands in terms of input value shares, $\theta_i \equiv p_i C_i / C$, and the Hessian of the cost function in terms of Allen–Uzawa elasticities of substitution, which are defined as $\sigma_{ij}^A \equiv C_{ij} C / (C_i C_j)$. The AUES is a dimensionless index of curvature, and is thus scale invariant; expressed in terms of AUESs, the Euler condition has the form $\sum_j \sigma_{ij}^A \theta_j = 0$.

A cost function is regular (well-behaved) at a point p if its value $C(p)$ is nonnegative, its first-derivatives $C_i(p)$ (which correspond to input demands) are nonnegative (monotonicity), and if the Hessian $[C_{ij}(p)]$ is negative semidefinite (implying concavity, a sufficient condition for this choice of inputs to minimize cost). Monotonicity implies $\theta_i(p) \geqslant 0$, and negative semidefiniteness of $[C_{ij}(p)]$ implies negative semidefiniteness for $[\sigma_{ij}^A]$.

We will refer to the range over which a function maintains monotonicity as the Monotonic Domain (MD). This can be expressed as

$$MD \equiv \{p \in S^N \mid C(p) \geqslant 0;\ \theta_i(p) \geqslant 0\}. \tag{1}$$

The range over which a function maintains concavity will be referred to as its Concave Domain (CD):

$$CD \equiv \{p \in S^N \mid C(p) \geqslant 0;\ [\sigma_{ij}^A(p)] \text{ negative semidefinite}\}. \tag{2}$$

The region of the price simplex over which a cost function is regular, which Despotakis (1986) termed the Outer Domain, is then simply

$$OD \equiv MD \cap CD. \tag{3}$$

As we noted above, the size of the Outer Domain is of crucial importance for equilibrium modelling; this is because the algorithms that are employed to compute equilibria can require function evaluation at price points which may lie arbitrarily far from the equilibrium point, and may thus fail to converge if functional forms are not globally well behaved. In applied general equilibrium exercises, however, regularity is not enough. When discrete changes in policy parameters are involved, we are also concerned with how well a functional form performs globally at approximating the underlying technological possibilities.

Despotakis defines the Inner Domain (ID) of a cost function as the region of the price simplex where the function remains close, in a sense to be defined by the user, to the ‘true’ technology. In applied modelling exercises, however, the information available to the modeller is typically limited to the calibration point p^0, and the ‘true’ technology is unknown. In the absence of global information on the technology to be approximated, the modeller must adopt, explicitly or implicitly, certain assumptions concerning the out-of-benchmark characteristics of functions, on the basis of the local information available. For example, when choosing a CES representation, a modeller implicitly assumes that, when moving away from the calibration point, the first and second derivatives of the cost function change in such a way as to ensure constancy of all cross AUESs. This corresponds to a specific set of conjectures about the third-order curvature properties of the cost function.

To make the notion of Inner Domain operational when only local information is available, we can employ a distance function Z:

$$Z(p) \equiv \|E(p), E(p^0)\|, \tag{4}$$

where $E(p)$ is a vector of curvature measures (e.g., elasticities) at p, and $E(p^0)$ represents the corresponding values at the calibration point. The definition of E can be left to the discretion of the modeller. The Inner Domain can then be defined as the region of the unit simplex where the value of Z is less than or equal to a pre-specified tolerance level:

$$ID(\delta) \equiv \{p \in S^N \mid Z(p) \leqslant \delta\}. \tag{5}$$

The choice of a particular curvature index implies certain assumptions about the global characteristics of the cost function. For example, if the modeller believes that the 'true' cost function exhibits constant value shares, then $E(p)$ will be chosen to represent a vector of value shares (in which case the best choice of functional form would naturally be a Cobb–Douglas cost function). In the following, we will restrict our discussion to a few second-order curvature indexes that have been proposed in the literature.

A well known dimensionless index of second-order curvature is the compensated price elasticity (CPE), which is defined as

$$\sigma_{ij}^C \equiv \frac{\partial \ln C_i}{\partial \ln p_j} = \frac{C_{ij} p_j}{C_i}. \tag{6}$$

A related measure of second-order curvature is the AUES, which has been already discussed; this can also be written as

$$\sigma_{ij}^A = \frac{\sigma_{ij}^C}{\theta_j}. \tag{7}$$

The AUES is a *one-input-one-price* elasticity of substitution (Mundlak, 1968), since, as (7) makes clear, it measures the responsiveness of the compensated demand for one input to a change in one input price. In contrast, the Morishima elasticity of substitution (MES; Morishima, 1967) constitutes a *two-input-one-price* elasticity measure, being defined as

$$\sigma_{ij}^M \equiv \frac{\partial \ln(C_i/C_j)}{\partial \ln(p_i/p_j)} = \sigma_{ij}^C - \sigma_{jj}^C. \tag{8}$$

Note that, in general, the MES is not symmetric, i.e. $\sigma_{ij}^M \neq \sigma_{ji}^M$.

A third type of curvature measure is represented by the class of *two-input-two-price* elasticities of substitution, which take the form $\partial \ln(C_i/C_j)/\partial \ln(p_j/p_i)$. One such index is the shadow elasticity of substitution (SES; Frenger, 1985), which is defined as

$$\sigma_{ij}^S \equiv \frac{\theta_i \sigma_{ij}^M + \theta_j \sigma_{ji}^M}{(\theta_i + \theta_j)}. \tag{9}$$

When technologies are of the CES type, σ_{ij}^A, σ_{ij}^M and σ_{ij}^S are all identical, but they are generally different otherwise.

In our tests we define the distance function Z as a weighted sum of the squared deviations from the benchmark elasticity matrix, where weights are chosen to be equal to the combined share of inputs i and j in total cost:

$$Z(p) \equiv \frac{\sum_{i \neq j} [\theta_i(p^0) + \theta_j(p^0)][\sigma_{ij}(p) - \sigma_{ij}(p^0)]^2}{\sum_{i \neq j} [\theta_i(p^0) + \theta_j(p^0)][\sigma_{ij}(p^0)]^2}. \tag{10}$$

Four different versions of the above norm are employed, respectively based on the CPE (σ^C), AUES (σ^A), MES (σ^M), and SES (σ^S).

In all of the tests we report, we select a tolerance level δ (in (5)) equal to 0.25. We have found that this choice of δ, though arbitrary, is effective for the purpose of our tests, which is not to quantify behaviour for individual functions, but rather to compare performance across functional forms. Given our definition of the norm Z, when $Z(p) \leqslant 0.25$ squared elasticity changes are less than 25% (i.e., elasticities change by less than 50% on average).

IV. Testing Methodology

This section describes a testing methodology which incorporates the notions of Inner and Outer Domain discussed above, and investigates the behaviour of the functions systematically over the price simplex.

The tests are designed as follows. The function under investigation is calibrated at a benchmark point p^0 to a given specification of derivatives up to the second order. In the tests reported here, we choose p^0 to be the center of the unit simplex and we consider two configurations of benchmark value shares: a symmetric configuration with $\theta = (0.33, 0.33, 0.33)$ and an asymmetric configuration with $\theta = (0.35, 0.60, 0.05)$. For each configuration of value shares, we examine a number of different benchmark configurations of second-order curvature conditions belonging to the *regular region*, i.e., the set of benchmark cross AUES configurations that are compatible with local concavity of the cost function (assuming that the zero-th and first-order regularity conditions are satisfied).

Because of symmetry and homogeneity, only $H = N(N-1)/2$ elements of the matrix $[\sigma_{ij}^A]$ are independent, which implies that we can ignore the diagonal terms. Thus, the regular region Q is a subset of $\mathbb{R}^H$, bounded by $N - 1$ conditions for negative semidefiniteness of the AUES matrix. Q is a convex set, since a convex combination of two negative semidefinite matrices is also negative semidefinite. Moreover, Q is a cone, since multiplication of a negative semidefinite matrix by a positive scalar results in a negative semidefinite matrix.

The latter property enables us to characterize the geometry of the regular region analyzing only the image of a projection of Q in $\mathbb{R}^{H-1}$. For this purpose we choose the following projection: the AUES matrix is divided by its largest positive off-diagonal element, so that the maximum off-diagonal element of the resulting matrix is always unity. In our testing, we focus on the case $N = 3$ and, without loss of generality, assume that the element (1,2) is the largest cross AUES. Then, the regular region Q lies in $\mathbb{R}^3$, and the image of the projection lies in $\mathbb{R}^2$, bounded by the following constraints ($[\sigma_{ij}^A]$ denotes the normalized AUES matrix):

$$\sigma_{13}^A \leqslant 1; \tag{11}$$

$$\sigma_{23}^A \leqslant 1; \tag{12}$$

$$\sigma_{13}^A \geqslant -(\theta_2/\theta_3); \tag{13}$$

$$\sigma_{23}^A \geqslant -(\theta_1/\theta_3); \tag{14}$$

$$(\theta_3/\theta_1)\sigma_{13}^A + (\theta_2/\theta_1) \geqslant 0; \tag{15}$$

$$\sigma_{13}^A\sigma_{23}^A + (\theta_1/\theta_3)\sigma_{13}^A + (\theta_2/\theta_3)\sigma_{23}^A \geqslant 0. \tag{16}$$

The first two constraints follow from normalization. The remaining four are the sign constraints on the first and second principal minors for negative semidefiniteness. Figure 1 depicts the projection of the regular region in $\mathbb{R}^2$.

Our testing procedure has been employed to explore four ‘slices’ through Q, those representing AUES matrices with maximum off-diagonal values respectively equal to $\frac{1}{2}$, 1, 2 and 4. For each of these sections we examined a uniform grid of AUES configurations with between 47 and 52 points depending on the benchmark value shares.

For each of these configurations, the properties of the function (monotonicity, concavity, and $Z(p)$ values) are systematically evaluated over an evenly-spaced triangular grid on the price simplex containing 325 points. The resulting discrete mapping is then contoured to derive piecewise-linear approximations of the various domains.

Figures 2–3 illustrate these contour set calculations for the TL form, with uniform input value shares and for elasticity configurations where the largest AUES value equals 2. Each price simplex refers to a different benchmark elasticity configuration, and its position on the page corresponds to the location of each configuration in the region Q as depicted in Figure 1. The center of each simplex represents calibration prices (all assumed equal to $\frac{1}{3}$).

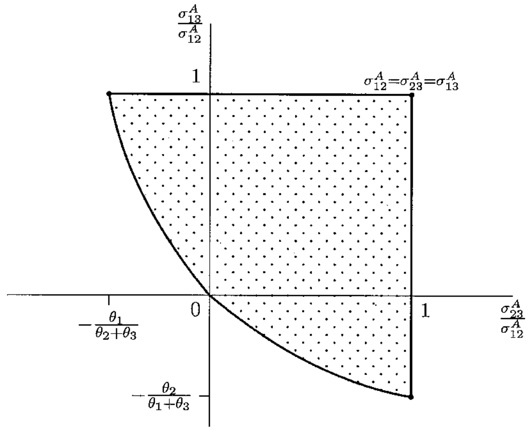

Figure 1. The projection of Q in $\mathbb{R}^2$.

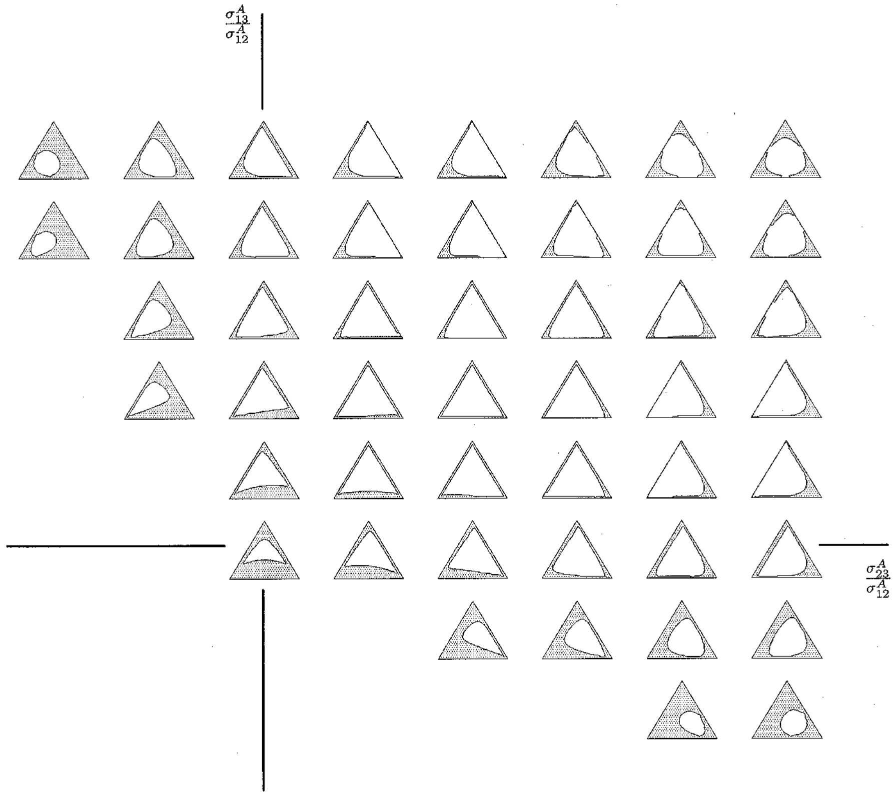

Figure 2. Calculated outer domain contour sets for the TL form with $\theta_1 = \theta_2 = \theta_3 = \frac{1}{3}$; maximum $\sigma_{12}^A = 2$.

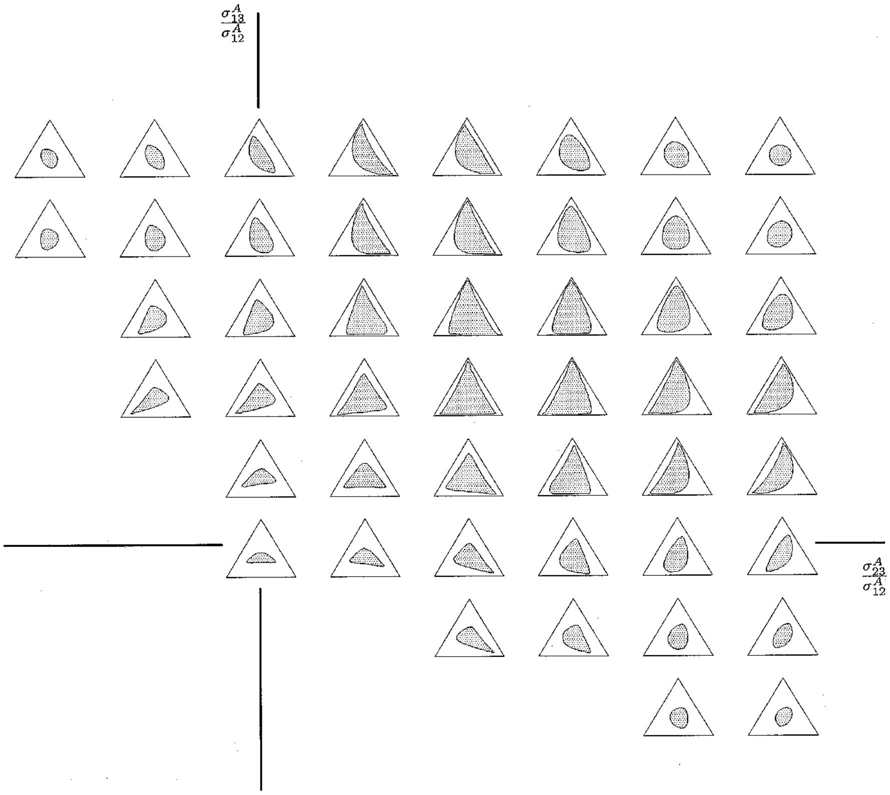

Figure 3. Calculated inner domain contour sets for the TL form with $\theta_1 = \theta_2 = \theta_3 = \frac{1}{3}$; maximum $\sigma_{12}^A = 2$; $\delta = 0.25$.

Shaded areas within each simplex in Figure 2 signify relative price changes for which the cost function loses concavity or monotonicity. Note that these regions can be very large as a proportion of the price simplex, especially when some of the inputs exhibit complementarity (in the upper left-hand corner and lower right-hand corner of Q). In some cases, the TL form loses regularity when very close to the center of the simplex, i.e., for price changes which could be well within the domain of a typical simulation experiment. The corresponding Inner Domains contour sets are shown in Figure 3; here, the shaded areas within each simplex signify relative prices for which $Z(p) \leqslant 0.25$. Such regions can be quite small, especially when inputs are complementary (the corners of the Q region).

Once the approximated contour sets are derived, they are used to compute the area of each domain, and to obtain synthetic indexes expressing the volume of these regions as a proportion of the volume of the unit price simplex. Synthetic indexes for the Monotonic and Concave Domains are denoted respectively as A_{MD} and A_{CD}:

$$A_{MD} = \text{Area}(MD)/\text{Area}(S^N); \tag{17}$$

and

$$A_{CD} = \text{Area}(CD)/\text{Area}(S^N). \tag{18}$$

For the Inner Domain we must also specify a tolerance level:

$$A_{ID}(\delta) = \text{Area}[ID(\delta)]/\text{Area}(S^N). \tag{19}$$

The resulting measures A_{MD}, A_{CD}, and $A_{ID}(\delta)$ (taking $\delta = 0.25$ to define the Inner Domain) are averaged over all sample points.

V. Test Results

We have applied our procedure to test the global behaviour of four different functional forms: TL, GL, NQ and NNCES. The TL and GL forms have been chosen because they are the best known among FFFs. The NQ form is included because it is globally concave (although it can lose monotonicity). The NNCES form is flexible and globally regular, and belongs to a family of functional forms that have been widely employed in the applied general equilibrium literature. All four functional forms and the formulae used for parameter calibration are described in the Appendix.

Results of our Outer Domain calculations are summarized in Table II, which reports average measures of MD, CD, and OD as a percentage of the price simplex for different specifications of maximum cross AUES values. Our tests confirm findings of earlier studies on the global properties of the TL and GL functional forms. The TL is prone to loss of concavity away from the calibration point whenever the benchmark cross-substitution elasticities depart from unity: in the symmetric case when the maximum cross AUES is 0.5, for example, the size of TL Concave Domain is only 27% of the price simplex. The size of the TL Concave Domain is largest when the maximum cross AUES is 2; in this case, cross AUES values range from 0 to 2, and thus ‘average’ around unity. When input value shares are asymmetric, the TL has somewhat better concavity properties, but it is more prone to lose monotonicity when cross elasticities depart from unity.

The GL has better global concavity properties than the TL, but tends to lose monotonicity as benchmark cross elasticities become large (above 1). When input value shares are asymmetric, the Outer Domain performance of the GL is slightly worse.

The NQ remains concave over its entire domain but tends to lose monotonicity at higher elasticities, as does the GL. The Outer Domain for the NQ (as well as the TL and GL) falls below 50% of the price simplex when the maximum cross

Table II. Average measured Outer Domain as a percentage of the price simplex.

Symmetric Value Shares: $\theta = (0.33, 0.33, 0.33)$

	Monotonic Domain				Concave Domain				Outer Domain			
$\sigma_{12}^{A} =$	0.5	1	2	4	0.5	1	2	4	0.5	1	2	4
TL	71	82	81	32	27	62	73	60	25	61	65	27
GL	100	99	69	30	93	93	93	93	93	92	63	27
NQ	100	100	73	33	100	100	100	100	100	100	72	31
NNCES	100	100	100	100	100	100	100	100	100	100	100	100

Asymmetric Value Shares: $\theta = (0.35, 0.60, 0.05)$

	Monotonic Domain				Concave Domain				Outer Domain			
$\sigma_{12}^{A} =$	0.5	1	2	4	0.5	1	2	4	0.5	1	2	4
TL	69	86	72	28	46	71	76	64	41	70	61	25
GL	100	98	61	25	95	95	94	92	94	92	57	24
NQ	100	99	67	27	100	100	100	100	100	99	66	26
NNCES	100	100	100	100	100	100	100	100	100	100	100	100

AUES is 4 and benchmark value shares are equal. When benchmark shares are asymmetric and the maximum cross AUES is 4, the TL, GL and NQ Outer Domain shrinks to roughly one quarter of the price simplex.

In contrast with the other functional forms, the NNCES is globally regular, and thus maintains concavity and monotonicity over the whole price simplex.

Table III reports results of Inner Domain calculations based respectively on the CPE, AUES, MES, and SES based norms. The CPE based norm appears to be more volatile than the other measures, but the ranking of functional forms is consistent across different norms.

In the symmetric case (upper panel), the TL form performs best for benchmark cross AUES values close to unity: the average TL Inner Domain size is more than 60% when the maximum AUES is equal to 1 (which implies an ‘average’ cross AUES value lower than 1). In the asymmetric case, the TL performs rather poorly, particularly for low cross AUES values. The GL form performs best for cross AUES values close to zero, with an average Inner Domain size in excess of 80% of the price simplex in some cases.

The NQ performs very poorly in all cases, with Inner Domain sizes often as small as 1% of the price simplex. The NNCES is the most consistent performer, with average Inner Domain measures above 50% in all cases with the exception of

Table III. Average measured Inner Domain as a percentage of the price simplex.

Symmetric Value Shares: $\theta = (0.33, 0.33, 0.33)$

	Compensated				Allen–Uzawa				Morishima				Shadow			
$\sigma_{12}^A =$	0.5	1	2	4	0.5	1	2	4	0.5	1	2	4	0.5	1	2	4
TL	3	43	20	5	6	62	35	8	12	61	50	12	13	58	47	12
GL	25	40	17	4	67	59	22	6	81	75	30	8	83	76	29	8
NQ	4	4	3	1	11	9	6	3	21	12	6	2	20	14	6	2
NNCES	14	40	13	3	67	71	59	41	71	67	61	52	70	67	59	47

Asymmetric Value Shares: $\theta = (0.35, 0.60, 0.05)$

	Compensated				Allen–Uzawa				Morishima				Shadow			
$\sigma_{12}^A =$	0.5	1	2	4	0.5	1	2	4	0.5	1	2	4	0.5	1	2	4
TL	7	63	18	4	9	59	36	9	16	66	44	12	13	57	43	12
GL	42	63	17	4	84	61	27	7	93	70	33	9	91	68	32	9
NQ	2	2	2	1	1	1	1	1	6	5	4	2	21	15	9	3
NNCES	35	78	20	4	66	74	68	50	92	90	83	73	88	87	83	66

the CPE based measure. As for the TL, the ID performance of the NNCES seems to be best when the maximum cross AUES value is 1. This finding may be due to the particular NNCES mapping we use in our tests (see the Appendix); alternative mappings may well exhibit different Inner Domain properties.

Comparison of Tables II and III suggests that the Outer Domain and the Inner Domain of a cost function might be correlated. To examine this conjecture more formally, we have computed the correlation coefficient between the sizes of the Outer Domain and the AUES Inner Domain over the range of benchmark elasticity values. If we denote with x_s the OD index calculated for each of the elasticity configurations we test, with y_s the corresponding ID index, and with $\overline{x}$ and $\overline{y}$ their respective averages, then the correlation coefficient can be expressed as $\sum_s(x_s - \overline{x})(y_s - \overline{y})\ /\ [\sum_s(x_s - \overline{x})^2 \sum_s(y_s - \overline{y})^2]^{1/2}$.

The results of these calculations (shown in Table IV) confirm the intuition that ‘instability’ is closely associated with loss of regularity. All computed coefficients are in excess of 0.5, and in many cases above 0.9. Thus, inherently regular functional forms such as the NNCES are not only preferable on computational grounds but they are also better at preserving local calibration information over the full domain of modelling exercises.

Table IV. Correlation between the AUES Inner and Outer Domains.

	Uniform Shares				Nonuniform Shares			
$\sigma_{12}^A =$	0.5	1	2	4	0.5	1	2	4
TL	0.92	0.92	0.85	0.90	0.96	0.97	0.93	0.77
GL	0.51	0.67	0.79	0.91	0.77	0.72	0.79	0.82
NQ	na **	na **	0.59	0.68	0.76	0.83	0.91	0.75

**The NQ form is globally regular for these shares and assumed elasticity values, and the correlation is therefore undefined.

VI. Summary and Conclusion

This paper has presented a procedure for testing the global properties of functional forms which explicitly recognizes their role in equilibrium modelling. We have used this procedure to test the regularity and third-order curvature properties of four flexible functional forms, and found that the Translog, Generalized Leontieff and Normalized Quadratic forms are all prone to loss of regularity, particularly when the benchmark cross elasticities are large. Globally regular functions, like the NNCES, are also better at preserving local calibration information over the domain of modelling exercises. For these reasons we conclude that globally regular functions like the NNCES are better suited for equilibrium analysis.

Further research should investigate the global behaviour of alternative specifications of the NNCES form so as to provide practitioners with some concrete guidance in the selection of nesting structure.

We believe that a better understanding of the properties of the functional forms used in applied general equilibrium exercises would ultimately increase transparency and improve users' understanding of model results.

Appendix: Cost Functions Definitions and Calibration

In this appendix we summarize formulae for recovering parameters of the functional forms we tested. The cost function is denoted by C, and the symbols $\bar{C}$, θ_i and σ_{ij}^A respectively denote the given unit cost level, input value shares and Allen–Uzawa elasticities of substitution at the calibration point.

A.1. Translog

The Translog unit cost function is defined as

$$\ln C(p) \equiv \ln\, b_0 + \sum_i b_i \ln\, p_i + \frac{1}{2} \sum_{ij} a_{ij} \ln\, p_i\, \ln\, p_j \equiv \ln\, b_0 + L(p). \quad (20)$$

Restrictions:

$$\sum_i b_i = 1; \tag{21}$$

$$a_{ij} = a_{ji}, \quad \forall i, \forall j; \tag{22}$$

$$\sum_j a_{ij} = 0, \quad \forall i. \tag{23}$$

Unit compensated demands:

$$C_i(p) = \left(b_i + \sum_j a_{ij} \ln\, p_j \right) C(p)/p_i, \quad \forall i. \tag{24}$$

Calibration:

$$a_{ij} = \theta_i \theta_j (\sigma_{ij}^A - 1), \quad i \neq j; \tag{25}$$

$$a_{ii} = -\sum_{j \neq i} a_{ij}, \quad \forall i; \tag{26}$$

$$b_i = \theta_i - \sum_j a_{ij} \ln\, p_j, \quad \forall i; \tag{27}$$

$$b_0 = \bar{C}\ \mathrm{e}^{-L(p)}. \tag{28}$$

A.2. Generalized Leontieff

The Generalized Leontieff unit cost function is defined as

$$C(p) \equiv \frac{1}{2} \sum_{ij} a_{ij} (p_i p_j)^{1/2}. \tag{29}$$

Restrictions:

$$a_{ij} = a_{ji}, \quad \forall i, \forall j. \tag{30}$$

Unit compensated demands:

$$C_i(p) = \frac{1}{2} \sum_j a_{ij} (p_j/p_i)^{1/2}, \quad \forall i. \tag{31}$$

Calibration:

$$a_{ij} = 4\theta_i\theta_j\bar{C}(p_ip_j)^{-(1/2)}\sigma^A_{ij}, \quad i \neq j; \tag{32}$$

$$a_{ii} = 2\theta_i\bar{C}/p_i - \sum_{j\neq i} a_{ij}(p_j/p_i)^{1/2}, \quad \forall i. \tag{33}$$

A.3. NORMALIZED QUADRATIC

The Normalized Quadratic unit cost function is defined as

$$C(p) \equiv \frac{1}{2}\left(\sum_{ij} a_{ij}p_ip_j\right) \Bigg/ \left(\sum_i b_ip_i\right). \tag{34}$$

Restrictions:

$$a_{ij} = a_{ji}, \quad \forall i, \forall j; \tag{35}$$

$$b_i \geqslant 0, \quad \forall i; \tag{36}$$

$$\sum_i b_i = 1. \tag{37}$$

Unit compensated demands:

$$C_i(p) = \left[\sum_j a_{ij}p_j - b_iC(p)\right] \Bigg/ \left(\sum_j b_jp_j\right), \quad \forall i. \tag{38}$$

Calibration:

$$a_{ij} = \bar{C}\theta_i\theta_j\left(\sigma^A_{ij}\sum_k b_kp_k + b_ip_i/\theta_i + b_jp_j/\theta_j\right)/(p_ip_j), \quad \forall i \neq j; \tag{39}$$

$$a_{ii} = \left[\theta_i\bar{C}\left(\sum_k b_kp_k + b_ip_i/\theta_i\right) - \sum_{j\neq i} a_{ij}p_ip_j\right]/(p_i)^2, \quad \forall i. \tag{40}$$

We examined two alternative specifications, one in which $b_i = \theta_i$, and another in which $b_i = 1/N$. The first specification produces larger Inner Domain estimates and is used in the tests we report on in Table III.

A.4. NONSEPARABLE NESTED CES

The mapping from benchmark elasticities to functional parameters is typically non-unique for nonseparable nested CES functions. Here, we restrict our attention to a particular nesting structure, the 'Lower Triangular Leontieff' mapping (LTL). (For a discussion of the general N-input LTL mapping, see Perroni and Rutherford, 1995.) Let us rearrange indices so that the maximum off-diagonal AUES element is σ_{12}^A. Then the three-input NNCES-LTL cost function can be defined as

$$C(p) \equiv$$

$$\phi \left[\epsilon(a_1 p_1 + a_3 p_3)^{1-\gamma} + (1-\epsilon)(b_2 p_2^{1-\mu} + b_3 p_3^{1-\mu})^{(1-\gamma)/(1-\mu)}\right]^{1/(1-\gamma)}. \tag{41}$$

Restrictions:

$$\gamma \geqslant 0; \tag{42}$$

$$\mu \geqslant 0; \tag{43}$$

$$\phi \geqslant 0; \tag{44}$$

$$1 \geqslant \epsilon \geqslant 0; \tag{45}$$

$$a_i \geqslant 0, \quad \forall i; \tag{46}$$

$$b_i \geqslant 0, \quad \forall i. \tag{47}$$

Unit compensated demands:

$$C_1(p) = \phi^{(1-\gamma)} \epsilon C(p)^\gamma a_1 (a_1 p_1 + a_3 p_3)^{-\gamma}; \tag{48}$$

$$C_2(p) = \phi^{(1-\gamma+\mu)} (1-\epsilon) C(p)^\gamma b_2 \left(b_2 p_2^{1-\mu} + b_3 p_3^{1-\mu}\right)^{(\mu-\gamma)/(1-\mu)} p_2^{-\mu}; \tag{49}$$

$$C_3(p) = \phi^{(1-\gamma)} \epsilon C(p)^\gamma a_3 (a_1 p_1 + a_3 p_3)^{-\gamma}$$

$$+\phi^{(1-\gamma+\mu)} (1-\epsilon) C(p)^\gamma b_3 \left(b_2 p_2^{1-\mu} + b_3 p_3^{1-\mu}\right)^{(\mu-\gamma)/(1-\mu)} p_3^{-\mu}. \tag{50}$$

Calibration:

Let us denote with s_3 the fraction of the total input of commodity 3 which enters the first subnest of the structure described by (41) – with $(1-s_3)$ representing the fraction entering the second subnest. If we select

$$\gamma = \sigma_{12}^A; \tag{51}$$

$$\mu = (\sigma_{12}^A \sigma_{13}^A - \sigma_{23}^A \sigma_{11}^A)/(\sigma_{13}^A - \sigma_{11}^A); \tag{52}$$

it can be shown that

$$s_3 = (\sigma_{12}^A - \sigma_{13}^A)/(\sigma_{12}^A - \sigma_{11}^A). \tag{53}$$

$$\phi = \bar{C}; \tag{54}$$

$$\epsilon = \theta_1 + s_3\theta_3; \tag{55}$$

$$a_1 = \theta_1/(\epsilon p_1); \tag{56}$$

$$a_3 = s_3\theta_3/(\epsilon p_3); \tag{57}$$

$$b_2 = \theta_2/[(1-\epsilon)p_2^{1-\mu}]; \tag{58}$$

$$b_3 = (1-s_3)\theta_3/[(1-\epsilon)p_3^{1-\mu}]. \tag{59}$$

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