A new perspective on the Stein–Tomas theorem

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One World Fractals

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Equivalently, for all $f \in L^p(\mu)$ and $q > 2$,

$$\|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mu)} \quad \text{(extension)}.$$

Some history

Let σ^{d-1} be the surface measure on S^{d-1}.

- Stein (1970's): Curvature causes Fourier transforms to decay.

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Why σ^{d-1}? Two reasons:

- $\sigma^{d-1}(B(e,r)) \lesssim r^{d-1}$, for all $e \in S^{d-1}$, $r > 0$.
- $\sup_{\xi \in \mathbb{R}^d} |\widehat{\sigma^{d-1}}(\xi)|^2 |\xi|^{d-1} < \infty$.

General measures

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Theorem (Stein–Tomas Theorem)

Let μ *be a non-zero, finite, compactly supported, Borel measure on* $\mathbb{R}^d$ *with*

$$\mu(B(x,r)) \lesssim r^{\alpha}, \quad \forall x \in \mathbb{R}^d,\ r > 0;$$

$$\sup_{\xi \in \mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{\beta} < \infty.$$

Then, if $q \geqslant 2 + \frac{4(d-\alpha)}{\beta}$*, for all* $f \in L^2(\mu)$*,*

$$\|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}.$$

Fourier and Frostman dimensions

Given μ on $\mathbb{R}^d$, define the Frostman and Fourier dimensions of μ respectively as

$$\dim_{\mathrm{Fr}} \mu = \sup\{\alpha \in \mathbb{R} : \mu(B(x,r)) \lesssim r^{\alpha}, \forall x \in \mathbb{R}^d, \forall r > 0\};$$

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Question

Can more information about the dimensionality of μ give us a better range for the restriction estimate to hold?

The Fourier spectrum

Given μ a finite, compactly supported, Borel measure on $\mathbb{R}^d$ and $\theta \in (0,1]$

$$\dim_{\mathrm{F}}^{\theta} \mu = \sup \left\{ s \in \mathbb{R} : \int_{\mathbb{R}^d} \left| \widehat{\mu}(\xi) \right|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta} - d} \, d\xi < \infty \right\}.$$

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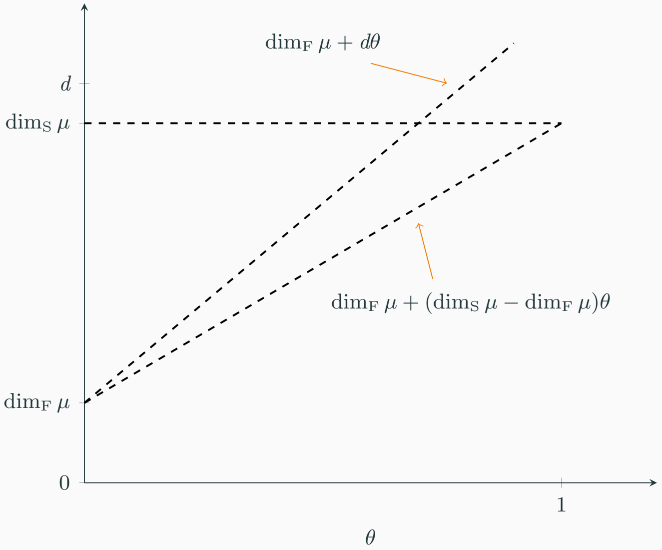
- $\theta \mapsto \dim_{\mathrm{F}}^{\theta} \mu$ is continuous, concave, and non-decreasing.
- $\dim_{\mathrm{F}}^{\theta} \mu \geqslant \dim_{\mathrm{F}} \mu + \theta(\dim_{\mathrm{S}} \mu - \dim_{\mathrm{F}} \mu)$.
- $\dim_{\mathrm{F}}^{\theta} \mu \leqslant \dim_{\mathrm{F}} \mu + d\theta$.



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A Stein–Tomas type result for the Fourier spectrum

Theorem (Carnovale–Fraser–dO, 24++)

Let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$, with $\dim_{\mathrm{Fr}} \mu = \alpha$ for some $0 < \alpha \leqslant d$. If

$$q > 2 + 2 \inf_{\substack{\theta \in [0,1] \\ \dim_{\mathrm{F}}^{\theta} \mu > d\theta}} \frac{(d-\alpha)(2-\theta)}{\dim_{\mathrm{F}}^{\theta} \mu - \alpha\theta},$$

then for every $f \in L^2(\mu)$,

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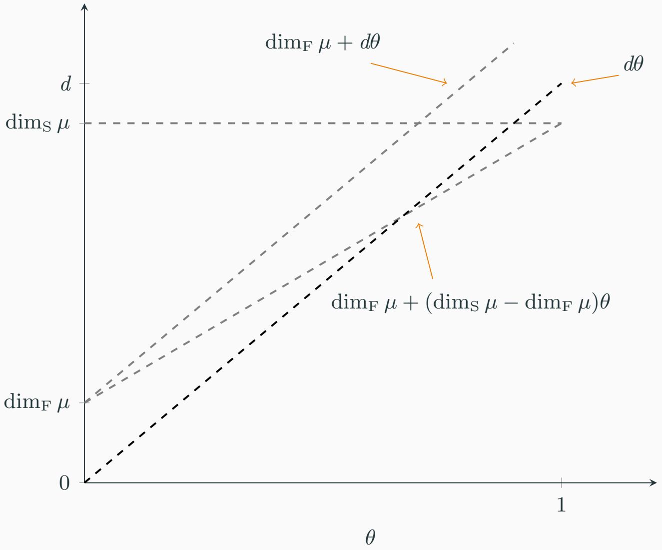

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What is more, restriction will not hold for

$$q < \sup\left\{\frac{2}{\theta} : \dim_{\mathrm{F}}^{\theta} \mu < d\theta\right\}.$$

Restriction for the cone

Let

$$C^{d-1} = \{(t, |t|) : t \in \mathbb{R}^{d-1}, \ |t| \leqslant 1\} \subseteq \mathbb{R}^d,$$

be the cone on $\mathbb{R}^d$ and ν^{d-1} the surface measure on C^{d-1}.

Conjecture (cone restriction conjecure)

The estimate $\|\widehat{f\nu^{d-1}}\|_{L^q(\mathbb{R}^d)} \lesssim_{p,q} \|f\|_{L^p(\nu^{d-1})}$ *holds for all* $f \in L^p(\nu^{d-1})$ *if and only if*

$$\frac{d-2}{p'} \geqslant \frac{d}{q} \text{ and } q > \frac{2(d-1)}{d-2},$$

and for $p = 2$ *(solved) if and only if*

$$q \geqslant \frac{2d}{d-2}.$$

- Cone restriction theorem: $q \geqslant \frac{2d}{d-2}$.
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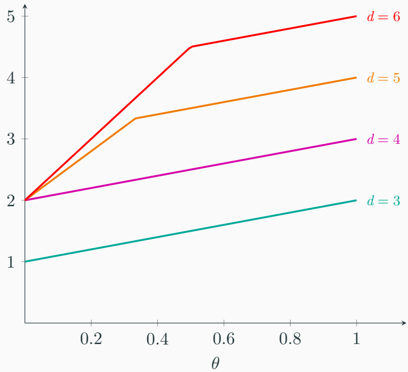

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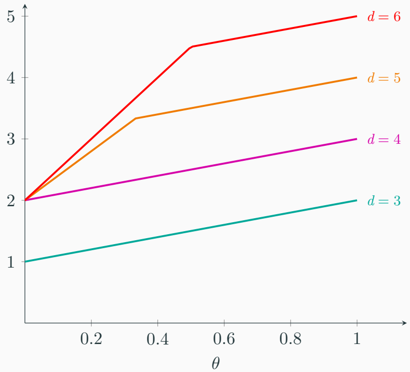

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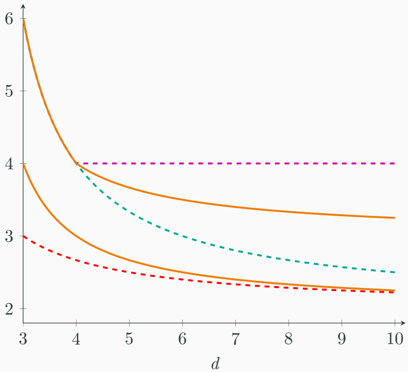

Thank you!