$$\delta_{xx} u_{j,n} = \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{h^2}$$

$$\delta_{x^+} \delta_{x^-} u_{j,n} = \delta_{x^+} \frac{u_{j,n} - u_{j-1,n}}{h}$$

$$= \frac{u_{j+1,n} - u_{j,n} - u_{j,n} + u_{j-1,n}}{h^2}$$

$$= \delta_{xx} u_{j,n}$$

$$\delta_{x^-} \delta_{x^+} u_{j,n} = \delta_{x^-} \frac{u_{j+1,n} - u_{j,n}}{h}$$

$$= \frac{u_{j+1,n} - u_{j,n} - u_{j,n} + u_{j-1,n}}{h^2}$$

$$= \delta_{xx} u_{j,n}$$

$$\langle \delta_{tt} u_{\cdot,n}, \delta_t u_{\cdot,n} \rangle =$$

$$= \sum_{j=1}^{m-1} \left(\frac{u_{j,n+1} - 2u_{j,n} + u_{j,n-1}}{k^2} \right) \left(\frac{u_{j,n+1} - u_{j,n-1}}{2k} \right)$$

$$\left[\frac{1}{2} \partial_t^+ \langle \partial_t^- u_{\cdot,2}, \partial_t^- u_{\cdot,2} \rangle = \right.$$

$$= \frac{1}{2} \partial_t^+ \sum_{j=1}^{m-1} \frac{(u_{j,2} - u_{j,2-1})^2}{k^2}$$

$$= \sum_{j=1}^{m-1} \frac{(u_{j,2+1} - u_{j,2})^2 - (u_{j,2} - u_{j,2-1})^2}{2k^3}$$

$$= \sum_{j=1}^{m-1} \frac{u_{j,2+1}^2 - 2\left[u_{j,2+1} u_{j,2} - 2 u_{j,2} u_{j,2-1}\right] - u_{j,2-1}^2}{2k^3}$$

$$\left. = \langle \partial_{tt} u_{\cdot,2}, \partial_t u_{\cdot,2} \rangle \right]$$

$$\langle \partial_{xx} u_{\cdot,2}, \partial_t u_{\cdot,2} \rangle \doteq \langle \partial_x^- \partial_x^+ u_{\cdot,2}, \partial_t u_{\cdot,2} \rangle =$$

$$= - \langle \partial_x^+ u_{\cdot,2}, \partial_x^+ \partial_t u_{\cdot,2} \rangle$$

$$- (\partial_t u_{1,2})(\partial_x^+ u_{0,2}) + \underbrace{(\partial_t u_{m,2})}_{=0}(\partial_x^+ u_{m-1,2})$$

$$= - \langle \partial_x^+ u_{\cdot,2}, \partial_x^+ \partial_t u_{\cdot,2} \rangle - (\partial_t u_{1,2})(\partial_x^- u_{1,2})$$

$$\langle \delta_{x^+} u_{\cdot,n}, \delta_{x^+} \delta_t u_{\cdot,n} \rangle =$$

$$= \sum_{j=1}^{m-1} \left(\frac{u_{j+1,n} - u_{j,n}}{h} \right) \cdot \delta_{x^+} \left(\frac{u_{j,n+1} - u_{j,n-1}}{2k} \right)$$

$$= \sum_{j=1}^{m-2} \left(\frac{u_{j+1,n} - u_{j,n}}{h} \right) \left(\frac{u_{j+1,n+1} - u_{j+1,n-1} - u_{j,n+1} + u_{j,n-1}}{2kh} \right)$$

$$\frac{1}{2} \delta_{t^+} \langle \delta_{x^+} u_{\cdot,n}, e_{t^-} \delta_{x^+} u_{\cdot,n} \rangle =$$

$$= \frac{1}{2} \delta_{t^+} \sum_{j=1}^{m-1} \left(\frac{u_{j+1,n} - u_{j,n}}{h} \right) \left(\frac{u_{j+1,n-1} - u_{j,n-1}}{h} \right)$$

$$= \sum_{j=1}^{m-1} \frac{1}{2k} \left[\left(\frac{u_{j+1,n+1} - u_{j,n+1}}{h} \right) \left(\frac{u_{j+1,n} - u_{j,n}}{h} \right) - \left(\frac{u_{j+1,n} - u_{j,n}}{h} \right) \left(\frac{u_{j+1,n-1} - u_{j,n-1}}{h} \right) \right]$$

$$= \sum_{j=1}^{m-1} \left(\frac{u_{j+1,n} - u_{j,n}}{h} \right) \left(\frac{u_{j+1,n+1} - u_{j+1,n-1} - u_{j,n+1} + u_{j,n-1}}{2kh} \right)$$

$$= \langle \delta_{x^+} u_{\cdot,n}, \delta_{x^+} \delta_t u_{\cdot,n} \rangle$$

Si sumamos de 0 a $m-1$:

$$< \delta_{xx} u_{\cdot,2} , \delta_t u_{\cdot,2} > \doteq < \delta_{x^-} \delta_{x^+} u_{\cdot,2} , \delta_t u_{\cdot,2} > =$$

$$= - < \delta_{x^+} u_{\cdot,2} , \delta_{x^+} \delta_t u_{\cdot,2} >$$

$$- (\delta_t u_{0,2})(\delta_{x^+} u_{-1,2}) + \underbrace{(\delta_t u_{m,2})}_{=0} (\delta_{x^+} u_{m-1,2})$$

$$= - < \delta_{x^+} u_{\cdot,2} , \delta_{x^+} \delta_t u_{\cdot,2} > - \underbrace{(\delta_t u_{0,2})}_{0} (\delta_{x^-} u_{0,2})$$

Aquí $u_{-1,2}$ es tal que se verifica el esquema:

$$\cancel{\delta_{tt} u_{0,2}}^{0} = c^2 \delta_{xx} u_{0,2}$$

$$\Rightarrow 0 = c^2 \frac{u_{1,2} - \cancel{2 u_{0,2}}^{0} + u_{-1,2}}{h^2}$$

$$\Rightarrow u_{-1,2} = u_{1,2}$$

En este caso: $\delta_{x^-} u_{0,2} = -\delta_{x^+} u_{0,2}$

Todo lo anterior se verifica sumando de 0 a $m-1$

En resumen, sumando de 0 a $n-1$

se verifica, $\forall r = 1, \ldots, n-1$:

$$\delta_t^+ \left[\frac{1}{2} \langle \delta_{t^-} u_{\cdot,r}, \delta_{t^-} u_{\cdot,r} \rangle + \frac{\varepsilon}{2} \langle \delta_{x^+}^2 u_{\cdot,r}, e_{t^-} \delta_{x^+} u_{\cdot,r} \rangle \right] = 0$$

donde, para llegar a esta expresión

tomamos $u_{-1,r} := u_{1,r}$.