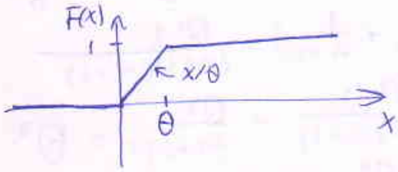
№1

$\xi \sim R(0,\theta)$ равном распр

выборка объема n $\vec{X_n}$



$\tilde\theta_1 = 2\bar{x} = 2\frac{1}{n}\sum_{i=1}^{n} x_i$

$\tilde\theta_2 = x_{min}$

$\tilde\theta_3 = x_{max}$

$\tilde\theta_5 = \left(x_1 + \frac{1}{n-1}\sum_{k=2}^{n} x_k\right)$

а) Проверить на несмещенность и состоятельность

$M\xi = \int_{-\infty}^{+\infty} x\,dF(x,\theta) = \int_0^\theta x\frac{1}{\theta}dx = \frac{\theta}{2}$

$p(x,\theta) = \frac{1}{\theta}\{(0;\theta)\}$

$M\xi^2 = \int_0^\theta x^2\frac{1}{\theta}dx = \frac{\theta^2}{3}$

$D\xi = M\xi^2 - (M\xi)^2 = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$

Парам. модель:

несмещенность: $\forall\theta\in\Theta \hookrightarrow M\hat\theta = \theta$

состоятельность: $\forall\theta\in\Theta\ \forall\varepsilon>0 \hookrightarrow P(|\hat\theta-\theta|\geq\varepsilon) \xrightarrow[n\to\infty]{} 0$

① $\measuredangle\ \tilde\theta_1$

$M\tilde\theta_1 = M(2\frac{1}{n}\sum x_i) = \frac{2}{n}\sum Mx_i = \frac{2}{n}nM\xi = \theta$ $\Rightarrow$ несмещен.

$D\tilde\theta_1 = D(2\frac{1}{n}\sum x_i) = \frac{4}{n^2}D\sum x_i = \frac{4}{n^2}\sum Dx_i = \frac{4}{n^2}nD\xi = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0$

$\Rightarrow$ явл состоят по дост усл

② $\measuredangle\ \tilde\theta_2$

$\forall\theta>0$ $M\tilde\theta_2 = Mx_{min}$

$\xi \sim F(x)$

$\xi_{min} \sim \underbrace{1-(1-F(x))^n}_{\Phi(x)}$ 1-ая порядк. стат.

$\varphi(x) = \Phi'(x) = n(1-F(x))^{n-1}F'(x) = n(1-\frac{x}{\theta})^{n-1}\frac{1}{\theta}\{(0,\theta)\} \Rightarrow$

$Mx_{min} = \int_0^\theta x\cdot n\cdot(1-\frac{x}{\theta})^{n-1}\frac{1}{\theta}dx = \left\{\begin{array}{l} t = 1-\frac{x}{\theta} \\ dt = -\frac{1}{\theta}dx \\ dx = -\theta dt \\ x = (1-t)\theta \end{array}\right\} = \int_1^0 n(1-t)t^{n-1}(-\theta dt) =$

$= n\theta\int_0^1 (t^{n-1} - t^n)dt = \frac{\theta}{n+1} \neq \theta \Rightarrow$ смещенная

Попробуем исправить оценку:

$\tilde\theta_2' = (n+1)x_{min}$

$M\tilde\theta_2' = (n+1)Mx_{min} = \theta$ теперь несмещ.

$D\tilde{\theta}_2' = D((n+1)x_{min}) = (n+1)^2 D x_{min}$

$M x_{min}^2 = \int_0^\theta x^2 n (1-\frac{x}{\theta})^{n-1}\frac{1}{\theta}dx = \int_0^1 \theta^2(1-t)^2 n t^{n-1}dt = n\theta^2\int_0^1 (t^{n-1}-2t^n+t^{n+1})dt =$

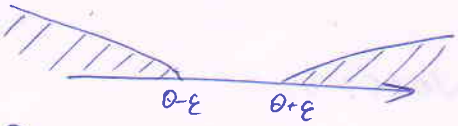
$= n\theta^2(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}) = \frac{\theta^2 2}{(n+1)(n+2)}$

$D x_{min} = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \theta^2\frac{n}{(n+1)^2(n+2)}$

$D\tilde{\theta}_2' = \frac{n\theta^2}{n+2} \underset{n\to\infty}{\not\to} 0$ нельзя использовать достат усл.

Δ определение:

$\forall\theta>0\ \forall\varepsilon>0\quad P(|\tilde{\theta}_2'-\theta|\geq\varepsilon)\xrightarrow[n\to\infty]{}0$



$P(|\tilde{\theta}_2'-\theta|\geq\varepsilon) \geq P(\tilde{\theta}_2'\geq\theta+\varepsilon) = P((n+1)x_{min}\geq\theta+\varepsilon) = P(x_{min}\geq\frac{\theta+\varepsilon}{n+1}) =$

$= 1-\Phi(\frac{\theta+\varepsilon}{n+1}) \underset{\text{подставили } \Phi(x)}{=} 1-(1-(1-\frac{\theta+\varepsilon}{n+1})^n) = (1-\frac{\theta+\varepsilon}{n+1})^n \xrightarrow[n\to\infty]{} e^{-\frac{\theta+\varepsilon}{\theta}} > 0$

$\tilde{\theta}_2'$ не явл сост. по опр

$P(|\tilde{\theta}_2-\theta|\geq\varepsilon) = P(x_{min}\leq\theta-\varepsilon) = \Phi(\theta-\varepsilon) = 1-(1-F(\theta-\varepsilon))^n = 1-(1-\frac{\theta-\varepsilon}{\theta})^n =$

$= 1-(\frac{\varepsilon}{\theta})^n \xrightarrow[n\to\infty]{} 1 \Rightarrow \tilde{\theta}_2$ не явл сост по опр

3) $\Delta\ \tilde{\theta}_3 = x_{max}$ $\qquad x_{max} \sim \underbrace{(F(x))^n}_{\Psi(x)}$

$M\tilde{\theta}_3 = M x_{max}$

$\psi(x) = \Psi'(x) = n(\frac{x}{\theta})^{n-1}\frac{1}{\theta}\{(0,\theta)\}$

$M x_{max} = \int_0^\theta x n (\frac{x}{\theta})^{n-1}\frac{1}{\theta}dx = \frac{n}{\theta^n}\frac{\theta^{n+1}}{n+1} = \frac{n}{n+1}\theta$ смещен.

Пробуем исправить оценку: $\tilde{\theta}_3' = \frac{n+1}{n}\tilde{\theta}_3$

$M\tilde{\theta}_3' = \frac{n+1}{n}M x_{max} = \theta$ несмещ.

$D\tilde{\theta}_3' = (\frac{n+1}{n})^2 D x_{max}$

$M x_{max}^2 = \int_0^\theta x^2 n(\frac{x}{\theta})^{n-1}\frac{1}{\theta}dx = \frac{n}{\theta^n}\frac{\theta^{n+2}}{n+2} = \frac{\theta^2 n}{n+2}$

$D x_{max} = \frac{\theta^2 n}{n+2} - (\frac{n}{n+1}\theta)^2 = \theta^2\frac{n(n+1)^2-n^2(n+2)}{(n+2)(n+1)^2} = \theta^2 n\frac{n^2+2n+1-(n^2+2n)}{(n+2)(n+1)^2} =$

$= \theta^2\frac{n}{(n+2)(n+1)^2}$

$D\tilde{\theta}_3' = \frac{\theta^2}{(n+2)n}\xrightarrow[n\to\infty]{}0$ состоят по достаточному усл.

к $\tilde{\theta}_3$ нельзя применить дост усл т.к. она смещен.

$\Delta\ \tilde{\theta}_3, \tilde{\theta}_3'$ на состоят по опред.

$\forall\theta>0, \forall\varepsilon>0$

$P(|\tilde{\theta}_3-\theta|\geq\varepsilon) = P(\tilde{\theta}_3\leq\theta-\varepsilon) = (\frac{\theta-\varepsilon}{\theta})^n = (1-\frac{\varepsilon}{\theta})^n \xrightarrow[n\to\infty]{} 0 \Rightarrow$ сост по опр

$P(|\tilde{\theta}_3'-\theta|\geq\varepsilon) = P(\frac{n+1}{n}\tilde{\theta}_3\leq\theta-\varepsilon) = P(\tilde{\theta}_3\leq\frac{(\theta-\varepsilon)n}{n+1}) = (\frac{n}{n+1})^n(1-\frac{\varepsilon}{\theta})^n \xrightarrow[n\to\infty]{} 0 \Rightarrow$ сост по опр

④ $\measuredangle$ $\tilde{\theta}_4 = x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i$

$M\tilde{\theta}_4 = Mx_1 + \frac{1}{n+1}\sum_{i=1}^{n} Mx_i = M\xi + \frac{1}{n-1}\cdot\sum_{i=2}^{n} M\xi = \frac{\theta}{2} + \frac{\theta}{2} = \theta$ несмещ

$D\tilde{\theta}_4 = Dx_1 + \frac{1}{(n+1)^2}\sum_{i=2}^{n} Dx_i = \frac{\theta^2}{12} + \frac{1}{n-1}\frac{\theta^2}{12} \xrightarrow{n\to\infty} 0$

$\tilde{\theta}_4 \xrightarrow{P} \theta$ не вып дост усл

$x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i \xrightarrow{P} \xi + \frac{\theta}{2}$ не сост.

(т.к. $x_1 \xrightarrow{P} \xi$, $\frac{1}{n-1}\sum x_i \to M\xi = \theta/2$)

b) Эффективность оценок исправл.

$\tilde{\theta}_3' = \frac{n+1}{n} X_{max}$, $\tilde{\theta}_1 = 2\bar{x}$

$D\tilde{\theta}_3' = \frac{\theta^2}{(n+2)n}$, $D\tilde{\theta}_1 = \frac{\theta^2}{3n}$

$\frac{1}{3n}$ $\quad$ $\frac{1}{n(n+2)}$

$n^2 + 2n$ $\quad$ $3n$

$n(n-1)$ $\quad$ 0

при $n = 0, 1$ равно, при $n \geq 2$ больше

$\Rightarrow \tilde{\theta}_3'$ более эфф, чем $\tilde{\theta}_1$

остальные оценки не явл одновр несмещ и сост.