Solutions - Topology Munkres

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These are my solutions to selected exercises of Munkres' Topology. **Be aware that this solutions might contain errors**. If you want to send comments, suggestions, corrections of errors/typos, etc, you can email me at anaemilia.deorellana@gmail.com or make a pull request on Github `https://github.com/AnadeOre/munkres-topology-solutions`

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Section 13: Basis for a Topology

Exercise 1. Let $\mathbb{X}$ be a topological space; let $A \subset \mathbb{X}$. Suppose that for each $x \in A$ there is an open set $U \ni x$ such that $U \subset A$. Show that A is open in $\mathbb{X}$.

Solution: Let $U = \bigcup_{x \in \mathbb{X}} U_x$, U is open, $A \subset U$ and $U \subset A$. Then $A = U$ and A is open. $\qquad \square$

Exercise 2. Consider the 9 topologies on the set $\mathbb{X} = \{a, b, c\}$. Compare them.

Solution:

$$\mathscr{T}_1 = \{\mathbb{X}, \varnothing\}$$
$$\mathscr{T}_2 = \{\{a, b\}, \{a\}, \mathbb{X}, \varnothing\}$$
$$\mathscr{T}_3 = \{\mathbb{X}, \varnothing, \{a, b\}, \{b\}, \{b, c\}\}$$
$$\mathscr{T}_4 = \{\mathbb{X}, \varnothing, \{b\}\}$$
$$\mathscr{T}_5 = \{\mathbb{X}, \varnothing, \{a\}, \{b, c\}\}$$
$$\mathscr{T}_6 = \{\mathbb{X}, \varnothing, \{a, b\}, \{b, c\}, \{b\}, \{c\}\}$$
$$\mathscr{T}_7 = \{\mathbb{X}, \varnothing, \{a, b\}\}$$
$$\mathscr{T}_8 = \{\mathbb{X}, \varnothing, \{a, b\}, \{a\}, \{b\}\}$$
$$\mathscr{T}_9 = \{\mathbb{X}, \varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

PHOTO $\qquad \square$

Exercise 3. Show that the collection $\mathscr{T}_c$ given in the example is a topology on the set $\mathbb{X}$. Is the collection
$$\mathscr{T}_\infty = \{U : \mathbb{X} \backslash U \text{ is infinite or empty or all of } \mathbb{X}\},$$
a topology on $\mathbb{X}$?

Solution:
$$\mathscr{T}_c = \{U \subset \mathbb{X} : U^c \text{ countable or } U^c = \mathbb{X}\}.$$

1. $\varnothing^c = \mathbb{X}$ and $\mathbb{X}^c = \varnothing$, then $\varnothing, \mathbb{X} \in \mathscr{T}_c$.

2. Let $\{U_\alpha\}_{\alpha \in A} \subset \mathscr{T}_c$ and let $\widetilde{A} \subset A$ such that for all $\widetilde{\alpha} \in \widetilde{A}$, $U_{\widetilde{\alpha}}^c = \mathbb{X}$ and for all $\alpha \in A \backslash \widetilde{A}$, U_α^c is countable. Then
$$\left(\bigcup_{\alpha \in A} U_\alpha \right)^c = \left(\bigcup_{\alpha \in A \backslash \widetilde{A}} U_\alpha \right)^c = \bigcap_{\alpha \in A \backslash \widetilde{A}} U_\alpha^c,$$
the latter is countable, therefore $\cup_{\alpha \in A} U_\alpha \in \mathscr{T}_c$.

3. Let $\{U_i\}_{i=1}^N \subset \mathscr{T}_c$, that is, for each i, $U_i = \varnothing$ or U_i^c is countable.
$$\left(\bigcap_{i=1}^N U_i \right)^c = \bigcup_{i=1}^N U_i^c = \begin{cases} \mathbb{X} & \text{if there exists } i \text{ such that } U_i = \varnothing; \\ \text{is countable} & \text{if } \forall\, i,\ U_i \text{ is countable.} \end{cases}$$

$\mathscr{T}_\infty$ is not a topology. In $\mathbb{R}$, $U_1 = (0, \infty)$ and $U_2 = (-\infty, 0]$ are in the topology, but their union is $\mathbb{R} \backslash \{0\}$ which is not infinite. $\qquad \square$

Exercise 4.

(a) If $\{\mathscr{T}_\alpha\}_\alpha$ is a family of topologies on $\mathbb{X}$, show that $\bigcap_\alpha \mathscr{T}_\alpha$ is a topology on $\mathbb{X}$. Is $\bigcup_\alpha \mathscr{T}_\alpha$ a topology on $\mathbb{X}$?

(b) Let $\{\mathcal{T}_\alpha\}_\alpha$ be a family of topologies on $\mathbb{X}$. Show that there is a unique smallest topology on $\mathbb{X}$ containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

Solution:

(a) Proving that $\bigcap_\alpha \mathcal{T}_\alpha$ is a topology is trivial. To see that the union of topologies is not a topology consider $\mathbb{X} = \{a, b, c\}$ and the two following topologies

$$\mathcal{T}_1 = \{\varnothing, \mathbb{X}, \{a\}, \{a, b\}\}$$
$$\mathcal{T}_2 = \{\varnothing, \mathbb{X}, \{c\}, \{b, c\}\}.$$

Then the union is $\mathcal{T} = \{\varnothing, \mathbb{X}, \{a\}, \{c\}, \{b, c\}, \{b, a\}\}$ and is not a topology because $\{b, c\} \cap \{b, a\} = \{b\} \notin \mathcal{T}$.

(b) The smallest topology that contains all the $\mathcal{T}_\alpha$ is $\bigcap \{\mathcal{T} \text{ topology} : \bigcup_\alpha \mathcal{T}_\alpha \subset \mathcal{T}\}$, and the largest topology that is contained in each $\mathcal{T}_\alpha$ is $\bigcup_\alpha \mathcal{T}_\alpha$.

(c) The smallest topology that contains $\mathcal{T}_1$ and $\mathcal{T}_2$ is $\mathcal{T} = \{\varnothing, \mathbb{X}, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and the greatest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$ is $\mathcal{T} = \{\varnothing, \mathbb{X}, \{a\}\}$.

$\square$

Exercise 5. Show that if $\mathscr{A}$ is a basis for a topology on $\mathbb{X}$, then the topology generated by $\mathscr{A}$ equals the intersection of all topologies on $\mathbb{X}$ that contain $\mathscr{A}$. Prove the same if $\mathscr{A}$ is a subbasis.

Solution: Let $(\mathbb{X}, \mathcal{T}')$ be a topological space, $\mathscr{A}$ basis of $\mathcal{T}'$ and define:

$$\mathcal{T}_{\mathscr{A}} = \{U : \forall\, x \in U\, \exists\, A \in \mathscr{A} : x \in A \subset U\}$$
$$\mathcal{T} = \bigcap\{\mathcal{T} : \mathscr{A} \subset \mathcal{T}\}.$$

As $\mathscr{A} \subset \mathcal{T}_{\mathscr{A}}$, $\mathcal{T} \subset \mathcal{T}_{\mathscr{A}}$.

Given $U \in \mathcal{T}_{\mathscr{A}}$, for each $x \in U$ there exists $A_x \in \mathscr{A}$ such that $x \in A_x \subset U$, then $U = \bigcup_{x \in U} A_x \in \mathscr{A}$. Therefore, $U \in \mathcal{T}$ and $\mathcal{T}_{\mathscr{A}} \subset \mathcal{T}$. With this we have $\mathcal{T} = \mathcal{T}_{\mathscr{A}}$.

If $\mathscr{A}$ is a subbasis, define

$$\mathcal{T}_{\mathscr{A}} = \{\text{unions of finite intersections of elements in } \mathscr{A}\}$$
$$\mathcal{T} = \bigcap\{\mathcal{T} : \mathscr{A} \subset \mathcal{T}\}.$$

Given that $\mathscr{A} \subset \mathcal{T}_{\mathscr{A}}$ we have $\mathcal{T} \subset \mathcal{T}_{\mathscr{A}}$ and given $U \in \mathcal{T}_{\mathscr{A}}$, U is union of finite intersections of elements in $\mathscr{A}$. Then $U \in \mathcal{T}$ and $\mathcal{T}_{\mathscr{A}} \subset \mathcal{T}$. $\square$

Exercise 6. Show that the topologies $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

Solution: Let $\mathcal{T}_\ell = \mathcal{T}\big(\{[a, b) : a < b\}\big)$ and $\mathcal{T}_K = \mathcal{T}\big(\{(a, b) : a < b\} \cup \{(a, b)\backslash K\}\big)$, with $K = \{1/n : n \in \mathbb{N}\}$.

Given $x \in \mathbb{R}$, $[x, b) \in \mathcal{T}_\ell$ is a neighbourhood of x, but there does not exist a neighbourhood of x, B in $\mathcal{T}_K$ such that $x \in B \subset [x, b)$.

Consider now the element $B = (0, 1)\backslash K$ in the basis of $\mathcal{T}''$ and take $0 \in B$. There does not exist any interval of the form $[a, b)$ that contains 0 and contained in B. $\square$

Exercise 7. Consider the following topologies on $\mathbb{R}$.

$$
\begin{aligned}
\mathscr{T}_1 &= \text{the standard topology,} \\
\mathscr{T}_2 &= \text{the topology of } \mathbb{R}_K, \\
\mathscr{T}_3 &= \text{the finite complement topology,} \\
\mathscr{T}_4 &= \text{the upper limit topology,} \\
\mathscr{T}_5 &= \text{the topology having left rays as basis.}
\end{aligned}
$$

Determine, for each of these topologies, which of the others it contains.

Solution:

$\mathscr{T}_1 \subset \mathscr{T}_4$ Trivial.

$\mathscr{T}_5 \subset \mathscr{T}_1$ Given $x \in (-\infty, b) \in \mathscr{T}_5$ there exists $(a,b) \in \mathscr{T}_1$ such that $\in (a,b) \subset (-\infty, b)$. But $\not\supset$ because given $x \in (a,b)$ $\not\exists \, (-\infty, c)$ such that $x \in (-\infty, c) \subset (a,b)$.

$\mathscr{T}_1 \subset \mathscr{T}_2$ Trivial.

$\mathscr{T}_2 \subset \mathscr{T}_4$ The problem is at 0. Now I can take $(0, x]$ or $(x, 0]$, because K only takes $1/n$ with $n \in \mathbb{N}$.

$\mathscr{T}_3 \subset \mathscr{T}_1$ If $B \in \mathscr{T}_3$ is such that $\#B^c < \infty$ then given $x \in B$ there exists (a,b) such that $x \in (a,b) \subset B$. But $\not\supset$ because given (a,b), if $B \subset (a,b)$ then $(a,b)^c \subset B^c$ and then $\#(a,b)^c = \infty$.

$\mathscr{T}_5 \not\subset \mathscr{T}_3$ Because given $(-\infty, a)$, if we had $B \subset (-\infty, a)$ then $(-\infty, a)^c \subset B^c$ and $\#(-\infty, a)^c = \infty$.

$\mathscr{T}_3 \not\subset \mathscr{T}_5$ If $B = \mathbb{R}\backslash\{0\}$, given $x \in B$ there does not exist $(-\infty, a) \in \mathscr{T}_5$ such that $x \in (-\infty, a) \subset B$.

$\square$

Exercise 8.

1. Apply Lemma 13.2 to show that the countable collection

$$
\mathscr{B} = \{(a,b) : a < b, a, b \in \mathbb{Q}\},
$$

is a basis that generates the standard topology on $\mathbb{R}$.

2. Show that the collection
$$
\mathscr{C} = \{[a,b) : a < b, a, b \in \mathbb{Q}\},
$$
is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Solution:

1. Let $x \in (a,b) \subset \mathbb{R}$, there exist $p, q \in \mathbb{Q}$ such that $a < p < x < q < b$, then $x \in (p,q) \subset (a,b)$ and then $\mathscr{B}$ is basis of $\mathscr{T}$.

2. Taking $[\pi, 4)$ and $x = \pi$, there doesn't exist $[p,q) \in \mathscr{C}$ such that $x \in [p,q) \subset [\pi, 4)$.

$\square$

Section 16: The Subspace Topology

Exercise 1. Show that if $\mathbb{Y}$ is a subspace of $\mathbb{X}$ and $A \subset \mathbb{Y}$ then the topology A inherits a subspace of $\mathbb{Y}$ is the same as the topology it inherits as a subspace of $\mathbb{X}$.

Solution: Call $\mathscr{T}_{\mathbb{X}}, \mathscr{T}_{\mathbb{Y}}$ the topology on $\mathbb{X}$ and the topology on $\mathbb{Y}$ inherited from $\mathbb{X}$, that is, $\mathscr{T}_{\mathbb{Y}} = \{\mathbb{Y} \cap U : U \in \mathscr{T}_{\mathbb{X}}\}$. Also, define

$$\mathscr{T}_A^{\mathbb{X}} = \{A \cap U : U \in \mathscr{T}_{\mathbb{X}}\}$$
$$\mathscr{T}_A^{\mathbb{Y}} = \{A \cap U : U \in \mathscr{T}_{\mathbb{Y}}\},$$

the topologies on A inherited from $\mathbb{X}$ and $\mathbb{Y}$ respectively.

Let $V \in \mathscr{T}_A^{\mathbb{X}}$, $V = A \cup U$ with $U \in \mathscr{T}_{\mathbb{X}}$, also, $A = A \cap \mathbb{Y}$, then $V = (A \cap \mathbb{Y}) \cap U = A \cap (\mathbb{Y} \cap U)$, given that $\mathbb{Y} \cap U \in \mathscr{T}_{\mathbb{Y}}$ then $V \in \mathscr{T}_A^{\mathbb{Y}}$.

Let $V \in \mathscr{T}_A^{\mathbb{Y}}$, $V = A \cap U$ with $U \in \mathscr{T}_{\mathbb{Y}}$, then $U = \mathbb{Y} \cap O$ with $O \in \mathscr{T}_{\mathbb{X}}$, then

$$V = A \cap U = A \cap (\mathbb{Y} \cap O) = (A \cap \mathbb{Y}) \cap O = A \cap O,$$

then $V \in \mathscr{T}_{\mathbb{X}}$. $\qquad\square$

Exercise 2. If $\mathscr{T}_{\mathbb{X}}$ and $\mathscr{T}_{\mathbb{X}}'$ are topologies on $\mathbb{X}$ and $\mathscr{T}_{\mathbb{X}}'$ is strictly finer that $\mathscr{T}_{\mathbb{X}}$, what can you say about the corresponding subspace topologies on the subset $\mathbb{Y}$ of $\mathbb{X}$.

Solution: Let $V \in \mathscr{T}_{\mathbb{Y}}$ and $y \in V$, $V = \mathbb{Y} \cap U$ with $U \in \mathscr{T}_X$, as $\mathscr{T}_{\mathbb{X}}' \supset \mathscr{T}_{\mathbb{X}}$ given $x \in U$ there exists $U' \in \mathscr{T}_{\mathbb{X}}'$ such that $x \in U' \subset U$. Taking $V' = \mathbb{Y} \cap U'$ we have $y \in V' \subset V$, therefore $\mathscr{T}_{\mathbb{Y}}' \supset \mathscr{T}_{\mathbb{Y}}$. $\qquad\square$

Exercise 3. Consider the set $\mathbb{Y} = [-1, 1]$ as a subspace of $\mathbb{R}$.

$$A = (1/2, 1) \to \text{ Open in } R \text{ and } \mathbb{Y}.$$
$$B = (1/2, 1] \to \text{ It is } (1/2, 2) \cap \mathbb{Y}, \text{ open in } \mathbb{Y}.$$
$$C = [1/2, 1) \to \text{ Not open, it has } 1/2 \text{ as border.}$$
$$D = [1/2, 1] \to \text{ Not open, it has } 1/2 \text{ as border.}$$
$$E = \{x : 0 < |x| < 1, 1/x \notin \mathbb{N}\} \to \text{ Open in } \mathbb{R}.$$

Exercise 4. A map $f : \mathbb{X} \to \mathbb{Y}$ is said to be an open map if for every open set U of $\mathbb{X}$, the set $f(U)$ is open in $\mathbb{Y}$. Show that $\pi_1 : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ and $\pi_2 : \mathbb{X} \times \mathbb{Y} \to \mathbb{Y}$ are open maps.

Solution: It suffices to prove it for the basis elements. Let $A = U \times V$ with $U \in \mathscr{T}_{\mathbb{X}}$, $V \in \mathscr{T}_{\mathbb{Y}}$. $\pi_1(A) = U \in \mathscr{T}_{\mathbb{X}}$ and $\pi_2(A) = V \in \mathscr{T}_{\mathbb{Y}}$.

If $A \in \mathscr{T}_{\mathbb{X} \times \mathbb{Y}}$, $A = \cup_\alpha (U_\alpha \times V_\alpha) = \left(\cup_\alpha U_\alpha\right) \times \left(\cup_\alpha V_\alpha\right)$. Then $\pi_1(A) = \cup_\alpha U_\alpha \in \mathscr{T}_{\mathbb{X}}$ and $\pi_2(A) = \cup_\alpha V_\alpha \in \mathscr{T}_{\mathbb{Y}}$ $\qquad\square$

Exercise 7. Let $\mathbb{X}$ be an ordered set. If $\mathbb{Y}$ is a proper subset of $\mathbb{X}$ that is convex in $\mathbb{X}$, does it follow that $\mathbb{Y}$ is an interval or a ray in $\mathbb{X}$?

Solution: Not necessarily, if $\mathbb{X} = \mathbb{Q}$ and $\mathbb{Y} = (-\pi, \pi) \cap \mathbb{Q}$ there doesn't exist $a, b \in \mathbb{Q}$ such that $(-\pi, \pi) \cap \mathbb{Q} = (a, b)$. $\qquad\square$

Exercise 8. If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. In each case it is a familiar topology.

Solution:

- If L is a horizontal line at height H then $[a, b) \times (c, d) \cap L = \{[x, H) : a \leq x < b\}$, then the topology is the same as $\mathbb{R}_\ell$.

- If L is a vertical line at H then $[a, b) \times (c, d) \cap L = \{(H, x) : a < x < b\}$ then the topology is the same as in $\mathbb{R}$.

- If L is inclined then

$$[a, b) \times (c, d) \cap L = \begin{cases} [a, b) \text{ in } L , & \text{if } a \leq c, b \leq d; \\ [a, d) \text{ in } L , & \text{if } a \leq c, d \leq b; \\ (c, b) \text{ in } L , & \text{if } c < a, b \leq d; \\ (c, d) \text{ in } L , & \text{if } c < a, d \leq b \end{cases}$$

then it is the same topology as $\mathbb{R}_\ell$.

As a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$ it is analogous in the first two cases, but if L is inclined, then the topology is the one generated by closed intervals $[a, b]$. $\quad\square$

Exercise 9. Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

Solution: Let $U \in \mathscr{T}_{\text{dic}}^{\mathbb{R} \times \mathbb{R}}$, $U = (a \times b, c \times d)$,

$$U = \big(\{a\} \times (b, \infty)\big) \cup \big((a, b) \times \mathbb{R}\big) \cup \big(\{b\} \times (-\infty, d)\big)$$

then $\mathscr{T}_{\text{dic}}^{\mathbb{R} \times \mathbb{R}} \subset \mathscr{T}_{\mathbb{R}_d \times \mathbb{R}}$.

Let $U \in \mathscr{T}_{\mathbb{R}_d \times \mathbb{R}}$, $U = \{x\} \times (a, b) \in \mathscr{T}_{\text{dic}}^{\mathbb{R} \times \mathbb{R}}$. Then $\mathscr{T}_{\text{dic}}^{\mathbb{R} \times \mathbb{R}} \subset \mathscr{T}_{\mathbb{R}_d \times \mathbb{R}}$. $\quad\square$

Section 17: Closed Sets and Limit Points

Exercise 1. Let $\mathscr{C}$ be a collection of subsets of the set $\mathbb{X}$. Suppose that $\varnothing$ and $\mathbb{X}$ are in $\mathscr{C}$, and that finite unions and arbitrary intersections of elements of $\mathscr{C}$ are in $\mathscr{C}$. Show that the collection

$$\mathscr{T} = \{\mathbb{X} \backslash C : C \in \mathscr{C}\},$$

is a topology on $\mathbb{X}$.

Solution:

1. $\varnothing, \mathbb{X} \in \mathscr{C}$, then $\mathbb{X} \backslash \mathbb{X} = \varnothing$ and $\mathscr{C} \backslash \varnothing = \mathbb{X}$ are in $\mathscr{T}$.

2. Let $\{U_\alpha\}_{\alpha \in I} \subset \mathscr{T}$, $U_\alpha = \mathbb{X} \backslash C_\alpha$ for each α and for some $C_\alpha \in \mathscr{C}$.

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} (\mathbb{X} \backslash C_\alpha) = \left(\bigcap_{\alpha \in I} C_\alpha\right)^c.$$

Given that $\cap_{\alpha \in I} C_\alpha \in \mathscr{C}$ then $\cup_{\alpha \in I} U_\alpha \in \mathscr{T}$.

3. Let $\{U_i\}_{i=1}^N \subset \mathscr{T}$, $U_i = \mathbb{X} \backslash C_i$ for each i with $C_i \in \mathscr{C}$.

$$\bigcap_{i=1}^N U_i = \bigcap_{i=1}^N (\mathbb{X} \backslash C_i) = \left(\bigcup_{i=1}^N C_i\right)^c.$$

Given that $\cup_{i=1}^N C_i \in \mathscr{C}$, then $\cap_{i=1}^N U_i \in \mathscr{T}$.

$\quad\square$

Exercise 2. Show that if A is closed in $\mathbb{Y}$ and $\mathbb{Y}$ is closed in $\mathbb{X}$, then A is closed in $\mathbb{X}$.

Solution: A is closed in $\mathbb{Y}$, then $A = F \cap \mathbb{Y}$ with F closed in $\mathbb{X}$, $\mathbb{Y}$ closed in $\mathbb{X}$ then $\mathbb{X}\backslash\mathbb{Y} = U$ is open in $\mathbb{X}$. With this

$$\mathbb{X}\backslash A = \mathbb{X}\backslash(F \cap \mathbb{Y}) = (\mathbb{X}\backslash F) \cup (\mathbb{X}\backslash\mathbb{Y}) = V \cup U \in \mathcal{T},$$

and A is closed in $\mathbb{X}$. $\qquad\square$

Exercise 3. Show that if A is closed in $\mathbb{X}$ and B is closed in $\mathbb{Y}$, then $A \times B$ is closed in $\mathbb{X} \times \mathbb{Y}$.

Solution: A and B are closed, then $\mathbb{X}\backslash A = U$ and $\mathbb{Y}\backslash B = V$ are open sets in $\mathbb{X}$ and $\mathbb{Y}$ respectively. Then

$$(\mathbb{X} \times \mathbb{Y})\backslash(A \times B) = U \times V,$$

proving the result. $\qquad\square$

Exercise 4. Show that if U is open in $\mathbb{X}$ and A is closed in $\mathbb{X}$, then $U\backslash A$ is open in $\mathbb{X}$ and $A\backslash U$ is closed in $\mathbb{X}$.

Solution: A is closed in $\mathbb{X}$, then $\mathbb{X}\backslash A = V$ open in $\mathbb{X}$. Then we have

$$U\backslash A = U \cap A^c = U \cap V$$

that is open in $\mathbb{X}$. And

$$\mathbb{X}\backslash(A\backslash U) = \mathbb{X} \cap (A\backslash U)^c = \mathbb{X} \cap (A \cap U^c)^c = \mathbb{X} \cap (A^c \cup U) = \mathbb{X} \cap (V \cup U) = V \cup U,$$

that is open in $\mathbb{X}$. $\qquad\square$

Exercise 5. Let $\mathbb{X}$ be an ordered set in the order topology $\mathcal{T}_<$. Show that $\overline{(a,b)} \subset [a,b]$. Under what conditions does equality hold?

Solution: $[a,b]$ is closed in $\mathcal{T}_<$ because $\mathbb{X}\backslash[a,b] = (-\infty, a) \cup (b, \infty)$ (or closed in one extreme if $\mathbb{X}$ has a maximum or minimum element). Given that $\overline{(a,b)} = \bigcap_{V \supset (a,b) \text{ closed}} V$ and $(a,b) \subset [a,b]$ we have $\overline{(a,b)} \subset [a,b]$.

It will be $\overline{(a,b)} = [a,b]$ if for every $\delta > 0$, $(a - \delta, a + \delta) \cap (a,b) \neq \varnothing$, for this to happen a must have infinite successors. $\qquad\square$

Exercise 6. Let A, B and A_α denote subsets of a space $\mathbb{X}$. Prove the following:

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(c) $\overline{\bigcup_\alpha A_\alpha} \supset \bigcup_\alpha \overline{A_\alpha}$; give an example where equality fails.

Solution:

(a) Let $A \subset B$, then

$$\{V \subset \mathbb{X} \text{ cerrado tal que } A \subset V\} \subset \{V \subset \mathbb{X} \text{ cerrado tal que } B \subset V\},$$

because if V is a closed set, if $B \subset V$ then $A \subset V$. Therefore, $\overline{A} \subset \overline{B}$.

(b) Let's first prove $\subset$, Let F, G be closed sets in $\mathbb{X}$ such that $F \supset A$ and $G \supset B$. Then $A \cup B \subset G \cup F$ (finite union of closed sets is closed) and $\overline{(A \cup B)} \subset \overline{A} \cup \overline{B}$.

To prove $\supset$ let H be a closed set in $\mathbb{X}$ such that $H \supset (A \cup B)$, then $H \supset A$ and $H \supset B$, therefore, $\overline{A} \cup \overline{B} \subset \overline{(A \cup B)}$.

(c) Let F be a closed set in $\mathbb{X}$ such that $F \supset \cup_\alpha A_\alpha$, then $F \supset A_\alpha$ for each α and therefore $\cup_\alpha \overline{A_\alpha} \subset \overline{\bigcup_\alpha A_\alpha}$.

To see that the other inclusion fails consider $A_n = \mathbb{R}\backslash[-1/n, 1/n]$, then $\bigcup_{n\in\mathbb{N}} \overline{A_n} = \mathbb{R}\backslash\{0\}$ and $\overline{\bigcup A_n} = \overline{\mathbb{R}\backslash\{0\}} = \mathbb{R}$.

$\square$

Exercise 7. Criticize the following "proof" that $\overline{\bigcup_\alpha A_\alpha} \subset \bigcup \overline{A_\alpha}$: If $\{A_\alpha\}$ is a collection of sets in $\mathbb{X}$ and if $x \in \overline{\bigcup_\alpha A_\alpha}$, then every neighbourhood U of x intersects $\bigcup_\alpha A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in \bigcup_\alpha \overline{A_\alpha}$.

Solution: The neighbourhood U of x intersects some A_α, but different neighbourhoods could intersect different A_α. $\square$

Exercise 8. Let A, B and A_α denote subspace of a space $\mathbb{X}$. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\supset$ or $\subset$ holds.

(a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

(b) $\overline{\bigcap_\alpha A_\alpha} = \bigcap_\alpha \overline{A_\alpha}$.

(c) $\overline{A\backslash B} = \overline{A}\backslash\overline{B}$.

Solution:

(a) Let F, G be closed sets such that $F \supset A$ and $G \supset B$, then $(A \cap B) \subset (F \cap G)$, as the intersection of closed sets is closed, $F \cap G$ is closed, therefore $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

To see that $\supset$ does not hold take $A = \{1/n : n \in \mathbb{N}\}$ and $B = \{-1/n : n \in \mathbb{N}\}$, then $\overline{A} = \{0\}$ and $\overline{B} = \{0\}$. $A \cap B = \varnothing$, then $\overline{A \cap B} = \varnothing$ but $\overline{A} \cap \overline{B} = \{0\}$.

(b) The inclusion $\supset$ does not hold for the same reason the previous one didn't. To prove $\subset$ let F_α be closed sets in $\mathbb{X}$ such that $A_\alpha \subset F_\alpha$. Then $\cap_\alpha A_\alpha \subset \cap_\alpha F_\alpha$ and the arbitrary intersection of closed sets is a closed set. Therefore, $\overline{\bigcap_\alpha A_\alpha} \subset \bigcap_\alpha \overline{A_\alpha}$.

(c) The inclusion $\subset$ does not hold, to see this let $A = \{1/n : n \in \mathbb{N}\}$, $B = \{0\}$, then $\overline{A} = \{0\} = \overline{B}$. $\overline{A}\backslash\overline{B} = \varnothing$ and $\overline{A\backslash B} = \overline{A} = \{0\}$.

To prove $\supset$ let $x \in \overline{A}\backslash\overline{B}$, then every neighbourhood U of x intersects A, but there exists V neighbourhood of x such that $V \cap B = \varnothing$.

$$(V \cap U) \cap (A\backslash B) \neq \varnothing,$$

therefore $x \in \overline{A\backslash B}$

$\square$

Exercise 10. Show that every order topology is Hausdorff.

Solution: Let $x, y \in \mathbb{X}$, $x \neq y$. Assume without any loss of generality that $x < y$. We have the following cases:

- If there exists z such that $x < z < y$ then $x \in (-\infty, z)$, $y \in (z, \infty)$ and $(-\infty, z) \cap (z, \infty) = \varnothing$.

- If there is no such z then $y = x + 1$ (the immediate successor), $x \in (-\infty, y)$, $y \in (x, \infty)$ and $(-\infty, y) \cap (x, \infty) = \varnothing$.

$\square$

Exercise 11. Show that the product of two Hausdorff spaces is Hausdorff

Solution: We know that the basis of $\mathscr{T}_{\mathbb{X} \times \mathbb{Y}}$ is

$$\{U \times V : U \in \mathscr{T}_\mathbb{X}, V \in \mathscr{T}_\mathbb{Y}\}.$$

Let $a \times b, c \times d \in \mathbb{X} \times \mathbb{Y}$, $a \times b \neq c \times d$.

- If $a \neq c$ and $b \neq d$ then there exist $U_a, U_c \in \mathscr{T}_\mathbb{X}$, $V_b, V_d \in \mathscr{T}_\mathbb{Y}$ neighbourhoods of a, b, c, d such that $U_a \cap U_c = \varnothing$, and $V_b \cap V_d = \varnothing$. Then $a \times b \in U_a \times V_b$, $c \times d \in U_c \times V_d$ and

$$(U_a \times V_b) \cap (U_c \times V_d) = (U_a \cap U_c) \times (V_c \cap V_d) = \varnothing.$$

- If $a = c$, $b \neq d$, let U be a neighbourhood of a and c, V_b, V_d disjoint neighbourhoods of b and d. Then $a \times b \in U \times V_b$, $c \times d \in U \times V_d$ and

$$(U \times V_b) \cap (U \times V_d) = U \times (V_b \cap V_d) = U \times \varnothing = \varnothing.$$

$\square$

Exercise 12. Show that a subspace of a Hausdorff space is Hausdorff.

Solution: Let $(\mathbb{X}, \mathscr{T}_\mathbb{X})$ be a Hausdorff space, $(\mathbb{Y}, \mathscr{T}_\mathbb{Y})$ subspace of $\mathbb{X}$. Let $x, y \in \mathbb{Y}$, $x \neq y$. As $\mathbb{X}$ is Hausdorff, there exist disjoint neighbourhoods $U, V \in \mathscr{T}_\mathbb{X}$ of x and y. Clearly $x \in U \cap \mathbb{Y}$, $y \in V \cap \mathbb{Y}$, $U \cap \mathbb{Y}, V \cap \mathbb{Y} \in \mathscr{T}_\mathbb{Y}$ and

$$(U \cap \mathbb{Y}) \cap (V \cap \mathbb{Y}) = (U \cap V) \cap \mathbb{Y} = \varnothing \cap \mathbb{Y} = \varnothing.$$

$\square$

Exercise 13. Show that $\mathbb{X}$ is Hausdorff if and only if the diagonal $\Delta = \{x \times x : x \in \mathbb{X}\}$ is closed in $\mathbb{X} \times \mathbb{X}$.

Solution:

($\Longrightarrow$) Let's see that $\Delta = \overline{\Delta}$. Obviously $\Delta \subset \overline{\Delta}$, suppose by contradiction that $\overline{\Delta} \not\subset \Delta$. Let $x \times y \in \overline{\Delta}$, $x \neq y$. Given $z \in \mathbb{X}$, consider the two following options:

– If $z \neq x$ there exist (because $\mathbb{X}$ is Hausdorff) disjoint neighbourhoods U, V of x and z.

– If $z \neq y$ then there exist disjoint neighbouthoods $\widetilde{U}, \widetilde{V}$ of y and z.

Therefore, for any $z \times z \in \mathbb{X} \times \mathbb{X}$ there exists a neighbourhood $U \times \widetilde{U}$ of $x \times y$ such that $U \times \widetilde{U} \cap \Delta = \varnothing$. Then $x \times y \notin \overline{\Delta}$, which contradicts our assumption.

($\Longleftarrow$) Δ is closed in $\mathbb{X} \times \mathbb{X}$, then $\Delta = \overline{\Delta}$. Let $x, y \in \mathbb{X}$, $x \neq y$. There must exist U, V neighbourhoods of x and y such that $U \cap V = \varnothing$, because if there existed $z \in U \cap V$ then $U \times V$ would be a neighbourhood of $z \times z$ and then $x \times y \in \overline{\Delta}$, which cannot happen.

$\square$

Exercise 14. In the finite complement topology, $\mathscr{T}_c$ on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?.

Solution: $\mathbb{R}$ with the finite complement topology is not Hausdorff, therefore, convergent sequence do not have a unique limit.

Suppose that $\{1/n\}$ converges to some $x_0 \in \mathbb{R}$. Then for every $U \in \mathscr{T}_c$, $U \ni x_0$, there exists $N \in \mathbb{N}$ such that $1/n \in U$ for every $n \geq N$. Because U is open, then U^c is finite. Suppose $U^c \supset \{1/n : N_0 \leq n < N_1\}$ then for every $n \geq N_1$, $U \supset \{1/n\}$. Therefore, every open set in $\mathscr{T}_c$ contain infinite elements of the sequence, so $\{1/n\}$ converges to every point. $\square$

Exercise 15. Show the T_1 axiom is equivalent to the condition that for each pair of points of $\mathbb{X}$, each has a neighbourhood not containing the other.

Solution:

($\Longrightarrow$) $\{x\}$ is closed for every $x \in \mathbb{X}$, then given $x, y \in \mathbb{X}$, the sets $V = \mathbb{X}\backslash\{x\}$, $U = \mathbb{X}\backslash\{y\}$ are disjoint neighbourhoods of y and x such that $x \notin V$ and $y \notin U$.

($\Longleftarrow$) For every $x \neq y$ there exist U_x, V_y neighbourhoods of x and y such that $x \notin V_y$ and $y \notin U_x$. Given $x_0 \in \mathbb{X}$, define $O = \cup_{y \neq x_0} V_y$, O is open and $\mathbb{X}\backslash x_0 = O$, then $\{x_0\}$ is closed.

$\square$

Exercise 16. Consider the five topologies on $\mathbb{R}$:

$$\begin{aligned}
\mathcal{T}_1 &= \text{ the standard topology,} \\
\mathcal{T}_2 &= \text{ the topology of } \mathbb{R}_K, \\
\mathcal{T}_3 &= \text{ the finite complement topology,} \\
\mathcal{T}_4 &= \text{ the upper limit topology,} \\
\mathcal{T}_5 &= \text{ the topology having left rays as basis.}
\end{aligned}$$

(a) Determine the closure of the set $K = \{1/n : n \in \mathbb{N}\}$ under each of these topologies.

(b) Which of these topologies satisfy the Hasudorff axiom? The T_1 axiom?

Solution:

(a)

- In $\mathcal{T}_1$, $\overline{K} = K \cup \{0\}$.
- In $\mathcal{T}_2$ the open sets are of the form (a, b) and $(a, b)\backslash K$. Given that the open sets (a, b) contain $\{0\}$ then $0 \in \overline{K}$. But if $0 \in (a, b)\backslash K$ then that open set does not have limit points of K. Therefore, in $\mathcal{T}_2$, $\overline{K} = K \cup \{0\}$.
- In $\mathcal{T}_3$ the complements of open sets contain at most finite elements of K. Then every open set contains infinite elements of K, that is, if $U \in \mathcal{T}_3$, $U \cap K \neq \varnothing$ and $(U\backslash\{x_0\}) \cap K \neq \varnothing$ for any $x_0 \in \mathbb{R}$. Then $\overline{K} = \mathbb{R}$.
- In $\mathcal{T}_4$ $K = \overline{K}$ because given $x_0 \in K$,
  - If $x_0 \leq 0$ then there exists $\delta > 0$ such that $(x_0 - \delta, x_0]$ is an neighbourhood of x_0 with $(x_0 - \delta, x_0] \cap K = \varnothing$.
  - If $x_0 > 1$ then given $1 < y < x_0$, $(y, x_0]$ is a neighbourhood of x_0 and $(y, x_0] \cap K = \varnothing$.
- In $\mathcal{T}_5$, $\overline{K} = [0, \infty)$.
  - If $(-\infty, a)$ is a neighbourhood of $x_0 \geq 0$, it contains infinite elements of K.
  - If $x_0 < 0$ then $(-\infty, 0)$ is a neighbourhood of x_0 and $(-\infty, 0) \cap K = \varnothing$.

(b)

- ✓ $\mathcal{T}_1$ is T_1 and Hausdorff.
- ✓ $\mathcal{T}_2$ is T_1 and Hausdorff because it includes $\mathcal{T}_1$.
- ✓ $\mathcal{T}_3$ is T_1 and Hausdorff: singletons are closed because $\mathbb{R}\backslash\{x\}$ is a set with finite complement, and therefore is open, so $\{x\}$ is closed. $\mathbb{X}\backslash\{x\}$, $\mathbb{X}\backslash\{y\}$ are disjoint neighbourhoods of any two points $x \neq y$.

✓ $\mathcal{T}_4$ is T_1 and Hausdorff: if $x \neq y$, suppose $x < y$ then $(x-1, x]$ is a neighbourhood of x, let $\varepsilon = \frac{y-x}{2}$, then $(x+\varepsilon, y]$ is a neighbourhood of y and $(x-1, x] \cap (x+\varepsilon, y] = \varnothing$.

✗ $\mathcal{T}_5$ is not T_1, if $x \neq y$, suppose $x < y$, then every neighbourhood of y contains x.

$\square$

Exercise 17. Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis $\mathscr{C} = \{[a, b) : a < b, a, b \in \mathbb{Q}\}$. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.

Solution:

$\mathbb{R}_\ell$:

A: $0 \in \overline{A}$ because if $[a, b) \ni 0$ then it contains infinite elements of A. $\sqrt{2} \notin \overline{A}$ because $[\sqrt{2}, \sqrt{2}+1)$ is a neighbourhood of $\sqrt{2}$ that does not intersect A.

B: $\sqrt{2} \in \overline{B}$ because if $[a, b) \ni \sqrt{2}$ then it contains infinite elements of B. $3 \notin \overline{B}$ because $[3, 4)$ is a neighbourhood of 3 that does not intersect B.

$\mathbb{R}_\mathscr{C}$

A: $0 \in \overline{A}$ because any set $[a, b) \ni 0$ with $a, b \in Q$ contains infinite elements of A. $\sqrt{2} \in \overline{A}$ because if $[a, b)$ is a neighbourhood of $\sqrt{2}$ then $a \in \mathbb{Q}$, $a < \sqrt{2}$, then the set contains infinite elements of A.

B: $\sqrt{2} \in \overline{B}$ because any neighbourhood $[a, b) \ni \sqrt{2}$ has $b \in Q$, $b > \sqrt{2}$, then it must contain infinite eleents of B. $3 \notin \overline{B}$ because $[3, 4)$ is an environment of 3 that does not intersect B.

$\square$

Exercise 18. Determine the closures of the following subsets of the ordered square:

$$A = \{(1/n) \times 0 : n \in \mathbb{N}\},$$
$$B = \{(1 - 1/n) \times \frac{1}{2} : n \in \mathbb{N}\},$$
$$C = \{x \times 0 : 0 < x < 1\},$$
$$D = \{x \times \frac{1}{2} : 0 < x < 1\},$$
$$E = \{\frac{1}{2} \times y : 0 < y < 2\}.$$

Solution:
$$A = \{(1/n) \times 0 : n \in \mathbb{N}\}.$$

0×1 is the limit point because its neighbourhoods are $(0 \times a, b \times c)$ with $a < 1$, $b > 0$ and $c < 1$ and that interval contains infinite elements of A.

$$B = \{(1 - 1/n) \times \frac{1}{2} : n \in \mathbb{N}\}.$$

Same as before but with 1×0, its neighbourhoods are $(a \times b, 1 \times c)$ with $\frac{1}{2} < a, b < 1$ and $c < \frac{1}{2}$.

$$C = \{x \times 0 : 0 < x < 1\}.$$

$0 \times 0 \notin \overline{C}$ because $(0 \times 0, 0 \times \frac{1}{2}) \cap C = \varnothing$. $1 \times 1 \notin \overline{C}$ because $(1 \times \frac{1}{2}, 1 \times 1] \cap C = \varnothing$. But $\{x \times 1 : 0 \le x < 1\} \subset \overline{C}$ and $1 \times 0 \in \overline{C}$.

$$D = \{x \times \frac{1}{2} : 0 < x < 1\}.$$

$$\overline{D} = \{x \times 1 : 0 \le x < 1\} \cup \{0 \times x : 0 < x \le 1\}.$$

$$E = \{\frac{1}{2} \times y : 0 < y < 2\}.$$

$\frac{1}{2} \times 0 \in \overline{E}$ because every neighbourhood is of the form $(a \times b, \frac{1}{2} \times c)$ with $a < \frac{1}{2}$, $b, c > 0$. $\frac{1}{2} \times 1 \in \overline{E}$ because every neighbourhood is of the form $(\frac{1}{2} \times a, b \times c)$, $b, c > 0$, $a < 1$. $\qquad \square$

Exercise 19. If $A \subset \mathbb{X}$, we define the boundary of A by the equation $\partial A = \overline{A} \cap \overline{(\mathbb{X} \backslash A)}$.

(a) Show that A° and ∂A are disjoint, and $\overline{A} = A^{\circ} \cup \partial A$.

(b) Show that $\partial A = \varnothing \iff A$ is both open and closed.

(c) Show that U is open $\iff \partial U = \overline{U} \backslash U$.

(d) If U is open, is it true that $U = (\overline{U})^{\circ}$? Justify your answer.

Solution:

(a) Let $x \in A^{\circ} \cap \partial A$.

  - Because $x \in A^{\circ}$ then $x \in V_0$ for some $V_0 \in \mathscr{T}$, $V_0 \subset A$.
  - Because $x \in \partial A$ then $x \in F$ for every $F \supset (\mathbb{X} \backslash A)$.

  Absurd, then $A^{\circ} \cap \partial A = \varnothing$.

  If $x \in \overline{A}$ then for every $U \in \mathscr{T}$, $U \ni x$, $U \cap A \ne \varnothing$. If $U \cap A \ne \varnothing$, then $U \not\subset A^c$ so $U \cap A^c \ne \varnothing$ and $x \in \overline{\mathbb{X} \backslash A}$. Also, if $U \cap A \ne \varnothing$ then $U \cap V \ne \varnothing$ with $V \in \mathscr{T}$, $V \in A$.

(b) Let's first prove $\implies$. If $\partial A = \varnothing$ then $\overline{A} = A^{\circ}$, as $A^{\circ} \subset A \subset \overline{A}$ then $\overline{A} = A = A^{\circ}$, then A is open and closed. Now $\impliedby$. If A is open then $A \in \mathscr{T}$, if A is closed then $\mathbb{X} \backslash A \in \mathscr{T}$. Then, given $x \in \mathbb{X}$ there exists U such that $x \in U \subset A$ or $x \in U \subset \mathbb{X} \backslash A$, then $x \notin \overline{A}$ and $x \notin \overline{\mathbb{X} \backslash A}$, so $\partial A = \varnothing$.

(c) Let's first prove $\implies$. If U is open, then

$$\partial U = \overline{U} \cap \overline{(\mathbb{X} \backslash U)} = \overline{U} \cap (\mathbb{X} \backslash U) = \overline{U} \cap \mathbb{X} \cap U^c = \overline{U} \backslash U.$$

For $\impliedby$, if $\partial U = \overline{U} \backslash U$ then

$$U^{\circ} = (U^{\circ} \cup \partial U) \backslash \partial U = \overline{U} \backslash \partial U = \overline{U} \backslash (\overline{U} \backslash U) = U.$$

(d) It is not true. If $U = \mathbb{R} \backslash \{0\}$, U is open, $\overline{U} = \mathbb{R}$ and $U^{\circ} = \mathbb{R}$, but $U \ne (\overline{U})^{\circ}$.

$\qquad \square$

Exercise 20. Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$.

(a) $A = \{x \times y : y = 0\}$

(b) $B = \{x \times y : x > 0, y \ne 0\}$

(c) $C = A \cup B$.

(d) $D = \{x \times y : x \in \mathbb{Q}\}$

(e) $E = \{x \times y : 0 < x^2 - y^2 \le 1\}$

(f) $F = \{x \times y : x \ne 0, y \le 1/x\}$.

Solution:

(a) $A = \overline{A}$, $A^\circ = \varnothing$.

(b) $B = B^\circ$, $\overline{B} = \{x \times y : x \ge 0, y \in \mathbb{R}\}$.

(c) $\overline{A \cup B} = A \cup B \cup \{x \times y : x = 0\}$, $(A \cup B)^\circ = \{x \times y : x > 0\}$.

(d) $\overline{D} = \mathbb{R} \times \mathbb{R}$, $D^\circ = \varnothing$.

(e) $\overline{E} = \{x \times y : 0 \le x^2 - y^2 \le 1\}$, $E^\circ = \{x \times y : 0 < x^2 - y^2 < 1\}$.

(f) $\overline{F} = \{x \times y : x \ne 0, y \le 1/x\} \cup \{x \times y : x = 0\}$, $F^\circ = \{x \times y : x \ne 0, y < 1/x\}$.

$\square$

Section 18: Continuous Functions

Exercise 2. Suppose that $f : \mathbb{X} \to \mathbb{Y}$ is continuous. If x is a limit point of the subset A of $\mathbb{X}$, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Solution: No, let $f : \mathbb{R} \to \mathbb{R}$, $f(x) \equiv 8$. Let $A = [0, 1)$, $x = 1$ is a limit point of A, $f(1) = 8$, $f([0, 1)) = 8$ and 8 is not an isolated point of $\{8\}$. $\square$

Exercise 3. Let $\mathbb{X}$ and $\mathbb{X}'$ denote a single set in two topologies $\mathscr{T}$ and $\mathscr{T}'$, respectively. Let $i : \mathbb{X}' \to \mathbb{X}$ be the identity function.

(a) Show that i is continuous $\iff \mathscr{T} \subset \mathscr{T}'$.

(b) Show that i is a homeomorphism $\iff \mathscr{T}' = \mathscr{T}$.

Solution:

(a)

($\implies$) i is continuous, then for every $U \in \mathscr{T}$, $f^{-1}(U) \in \mathscr{T}'$, but $f^{-1}(U) = U$, then $U \in \mathscr{T}$ implies $U \in \mathscr{T}'$, so $\mathscr{T} \subset \mathscr{T}'$.

($\implies$) If $\mathscr{T} \subset \mathscr{T}'$ then for every $U \in \mathscr{T}$, $U \in \mathscr{T}'$. Given $U \in \mathscr{T}$, $f^{-1}(U) = U \in \mathscr{T}'$, then i is continuous.

(b)

$\square$

Exercise 4. Given $x_0 \in \mathbb{X}$ and $y_0 \in \mathbb{Y}$, show that the maps $f : \mathbb{X} \to \mathbb{X} \times \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{X} \times \mathbb{Y}$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y,$$

are imbeddings.

Solution: To see that f is an imbedding we must see that $f : \mathbb{X} \to f(\mathbb{X})$ is a homeomorphism. $f(\mathbb{X}) = \mathbb{X} \times \{y_0\}$, given $U \times V$ open in $\mathbb{X} \times \{y_0\}$, it must be $V = \{y_0\} \cap \widetilde{V}$, then U is open in $\mathbb{X}$ and $f^{-1}(U \times \{y_0\}) = U \in \mathscr{T}_{\mathbb{X}}$.

Likewise, if $U \in \mathscr{T}_{\mathbb{X}}$, $f(U) = U \times \{x_0\}$, that is open in $\mathbb{X} \times \{y_0\}$. Then f and f^{-1} are continuous, given that f is bijective, it is an imbedding.

The proof for g is analogous. $\qquad\square$

Exercise 5. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

Solution: The function $f(x) = \frac{x-a}{b-a}$ is a homeomorphism between those spaces. $\qquad\square$

Exercise 7.

(a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is "continuous from the right", that is,

$$\lim_{x \to a^+} f(x) = f(a),$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.

(b) Can you conjecture what functions $f : \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_\ell$? As maps from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$?

Solution:

(a) Let $\mathscr{T}_\ell$ and $\mathscr{T}_s$ be the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}$ respectively. Because f is continuous from the right then
$$f([x_0, x_0 + \delta)) \subset \big(f(x_0) - \varepsilon, f(x_0) + \varepsilon\big).$$

Then $[x_0, x_0 + \delta) \subset f^{-1}\big(f(x_0) - \varepsilon, f(x_0) + \varepsilon\big)$. Given that $[x_0, x_0 + \delta)$ is open in $\mathscr{T}_\ell$ then f is continuous.

(b)

$\qquad\square$

Exercise 8. Let $\mathbb{Y}$ be an ordered set in the order topology. Let $f, g : \mathbb{X} \to \mathbb{Y}$ be continuous.

(a) Show that the set $\{x : f(x) \leq g(x)\}$ is closed in $\mathbb{X}$.

(b) Let $h : \mathbb{X} \to \mathbb{Y}$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous. [Hint: Use the pasting lemma.]

Solution:

(a) Let $A = \{x : f(x) \leq g(x)\}$, we will see that $\mathbb{X} \backslash A$ is open. f and g are continuous, then for every $V \in \mathscr{T}_<$, $f^{-1}(V), g^{-1}(V) \in \mathscr{T}_{\mathbb{X}}$

$$f^{-1}\big((g(x), \infty)\big) = \{x \in \mathbb{X} : f(x) \in (g(x), \infty)\} = \{x \in \mathbb{X} : f(x) > g(x)\}.$$

Then $\mathbb{X} \backslash A$ is the preimage of $(g(x), \infty)$, which is open in $\mathscr{T}_<$, because f is continuous, $\mathbb{X} \backslash A$ is open and A closed.

(b) We can write the function h as

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \le g(x) \\ g(x) & \text{if } g(x) \le f(x). \end{cases}$$

Also

$$\mathbb{X} = A \cup B = \{x : f(x) \le g(x)\} \cup \{x : g(x) \le f(x)\},$$

the sets A and B are closed and $f(x) = g(x)$ if $x \in A \cap B$, then h is continuous.

$\square$

Exercise 10. Let $f : A \to B$ and $g : C \to D$ be continuous functions. Let us define a map $f \times g : A \times C \to B \times D$ by the equation

$$(f \times g)(a \times c) = f(a) \times g(c).$$

Show that $f \times g$ is continuous.

Solution: Let $U \times V$ be an open set in $B \times D$,

$$(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V).$$

Given that f and g are continuous then $f^{-1}(U)$ and $g^{-1}(V)$ are open sets in A and C respectively, and $f^{-1}(U) \times f^{-1}(V)$ is open in $A \times C$. $\square$

Exercise 13. Let $A \subset \mathbb{X}$, let $f : A \to \mathbb{Y}$ be continuous, let $\mathbb{Y}$ be Hausdorff. Show that if f may be extended to a continuous function $g : \widetilde{A} \to \mathbb{Y}$, then g is uniquely determined by f.

Solution: Let $x \in A'$, for every neighbourhood U of x, $U \cap (A \backslash \{x\}) \ne \varnothing$. Suppose $g(x) \ne \widetilde{g}(x)$. Because $\mathbb{Y}$ is Hausdorff there exist V and $\widetilde{V}$, neighbourhoods of $g(x)$ and $\widetilde{g}(x)$ such that $V \cap \widetilde{V} = \varnothing$.

$g^{-1}(V)$ and $\widetilde{g}^{-1}(\widetilde{V})$ are neighbourhoods of x in $\mathbb{X}$, then it also is $g^{-1}(V) \cap \widetilde{g}^{-1}(\widetilde{V})$, and as $x \in A'$ then

$$(g^{-1}(V) \cap \widetilde{g}^{-1}(\widetilde{V})) \cap (A \backslash \{x\}) \ne \varnothing.$$

But in A, $g = \widetilde{g}$, then there exists $y \in g^{-1}(V) \cap g^{-1}(\widetilde{V}) \cap (A \backslash \{x\})$ and $g^{-1}(V) \cap \widetilde{g}^{-1}(\widetilde{V}) = g^{-1}(V \cap \widetilde{V})$.

Then $y \in g^{-1}(V \cap \widetilde{V})$, but $V \cap \widetilde{V} = \varnothing$, absurd. The absurd follows from assuming that $\widetilde{g} = g$ in A'. $\square$

Section 19: The Product Topology

Exercise 1. Prove Theorem 19.2: Let $\{(\mathbb{X}_\alpha, \mathscr{T}_\alpha)\}_{\alpha \in J}$ be a sequence of topological spaces such that each $\mathscr{T}_\alpha$ is generated by the basis $\mathscr{B}_\alpha$. The collection B of all sets of the form $\prod_{\alpha \in J} B_\alpha$, where $B_\alpha \in \mathscr{B}_\alpha$ for each α is a basis for the $\mathscr{T}_{\text{box}}$ in $\prod_{\alpha \in J} \mathbb{X}_\alpha$.

Solution: We need to see that the two properties for a basis are satisfied.

1. Let $(x_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} \mathbb{X}_\alpha$. Given that $\mathscr{T}_\alpha$ is generated by $\mathscr{B}_\alpha$ then there exists $B_\alpha \in \mathscr{B}_\alpha$ such that $B_\alpha \ni x_\alpha$. Doing this for each α yields that $(x_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} B_\alpha \in B$.

2. Let $(x_\alpha)_{\alpha \in J} \in \left(\prod_{\alpha \in J} B_\alpha \right) \cap \left(\prod_{\gamma \in J} B_\gamma \right)$. This intersection also belongs to B, so the property is trivial.

$\square$

Exercise 2. Prove Theorem 19.3: Let A_α be a subspace of $\mathbb{X}_\alpha$, for each $\alpha \in J$. Then $\prod_\alpha A_\alpha$ is a subspace of $\prod_\alpha \mathbb{X}_\alpha$ if both products are given the box topology or if both products are given the product topology.

Solution:

Box Topology: Let $\mathscr{T}_{\text{box}}$ be the box topology in $\prod_{\alpha \in J} A_\alpha$, that is, the topology generated by $\prod_{\alpha \in J} U_\alpha$ with $U_\alpha = V_\alpha \cap A_\alpha$, V_α open in $\mathbb{X}_\alpha$.

$$\prod_{\alpha \in J} U_\alpha = \prod_{\alpha \in J}(V_\alpha \cap A_\alpha) = \left(\prod_{\alpha \in J} V_\alpha\right) \cap \left(\prod_{\alpha \in J} A_\alpha\right).$$

Product Topology: Let $\mathscr{T}_{\text{prod}}$ be the product topology in $\prod_{\alpha \in J} A_\alpha$. Let $\prod_{\alpha \in J} U_\alpha \in \mathscr{T}_{\text{prod}}$, that is, there are finite U_α such that $U_{\alpha_j} = V_{\alpha_j} \cap A_{\alpha_j}$ and the rest are A_{α_i}, then

$$\prod_{\alpha \in J} U_\alpha = \prod_{\alpha_j}(V_{\alpha_j} \cap A_{\alpha_j}) \times \prod_{\alpha_i} A_{\alpha_i} = \left(\prod_{\alpha_j} V_{\alpha_j} \times \prod_{\alpha_i} \mathbb{X}_{\alpha_i}\right) \cap \left(\prod_\alpha A_\alpha\right).$$

$\square$

Exercise 3. Prove Theorem 19.4: If each space $\mathbb{X}_\alpha$ is a Hausdorff space, then $\prod_{\alpha \in J} \mathbb{X}_\alpha$ is a Hausdorff space in both the box and product topologies.

Solution: Let $(x_\alpha)_{\alpha \in J}, (y_\alpha)_{\alpha \in J} \in \prod_\alpha \mathbb{X}_\alpha$, suppose without any loss of generality that $x_\alpha \neq y_\alpha$ for every α. Given that for each α, $\mathbb{X}_\alpha$ is a Hausdorff space there exists for every α disjoint sets U_α, V_α such that $U_\alpha \ni x_\alpha$ and $V_\alpha \ni y_\alpha$.
Box Topology: $\prod_{\alpha \in J} U_\alpha$, and $\prod_{\alpha \in J} V_\alpha$ are environments of $(x_\alpha)_\alpha$ and $(y_\alpha)_\alpha$ and

$$\left(\prod_\alpha U_\alpha\right) \cap \left(\prod_\alpha V_\alpha\right) = \prod_{\alpha \in J}(U_\alpha \cap V_\alpha) = \varnothing.$$

Product Topology: $\pi_\alpha^{-1}(U_\alpha) = \prod_{\beta \in J} O_\beta$, with $O_\beta = U_\alpha$ if $\alpha = \beta$ or $\mathbb{X}_\beta$ if $\beta \neq \alpha$. Likewise, $\pi_\alpha^{-1}(V_\alpha) = \prod_{\beta \in J} P_\beta$. $\prod_{\beta \in J} O_\beta$ is a neighbourhood of $(x_\alpha)_\alpha$ and $\prod_{\beta \in J} P_\beta$ is a neighbourhood of $(y_\alpha)_{\alpha \in J}$ and they are disjoint. $\square$

Exercise 4. Show that $(\mathbb{X}_1 \times \cdots \times \mathbb{X}_{n-1}) \times \mathbb{X}_n$ is homeomorphic with $\mathbb{X}_1 \times \cdots \times \mathbb{X}_n$.

Solution: Let $f : (\mathbb{X}_1 \times \cdots \times \mathbb{X}_{n-1}) \times \mathbb{X}_n \to \mathbb{X}_1 \times \cdots \times \mathbb{X}_n$, be the mapping $(x_1, \ldots, x_{n-1}) \times x_n \mapsto x_1 \times \cdots \times x_n$. The function f is clearly a homeomorphism. $\square$

Exercise 5. One of the implications stated in Theorem 19.6 holds for the box topology. Which one?

Solution: In the box topology, if $f : A \to \prod_{\alpha \in J} \mathbb{X}_\alpha$, $f(a) = (f_\alpha(a))$ is continuous, then each f_α is continuous. This is because the projection π_α is continuous. $\square$

Exercise 6. Let $x_1, x_2, \ldots$ be a sequence of the points of the product space $\prod_\alpha \mathbb{X}_\alpha$. Show that this sequence converges to the point x if and only if the sequence $\pi_\alpha(x_1), \pi_\alpha(x_2), \ldots$ converges to $\pi_\alpha(x)$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Solution:

($\Longrightarrow$) $x_n \to x$, then for every $U \in \mathscr{T}_{\text{prod}}$, $U \ni x$, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for every $n \geq N$. We may assume without any loss of generality that U is one of the basis, $U = \prod_\alpha U_\alpha$, then $x_\alpha \in U_\alpha$ for each α, and there exists $N \in \mathbb{N}$ such that $x_{n_\alpha} \in U_\alpha$ for every $n \geq N$ for every α. Then $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for each α.

($\Longleftarrow$) $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for each α. Let $U = \prod_\alpha U_\alpha$ be a neighbourhood of x, then $\{U_\alpha\}$ are only finite $\{U_{\alpha_i}\}_{i=1}^m$, and the rest are all $\mathbb{X}_\alpha$. For each U_α there exists N_α such that $\pi_\alpha(x_n) \in U_\alpha$ for every $n \geq N$. Taking $N = \max_\alpha N_\alpha$, $x_n \in U$ for every $n \geq N$. Notice that N is finite because if $U_\alpha = \mathbb{X}_\alpha$ then $N_\alpha = 1$, then, there are only finite $N_{\alpha_i} > 1$ for $i = 1, \dots, m$.

$\square$

Exercise 7. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero", that is, all sequences $(x_1, x_2, \dots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies?

Solution:

Product Topology: Let $x \in \mathbb{R}^\omega$, U neighbourhood of x, that is $U = \prod_{\alpha \in J} U_\alpha$ with $U_\alpha = \mathbb{R}$ for infinitely many α and $U_\alpha \subset \mathbb{R}$ for finitely many α. Then, only finitely many U_α may not contain $\{0\}$ and any neighbourhood of $x \in \mathbb{R}^\omega$ will have elements of $\mathbb{R}^\infty$. Therefore $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$.
Box Topology: Let $x \in \mathbb{R}^\omega$, U neighbourhood of x. $U = \prod_{\alpha \in J} U_\alpha$, with U_α neighbourhood of x_α. If $x_\alpha \notin \mathbb{R}^\infty$ then there may be infinitely U_α that do not contain 0. Then $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$. $\square$

Section 20: The Metric Topology

Exercise 2. Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

Solution: Define the distance $\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_0^+$ as

$$\rho(x_1 \times x_2, y_1 \times y_2) = \begin{cases} 1 & \text{if } x_1 = y_1; \\ \bar{d}(x_2, y_2) & \text{otherwise.} \end{cases}$$

$\square$

Exercise 3. Let $\mathbb{X}$ be a metric space with metric d.

(a) Show that $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ is continuous.

(b) Let $\mathbb{X}'$ denote a space having the same underlying set as $\mathbb{X}$. Show that if $d : \mathbb{X}' \times \mathbb{X}' \to \mathbb{R}$ is continuous, then the topology of $\mathbb{X}'$ is finer than the topology of $\mathbb{X}$.

Solution:

(a) Let $(a, b) \subset \mathbb{R}$, $d^{-1}((a, b)) = \{x \times y \in \mathbb{X} \times \mathbb{X} : d(x, y) \in (a, b)\}$. Given $x \times y \in d^{-1}(a, b)$ there exists $\varepsilon > 0$ sufficiently small such that

$$\big(d(x, y) - \varepsilon, d(x, y) + \varepsilon\big) \subset (a, b).$$

Let $B = B(x, \varepsilon/2) \times B(y, \varepsilon/2)$. B is open in $\mathbb{X} \times \mathbb{X}$, we want to see that $B \subset d^{-1}((a, b))$. If $z \times w \in B$ then

$$d(z, w) \leq d(z, x) + d(x, w) \leq d(z, x) + d(x, y) + d(y, w) < \frac{\varepsilon}{2} + d(x, y) + \frac{\varepsilon}{2} = d(x, y) + \varepsilon,$$

then $d(z, w) \in (a, b)$ and $B \subset d^{-1}\big((a, b)\big)$.

(b) We will see that given $U \subset \mathscr{T}$ there exists $V \in \mathscr{T}'$ such that $V \subset U$. Let $U = B(x_0, \varepsilon)$ with $x_0 \in \mathbb{X}$, as $d(x_0, x_0) = 0$ there exist U, V open in $\mathbb{X}'$ such that $d(U \times V) \subset (-\varepsilon, \varepsilon)$. Then $V \subset U$.

$\square$

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Exercise 4. Consider the product, uniform and box topologies on $\mathbb{R}^\omega$.

(a) In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$f(t) = (t, 2t, 3t, \dots),$$
$$g(t) = (t, t, t, \dots),$$
$$h(t) = (t, \tfrac{1}{2}t, \tfrac{1}{3}t, \dots).$$

(b) In which topologies do the following sequences converge?

$$
\begin{array}{llll}
w_1 &= (1, 1, 1, 1, \dots), & x_1 &= (1, 1, 1, 1, \dots), \\
w_2 &= (0, 2, 2, 2, \dots), & x_2 &= (0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots), \\
w_3 &= (0, 0, 3, 3, \dots), & x_3 &= (0, 0, \tfrac{1}{3}, \tfrac{1}{3}, \dots), \\
& \cdots & & \cdots \\
y_1 &= (1, 0, 0, 0, \dots), & z_1 &= (1, 1, 0, 0, \dots), \\
y_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0), & z_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \dots), \\
y_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, \dots), & z_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \dots), \\
& \cdots & & \cdots
\end{array}
$$

Solution:

(a) Let's first consider f.

Box: It's not continuous. $U = (-1, 1) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times \cdots \times (-\tfrac{1}{n}, \tfrac{1}{n})$ is open in $\mathscr{T}_{\text{box}}$ but $f^{-1}(U) = \{0\}$ is closed.
Uniform: It's not continuous. $U = B_\rho(0, 1/2) = \prod_{\alpha \in J}(-\tfrac{1}{2}, \tfrac{1}{2})$ is an element in the basis, but $f^{-1}(U) = \{0\}$ that is closed.
Product: It is continuous because each f_α is continuous.

Now let's study the continuity of g.

Box: Not continuous. For the same U as before, $g^{-1}(U) = \{0\}$ which is not open.
Uniform: It is continuous. Consider for $\varepsilon < 1$, $B_{\bar\rho}(x, \varepsilon) = \bigcup_{\delta < \varepsilon} \prod_{\alpha \in J}(x_\alpha - \delta, x_\alpha + \delta)$. Then $g^{-1}(B_{\bar\rho}(x, \varepsilon)) \supset (x_\alpha - \delta, x_\alpha + \delta)$ for some $\alpha \in J$ and $\delta < \varepsilon$.
Product: It is continuous. f_α is continuous for every α.

Finally, let's study h.

Box: It's not continuous. $U = (-1, 1) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times (-\tfrac{1}{4}, \tfrac{1}{4}) \times \cdots$ is open but $h^{-1}(U) = \{0\}$.
Uniform: Consider $B(x, \varepsilon)$ and $y \in B(x, \varepsilon)$, if for every α there exists δ_α such that $|x_\alpha - y_\alpha| < \delta_\alpha < \varepsilon$. Then

$$\left| x_n - \frac{1}{n}s \right| < \left| x_n - \frac{t}{n} \right| + \left| \frac{t}{n} - \frac{s}{n} \right|.$$

Product: It is continuous because each f_α is continuous.

(b) Let's start with w_n.

Box: It doesn't converge. $U = (-1, 1) \times (-1, 1) \times \cdots$ is a neighbourhood of 0, but it does not contain any w_n for $n \geq 1$.
Uniform: It doesn't converge. $B_{\bar\rho}(0, \varepsilon)$, $\bar\rho(0, w_n) = n \geq 1$, then, if $\varepsilon < 1$ the ball does

not contain any w_n.

Product: It converges to 0. Given $U = \prod_{\alpha \in J} U_\alpha$ with $U_\alpha = \mathbb{R}$ except for finite α. If U is a neighbourhood of 0, let $N = \max\{\alpha : U_\alpha \neq \mathbb{R}\} + 1$, then $U \ni w_n$ for every $n \geq N$.

Now let's study x_n.

Box: It doesn't converge, the set $U = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times \cdots$ is a neighbourhood of 0 but doesn't contain any x_n.

Uniform: It converges to 0, $\bar{\rho}(0, x_1) = 1/n$, then $B(0, \varepsilon)$ contains all x_n for $n \geq N$ with $N = \lceil 1/\varepsilon \rceil$.

Product: It converges to 0, the proof is the same as in the previous sequence.

Now let's study the convergence of y_n.

Box: It doesn't converge, the set $U = \prod_n (-\frac{1}{n^2}, \frac{1}{n^2})$ is a neighbourhood of 0 but does not contain any y_n.

Uniform: It converges to 0, $\bar{\rho}(0, y_n) = 1/n$, then $B(0, \varepsilon)$ contains y_n for every $n \geq N$ with $N = \lceil \frac{1}{n} \rceil$.

Product: Same as before.

Finally, let's study z_n.

Box: It converges to 0. Given $U = (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \times \cdots$, it suffices to take $N = \lceil \frac{1}{\max\{\varepsilon_1, \varepsilon_2\}} \rceil$, then U contains z_n for every $n \geq N$.

Uniform: The same as before.

Product: The same as before.

$\square$

Exercise 5. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology?

Solution: If $x \to 0$ then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, $|x_n| < \varepsilon$. Then the sequence that has zeros from one N, $y = (y_1, \ldots, y_N, 0, \ldots) \in B(x, \varepsilon)$ and $y \in \mathbb{R}^\infty$, therefore $x \in \overline{\mathbb{R}^\infty}$.

If $x \notin \mathbb{R}^\infty$ and $x \not\to 0$ for every $N \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $|x_n| \geq \varepsilon$ if $n \geq N$. Then $x \notin \overline{\mathbb{R}^\infty}$. $\square$

Exercise 6. Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $x = (x_1, x_2, \cdots) \in \mathbb{R}^\omega$ and given $0 < \varepsilon < 1$, let
$$U(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon) \times \cdots,$$

(a) Show that $U(x, \varepsilon)$ is not equal to the ε-ball $B_{\bar{\rho}}(x, \varepsilon)$.

(b) Show that $U(x, \varepsilon)$ is not even open in the uniform topology.

(c) Show that
$$B_{\bar{\rho}}(x, \varepsilon) = \bigcup_{\delta < \varepsilon} U(x, \delta).$$

Solution:

(a) Let $y = \big(x_1 + (1-1)\varepsilon, x_2 + (1 - \frac{1}{2}\varepsilon), \ldots, x_n + (1 - \frac{1}{n}\varepsilon), \ldots \big)$, $y \in U(x, \varepsilon)$ but $y \notin B_{\bar{\rho}}(x, \varepsilon)$ because $\bar{\rho}(x, y) = \varepsilon$.

(b) Let $y_n = x_n + (1 - \frac{1}{n})\varepsilon$ (the one we had before), $y = (y_n)_n \in U(x, \varepsilon)$. Let $0 < \delta < 1$, we want to see that $B_{\bar{\rho}}(y, \delta) \not\subset U(x, \varepsilon)$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N}\varepsilon < \delta$ and

$$z = \begin{cases} y_n & \text{if } n < N; \\ x_n + \varepsilon & \text{if } n \geq N. \end{cases}$$

If $n < N$ then $|z_n - y_n| = 0 < \delta$ and if $n \geq N$, $|z_n - y_n| = \frac{1}{n}\varepsilon < \delta$. Then $z \in B(y, \delta)$ but $z \notin U(x, \varepsilon)$.

(c) To prove $\subset$ take $y \in B_{\bar{\rho}}(x, \varepsilon)$, $\bar{\rho}(x, y) = \sup\{|x_\alpha - y_\alpha|\} < \varepsilon$, then there exists $\delta < \varepsilon$ such that $|x_\alpha - y_\alpha| < \delta$ for every $\alpha \in J$. Then $y_\alpha \in (x_\alpha - \delta, x_\alpha + \delta)$. The other inclusion is trivial.

$\square$

Section 21: The Metric Topology (continued)

Exercise 2. Let $\mathbb{X}$ and $\mathbb{Y}$ be metric spaces with metrics $d_{\mathbb{X}}$ and $d_{\mathbb{Y}}$, respectively. Let $f : \mathbb{X} \to \mathbb{Y}$ have the property that for every pair of points x_1, x_2 of $\mathbb{X}$,

$$d_{\mathbb{Y}}\big(f(x_1), f(x_2)\big) = d_{\mathbb{X}}(x_1, x_2).$$

Show that f is an imbedding. It is called an isometric imbedding of $\mathbb{X}$ in $\mathbb{Y}$.

Solution: We must check that f and f^{-1} are continuous. Let $B^{\mathbb{Y}}(f(x), r)$ be an open set in $\mathbb{Y}$.

$$B^{\mathbb{Y}}(f(x), r) = \{f(y) : d_{\mathbb{Y}}\big(f(x), f(y)\big) < r\} = \{f(y) : d_{\mathbb{X}}(x, y) < r\},$$

then $f^{-1}\big(B^{\mathbb{Y}}(f(x), r)\big) = B^{\mathbb{X}}(x, r)$, that is open.

Likewise, $f(B^{\mathbb{X}}(x, r)) = f(B^{\mathbb{Y}}(f(x), r))$, then f^{-1} is continuous. Given that f is bijective, we have that it is an imbedding. $\square$

Exercise 3. Let $\mathbb{X}_n$ be a metric space with the metric d_n for $n \in \mathbb{N}$.

(a) Show that
$$\rho(x, y) = \max\{d_1(x_1, 1_1), \ldots, d_n(x_n, y_n)\},$$
is a metric for the product space $\mathbb{X}_1 \times \cdots \mathbb{X}_n$.

(b) Let $\bar{d}_i = \min\{d_i, 1\}$. Show that

$$D(x, y) = \sup\{\bar{d}_i(x_i, y_i)/i\},$$

is a metric for the product space $\prod_i \mathbb{X}_i$.

Solution: (a) Let $x \in \mathbb{X}_1 \times \cdots \times \mathbb{X}_n$, $x = (x_i)$,

$$\begin{aligned}
B_\rho(x, r) &= \{y \in \mathbb{X}_1 \times \cdots \times \mathbb{X}_n : \rho(x, y) < r\} \\
&= \{y \in \mathbb{X}_1 \times \cdots \times \mathbb{X}_n : \max\{d_1, \ldots\} < r\} \\
&= \{y \in \mathbb{X}_1 \times \cdots \times \mathbb{X}_n : \forall 1 \leq i \leq n : d_i(x_i, y_i) < r\} \\
&= B_{d_1}(x_1, r) \times \cdots \times B_{d_n}(x_n, r).
\end{aligned}$$

Then $B_\rho(x, r)$ is open in the product topology.

Let $\prod_{i=1}^n U_i$ be open in $\mathcal{T}_{\mathrm{prod}}$. Then U_i is open in $\mathbb{X}_i$, that is

$$\prod_{i=1}^n U_i = \prod_{i=1}^n B_{d_i}(x_i, r_i),$$

Let $r = \max_{1 \leq i \leq n} r_i$, then

$$B_\rho(x, r) = \prod_{i=1}^n B_{d_i}(x_i, r_i).$$

(b) Let $\prod_\alpha U_\alpha$ be open in $\mathcal{T}_{\mathrm{prod}}$, that is $U_\alpha = \mathbb{X}_\alpha$ for all but finite indexes $\alpha_1, \ldots, \alpha_m$. For those indexes, $U_{\alpha_i} = B_{d_{\alpha_i}}(x_{\alpha_i}, r_{\alpha_i})$ with $r_i < 1$ for every i. Let $r = \min r_{\alpha_i}$, $r < 1$ and $B_D(x, r) \subset \prod_\alpha U_\alpha$. For the other inclusion let $B_D(x, r)$ for $r < 1$. Let $N \in \mathbb{N}$ be such that $1/N < r$, define $U = \prod_{i=1}^N B_{d_i}(x_i, r) \times \prod_{i=N+1}^\infty \mathbb{X}_i$. Let $y \in U$, $y = (y_1, y_2, \ldots)$

$$\frac{\overline{d}(x_i, y_i)}{i} = \frac{d(x_i, y_i)}{i} < r \quad \text{for each } i = 1, \ldots, N.$$

And

$$\frac{\overline{d}(x_i, y_i)}{i} = \frac{1}{i} < r \quad \text{for each } i = N + 1, \ldots,$$

then $\overline{d}(x_i, y_i)/i < r$ for each i and $y \in B_D(x, r)$ and with this $U \subset B_D(x, r)$. $\qquad\square$

Exercise 4. Show that $\mathbb{R}_\ell$ and the ordered square satisfy the first countability axiom.

Solution: $\mathbb{R}_\ell$: Given $x \in \mathbb{R}_\ell$, $\{[x, x + 1/n)\}$ is a countable base at x because if $[a, b) \ni x$ then $a \leq x$ and as $x \neq b$ there exists N sufficiently big such that $x + 1/N < b$, then $[x, x + 1/N) \subset [a, b)$.

Ordered square: Let $x \times y \in [0, 1] \times [0, 1]$.

- If $0 < y < 1$ let $d = \min\{y, 1 - y\}$ and take $\left\{ \left(x \times (y - d/n), x \times (y + d/n)\right) \right\}$.

- If $y = 1$ $0 \leq x < 1$ take $\left\{ \left(x \times (1 - 1/n), (x + 1/n) \times 1/n\right) \right\}$.

- If $y = 0$ and $0 < x \leq 1$ take $\left\{ \left((x - 1/n) \times (1 - 1/n), x \times 1/n)\right) \right\}$.

- If $x \times y = (0, 0)$ take $\left\{ [0 \times 0, 0 \times 1/n) \right\}$.

- If $x \times y = (1, 1)$ take $\left\{ \left(1 \times (1 - 1/n), 1 \times 1\right] \right\}$.

$\qquad\square$

Exercise 7. Let $\mathbb{X}$ be a set, and let $f_n : \mathbb{X} \to \mathbb{R}$ be a sequence of functions. Let $\overline{\rho}$ be the uniform metric on the space $\mathbb{R}^{\mathbb{X}}$. Show that the sequence $(f_n)_n$ converges uniformly to the function $f : \mathbb{X} \to \mathbb{R}$ if and only if the sequence (f_n) converges to f as elements of the metric space $(\mathbb{R}^{\mathbb{X}}, \overline{\rho})$.

Solution: If $f_n \to f$ with $\overline{\rho}$ then given $r < 1$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $f_n \in B_{\overline{\rho}}(f, r)$ and then for every $n \geq N$, $|f_n(x) - f(x)| < r$ and so $f_n \overset{\mathrm{unif}}{\to} f$.

If $f_n \overset{\mathrm{unif}}{\to} f$, given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $x \in \mathbb{X}$ and $n \geq N$, $|f_n(x) - f(x)| < \varepsilon/2$, then

$$\overline{\rho}(f_n, f) = \min\{\sup |f_n(x) - f(x)|, 1\} \leq \frac{\varepsilon}{2} < \varepsilon,$$

then $f_n \to f$ in $\overline{\rho}$. $\qquad\square$

Exercise 8. Let $\mathbb{X}$ be a topological space and let $\mathbb{Y}$ be a metric space. Let $f_n : \mathbb{X} \to \mathbb{Y}$ be a sequence of continuous functions. Let x_n be a sequence of points of $\mathbb{X}$ converging to x. Show that if the sequence (f_n) converges uniformly to f; then $(f_n(x_n))$ converges to $f(x)$.

Solution: Given that $f_n \overset{\text{unif}}{\to} f$, given $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \varepsilon/2$ for every $n \geq N_1$ for every $x \in \mathbb{X}$. Also, given that f is continuous and $x_n \to x$ we have $f(x_n) \to f(x)$, that is, there exists $N_2 \in \mathbb{N}$ such that $d(f(x_n), f(x)) < \varepsilon/2$ for every $n \geq N_2$.

Let $N = \max\{N_1, N_2\}$, then

$$d(f_n(x_n), f(x)) \leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

Exercise 10. Using the closed set formulation of continuity, show that the following are closed subsets of $\mathbb{R}^2$.

$$A = \{x \times y : xy = 1\},$$
$$S^1 = \{x \times y : x^2 + y^2 = 1\},$$
$$B^2 = \{x \times y : x^2 + y^2 \leq 1\}.$$

Solution: Defining $f(x, y) = xy$, and $g(x, y) = x^2 + y^2$, we have $A = f^{-1}(\{1\})$, $S^1 = g^{-1}(\{1\})$ and $B = g^{-1}([0, 1])$.

$\square$

Section 23: Connected spaces

Exercise 1. Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on $\mathbb{X}$. If $\mathscr{T}' \supset \mathscr{T}$, what does connectedness of $\mathbb{X}$ in one topology imply about connectedness in the other?

Solution: $\mathbb{X}$ connected in $\mathscr{T}'$ implies $\mathbb{X}$ connected in $\mathscr{T}$ because if there do not exist sets A and B that separate $\mathbb{X}$ in $\mathscr{T}'$ then they also don't exist in $\mathscr{T}$.

$\square$

Exercise 3. Let $\{A_\alpha\}$ be a sequence of connected subspaces of $\mathbb{X}$; let A be a connected subspace of $\mathbb{X}$. Show that if $A \cap A_\alpha \neq \varnothing$ for all α then $A \cup \left(\bigcup_\alpha A_\alpha \right)$ is connected.

Solution: Suppose that $A \cup \left(\bigcup_\alpha A_\alpha \right)$ is disconnected, then there exist sets U, V that are both open and closed, such that $A \cup \left(\bigcup_\alpha A_\alpha \right) = U \cup V$. Given that A is connected, it must be $A \subset U$ and because $A \cap A_\alpha \neq \varnothing$ for each α and A_α are connected, then $A_\alpha \in U$ for each α. Therefore, $V = \varnothing$.

$\square$

Exercise 4. Show that if $\mathbb{X}$ is an infinite set, it is connected in the finite complement topology.

Solution: If it was disconnected then there would exist open sets U, V such that $U \cap V = \varnothing$ and $\mathbb{X} = U \cup V$. But because V is open, then $V^c = U$ is finite, and also U is open so $U^c = V$ is finite, which is a contradiction. Then $\mathbb{X}$ is connected.

$\square$

Exercise 6. Let $A \subset \mathbb{X}$. Show that if C is a connected subspace of $\mathbb{X}$ that intersects both A and $\mathbb{X} \backslash A$, then C intersects ∂A.

Solution: Assume $C \cap \partial A = \varnothing$ and let $U = C \cap \overline{A}$, $V = C \cap \overline{(\mathbb{X} \backslash A)}$. U and V are open sets because

$$C \cap \overline{A} = C \cap (A^\circ \cup \partial A) = C \cap A^\circ,$$

and

$$C \cap \overline{(\mathbb{X} \backslash A)} = C \cap \left((\mathbb{X} \backslash A)^\circ \cup \partial(\mathbb{X} \backslash A) \right).$$

Because C is connected and $C = U \cup V$ then it cannot be $U \cap V = \varnothing$, so

$$U \cap V = C \cap \partial A \neq \varnothing U \cap V = C \cap \partial A \neq \varnothing.$$

$\square$

Exercise 7. Is the space $\mathbb{R}_\ell$ connected?

Solution: No, because $\mathbb{R}_\ell = (-\infty, 0) \cup [0, \infty)$. ☐

Exercise 8. Determine whether $\mathbb{R}^\omega$ is connected in the uniform topology.

Solution: It is not connected. Let A be the set of bounded sequences and B the set of unbounded sequences. $\mathbb{R}^\omega = A \cup B$, so we just have to prove that A and B are open sets.

Let $\{a_n\} \in A$, given $x \in B_\rho(a, r)$ we have $|x_n - a_n| < r$ for every n, then $\{x_n\}$ is bounded and A is open. Likewise, B is open. ☐

Section 24: Connected Subspaces of the Real Line

Exercise 1.

(a) Show that no two of the spaces $(0, 1), (0, 1]$, and $[0, 1]$ are homeomorphic. [Hint: what happens if you remove a point from each of these spaces?].

(b) Suppose that there exist imbeddings $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{X}$. Show by means of an example that $\mathbb{X}$ and $\mathbb{Y}$ need not be homeomorphic.

(c) Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

Solution:

(a) If there existed a function $f : [0, 1] \to (0, 1]$ (or any other pair of two), suppose that $f(0) \neq 1$, then $f(0) = a$. Then $(0, a), (a, 1]$ is a separation of $(0, 1]$, which is absurd because $(0, 1]$ is connected.

(b) Let $\mathbb{X} = [0, 1]$, $\mathbb{Y} = (0, 1)$, $f(x) = \frac{x}{4} + \frac{1}{4}$, then $f([0, 1]) = [\frac{1}{4}, \frac{1}{2}]$ and g the identity function. Clearly f and g are imbeddings, but $[0, 1]$ is not homeomorphic to $(0, 1)$.

(c) Suppose there existed a homeomorphism $f : \mathbb{R} \to \mathbb{R}^n$, $\mathbb{R}\backslash\{0\} = (-\infty, 0) \cup (0, \infty)$ is disconnected, but $\mathbb{R}^n\backslash\{f(0)\}$ is connected, which is a contradiction.

☐

Exercise 2. Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show there exists a point $x \in S^1$ such that $f(x) = f(-x)$.

Solution: Let $g(x) = f(x) - f(-x)$, g is continuous and for every $x \in S^1$

$$-g(x) = f(-x) - f(x) = g(-x),$$

then $-g(x) < 0 < g(x)$ and there must exist x_0 such that $g(x_0) = 0$, for such x_0 we have $f(x_0) = f(-x_0)$. ☐

Exercise 5. Consider the following sets in the dictionary order. Which are linear continua (LC)?

(a) $\mathbb{N} \times [0, 1)$.

(b) $[0, 1) \times \mathbb{N}$.

(c) $[0, 1) \times [0, 1]$.

(d) $[0, 1] \times [0, 1)$.

Solution:

(a) It is LC. The second condition for LC is satisfied, to check the first one let $A \subset \mathbb{N} \times [0,1)$, $b = \sup \pi_1(A)$ and $c = \sup \pi_2(A)$. If $c \neq 1$, $b \times c$ is the least upper bound in A, and if $c = 1$ then $b \times c$ cannot be upper bound because $c \notin [0,1)$, then the upper bound is $b + 1 \times c$.

(b) It isn't LC. The least upper bound property is not satisfied because between $x \times 1$ and $x \times 2$ there are no elements from the space.

(c) It is LC. The second property is trivial, to check the first property one has to use the method found in Example 1 in the book.

(d) It is not LC. The set $A = \{0 \times x : x \in [0,1)\}$ does not have the least upper bound property.

□

Exercise 6. Show that if $\mathbb{X}$ is a well-ordered set, then $\mathbb{X} \times [0,1)$ in the dictionary order is a linear continuum.

Solution: The second property is trivial. ... □

Exercise 8.

(a) Is a product of path-connected spaces necessarily path connected?

(b) If $A \subset \mathbb{X}$ and A is path connected, is $\overline{A}$ necessarily path connected?

(c) If $f : \mathbb{X} \to \mathbb{Y}$ is continuous and $\mathbb{X}$ is path connected, is $f(\mathbb{X})$ necessarily path connected?

(d) If $\{A_\alpha\}$ is a collectieon of path-connected subspaces of $\mathbb{X}$ and if $\bigcap_\alpha A_\alpha \neq \varnothing$, is $\bigcup_\alpha A_\alpha$ necessarily path connected?.

Solution:

(a) Let $x_1 \times y_1, x_2 \times y_2 \in \mathbb{X} \times \mathbb{Y}$, we know that there exist functions $f : [0,1] \to \mathbb{X}$ and $g : [0,1] \to \mathbb{Y}$ such that $f(0) = x_1$, $f(1) = x_2$, $g(0) = y_1$ and $g(1) = y_2$. The function $h : [0,1] \to \mathbb{X} \times \mathbb{Y}$ defined as $t \mapsto f(t) \times g(t)$ is continuous and satisfies $h(0) = x_1 \times y_1$ and $h(1) = x_2 \times y_2$.

(b) No, the toplogist sine curve is a counterexample.

(c) Yes, let $y_1, y_2 \in f(\mathbb{X})$, there exist $x_1, x_2 \in \mathbb{X}$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$, Because $\mathbb{X}$ is path connected there exists $g : [0,1] \to \mathbb{X}$ such that $g(0) = x_1$ and $g(1) = x_2$. The composition is $f \circ g : [0,1] \to f(\mathbb{X})$, it is continuous and satisfies

$$f \circ g(0) = f(x_1) = y_1;$$
$$f \circ g(1) = f(x_2) = y_2.$$

(d) Yes, Given $x, y \in \bigcup_\alpha A_\alpha$, let $\widetilde{x} \in \bigcap_\alpha A_\alpha$, there exist functions $f : [0,1] \to \bigcup_\alpha A_\alpha$ and $g : [1,2] \to \bigcup_\alpha A_\alpha$ such that $f(0) = x$, $f(1) = \widetilde{x}$, $g(1) = \widetilde{x}$ and $g(2) = y$. The pasting of those functions is a new function h that connects x and y.

□

Exercise 10. Show that if U is an open, connected subspace of $\mathbb{R}^2$, then U is path connected. [Hint: Show that given $x_0 \in U$, the set of points that can be joined to x_0 by a path in U is both open and closed in U.]

Solution: We will show that $A = \{x \in U : \exists f : [0,1] \to U : f(0) = x_0, f(1) = x\}$ is both open and closed, then it will be $A = U$.

Let $x \in A$, $x \in U$ and because U is open there exists $B \ni x$ such that every $y \in B$ can be connected to x_0 (balls are path connected), then $B \subset A$ and A is open.

To see that A is closed let $x \in U$, $x \notin A$, consider the open ball $B(x,r)$ for some $r > 0$, every $y \in B(x,r)$ can be connected with x but cannot be connected with x_0 because then x could, which would be absurd, therefore, $y \notin A$. $\qquad\square$

Section 25: Components and Local Connectedness

Exercise 1. What are the components and path components of $\mathbb{R}_\ell$? What are the continuous maps $f : \mathbb{R} \to \mathbb{R}_\ell$?

Solution: Let $A \subset \mathbb{R}_\ell$, $a, b \in A$, $a < b$. If $a \sim b$ then there exists a component $C \ni a, b$, but $a \in C \cap (-\infty, b)$ and $b \in C \cap [b, \infty]$, then C is not connected. Such C cannot exist, therefore the components of $\mathbb{R}_\ell$ are singletons $\{x\}$. The path components are the same, given that A path connected implies A connected, so they cannot be more than singletons as well.

Given that $\mathbb{R}$ is connected and the continuous map of a connected set is connected, a function $f : \mathbb{R} \to \mathbb{R}_\ell$ is continuous if and only if it is constant. $\qquad\square$

Exercise 4. Let $\mathbb{X}$ be locally path connected. Show that every connected open set in $\mathbb{X}$ is path connected.

Solution: Let U be an open set in $\mathbb{X}$, $x_0 \in U$ fixed, consider the set

$$A = \{x \in U : \exists f : [0,1] \to U, f(0) = x_0, f(1) = x\}.$$

A is open: Given $x \in A$, we know there exists V path connected such that $x \in V \subset U$. But because V is path connected then it must be $V \subset A$ because of the pasting lemma.

A is closed: Given $x \notin A$, we know there exists V path connected such that $x \in V \subset U$. Then it must be $V \cap A = \varnothing$, otherwise using pasting lemma we'd have $x \in A$.

Therefore, $A \subset U$ is open and closed in U, then $U = A$ and is path connected. $\qquad\square$

Exercise 5. Let $\mathbb{X}$ denote the rational points of the interval $[0,1] \times \{0\}$ of $\mathbb{R}^2$. Let T denote the union of all line segments joining the point $p = 0 \times 1$ to points of $\mathbb{X}$.

(a) Show that T is path connected, but it is locally connected only at the point p.

(b) Find a subset of $\mathbb{R}^2$ that is path connected but is locally connected at none of its points.

Solution:

(a) Given $x, y \in T$, if x and y are in the same line segment then there is nothing we need to do. If they aren't in the same line segment just paste the two segments together with p as intermediate point. Therefore, T is path connected.
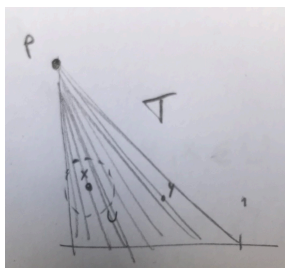
Now let's prove that it is only locally connected at p. The ball in p is path connected, given $x \in T$, $x \neq p$, the ball $B(x, r)$ is formed by disjoint line segments, so we can separate the ball into two parts using a line that is not in T for which its preimage is an irrational number.

(b) Let T' be union of all line segments joining 1×0 to the rational points of $[0, 1] \times \{1\}$, the union $T \cup T'$ is path connected but locally connected at none of its points.

$\square$
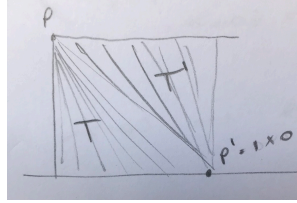
Section 26: Connectedness and Compactness

Exercise 1.

(a) Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on the set $\mathbb{X}$; suppose that $\mathscr{T}' \supset \mathscr{T}$. What does compactness of $\mathbb{X}$ under one of these topologies imply about compactness about the other?

(b) Show that if $\mathbb{X}$ is compact Hausdorff under both $\mathscr{T}$ and $\mathscr{T}'$, then either $\mathscr{T}$ and $\mathscr{T}'$ are equal or they are not comparable.

Solution:

(a) If $\mathbb{X}$ is compact in $\mathscr{T}'$ then it is compact in $\mathscr{T}$ because if $\{A_\alpha\}_\alpha \subset \mathscr{T}$ is a convering of $\mathbb{X}$ then $\{A_\alpha\}_\alpha \subset \mathscr{T}'$ and there exists a finite covering $\{A_{\alpha_i}\}_{i=1}^m$, and $\{A_{\alpha_i}\}_{i=1}^m \subset \mathscr{T}$. The other inclusion does not work, in $\mathbb{R}$, $\mathscr{T}_{\text{finite complement}} \subset \mathscr{T}_{\text{standard}}$. In the finite complement topology (a, b) is compact but not in the standard

(b) Suppose $\mathscr{T} \subset \mathscr{T}'$, the identity function $f : (\mathbb{X}, \mathscr{T}) \to (\mathbb{X}, \mathscr{T}')$ is bijective and continuous and maps compact sets into compact sets. Given $U \subset \mathbb{X}$, $f(U)$ is open if and only if U is open.

$\square$

Exercise 2.

(a) Show that in the finite complement topology on $\mathbb{R}$, every subspace is compact.

(b) If $\mathbb{R}$ has the topology consisting of all sets A such that $\mathbb{R} \backslash A$ is either countable or all of $\mathbb{R}$, is $[0, 1]$ a compact subspace?

Solution: (a) Let $\mathbb{Y} \subset \mathbb{R}$ with the finite complement topology, $\{A_\alpha\}_\alpha$ a covering of $\mathbb{Y}$. Given that each A_α is open, it covers all but finite elements of $\mathbb{Y}$. Therefore, take one A_{α_0} and then finite A_{α_i} for $i = 1, \ldots, m$ such that those cover the ones A_α misses. The family $\{A_{\alpha_i}\}_{i=0}^m$ is a finite covering of $\mathbb{Y}$.

(b) $[0, 1]$ is not compact in the countable complement topology, let $A_N = \{1/n : n \geq N\}$. The sets $B_N = [0, 1] \backslash A_N$, $\mathbb{R} \backslash B_N = A_N$ is countable, therefore $B_N \in \mathscr{T}$. $\bigcup B_N$ covers $[0, 1]$ but there is no finite subcover.

$\square$

Exercise 3. Show that a finite union of compact subspaces of $\mathbb{X}$ is compact.

Solution: Let $\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_N$ be compact subspaces of $\mathbb{X}$ and $\{A_\alpha\}_\alpha$ a covering of $\bigcup_{i=1}^N \mathbb{X}_i$. Then $\{A_\alpha\}$ is a cover of each $\mathbb{X}_i$, therefore there exists for each i a finite subcover $\{A_{\alpha_j}^i\}_{j=1}^{m_i}$, $\bigcup_{i=1}^N \bigcup_{j=1}^{m_i} A_{\alpha_j}^i$ is a finite cover of the union. $\square$

Exercise 4. Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed bounded subspace is compact.

Solution: Metric spaces are Hausdorff, therefore, $\mathbb{X}$ is Hausdorff and $\mathbb{Y} \subset \mathbb{X}$ is compact, then it is closed. Suppose that $\mathbb{Y}$ is not bounded, then for every $x \in \mathbb{X}$ and $r > 0$ there exists $y \in \mathbb{Y}$ such that $y \notin B(x, r)$. The union of all such balls is a covering of $\mathbb{X}$, but there does not exist a finite covering of $\mathbb{Y}$. Absurd, then $\mathbb{Y}$ is bounded.

Consider the space $(\mathbb{R}, \overline{d})$, $\overline{d} \not\sim d$, but $\mathscr{T}_{\overline{d}} = \mathscr{T}_d$, then they have the same compact sets, but with $\overline{d}$, $\mathbb{R}$ is bounded, but it is not compact. $\square$

Exercise 5. Let A and B be disjoint compact subspaces of the Hausdorff space $\mathbb{X}$. Show that there exist disjoint open sets U and V containing A and B, respectively.

Solution: We know (Lemma 26.4) that for each $a \in A$, as B is compact and $A \cap B = \varnothing$, there exist disjoint open sets $U_\alpha \ni a$ and $V_\alpha \supset B$. Clearly $\{U_a\}_{a \in A}$ is a covering of A, and given that A is compact, there exists a finite subcover $\{U_{a_i}\}_{i=1}^m$. Defining $V = \bigcap_{i=1}^m V_{a_i}$ and $U = \bigcup_{i=1}^m U_{a_i}$ we have $U \cap V = \varnothing$, $U \supset A$, $V \supset V$ and U, V open sets. $\square$

Exercise 7. Show that if $\mathbb{Y}$ is compact, then the projection $\pi_1 : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ is a closed map.

Solution: Define $B = \pi_1(A)$. Given $x_0 \notin B$, $x_0 \times \mathbb{Y} \cap A = \varnothing$. $(\mathbb{X} \times \mathbb{Y}) \backslash A$ is open and contains $x_0 \times \mathbb{Y}$, then (tube lemma) it contains $W \times \mathbb{Y}$ with W an open set in $\mathbb{X}$ that contains x_0.

Doing this for each $x_0 \notin B$ we obtain $B = \left(\bigcup_{x \notin B} W_x \right)^c$, therefore B is closed. $\square$

Exercise 8. Theorem. Let $f : \mathbb{X} \to \mathbb{Y}$ be compact Hausdorff. Then f is continuous if and only if the graph of f,
$$G_f = \{x \times f(x) : x \in \mathbb{X}\},$$
is closed in $\mathbb{X} \times \mathbb{Y}$. [Hint: If G_f is closed and V is a neighourhood of $f(x_0)$, then the intersection of G_f and $\mathbb{X} \times (\mathbb{Y} \backslash V)$ is closed. Apply Exercise 7.]

Solution:

($\Longrightarrow$) f is continuous, if $x \times y \in \mathbb{X} \times \mathbb{Y}$ is such that $x \times y \notin G_f$ then $y \neq f(x)$ and because $\mathbb{Y}$ is Hausdorff there exist open sets in $\mathbb{Y}$, U, V such that $U \cap V = \varnothing$, $U \ni f(x)$ and $V \ni y$. Given that f is continuous, $f^{-1}(U)$ is open and $x \in f^{-1}(U)$.

$f^{-1}(U) \times V$ is a neighourhood of $x \times y$ and $(f^{-1}(U) \times V) \cap G_f = \varnothing$, because if it wasn't then there would exist $x_0 \in f^{-1}(U)$ such that $f(x_0) \in V$, but $U \cap V = \varnothing$.

- Let $x_0 \in \mathbb{X}$, V neighbourhood of $f(x_0)$, $\mathbb{X} \times V$ is open, $\mathbb{Y} \backslash V$ is closed and $G_f \cap (\mathbb{X} \times (\mathbb{Y} \backslash V))$ is closed. Using exercise 7, as $\mathbb{Y}$ is compact, $\pi_1 : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ is closed, then $\pi_1(G_f \cap (\mathbb{X} \times (\mathbb{Y} \backslash V)))$ is closed.

$$\pi_1(G_f \cap (\mathbb{X} \times (\mathbb{Y} \backslash V))) = (f^{-1}(V))^c,$$

then $f^{-1}(V)$ is open and f is continuous.

$\square$

Section 27: Compact Subspaces of the Real Line

Exercise 1. Prove that if $\mathbb{X}$ is an ordered set in which every closed interval is compact, then $\mathbb{X}$ has the least upper bound property.

Solution: Let $A \subset \mathbb{X}$ be a set bounded by some $x \in \mathbb{X}$, that is, for every $a \in A$, $a \leq x$. If $x \in A$, then x is the least upper bound and there is nothing to prove. If $x \notin A$ but $x = a + 1$ for some $a \in A$, then a is the least upper bound. If none of these situations occur, define the set B formed by all the upper bounds of A and consider the family $\mathscr{C}$ of intervals $[a, b]$ with $a \in A$ and $b \in B$. $\mathscr{C}$ has the finite intersection property, then there exists $c \in \bigcap_{[a,b] \in \mathscr{C}} [a, b]$, c is the least upper bound. $\square$

Exercise 2. Let $\mathbb{X}$ be a metric space with metric d; let $A \subset \mathbb{X}$ be nonempty.

  (a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

  (b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

  (c) Define the ε-neighbourhood of A in $\mathbb{X}$ to be the set

$$U(A, \varepsilon) = \{x : d(x, A) < \varepsilon\}.$$

     Show that $U(A, \varepsilon)$ equals the union of the open balls $B_d(a, \varepsilon)$ for $a \in A$.

  (d) Assume that A is compact; let U be an open set containing A. Show that some ε-neighbourhood of A is contained in U.

  (e) Show the result in (d) need not hold if A is closed but not compact.

Solution: (a)

    ($\Longleftarrow$) If $x \in \overline{A}$ then for every neighbourhood U of x, $(U \cap A) \backslash \{x\} = \varnothing$. $\{B(x, 1/n)\}$ is a family of neighbourhoods of x and for every $n \in \mathbb{N}$ there exists $y \in (B(x, 1/n) \cap A) \backslash \{x\}$, that is, $d(y, x) < n$, therefore, $d(x, A) = 0$.

    ($\Longrightarrow$) If $d(x, A) = 0$ then for every n there exists $y \in A$ such that $d(x, y) < 1/n$, then $y \in B(x, 1/n)$ and with this $(B(x, 1/n) \cap A) \backslash \{x\} \neq \varnothing$ for every n, therefore, $x \in \overline{A}$.

  (b) Let $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$, be the distance function $d(x, A)$. We know that d is continuous and because A is compact, it reaches its minimum value at some point $a \in A$, then $d(x, a) = d(x, A)$.

  (c) Let's prove the double inclusion.

    ($\supset$) Let $x \in \bigcup_{a \in A} B_d(a, \varepsilon)$, then $x \in B_d(a, \varepsilon)$ for some $a \in A$ and $d(x, a) < \varepsilon$, then $d(x, A) < \varepsilon$ and $x \in U(A, \varepsilon)$.

    ($\subset$) Let $x \in U(A, \varepsilon)$, $d(x, A) < \varepsilon$, this happens if and only if $d(x, a) < \varepsilon$ for some $a \in A$, then $x \in B_d(a, \varepsilon)$.

  (d) Let $\{B(a, d(a, \mathbb{X} \backslash U))\}$ be a covering of A. Because A is compact, there exists a finite sub-covering $\{B(a_i, d(a_i, \mathbb{X} \backslash U))\}$, let $\varepsilon = \min_{1 \leq i \leq N} d(a_i, \mathbb{X} \backslash U)$, then $U(A, \varepsilon) = \bigcup_{a \in A} B(a, \varepsilon) \subset U$.

  (e) In $\mathbb{R} \times \mathbb{R}$ the set $\{x \times \frac{1}{x} : x > 0\}$ is closed but not compact. $\square$

Exercise 3. Recall that $\mathbb{R}_K$ denotes $\mathbb{R}$ in the K-topology.

(a) Show that $[0, 1]$ is not compact as a subspace of $\mathbb{R}_K$.

(b) Show that $\mathbb{R}_K$ is connected. [Hint: $(-\infty, 0)$ and $(0, \infty)$ inherit their usual topologies as subspaces of $\mathbb{R}_K$.]

(c) Show that $\mathbb{R}_K$ is not path connected.

Solution:

(a) $(-1, 2) \backslash K \cup \{(1/n, 1)\}_{n=1}^\infty$ is a cover of $[0, 1]$, but there does not exist a finite subcovering.

(b) Suppose that $\mathbb{R}_K$ is not connected, then there exist open sets U, V such that $\mathbb{R}_K = U \cup V$. The sets $(-\infty, 0)$ and $(0, \infty)$ are connected, so let suppose $(0, \infty) \subset V$ and $(-\infty, 0) \subset U$, if $0 \in U$ then $(-\infty, 0]$ is open and closed at the same time, absurd, then $\mathbb{R}_K$ is connected.

(c) Suppose there exists $f : [a, b] \to [0, 1]$ such that $f(a) = 0$, $f(b) = 1$. Given that the usual topology is contained in the K-topology then f is surjective, then $f([a, b]) = [0, 1]$, then $[0, 1]$ should be compact in $\mathbb{R}_K$, which does not happen. Therefore, $\mathbb{R}_K$ is not path connected.

$\square$

Exercise 4. Show that a connected metric space having more than one point in uncountable.

Solution: Given that the distance function is continuous, it maps connected sets onto connected sets, given $x \in \mathbb{X}$, the function $d_x : \mathbb{X} \to \mathbb{R}$ given by $d_x(y) = d(x, y)$ is continuous. Since $\mathbb{X}$ is connected $d_x(\mathbb{X})$ is connected and contains 0 because $d_x(x) = 0$. Thus, if $y \in \mathbb{X}$ and $y \neq x$, then $d_x(\mathbb{X})$ contains the set $[0, \delta]$, where $\delta = d_x(y) > 0$. Then $\mathbb{X}$ must be uncountable. $\square$

Section 28: Limit Point Compactness

Exercise 1. Give $[0, 1]^\omega$ the uniform topology. Find an infinite subset of this space that has no limit point.

Solution: Each point of the point $x_n = (0, \ldots, 1, 0, \ldots)$ with 1 in the n-th component is isolated in the sequence, because $\overline{\rho}(x_n, x_m) = 1$ for all $n \neq m$, therefore it cannot have a limit point. $\square$

Exercise 2. Show that $[0, 1]$ is not limit point compact as a subspace of $\mathbb{R}_\ell$.

Solution: The set $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ has no limit points in $\mathbb{R}_\ell$, there exists a neighbourhood $[1, 2) \cap [0, 1]$ that does not contain points of A, and if $x \in [0, 1]$, $x \neq 1$ then there exists $N \in \mathbb{N}$ such that

$$\frac{1}{N - 1} \leq x < 1 - \frac{1}{N},$$

is a neighbourhood of x that does not intersect A. $\square$

Exercise 3. Let $\mathbb{X}$ be limit point compact.

(a) If $f : \mathbb{X} \to \mathbb{Y}$ is continuous, does it follow that $f(\mathbb{X})$ is limit point compact?

(b) If A is a closed subset of $\mathbb{X}$, does it follow that A is limit point compact?

Solution:

(a) No, we know that $\mathbb{X} = \mathbb{Z} \times \{a, b\}$ is limit point compact and the projection π_1 is continuous, but $\pi_1(\mathbb{X}) = \mathbb{Z}$ is not limit point compact.

(b) Yes, if $B \subset A$ is an infinite set has limit points in $\mathbb{X}$ then they must be in A because it is closed.

$\square$

Exercise 4. A space $\mathbb{X}$ is said to be countably compact if every countable open covering of $\mathbb{X}$ contains a finite subcollection that covers $\mathbb{X}$. Show that for a T_1 space $\mathbb{X}$, countable compactness is equivalent to limit point compactness. [Hint: If no finite subcollection of U_n covers $\mathbb{X}$, choose $x \notin U_1 \cup \cdots U_n$, for each n.]

Solution:

($\Longrightarrow$) Let $\mathbb{Y} \subset \mathbb{X}$ be an infinite set and $Z \subset \mathbb{Y}$ countable, $Z = \{x_n\}_n$. Suppose that no point in Z is limit point, then for every n_0 there exists a neighbourhood U_{n_0} of x_{n_0} such that $U_{n_0} \cap Z = \{x_{n_0}\}$. Because Z is closed, $\{U_n\} \cup (\mathbb{X} \backslash Z)$ is open and countable, but it does not contain a finite subcovering. Thus, Z cannot be closed and this means that there exists a limit point of Z which is not in Z.

($\Longleftarrow$) Let $\{U_n\}_n$ be a countable covering of $\mathbb{X}$, define $V_n = \bigcup_{i=1}^n U_i$. If there is no finite subcover of $\{U_n\}$ then for each n there exists $x_n \notin V_n$. Let $\{x_n\}$ the sequence formed this way, this sequence has a limit point a, $a \in V_{n_0}$ for some n_0 but V_{n_0} contains only finite points in the sequence. Given that a is a limit point in a T_1 space, every neighbourhood of a has infinite points, therefore, there exists N such that V_N covers $\mathbb{X}$, thus, $\mathbb{X}$ is countably compact.

$\square$

Exercise 5. Show that $\mathbb{X}$ is countably compact if and only if every nested sequence $C_1 \supset C_2 \supset \cdots$ of closed nonempty sets of $\mathbb{X}$ has a nonempty intersection.

Solution:

($\Longrightarrow$) Let $\mathscr{A} = \{\mathbb{X} \backslash C_i : i \in \mathbb{N}\}$, $\mathscr{A}$ covers $\mathbb{X}$ if and only if $\bigcap C_i = \varnothing$. Therefore, $\mathscr{A}$ does not contain any finite subcover, because if it did, then it would be $\bigcap_{i=1}^m C_i = \varnothing$ and that cannot happen because $C_1 \supset C_2 \supset \cdots$. Because $\mathscr{A}$ does not cover $\mathbb{X}$ there exists $x \in \mathbb{X}$ such that $x \notin \mathscr{A}$, then $x \in \bigcap_{i=1}^\infty C_i$.

($\Longleftarrow$) Let $\{A_i\}_{i \in \mathbb{N}}$ be a covering of $\mathbb{X}$. Define the closed sets $B_n = \mathbb{X} \backslash \left(\bigcup_{i \le n} A_i \right)$, which satisfy $B_1 \supset B_2 \supset \cdots$, then $\bigcap B_i \ne \varnothing$,

$$\left(\bigcap B_n \right)^c = \bigcup A_i = \mathbb{X},$$

then there exists $N \in \mathbb{N}$ such that $B_N \ne \varnothing$, thus, $\bigcup_{i=1}^N A_i = \mathbb{X}$

$\square$

Section 30: The Countability Axioms

Exercise 1.

(a) A G_δ set in a space $\mathbb{X}$ is a set A that equals a countable intersection of open sets on $\mathbb{X}$. Show that in a first-countable T_1 space, every one-point set is a G_δ set.

(b) There is a familiar space in which every one-point set is a G_δ set, which nevertheless does not satisfy the first countability axiom. What is it?

Solution:

(a) Let $x \in \mathbb{X}$ and let $\mathscr{B} = \{B_n\}_{n\in\mathbb{N}}$ be a countable base for x, let's see that $x = \bigcap_n B_n$.

Let $y \in \cap_n B_n$, suppose by contradiction that $y \neq x$. Given that $\mathbb{X}$ is T_1 we know that there exist neighbourhoods U, V of x and y respectively such that $y \notin U$ and $x \notin V$. Because $\mathbb{X}$ is 1° countable then there exists n_0 such that $B_{n_0} \subset U$ and $y \in B_{n_0}$, which is a contradiction. Then $x = \bigcap_n B_n$.

(b) $\mathbb{R}^\omega$ with $\mathscr{T}_{\text{box}}$ is not 1° countable, but given $x \in \mathbb{R}^\omega$ let $U_n^i = (x_i - 1/n, x_i + 1/n)$, $\{U_n^i\}_n$ is a local basis of each x_i such that $\bigcap_n U_n^i = \{x_i\}$, therefore, $\bigcap_n \left(\prod_i U_n^i\right) = \{x\}$.

$\square$

Exercise 4. Show that every compact metrizable space $\mathbb{X}$ has a countable basis. [Hint: Let $\mathscr{A}_n$ be a finite covering of $\mathbb{X}$ by $1/n$ balls.]

Solution: Let $\{B(x, 1/n)\}_{x\in\mathbb{X}}$ be a covering of $\mathbb{X}$, because it is compact, there exists a finite subcover $\{B(x_i, 1/n)\}_{i=1}^{m(n)}$. Consider $\mathscr{B} = \bigcup_{n\in\mathbb{N}}\{B(x_i, 1/n)\}_{i=1}^{m(n)}$, $\mathscr{B}$ is a basis:

1. If $x \in \mathbb{X}$ then $x \in B(x_i, 1/n)$ for some i and n, because $\{B(x_i, 1/n)\}$ was a cover of $\mathbb{X}$.

2. If $x \in B(x_i, 1/n_i) \cap B(x_j, 1/n_j)$ then there exist r_i, r_j and y such that $B(y, r_i) \subset B(x_i, 1/n_i)$ and $B(y, r_j) \subset B(x_j, 1/n_j)$. Taking $r = \min(r_1, r_2)$ we have that there exists $B(x_{n_0}, 1/n_0) \subset B(y, r)$ that contains x.

$\square$

Exercise 5.

(a) Show that every metrizable space with a countable dense subset has a countable basis.

(b) Show that every metrizable Lindelöf space has a countable basis.

Solution:

(a) If $D \subset \mathbb{X}$ is a countable dense set then $\mathscr{B} = \bigcup_{n\in\mathbb{N}}\{B(d, 1/n)\}_{d\in D}$, is a countable basis of $\mathbb{X}$.

(b) Given $n \in \mathbb{N}$, $\{B(x, 1/n)\}_{x\in\mathbb{X}}$ is a covering of $\mathbb{X}$, and because $\mathbb{X}$ is Lindelöf, there exists a countable subcovering $\{B(x_i, 1/n)\}_{i\in\mathbb{N}}$. The set $\mathscr{B} = \bigcup_{n\in\mathbb{N}}\{B(x_i, 1/n)\}_{i\in\mathbb{N}}$ is a countable basis of $\mathbb{X}$.

$\square$

Exercise 6. Show that $\mathbb{R}_\ell$ and I_0^2 are not metrizable.

Solution: We know that Lindelöf + Metrizable $\implies$ 2° countable. As $\mathbb{R}_\ell$ is not 2° countable and is Lindelöf, then it cannot be metrizable. Also, Compact + Metrizable $\implies$ Separable, I_0^2 is compact, because us it is a linear continuum, then it also cannot be metrizable, because it is not separable.

$\square$

Exercise 8. Which of our four countability axioms does $\mathbb{R}^\omega$ in the uniform topology satisfy?

Solution: We've seen in the book that $\mathbb{R}^\omega$ with the uniform topology is 1° countable, it is not 2° countable (done in the book), It is also not Lindelöf nor separable because of what we've proven in Exercise 5.

$\square$

Exercise 9. Let A be a closed subspace of $\mathbb{X}$. Show that if $\mathbb{X}$ is Lindelöf, then A is Lindelöf. Show by example that if $\mathbb{X}$ has a countable dense subset, A need not have a countable dense subset.

Solution: Let $\{A_\alpha\}_\alpha$ be an open cover of A. $\{A_\alpha\}_\alpha \cup \{\mathbb{X}\backslash A\}$ is a covering of $\mathbb{X}$, and because it is Lindelöf then there exists a countable subcover $\{A_{\alpha_i}\}_{i\in\mathbb{N}} \cup \{\mathbb{X}\backslash A\}$, the family $\{A_{\alpha_i}\}_{i\in\mathbb{N}}$ is a countable cover of A.

Example: The Sorgenfray plane, $A = L = \{x \times (-x) : x \in \mathbb{R}_\ell\}$ $\qquad\square$

Exercise 11. Let $f : \mathbb{X} \to \mathbb{Y}$ be continuous. Show that if $\mathbb{X}$ is Lindelöf, or if $\mathbb{X}$ has a countable dense subset, then $f(\mathbb{X})$ satisfies the same condition.

Solution:

- $\mathbb{X}$ Lindelöf. Let $\{B_\alpha\}_\alpha$ be a covering of $f(\mathbb{X})$, the set $A_\alpha = f^{-1}(B_\alpha)$ is open in $\mathbb{X}$ and covers it. Then there exists a countable subcover $\{f^{-1}(B_{\alpha_i})\}_{i\in\mathbb{N}}$, the family $\{B_{\alpha_i}\}_{i\in\mathbb{N}}$ is a countable subcover of $f(\mathbb{X})$.

- $\mathbb{X}$ separable. Let $D \subset \mathbb{X}$ be the countable dense set in $\mathbb{X}$, then $\overline{f(D)} = f(\overline{D}) == f(\mathbb{X})$.

$\qquad\square$

Exercise 14. Show that if $\mathbb{X}$ is Lindelöf and $\mathbb{Y}$ is compact, then $\mathbb{X} \times \mathbb{Y}$ is Lindelöf.

Solution: We will take a covering formed by elements of the basis $\{U_\alpha \times V_\alpha\}_\alpha$ of $\mathbb{X} \times \mathbb{Y}$. Given $x_0 \in \mathbb{X}$, $x_0 \times \mathbb{Y}$ is compact and there exists $\{U_{\alpha_i} \times V_{\alpha_i}\}_{i=1}^m$, a finite subcover of $x_0 \times \mathbb{Y}$. Let $U_{x_0} = \bigcap_{i=1}^m U_{\alpha_i}$. The family $\{U_x\}_{x\in\mathbb{X}}$ is a covering of $\mathbb{X}$ and there exists a countable subcover $\{U_{x_n}^i\}_{i\in\mathbb{N}}$. Consider the countable family $\{U_{x_n}^i \times V_{x_n}^i\}_{i\in\mathbb{N}}$. Let $x \times y \in \mathbb{X} \times \mathbb{Y}$, $x \in U_{x_n}$ for some n, then $x \times y \in U_{x_n}^i \times V_{x_n}^i$ for some i, and so the family in consideration is a countable subcover of $\mathbb{X} \times \mathbb{Y}$. $\qquad\square$

Section 31: The Separation Axioms

Exercise 1. Show that if $\mathbb{X}$ is regular, every pair of points of $\mathbb{X}$ have neighbourhoods whose closures are disjoint.

Solution: Let $x, y \in \mathbb{X}$ be distinct points, given that singletons are closed sets, there exist U, V neighbourhoods of x and y such that $U \cap V = \varnothing$. Because $\mathbb{X}$ is regular, there exist U' and V' such that $x \in U' \subset \overline{U'} \subset U$ and $y \in V' \subset \overline{V'} \subset V$, thus $\overline{U'} \cap \overline{V'} = \varnothing$. $\qquad\square$

Exercise 2. Show that if $\mathbb{X}$ is normal, every pair of disjoint closed sets have neighbourhoods whose closures are disjoint.

Solution: Let A, B be closed disjoint subsets of $\mathbb{X}$. There exist open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$. Because $\mathbb{X}$ is normal we know that there exist U', V' such that $A \subset U' \subset \overline{U'} \subset U$ y $B \subset V' \subset \overline{V'} \subset V$, then U' and V' are neighbourhoods of A and B such that $\overline{U'} \cap \overline{V'} = \varnothing$. $\qquad\square$

Exercise 3. Show that every order topology is regular.

Solution:

- $\{x\} = \big((-\infty, x) \cup (x, \infty)\big)^c$.

- Let $x \in \mathbb{X}$, $A \subset \mathbb{X}$ closed sch that $x \notin A = \overline{A}$. Then there exists $(a, b) \ni x$ open set such that $(a, b) \cap A = \varnothing$. The set $V = (-\infty, a) \cup (b, \infty)$ is a neighbourhood of A and $(a, b) \cap V = \varnothing$.

Exercise 4. Let $(\mathbb{X}, \mathscr{T})$ and $(\mathbb{X}, \mathscr{T}')$ be two topological spaces with the same set such that $T \subset \mathscr{T}'$. If one of the spaces is Hausdorff (or regular, or normal), what does that imply about the other?

Solution: If $(\mathbb{X}, \mathscr{T})$ is Hausdorff then $(\mathbb{X}, \mathscr{T}')$ is Hausdorff, given $x, y \in \mathbb{X}$, $x \neq y$ there exist U, V neighbourhoods of x and y such that $U \cap V = \varnothing$ and $U, V \in \mathscr{T}'$, so $(\mathbb{X}, \mathscr{T}')$ is Hausdorff. The other implication is not necessarily valid, in $\mathbb{R}$, $\mathscr{T} = \{\varnothing, \mathbb{R}\}$ is contained in $\mathscr{T}' = \mathcal{P}(\mathbb{R})$, $\mathscr{T}'$ is Hausdorff but $\mathscr{T}$ is not.

For normal and regular sets none of the implications are valid. The standard topology is regular and normal but the K-topology is neither regular nor normal. For the other direction take the K-topology, and $\mathscr{T}' = \mathcal{P}(\mathbb{R})$. $\qquad\square$

Exercise 5. Let $f, g : \mathbb{X} \to \mathbb{Y}$ be continuous; assume that $\mathbb{Y}$ is Hausdorff. Show that $\{x : f(x) = g(x)\}$ is closed in $\mathbb{X}$.

Solution: Let $A = \{x : f(x) = g(x)\}$, if $x \notin A$ then $f(x) \neq g(x)$ and because $\mathbb{Y}$ is Hausdorff there exist $U, V \subset \mathbb{Y}$ neighbourhoods of $f(x)$ and $g(x)$ such that $U \cap V = \varnothing$. The set $B = f^{-1}(U) \cap g^{-1}(V)$, is a neighbourhood of x and $B \subset A^c$, then A^c is open and A is closed. $\qquad\square$

Exercise 6. Let $p : \mathbb{X} \to \mathbb{Y}$ be a closed continuous surjective map. Show that if $\mathbb{X}$ is normal, then so is $\mathbb{Y}$. [Hint: If U is an open set containing $p^{-1}(y)$, show there is a neighbourhood W of y such that $p^{-1}(W) \subset U$.]

Solution: Singletons are closed because for every $x \in \mathbb{X}$, $\{x\}$ is closed and given that p is a closed surjective function, we have that $p(\{x\}) = \{y\}$ is closed.

Let $B \subset \mathbb{Y}$ be a closed set and V a neighbourhood of B. We want to prove that there exists an open set Z such that $B \subset Z \subset \overline{Z} \subset V$. The sets $A = p^{-1}(B)$ and $U = p^{-1}(U)$ are closed and open respectively because p is continuous. Given that $\mathbb{X}$ is normal, there exists an open set W such that $A \subset W \subset \overline{W} \subset U$, then $\mathbb{X}\backslash W$ is closed and using that p is surjective yields $P(W) \subset \mathbb{Y}\backslash p(\mathbb{X}\backslash W)$. Denoting $p(W)$ by Z we have

$$B = p(A) \subset Z \subset \mathbb{Y}\backslash p(\mathbb{X}\backslash W) \subset p(V),$$
$$\overline{Z} = \overline{p(W)} = p(\overline{W}) \subset p(U) = V.$$

Therefore, $\mathbb{Y}$ is normal. $\qquad\square$

Section 32: Normal Spaces

Exercise 1. Show that a closed subspace of a normal space is normal.

Solution: Let $\mathbb{X}$ be a normal space, $\mathbb{Y} \subset \mathbb{X}$ a closed subspace. Let $A, B \subset \mathbb{Y}$ be closed sets, then $A = \widetilde{A} \cap \mathbb{Y}$ and $B = \widetilde{B} \cap \mathbb{Y}$ with $\widetilde{A}, \widetilde{B}$ closed sets in $\mathbb{X}$. Given that $\mathbb{X}$ is normal, there exist disjoint neighbourhoods U, V of $\widetilde{A}$ and $\widetilde{B}$, then $(U \cap \mathbb{Y}) \cap (V \cap \mathbb{Y}) = \varnothing$, $U \cap Y \supset A$ and $V \cap \mathbb{Y} \supset B$. $\qquad\square$

Exercise 4. Show that every regular Lindelöf space is normal.

Solution: It's very similar to the theorem, Regular and 2° countable imply normal, so we just need to know that it has a countable basis, for this consider a cover $a \in B_n \subset U_n$ and use the Lindelöf condition. $\qquad\square$

Exercise 5. Is $\mathbb{R}^\omega$ normal in the product topology? In the uniform topology?

Solution: $\mathbb{R}^\omega$ with both the product and uniform topology are metrizable, then they are normal. $\qquad\square$

Section 33: The Urysohn Lemma

Exercise 1. Examine the proof of the Urysohn lemma, and show that for given r,

$$f^{-1}(r) = \bigcap_{p>r} U_p \setminus \left(\bigcup_{q<r} U_q \right),$$

for p, q rationals.

Solution: We will prove it by double inclusion.

$\subset$: Let $x \in f^{-1}(r)$, then $r = \inf \mathbb{Q}(x) = \inf\{p : x \in U_p\}$, if $t \in \mathbb{Q}$ is such that $r < t$ then there exists $p < t$, $p \in \mathbb{Q}(x)$ such that $r < p < t$, then $x \in U_p$. Doing this for every $t > r$ we have $x \in \bigcap_{p>r} U_p$.

If $q \in \mathbb{Q}$ is such that $r > q$, then $x \notin U_q$ because r is the infimum, then $x \notin \bigcup_{q<r} U_q$.

So $f^{-1}(r) \subset \bigcap_{p>r} U_p \setminus \left(\bigcup_{q<r} U_q \right)$.

$\supset$: If $x \in \bigcap_{p>r} U_p \setminus \left(\bigcup_{q<r} U_q \right)$, then

- $x \in U_p \subset \overline{U}_p$ for every $p > r$, then $f(x) \le r$,
- $x \notin U_q$ for every $q < r$, then $f(x) \ge r$.

Then $f(x) = r$ and $x \in f^{-1}(r)$.

$\square$

Exercise 3. Give a direct proof of the Urysohn lemma for a metric space $(\mathbb{X}, d)$ by setting

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Solution: Given that $(\mathbb{X}, d)$ is a metric space, it is normal. If $x \in A$ $f(x) = 0$ and if $x \in B$, $f(x) = 1$. We only need to see that f is continuous, but this is direct because we know that $d(x, A)$ is continuous. $\square$

Exercise 8. Let $\mathbb{X}$ be completely regular; let A and B be disjoint closed subsets of $\mathbb{X}$. Show that if A is compact, there is a continuous function $f : \mathbb{X} \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Solution: Given that $\mathbb{X}$ is completely regular, for every $a \in A$, as $a \notin B$ there exists $f_a : \mathbb{X} \to [0, 1]$ such that $f_a(a) = 0$ and $f_a(B) = \{1\}$. The family $\{f_a^{-1}([0, 1/2))\}_{a \in A}$ is a cover by open sets of A (because each f_a is continuous) and because A is compact there exists a finite subcover $\{f_{a_i}^{-1}([0, 1/2))\}_{i=1}^m$. Define the functions

$$g(x) = \min_{i=1,\ldots,m} f_{a_i}(x); \quad h(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2) \\ 2x - 1 & \text{if } x \in [1/2, 1]. \end{cases}$$

The function $f = h \circ g$ is continuous and satisfies what we desired. $\square$

Section 34: The Urysohn Metrization Theorem

Exercise 1. Give an example showing that a Hausdorff space with a countable basis need not be metrizable.

Solution: $\mathbb{R}_K$ is Hausdorff and 2° countable but not metrizable, because if it was metrizable then it would be normal, which is not. $\qquad\square$

Exercise 3. Let $\mathbb{X}$ be a compact Hausdorff space. Show that $\mathbb{X}$ is metrizable if and only if $\mathbb{X}$ has a countable basis.

Solution:

($\Longleftarrow$) If $\mathbb{X}$ is compact and Hausdorff, then it is normal and therefore regular. Given that it is 2° countable we have by Urysohn metrization theorem that it is metrizable.

($\Longrightarrow$) Compact metric spaces are 2° countable.

$\qquad\square$

Exercise 6. Check the details of the proof of Theorem 34.2.

Theorem 34.2: Let $\mathbb{X}$ be a space in which one one-point sets are closed. Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of continuous functions, $f_\alpha : \mathbb{X} \to \mathbb{R}$ satisfying the requirement that for each point $x_0 \in \mathbb{X}$, and each neighbourhood U of x_0, there is an index α such that $f_\alpha(x_0) > 0$ and $f_\alpha \equiv 0$ ourside U. Then the function $F : \mathbb{X} \to \mathbb{R}^J$ defined by $F(x) = (f_\alpha(x))_{\alpha \in J}$ is an imbedding of $\mathbb{X}$ in $\mathbb{R}^J$.

Solution: F is continuous because $\mathbb{R}^J$ has the product topology and each f_α is continuous.

F is injective because given $x \neq y$ there exists $\alpha \in J$ such that $f_\alpha(x) > 0$ and $f_\alpha(y) = 0$, then $F(x) \neq F(y)$.

F is an imbedding, clearly it is bijective as a function from $\mathbb{X}$ to $F(\mathbb{X})$, we need to see that for every open set U in $\mathbb{X}$, $F(U)$ is open in $F(\mathbb{X})$. Let $x \in F(U)$ and $x_0 \in U$ be such that $F(x_0) = z_0$. Let α_0 be such that $f_{\alpha_0}(x_0) > 0$ and $f_{\alpha_0} \equiv 0$ outside U. Define V as $\pi_{\alpha_0}^{-1}((0, \infty)) \in \mathbb{R}^J$ and $W = F(\mathbb{X}) \cap V$, clearly W is open in $F(\mathbb{X})$, let's see that $z_0 \in W \subset F(U)$.

$$\pi_{\alpha_0}(z_0) = \pi_{\alpha_0}\big(F(x_0)\big) = f_{\alpha_0}(x_0) > 0,$$

thus $z_0 \in W$. To prove $W \subset F(U)$ let $z \in W$, $z = F(x)$ for some $x \in \mathbb{X}$ and $\pi_{\alpha_0}(z) \in (0, \infty)$. Given that $\pi_{\alpha_0}(z) = \pi_{\alpha_0}\big(F(x)\big) = f_{\alpha_0}(x)$ and $f_{\alpha_0} \equiv 0$ outside U, then $x \in U$ and $z = F(x) \in F(U)$. Therefore, $W \subset F(U)$.

Thus F is an imbedding from $\mathbb{X}$ onto $\mathbb{R}^J$. $\qquad\square$

Exercise 9. Let $\mathbb{X}$ be a compact Hausdorff space that is the union of the closed subspaces $\mathbb{X}_1$ and $\mathbb{X}_2$. If $\mathbb{X}_1$ and $\mathbb{X}_2$ are metrizable, show that $\mathbb{X}$ is metrizable. [Hint: Construct a countable collection $\mathscr{A}$ of open sets of $\mathbb{X}$ whose intersections with $\mathbb{X}_i$ form a basis for $\mathbb{X}_i$, for $i = 1, 2$. Assume $\mathbb{X}_1 \backslash X_2$ and $\mathbb{X}_2 \backslash \mathbb{X}_1$ belong to $\mathscr{A}$. Let $\mathscr{B}$ consist of finite intersections of elements of $\mathscr{A}$.]

Solution: $\mathbb{X}$ is compact and Hausdorff, therefore it is regular, to prove that it is metrizable we need only to show that it is 2° countable.

$\mathbb{X}_1$ and $\mathbb{X}_2$ are closed, then because $\mathbb{X}$ is compact, they are compact, and because they are also metric spaces, they are 2° countable. Let $\mathscr{B}_i$ be the countable basis for each $\mathbb{X}_i$, $i = 1, 2$ and let

$$\mathscr{A} = \{U \in \mathscr{T} : U \cap \mathbb{X}_i \in \mathscr{B}_i\},$$

Suppose $\mathbb{X}_1 \backslash X_2$ and $\mathbb{X}_2 \backslash \mathbb{X}_1$ belong to $\mathscr{A}$. $\mathscr{A}$ is countable because $\mathscr{B}_1$ and $\mathscr{B}_2$ are countable. Let

$$\mathscr{B} = \{\bigcap_{i \in J} U_i : J \subset \mathbb{N} \text{ finite } U_i \in \mathscr{A}\},$$

let's see that $\mathscr{B}$ is a basis for $\mathbb{X}$.

- Given $x \in \mathbb{X}$, assume $x \in \mathbb{X}_1$, then given that $\mathscr{B}_1$ is a basis for $\mathbb{X}_1$, there exists $B \in \mathscr{B}_1$ such that $x \in B$. But $B = U \cap \mathbb{X}_1$ for some open set $U \subset \mathbb{X}$, therefore, $x \in U \in \mathscr{B}$.

- If $x \in U_1 \cap U_2$, with $U_1, U_2 \in \mathscr{B}$, then $U_1 = \bigcap_{i \in J_1} B_i$, $U_2 = \bigcap_{i \in J_2} B_i$, $J_1, J_2 \subset \mathbb{N}$ finite. Let $U_1 \cap U_2 = \bigcap_{i \in J_1 \cup J_2} B_i \in \mathscr{B}$.

Then $\mathscr{B}$ is a basis for $\mathbb{X}$, and clearly it defines the original topology. $\square$

Section 37: The Tychonoff Theorem

Exercise 3. Consider the three statements:

(I) If $\mathbb{X}$ is a set and $\mathscr{A}$ is a collection of subsets of $\mathbb{X}$ having the countable intersection property, then there is a collection $\mathscr{D}$ of subsets of $\mathbb{X}$ such that $\mathscr{D} \supset \mathscr{A}$ and $\mathscr{D}$ is maximal with respect to the countable intersection property.

(II) Suppose $\mathscr{D}$ is maximal with respect to the countable intersection property (CIP). Then countable intersections of elements of $\mathscr{D}$ are in $\mathscr{D}$. Furthermore, if A is a subset of $\mathbb{X}$ that intersects every element of $\mathscr{D}$, then A is an element of $\mathscr{D}$.

(III) Products of Lindelöf spaces are Lindelöf.

(a) Show that (I) and (II) together imply (III).

(b) Show that (II) holds.

(c) Products of Lindelöf spaces need not be Lindelöf. Therefore, (I) does not hold. If one attempts to generalize the proof of Lemma 37.1 to the countable intersection property, at what point does the proof break down?

Solution:

(a) Let $\mathscr{A} = \{A_\alpha\}_\alpha$ be a family of sets in $\prod_\alpha \mathbb{X}_\alpha$ that satisfies the CIP. Let $\mathscr{D} = \{D_\beta\}$ be the corresponding maximal family, $\overline{\pi_{\beta_0}(D_\beta)}$ are closed and satisfy CIP. Given that D_{β_0} is Lindelöf then $\bigcap_\beta \overline{(\pi_{\beta_0}(D_\beta))} \neq \varnothing$. Let $x_{\beta_0} \in \bigcap_\beta \overline{(\pi_{\beta_0}(D_\beta))}$, we do this for each β_0 and obtain $x = \prod_\beta x_\beta$, we want to see that $x \in \overline{D}$ for every $D \in \mathscr{D}$.

Let U_β be a neighbourhood of x_β in $\mathbb{X}_\beta$, $x_\beta \in \prod_\beta(D)$ for every D, then $U_\beta \cap \prod_\beta(D)$ for every D, then because of (II), $\pi_\beta^{-1}(U_\beta) \in \mathscr{D}$ and the same happens for countable intersections, $\bigcap_{n \in \mathbb{N}} \pi_{\beta_n}^{-1}(U_{\beta_n}) \in \mathscr{D}$. Then $\mathscr{D}$ has every element in the basis that contain x. Let U be an open set in $\prod_\alpha \mathbb{X}_\alpha$, such that $x \in U$ and $U \cap D \neq \varnothing$ for every $D \in \mathscr{D}$. Then $x \in \overline{D}$ for every $D \in \mathscr{D}$, as we wanted to prove. We need only to prove that a space is Lindelöf if an only if every collection $\{C_\alpha\}_\alpha = \mathscr{C}$ that satisfies CIP has nonempty intersection, but this is proven like in Theorem 26.9.

(b) The existance of the maximum element in the sequence is what fails in the countable space.

$\square$