NONLINEAR
PHYSICAL
SCIENCE

Abdul-Majid Wazwaz

Partial Differential Equations and Solitary Waves Theory

高等教育出版社
HIGHER EDUCATION PRESS

Springer

NONLINEAR PHYSICAL SCIENCE

NONLINEAR PHYSICAL SCIENCE

Nonlinear Physical Science focuses on recent advances of fundamental theories and principles, analytical and symbolic approaches, as well as computational techniques in nonlinear physical science and nonlinear mathematics with engineering applications.

Topics of interest in *Nonlinear Physical Science* include but are not limited to:

- New findings and discoveries in nonlinear physics and mathematics
- Nonlinearity, complexity and mathematical structures in nonlinear physics
- Nonlinear phenomena and observations in nature and engineering
- Computational methods and theories in complex systems
- Lie group analysis, new theories and principles in mathematical modeling
- Stability, bifurcation, chaos and fractals in physical science and engineering
- Nonlinear chemical and biological physics
- Discontinuity, synchronization and natural complexity in the physical sciences

SERIES EDITORS

INTERNATIONAL ADVISORY BOARD

Abdul-Majid Wazwaz

Partial Differential Equations and Solitary Waves Theory

With 14 figures

高等教育出版社
HIGHER EDUCATION PRESS

Springer

Author
Abdul-Majid Wazwaz
Department of Mathematics
Saint Xavier University
Chicago, IL 60655,USA
E-mail: Wazwaz@sxu.edu

ISSN 1867-8440 e-ISSN 1867-8459
Nonlinear Physical Science

ISBN 978-7-04-025480-8
Higher Education Press, Beijing

ISBN 978-3-642-00250-2 e-ISBN 978-3-642-00251-9
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2009920462

Cover design: Frido Steinen-Broo, EStudio Calamar, Spain

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

This book is dedicated to my wife, our son, and our three daughters for supporting me in all my endeavors

Preface

Partial Differential Equations and Solitary Waves Theory is designed to serve as a text and a reference. The book is designed to be accessible to advanced undergraduate and beginning graduate students as well as research monograph to researchers in applied mathematics, science and engineering. This text is different from other texts in that it explains classical methods in a non abstract way and it introduces and explains how the newly developed methods provide more concise methods to provide efficient results.

Partial Differential Equations and Solitary Waves Theory is designed to focus readers' attentions on these recently developed valuable techniques that have proven their effectiveness and reliability over existing classical methods. Moreover, this text also explains the necessary classical methods because the aim is that new methods would complement the traditional methods in order to improve the understanding of the material.

The book avoids approaching the subject through the compact and classical methods that make the material impossible to be grasped, especially by students who do not have the background in these abstract concepts. Compact theorems and abstract handling of the material are not presented in this text.

The book was developed as a result of many years of experience in teaching partial differential equations and conducting research work in this field. The author has taken account on his teaching experience, research work as well as valuable suggestions received from students and scholars from a wide variety of audience. Numerous examples and exercises, ranging in level from easy to difficult, but consistent with the material, are given in each section to give the reader the knowledge, practice and skill in partial differential equations and solitary waves theory. There is plenty of material in this text to be covered in two semesters for senior undergraduates and beginning graduates of Mathematics, Science, and Engineering.

The content of the book is divided into two distinct parts, each is a self-contained and practical part. Part I contains eleven chapters that handle the partial differential equations by using the newly developed methods, namely, *Adomian decomposition method* and *Variational Iteration Method*. Some of the traditional methods are used in this part. With a diverse readership and interdisciplinary audience of applied mathematics, science, and engineering, attempts are made so that part I presents both analytical and numerical approaches in a clear and systematic fashion to make this book accessible to many who work in this field.

Part II contains seven chapters devoted to thoroughly examine solitary waves theory. Since the discovery of solitons in 1965, mathematicians, engineers, and physicists have been intrigued by the rich mathematical structure of solitons. Solitons play a prevalent role in propagation of light in fibers, surface waves in nonlinear dielectrics, optical bistability, optical switching in slab wave guides, and many other phenomena in plasma and fluid dynamics.

Chapter 1 provides the basic definitions and introductory concepts. Initial value problems and boundary value problems are discussed. In Chapter 2, the first order partial differential equations are handled by the newly developed methods, namely, the *Adomian decomposition method* (ADM) and the *variational iteration method* (VIM). The method of characteristics is introduced and explained in detail. Chapter 3 deals with the one-dimensional heat flow where homogeneous and inhomogeneous initial-boundary value problems are approached by using the decomposition method, the variational iteration method and the method of separation of variables. Chapter 4 is entirely devoted to the two-dimensional and three-dimensional heat flow. Chapter 5 provides the reader with a comprehensive discussion of the literature related to the one-dimensional wave equation. The decomposition method and the variational iteration method are used in handling the wave equations in a bounded and an unbounded domain. Moreover, the method of separation of variables and the D'Alembert method are also used. Chapter 6 presents a comprehensive study on wave equations in two-dimensional and three-dimensional spaces. Chapter 7 is devoted to the Laplace's equation in two- and three-dimensional rectangular coordinates and in polar coordinates. Moreover, the Laplace's equation in annulus form is also investigated by using the decomposition method and the separation of variables method. Chapter 8 introduces a comprehensive study on nonlinear partial differential equations. Even though the subject is considered difficult and mostly addressed in distinct books independent of linear PDEs, but it will be handled successfully and elegantly by using the newly developed decomposition method and the variational iteration method. Chapter 9 provides the reader with a variety of linear and nonlinear applications selected from mathematical physics, population growth models and evolution concepts. The useful concept of solitons and the recently developed concept of *Compactons* are thoroughly examined by using both traditional and new methods. Chapter 10 is concerned with the numerical techniques. Emphasis in this chapter will be on combining the decomposition series solution, the variational iteration method, and the Padé approximants to provide a promising tool that can be applied for further applications. Chapter 11 is concerned with the concepts of solitons and compactons. In this chapter, the solitons and compactons are determined by using prescribed conditions, a necessary condition for the applicability of the decomposition method.

Part II of this book gives a self-contained, practical and realistic approach to solitary wave theory. The dissipation and the dispersion effects are thoroughly investigated. Solitons play a prevalent role in many scientific and engineering phenomena.

The newly discovered *compactons*: solitons with a compact support are also studied. Part II of this book is devoted to use mainly the Hirota's bilinear method, combined with simplified version developed by Hereman and the tanh-coth method. Chapter 12 presents discussions about the dissipation and dispersion effects, analytic and nonanalytic solutions, conservation laws and multiple-soliton solutions, tanh-coth method, and Hirota's bilinear method combined with the Hereman's simplified form of the Hirota's method. In Chapter 13, the family of the KdV equations is studied. Multi-soliton solutions are obtained for only completely integrable equations of this family. Compactons solutions are also examined. Chapter 14 is concerned with KdV and mKdV equations of higher orders. The single solitons and the multiple-soliton solutions for completely integrable equations are addressed by using the Hirota's bilinear method. In addition, the Hirota-Satsuma equations and the generalized short wave equations were investigated for multiple-soliton solutions.

Chapter 15 investigates many KdV-type of equations where soliton solutions and multi-soliton solutions are obtained by using tanh-coth method and Hirota's method respectively. Chapter 16 is entirely devoted to study a family of well-known physical models for solitons and multi-soliton solutions as well. Some of these equations are Boussinesq equation, Klein-Gordon equation, Liouville equation, sine-Gordon equation, DBM equation, and others. Chapter 17 provides the reader with a comprehensive discussion of the literature related to Burgers, Fisher, Huxley, FitzHugh-Nagumo equations and related equations. Most of these equations are characterized by the dissipation phenomena that give kinks solutions. Chapter 18 presents a comprehensive study on two distinct types of equations that appear in solitary wave theory. The family of Camassa-Holm equations is examined to obtain the nonanalytic solution of peakons. On the other hand, the Schrodinger and Ginzburg-Landau equations of different orders are studied in this chapter.

The book concludes with six useful appendices. Moreover, the book introduces the traditional methods in the same amount of concern to provide the reader with the knowledge needed to make a comparison.

I deeply acknowledge Professor Louis Pennisi who made very valuable suggestions that helped a great deal in directing this book towards its main goal. I also deeply acknowledge Professor Masaaki Ito and Professor Willy Hereman for many helpful discussions and useful remarks. I owe them my deepest thanks.

I am deeply indebted to my wife, my son and my daughters who provided me with their continued encouragement, patience and support during the long days of preparing this book.

The author would highly appreciate any note concerning any constructive suggestion.

Saint Xavier University
Chicago, IL 60655
2009

Abdul-Majid Wazwaz
E-mail: wazwaz@sxu.edu

Contents

Part I
Partial Differential Equations

Chapter 1
Basic Concepts

1.1 Introduction

It is well known that most of the phenomena that arise in mathematical physics and engineering fields can be described by partial differential equations (PDEs). In physics for example, the heat flow and the wave propagation phenomena are well described by partial differential equations [1–4]. In ecology, most population models are governed by partial differential equations [5–6]. The dispersion of a chemically reactive material is characterized by partial differential equations. In addition, most physical phenomena of fluid dynamics, quantum mechanics, electricity, plasma physics, propagation of shallow water waves, and many other models are controlled within its domain of validity by partial differential equations.

Partial differential equations have become a useful tool for describing these natural phenomena of science and engineering models. Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving partial differential equations, and the implementation of these methods.

However, in this text, we will restrict our analysis to solve partial differential equations along with the given conditions that characterize the initial conditions and the boundary conditions of the dependent variable [7]. We fill focus our concern on deriving solutions to PDEs and not on the derivation of these equations. In this text, our presentation will be based on applying the recent developments in this field and on applying some of the traditional methods as well. The formulation of partial differential equations and the scientific interpretation of the models will not be discussed.

It is to be noted that several methods are usually used in solving PDEs. The newly developed *Adomian decomposition method* and the related improvements of the *modified technique* and the *noise terms* phenomena will be effectively used. Moreover, the *variational iteration method* that was recently developed will be used as well. The recently developed techniques have been proved to be reliable, accurate and effective in both the analytic and the numerical purposes. The Adomian decomposition method and the variational iteration method were formally proved to provide the solution in terms of a rapid convergent infinite series that may yield the exact solution in many cases. As will be seen in part I of this text, both methods require the use of conditions such as initial conditions. The other related modifications were shown to be powerful in that it accelerate the rapid convergence of the solution. However, some of the traditional methods, such as the separation of variables method and the method of characteristics will be applied as well. Moreover, the other techniques, such as integral transforms, perturbation methods, numerical methods and other traditional methods, that are usually used in other texts, will not be used in this text.

In Part II of this text, we will focus our work on nonlinear evolution equations that describe a variety of physical phenomena. The Hirota's bilinear formalism and the tanh-coth method will be employed in the second part. These methods will be used to determine soliton solutions and multiple-soliton solutions, for completely integrable equations, as well. Several well-known nonlinear evolution equations such as the KdV equation, Burgers equation, Boussinesq equation, Camassa-Holm equation, sine-Gordon equation, and many others will be investigated.

1.2 Definitions

1.2.1 Definition of a PDE

A partial differential equation (PDE) is an equation that contains the dependent variable (the unknown function), and its partial derivatives. It is known that in the ordinary differential equations (ODEs), the dependent variable $u = u(x)$ depends only on one independent variable x. Unlike the ODEs, the dependent variable in the PDEs, such as $u = u(x,t)$ or $u = u(x,y,t)$, must depend on more than one independent variable. If $u = u(x,t)$, then the function u depends on the independent variable x, and on the time variable t. However, if $u = u(x,y,t)$, then the function u depends on the space variables x,y, and on the time variable t.

Examples of the PDEs are given by

$$u_t = ku_{xx}, \tag{1.1}$$

$$u_t = k(u_{xx} + u_{yy}), \tag{1.2}$$

$$u_t = k(u_{xx} + u_{yy} + u_{zz}), \tag{1.3}$$

that describe the heat flow in one dimensional space, two dimensional space, and three dimensional space respectively. In (1.1), the dependent variable $u = u(x,t)$ depends on the position x and on the time variable t. However, in (1.2), $u = u(x,y,t)$ depends on three independent variables, the space variables x,y and the time variable t. In (1.3), the dependent variable $u = u(x,y,z,t)$ depends on four independent variables, the space variables x,y, and z, and the time variable t.

Other examples of PDEs are given by

$$u_{tt} = c^2 u_{xx}, \tag{1.4}$$
$$u_{tt} = c^2 (u_{xx} + u_{yy}), \tag{1.5}$$
$$u_{tt} = c^2 (u_{xx} + u_{yy} + u_{zz}), \tag{1.6}$$

that describe the wave propagation in one dimensional space, two dimensional space, and three dimensional space respectively. Moreover, the unknown functions in (1.4), (1.5), and (1.6) are defined by $u = u(x,t)$, $u = u(x,y,t)$, and $u = u(x,y,z,t)$ respectively.

The well known Laplace equation is given by

$$u_{xx} + u_{yy} = 0, \tag{1.7}$$
$$u_{xx} + u_{yy} + u_{zz} = 0, \tag{1.8}$$

where the function u does not depend on the time variable t. As will be seen later, the Laplace's equation in polar coordinates is given by

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \tag{1.9}$$

where $u = u(r,\theta)$.

Moreover, the Burgers equation and the KdV equation are given by

$$u_t + uu_x - \nu u_{xx} = 0, \tag{1.10}$$
$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.11}$$

respectively, where the function u depends on x and t.

1.2.2 Order of a PDE

The order of a PDE is the order of the highest partial derivative that appears in the equation. For example, the following equations

$$\begin{aligned} u_x - u_y &= 0, \\ u_{xx} - u_t &= 0, \\ u_y - uu_{xxx} &= 0, \end{aligned} \tag{1.12}$$

are PDEs of first order, second order, and third order respectively.

Example 1. Find the order of the following PDEs:

(a) $u_t = u_{xx} + u_{yy}$

(b) $u_x + u_y = 0$

(c) $u^4 u_{xx} + u_{xxy} = 2$

(d) $u_{xx} + u_{yyyy} = 1$

Solution.

(a) The highest partial derivative contained in this equation is u_{xx} or u_{yy}. The PDE is therefore of order two.

(b) The highest partial derivative contained in this equation is u_x or u_y. The PDE is therefore of order one.

(c) The highest partial derivative contained in this equation is u_{xxy}. The PDE is therefore of order three.

(d) The highest partial derivative contained in this equation is u_{yyyy}. The PDE is therefore of order four.

1.2.3 Linear and Nonlinear PDEs

Partial differential equations are classified as **linear** or **nonlinear**. A partial differential equation is called linear if:
(1) the power of the dependent variable and each partial derivative contained in the equation is one, and
(2) the coefficients of the dependent variable and the coefficients of each partial derivative are constants or independent variables. However, if any of these conditions is not satisfied, the equation is called nonlinear.

Example 2. Classify the following PDEs as *linear* or *nonlinear*:

(a) $xu_{xx} + yu_{yy} = 0$

(b) $uu_t + xu_x = 2$

(c) $u_x + \sqrt{u} = x$

(d) $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

Solution.

(a) The power of each partial derivative u_{xx} and u_{yy} is one. In addition, the coefficients of the partial derivatives are the independent variables x and y respectively. Hence, the PDE is linear.

(b) Although the power of each partial derivative is one, but u_t has the dependent variable u as its coefficient. Therefore, the PDE is nonlinear.

(c) The equation is nonlinear because of the term $\sqrt{u}$.

(d) The equation is linear because it satisfies the two necessary conditions.

1.2.4 Some Linear Partial Differential Equations

As stated before, linear partial differential equations arise in many areas of scientific applications, such as the diffusion equation and the wave equation. In what follows, we list some of the well-known models that are of important concern:
1. The *heat equation* in one dimensional space is given by

$$u_t = k u_{xx}, \tag{1.13}$$

where k is a constant.
2. The *wave equation* in one dimensional space is given by

$$u_{tt} = c^2 u_{xx}, \tag{1.14}$$

where c is a constant.
3. The *Laplace equation* is given by

$$u_{xx} + u_{yy} = 0. \tag{1.15}$$

4. The *Klein-Gordon equation* is given by

$$\nabla^2 u - \frac{1}{c^2} u_{tt} = \mu^2 u, \tag{1.16}$$

where c and μ are constants.
5. The *Linear Schrodinger's equation* is given by

$$i u_t + u_{xx} = 0, \quad i = \sqrt{-1}. \tag{1.17}$$

6. The *Telegraph equation* is given by

$$u_{xx} = a u_{tt} + b u_t + c u, \tag{1.18}$$

where a, b and c are constants. It is to be noted that these linear models and others will be studied in details in the forthcoming chapters.

1.2.5 Some Nonlinear Partial Differential Equations

It was mentioned earlier that partial differential equations arise in different areas of mathematical physics and engineering, including fluid dynamics, plasma physics, quantum field theory, nonlinear wave propagation and nonlinear fiber optics [8]. In what follows we list some of the well-known nonlinear models that are of great interest:
1. The *Advection equation* is given by

$$u_t + uu_x = f(x,t). \tag{1.19}$$

2. The *Burgers equation* is given by

$$u_t + uu_x = \alpha u_{xx}. \tag{1.20}$$

3. The *Korteweg de-Vries* (*KdV*) *equation* is given by

$$u_t + auu_x + bu_{xxx} = 0. \tag{1.21}$$

4. The *modified KdV equation* (*mKdV*) is given by

$$u_t - 6u^2u_x + u_{xxx} = 0. \tag{1.22}$$

5. The *Boussinesq equation* is given by

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0. \tag{1.23}$$

6. The *sine-Gordon equation* is given by

$$u_{tt} - u_{xx} = \alpha \sin u. \tag{1.24}$$

7. The *sinh-Gordon equation* is given by

$$u_{tt} - u_{xx} = \alpha \sinh u. \tag{1.25}$$

8. The *Liouville equation* is given by

$$u_{tt} - u_{xx} = e^{\pm u}. \tag{1.26}$$

9. The *Fisher equation* is

$$u_t = Du_{xx} + u(1-u). \tag{1.27}$$

10. The *Kadomtsev-Petviashvili* (*KP*)*equation* is given by

$$(u_t + auu_x + bu_{xxx})_x + u_{yy} = 0. \tag{1.28}$$

11. The *K*(*n,n*)*equation* is given by

$$u_t + a(u^n)_x + b(u^n)_{xx} = 0, \quad n > 1. \tag{1.29}$$

12. The *Nonlinear Schrodinger* (*NLS*) *equation* is

$$iu_t + u_{xx} + \gamma|u|^2u = 0. \tag{1.30}$$

13. The *Camassa-Holm*(*CH*)*equation* is given by

$$u_t - u_{xxt} + au_x + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \tag{1.31}$$

14. The *Degasperis-Procesi* (*DP*) *equation* is given by

$$u_t - u_{xxt} + au_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.32}$$

The above-mentioned nonlinear partial differential equations and many others will be examined in the forthcoming chapters. These equations are important and many give rise to solitary wave solutions.

1.2.6 Homogeneous and Inhomogeneous PDEs

Partial differential equations are also classified as **homogeneous** or **inhomogeneous**. A partial differential equation of any order is called homogeneous if every term of the PDE contains the dependent variable u or one of its derivatives, otherwise, it is called an inhomogeneous PDE [7]. This can be illustrated by the following example.

Example 3. Classify the following partial differential equations as homogeneous or inhomogeneous:

(a) $u_t = 4u_{xx}$

(b) $u_t = u_{xx} + x$

(c) $u_{xx} + u_{yy} = 0$

(d) $u_x + u_y = u + 4$

Solution.

(a) The terms of the equation contain partial derivatives of u only, therefore it is a homogeneous PDE.

(b) The equation is an inhomogeneous PDE, because one term contains the independent variable x.

(c) The equation is a homogeneous PDE.

(d) The equation is an inhomogeneous PDE.

1.2.7 Solution of a PDE

A solution of a PDE is a function u such that it satisfies the equation under discussion and satisfies the given conditions as well. In other words, for u to satisfy the equation, the left hand side of the PDE and the right hand side should be the same upon substituting the resulting solution. This concept will be illustrated by examining the following examples. Examples of partial differential equations subject to specific conditions will be examined in the coming chapters.

Example 4. Show that $u(x,t) = \sin x e^{-4t}$ is a solution of the following PDE

$$u_t = 4u_{xx}. \tag{1.33}$$

Solution.

Left Hand Side (LHS)= $u_t = -4\sin x e^{-4t}$
Right Hand Side (RHS)=$4u_{xx} = -4\sin x e^{-4t}$=LHS

Example 5. Show that $u(x,y) = \sin x \sin y + x^2$ is a solution of the following PDE

$$u_{xx} = u_{yy} + 2. \tag{1.34}$$

Solution.

Left Hand Side (LHS)= $u_{xx} = -\sin x \sin y + 2$
Right Hand Side (RHS)=$u_{yy} + 2 = -\sin x \sin y + 2$=LHS

Example 6. Show that $u(x,t) = \cos x \cos t$ is a solution of the following PDE

$$u_{tt} = u_{xx}. \tag{1.35}$$

Solution.

Left Hand Side (LHS)= $u_{tt} = -\cos x \cos t$
Right Hand Side (RHS)=$u_{xx} = -\cos x \cos t$=LHS

Example 7. Show that

(a) $u(x,y) = xy$

(b) $u(x,y) = x^2y^2$

(c) $u(x,y) = \sin(xy)$

are solutions of the equation

$$xu_x - yu_y = 0. \tag{1.36}$$

Solution.

(a) $u = xy,\ u_x = y,\ \ u_y = x,$
LHS $= xy - yx$=0

(b) $u = x^2y^2,\ u_x = 2xy^2,\ \ u_y = 2x^2y,$
LHS $= 2x^2y^2 - 2x^2y^2$=0

(c) $u = \sin(xy),\ u_x = y\cos(xy),\ \ u_y = x\cos(xy),$
LHS $= xy\cos(xy) - xy\cos(xy)$=0

Consequently, we conclude that a general solution is of the form

$$u = f(xy). \tag{1.37}$$

Remarks:

The following remarks can be drawn here in discussing the concept of a solution of a PDE.

1. For a linear homogeneous ordinary differential equation, it is well-known that if $u_1, u_2, u_3, \cdots, u_n$ are solutions of the equation, then a linear combination of $u_1, u_2, u_3, \cdots$ given by

$$u = c_1u_1 + c_2u_2 + c_3u_3 + \cdots + c_nu_n, \tag{1.38}$$

is also a solution. The concept of combining two or more of these solutions is called the **superposition principle**.

It is interesting to note that the superposition principle works effectively for linear homogeneous PDEs in a given domain. The concept will be explained in Chapter 3 when using the method of separation of variables.

2. For a linear ordinary differential equation, the *general* solution depends mainly on *arbitrary constants*. Unlike ODEs, in linear partial differential equations, the *general* solution depends on *arbitrary functions*. This can be easily examined by noting that the PDE

$$u_x + u_y = 0 \tag{1.39}$$

has its solution given by

$$u = f(x - y), \tag{1.40}$$

where $f(x-y)$ is an arbitrary differentiable function. This means that the solution of (1.39) can be any of the following functions:

$$\begin{aligned} u &= x - y, \\ u &= e^{x-y}, \\ u &= \sinh(x-y), \\ u &= \ln(x-y), \end{aligned} \tag{1.41}$$

and any function of the form $f(x-y)$. However, the general solution of a PDE is of little use. In fact a particular solution is always required that will satisfy prescribed conditions.

1.2.8 Boundary Conditions

As stated above, the general solution of a PDE is of little use. A particular solution is frequently required that will satisfy prescribed conditions. Given a PDE that controls the mathematical behavior of physical phenomenon in a bounded domain D, the dependent variable u is usually prescribed at the boundary of the domain D. The boundary data is called *boundary conditions*. The boundary conditions are given in three types [2,7] defined as follows:

1. **Dirichlet Boundary Conditions**: In this case, the function u is usually prescribed on the boundary of the bounded domain. For a rod of length L, where $0 < x < L$, the boundary conditions are defined by $u(0) = \alpha$, $u(L) = \beta$, where α and β are constants. For a rectangular plate, $0 < x < L_1$, $0 < y < L_2$, the boundary conditions $u(0,y), u(L_1,y), u(x,0)$, and $u(x,L_2)$ are usually prescribed. The boundary conditions are called homogeneous if the dependent variable u at any point on the boundary is zero, otherwise the boundary conditions are called inhomogeneous.

2. **Neumann Boundary Conditions**: In this case, the normal derivative $\frac{du}{dn}$ of u along the outward normal to the boundary is prescribed. For a rod of length L, Neumann boundary conditions are of the form $u_x(0,t) = \alpha, u_x(L,t) = \beta$.

3. **Mixed Boundary Conditions**: In this case, a linear combination of the dependent variable u and the normal form $\frac{du}{dn}$ is prescribed on the boundary.

It is important to note that it is not always necessary for the domain to be bounded, however one or more parts of the boundary may be at infinity. This type of problems will be discussed in the coming chapters.

1.2.9 Initial Conditions

It was indicated before that the PDEs mostly arise to govern physical phenomenon such as heat distribution, wave propagation phenomena and phenomena of quantum mechanics. Most of the PDEs, such as the diffusion equation and the wave equation, depend on the time t. Accordingly, the initial values of the dependent variable u at the starting time $t = 0$ should be prescribed. It will be discussed later that for the heat case, the initial value $u(t = 0)$, that defines the temperature at the starting time, should be prescribed. For the wave equation, the initial conditions $u(t = 0)$ and $u_t(t = 0)$ should also be prescribed.

1.2.10 Well-posed PDEs

A partial differential equation is said to be *well-posed* if a unique solution that satisfies the equation and the prescribed conditions exists, and provided that the unique solution obtained is *stable*. The solution to a PDE is said to be stable if a small change in the conditions or the coefficients of the PDE results in a small change in the solution.

Exercises 1.2

1. Find the order of the following PDEs:

(a) $u_{xx} = u_{xxx} + u + 1$
(b) $u_{tt} = u_{xx} + u_{yy} + u_{zz}$
(c) $u_x + u_y = 0$
(d) $u_t + u_{xxyy} = u$

2. Classify the following PDEs as linear or nonlinear:

(a) $u_t = u_{xx} - u$
(b) $u_{tt} = u_{xx} + u^2$
(c) $u_x + u_y = u$
(d) $u_t + uu_{xxyy} = 0$

3. Classify the following PDEs as homogeneous or inhomogeneous:

(a) $u_t = u_{xx} + x$
(b) $u_{tt} = u_{xx} + u_{yy} + u_{zz}$
(c) $u_x + u_y = 1$
(d) $u_t + u_{xxy} = u$

4. Verify that the given function is a particular solution of the corresponding PDE:

(a) $u_x + u_y = x + y$, $u(x,y) = xy$
(b) $u_x - u_y = 0$, $u(x,y) = x + y$
(c) $u_x + u_y = u$, $u(x,y) = e^x + e^y$
(d) $xu_x + u_y = u$, $u(x,y) = x + e^y$

5. Verify that the given function is a particular solution of the corresponding PDE:

(a) $u_t = u_{xx}$, $u(x,t) = x + e^{-t}\sin x$
(b) $u_t = u_{xx} - 2u$, $u(x,t) = e^{-t}\sinh x$
(c) $u_{tt} = u_{xx}$, $u(x,t) = \sin x \sin t$
(d) $u_{tt} = 2(u_{xx} + u_{yy})$, $u(x,y,t) = \cos x \cos y \cos(2t)$

6. Show that the functions

(a) $u(x,y) = \dfrac{x}{y}$

(b) $u(x,y) = \sin\left(\dfrac{x}{y}\right)$

(c) $u(x,y) = \cosh\left(\dfrac{x}{y}\right)$

are solutions of the equation $xu_x + yu_y = 0$. Show that $u = f(\frac{x}{y})$ is a general solution of the equation where f is an arbitrary differentiable function.

7. Show that the functions

(a) $u(x,y) = x + y^2$

(b) $u(x,y) = \sin(x+y^2)$

(c) $u(x,y) = e^{x+y^2}$

are solutions of the equation

$$2yu_x - u_y = 0.$$

Show that $u = f(x+y^2)$ is a general solution of the equation where f is an arbitrary differentiable function.

8. Verify that the given function is a general solution of the corresponding PDE:

(a) $u_{tt} = u_{xx}$, $u(x,t) = f(x+t) + g(x-t)$

(b) $u_x - u_y = 0$, $u(x,y) = f(x+y)$

(c) $4y^2 u_{xx} + \frac{1}{y} u_y - u_{yy} = 0, u(x,y) = f(x+y^2) + g(x-y^2)$ given that f and g are twice differentiable functions.

(d) $u_{xx} - \frac{1}{x} u_x - 4x^2 u_{yy} = 0, u(x,y) = f(x^2+y) + g(x^2-y)$ given that f and g are twice differentiable functions.

1.3 Classifications of a Second-order PDE

A second order linear partial differential equation in two independent variables x and y in its general form is given by

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{1.42}$$

where A, B, C, D, E, F, and G are constants or functions of the variables x and y. A second order partial differential equation (1.42) is usually classified into three basic classes of equations, namely:

1. **Parabolic.** Parabolic equation is an equation that satisfies the property

$$B^2 - 4AC = 0. \tag{1.43}$$

Examples of parabolic equations are heat flow and diffusion processes equations. The heat transfer equation

$$u_t = ku_{xx} \tag{1.44}$$

will be discussed in details in Chapters 3 and 4.

2. **Hyperbolic.** Hyperbolic equation is an equation that satisfies the property

$$B^2 - 4AC > 0. \tag{1.45}$$

Examples of hyperbolic equations are wave propagation equations. The wave equation

$$u_{tt} = c^2 u_{xx} \tag{1.46}$$

will be discussed in details in Chapters 5 and 6.
3. **Elliptic.** Elliptic equation is an equation that satisfies the property

$$B^2 - 4AC < 0. \tag{1.47}$$

Examples of elliptic equations are Laplace's equation and Schrodinger equation. The Laplace equation in a two dimensional space

$$u_{xx} + u_{yy} = 0 \tag{1.48}$$

will be discussed in details in Chapter 7. The Laplace's equation is often called the potential equation because $u(x,y)$ defines the potential function.

Example 1. Classify the following second order partial differential equations as *parabolic, hyperbolic* or *elliptic*:

(a) $u_t = 4u_{xx}$

(b) $u_{tt} = 4u_{xx}$

(c) $u_{xx} + u_{yy} = 0$

Solution.

(a) $A = 4, B = C = 0$
This means that

$$B^2 - 4AC = 0. \tag{1.49}$$

Hence, the equation in (a) is parabolic.

(b) $A = 4,\ B = 0,\ C = -1$
This means that

$$B^2 - 4AC = 16 > 0. \tag{1.50}$$

Hence, the equation in (b) is hyperbolic.

(c) $A = 1, B = 0, C = 1$
This means that

$$B^2 - 4AC = -4 < 0. \tag{1.51}$$

Hence, the equation in (c) is elliptic.

Example 2. Classify the following second order partial differential equations as *parabolic, hyperbolic* or *elliptic*:

(a) $u_{tt} = u_{xx} - u_t$

(b) $u_t = u_{xx} - u_x$

(c) $u_{xx} + xu_{yy} = 0$

Solution.

(a) $A = 1, B = 0, C = -1$
This means that

$$B^2 - 4AC = 4 > 0. \tag{1.52}$$

Hence, the equation in (a) is hyperbolic.

(b) $A = 1, B = C = 0$
This means that

$$B^2 - 4AC = 0. \tag{1.53}$$

Hence, the equation in (b) is parabolic.

(c) $A = 1, B = 0, C = x$
This means that

$$B^2 - 4AC = -4x. \tag{1.54}$$

The equation in (c) is parabolic if $x = 0$, hyperbolic if $x < 0$, and elliptic if $x > 0$.

Exercises 1.3

Classify the following second order partial differential equations as *parabolic, hyperbolic* or *elliptic*:

1. $u_{tt} = c^2 u_{xx}$
2. $u_{xx} + u_{yy} + u = 0$
3. $u_t = 4u_{xx} + xt$
4. $u_{tt} = u_{xx} + xt$
5. $u_t = u_{xx} + 2u_x + u$
6. $u_{xy} = 0$
7. $u_{xx} + u_{yy} = 4$
8. $u_{xx} + u = 0$
9. $u_{tt} = u_{xx} - u_t$
10. $yu_{xx} + u_{yy} = 0$

References

1. D. Bleecker and G. Csordas, Basic Partial Differential Equations, Chapman and Hall, New York, (1995).
2. S.J. Farlow, Partial Differential Equations for Scientists and Engineers, Dover, New York, (1993).
3. F. John, Partial Differential Equations, Springer-Verlag, New York, (1982).
4. L. Lam, Nonlinear Physics for Beginners, World Scientific, Singapore, (1998).
5. J.D. Logan, An Introduction to Nonlinear Partial Differential Equations, John Wiley, New York, (1994).
6. A.M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema, Leiden, (2002).
7. G.B. Whitham, Linear and Nonlinear Waves, John Wiley, New York, (1976).
8. D. Zwillinger, Handbook of Differential Equations, Academic Press, New York, (1992).

Chapter 2
First-order Partial Differential Equations

2.1 Introduction

In this chapter we will discuss the first order linear partial differential equations, homogeneous and inhomogeneous. Partial differential equations of first order are used to model traffic flow on a crowded road, blood flow through an elastic-walled tube, shock waves and as special cases of the general theories of gas dynamics and hydraulics.

It is the concern of this text to introduce the recently developed methods to handle partial differential equations in an accessible manner. Some of the traditional techniques will be used as well. In this text we will apply the *Adomian decomposition method* [1–4] and the related phenomenon of the noise terms [7–10] that will accelerate the rapid convergence of the solution. The decomposition method and the improvements made by the noise terms phenomenon and the modified decomposition method [8] are reliable and effective techniques of promising results. Moreover, the *variational iteration method* [5] will be applied as well. These two methods provide the solution in an infinite series form. The obtained series may converge to a closed form solution if exact solution exists. For concrete problems where exact solution does not exist, the truncated series may be used for numerical purposes.

In addition to Adomian decomposition method and the variational iteration method, the classic *method of characteristics* will be used in this chapter. A comparative study between the method of characteristics and the other two methods will be carried out through illustrative examples.

2.2 Adomian Decomposition Method

In this section we will discuss the Adomian decomposition method. The Adomian decomposition method has been receiving much attention in recent years in applied mathematics in general, and in the area of series solutions in particular. The method proved to be powerful, effective, and can easily handle a wide class of linear or nonlinear, ordinary or partial differential equations, and linear and nonlinear integral equations. The decomposition method demonstrates fast convergence of the solution and therefore provides several significant advantages. In this text, the method will be successfully used to handle most types of partial differential equations that appear in several physical models and scientific applications. The method attacks the problem in a direct way and in a straightforward fashion without using linearization, perturbation or any other restrictive assumption that may change the physical behavior of the model under discussion.

The Adomian decomposition method was introduced and developed by George Adomian in [1–2] and is well addressed in the literature. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations and integral equations as well. For more details, the reader is advised to see the references [1–4, 7–10] and the references therein.

The Adomian decomposition method consists of decomposing the unknown function $u(x,y)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y), \tag{2.1}$$

where the components $u_n(x,y), n \geqslant 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components $u_0, u_1, u_2, \cdots$ individually. As will be seen through the text, the determination of these components can be achieved in an easy way through a recursive relation that usually involve simple integrals.

To give a clear overview of Adomian decomposition method, we first consider the linear differential equation written in an operator form by

$$Lu + Ru = g, \tag{2.2}$$

where L is, mostly, the lower order derivative which is assumed to be invertible, R is other linear differential operator, and g is a source term. It is to be noted that the nonlinear differential equations will be presented in Chapter 8. We next apply the inverse operator L^{-1} to both sides of equation (2.2) and using the given condition to obtain

$$u = f - L^{-1}(Ru), \tag{2.3}$$

where the function f represents the terms arising from integrating the source term g and from using the given conditions that are assumed to be prescribed. As indicated before, Adomian method defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n, \tag{2.4}$$

where the components $u_0, u_1, u_2, \cdots$ are usually recurrently determined. Substituting (2.4) into both sides of (2.3) leads to

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{2.5}$$

For simplicity, Equation (2.5) can be rewritten as

$$u_0 + u_1 + u_2 + u_3 + \cdots = f - L^{-1}\left(R(u_0 + u_1 + u_2 + \cdots)\right). \tag{2.6}$$

To construct the recursive relation needed for the determination of the components $u_0, u_1, u_2, \cdots$, it is important to note that Adomian method suggests that the zeroth component u_0 is usually defined by the function f described above, i.e. by all terms, that are not included under the inverse operator L^{-1}, which arise from the initial data and from integrating the inhomogeneous term. Accordingly, the formal recursive relation is defined by

$$\begin{aligned} u_0 &= f, \\ u_{k+1} &= -L^{-1}\left(R(u_k)\right), \; k \geqslant 0, \end{aligned} \tag{2.7}$$

or equivalently

$$\begin{aligned} u_0 &= f, \\ u_1 &= -L^{-1}\left(R(u_0)\right), \\ u_2 &= -L^{-1}\left(R(u_1)\right), \\ u_3 &= -L^{-1}\left(R(u_2)\right), \\ &\;\vdots \end{aligned} \tag{2.8}$$

It is clearly seen that the relation (2.8) reduced the differential equation under consideration into an elegant determination of computable components. Having determined these components, we then substitute it into (2.4) to obtain the solution in a series form.

It was formally shown by many researchers that if an exact solution exists for the problem, then the obtained series converges very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. Cherruault examined the convergence of Adomian's method in [3]. In addition, Cherruault and Adomian presented a new proof of convergence of the method in [4]. For more details about the proofs presented to discuss the rapid convergence, the reader is advised to see the references mentioned above and the references therein.

However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. It was also shown by many that the series obtained by evaluating few terms gives an approximation of high degree of accuracy if compared with other numerical techniques.

It seems reasonable to give a brief outline about the works conducted by Adomian and other researchers in applying Adomian's method. Adomian in [1–2] and in many other works introduced his method and applied it to many deterministic and stochastic problems. He implemented his method to solve frontier problems of physics. The Adomian's achievements in this regard are remarkable and of promising results.

Adomian's method has attracted a considerable amount of research work. A comparison between the decomposition method and the perturbation technique showed the efficiency of the decomposition method compared to the tedious work required by the perturbation method. The advantage of the decomposition method over Picard's method has been emphasized in many works. Also, a comparative study between Adomian's method and Taylor series method has been examined to show that the decomposition method requires less computational work if compared with Taylor series. Other comparisons with traditional methods such as finite difference method have been conducted in the literature.

It is to be noted that many studies have shown that few terms of the decomposition series provide a numerical result of a high degree of accuracy. Rach *et. al.* [6] employed Adomian's method to solve differential equations with singular coefficients such as Legendre's equation, Bessel's equation, and Hermite's equation. Moreover, in [10], a suitable definition of the operator was used to overcome the difficulty of singular points of Lane-Emden equation. In [10], a new definition of the operator was introduced to overcome the singularity behavior for the Lane-Emden type of equations. Many other studies implement the decomposition method for differential equations, ordinary and partial, and for integral equations, linear and nonlinear.

It is normal in differential equations that we seek a closed form solution or a series solution with a proper number of terms. Although this book is devoted to handle partial differential equations, but it seems reasonable to use the decomposition method to discuss two ordinary differential equations where an exact solution is obtained for the first equation and a series approximation is determined for the second equation. For the first problem we consider the equation

$$u'(x) = u(x),\ u(0) = A. \tag{2.9}$$

In an operator form, Equation (2.9) becomes

$$Lu = u, \tag{2.10}$$

where the differential operator L is given by

$$L = \frac{d}{dx}, \tag{2.11}$$

and therefore the inverse operator L^{-1} is defined by

$$L^{-1}(\cdot) = \int_0^x (\cdot)dx. \tag{2.12}$$

Applying L^{-1} to both sides of (2.10) and using the initial condition we obtain

$$L^{-1}(Lu) = L^{-1}(u), \tag{2.13}$$

so that

$$u(x) - u(0) = L^{-1}(u), \tag{2.14}$$

or equivalently

$$u(x) = A + L^{-1}(u). \tag{2.15}$$

Substituting the series assumption (2.5) into both sides of (2.15) gives

$$\sum_{n=0}^{\infty} u_n(x) = A + L^{-1}\left(\sum_{n=0}^{\infty} u_n(x)\right). \tag{2.16}$$

In view of (2.16), the following recursive relation

$$\begin{aligned} u_0(x) &= A, \\ u_{k+1}(x) &= L^{-1}(u_k(x)),\ k \geqslant 0, \end{aligned} \tag{2.17}$$

follows immediately. Consequently, we obtain

$$\begin{aligned} u_0(x) &= A, \\ u_1(x) &= L^{-1}(u_0(x)) = Ax, \\ u_2(x) &= L^{-1}(u_1(x)) = A\frac{x^2}{2!}, \\ u_3(x) &= L^{-1}(u_2(x)) = A\frac{x^3}{3!}, \\ &\vdots \end{aligned} \tag{2.18}$$

Substituting (2.18) into (2.5) gives the solution in a series form by

$$u(x) = A(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots), \tag{2.19}$$

and in a closed form by

$$u(x) = Ae^x. \tag{2.20}$$

We next consider the well-known Airy's equation

$$u''(x) = xu(x),\ u(0) = A, u'(0) = B. \tag{2.21}$$

In an operator form, Equation (2.21) becomes

$$Lu = xu, \tag{2.22}$$

where the differential operator L is given by

$$L = \frac{d^2}{dx^2}, \tag{2.23}$$

and therefore the inverse operator L^{-1} is defined by

$$L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)dx\,dx. \tag{2.24}$$

Operating with L^{-1} on both sides of (2.21) and using the initial conditions we obtain

$$L^{-1}(Lu) = L^{-1}(xu), \tag{2.25}$$

so that

$$u(x) - xu'(0) - u(0) = L^{-1}(xu), \tag{2.26}$$

or equivalently

$$u(x) = A + Bx + L^{-1}(xu). \tag{2.27}$$

Substituting the series assumption (2.5) into both sides of (2.27) yields

$$\sum_{n=0}^{\infty} u_n(x) = A + Bx + L^{-1}(x \sum_{n=0}^{\infty} (u_n(x)). \tag{2.28}$$

Following the decomposition method we obtain the following recursive relation

$$\begin{aligned} u_0(x) &= A + Bx, \\ u_{k+1}(x) &= L^{-1}(xu_k(x)),\, k \geqslant 0. \end{aligned} \tag{2.29}$$

Consequently, we obtain

$$\begin{aligned} u_0(x) &= A + Bx, \\ u_1(x) &= L^{-1}(xu_0(x)) = A\frac{x^3}{6} + B\frac{x^4}{12}, \\ u_2(x) &= L^{-1}(xu_1(x)) = A\frac{x^6}{180} + B\frac{x^7}{504}, \\ &\vdots \end{aligned} \tag{2.30}$$

Substituting (2.30) into (2.5) gives the solution in a series form by

$$u(x) = A(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots) + B(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots). \tag{2.31}$$

Other components can be easily computed to enhance the accuracy of the approximation.

It seems now reasonable to apply Adomian decomposition method to first-order partial differential equations. For the convenience of the reader, and without loss of generality, we consider the inhomogeneous partial differential equation:

$$u_x + u_y = f(x,y),\, u(0,y) = g(y),\, u(x,0) = h(x). \tag{2.32}$$

In an operator form, Eq. (2.32) can be written as

$$L_x u + L_y u = f(x,y), \tag{2.33}$$

where

$$L_x = \frac{\partial}{\partial x}, L_y = \frac{\partial}{\partial y}, \tag{2.34}$$

where each operator is assumed easily invertible, and thus the inverse operators L_x^{-1} and L_y^{-1} exist and given by

$$\begin{aligned} L_x^{-1}(\cdot) &= \int_0^x (\cdot)dx, \\ L_y^{-1}(\cdot) &= \int_0^y (\cdot)dy. \end{aligned} \tag{2.35}$$

This means that

$$L_x^{-1} L_x u(x,y) = u(x,y) - u(0,y). \tag{2.36}$$

Applying L_x^{-1} to both sides of (2.33) gives

$$L_x^{-1} L_x u = L_x^{-1}(f(x,y)) - L_x^{-1}(L_y u), \tag{2.37}$$

or equivalently

$$u(x,y) = g(y) + L_x^{-1}(f(x,y)) - L_x^{-1}(L_y u), \tag{2.38}$$

obtained by using (2.36) and by using the condition $u(0,y) = g(y)$. As stated above, the decomposition method sets

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y). \tag{2.39}$$

Substituting (2.39) into both sides of (2.38) we find

$$\sum_{n=0}^{\infty} u_n(x,y) = g(y) + L_x^{-1}(f(x,y)) - L_x^{-1}\left(L_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right). \tag{2.40}$$

This can be rewritten as

$$u_0 + u_1 + u_2 + \cdots = g(y) + L_x^{-1}(f(x,y)) - L_x^{-1} L_y(u_0 + u_1 + u_2 + \cdots). \tag{2.41}$$

The zeroth component u_0, as suggested by Adomian method is always identified by the given initial condition and the terms arising from $L_x^{-1}(f(x,y))$, both of which are assumed to be known. Accordingly, we set

$$u_0(x,y) = g(y) + L_x^{-1}(f(x,y)). \tag{2.42}$$

Consequently, the other components $u_{k+1}, k \geqslant 0$ are defined by using the relation

$$u_{k+1}(x,y) = -L_x^{-1} L_y(u_k), \; k \geqslant 0. \tag{2.43}$$

Combining Eqs. (2.42) and (2.43), we obtain the recursive scheme

$$\begin{aligned} u_0(x,y) &= g(y) + L_x^{-1}(f(x,y)), \\ u_{k+1}(x,y) &= -L_x^{-1}L_y(u_k),\, k \geqslant 0, \end{aligned} \tag{2.44}$$

that forms the basis for a complete determination of the components $u_0, u_1, u_2, \cdots$. Therefore, the components can be easily obtained by

$$\begin{aligned} u_0(x,y) &= g(y) + L_x^{-1}(f(x,y)), \\ u_1(x,y) &= -L_x^{-1}\left(L_y u_0(x,y)\right), \\ u_2(x,y) &= -L_x^{-1}(L_y u_1(x,y)), \\ u_3(x,y) &= -L_x^{-1}(L_y u_2(x,y)), \end{aligned} \tag{2.45}$$

and so on. Thus the components u_n can be determined recursively as far as we like. It is clear that the accuracy of the approximation can be significantly improved by simply determining more components. Having established the components of $u(x,y)$, the solution in a series form follows immediately. However, the expression

$$\phi_n = \sum_{r=0}^{n-1} u_r(x,y) \tag{2.46}$$

is considered the n-term approximation to u. For concrete problems, where exact solution is not easily obtainable, we usually use the truncated series (2.46) for numerical purposes. As indicated earlier, the convergence of Adomian decomposition method has been established by many researchers, but will not be discussed in this text.

It is important to note that the solution can also be obtained by finding the y-solution by applying the inverse operator L_y^{-1} to both sides of the equation

$$L_y = f(x,y) - L_x u. \tag{2.47}$$

The equality of the x-solution and the y-solution is formally justified and will be examined through the coming examples.

It should be noted here that the series solution (2.39) has been proved by many researchers to converge rapidly, and a closed form solution is obtainable in many cases if a closed form solution exists.

It was found, as will be seen later, that very few terms of the series obtained in (2.39) provide a high degree of accuracy level which makes the method powerful when compared with other existing numerical techniques. In many cases the series representation of $u(x,y)$ can be summed to yield the closed form solution. Several works in this direction have demonstrated the power of the method for analytical and numerical applications.

The essential features of the decomposition method for linear and nonlinear equations, homogeneous and inhomogeneous, can be outlined as follows:

1. Express the partial differential equation, linear or nonlinear, in an operator form.
2. Apply the inverse operator to both sides of the equation written in an operator form.
3. Set the unknown function $u(x,y)$ into a decomposition series

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y), \tag{2.48}$$

whose components are elegantly determined. We next substitute the series (2.48) into both sides of the resulting equation.

4. Identify the zeroth component $u_0(x,y)$ as the terms arising from the given conditions and from integrating the source term $f(x,y)$, both are assumed to be known.

5. Determine the successive components of the series solution $u_k, k \geqslant 1$ by applying the recursive scheme (2.44), where each component u_k can be completely determined by using the previous component u_{k-1}.

6. Substitute the determined components into (2.48) to obtain the solution in a series form. An exact solution can be easily obtained in many equations if such a closed form solution exists.

It is to be noted that Adomian decomposition method approaches any equation, homogeneous or inhomogeneous, and linear or nonlinear in a straightforward manner without any need to restrictive assumptions such as linearization, discretization or perturbation. There is no need in using this method to convert inhomogeneous conditions to homogeneous conditions as required by other techniques.

The essential steps of the Adomian decomposition method will be illustrated by discussing the following examples.

Example 1. Use Adomian decomposition method to solve the following inhomogeneous PDE

$$u_x + u_y = x + y, \; u(0,y) = 0, \; u(x,0) = 0. \tag{2.49}$$

Solution.

In an operator form, Eq. (2.49) can be written as

$$L_x u = x + y - L_y u, \tag{2.50}$$

where

$$L_x = \frac{\partial}{\partial x}, \; L_y = \frac{\partial}{\partial y}. \tag{2.51}$$

It is clear that L_x is invertible, hence L_x^{-1} exists and given by

$$L_x^{-1}(\cdot) = \int_0^x (\cdot)\, dx. \tag{2.52}$$

The x-solution:

This solution can be obtained by applying L_x^{-1} to both sides of (2.50), hence we find

$$L_x^{-1} L_x u = L_x^{-1}(x + y) - L_x^{-1}(L_y u), \tag{2.53}$$

or equivalently

$$u(x,y) = u(0,y) + \frac{1}{2}x^2 + xy - L_x^{-1}(L_y u) = \frac{1}{2}x^2 + xy - L_x^{-1}(L_y u), \tag{2.54}$$

obtained upon using the given condition $u(0,y)=0$, Eq. (2.36) and by integrating $f(x,y)=x+y$ with respect to x. As stated above, the decomposition method identifies the unknown function $u(x,y)$ as an infinite number of components $u_n(x,y), n \geqslant 0$ given by

$$u(x,y)=\sum_{n=0}^{\infty} u_n(x,y). \tag{2.55}$$

Substituting (2.55) into both sides of (2.54) we find

$$\sum_{n=0}^{\infty} u_n(x,y)=\frac{1}{2}x^2+xy-L_x^{-1}\left(L_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right). \tag{2.56}$$

Using few terms of the decomposition (2.55) we obtain

$$u_0+u_1+u_2+\cdots=\frac{1}{2}x^2+xy-L_x^{-1}\left(L_y(u_0+u_1+u_2+\cdots)\right). \tag{2.57}$$

As presented before, the decomposition method identifies the zeroth component u_0 by all terms arising from the given condition and from integrating $f(x,y)=x+y$, therefore we set

$$u_0(x,y)=\frac{1}{2}x^2+xy. \tag{2.58}$$

Consequently, the recursive scheme that will enable us to completely determine the successive components is thus constructed by

$$\begin{aligned} u_0(x,y) &= \frac{1}{2}x^2+xy, \\ u_{k+1}(x,y) &= -L_x^{-1}(L_y(u_k)),\ k\geqslant 0. \end{aligned} \tag{2.59}$$

This in turn gives

$$\begin{aligned} u_1(x,y) &= -L_x^{-1}(L_y u_0)=-L_x^{-1}\left(L_y\left(\frac{1}{2}x^2+xy\right)\right)=-\frac{1}{2}x^2, \\ u_2(x,y) &= -L_x^{-1}(L_y u_1)=-L_x^{-1}\left(L_y\left(-\frac{1}{2}x^2\right)\right)=0. \end{aligned} \tag{2.60}$$

Accordingly, $u_k=0, k\geqslant 2$. Having determined the components of $u(x,y)$, we find

$$u=u_0+u_1+u_2+\cdots=\frac{1}{2}x^2+xy-\frac{1}{2}x^2=xy, \tag{2.61}$$

the exact solution of the equation under discussion.

The y-solution:

It is important to note that the exact solution can also be obtained by finding the y-solution. In an operator form we can write the equation by

$$L_y=x+y-L_x u. \tag{2.62}$$

Assume that L_y^{-1} exists and defined by

$$L_y^{-1}(\cdot) = \int_0^y (\cdot)dy. \tag{2.63}$$

Applying L_y^{-1} to both sides of the Eq. (2.62) gives

$$u(x,y) = xy + \frac{1}{2}y^2 - L_y^{-1}(L_x u). \tag{2.64}$$

As mentioned above, the decomposition method sets the solution $u(x,y)$ in a series form by

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y). \tag{2.65}$$

Inserting (2.65) into both sides of (2.64) we obtain

$$\sum_{n=0}^{\infty} u_n(x,y) = xy + \frac{1}{2}y^2 - L_y^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right). \tag{2.66}$$

Using few terms only for simplicity reasons, we obtain

$$u_0 + u_1 + u_2 + \cdots = xy + \frac{1}{2}y^2 - L_y^{-1}(L_x(u_0 + u_1 + u_2 + \cdots)). \tag{2.67}$$

The decomposition method identifies the zeroth component u_0 by all terms arising from the given condition and from integrating $f(x,y) = x + y$, therefore we set

$$u_0(x,y) = xy + \frac{1}{2}y^2. \tag{2.68}$$

To completely determine the successive components of $u(x,y)$, the recursive scheme is thus defined by

$$\begin{aligned} u_0(x,y) &= xy + \frac{1}{2}y^2, \\ u_{k+1}(x,y) &= -L_y^{-1}(L_x(u_k)),\ k \geqslant 0. \end{aligned} \tag{2.69}$$

This gives

$$\begin{aligned} u_1(x,y) &= -L_y^{-1}(L_x u_0) = -L_y^{-1}\left(L_x\left(xy + \frac{1}{2}y^2\right)\right) = -\frac{1}{2}y^2, \\ u_2(x,y) &= -L_y^{-1}(L_x u_1) = -L_y^{-1}\left(L_x\left(-\frac{1}{2}y^2\right)\right) = 0. \end{aligned} \tag{2.70}$$

Consequently, $u_k = 0, k \geqslant 2$. Having determined the components of $u(x,y)$, we find

$$u(x,y) = u_0 + u_1 + u_2 + \cdots = xy + \frac{1}{2}y^2 - \frac{1}{2}y^2 = xy, \tag{2.71}$$

the exact solution of the equation under discussion.

Example 2. Solve the following homogeneous partial differential equation

$$\begin{aligned} u_x - u_y &= 0, \\ u(0,y) &= y,\, u(x,0) = x. \end{aligned} \tag{2.72}$$

Solution.

In an operator form, Eq. (2.72) becomes

$$L_x u(x,y) = L_y u(x,y), \tag{2.73}$$

where the operators L_x and L_y are defined by

$$L_x = \frac{\partial}{\partial x},\, L_y = \frac{\partial}{\partial y}. \tag{2.74}$$

Applying the inverse operator L_x^{-1} to both sides of (2.73) and using the given condition $u(0,y) = y$ yields

$$u(x,y) = y + L_x^{-1}(L_y u). \tag{2.75}$$

We next define the unknown function $u(x,y)$ by the decomposition series

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y). \tag{2.76}$$

Inserting (2.76) into both sides of (2.75) gives

$$\sum_{n=0}^{\infty} u_n(x,y) = y + L_x^{-1}\left(L_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right). \tag{2.77}$$

By considering few terms of the decomposition of $u(x,y)$, Eq. (2.77) becomes

$$u_0 + u_1 + u_2 + \cdots = y + L_x^{-1}\left(L_y(u_0 + u_1 + u_2 + \cdots)\right). \tag{2.78}$$

Proceeding as before, we identify the zeroth component u_0 by

$$u_0(x,y) = y. \tag{2.79}$$

Having identified the zeroth component $u_0(x,y)$, we obtain the recursive scheme

$$\begin{aligned} u_0(x,y) &= y, \\ u_{k+1}(x,y) &= L_x^{-1} L_y(u_k),\, k \geqslant 0. \end{aligned} \tag{2.80}$$

The components $u_0, u_1, u_2, \cdots$ are thus determined as follows:

$$\begin{aligned} u_0(x,y) &= y, \\ u_1(x,y) &= L_x^{-1}L_y u_0 = L_x^{-1}L_y(y) = x, \\ u_2(x,y) &= L_x^{-1}L_y u_1 = L_x^{-1}L_y(x) = 0. \end{aligned} \tag{2.81}$$

It is obvious that all components $u_k(x,y) = 0, k \geqslant 2$. Consequently, the solution is given by

$$u(x,y) = u_0(x,y) + u_1(x,y) + \cdots = u_0(x,y) + u_1(x,y) = y + x, \tag{2.82}$$

the exact solution obtained by using the decomposition series (2.76).

It is important to note here that the exact solution given by (2.82) can also be obtained by determining the y-solution as discussed above. This is left as an exercise to the reader.

Example 3. Solve the following homogeneous partial differential equation

$$xu_x + u_y = 3u, \quad u(x,0) = x^2, \quad u(0,y) = 0. \tag{2.83}$$

Solution.

In an operator form, Eq. (2.83) becomes

$$L_y u(x,y) = 3u(x,y) - xL_x u(x,y). \tag{2.84}$$

Applying the inverse operator L_y^{-1} to both sides of (2.84) and using the given condition $u(x,0) = x^2$ yields

$$u(x,y) = x^2 + L_y^{-1}(3u - xL_x u). \tag{2.85}$$

Substituting $u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$ into both sides of (2.85) gives

$$\sum_{n=0}^{\infty} u_n(x,y) = x^2 + L_y^{-1}\left(3\left(\sum_{n=0}^{\infty} u_n(x,y)\right) - xL_x\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right). \tag{2.86}$$

By considering few terms of the decomposition of $u(x,y)$, Eq. (2.86) becomes

$$u_0 + u_1 + u_2 + \cdots = x^2 + L_y^{-1}\left(3(u_0 + u_1 + \cdots) - xL_x(u_0 + u_1 + u_2 + \cdots)\right). \tag{2.87}$$

Proceeding as before, we identify the recursive scheme

$$\begin{aligned} u_0(x,y) &= x^2, \\ u_{k+1}(x,y) &= L_y^{-1}(3u_k - xL_x u_k),\ k \geqslant 0. \end{aligned} \tag{2.88}$$

The components $u_0, u_1, u_2, \cdots$ are thus determined as follows:

$$\begin{aligned} u_0(x,y) &= x^2, \\ u_1(x,y) &= L_y^{-1}(3u_0 - xL_x u_0) = x^2 y, \end{aligned}$$

$$\begin{aligned} u_2(x,y) &= L_y^{-1}(3u_1 - xL_x u_1) = \frac{x^2y^2}{2!}, \\ u_3(x,y) &= L_y^{-1}(3u_2 - xL_x u_2) = \frac{x^2y^3}{3!}, \\ &\vdots \end{aligned} \tag{2.89}$$

Consequently, the solution is given by

$$u(x,y) = u_0 + u_1 + u_2 + \cdots = x^2(1 + y + \frac{y^2}{2!} + \cdots) = x^2e^y. \tag{2.90}$$

Example 4. Solve the following homogeneous partial differential equation

$$u_x - yu = 0, \quad u(0,y) = 1. \tag{2.91}$$

Solution.

In an operator form, Eq. (2.91) becomes

$$L_x u(x,y) = yu(x,y), \tag{2.92}$$

where the operator L_x is defined as

$$L_x = \frac{\partial}{\partial x}. \tag{2.93}$$

Applying the integral operator L_x^{-1} to both sides of (2.92) and using the given condition that $u(0,y) = 1$ gives

$$u(x,y) = 1 + L_x^{-1}(yu(x,y)). \tag{2.94}$$

Following the discussion presented above, we define the unknown function $u(x,y)$ by the decomposition series

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y). \tag{2.95}$$

Inserting (2.95) into both sides of (2.94) gives

$$\sum_{n=0}^{\infty} u_n(x,y) = 1 + L_x^{-1}\left(y\sum_{n=0}^{\infty} u_n(x,y)\right), \tag{2.96}$$

or equivalently

$$u_0 + u_1 + u_2 + \cdots = 1 + L_x^{-1}\left(y\left(u_0 + u_1 + u_2 + \cdots\right)\right), \tag{2.97}$$

by considering few terms of the decomposition of $u(x,y)$. The components $u_0, u_1, u_2, \cdots$ are thus determined by using the recursive relationship as follows:

$$
\begin{aligned}
u_0(x,y) &= 1, \\
u_1(x,y) &= L_x^{-1}(yu_0) = xy, \\
u_2(x,y) &= L_x^{-1}(yu_1) = \frac{1}{2!}x^2y^2, \\
u_3(x,y) &= L_x^{-1}(yu_2) = \frac{1}{3!}x^3y^3,
\end{aligned}
\tag{2.98}
$$

and so on for other components. Consequently, the solution in a series form is given by

$$
\begin{aligned}
u(x,y) &= u_0(x,y) + u_1(x,y) + u_2(x,y) + \cdots, \\
&= 1 + xy + \frac{1}{2!}x^2y^2 + \frac{1}{3!}x^3y^3 + \cdots,
\end{aligned}
\tag{2.99}
$$

and in a closed form

$$u(x,y) = e^{xy}. \tag{2.100}$$

Example 5. Solve the following homogeneous PDE

$$u_t + cu_x = 0, \quad u(x,0) = x, \tag{2.101}$$

where c is a constant.

Solution.

In an operator form, Eq. (2.101) can be rewritten as

$$L_t u(x,t) = -cL_x u, \tag{2.102}$$

where the operator L_t is defined as

$$L_t = \frac{\partial}{\partial t}. \tag{2.103}$$

It is clear that the operator L_t is invertible, and the inverse operator L_t^{-1} is an indefinite integral from 0 to t. Applying the integral operator L_t^{-1} to both sides of (2.102) and using the given condition that $u(x,0) = x$ yields

$$u(x,t) = x - cL_t^{-1}(L_x u(x,t)). \tag{2.104}$$

Proceeding as before, we substitute the decomposition series for $u(x,t)$ into both sides of (2.104) to obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = x - cL_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{2.105}$$

Using few terms of the decomposition of $u(x,y)$, Eq. (2.105) becomes

$$u_0+u_1+u_2+\cdots = x - cL_t^{-1}\left(L_x(u_0+u_1+u_2+\cdots)\right). \tag{2.106}$$

The components $u_0, u_1, u_2, \cdots$ can be determined by using the recursive relationship as follows:

$$\begin{aligned} u_0(x,t) &= x, \\ u_1(x,t) &= -cL_t^{-1}(L_x u_0) = -ct, \\ u_2(x,t) &= -cL_t^{-1}(L_x u_1) = 0. \end{aligned} \tag{2.107}$$

We can easily observe that $u_k = 0,\ k \geqslant 2$. It follows that the solution in a closed form is given by

$$u(x,t) = x - ct. \tag{2.108}$$

Example 6. Solve the following partial differential equation

$$\begin{aligned} u_x + u_y + u_z &= u, \quad u(0,y,z) = 1+e^y+e^z, \\ u(x,0,z) &= 1+e^x+e^z, \quad u(x,y,0) = 1+e^x+e^y, \end{aligned} \tag{2.109}$$

where $u = u(x,y,z)$.

Solution.

In an operator form, Eq. (2.109) can be rewritten as

$$L_x u(x,y,z) = u - L_y u - L_z u, \tag{2.110}$$

where the operators L_x, L_y and L_z are defined by

$$L_x = \frac{\partial}{\partial x}, \quad L_y = \frac{\partial}{\partial y}, \quad L_z = \frac{\partial}{\partial z}. \tag{2.111}$$

Assume that the operator L_x is invertible, and the inverse operator L_x^{-1} is an indefinite integral from 0 to x. Applying the integral operator L_x^{-1} to both sides of (2.110) and using the given condition that $u(0,y,z) = 1+e^y+e^z$ yields

$$u(x,y,z) = 1+e^y+e^z+L_x^{-1}\left(u - L_y u - L_z u\right). \tag{2.112}$$

Proceeding as before, we substitute the decomposition

$$u(x,y,z) = \sum_{n=0}^{\infty} u_n(x,y,z) \tag{2.113}$$

into both sides of (2.112) to find

$$\sum_{n=0}^{\infty} u_n(x,y,z) = 1+e^y+e^z+L_x^{-1}\left(\sum_{n=0}^{\infty} u_n - L_y(\sum_{n=0}^{\infty} u_n) - L_z(\sum_{n=0}^{\infty} u_n)\right). \tag{2.114}$$

Using few terms of the decomposition of $u(x,y,z)$, Eq. (2.114) becomes

$$
\begin{aligned}
u_0+u_1+u_2+\cdots = & \; 1+e^y+e^z+L_x^{-1}(u_0+u_1+u_2+\cdots) \\
& -L_x^{-1}\left(L_y(u_0+u_1+u_2+\cdots)\right) \\
& -L_x^{-1}\left(L_z(u_0+u_1+u_2+\cdots)\right).
\end{aligned} \tag{2.115}
$$

The components $u_0, u_1, u_2, \cdots$ can be determined recurrently as follows

$$
\begin{aligned}
u_0(x,y,z) &= 1+e^y+e^z, \\
u_1(x,y,z) &= L_x^{-1}(u_0-L_y u_0-L_z u_0)=x, \\
u_2(x,y,z) &= L_x^{-1}(u_1-L_y u_1-L_z u_1)=\frac{1}{2!}x^2, \\
u_3(x,y,z) &= L_x^{-1}(u_2-L_y u_2-L_z u_2)=\frac{1}{3!}x^3,
\end{aligned} \tag{2.116}
$$

and so on. Consequently, the solution in a series form is given by

$$
u(x,y,z)=(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots)+e^y+e^z, \tag{2.117}
$$

and in a closed form

$$
u(x,y,z)=e^x+e^y+e^z. \tag{2.118}
$$

It is interesting to note that we can easily show that the y-solution and the z-solution will also give the same solution as in (2.118).

In closing this section, we point out that Adomian decomposition method works effectively for nonlinear differential equations. However, an algorithm is needed to express the nonlinear terms contained in the nonlinear equation. The implementation of the decomposition method to handle the nonlinear differential equations will be explained in details in Chapter 8.

Exercises 2.2

In exercises 1–4, use the decomposition method to show that the exact solution can be obtained by determining the x-solution or the y-solution:

1. $u_x+u_y=2xy^2+2x^2y,\ u(x,0)=0,\ u(0,y)=0$

2. $u_x+u_y=2x+2y,\ u(x,0)=x^2, u(0,y)=y^2$

3. $u_x+yu=0,\ u(x,0)=1,\ u(0,y)=1$

4. $u_x+u_y=u,\ u(x,0)=1+e^x,\ u(0,y)=1+e^y$

In exercises 5–12, use the decomposition method to solve the following partial differential equations:

5. $u_x+u_y=2u,\ u(x,0)=e^x,\ u(0,y)=e^y$

6. $u_x - u_y = 2$, $u(x,0) = x$, $u(0,y) = -y$

7. $u_x + u_y = x^2 + y^2$, $u(x,0) = \frac{1}{3}x^3$, $u(0,y) = \frac{1}{3}y^3$

8. $xu_x + u_y = u$, $u(x,0) = 1 + x$, $u(0,y) = e^y$

9. $u_x + yu_y = u$, $u(x,0) = e^x$, $u(0,y) = 1 + y$

10. $xu_x + u_y = 2u$, $u(x,0) = x$, $u(0,y) = 0$

11. $u_x + yu_y = 2u$, $u(x,0) = 0$, $u(0,y) = y$

12. $u_x + u_y = 0$, $u(x,0) = e^x$, $u(0,y) = e^{-y}$

In exercises 13–16, use the decomposition method to solve the following partial differential equations:

13. $u_x + u_y + u_z = 3$, $u(0,y,z) = y + z$, $u(x,0,z) = x + z$, $u(x,y,0) = x + y$

14. $u_x + u_y + u_z = 3u$, $u(0,y,z) = e^{y+z}$, $u(x,0,z) = e^{x+z}$, $u(x,y,0) = e^{x+y}$

15. $u_x + yu_y + zu_z = 3u$, $u(0,y,z) = yz$, $u(x,0,z) = 0$, $u(x,y,0) = 0$

16. $u_x + yu_y + zu_z = u$, $u(0,y,z) = 1 + y + z$, $u(x,0,z) = z + e^x$, $u(x,y,0) = y + e^x$

In exercises 17–20, use the decomposition method to solve the following partial differential equations:

17. $u_x - u_y = 1 + 2x + 2y$, $u(0,y) = y + y^2$, $u(x,0) = 2x + 3x^2$

18. $u_x - u_y = 0$, $u(0,y) = y + y^2$, $u(x,0) = x + x^2$

19. $u_x - u_y = 0$, $u(0,y) = \sin y$, $u(x,0) = \sin x$

20. $u_x - u_y = 0$, $u(0,y) = \cosh y$, $u(x,0) = \cosh x$

2.3 The Noise Terms Phenomenon

In this section, we will present a useful tool that will accelerate the convergence of the Adomian decomposition method. The noise terms phenomenon provides a major advantage in that it demonstrates a fast convergence of the solution. It is important to note here that the **noise terms** phenomenon, that will be introduced in this section, may appear only for inhomogeneous PDEs. In addition, this phenomenon is applicable to all inhomogeneous PDEs of any order and will be used where appropriate in the coming chapters. The noise terms, if existed in the components u_0 and u_1, will provide, in general, the solution in a closed form with only two successive iterations.

In view of these remarks, we now outline the ideas of the noise terms :
1. The **noise terms** are defined as the identical terms with opposite signs that arise in the components u_0 and u_1. As stated above, these identical terms with opposite signs may exist only for inhomogeneous differential equations.
2. By canceling the noise terms between u_0 and u_1, even though u_1 contains further terms, the remaining non-canceled terms of u_0 may give the exact solution of the PDE. Therefore, it is necessary to verify that the non-canceled terms of u_0 satisfy the PDE under discussion.

On the other hand, if the non-canceled terms of u_0 did not satisfy the given PDE, or the noise terms did not appear between u_0 and u_1, then it is necessary to determine more components of u to determine the solution in a series form.
3. It was formally shown that the noise terms appear for specific cases of inhomogeneous equations, whereas homogeneous equations do not show noise terms. The conclusion about the self-canceling noise terms was based on observations drawn from solving specific models where no proof was presented. For further readings about the noise terms phenomenon, see [8,10].
4. It was formally proved by researchers that a necessary condition for the appearance of the noise terms is required. The conclusion made in [8,10] is that the zeroth component u_0 must contain the exact solution u among other terms. Moreover, it was shown that the nonhomogenity condition does not always guarantee the appearance of the noise terms as examined in [8,10].

A useful summary about the noise terms phenomenon can be drawn as follows:
1. The noise terms are defined as the identical terms with opposite signs that may appear in the components u_0 and u_1.
2. The noise terms appear only for specific types of inhomogeneous equations whereas noise terms do not appear for homogeneous equations.
3. Noise terms may appear if the exact solution is part of the zeroth component u_0.
4. Verification that the remaining non-canceled terms satisfy the equation is necessary and essential.

The phenomenon of the useful noise terms will be explained by the following illustrative examples.

Example 1. Use the decomposition method and the noise terms phenomenon to solve the following inhomogeneous PDE

$$u_x + u_y = (1+x)e^y,\ u(0,y) = 0,\ u(x,0) = x. \tag{2.119}$$

Solution.

The inhomogeneous PDE can be rewritten in an operator form by

$$L_x u = (1+x)e^y - L_y u. \tag{2.120}$$

Clearly L_x is invertible and therefore the inverse operator L_x^{-1} exists. Applying L_x^{-1} to both sides of (2.120) and using the given condition leads to

$$u(x,y) = (x + \frac{x^2}{2!})e^y - L_x^{-1}(L_y u). \tag{2.121}$$

Using the decomposition series $u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$ into (2.121) gives

$$\sum_{n=0}^{\infty} u_n(x,y) = (x + \frac{x^2}{2!})e^y - L_x^{-1}\left(L_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right), \tag{2.122}$$

or equivalently

$$u_0 + u_1 + u_2 + \cdots = (x + \frac{x^2}{2!})e^y - L_x^{-1}\left(L_y(u_0 + u_1 + u_2 + \cdots)\right). \tag{2.123}$$

Proceeding as before, the components $u_0, u_1, u_2, \cdots$ are determined in a recursive manner by

$$\begin{aligned} u_0(x,y) &= \left(x + \frac{x^2}{2!}\right)e^y, \\ u_1(x,y) &= -L_x^{-1}(L_y u_0) = -\left(\frac{x^2}{2!} + \frac{x^3}{3!}\right)e^y, \\ u_2(x,y) &= -L_x^{-1}(L_y u_1) = \left(\frac{x^3}{3!} + \frac{x^4}{4!}\right)e^y. \end{aligned} \tag{2.124}$$

Considering the first two components u_0 and u_1 in (2.124), it is easily observed that the noise terms $\frac{x^2}{2!}e^y$ and $-\frac{x^2}{2!}e^y$ appear in u_0 and u_1 respectively. By canceling the noise term $\frac{x^2}{2!}e^y$ in u_0, and by verifying that the remaining non-canceled terms of u_0 satisfy Eq. (2.119), we find that the exact solution is given by

$$u(x,y) = xe^y. \tag{2.125}$$

Notice that the exact solution is verified through substitution in the equation (2.119) and not only upon the appearance of the noise terms. In addition, the other noise terms that appear between other components will vanish in the limit.

Example 2. Use the decomposition method and the noise terms phenomenon to solve the following inhomogeneous PDE

$$u_x + yu_y = y(\cosh x + \sinh x),\ u(0,y) = y,\ u(x,0) = 0. \tag{2.126}$$

Solution.

The inhomogeneous PDE can be rewritten in an operator form by

$$L_x u(x,y) = y(\cosh x + \sinh x) - yL_y u. \tag{2.127}$$

Applying L_x^{-1} to both sides of (2.127) and using the given condition gives

$$u(x,y) = 1 + y\sinh x + y\cosh x - y - L_x^{-1}(yL_y u). \tag{2.128}$$

Substituting the decomposition series of $u(x,y)$ into (2.128) gives

$$\sum_{n=0}^{\infty} u_n(x,y) = 1 + y\sinh x + y\cosh x - y - L_x^{-1}\left(yL_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right), \tag{2.129}$$

or equivalently

$$\begin{aligned} u_0 + u_1 + u_2 + \cdots = {} & 1 + y\sinh x + y\cosh x - y \\ & - L_x^{-1}\left(yL_y(u_0 + u_1 + u_2 + \cdots)\right). \end{aligned} \tag{2.130}$$

Identifying the zeroth component u_0 as discussed before, the components $u_0, u_1, u_2, \cdots$ can be determined in a recursive manner by

$$\begin{aligned} u_0(x,y) &= 1 + y\sinh x + y\cosh x - y, \\ u_1(x,y) &= -L_x^{-1}(yL_y u_0) = -y(\cosh x + \sinh x) + y(x+1). \end{aligned} \tag{2.131}$$

It is easily observed that three noise terms $\pm y\cosh x, \mp y$ and $\pm y\sinh x$ appear in u_0 and u_1. By canceling the noise terms in u_0, the remaining non-canceled term of u_0 satisfies the given conditions, but does not satisfy the PDE. However, canceling the first two noise terms gives the exact solution

$$u(x,y) = 1 + y\sinh x. \tag{2.132}$$

This can be verified through substitution in equation (2.126).

Example 3. Use the decomposition method and the noise terms phenomenon to solve the following PDE:

$$u_x + u_y = x^2 + 4xy + y^2,\ u(0,y) = 0,\ u(x,0) = 0. \tag{2.133}$$

Solution.

We first rewrite the inhomogeneous PDE (2.133) in an operator form

$$L_x u = x^2 + 4xy + y^2 - L_y u. \tag{2.134}$$

Proceeding as before and applying the inverse operator L_x^{-1} to both sides of (2.134) and using the given condition we obtain

$$u(x,y) = \frac{1}{3}x^3 + 2x^2y + xy^2 - L_x^{-1}\left(L_y u(x,y)\right). \tag{2.135}$$

Proceeding as before, the first two components u_0 and u_1 are given by

$$\begin{aligned} u_0(x,y) &= \frac{1}{3}x^3 + 2x^2y + xy^2, \\ u_1(x,y) &= -L_x^{-1}(L_y u_0) = -x^2y - \frac{2}{3}x^3. \end{aligned} \tag{2.136}$$

We can easily observe that the two components u_0 and u_1 do not contain noise terms. This confirms our belief that although the PDE is an inhomogeneous equation, but the noise terms between the first two components did not exist in this problem. Unlike the previous examples, we should determine more components to obtain an insight through the solution. Therefore, other components should be determined. Hence we find

$$\begin{aligned} u_2(x,y) &= -L_x^{-1}(-x^2) = \frac{1}{3}x^3, \\ u_k(x,y) &= 0, \quad k \geqslant 3. \end{aligned} \tag{2.137}$$

Based on the result we obtained for u_2, other components of $u(x,y)$ will vanish. Consequently, we find that

$$\begin{aligned} u(x,y) &= u_0 + u_1 + u_2 + \cdots, \\ &= \frac{1}{3}x^3 + 2x^2y + xy^2 - x^2y - \frac{2}{3}x^3 + \frac{1}{3}x^3 = xy^2 + x^2y, \end{aligned} \tag{2.138}$$

the exact solution of the equation even though we do not have noise terms.

Exercises 2.3

In Exercises 1–12, use the decomposition method and the noise terms phenomenon to solve the following partial differential equations:

1. $u_x + u_y = 3x^2 + 3y^2$, $u(x,0) = x^3$, $u(0,y) = y^3$

2. $u_x + u_y = \sinh x + \sinh y$, $u(x,0) = 1 + \cosh x, u(0,y) = 1 + \cosh y$

3. $u_x + u_y = x + y$, $u(x,0) = u(0,y) = 0$

4. $u_x - u_y = \cos x + \sin y$, $u(x,0) = 1 + \sin x$, $u(0,y) = \cos y$

5. $u_x + u_y = \sin x + \sin y + x\cos y + y\cos x$, $u(x,0) = u(0,y) = 0$

6. $u_x - u_y = \cos x + \cos y + x\sin y + y\sin x$, $u(x,0) = x$, $u(0,y) = -y$

7. $u_x + u_y = (1+y)e^x + (1+x)e^y$, $u(x,0) = x$, $u(0,y) = y$

8. $u_x - u_y = (1+y)e^{-x} + (1+x)e^{-y}$, $u(x,0) = x$, $u(0,y) = -y$

9. $u_x + yu_y - u = 2xy^2 + x^2y^2$, $u(x,0) = u(0,y) = 0$

10. $u_x + yu_y - u = xy^2 + y^2 + 2xy$, $u(x,0) = u(0,y) = 0$

11. $u_x + u_y = \cos x + \sinh y$, $u(x,0) = 1 + \sin x$, $u(0,y) = \cosh y$

12. $u_x + u_y = u + e^y$, $u(x,0) = x$, $u(0,y) = 0$

In Exercises 13–18, show that the noise terms do not appear in the first two components of the solution of the inhomogeneous partial differential equations. Find the exact solution.

13. $u_x + u_y = 2xy^3 + 6x^2y^2 + 2x^3y$, $u(x,0) = u(0,y) = 0$

14. $u_x - u_y = 3x^2y^4 - 3x^4y^2$, $u(x,0) = u(0,y) = 0$

15. $u_x - u_y = 0$, $u(x,0) = x^2, u(0,y) = y^2$

16. $u_x + u_y = 4x + 4y$, $u(x,0) = x^2, u(0,y) = y^2$

17. $u_x + u_y = x + y$, $u(x,0) = x^2, u(0,y) = y^2$

18. $u_x + u_y = 1 + u - x$, $u(x,0) = 1 + x + e^x, u(0,y) = 1 + e^y$

2.4 The Modified Decomposition Method

In this section we will introduce a reliable modification of the Adomian decomposition method developed by Wazwaz and presented in [7]. The modified decomposition method will further accelerate the convergence of the series solution. It is to be noted that the modified decomposition method will be applied, wherever it is appropriate, to all partial differential equations of any order. The modification will be outlined in this section and will be employed in this section and in other chapters as well.

To give a clear description of the technique, we consider the partial differential equation in an operator form

$$Lu + Ru = g, \tag{2.139}$$

where L is the highest order derivative, R is a linear differential operator of less order or equal order to L, and g is the source term. Operating with the inverse operator L^{-1} on (2.139) we obtain

$$u = f - L^{-1}(Ru), \tag{2.140}$$

where f represents the terms arising from the given initial condition and from integrating the source term g. We then proceed as discussed in Section 2.2 and define the solution u as an infinite sum of components defined by

$$u = \sum_{n=0}^{\infty} u_n. \tag{2.141}$$

The aim of the decomposition method is to determine the components $u_n, n \geqslant 0$ recurrently and elegantly. To achieve this goal, the decomposition method admits the use of the recursive relation

$$\begin{aligned} u_0 &= f, \\ u_{k+1} &= -L^{-1}(Ru_k), \quad k \geqslant 0. \end{aligned} \tag{2.142}$$

In view of (2.142), the components u_n, $n \geqslant 0$ are readily obtained.

The modified decomposition method introduces a slight variation to the recursive relation (2.142) that will lead to the determination of the components of u in a faster and easier way. For specific cases, the function f can be set as the sum of two partial functions, namely f_1 and f_2. In other words, we can set

$$f = f_1 + f_2. \tag{2.143}$$

Using (2.143), we introduce a qualitative change in the formation of the recursive relation (2.142). To reduce the size of calculations, we identify the zeroth component u_0 by one part of f, namely f_1 or f_2. The other part of f can be added to the component u_1 among other terms. In other words, the modified recursive relation can be identified by

$$\begin{aligned} u_0 &= f_1, \\ u_1 &= f_2 - L^{-1}(Ru_0), \\ u_{k+1} &= -L^{-1}(Ru_k), \quad k \geqslant 1. \end{aligned} \tag{2.144}$$

An important point can be made here in that we suggest a change in the formation of the first two components u_0 and u_1 only. Although this variation in the formation of u_0 and u_1 is slight, however it plays a major role in accelerating the convergence of the solution and in minimizing the size of calculations.

Two important remarks related to the modified method [7] can be made here. First, by proper selection of the functions f_1 and f_2, the exact solution u may be obtained by using very few iterations, and sometimes by evaluating only two components. The success of this modification depends only on the choice of f_1 and f_2, and this can be made through trials. Second, if f consists of one term only, the standard decomposition method should be employed in this case.

It is worth mentioning that the modified decomposition method will be used for linear and nonlinear equations of any order. In the coming chapters, it will be used wherever it is appropriate.

The modified decomposition method will be illustrated by discussing the following examples.

Example 1. Use the modified decomposition method to solve the first order partial differential equation:

$$u_x + u_y = 3x^2y^3 + 3x^3y^2, \quad u(0,y) = 0. \tag{2.145}$$

Solution.

In an operator form, Eq. (2.145) becomes

$$L_x u = 3x^2y^3 + 3x^3y^2 - u_y, \tag{2.146}$$

where L_x is a first order partial derivative with respect to x. Applying the inverse operator L_x^{-1} to both sides of (2.146) gives

$$u(x,y) = x^3y^3 + \frac{3}{4}x^4y^2 - L_x^{-1}(u_y). \tag{2.147}$$

The function $f(x,y)$ consists of two terms, hence we set

$$\begin{aligned} f_1(x,y) &= x^3y^3, \\ f_2(x,y) &= \frac{3}{4}x^4y^2. \end{aligned} \tag{2.148}$$

In view of (2.148) we introduce the modified recursive relation

$$\begin{aligned} u_0(x,y) &= x^3y^3, \\ u_1(x,y) &= \frac{3}{4}x^4y^2 - L_x^{-1}(u_0)_y, \\ u_{k+1}(x,y) &= -L_x^{-1}(u_k)_y,\ k \geqslant 1. \end{aligned} \tag{2.149}$$

This gives

$$\begin{aligned} u_0(x,y) &= x^3y^3, \\ u_1(x,y) &= \frac{3}{4}x^4y^2 - L_x^{-1}(3x^3y^2) = 0, \\ u_{k+1}(x,y) &= 0, \quad k \geqslant 1. \end{aligned} \tag{2.150}$$

It then follows that the solution is

$$u(x,y) = x^3y^3. \tag{2.151}$$

This example clearly shows that the solution can be obtained by using two iterations, and hence the volume of calculations is reduced.

Example 2. Use the modified decomposition method to solve the first order partial differential equation:

$$u_x - u_y = x^3 - y^3, \quad u(0,y) = \frac{1}{4}y^4. \tag{2.152}$$

Solution.

In an operator form, Eq. (2.152) becomes

$$L_x u = x^3 - y^3 + u_y, \tag{2.153}$$

where L_x is a first order partial derivative with respect to x. Proceeding as before we obtain

$$u(x,y) = \frac{1}{4}y^4 + \frac{1}{4}x^4 - xy^3 + L_x^{-1}(u_y). \tag{2.154}$$

We next split the function $f(x,y)$ as follows

$$\begin{aligned} f_1(x,y) &= \frac{1}{4}y^4 + \frac{1}{4}x^4, \\ f_2(x,y) &= -xy^3. \end{aligned} \tag{2.155}$$

Consequently, we set the modified recursive relation

$$\begin{aligned} u_0(x,y) &= \frac{1}{4}y^4 + \frac{1}{4}x^4, \\ u_1(x,y) &= -xy^3 + L_x^{-1}(u_{0_y}), \\ u_{k+1}(x,y) &= L_x^{-1}(u_{k_y}), \quad k \geqslant 1. \end{aligned} \tag{2.156}$$

This gives

$$\begin{aligned} u_0(x,y) &= \frac{1}{4}y^4 + \frac{1}{4}x^4, \\ u_1(x,y) &= -xy^3 + L_x^{-1}(y^3) = 0, \\ u_{k+1}(x,y) &= 0, \quad k \geqslant 1. \end{aligned} \tag{2.157}$$

The exact solution

$$u(x,y) = \frac{1}{4}x^4 + \frac{1}{4}y^4, \tag{2.158}$$

follows immediately.

Example 3. Use the modified decomposition method to solve the first order partial differential equation:

$$u_x + u_y = u, \quad u(0,y) = 1 + e^y. \tag{2.159}$$

Solution.

Operating with the inverse operator L_x^{-1} on (2.159) and using the given condition gives

$$u(x,y) = 1 + e^y + L_x^{-1}(u - u_y). \tag{2.160}$$

We next split function $f(x,y)$ as follows

$$\begin{aligned} f_1(x,y) &= e^y, \\ f_2(x,y) &= 1. \end{aligned} \tag{2.161}$$

To determine the components of $u(x,y)$, we set the modified recursive relation

$$\begin{aligned} u_0(x,y) &= e^y, \\ u_1(x,y) &= 1 + L_x^{-1}\left(u_0 - (u_0)_y\right), \\ u_{k+1}(x,y) &= L_x^{-1}\left(u_k - (u_k)_y\right), \quad k \geqslant 1. \end{aligned} \tag{2.162}$$

This gives

$$\begin{aligned} u_0(x,y) &= e^y, \\ u_1(x,y) &= 1, \\ u_2(x,y) &= x, \\ u_3(x,y) &= \frac{x^2}{2!}, \end{aligned} \tag{2.163}$$

and so on. The solution in a series form is given by

$$u(x,y) = e^y + (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots), \tag{2.164}$$

and in a closed form by

$$u(x,y) = e^x + e^y. \tag{2.165}$$

Example 4. Use the modified decomposition method to solve the first order partial differential equation:

$$u_x + u_y = \cosh x + \cosh y, \quad u(x,0) = \sinh x. \tag{2.166}$$

Solution.

To effectively use the given condition, we rewrite (2.166) in an operator form by

$$L_y u = \cosh x + \cosh y - u_x. \tag{2.167}$$

Applying the inverse operator L_y^{-1} on (2.167) and using the given condition gives

$$u(x,y) = \sinh x + \sinh y + y\cosh x - L_y^{-1}(u_x). \tag{2.168}$$

The function $f(x,y)$ can be written as $f_1 + f_2$ where

$$\begin{aligned} f_1(x,y) &= \sinh x + \sinh y, \\ f_2(x,y) &= y\cosh x. \end{aligned} \tag{2.169}$$

To determine the components of $u(x,y)$, we set the modified recursive relation

$$\begin{aligned} u_0(x,y) &= \sinh x + \sinh y, \\ u_1(x,y) &= y\cosh x - L_y^{-1}((u_0)_x) = 0, \\ u_{k+1}(x,y) &= -L_y^{-1}((u_k)_x) = 0, \quad k \geqslant 1. \end{aligned} \tag{2.170}$$

The exact solution

$$u(x,y) = \sinh x + \sinh y, \tag{2.171}$$

follows immediately.

It is interesting to point out that two iterations only were used to determine the exact solution. However, using the following formation

$$\begin{aligned} f_1(x,y) &= \sinh x, \\ f_2(x,y) &= \sinh y + y\cosh x, \end{aligned} \tag{2.172}$$

for $f(x,y)$ will give the following recursive relation

$$\begin{aligned} u_0(x,y) &= \sinh x, \\ u_1(x,y) &= \sinh y + y\cosh x - L_y^{-1}(\cosh x) = \sinh y, \\ u_{k+1}(x,y) &= 0, \quad k \geqslant 1. \end{aligned} \tag{2.173}$$

It is obvious from (2.173) that all components $u_j = 0, j \geqslant 2$.

Consequently, the exact solution is

$$u(x,y) = \sinh x + \sinh y, \tag{2.174}$$

obtained by using the first two components only.

Exercises 2.4

Use the modified decomposition method to solve the following first order partial differential equations:

1. $u_x + u_y = 3x^2 + 3y^2$, $u(0,y) = y^3$

2. $u_x - u_y = 2x + 2y$, $u(0,y) = -y^2$

3. $u_x + u_y = 4x + 4y$, $u(0,y) = y^2$

4. $u_x + u_y = \sinh x + \sinh y$, $u(0,y) = 1 + \cosh y$

5. $u_x - u_y = \cos x + \sin y$, $u(0,y) = \cos y$

6. $u_x + yu_y - u = 2xy^2 + x^2y^2$, $u(0,y) = 0$

7. $u_x - u_y = \cos x - \cos y$, $u(0,y) = \sin y$

8. $u_x + u_y = u$, $u(0,y) = 1 - e^y$

9. $xu_x + u_y = 2x^2 + 3y^2$, $u(0,y) = y^3$

10. $u_x + u_y = 2x + \cos y$, $u(0,y) = 1 + \sin y$

11. $u_x + xu_y = 1 + x\cosh y$, $u(0,y) = 1 + \sinh y$

12. $u_x - xu_y = \cos x - x\cosh y$, $u(0,y) = \sinh y$

2.5 The Variational Iteration Method

It was stated before that Adomian decomposition method, with its modified form and the noise terms phenomenon, and some of the traditional methods will be used in this text. The other well-known methods, such as the inverse scattering method, the pseudo spectral method, Backlund transformation method and other traditional methods will not be used here because it can be found in many other texts.

In addition to Adomian decomposition method, the newly developed variational iteration method will be applied. The variational iteration method (VIM) established by Ji-Huan He [5] is thoroughly used by mathematicians to handle a wide variety of scientific and engineering applications: linear and nonlinear, and homogeneous and inhomogeneous as well. It was shown that this method is effective and reliable for analytic and numerical purposes. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists. The VIM does not require specific treatments for nonlinear problems as in Adomian method, perturbation techniques, etc. In what follows, we present the main steps of the method.

Consider the differential equation

$$Lu + Nu = g(t), \tag{2.175}$$

where L and N are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term.

The variational iteration method presents a correction functional for Eq. (2.175) in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi)\left(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\right) d\xi, \tag{2.176}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$.

It is obvious now that the main steps of the He's variational iteration method require first the determination of the Lagrange multiplier $\lambda(\xi)$ that will be identified optimally. Integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\xi)$. In other words we can use

$$\begin{aligned} \int \lambda(\xi)u_n'(\xi)d\xi &= \lambda(\xi)u_n(\xi) - \int \lambda'(\xi)u_n(\xi)d\xi, \\ \int \lambda(\xi)u_n''(\xi)d\xi &= \lambda(\xi)u_n'(\xi) - \lambda'(\xi)u_n(\xi) + \int \lambda''(\xi)u_n(\xi)d\xi, \end{aligned} \tag{2.177}$$

and so on. The last two identities can be obtained by integrating by parts.

Having determined the Lagrange multiplier $\lambda(\xi)$, the successive approximations $u_{n+1}, n \geqslant 0$, of the solution u will be readily obtained upon using any selective function u_0. Consequently, the solution

$$u = \lim_{n\to\infty} u_n. \tag{2.178}$$

In other words, the correction functional (2.176) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations.

The variational iteration method will be used now to study the same examples used before in section 2.2 to help for comparison reasons.

Example 1. Use variational iteration method to solve the following inhomogeneous PDE

$$u_x + u_y = x + y, \quad u(0,y) = 0, \quad u(x,0) = 0. \tag{2.179}$$

Solution.

The correction functional for equation (2.179) is

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \lambda(\xi)\left(\frac{\partial u_n(\xi,y)}{\partial \xi} + \frac{\partial \tilde{u}_n(\xi,y)}{\partial y} - \xi - y\right) d\xi. \tag{2.180}$$

Using (2.177), the stationary conditions

$$\begin{aligned} 1 + \lambda|_{\xi=x} &= 0, \\ \lambda'|_{\xi=x} &= 0, \end{aligned} \tag{2.181}$$

follow immediately. This in turn gives

$$\lambda = -1. \tag{2.182}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (2.180) gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) - \int_0^x \left(\frac{\partial u_n(\xi,y)}{\partial \xi} + \frac{\partial u_n(\xi,y)}{\partial y} - \xi - y\right) d\xi, \quad n \geqslant 0. \tag{2.183}$$

As stated before, we can select $u_0(x,y) = u(0,y) = 0$ from the given conditions. Using this selection into (2.183) we obtain the following successive approximations

$$\begin{aligned} u_0(x,y) &= 0, \\ u_1(x,y) &= 0 - \int_0^x \left(\frac{\partial u_0(\xi,y)}{\partial \xi} + \frac{\partial u_0(\xi,y)}{\partial y} - \xi - y\right) d\xi = \frac{1}{2}x^2 + xy, \\ u_2(x,y) &= \frac{1}{2}x^2 + xy - \int_0^x \left(\frac{\partial u_1(\xi,y)}{\partial \xi} + \frac{\partial u_1(\xi,y)}{\partial y} - \xi - y\right) d\xi = xy, \\ u_3(x,y) &= xy - \int_0^x \left(\frac{\partial u_2(\xi,y)}{\partial \xi} + \frac{\partial u_2(\xi,y)}{\partial y} - \xi - y\right) d\xi = xy, \\ &\vdots \\ u_n(x,y) &= xy. \end{aligned} \tag{2.184}$$

The VIM admits the use of

$$u(x,y) = \lim_{n\to\infty} u_n(x,y), \tag{2.185}$$

that gives the exact solution by

$$u(x,y) = xy. \tag{2.186}$$

Example 2. Solve the following homogeneous partial differential equation by the variational iteration method

$$u_x - u_y = 0, \quad u(0,y) = y, \quad u(x,0) = x. \tag{2.187}$$

Solution.

The correction functional for Eq. (2.187) is

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \lambda(\xi)\left(\frac{\partial u_n(\xi,y)}{\partial \xi} - \frac{\partial \tilde{u}_n(\xi,y)}{\partial y}\right) d\xi. \tag{2.188}$$

This gives the stationary conditions

$$\begin{aligned} 1+\lambda|_{\xi=x} &= 0, \\ \lambda'|_{\xi=x} &= 0. \end{aligned} \tag{2.189}$$

This gives

$$\lambda = -1. \tag{2.190}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (2.188) gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) - \int_0^x \left(\frac{\partial u_n(\xi,y)}{\partial \xi} - \frac{\partial u_n(\xi,y)}{\partial y}\right) d\xi, \quad n \geqslant 0. \tag{2.191}$$

We now select $u_0(x,y) = u(0,y) = y$ from the given conditions. Using this selection into (2.191) we obtain the following successive approximations

$$\begin{aligned} u_0(x,y) &= y, \\ u_1(x,y) &= y - \int_0^x \left(\frac{\partial u_0(\xi,y)}{\partial \xi} - \frac{\partial u_0(\xi,y)}{\partial y}\right) d\xi = x+y, \\ u_2(x,y) &= x+y - \int_0^x \left(\frac{\partial u_1(\xi,y)}{\partial \xi} - \frac{\partial u_1(\xi,y)}{\partial y}\right) d\xi = x+y, \\ &\vdots \\ u_n(x,y) &= x+y. \end{aligned} \tag{2.192}$$

The VIM gives the exact solution by

$$u(x,y) = x+y. \tag{2.193}$$

Example 3. Use the variational iteration method to solve the following homogeneous partial differential equation

$$u_y + xu_x = 3u, \quad u(x,0) = x^2, \quad u(0,y) = 0. \tag{2.194}$$

Solution.

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + x\frac{\partial \tilde{u}_n(x,\xi)}{\partial x} - 3\tilde{u}_n(x,\xi)\right) d\xi. \tag{2.195}$$

As presented before, the stationary conditions are

$$\begin{aligned} 1 + \lambda|_{\xi=x} &= 0, \\ \lambda'|_{\xi=x} &= 0, \end{aligned} \tag{2.196}$$

and this gives

$$\lambda = -1. \tag{2.197}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (2.195) gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) - \int_0^y \left(\frac{\partial u_n(\xi,y)}{\partial \xi} + x\frac{\partial u_n(\xi,y)}{\partial x} - 3u_n\right) d\xi, \quad n \geqslant 0. \tag{2.198}$$

We can select $u_0(x,y) = x^2$ from the given conditions. Using this selection into (2.198) we obtain the following successive approximations

$$\begin{aligned}
u_0(x,y) &= x^2, \\
u_1(x,y) &= x^2 - \int_0^y \left(\frac{\partial u_0(x,\xi)}{\partial \xi} + x\frac{\partial u_0(x,\xi)}{\partial x} - 3u_0(x,\xi)\right) d\xi = x^2 + x^2 y, \\
u_2(x,y) &= x^2 + x^2 y - \int_0^y \left(\frac{\partial u_1(x,\xi)}{\partial \xi} + x\frac{\partial u_1(x,\xi)}{\partial x} - 3u_1(x,\xi)\right) d\xi \\
&= x^2 + x^2 y + \frac{1}{2!}x^2 y^2, \\
u_3(x,t) &= x^2 + x^2 y + \frac{1}{2!}x^2 y^2 - \int_0^y \left(\frac{\partial u_2(x,\xi)}{\partial \xi} + x\frac{\partial u_2(x,\xi)}{\partial x} - 3u_2(x,\xi)\right) d\xi \\
&= x^2 + x^2 y + \frac{1}{2!}x^2 y^2 + \frac{1}{3!}x^2 y^3, \\
&\vdots \\
u_n(x,y) &= x^2(1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \frac{1}{4!}y^4 + \cdots).
\end{aligned} \tag{2.199}$$

The VIM admits the use of

$$u(x,y) = \lim_{n\to\infty} u_n(x,y), \tag{2.200}$$

that gives the exact solution by

$$u(x,y) = x^2 e^y. \tag{2.201}$$

The last result is consistent with the result obtained before by Adomian method.

Example 4. Solve the following homogeneous partial differential equation

$$u_x - yu = 0, \quad u(0,y) = 1. \tag{2.202}$$

Solution.

The correction functional for Eq. (2.202) is

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \lambda(\xi)\left(\frac{\partial u_n(\xi,y)}{\partial \xi} - y\tilde{u}_n(\xi,y)\right) d\xi. \tag{2.203}$$

As concluded before we find

$$\lambda = -1. \tag{2.204}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (2.203) gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) - \int_0^x \left(\frac{\partial u_n(\xi,y)}{\partial \xi} - yu_n(\xi,y)\right) d\xi, \quad n \geqslant 0. \tag{2.205}$$

As stated before, we can select $u_0(x,y) = 1$ from the given conditions. Using this selection into (2.205) we obtain the following successive approximations

$$\begin{aligned}
u_0(x,y) &= 1,\\
u_1(x,y) &= 1 - \int_0^x \left(\frac{\partial u_0(\xi,y)}{\partial \xi} - yu_0(\xi,y)\right) d\xi = 1 + xy,\\
u_2(x,y) &= 1 + xy - \int_0^x \left(\frac{\partial u_1(\xi,y)}{\partial \xi} - yu_1(\xi,y)\right) d\xi = 1 + xy + \frac{1}{2!}x^2y^2,\\
u_3(x,t) &= 1 + xy + \frac{1}{2!}x^2y^2 - \int_0^x \left(\frac{\partial u_2(\xi,y)}{\partial \xi} - yu_2(\xi,y)\right) d\xi,\\
&= 1 + xy + \frac{1}{2!}x^2y^2 + \frac{1}{3!}x^3y^3,\\
&\ \vdots\\
u_n(x,y) &= 1 + xy + \frac{1}{2!}x^2y^2 + \frac{1}{3!}x^3y^3 + \cdots.
\end{aligned} \tag{2.206}$$

Using the identity

$$u(x,y) = \lim_{n\to\infty} u_n(x,y), \tag{2.207}$$

we obtain the exact solution by

$$u(x,y) = e^{xy}. \tag{2.208}$$

The last result is consistent with the result obtained before by Adomian method for Example 4.

Example 5. Solve the following homogeneous PDE

$$u_t + cu_x = 0, \quad u(x,0) = x, \tag{2.209}$$

where c is a constant.

Solution.

The correction functional for Eq. (2.209) is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + c\frac{\partial \tilde{u}_n(x,\xi)}{\partial x}\right) d\xi. \tag{2.210}$$

This also gives

$$\lambda = -1. \tag{2.211}$$

Consequently, we obtain the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + c\frac{\partial u_n(x,\xi)}{\partial x}\right) d\xi, \quad n \geqslant 0. \tag{2.212}$$

As stated before, we can select $u_0(x,t) = x$ from the given conditions. Using this selection into (2.212) we obtain the following successive approximations

$$\begin{aligned}
u_0(x,t) &= x,\\
u_1(x,t) &= x - \int_0^t \left(\frac{\partial u_0(x,\xi)}{\partial \xi} + c\frac{\partial u_0(x,\xi)}{\partial x}\right) d\xi = x - ct,\\
u_2(x,t) &= x - ct - \int_0^t \left(\frac{\partial u_1(x,\xi)}{\partial \xi} + c\frac{\partial u_1(x,\xi)}{\partial x}\right) d\xi = x - ct,\\
u_3(x,t) &= x - ct - \int_0^t \left(\frac{\partial u_2(x,\xi)}{\partial \xi} + c\frac{\partial u_2(x,\xi)}{\partial x}\right) d\xi = x - ct,\\
&\vdots\\
u_n(x,y) &= x - ct.
\end{aligned} \tag{2.213}$$

This gives the exact solution by

$$u(x,t) = x - ct. \tag{2.214}$$

Example 6. Solve the following partial differential equation

$$\begin{aligned}
u_x + u_y + u_z &= u,\\
u(0,y,z) &= 1 + e^y + e^z,
\end{aligned}$$

$$\begin{aligned} u(x,0,z) &= 1+e^x+e^z, \\ u(x,y,0) &= 1+e^x+e^y, \end{aligned} \tag{2.215}$$

where $u = u(x,y,z)$.

Solution.

The correction functional for equation (2.215) is

$$\begin{aligned} u_{n+1}(x,y,z) &= u_n(x,y,z) \\ &+ \int_0^x \lambda(\xi)\left(\frac{\partial u_n(\xi,y,z)}{\partial \xi} + \frac{\partial \tilde{u}_n(\xi,y,z)}{\partial y} + \frac{\partial \tilde{u}_n(\xi,y,z)}{\partial z} - \tilde{u}_n(\xi,y,z)\right) d\xi. \end{aligned} \tag{2.216}$$

Proceeding as before, we find

$$\lambda = -1. \tag{2.217}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (2.216) gives the iteration formula

$$\begin{aligned} u_{n+1}(x,y,z) &= u_n(x,y,z) \\ &- \int_0^x \left(\frac{\partial u_n(\xi,y,z)}{\partial \xi} + \frac{\partial u_n(\xi,y,z)}{\partial y} + \frac{\partial u_n(\xi,y,z)}{\partial z} - u_n(\xi,y,z)\right) d\xi, \quad n \geqslant 0. \end{aligned} \tag{2.218}$$

We can select $u_0(x,y,z) = 1+e^y+e^z$ from the given conditions. Using this selection into (2.218) we obtain the following successive approximations

$$\begin{aligned} u_0(x,y,z) &= 1+e^y+e^z, \\ u_1(x,y,z) &= 1+e^y+e^z \\ &\quad - \int_0^x \left(\frac{\partial u_0(\xi,y,z)}{\partial \xi} + \frac{\partial u_0(\xi,y,z)}{\partial y} + \frac{\partial u_0(\xi,y,z)}{\partial z} - u_0(\xi,y,z)\right) d\xi \\ &= 1+x+e^y+e^z, \\ u_2(x,y,z) &= 1+x+e^y+e^z \\ &\quad - \int_0^x \left(\frac{\partial u_1(\xi,y,z)}{\partial \xi} + \frac{\partial u_1(\xi,y,z)}{\partial y} + \frac{\partial u_1(\xi,y,z)}{\partial z} - u_1(\xi,y,z)\right) d\xi \\ &= 1+x+\frac{1}{2!}x^2+e^y+e^z, \\ u_3(x,y,z) &= 1+x+\frac{1}{2!}x^2+e^y+e^z \\ &\quad - \int_0^x \left(\frac{\partial u_2(\xi,y,z)}{\partial \xi} + \frac{\partial u_2(\xi,y,z)}{\partial y} + \frac{\partial u_2(\xi,y,z)}{\partial z} - u_2(\xi,y,z)\right) d\xi \end{aligned}$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + e^y + e^z,$$

$$\vdots$$

$$u_n(x,y,z) = (1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots) + e^y + e^z. \tag{2.219}$$

As a result, the exact solution is given by

$$u(x,y,z) = e^x + e^y + e^z. \tag{2.220}$$

This result was obtained before by Adomian method for Example 6.

Exercises 2.5

Use the variational iteration method to solve 1–20 from Exercises 2.2.

2.6 Method of Characteristics

In this section, the first order partial differential equation

$$au_x + bu_y = f(x,y) + ku, \quad u(0,y) = h(y) \tag{2.221}$$

will be investigated by using the traditional method of characteristics. It is important to note that a, b and f depend on x, y and u but not on the derivatives of u. In addition, we also assume that a, b and f are continuously differentiable of their arguments.

Assuming that $u(x,y)$ is a solution of (2.221), then by using the chain rule we obtain

$$du = u_x\,dx + u_y\,dy. \tag{2.222}$$

A close examination of (2.221) and (2.222) leads to the system of equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f(x,y) + ku}. \tag{2.223}$$

The pair

$$\frac{dx}{a} = \frac{dy}{b}, \tag{2.224}$$

gives the solution

$$bx - ay = c, \tag{2.225}$$

where c is a constant. We next consider the pair

$$\frac{dx}{a} = \frac{du}{f(x,y)+ku}, \tag{2.226}$$

or equivalently

$$\frac{dx}{a} = \frac{du}{f(x,\frac{bx-c}{a})+ku}. \tag{2.227}$$

Equation (2.227) can be rewritten as

$$\frac{du}{dx} - \frac{k}{a}u = \frac{1}{a}f(x,\frac{bx-c}{a}), \tag{2.228}$$

a first order linear ordinary differential equation. The integrating factor of (2.228) is given by

$$\mu = e^{-\frac{k}{a}x}. \tag{2.229}$$

Accordingly, the solution of (2.228) can be expressed in the form

$$u = G(x,c)+c_1, \tag{2.230}$$

where

$$c_1 = g(c), \tag{2.231}$$

where g is an arbitrary function. Eq. (2.230) can be rewritten as

$$u = G(x,c)+g(c). \tag{2.232}$$

Using the given condition leads to the determination of $g(c)$. Based on this and using (2.225), the solution $u(x,y)$ is readily obtained.

It is to be noted that first order partial differential equations in higher dimensions will not be discussed in this section. The decomposition method can handle such problems elegantly and easily if compared with the method of characteristics.

To give a clear overview of the method of characteristics, we will discuss some of the examples presented before in Section 2.2 and Section 2.3. The illustration can be used as a comparative study between the method of characteristics and the decomposition method.

Example 1. Use the method of characteristics to solve the first order partial differential equation

$$u_x+u_y = x+y, \quad u(x,0) = 0. \tag{2.233}$$

Solution.

Following the discussion presented above we set the system of equations

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{x+y}. \tag{2.234}$$

The left pair of (2.234) gives the solution

$$x - y = c, \tag{2.235}$$

where c is a constant. Using (2.235) into the right pair of (2.234) gives

$$\frac{dy}{1} = \frac{du}{2y + c}, \tag{2.236}$$

a separable differential equation that gives the solution

$$u(x,y) = y^2 + cy + c_1. \tag{2.237}$$

Recall that

$$c_1 = g(c), \tag{2.238}$$

then (2.237) becomes

$$u(x,y) = y^2 + y(x - y) + g(x - y), \tag{2.239}$$

where $g(x-y)$ is an arbitrary function. To determine $g(x-y)$ we substitute the given condition into (2.239) to obtain

$$g(x) = 0, \tag{2.240}$$

and therefore

$$g(x - y) = 0. \tag{2.241}$$

Consequently, the solution is given by

$$u(x,y) = xy. \tag{2.242}$$

Example 2. Use the method of characteristics to solve the first order partial differential equation

$$u_x - u_y = 0, \quad u(x,0) = x. \tag{2.243}$$

Solution.

Following Example 1, we set the system of equations

$$\frac{dx}{1} = \frac{dy}{-1}, \quad du = 0. \tag{2.244}$$

The first equation of (2.244) gives the solution

$$x + y = c, \tag{2.245}$$

where c is a constant. The second equation of (2.244) gives the solution

$$u(x,y) = c_1. \tag{2.246}$$

This means that

$$u(x,y) = c_1 = g(c) = g(x + y), \tag{2.247}$$

where $g(x+y)$ is an arbitrary function. Substituting the given condition into (2.247) gives

$$g(x)=x, \tag{2.248}$$

and therefore

$$g(x+y)=x+y. \tag{2.249}$$

The solution is therefore given by

$$u(x,y)=x+y, \tag{2.250}$$

obtained upon substituting (2.249) into (2.247).

Example 3. Use the method of characteristics to solve the first order partial differential equation

$$xu_x+u_y=x\sinh y+u, \quad u(0,y)=0. \tag{2.251}$$

Solution.

Following the examples discussed above we set the system of equations

$$\frac{dx}{x}=\frac{dy}{1}=\frac{du}{x\sinh y+u}. \tag{2.252}$$

The left pair of (2.252) gives the solution

$$x=ce^y. \tag{2.253}$$

Substituting $x=ce^y$ into the right pair of (2.252) gives

$$\frac{dy}{1}=\frac{du}{ce^y\sinh y+u}, \tag{2.254}$$

that can be reduced to the first order linear ordinary differential equation

$$\frac{du}{dy}-u=ce^y\sinh y, \tag{2.255}$$

that gives the solution

$$u(x,y)=e^y\left(c\cosh y+c_1\right). \tag{2.256}$$

Using the given condition and noting that $c=xe^{-y}$ gives the solution

$$u(x,y)=x\cosh y. \tag{2.257}$$

Example 4. Use the method of characteristics to solve the first order partial differential equation

$$u_x-yu=0, \quad u(0,y)=1. \tag{2.258}$$

Solution.

First we set the system of equations

$$\frac{dx}{1} = \frac{du}{yu}, \quad dy = 0. \tag{2.259}$$

The second equation gives the solution

$$y = c. \tag{2.260}$$

Substituting $y = c$ into the first equation of (2.259) gives

$$\frac{du}{cu} = \frac{dx}{1}, \tag{2.261}$$

which gives the solution

$$\ln u = cx + g(c), \tag{2.262}$$

or equivalently

$$\ln u = cx + g(y). \tag{2.263}$$

To determine $g(y)$ we substitute the given condition into (2.263) to obtain

$$g(y) = 0, \tag{2.264}$$

and therefore the solution is given by

$$u(x,y) = e^{xy}. \tag{2.265}$$

Exercises 2.6

Use the method of characteristics to solve the following first order partial differential equations:

1. $u_x + u_y = 2x + 2y$, $u(x,0) = x^2$

2. $u_x + u_y = u$, $u(x,0) = 1 + e^x$

3. $u_x + yu = 0$, $u(0,y) = 1$

4. $3u_x - 2u_y = 3\sin x$, $u(0,y) = 3y - 1$

5. $u_x + u_y = 2u$, $u(x,0) = e^x$

6. $xu_x + u_y = 2u$, $u(x,0) = x$

7. $u_x + yu_y = 2u$, $u(0,y) = y$

8. $u_x + u_y = \sinh x + \sinh y$, $u(0,y) = 1 + \cosh y$

9. $u_x + u_y = 2x + 2y$, $u(x,0) = 0$

10. $u_x + yu_y = 2xy^2 + 2x^2y^2$, $u(0,y) = 0$

11. $u_x + ku_y = 0$, $u(0,y) = y$

12. $u_x - u_y = 2$, $u(0,y) = -y$

13. $u_x + u_y = 1 + \cos y$, $u(0,y) = \sin y$

14. $xu_x + u_y = 2u$, $u(0,y) = 0$

15. $xu_x + yu_y = 2u$, $u(0,y) = 0$

16. $2u_x + 3u_y = 2u + e^y$, $u(0,y) = 1 + e^y$

17. $xu_x + yu_y = 2xyu$, $u(0,y) = 1$

18. $u_x + 4u_y = 5u$, $u(0,y) = e^y$

19. $xu_x + u_y = x\cosh y + u$, $u(0,y) = 0$

20. $u_x + yu_y = y\sinh x + u$, $u(0,y) = y$

2.7 Systems of Linear PDEs by Adomian Method

Systems of partial differential equations, linear or nonlinear, have attracted much concern in studying evolution equations that describe wave propagation, in investigating shallow water waves, and in examining the chemical reaction-diffusion model of Brusselator. The general ideas and the essential features of these systems are of wide applicability. The commonly used methods are the method of characteristics and the Riemann invariants among other methods. The existing techniques encountered some difficulties in terms of the size of computational work needed, especially when the system involves several partial differential equations.

To avoid the difficulties that usually arise from traditional strategies, the Adomian decomposition method will form a reasonable basis for studying systems of partial differential equations. The method, as we have seen before, has a useful attraction in that it provides the solution in a rapidly convergent power series with elegantly computable terms. The Adomian decomposition method transforms the system of partial differential equations into a set of recursive relations that can be easily examined. Due to simplicity reasons, we will use in this section Adomian decomposition method.

We first consider the system of partial differential equations written in an operator form

$$\begin{aligned} L_t u + L_x v &= g_1, \\ L_t v + L_x u &= g_2, \end{aligned} \tag{2.266}$$

with initial data

$$\begin{aligned} u(x,0) &= f_1(x), \\ v(x,0) &= f_2(x), \end{aligned} \tag{2.267}$$

where L_t and L_x are considered, without loss of generality, first order partial differential operators, and g_1 and g_2 are inhomogeneous terms. Applying the inverse operator L_t^{-1} to the system (2.266) and using the initial data (2.267) yields

$$\begin{aligned} u(x,t) &= f_1(x) + L_t^{-1} g_1 - L_t^{-1} L_x v, \\ v(x,t) &= f_2(x) + L_t^{-1} g_2 - L_t^{-1} L_x u. \end{aligned} \tag{2.268}$$

The Adomian decomposition method suggests that the linear terms $u(x,t)$ and $v(x,t)$ be decomposed by an infinite series of components

$$\begin{aligned} u(x,t) &= \textstyle\sum_{n=0}^{\infty} u_n(x,t), \\ v(x,t) &= \textstyle\sum_{n=0}^{\infty} v_n(x,t), \end{aligned} \tag{2.269}$$

where $u_n(x,t)$ and $v_n(x,t), n \geqslant 0$ are the components of $u(x,t)$ and $v(x,t)$ that will be elegantly determined in a recursive manner.

Substituting (2.269) into (2.268) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) &= f_1(x) + L_t^{-1} g_1 - L_t^{-1} \left(L_x (\sum_{n=0}^{\infty} v_n) \right), \\ \sum_{n=0}^{\infty} v_n(x,t) &= f_2(x) + L_t^{-1} g_2 - L_t^{-1} \left(L_x (\sum_{n=0}^{\infty} u_n) \right). \end{aligned} \tag{2.270}$$

Following Adomian analysis, the system (2.266) is transformed into a set of recursive relations given by

$$\begin{aligned} u_0(x,t) &= f_1(x) + L_t^{-1} g_1, \\ u_{k+1}(x,t) &= -L_t^{-1} \left(L_x v_k \right), \quad k \geqslant 0, \end{aligned} \tag{2.271}$$

and

$$\begin{aligned} v_0(x,t) &= f_2(x) + L_t^{-1} g_2, \\ v_{k+1}(x,t) &= -L_t^{-1} \left(L_x u_k \right), \quad k \geqslant 0. \end{aligned} \tag{2.272}$$

The zeroth components $u_0(x,t)$ and $v_0(x,t)$ are defined by all terms that arise from initial data and from integrating the inhomogeneous terms. Having defined the zeroth pair (u_0, v_0), the pair (u_1, v_1) can be determined recurrently by using (2.271) and (2.272). The remaining pairs (u_k, v_k), $k \geqslant 2$ can be easily determined in a parallel manner. Additional pairs for the decomposition series normally account for higher accuracy. Having determined the components of $u(x,t)$ and $v(x,t)$, the solution (u, v) of the system follows immediately in the form of a power series expansion upon using (2.269). The series obtained can be summed up in many cases to give a closed form solution. For concrete problems, the n-term approximants can be used for numerical purposes.

To give a clear overview of the content of this work, several illustrative examples have been selected to demonstrate the efficiency of the method.

Example 1. We first consider the linear system:

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x &= 0, \end{aligned} \tag{2.273}$$

with the initial data

$$\begin{aligned} u(x,0) &= e^x, \\ v(x,0) &= e^{-x}, \end{aligned} \tag{2.274}$$

Solution.

To derive the solution by using the decomposition method, we follow the recursive relations (2.271) and (2.272) to obtain

$$\begin{aligned} u_0(x,t) &= e^x, \\ u_{k+1}(x,t) &= -L_t^{-1}L_x(v_k), \quad k \geqslant 0, \end{aligned} \tag{2.275}$$

and

$$\begin{aligned} v_0(x,t) &= e^{-x}, \\ v_{k+1}(x,t) &= -L_t^{-1}L_x(u_k), \quad k \geqslant 0. \end{aligned} \tag{2.276}$$

The remaining components are thus determined by

$$\begin{aligned} u_1(x,t) &= te^{-x}, \quad v_1(x,t) = \ -te^x, \\ u_2(x,t) &= \frac{t^2}{2!}e^x, \quad v_2(x,t) = \frac{t^2}{2!}e^{-x}, \\ u_3(x,t) &= \frac{t^3}{3!}e^{-x}, \ v_3(x,t) = -\frac{t^3}{3!}e^x, \end{aligned} \tag{2.277}$$

and so on. Using (2.277) we obtain

$$\begin{aligned} u(x,t) &= e^x(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots) + e^{-x}(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots), \\ v(x,t) &= e^{-x}(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots) - e^x(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots), \end{aligned} \tag{2.278}$$

which has an exact analytical solution of the form

$$(u,v) = (e^x \cosh t + e^{-x} \sinh t, e^{-x} \cosh t - e^x \sinh t). \tag{2.279}$$

Example 2. Consider the linear system of partial differential equations

$$\begin{aligned} u_t + u_x + 2v &= 0, \\ v_t + v_x - 2u &= 0, \end{aligned} \tag{2.280}$$

with the initial data

$$\begin{aligned} u(x,0) &= \cos x, \\ v(x,0) &= \sin x. \end{aligned} \tag{2.281}$$

Solution.

Operating with L_t^{-1} on (2.280) and using (2.281) we obtain

$$\begin{aligned} u(x,t) &= \cos x - L_t^{-1}(2v + L_x u), \\ v(x,t) &= \sin x + L_t^{-1}(2u - L_x v). \end{aligned} \tag{2.282}$$

Using the series representation (2.269) into (2.282) admits the use of the system of recursive relations

$$\begin{aligned} u_0(x,t) &= \cos x, \\ u_{k+1}(x,t) &= -L_t^{-1}\left(2v_k + L_x(u_k)\right), \quad k \geqslant 0, \end{aligned} \tag{2.283}$$

and

$$\begin{aligned} v_0(x,t) &= \sin x, \\ v_{k+1}(x,t) &= L_t^{-1}\left(2u_k - L_x(v_k)\right), \quad k \geqslant 0. \end{aligned} \tag{2.284}$$

Consequently, the pair of zeroth components is defined by

$$(u_0, v_0) = (\cos x, \sin x). \tag{2.285}$$

Using (2.285) into (2.283) and (2.284) gives

$$\begin{aligned} u_1(x,t) &= -t\sin x, \\ v_1(x,t) &= t\cos x. \end{aligned} \tag{2.286}$$

In a like manner we obtain the pairs

$$\begin{aligned} (u_2, v_2) &= (-\frac{t^2}{2!}\cos x, -\frac{t^2}{2!}\sin x), \\ (u_3, v_3) &= (\frac{t^3}{3!}\sin x, -\frac{t^3}{3!}\cos x). \end{aligned} \tag{2.287}$$

Combining the results obtained above we obtain

$$\begin{aligned} u(x,t) &= \cos x(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots) - \sin x(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots), \\ v(x,t) &= \sin x(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots) + \cos x(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots), \end{aligned} \tag{2.288}$$

so that the pair (u, v) is known in a closed form by

$$(u, v) = \left(\cos(x+t), \sin(x+t)\right), \tag{2.289}$$

obtained upon using Taylor series and trigonometric identities.

Exercises 2.7

Use Adomian decomposition method to solve the following systems of first order partial differential equations:

1. $u_t + v_x - (u+v) = 0,\ v_t + u_x - (u+v) = 0,$
   $u(x,0) = \sinh x,\ v(x,0) = \cosh x$
2. $u_t + u_x - 2v = 0,\ v_t + v_x + 2u = 0,$
   $u(x,0) = \sin x,\ v(x,0) = \cos x$
3. $u_t + u_x - 2v_x = 0,\ v_t + v_x - 2u_x = 0,$
   $u(x,0) = \cos x,\ v(x,0) = \cos x$
4. $u_x - v_t = 2,\ v_x + u_t = 2,$
   $u(x,0) = x,\ v(x,0) = x$
5. $u_x + v_x = 2\cos x,\ u_t - v_t = 2\cos t,$
   $u(x,0) = \sin x,\ v(x,0) = \sin x$
6. $u_t - v_x + (u+v) = 0,\ v_t - u_x + (u+v) = 0,$
   $u(x,0) = \sinh x,\ v(x,0) = \cosh x$
7. $u_t + v_x - w_y = w,\ v_t + w_x + u_y = u,\ w_t + v_x - v_y = v,$
   $u(x,y,0) = -w(x,y,0) = \sin(x+y),\ v(x,y,0) = \cos(x+y)$
8. $u_t + u_x + 2w = 0,\ v_t + v_x + 2u = 0,\ w_t + w_x - 2u = 0,$
   $u(x,y,0) = \sin(x+y),\ v(x,y,0) = -w(x,y,0) = \cos(x+y)$
9. $u_x + v_y - w_t = 1,\ v_x + w_y + u_t = 1,\ w_x + u_y + v_t = 1,$
   $u(x,y,0) = x+y,\ v(x,y,0) = x+y,\ w(x,y,0) = x-y$
10. $u_x + v_t + w_y = e^x,\ v_y + w_x + u_t = e^y,\ w_t + u_y + v_x = e^t,$
    $u(x,y,0) = e^x,\ v(x,y,0) = e^y,\ w(x,y,0) = 1$

2.8 Systems of Linear PDEs by Variational Iteration Method

In this section we will apply the variational iteration method for solving systems of linear partial differential equations. We write a system in an operator form by

$$\begin{aligned} L_t u + R_1(u,v) &= g_1, \\ L_t v + R_2(u,v) &= g_2, \end{aligned} \tag{2.290}$$

where $u = u(x,t)$, with initial data

$$\begin{aligned} u(x,0) &= f_1(x), \\ v(x,0) &= f_2(x), \end{aligned} \tag{2.291}$$

where L_t is considered a first order partial differential operator, and $R_j, 1 \leqslant j \leqslant 3$ are linear operators, and g_1, and g_2 are source terms. Following the discussion presented above for variational iteration method, the following correction functionals for the system (2.290) can be set in the form

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda_1 \left(Lu_n(\xi) + R_1(\tilde{u}_n, \tilde{v}_n) - g_1(\xi)\right) d\xi, \\ v_{n+1}(x,t) &= v_n(x,t) + \int_0^t \lambda_2 \left(Lv_n(\xi) + R_2(\tilde{u}_n, \tilde{v}_n) - g_2(\xi)\right) d\xi, \end{aligned} \tag{2.292}$$

where $\lambda_j, j = 1,2$ are general Lagrange multipliers, which can be identified optimally via the variational theory, and $\tilde{u}_n$, and $\tilde{v}_n$ as restricted variations which means $\delta\tilde{u}_n = 0$, and $\delta\tilde{v}_n = 0$. The Lagrange multipliers $\lambda_j, j = 1,2$ will be identified optimally via integration by parts as introduced before. The successive approximations $u_{n+1}(x,t)$ and $v_{n+1}(x,t), n \geqslant 0$, of the solutions $u(x,t)$ and $v(x,t)$ will follow immediately upon using the obtained Lagrange multipliers and by using selective functions u_0 and v_0. The initial values may be used for the selective zeroth approximations. With the Lagrange multipliers λ_j determined, several approximations $u_j(x,t), v_j(x,t), j \geqslant 0$ can be computed. Consequently, the solutions are given by

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t), \\ v(x,t) &= \lim_{n\to\infty} v_n(x,t). \end{aligned} \tag{2.293}$$

To give a clear overview of the analysis introduced above, the two examples that were studied before will be used to explain the technique that we summarized before, therefore we will keep the same numbers.

Example 1. We first consider the linear system:

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x &= 0, \end{aligned} \tag{2.294}$$

with the initial data

$$\begin{aligned} u(x,0) &= e^x, \\ v(x,0) &= e^{-x}, \end{aligned} \tag{2.295}$$

where $u = u(x,t)$ and $v = v(x,t)$.

Solution.

The correction functionals for (2.294) read

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda_1(\xi) \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{v}_n(x,\xi)}{\partial x}\right) d\xi, \\ v_{n+1}(x,t) &= v_n(x,t) + \int_0^t \lambda_2(\xi) \left(\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{u}_n(x,\xi)}{\partial x}\right) d\xi. \end{aligned} \tag{2.296}$$

This gives the stationary conditions

$$\begin{aligned} 1+\lambda_1|_{\xi=t} &= 0, \\ \lambda_1'(\xi=t) &= 0, \end{aligned} \tag{2.297}$$

and

$$\begin{aligned} 1+\lambda_2|_{\xi=t} &= 0, \\ \lambda_2'(\xi=t) &= 0. \end{aligned} \tag{2.298}$$

As a result we find

$$\lambda_1 = \lambda_2 = -1. \tag{2.299}$$

Substituting these values of the Lagrange multipliers into the functionals (2.296) gives the iteration formulas

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial v_n(x,\xi)}{\partial x} \right) d\xi, \\ v_{n+1}(x,t) &= v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} \right) d\xi, \quad n \geqslant 0. \end{aligned} \tag{2.300}$$

We can select $u_0(x,t) = e^x, v_0(x,t) = e^{-x}$ by using the given initial values. Accordingly, we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= e^x, \\ v_0(x,t) &= e^{-x}, \\ u_1(x,t) &= e^x + te^{-x}, \\ v_1(x,t) &= e^{-x} - te^x, \\ u_2(x,t) &= e^x + te^{-x} + \frac{1}{2!}t^2e^x, \\ v_2(x,t) &= e^{-x} - te^x + \frac{1}{2!}t^2e^{-x}, \\ u_3(x,t) &= e^x + te^{-x} + \frac{1}{2!}t^2e^x + \frac{1}{3!}t^3e^{-x}, \\ v_3(x,t) &= e^{-x} - te^x + \frac{1}{2!}t^2e^{-x} - \frac{1}{3!}t^3e^x, \\ &\vdots \\ u_n(x,t) &= e^x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) + e^{-x}\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right), \\ v_n(x,t) &= e^{-x}\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - e^x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right). \end{aligned} \tag{2.301}$$

Recall that

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t), \\ v(x,t) &= \lim_{n\to\infty} v_n(x,t). \end{aligned} \tag{2.302}$$

Consequently, the exact analytical solutions are of the form

$$(u,v) = (e^x\cosh t + e^{-x}\sinh t, e^{-x}\cosh t - e^x\sinh t), \tag{2.303}$$

obtained by using Taylor series for $\sinh t$ and $\cosh t$.

Example 2. Consider the linear system of partial differential equations

$$\begin{aligned} u_t + u_x + 2v &= 0, \\ v_t + v_x - 2u &= 0, \end{aligned} \tag{2.304}$$

with the initial data

$$\begin{aligned} u(x,0) &= \cos x, \\ v(x,0) &= \sin x. \end{aligned} \tag{2.305}$$

Solution.

The correction functionals for (2.304) read

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda_1(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{u}_n(x,\xi)}{\partial x} + 2\tilde{v}_n(x,\xi)\right) d\xi, \\ v_{n+1}(x,t) &= v_n(x,t) + \int_0^t \lambda_2(\xi)\left(\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{v}_n(x,\xi)}{\partial x} - 2\tilde{u}_n(x,\xi)\right) d\xi. \end{aligned} \tag{2.306}$$

As a result, the stationary conditions are given by

$$\begin{aligned} 1 + \lambda_1|_{\xi=t} &= 0, \quad \lambda_1'(\xi = t) = 0, \\ 1 + \lambda_2|_{\xi=t} &= 0, \quad \lambda_2'(\xi = t) = 0. \end{aligned} \tag{2.307}$$

As a result we find

$$\lambda_1 = \lambda_2 = -1. \tag{2.308}$$

Substituting these values of the Lagrange multipliers into the functionals (2.306) gives the iteration formulas

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} + 2v_n(x,\xi)\right) d\xi, \\ v_{n+1}(x,t) &= v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial v_n(x,\xi)}{\partial x} - 2u_n(x,\xi)\right) d\xi, \quad n \geqslant 0. \end{aligned} \tag{2.309}$$

We can select $u_0(x,t) = \cos x, v_0(x,t) = \sin x$ by using the given initial values. Accordingly, we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= \cos x, \\ v_0(x,t) &= \sin x, \\ u_1(x,t) &= \cos x - t\sin x, \\ v_1(x,t) &= \sin x + t\cos x, \\ u_2(x,t) &= \cos x - t\sin x - \frac{t^2}{2!}\cos x, \end{aligned}$$

$$
\begin{aligned}
v_2(x,t) &= \sin x + t\cos x - \frac{t^2}{2!}\cos x,\\
u_3(x,t) &= \cos x - t\sin x - \frac{t^2}{2!}\cos x + \frac{t^3}{3!}\sin x,\\
v_3(x,t) &= \sin x + t\cos x - \frac{t^2}{2!}\cos x - \frac{t^3}{3!}\cos x,\\
&\ \vdots\\
u_n(x,t) &= \cos x\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\cdots\right) - \sin x\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots\right),\\
v_n(x,t) &= \sin x\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots\right) + \cos x\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\right).
\end{aligned}
\tag{2.310}
$$

Recall that

$$
\begin{aligned}
u(x,t) &= \lim_{n\to\infty} u_n(x,t),\\
v(x,t) &= \lim_{n\to\infty} v_n(x,t).
\end{aligned}
\tag{2.311}
$$

This gives the pair of solutions (u,v) in a closed form by

$$
(u,v) = (\cos(x+t), \sin(x+t)). \tag{2.312}
$$

Example 3. Consider the linear system of partial differential equations

$$
\begin{aligned}
u_x + v_y - w_t &= 1,\\
v_x + w_y + u_t &= 1,\\
w_x + u_y + v_t &= 1,
\end{aligned}
\tag{2.313}
$$

with the given data

$$
\begin{aligned}
u(0,y,t) &= y+t,\\
v(0,y,t) &= y-t,\\
w(0,y,t) &= -y+t.
\end{aligned}
\tag{2.314}
$$

Solution.

The correction functionals for this system read

$$
\begin{aligned}
u_{n+1}(x,y,t) &= u_n(x,y,t) + \int_0^x \lambda_1(\xi)\left(\frac{\partial u_n}{\partial \xi} + \frac{\partial \tilde{v}_n}{\partial y} - \frac{\partial \tilde{w}_n}{\partial t} - 1\right) d\xi,\\
v_{n+1}(x,y,t) &= v_n(x,y,t) + \int_0^x \lambda_2(\xi)\left(\frac{\partial v_n}{\partial \xi} + \frac{\partial \tilde{w}_n}{\partial y} + \frac{\partial \tilde{u}_n}{\partial t} - 1\right) d\xi,\\
w_{n+1}(x,y,t) &= w_n(x,y,t) + \int_0^x \lambda_3(\xi)\left(\frac{\partial w_n}{\partial \xi} + \frac{\partial \tilde{u}_n}{\partial y} + \frac{\partial \tilde{v}_n}{\partial t} - 1\right) d\xi.
\end{aligned}
\tag{2.315}
$$

As a result, the stationary conditions are given by

$$
\begin{aligned}
1+\lambda_1|_{\xi=x} &= 0, \quad \lambda_1'(\xi=x)=0,\\
1+\lambda_2|_{\xi=x} &= 0, \quad \lambda_2'(\xi=x)=0,\\
1+\lambda_3|_{\xi=x} &= 0, \quad \lambda_3'(\xi=x)=0,
\end{aligned}
\tag{2.316}
$$

As a result we find

$$
\lambda_1=\lambda_2=\lambda_3=-1. \tag{2.317}
$$

We can select $u_0(x,y,t)=y+t, v_0(x,y,t)=y-t, w_0(x,y,t)=-y+t$ by using the given initial values. Accordingly, we obtain the following successive approximations

$$
\begin{aligned}
u_0(x,y,t)&=y+t, \quad v_0(x,y,t)=y-t, \quad w_0(x,y,t)=-y+t,\\
u_1(x,y,t)&=x+y+t,\ v_1(x,y,t)=x+y-t,\ w_1(x,y,t)=x-y+t,\\
&\vdots\\
u_n(x,y,t)&=x+y+t,\ v_n(x,y,t)=x+y-t,\ w_n(x,y,t)=x-y+t,
\end{aligned}
\tag{2.318}
$$

where $n \geqslant 2$. This gives the following solutions

$$
u(x,y,t)=x+y+t, \quad v(x,y,t)=x+y-t, \quad w(x,y,t)=x-y+t. \tag{2.319}
$$

Exercises 2.8

Use the variational iteration method to solve the systems of first order partial differential equations of Exercises 2.7.

References

1. G. Adomian, Nonlinear Stochastic Operator Equations, Academic Press, San Diego, (1986).
2. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, (1994).
3. Y. Cherruault, Convergence of Adomian's method, *Kybernotes*, 18(20), 31–38, (1990).
4. Y. Cherruault and G. Adomian, Decomposition methods: a new proof of convergence, *Math. Comput. Modelling*, 18(12), 103–106, (1993).
5. J. H. He, A variational iteration method-a kind of nonlinear analytical technique: Some examples, *Int. J. Nonlinear Mech.*, 34(4), 699–708, (1999).
6. R. Rach, A. Baghdasarian, and G. Adomian, Differential equations with singular coefficients, *Appl. Math. Lett.*, 47(2/3), 179-184, (1992).
7. A.M. Wazwaz, A reliable modification of Adomian's decomposition method, *Appl. Math. Comput.*, 92(1), 1–7, (1998).
8. A.M.Wazwaz, Partial Differential Equations: Methods and Applications, Balkema Publishers, Leiden, (2002).
9. A.M. Wazwaz, The modified decomposition method for analytic treatment of differential equations, *Appl. Math. Comput.*, 173(1), 165–176, (2006).
10. A.M. Wazwaz, A new algorithm for solving differential equations of the Lane-Emden type, *Appl. Math. Comput.*, 118(2/3), 287–310, (2001).

Chapter 3
One Dimensional Heat Flow

3.1 Introduction

In Chapter 1, it was indicated that many phenomena of physics and engineering are expressed by partial differential equations PDEs. The PDE is termed a *Boundary Value Problem* (BVP) if the boundary conditions of the dependent variable u and some of its partial derivatives are often prescribed. However, the PDE is called an *Initial Value Problem* (IVP) if the initial conditions of the dependent variable u are prescribed at the starting time $t = 0$. Moreover, the PDE is termed *Initial-Boundary Value Problem* (IBVP) if both initial conditions and boundary conditions are prescribed.

In this chapter, we will study the one dimensional heat flow. Our concern will be focused on solving the PDE in conjunction with the prescribed initial and boundary conditions. The Adomian decomposition method [1–2] and the variational iteration method [4–5] will be used to handle the heat flow PDEs. Moreover, the well-known traditional method of the separation of variables will be used as well.

In this section we will study the physical problem of heat conduction in a rod of length L. The temperature distribution of a rod is governed by an initial-boundary value problem [3,6,8] that is often defined by:

1. **Partial Differential Equation** (PDE) that governs the heat flow in a rod. The PDE can be formally shown to satisfy

$$u_t = \overline{k} u_{xx}, \quad 0 < x < L, \quad t > 0, \tag{3.1}$$

where $u \equiv u(x,t)$ represents the temperature of the rod at the position x at time t, and $\overline{k}$ is the thermal diffusivity of the material that measures the rod ability to heat conduction.

2. **Boundary Conditions** (BC) that describe the temperature u at both ends of the rod. One form of the BC is given by the Dirichlet boundary conditions

$$u(0,t) = 0, \quad t \geqslant 0,$$

$$u(L,t) = 0, \quad t \geqslant 0. \tag{3.2}$$

The given boundary conditions in (3.2) indicate that the ends of the rod are kept at 0 temperature. As indicated in Chapter 1, boundary conditions are given in three types, namely: Dirichlet boundary conditions, Neumann boundary conditions, and mixed boundary conditions. In addition, the boundary conditions come in a homogeneous or inhomogeneous type.

3. **Initial Condition** (IC) that describes the initial temperature u at time $t = 0$. The IC is usually defined by

$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant L. \tag{3.3}$$

Based on these definitions, the initial-boundary value problem that controls the heat conduction in a rod is given by

$$\begin{array}{lll} \text{PDE} & u_t = \bar{k}u_{xx}, & 0 < x < L,\ t > 0, \\ \text{BC} & u(0,t) = 0, & t \geqslant 0, \\ & u(L,t) = 0, & t \geqslant 0, \\ \text{IC} & u(x,0) = f(x), & 0 \leqslant x \leqslant L. \end{array} \tag{3.4}$$

As stated before we will focus our discussions on determining a particular solution of the heat equation (3.4), recalling that the general solution is of little use.

It is of interest to note that the PDE in (3.4) arises in two different types, namely:

1. **Homogeneous Heat Equation**: This type of equations is often given by

$$u_t = \bar{k}u_{xx}, \quad 0 < x < L,\ t > 0. \tag{3.5}$$

Further, heat equation with a lateral heat loss is formally derived as a homogeneous PDE of the form

$$u_t = \bar{k}u_{xx} - u, \quad 0 < x < L,\ t > 0. \tag{3.6}$$

2. **Inhomogeneous Heat Equation**: This type of equations is often given by

$$u_t = \bar{k}u_{xx} + g(x), \quad 0 < x < L,\ t > 0, \tag{3.7}$$

where $g(x)$ is called the heat source which is independent of time.

3.2 The Adomian Decomposition Method

In this chapter the Adomian decomposition method will be used in a similar way to that used in the previous chapter. As shown before, the method introduces the solution of any equation in a series form, where the components of the solution are elegantly computed by a recursive manner. Further, the resulting series may converge to a closed form solution if exact solution exists [7–8]. In the case where a closed form solution is not obtainable, a truncated n-term approximation is usually used for approximations and numerical purposes. It was formally proved by many researchers that the method provides the solution in a rapidly convergent power series.

An important point can be made here in that the method attacks the problem, homogeneous or inhomogeneous, in a straightforward manner without any need for transformation formulas. Further, there is no need to change the inhomogeneous boundary conditions to homogeneous conditions as required by the method of separation of variables that will be discussed later. The formal steps of the decomposition method have been outlined before in Chapter 2. In what follows, we introduce a framework for implementing this method to solve the one dimensional heat equation.

Without loss of generality, we study the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_t = u_{xx}, & 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = 0, & t \geqslant 0, \\ & u(L,t) = 0, & t \geqslant 0, \\ \text{IC} & u(x,0) = f(x), & 0 \leqslant x \leqslant \pi, \end{array} \tag{3.8}$$

to achieve our goal.

To begin our analysis, we first rewrite (3.8) in an operator form by

$$L_t\, u(x,t) = L_x\, u(x,t), \tag{3.9}$$

where the differential operators L_t and L_x are defined by

$$L_t = \frac{\partial}{\partial t}, \quad L_x = \frac{\partial^2}{\partial x^2}. \tag{3.10}$$

It is obvious that the integral operators L_t^{-1} and L_x^{-1} exist and may be regarded as one and two-fold definite integrals respectively defined by

$$L_t^{-1}(\cdot) = \int_0^t (\cdot)dt, \quad L_x^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)\, dx\, dx. \tag{3.11}$$

This means that

$$L_t^{-1} L_t\, u(x,t) = u(x,t) - u(x,0). \tag{3.12}$$

Applying L_t^{-1} to both sides of (3.9) and using the initial condition we find

$$u(x,t) = f(x) + L_t^{-1}(L_x\, u(x,t)). \tag{3.13}$$

The decomposition method defines the unknown function $u(x,t)$ into a sum of components defined by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{3.14}$$

where the components $u_0(x,t), u_1(x,t), u_2(x,t), \cdots$ are to be determined. Substituting (3.14) into both sides of (3.13) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right), \tag{3.15}$$

or equivalently

$$u_0 + u_1 + u_2 + \cdots = f(x) + L_t^{-1}\left(L_x(u_0 + u_1 + u_2 + \cdots)\right). \tag{3.16}$$

The decomposition method suggests that the zeroth component $u_0(x,t)$ is identified by the terms arising from the initial/boundary conditions and from source terms. The remaining components of $u(x,t)$ are determined in a recursive manner such that each component is determined by using the previous component. Accordingly, we set the recurrence scheme

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_{k+1}(x,t) &= L_t^{-1}\left(L_x\left(u_k(x,t)\right)\right), \quad k \geqslant 0, \end{aligned} \tag{3.17}$$

for the complete determination of the components $u_n(x,t), n \geqslant 0$. In view of (3.17), the components $u_0(x,t), u_1(x,t), u_2(x,t), \cdots$ are determined individually by

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_1(x,t) &= L_t^{-1}L_x(u_0) = f''(x)t, \\ u_2(x,t) &= L_t^{-1}L_x(u_1) = f^{(4)}(x)\frac{t^2}{2!}, \\ u_3(x,t) &= L_t^{-1}L_x(u_2) = f^{(6)}(x)\frac{t^3}{3!}, \\ &\vdots \end{aligned} \tag{3.18}$$

Other components can be determined in a like manner as far as we like. The accuracy level can be effectively improved by increasing the number of components determined. Having determined the components $u_0, u_1, \cdots$, the solution $u(x,t)$ of the PDE is thus obtained in a series form given by

$$u(x,t) = \sum_{n=0}^{\infty} f^{(2n)}(x)\frac{t^n}{n!}, \tag{3.19}$$

obtained by substituting (3.18) into (3.14).

An important conclusion can be made here; the solution (3.19) is obtained by using the initial condition only without using the boundary conditions. This solution is obtained by using the inverse operator L_t^{-1}. The obtained solution can be used to show that it satisfies the given boundary conditions.

However, the solution (3.19) can also be obtained by using the inverse operator L_x^{-1}. In fact, the solution obtained in this way requires the use of boundary conditions and initial condition as well. This leads to an important conclusion that solving the PDE in the t direction reduces the size of computational work. This important observation will be confirmed through examples that will be discussed later.

To give a clear overview of the content of the decomposition method, we have chosen several examples, homogeneous and inhomogeneous, to illustrate the discussion given above.

3.2.1 Homogeneous Heat Equations

The Adomian decomposition method will be used to solve the following homogeneous heat equations [6] where the boundary conditions are also homogeneous.

Example 1. Use the Adomian decomposition method to solve the initial-boundary value problem

$$\begin{array}{lll} \text{PDE} & u_t = u_{xx}, & 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = 0, & t \geqslant 0, \\ & u(\pi,t) = 0, & t \geqslant 0, \\ \text{IC} & u(x,0) = \sin x. & \end{array} \tag{3.20}$$

Solution.

In an operator form, Equation (3.20) can be written as

$$L_t u(x,t) = L_x u(x,t). \tag{3.21}$$

Applying L_t^{-1} to both sides of (21) and using the initial condition we find

$$u(x,t) = \sin x + L_t^{-1}\left(L_x u(x,t)\right). \tag{3.22}$$

We next define the unknown function $u(x,t)$ by a sum of components defined by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{3.23}$$

Substituting the decomposition (3.23) into both sides of (3.22) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right), \tag{3.24}$$

or equivalently

$$u_0 + u_1 + u_2 + \cdots = \sin x + L_t^{-1}\left(L_x\left(u_0 + u_1 + u_2 + \cdots\right)\right). \tag{3.25}$$

Identifying the zeroth component $u_0(x,t)$ as assumed before and following the recursive algorithm (3.17) we obtain

$$
\begin{aligned}
u_0(x,t) &= \sin x,\\
u_1(x,t) &= L_t^{-1}\left(L_x\left(u_0\right)\right) = -t\sin x,\\
u_2(x,t) &= L_t^{-1}\left(L_x\left(u_1\right)\right) = \frac{1}{2!}t^2\sin x,\\
&\ \ \vdots
\end{aligned}
\tag{3.26}
$$

Consequently, the solution $u(x,t)$ in a series form is given by

$$
\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots\\
&= \sin x\left(1 - t + \frac{1}{2!}t^2 - \cdots\right),
\end{aligned}
\tag{3.27}
$$

and in a closed form by

$$
u(x,t) = e^{-t}\sin x, \tag{3.28}
$$

obtained upon using the Taylor expansion of e^{-t}. The solution (3.28) satisfies the PDE, the boundary conditions and the initial condition.

Example 2. Use the Adomian decomposition method to solve the initial-boundary value problem

$$
\begin{array}{lrl}
\text{PDE} & u_t = u_{xx}, & 0 < x < \pi,\ t > 0,\\
\text{BC} & u(0,t) = e^{-t}, & t \geqslant 0,\\
 & u(\pi,t) = \pi - e^{-t}, & t \geqslant 0,\\
\text{IC} & u(x,0) = x + \cos x. &
\end{array}
\tag{3.29}
$$

Solution.

It is important to note that the boundary conditions in this example are inhomogeneous. The decomposition method does not require any restrictive assumption on boundary conditions when approaching the problem in the t direction or in the x direction.

Applying L_t^{-1} to both sides of the operator form

$$
L_t u(x,t) = L_x u(x,t), \tag{3.30}
$$

and using the initial condition we find

$$
u(x,t) = x + \cos x + L_t^{-1}\left(L_x(u(x,t))\right). \tag{3.31}
$$

Substituting the decomposition series

$$
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{3.32}
$$

into both sides of (3.31) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = x + \cos x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{3.33}$$

Identifying the component $u_0(x,t)$ and following the recursive algorithm (3.17) we obtain

$$\begin{aligned}
u_0(x,t) &= x + \cos x,\\
u_1(x,t) &= L_t^{-1}\left(L_x\left(u_0\right)\right) = -t\cos x,\\
u_2(x,t) &= L_t^{-1}\left(L_x\left(u_1\right)\right) = \frac{1}{2!}t^2\cos x,\\
u_3(x,t) &= L_t^{-1}\left(L_x\left(u_2\right)\right) = -\frac{1}{3!}t^3\cos x,\\
&\ \,\vdots
\end{aligned} \tag{3.34}$$

Consequently, the solution $u(x,t)$ in a series form is given by

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots\\
&= x + \cos x\left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \cdots\right),
\end{aligned} \tag{3.35}$$

and in a closed form by

$$u(x,t) = x + e^{-t}\cos x, \tag{3.36}$$

obtained upon using the Taylor expansion for e^{-t}.

It is important to point out that the decomposition method has been used in the last two examples in the t-dimension by using the differential operator L_t and by operating with the inverse operator L_t^{-1}. However, the method can also be used in the x-dimension. Although the x-solution can be obtained in a similar fashion, however it requires more computational work if compared with the solution in the t-dimension. This can be attributed to the fact that we use the initial condition IC only in using the t-dimension, whereas a boundary condition and an initial condition are used to obtain the solution in the x-direction. This can be clearly illustrated by discussing the following examples.

Example 3. Use the decomposition method in the x-direction to solve the initial-boundary value problem of Example 1 given by

$$\begin{aligned}
&\text{PDE} && u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0,\\
&\text{BC} && u(0,t) = 0, \quad t \geqslant 0,\\
& && u(\pi,t) = 0, \quad t \geqslant 0,\\
&\text{IC} && u(x,0) = \sin x.
\end{aligned} \tag{3.37}$$

Solution.

In an operator form, Eq. (3.37) can be written by

$$L_x u(x,t) = L_t u(x,t), \quad 0 < x < \pi, \quad t > 0 \tag{3.38}$$

where

$$L_x = \frac{\partial^2}{\partial x^2}, \tag{3.39}$$

so that L_x^{-1} is a two-fold integral operator defined by

$$L_x^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)\,dx\,dx. \tag{3.40}$$

This means that

$$L_x^{-1} L_x u = u(x,t) - u(0,t) - x u_x(0,t) = u(x,t) - x u_x(0,t). \tag{3.41}$$

Applying L_x^{-1} to both sides of (3.38) and using the proper boundary condition we obtain

$$\begin{aligned} u(x,t) &= x u_x(0,t) + L_x^{-1}\left(L_t\, u(x,t)\right), \\ &= x h(t) + L_x^{-1}\left(L_t\, u(x,t)\right), \end{aligned} \tag{3.42}$$

where

$$h(t) = u_x(0,t). \tag{3.43}$$

Substituting the decomposition (3.23) into both sides of (3.42) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = x h(t) + L_x^{-1}\left(L_t\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{3.44}$$

Proceeding as before, the components of $u(x,t)$ are determined by

$$\begin{aligned} u_0(x,t) &= x h(t), \\ u_1(x,t) &= L_x^{-1}\left(L_t u_0\right) = \frac{1}{3!} x^3 h'(t) \\ u_2(x,t) &= L_x^{-1}\left(L_t u_1\right) = \frac{1}{5!} x^5 h''(t) \\ &\vdots \end{aligned} \tag{3.45}$$

Accordingly, the solution in a series form is given by

$$u(x,t) = x h(t) + \frac{1}{3!} x^3 h'(t) + \frac{1}{5!} x^5 h''(t) + \cdots. \tag{3.46}$$

The unknown function $h(t)$ should be derived so that the solution $u(x,t)$ is completely determined. This can be achieved by using the initial condition

$$u(x,0) = \sin x. \tag{3.47}$$

Substituting $t = 0$ into (3.46), using the initial condition (3.47), and using the Taylor expansion of $\sin x$ we find

$$xh(0)+\frac{1}{3!}x^3h'(0)+\frac{1}{5!}x^5h''(0)+\cdots=\sin x=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots \tag{3.48}$$

Equating the coefficients of like powers of x in both sides gives

$$h(0)=1,\, h'(0)=-1,\, h''(0)=1,\cdots \tag{3.49}$$

Using the Taylor expansion of $h(t)$ and the result (3.49) we obtain

$$\begin{aligned} h(t) &= h(0)+h'(0)t+\frac{1}{2!}h''(0)t^2-\frac{1}{3!}h'''(0)t^3+\cdots \\ &= 1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\cdots \\ &= e^{-t}. \end{aligned} \tag{3.50}$$

Combining (3.46) and (3.50), the solution $u(x,t)$ in a series form is

$$u(x,t)=e^{-t}\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots\right), \tag{3.51}$$

and in a closed form is given by

$$u(x,t)=e^{-t}\sin x. \tag{3.52}$$

Example 4. Use the decomposition method in the x-direction to solve the initial-boundary value problem of Example 2.

Solution.

In an operator form we set

$$L_x\, u(x,t)=L_t\, u(x,t),\quad 0<x<\pi,\, t>0 \tag{3.53}$$

where

$$L_x=\frac{\partial^2}{\partial x^2}, \tag{3.54}$$

so that L_x^{-1} is a two-fold integral operator defined by

$$L_x^{-1}(\cdot)=\int_0^x\int_0^x(\cdot)dx\,dx. \tag{3.55}$$

Applying L_x^{-1} to both sides of (3.53) and using the first boundary condition

$$\begin{aligned} u(x,t) &= u(0,t)+xu_x(0,t)+L_x^{-1}\left(L_tu(x,t)\right), \\ &= e^{-t}+xh(t)+L_x^{-1}\left(L_tu(x,t)\right), \end{aligned} \tag{3.56}$$

where

$$h(t)=u_x(0,t). \tag{3.57}$$

Substituting the decomposition (3.23) into both sides of (3.56) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = e^{-t} + xh(t) + L_x^{-1}\left(L_t\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{3.58}$$

Proceeding as before, the components of $u(x,t)$ are determined by

$$\begin{aligned} u_0(x,t) &= e^{-t} + xh(t), \\ u_1(x,t) &= L_x^{-1}\left(L_t u_0\right) = -\frac{1}{2!}x^2 e^{-t} + \frac{1}{3!}x^3 h'(t) \\ u_2(x,t) &= L_x^{-1}\left(L_t u_1\right) = \frac{1}{4!}x^4 e^{-t} + \frac{1}{5!}x^5 h''(t) \\ &\ \vdots \end{aligned} \tag{3.59}$$

Accordingly, the solution in a series form is given by

$$\begin{aligned} u(x,t) &= e^{-t}\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) \\ &\quad + xh(t) + \frac{1}{3!}x^3 h'(t) + \frac{1}{5!}x^5 h''(t) + \cdots, \\ &= e^{-t}\cos x + xh(t) + \frac{1}{3!}x^3 h'(t) + \frac{1}{5!}x^5 h''(t) + \cdots. \end{aligned} \tag{3.60}$$

It remains to determine the function $h(t)$ in order to completely determine $u(x,t)$. This can be done by using the initial condition

$$u(x,0) = x + \cos x. \tag{3.61}$$

Using initial condition (3.61) into (3.60) we find

$$x + \cos x = \cos x + xh(0) + \frac{1}{3!}x^3 h'(0) + \frac{1}{5!}x^5 h''(0) + \cdots. \tag{3.62}$$

Equating the coefficients of like powers of x in both sides gives

$$\begin{aligned} h(0) &= 1, \\ h^{(n)}(0) &= 0, \quad n \geqslant 1. \end{aligned} \tag{3.63}$$

Using the Taylor expansion of $h(t)$ and the result (3.63) we obtain

$$h(t) = 1. \tag{3.64}$$

Combining (3.60) and (3.64), the solution $u(x,t)$ in a closed form is given by

$$u(x,t) = x + e^{-t}\cos x. \tag{3.65}$$

For simplicity reasons, we will apply the inverse operator L_t^{-1} to obtain the solution in the following homogeneous PDEs.

Example 5. Use the Adomian decomposition method to solve the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_t = u_{xx} - u, & 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = 0, & t \geqslant 0, \\ & u(\pi,t) = 0, & t \geqslant 0, \\ \text{IC} & u(x,0) = \sin x. & \end{array} \tag{3.66}$$

Solution.

We point out here that the homogeneous PDE in (3.66) defines a heat equation with a lateral heat loss. This can be attributed to the additional term $-u(x,t)$ at the right hand side of the standard heat equation.

In an operator form, Equation (3.66) can be written as

$$L_t u(x,t) = L_x u(x,t) - u(x,t). \tag{3.67}$$

Applying L_t^{-1} to both sides of (3.67) gives

$$u(x,t) = \sin x + L_t^{-1}\left(L_x u(x,t) - u(x,t)\right). \tag{3.68}$$

Substituting the decomposition (3.23) into both sides of (3.68) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right) - \sum_{n=0}^{\infty} u_n(x,t)\right). \tag{3.69}$$

Proceeding as before we obtain

$$\begin{aligned} u_0(x,t) &= \sin x, \\ u_1(x,t) &= L_t^{-1}\left(L_x(u_0) - u_0\right) = -2t\sin x, \\ u_2(x,t) &= L_t^{-1}\left(L_x(u_1) - u_1\right) = \frac{1}{2!}(2t)^2\sin x, \\ &\ \vdots \end{aligned} \tag{3.70}$$

Consequently, the solution $u(x,t)$ in a series form is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\ &= \sin x\left(1 - 2t + \frac{1}{2!}(2t)^2 - \cdots\right), \end{aligned} \tag{3.71}$$

and in a closed form by

$$u(x,t) = e^{-2t}\sin x, \tag{3.72}$$

obtained upon using the Taylor expansion of e^{-2t}.

3.2.2 Inhomogeneous Heat Equations

A great advantage of the decomposition method is that it can provide solutions to PDE, homogeneous or inhomogeneous, without any need to use any transformation formula as required by the method of separation of variables. The advantage lies in the fact that the method is computationally convenient and provides the solution in a rapid convergent series. The method attacks the inhomogeneous problem in a similar way to that used in the homogeneous type of problems [1,8].

Example 6. Use the Adomian decomposition method to solve the inhomogeneous PDE

$$\begin{array}{lrl} \text{PDE} & u_t = & u_{xx} + \sin x, \quad 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = & e^{-t}, \quad t \geqslant 0, \\ & u(\pi,t) = & -e^{-t}, \quad t \geqslant 0, \\ \text{IC} & u(x,0) = & \cos x. \end{array} \tag{3.73}$$

Solution.

In an operator form, Equation (3.73) becomes

$$L_t u(x,t) = L_x u(x,t) + \sin x. \tag{3.74}$$

Operating with L_t^{-1} on both sides of (3.74) gives

$$u(x,t) = t\,\sin x + \cos x + L_t^{-1}\left(L_x u(x,t)\right). \tag{3.75}$$

Using the decomposition (3.23) we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = t\sin x + \cos x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{3.76}$$

It should be noted here that the zeroth component u_0 will be defined as the sum of all terms that are not included in the operator L_t^{-1}. In fact the zeroth component is assigned the terms that arise from integrating the source term $\sin x$ and from using the initial condition.

To determine the components of $u(x,t)$, we proceed as before, hence we set

$$\begin{aligned} u_0(x,t) &= t\sin x + \cos x, \\ u_1(x,t) &= L_t^{-1}\left(L_x\left(u_0\right)\right), \\ &= L_t^{-1}\left(-\cos x - t\sin x\right) = -t\cos x - \frac{1}{2!}t^2\sin x, \\ u_2(x,t) &= L_t^{-1}\left(L_x\left(u_1\right)\right), \\ &= L_t^{-1}\left(t\cos x + \frac{1}{2!}t^2\sin x\right) = \frac{1}{2!}t^2\cos x + \frac{1}{3!}t^3\sin x, \end{aligned} \tag{3.77}$$

and so on. Consequently, the solution $u(x,t)$ in a series form is given by

$$
\begin{aligned}
u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots \\
&= \sin x\left(t-\frac{1}{2!}t^2+\frac{1}{3!}t^3\cdots\right)+\cos x\left(1-t+\frac{1}{2!}t^2\cdots\right),
\end{aligned}
\tag{3.78}
$$

and in a closed form by

$$
u(x,t) = \left(1-e^{-t}\right)\sin x + e^{-t}\cos x. \tag{3.79}
$$

Example 7. Use the Adomian decomposition method to solve the inhomogeneous PDE

$$
\begin{array}{lll}
\text{PDE} & u_t = u_{xx}+\cos x, & 0<x<\pi,\ t>0, \\
\text{BC} & u(0,t) = 1-e^{-t}, & t\geqslant 0, \\
 & u(\pi,t) = e^{-t}-1, & t\geqslant 0, \\
\text{IC} & u(x,0) = 0, & 0\leqslant x\leqslant \pi.
\end{array}
\tag{3.80}
$$

Solution.

Proceeding as before we find

$$
u(x,t) = t\cos x + L_t^{-1}\left(L_x u(x,t)\right). \tag{3.81}
$$

This gives

$$
\sum_{n=0}^{\infty} u_n(x,t) = t\cos x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{3.82}
$$

We next use the recurrence relation

$$
\begin{aligned}
u_0(x,t) &= t\cos x, \\
u_1(x,t) &= L_t^{-1}\left(L_x\left(u_0\right)\right) = -\frac{1}{2!}t^2\cos x, \\
u_2(x,t) &= L_t^{-1}\left(L_x\left(u_1\right)\right) = \frac{1}{3!}t^3\cos x,
\end{aligned}
\tag{3.83}
$$

and so on. In view of (3.83), the solution $u(x,t)$ in a series form is given by

$$
\begin{aligned}
u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots \\
&= \cos x\left(t-\frac{1}{2!}t^2+\frac{1}{3!}t^3-\cdots\right),
\end{aligned}
\tag{3.84}
$$

and in a closed form by

$$
u(x,t) = \left(1-e^{-t}\right)\cos x. \tag{3.85}
$$

Exercises 3.2

In Exercises 1–6, use the decomposition method to solve the following homogeneous partial differential equations:

1. $u_t = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = \pi,\ t \geqslant 0$
   $u(x,0) = x + \sin x$

2. $u_t = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 4 + e^{-t},\ u(\pi,t) = 4 - e^{-t},\ t \geqslant 0$
   $u(x,0) = 4 + \cos x$

3. $u_t = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = 0,\ t \geqslant 0$
   $u(x,0) = \sin x$

4. $u_t = u_{xx} - 4u,\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = 0,\ t \geqslant 0$
   $u(x,0) = \sin x$

5. $u_t = u_{xx} - 2u,\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = e^{-t}\sinh\pi,\ t \geqslant 0$
   $u(x,0) = \sinh x$

6. $u_t = u_{xx} - 2u,\ 0 < x < \pi,\ t > 0$
   $u(0,t) = e^{-t},\ u(\pi,t) = e^{-t}\cosh\pi,\ t \geqslant 0$
   $u(x,0) = \cosh x$

In Exercises 7–12, solve the inhomogeneous initial-boundary value problems:

7. $u_t = u_{xx} + \sin(2x),\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = 0,\ t \geqslant 0$
   $u(x,0) = \sin x + \frac{1}{4}\sin(2x)$

8. $u_t = u_{xx} - 2,\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = \pi^2,\ t \geqslant 0$
   $u(x,0) = x^2 + \sin x$

9. $u_t = u_{xx} - 6x,\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = \pi^3,\ t \geqslant 0$
   $u(x,0) = x^3 + \sin x$

10. $u_t = u_{xx} - 6,\ 0 < x < \pi,\ t > 0$
    $u(0,t) = e^{-t},\ u(\pi,t) = 3\pi^2 - e^{-t},\ t \geqslant 0$
    $u(x,0) = 3x^2 + \cos x$

11. $u_t = u_{xx} - 2,\ 0 < x < \pi,\ t > 0$
    $u(0,t) = e^{-t},\ u(\pi,t) = \pi^2 - e^{-t},\ t \geqslant 0$

$u(x,0) = x^2 + \cos x$

12. $u_t = u_{xx} - 6x,\ 0 < x < \pi,\ t > 0$
$u(0,t) = e^{-t},\ u(\pi,t) = \pi^3 - e^{-t},$
$u(x,0) = x^3 + \cos x$

In Exercises 13–18, solve the initial-boundary value problems:

13. $u_t = u_{xx},\ 0 < x < 1,\ t > 0$
$u(0,t) = 1,\ u(1,t) = 1,\ t \geqslant 0$
$u(x,0) = 1 + \sin(\pi x)$

14. $u_t = 4u_{xx},\ 0 < x < 1,\ t > 0$
$u(0,t) = 1,\ u(1,t) = 1,\ t \geqslant 0$
$u(x,0) = 1 + \sin(\pi x)$

15. $u_t = 4u_{xx},\ 0 < x < \frac{\pi}{2},\ t > 0$
$u(0,t) = e^{-4t},\ u(\frac{\pi}{2},t) = 0,\ t \geqslant 0$
$u(x,0) = \cos x$

16. $u_t = 2u_{xx},\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = \pi,\ t \geqslant 0$
$u(x,0) = x + \sin x$

17. $u_t = u_{xx},\ 0 < x < \pi,\ t > 0$
$u_x(0,t) = 0,\ u_x(\pi,t) = 0,\ t \geqslant 0$
$u(x,0) = \cos x$

18. $u_t = u_{xx},\ 0 < x < \pi,\ t > 0$
$u_x(0,t) = 0,\ u_x(\pi,t) = 0,\ t \geqslant 0$
$u(x,0) = 2 + \cos x$

3.3 The Variational Iteration Method

The variational iteration method (VIM), established by Ji-Huan He [4–5] was presented before in Chapter 2. It is thoroughly used by many researchers to handle linear and nonlinear models. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists. In what follows, we summarize the main steps of this method. For the differential equation

$$Lu + Nu = g(x,t), \tag{3.86}$$

where L and N are linear and nonlinear operators respectively, and $g(x,t)$ is the source inhomogeneous term, the variational iteration method admits the use of the correction functional for equation (3.86) which can be written as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\right) d\xi, \quad n \geqslant 0. \tag{3.87}$$

It is obvious that the successive approximations $u_j, j \geqslant 0$ can be established by determining $\lambda(\xi)$, a general Lagrange multiplier, which can be identified optimally via the variational theory. The function $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. Using the obtained $\lambda(\xi)$ and selecting $u_0(x,t)$, the successive approximations $u_{n+1}(x,t), n \geqslant 0$, of the solution $u(x,t)$ will follow immediately.

3.3.1 Homogeneous Heat Equations

In what follows, we will apply the VIM to the some examples that were examined before.

Example 1. Use the variational iteration method to solve the initial-boundary value problem

$$\begin{array}{lll} \text{PDE} & u_t = u_{xx}, & 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = 0, & u(\pi,t) = 0,\ t \geqslant 0, \\ \text{IC} & u(x,0) = \sin x. & \end{array} \tag{3.88}$$

Solution.

The correction functional for (3.88) reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2}\right) d\xi. \tag{3.89}$$

The stationary conditions

$$\begin{aligned} 1 + \lambda|_{\xi=t} &= 0, \\ \lambda'|_{\xi=t} &= 0, \end{aligned} \tag{3.90}$$

gives

$$\lambda = -1. \tag{3.91}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (3.89) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right) d\xi, \quad n \geqslant 0. \tag{3.92}$$

As stated before, we can select $u_0(x,0) = \sin x$ from the given initial condition. Using this selection into (3.92) we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= \sin x, \\ u_1(x,t) &= \sin x - t\sin x, \\ u_2(x,t) &= \sin x - t\sin x + \frac{1}{2!}t^2 \sin x, \end{aligned}$$

$$\begin{aligned} u_3(x,t) &= \sin x - t\sin x + \frac{1}{2!}t^2\sin x - \frac{1}{3!}t^3\sin x,\\ &\vdots \\ u_n(x,t) &= \sin x\left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots\right). \end{aligned} \tag{3.93}$$

The VIM introduces the use of

$$u(x,t) = \lim_{n\to\infty} u_n(x,t), \tag{3.94}$$

that gives the exact solution by

$$u(x,t) = e^{-t}\sin x, \tag{3.95}$$

obtained upon using the Taylor expansion of e^{-t}.

Example 2. Use the variational iteration method to solve the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_t = u_{xx}, & 0 < x < \pi,\ t > 0,\\ \text{BC} & u(0,t) = e^{-t}, & t \geqslant 0,\\ & u(\pi,t) = \pi - e^{-t}, & t \geqslant 0,\\ \text{IC} & u(x,0) = x + \cos x. & \end{array} \tag{3.96}$$

Solution.

The correction functional for (3.96) reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2}\right)d\xi. \tag{3.97}$$

As discussed before in Example 1 we find

$$\lambda = -1. \tag{3.98}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (3.97) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right)d\xi, \quad n \geqslant 0. \tag{3.99}$$

As stated before, we can select $u_0(x,0) = x + \cos x$ from the given initial condition. Using this selection into (3.99) we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= x + \cos x,\\ u_1(x,t) &= x + \cos x - t\cos x,\\ u_2(x,t) &= x + \cos x - t\cos x + \frac{1}{2!}t^2\cos x, \end{aligned}$$

$$
\begin{aligned}
u_3(x,t) &= x+\cos x - t\cos x + \frac{1}{2!}t^2\cos x - \frac{1}{3!}t^3\cos x,\\
&\ \vdots \\
u_n(x,t) &= x+\cos x\left(1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4+\cdots\right).
\end{aligned} \tag{3.100}
$$

Consequently, the exact solution is

$$u(x,t) = x+e^{-t}\cos x, \tag{3.101}$$

obtained upon using the Taylor expansion of e^{-t}.

Example 3. Use the variational iteration method to solve the initial-boundary value problem

$$
\begin{array}{lrl}
\text{PDE} & u_t = & u_{xx}-u, \quad 0<x<\pi,\ t>0,\\
\text{BC} & u(0,t) = & 0, \quad t\geqslant 0,\\
 & u(\pi,t) = & 0, \quad t\geqslant 0,\\
\text{IC} & u(x,0) = & \sin x.
\end{array} \tag{3.102}
$$

Solution.

The correction functional for (3.102) is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} + \tilde{u}_n(x,\xi)\right)d\xi. \tag{3.103}$$

Following the discussion presented before we find

$$\lambda = -1. \tag{3.104}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (3.103) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + u_n(x,\xi)\right)d\xi, \quad n\geqslant 0. \tag{3.105}$$

As stated before, we can select $u_0(x,0) = \sin x$ from the given initial condition. Using this selection into (3.105) we obtain the successive approximations

$$
\begin{aligned}
u_0(x,t) &= \sin x,\\
u_1(x,t) &= \sin x - 2t\sin x,\\
u_2(x,t) &= \sin x - 2t\sin x + \frac{1}{2!}(2t)^2\sin x,\\
u_3(x,t) &= \sin x - 2t\sin x + \frac{1}{2!}(2t)^2\sin x - \frac{1}{3!}(2t)^3\sin x,\\
&\ \vdots \\
u_n(x,t) &= \sin x\left(1-2t+\frac{1}{2!}(2t)^2-\frac{1}{3!}(2t)^3+\cdots\right).
\end{aligned} \tag{3.106}
$$

Hence, the exact solution is given by

$$u(x,t) = e^{-2t}\sin x. \tag{3.107}$$

3.3.2 Inhomogeneous Heat Equations

The variational iteration method handles the inhomogeneous heat problem in a similar way to that used in the homogeneous type of equations. This will be illustrated by solving the same examples studied before in the previous section by using Adomian method.

Example 4. Use the variational iteration method to solve the inhomogeneous PDE

$$\begin{array}{lll} \text{PDE} & u_t = u_{xx} + \sin x, & 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = e^{-t}, & t \geqslant 0, \\ & u(\pi,t) = -e^{-t}, & t \geqslant 0, \\ \text{IC} & u(x,0) = \cos x. & \end{array} \tag{3.108}$$

Solution.

The correction functional for (3.108) reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} - \sin x\right) d\xi. \tag{3.109}$$

Solving the stationary conditions gives $\lambda = -1$. Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (3.109) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - \sin x\right) d\xi, \quad n \geqslant 0. \tag{3.110}$$

We next select $u_0(x,0) = \cos x$. Consequently, we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= \cos x, \\ u_1(x,t) &= \cos x - t\cos x + t\sin x, \\ u_2(x,t) &= \cos x - t\cos x + t\sin x - \frac{1}{2!}t^2\sin x + \frac{1}{2!}t^2\cos x, \\ u_3(x,t) &= \cos x\left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3\right) + \sin x\left(t - \frac{1}{2!}t^2 + \frac{1}{3!}t^3\right), \\ &\ \ \vdots \\ u_n(x,t) &= \cos x\left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \cdots\right) + \sin x\left(t - \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots\right). \end{aligned} \tag{3.111}$$

Accordingly, the exact solution

$$u(x,t) = e^{-t}\cos x + (1-e^{-t})\sin x, \tag{3.112}$$

is readily obtained.

Example 5. Use the variational iteration method to solve the inhomogeneous PDE

$$\begin{array}{ll} \text{PDE} & u_t = u_{xx} + \cos x, \quad 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = 1 - e^{-t}, \quad t \geqslant 0. \\ & u(\pi,t) = e^{-t} - 1, \quad t \geqslant 0. \\ \text{IC} & u(x,0) = 0, \quad 0 \leqslant x \leqslant \pi. \end{array} \tag{3.113}$$

Solution.

The correction functional for this equation is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} - \cos x\right) d\xi. \tag{3.114}$$

Substituting the value of the Lagrange multiplier $\lambda = -1$ into the functional (3.114) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - \cos x\right) d\xi, \quad n \geqslant 0. \tag{3.115}$$

We next select $u_0(x,0) = 0$. Consequently, we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= 0, \\ u_1(x,t) &= t\cos x, \\ u_2(x,t) &= t\cos x - \frac{1}{2!}t^2\cos x, \\ u_3(x,t) &= t\cos x - \frac{1}{2!}t^2\cos x + \frac{1}{3!}t^3\cos x, \\ &\vdots \\ u_n(x,t) &= \cos x\left(t - \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots\right). \end{aligned} \tag{3.116}$$

Hence, the exact solution

$$u(x,t) = \left(1 - e^{-t}\right)\cos x, \tag{3.117}$$

follows immediately.

Exercises 3.3

Use the variational iteration method to solve the problems 1–18 in Exercises 3.2.

3.4 Method of Separation of Variables

In this section the homogeneous partial differential equation that describes the heat flow in a rod will be discussed by using a well-known method called the *method of separation of variables*. The method is commonly used to solve heat conduction problems and other types of problems such as the wave equation and the Laplace equation [3,6,8].

The most important feature of the method of separation of variables is that it successively replaces the partial differential equation by a system of ordinary differential equations that are usually easy to handle. Unlike the decomposition method, the method of separation of variables employs specific assumptions and transformation formulas in handling partial differential equations. In particular, the method of separation of variables requires that the boundary conditions be homogeneous. For inhomogeneous boundary conditions, a transformation formula should be employed to transform inhomogeneous boundary conditions to homogeneous boundary conditions.

3.4.1 Analysis of the Method

We begin our analysis by writing the homogeneous partial differential equation, with homogeneous boundary conditions, that describes the heat flow by the partial differential equation

$$\begin{array}{lrl} \text{PDE} & u_t = & \overline{k}u_{xx}, \quad 0<x<L,\ t>0, \\ \text{BC} & u(0,t) = & 0, \quad u(L,t)=0, \\ \text{IC} & u(x,0) = & f(x). \end{array} \tag{3.118}$$

The method of separation of variables consists of assuming that the temperature $u(x,t)$ is identified as the product of two distinct functions $F(x)$ and $T(t)$, where $F(x)$ depends on the space variable x and $T(t)$ depends on the time variable t. In other words, this assumption allows us to set

$$u(x,t) = F(x)T(t). \tag{3.119}$$

Differentiating both sides of (3.119) with respect to t and twice with respect to x we obtain

$$\begin{aligned} u_t(x,t) &= F(x)T'(t), \\ u_{xx}(x,t) &= F''(x)T(t). \end{aligned} \tag{3.120}$$

Substituting (3.120) into (3.118) yields

$$F(x)T'(t) = \overline{k}F''(x)T(t). \tag{3.121}$$

Dividing both sides of (3.121) by $\overline{k}F(x)T(t)$ gives

$$\frac{T'(t)}{\overline{k}T(t)} = \frac{F''(x)}{F(x)}. \tag{3.122}$$

It is clear from (3.122) that the left hand side depends only on t and the right hand side depends only on x. This means that the equality holds only if both sides are equal to the same constant. Therefore, we set

$$\frac{T'(t)}{\overline{k}T(t)} = \frac{F''(x)}{F(x)} = -\lambda^2. \tag{3.123}$$

The selection of $-\lambda^2$, and not λ^2, in (3.123) is the only selection for which non-trivial solutions exist. However, we can easily show that selecting the constant to be zero or a positive value will lead to the trivial solution $u(x,t) = 0$.

It is clear that (3.123) gives two distinct ordinary differential equations given by

$$\begin{aligned} T'(t) + \overline{k}\lambda^2 T(t) &= 0, \\ F''(x) + \lambda^2 F(x) &= 0. \end{aligned} \tag{3.124}$$

This means that the partial differential equation (3.118) is reduced to the more familiar ordinary differential equations (3.124) where each equation relies only on one variable.

To determine $T(t)$, we solve the first order linear ODE

$$T'(t) + \overline{k}\lambda^2 T(t) = 0, \tag{3.125}$$

to find that

$$T(t) = Ce^{-\overline{k}\lambda^2 t}, \tag{3.126}$$

where C is a constant.

On the other hand, the function $F(x)$ can be easily determined by solving the second order linear ODE

$$F''(x) + \lambda^2 F(x) = 0, \tag{3.127}$$

to find that

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{3.128}$$

where A and B are constants.

To determine A, B, and λ we use the homogeneous boundary conditions

$$\begin{aligned} u(0,t) &= 0, \\ u(L,t) &= 0, \end{aligned} \tag{3.129}$$

as given above by (3.118). Substituting (3.129) into the assumption (3.119) gives

$$\begin{aligned} F(0)T(t) &= 0, \\ F(L)T(t) &= 0, \end{aligned} \tag{3.130}$$

which gives

$$\begin{aligned} F(0) &= 0, \\ F(L) &= 0. \end{aligned} \tag{3.131}$$

Using $F(0) = 0$ into (3.128) leads to

$$A = 0, \tag{3.132}$$

hence Eq. (3.128) becomes

$$F(x) = B\sin(\lambda x). \tag{3.133}$$

Substituting the condition $F(L) = 0$ of (3.131) into (3.133) yields

$$B\sin(\lambda L) = 0. \tag{3.134}$$

This means that

$$B = 0, \tag{3.135}$$

or

$$\sin(\lambda L) = 0. \tag{3.136}$$

We ignore $B = 0$ since it gives the trivial solution $u(x,t) = 0$. It remains that

$$\sin(\lambda L) = 0. \tag{3.137}$$

This gives an infinite number of values for λ_n given by

$$\lambda_n L = n\pi,\ n = 1, 2, 3, \cdots, \tag{3.138}$$

or equivalently

$$\lambda_n = \frac{n\pi}{L},\ n = 1, 2, 3, \cdots. \tag{3.139}$$

We exclude $n = 0$ since it gives the trivial solution $u(x,t) = 0$.

In view of the infinite number of values for λ_n, we therefore write

$$\begin{aligned} F_n(x) &= \sin(\frac{n\pi}{L}x), \\ T_n(t) &= e^{-\overline{k}(\frac{n\pi}{L})^2 t}, n = 1, 2, 3, \cdots. \end{aligned} \tag{3.140}$$

Ignoring the constants B and C, we conclude that the functions, called the fundamental solutions

$$\begin{aligned} u_n(x,t) &= F_n(x)T_n(t), \\ &= \sin(\tfrac{n\pi}{L}x)e^{-\overline{k}(\frac{n\pi}{L})^2 t}, \quad n = 1, 2, \cdots, \end{aligned} \tag{3.141}$$

that satisfy Eq. (3.118) and the given boundary conditions.

Recall that the superposition principle admits that a linear combination of the functions $u_n(x,t)$ also satisfy the given equation and the boundary conditions. Therefore, using this principle gives the general solution by

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\overline{k}(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi}{L}x), \tag{3.142}$$

where the arbitrary constants $B_n, n \geqslant 1$, are as yet undetermined.

To determine $B_n, n \geqslant 1$, we substitute $t = 0$ in (3.142) and by using the initial condition we find

$$\sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L}x) = f(x). \tag{3.143}$$

The constants B_n can be determined in this case by using Fourier coefficients given by the formula

$$B_n = \frac{2}{L}\int_0^L f(x)\,\sin(\frac{n\pi}{L}x)\,dx. \tag{3.144}$$

Having determined the constants B_n, the particular solution $u(x,t)$ follows immediately.

On the other hand, if the initial condition $f(x)$ is given in terms of $\sin(\frac{n\pi}{L}x)$, $n \geqslant 1$, the constants B_n can be completely determined by expanding (3.142), using the initial condition, and by equating the coefficients of like terms on both sides. The initial condition in the first two examples will be trigonometric functions.

To give a clear overview of the method of separation of variables, we have selected several examples to illustrate the analysis presented above.

Example 1. Use the method of separation of variables to solve the following initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_t = u_{xx}, & 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = 0, & t \geqslant 0, \\ & u(\pi,t) = 0, & t \geqslant 0, \\ \text{IC} & u(x,0) = \sin x + 3\sin(2x). & \end{array} \tag{3.145}$$

Solution.

We first set

$$u(x,t) = F(x)T(t). \tag{3.146}$$

Differentiating (3.146) once with respect to t and twice with respect to x and proceeding as before we obtain the two distinct ODEs given by

$$T'(t) + \lambda^2 T(t) = 0, \tag{3.147}$$

and

$$F''(x) + \lambda^2 F(x) = 0, \tag{3.148}$$

so that

$$T(t) = Ce^{-\lambda^2 t}, \tag{3.149}$$

and

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x). \tag{3.150}$$

To determine the constants A, B and λ, we first use the boundary conditions to obtain

$$\begin{aligned} u(0,t) &= F(0)T(t) = 0, \Longrightarrow F(0) = 0, \\ u(\pi,t) &= F(\pi)T(t) = 0, \Longrightarrow F(\pi) = 0. \end{aligned} \tag{3.151}$$

Using (3.151) into (3.150) we find

$$A = 0, \tag{3.152}$$

and

$$\sin(\pi\lambda) = 0, \tag{3.153}$$

which gives λ_n by

$$\lambda_n = n, \quad n = 1,2,3,\cdots. \tag{3.154}$$

Recall that $n = 0$ gives the trivial solution $u(x,t) = 0$, and therefore it is excluded from the values of λ_n. In accordance with the infinite number of values of λ_n, we therefore write

$$\begin{aligned} F_n(x) &= \sin(nx), \\ T_n(t) &= e^{-n^2 t}, \quad n = 1,2,\cdots. \end{aligned} \tag{3.155}$$

This gives the fundamental set of solutions

$$u_n(x,t) = F_n(x)T_n(t) = \sin(nx)e^{-n^2 t}, \quad n = 1,2,3,\cdots, \tag{3.156}$$

where these solutions satisfy Eq. (3.145) and the given boundary conditions. Using the superposition principle we obtain

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t}\sin(nx), \tag{3.157}$$

or in a series form by

$$u(x,t) = B_1 e^{-t}\sin(x) + B_2 e^{-4t}\sin(2x) + B_3 e^{-9t}\sin(3x) + \cdots, \tag{3.158}$$

where the arbitrary constants $B_n, n \geqslant 1$ are as yet undetermined. To determine the constants B_n we use the initial condition and substitute $t = 0$ in (3.158) to find

$$B_1\sin(x) + B_2\sin(2x) + B_3\sin(3x) + \cdots = \sin(x) + 3\sin(2x). \tag{3.159}$$

Equating the coefficients of like terms of both sides we obtain

$$B_1 = 1, \quad B_2 = 3, \quad B_k = 0, \quad k \geqslant 3. \tag{3.160}$$

In view of (3.160), it is clear that the particular solution consists of two terms only and can be obtained by substituting $n = 1,2$ into (3.157) to find

$$u(x,t) = e^{-t}\sin(x) + 3e^{-4t}\sin(2x). \tag{3.161}$$

Example 2. Use the method of separation of variables to solve the following initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_t = & u_{xx}, \quad 0 < x < \pi,\ t > 0, \\ \text{BC} & u_x(0,t) = & 0, \quad t \geqslant 0, \\ & u_x(\pi,t) = & 0, \quad t \geqslant 0, \\ \text{IC} & u(x,0) = & 2 + 3\cos x. \end{array} \tag{3.162}$$

Solution.

It is interesting to note that the problem uses the Neumann boundary conditions, i.e the rates of flow $u_x(0,t) = 0$ and $u_x(\pi,t) = 0$ at the boundaries instead of the temperatures at both ends of the rod. This case arises when both ends of the rod are insulated. This means that no heat flows in or out at the ends of the rod.

We first set

$$u(x,t) = F(x)T(t). \tag{3.163}$$

Proceeding as before we find

$$T'(t) + \lambda^2 T(t) = 0, \tag{3.164}$$

and

$$F''(x) + \lambda^2 F(x) = 0, \tag{3.165}$$

so that

$$T(t) = Ce^{-\lambda^2 t}, \tag{3.166}$$

and

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x). \tag{3.167}$$

To determine the constants A, B and λ, we apply the boundary conditions in (3.162) that can be expressed as

$$\begin{array}{l} u_x(0,t) = F'(0)T(t) = 0, \Longrightarrow F'(0) = 0, \\ u_x(\pi,t) = F'(\pi)T(t) = 0, \Longrightarrow F'(\pi) = 0. \end{array} \tag{3.168}$$

Using (3.168) into (3.167) we find

$$B = 0, \tag{3.169}$$

and

$$\lambda \sin(\pi\lambda) = 0, \tag{3.170}$$

which gives λ_n by

$$\lambda = 0, \text{ or } \lambda_n = n, \quad n = 1,2,3,\cdots, \tag{3.171}$$

and therefore

$$\lambda_n = n, \quad n = 0, 1, 2, 3, \cdots, \tag{3.172}$$

where unlike the case of Example 1, $\lambda = 0$ is included because it will not give the trivial solution $u(x,t) = 0$.

In accordance with the infinite number of values of λ_n, we therefore write

$$\begin{aligned} F_n(x) &= \cos(nx), \\ T_n(t) &= e^{-n^2 t}, \quad n = 0, 1, 2, \cdots. \end{aligned} \tag{3.173}$$

Using the superposition principle, the general solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-n^2 t} \cos(nx), \tag{3.174}$$

or in a series form by

$$u(x,t) = A_0 + A_1 e^{-t} \cos(x) + A_2 e^{-4t} \cos(2x) + A_3 e^{-9t} \cos(3x) + \cdots. \tag{3.175}$$

To determine the constants A_n we use the initial condition and replace t by zero in (3.175) to find

$$A_0 + A_1 \cos(x) + A_2 \cos(2x) + \cdots = 2 + 3\cos(x). \tag{3.176}$$

Equating the coefficients of like terms on both sides we obtain

$$A_0 = 2, \quad A_1 = 3, \quad A_k = 0,\ k \geqslant 2. \tag{3.177}$$

In view of (3.177), the particular solution is given by

$$u(x,t) = 2 + 3e^{-t} \cos(x). \tag{3.178}$$

Example 3. Use the method of separation of variables to solve the following initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_t = & u_{xx}, \quad 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = & 0,\ t \geqslant 0, \\ & u(\pi,t) = & 0,\ t, \geqslant 0 \\ \text{IC} & u(x,0) = & 1. \end{array} \tag{3.179}$$

Solution.

We first set

$$u(x,t) = F(x)T(t). \tag{3.180}$$

Proceeding as before, we find

$$T(t) = Ce^{-\lambda^2 t}, \tag{3.181}$$

and

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x). \tag{3.182}$$

Using the boundary conditions gives

$$A = 0, \tag{3.183}$$

and

$$\lambda_n = n, \quad n = 1, 2, 3, \cdots. \tag{3.184}$$

Using the superposition principle, the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin(nx). \tag{3.185}$$

To determine the constants B_n we use the initial condition to find

$$\sum_{n=1}^{\infty} B_n \sin(nx) = 1. \tag{3.186}$$

The arbitrary constants are determined by using the Fourier method, therefore we find

$$\begin{aligned} B_n &= \frac{2}{\pi}\int_0^{\pi} \sin(nx)dx, \\ &= \frac{2}{n\pi}(1-\cos(n\pi)), \end{aligned} \tag{3.187}$$

so that

$$B_n = \begin{cases} 0 & \text{if } n \text{ is even}, \\ \dfrac{4}{n\pi} & \text{if } n \text{ is odd}. \end{cases} \tag{3.188}$$

This means that we can express B_n by

$$\begin{aligned} B_{2m} &= 0, \\ B_{2m+1} &= \frac{4}{(2m+1)\pi}, \; m = 0, 1, 2, \cdots. \end{aligned} \tag{3.189}$$

Combining (3.185) and (3.189), the particular solution is given by

$$u(x,t) = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{1}{2m+1} e^{-(2m+1)^2 t} \sin(2m+1)x. \tag{3.190}$$

The initial condition $u(x,0) = 1$ can be justified by using Appendix F.

Example 4. Use the method of separation of variables to solve the following initial-boundary value problem

$$
\begin{array}{lrl}
\text{PDE} & u_t = u_{xx}, & 0 < x < \pi,\ t > 0, \\
\text{BC} & u_x(0,t) = 0,\ t \geqslant 0, & \\
 & u_x(\pi,t) = 0,\ t \geqslant 0, & \\
\text{IC} & u(x,0) = x. &
\end{array}
\tag{3.191}
$$

Solution.

We first set

$$u(x,t) = F(x)T(t). \tag{3.192}$$

Following the previous discussions we find

$$T(t) = Ce^{-\lambda^2 t}, \tag{3.193}$$

and

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x). \tag{3.194}$$

The Neumann boundary conditions give

$$B = 0, \tag{3.195}$$

and

$$\lambda_n = n, \quad n = 0,1,2,3,\cdots. \tag{3.196}$$

Using the results we obtained for λ_n, we write

$$
\begin{aligned}
F_n(x) &= \cos(nx), \\
T_n(t) &= e^{-n^2 t}, \quad n = 0,1,2,\cdots.
\end{aligned}
\tag{3.197}
$$

Using the superposition principle, the general solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-n^2 t}\cos(nx). \tag{3.198}$$

To determine the constants A_n we use the initial condition to find

$$\sum_{n=0}^{\infty} A_n \cos(nx) = x. \tag{3.199}$$

The arbitrary constants A_n are determined by using the Fourier method, therefore we find

$$
\begin{aligned}
A_0 &= \frac{1}{\pi}\int_0^{\pi} x dx = \frac{\pi}{2} \\
A_n &= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx = \frac{2}{\pi}\left(\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\right), n = 1,2,\cdots,
\end{aligned}
\tag{3.200}
$$

so that

$$A_n = \begin{cases} 0 & \text{if } n \text{ is even, } n \neq 0, \\ -\dfrac{4}{\pi n^2} & \text{if } n \text{ is odd.} \end{cases} \tag{3.201}$$

Based on these results for the constants A_n, the particular solution is given by

$$u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} e^{-(2m+1)^2 t} \cos(2m+1)x. \tag{3.202}$$

Exercises 3.4.1

Solve the following initial-boundary value problems by the method of separation of variables:

1. $u_t = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = 0$
   $u(x,0) = \sin x + 2\sin(3x)$

2. $u_t = u_{xx},\ 0 < x < 1,\ t > 0$
   $u(0,t) = 0,\ u(1,t) = 0$
   $u(x,0) = \sin(\pi x) + \sin(2\pi x)$

3. $u_t = 4u_{xx},\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = 0$
   $u(x,0) = \sin(2x)$

4. $u_t = 2u_{xx},\ 0 < x < 1,\ t > 0$
   $u(0,t) = 0,\ u(1,t) = 0$
   $u(x,0) = \sin(\pi x)$

5. $u_t = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u_x(0,t) = 0,\ u_x(\pi,t) = 0$
   $u(x,0) = 1 + \cos x$

6. $u_t = 2u_{xx},\ 0 < x < \pi,\ t > 0$
   $u_x(0,t) = 0,\ u_x(\pi,t) = 0$
   $u(x,0) = 3 + 4\cos x$

7. $u_t = 3u_{xx},\ 0 < x < \pi,\ t > 0$
   $u_x(0,t) = 0,\ u_x(\pi,t) = 0$
   $u(x,0) = 1 + \cos x + \cos(2x)$

8. $u_t = 4u_{xx},\ 0 < x < 1,\ t > 0$
   $u_x(0,t) = 0,\ u_x(1,t) = 0$
   $u(x,0) = 2 + 2\cos(2\pi x)$

9. $u_t = 4u_{xx},\ 0 < x < \pi,\ t > 0$

$u(0,t) = 0,\ u(\pi,t) = 0$
$u(x,0) = 2$

10. $u_t = 2u_{xx},\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = 0$
$u(x,0) = 3$

11. $u_t = 3u_{xx},\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = 0$
$u(x,0) = 2x$

12. $u_t = u_{xx},\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = 0$
$u(x,0) = 1 + x$

3.4.2 Inhomogeneous Boundary Conditions

In this section we will consider the case where the ends of a rod are kept at constant temperatures different from zero. It is well known that the method of separation of variables is applicable if the equation and the boundary conditions are linear and homogeneous. Consequently, a transformation formula is needed that will enable us to convert the inhomogeneous boundary conditions to homogeneous boundary conditions. This is necessary in order to apply the method of separation of variables in a parallel way to that used above.

We begin our analysis by considering the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_t = & u_{xx},\ 0 < x < L,\ t > 0, \\ \text{BC} & u(0,t) = & \alpha,\ t \geqslant 0, \\ & u(L,t) = & \beta,\ t \geqslant 0, \\ \text{IC} & u(x,0) = & f(x). \end{array} \tag{3.203}$$

To convert the boundary conditions from inhomogeneous to homogeneous we simply use the following transformation formula

$$u(x,t) = \left(\alpha + \frac{x}{L}(\beta - \alpha)\right) + v(x,t). \tag{3.204}$$

This means that $u(x,t)$ consists of a **steady-state solution** [3], that does not depend on time, defined by

$$w(x) = \alpha + \frac{x}{L}(\beta - \alpha), \tag{3.205}$$

that satisfies the boundary conditions, and a **transient solution** given by $v(x,t)$. We can easily show that $v(x,t)$ will be governed by the initial-boundary value problem

$$
\begin{array}{lrl}
\text{PDE} & v_t = v_{xx}, & 0 < x < L,\ t > 0, \\
\text{BC} & v(0,t) = 0,\ t \geqslant 0, & \\
 & v(L,t) = 0,\ t \geqslant 0, & \\
\text{IC} & v(x,0) = f(x) - \left(\alpha + \frac{x}{L}(\beta - \alpha)\right). &
\end{array}
\tag{3.206}
$$

Consequently, the method of separation of variables can be used in a similar way to that used in the previous section. Recall that Adomian decomposition method can be implemented directly. To get a better understanding of converting inhomogeneous boundary conditions to homogeneous boundary conditions, the following illustrative example will be discussed.

Example 5. Solve the following initial-boundary value problem

$$
\begin{array}{lrl}
\text{PDE} & u_t = u_{xx}, & 0 < x < 1,\ t > 0, \\
\text{BC} & u(0,t) = 1,\ t \geqslant 0, & \\
 & u(1,t) = 2,\ t \geqslant 0, & \\
\text{IC} & u(x,0) = 1 + x + 2\sin(\pi x). &
\end{array}
\tag{3.207}
$$

Solution.

Using the transformation (3.204) we obtain

$$
u(x,t) = (1+x) + v(x,t). \tag{3.208}
$$

In view of (3.208), Equation (3.207) is transformed into

$$
\begin{array}{lrl}
\text{PDE} & v_t = v_{xx}, & 0 < x < 1,\ t > 0, \\
\text{BC} & v(0,t) = 0,\ t \geqslant 0, & \\
 & v(1,t) = 0,\ t \geqslant 0, & \\
\text{IC} & v(x,0) = 2\sin(\pi x). &
\end{array}
\tag{3.209}
$$

Assuming that

$$
v(x,t) = F(x)T(t), \tag{3.210}
$$

and proceeding as before we obtain

$$
T(t) = Ce^{-\lambda^2 t}, \tag{3.211}
$$

and

$$
F(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{3.212}
$$

where A, B, and C are constants. Using the boundary conditions gives

$$
v(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t} \sin(n\pi x). \tag{3.213}
$$

Using the initial condition in (3.209) and expanding (3.213) we obtain

$$B_1 = 2, \quad B_k = 0, \quad k \geqslant 2. \tag{3.214}$$

This gives the solution for $v(x,t)$ by

$$v(x,t) = 2e^{-\pi^2 t}\sin(\pi x), \tag{3.215}$$

so that the particular solution

$$u(x,t) = 1 + x + 2e^{-\pi^2 t}\sin(\pi x), \tag{3.216}$$

follows immediately.

At this point, it seems reasonable to use the Adomian decomposition method to solve the initial-boundary value problem of Example 5. The newly developed approach can be used to examine the performance of the decomposition method if compared with the classical method of separation of variables.

Applying the inverse operator L_t^{-1} to the operator form of (3.207) and using the initial condition, we obtain

$$u(x,t) = 1 + x + 2\sin(\pi x) + L_t^{-1}\left(L_x u(x,t)\right). \tag{3.217}$$

Using the decomposition series (3.23) of $u(x,t)$ yields

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 + x + 2\sin(\pi x) + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{3.218}$$

Using the recursive algorithm we obtain

$$\begin{aligned} u_0(x,t) &= 1 + x + 2\sin(\pi x), \\ u_1(x,t) &= L_t^{-1}(L_x u_0) = -2\pi^2 t\sin(\pi x), \\ u_2(x,t) &= L_t^{-1}(L_x u_1) = 2\pi^4\frac{t^2}{2!}t\sin(\pi x), \end{aligned} \tag{3.219}$$

and so on. Consequently, the solution in a series form is given by

$$u(x,t) = 1 + x + 2\sin(\pi x)\left(1 - \pi^2 t + \pi^4\frac{t^2}{2!} - \cdots\right), \tag{3.220}$$

and in a closed form

$$u(x,t) = 1 + x + 2\sin(\pi x)e^{-\pi^2 t}. \tag{3.221}$$

Exercises 3.4.2

In Exercises 1–4, use the method of separation of variables to solve the initial-boundary value problems:

1. $u_t = u_{xx},\ 0 < x < 1,\ t > 0$
   $u(0,t) = 1,\ u(1,t) = 3$
   $u(x,0) = 1 + 2x + 3\sin(\pi x)$

2. $u_t = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 1,\ u(\pi,t) = 1$
   $u(x,0) = 1 + 4\sin x$

3. $u_t = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = \pi$
   $u(x,0) = x + \sin(2x)$

4. $u_t = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 4,\ u(\pi,t) = 4 - 4\pi$
   $u(x,0) = 4 - 4x + \sin(3x)$

5. Solve the following initial-boundary value problem by the decomposition method and by the separation of variables method:

$$u_t = u_{xx},\ 0 < x < \pi,\ t > 0$$
$$u(0,t) = 2,\ u(\pi,t) = 2 + 3\pi$$
$$u(x,0) = 2 + 3x + \sin x$$

6. Solve the following initial-boundary value problem by the decomposition method and by the separation of variables method:

$$u_t = u_{xx},\ 0 < x < 1,\ t > 0$$
$$u(0,t) = 1,\ u(1,t) = 3$$
$$u(x,0) = 1 + 2x + 3\sin(2\pi x)$$

3.4.3 Equations with Lateral Heat Loss

For a rod with a lateral heat loss, it can be proved that the heat flow is controlled by the homogeneous PDE

$$\begin{array}{llr} \text{PDE} & u_t = \overline{k}u_{xx} - cu, \quad 0 < x < L, t > 0, & \\ \text{BC} & u(0,t) = 0, & \\ & u(L,t) = 0, & (3.222) \\ \text{IC} & u(x,0) = f(x). & \end{array}$$

It is easily observed that this equation is not the standard heat equation we discussed so far. Instead, it includes the term $-cu(x,t)$ due to the lateral heat loss.

We will focus our attention on converting Eq. (3.222) to a standard heat equation. Thereafter, we can implement the separation of variables method in a straightforward way. This goal can be achieved by using the transformation formula

$$u(x,t) = e^{-ct}w(x,t). \tag{3.223}$$

Accordingly, $w(x,t)$ will be governed by the IBVP

$$\begin{array}{lrl} \text{PDE} & w_t &= \bar{k}w_{xx},\ 0<x<L, t>0, \\ \text{BC} & w(0,t) &= 0, \\ & w(L,t) &= 0, \\ \text{IC} & w(x,0) &= f(x). \end{array} \tag{3.224}$$

where $w(x,t)$ can be easily obtained in a similar manner to the discussion stated above. The following example illustrates the use of the transformation formula (3.223).

Example 6. Solve the following initial boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_t &= u_{xx} - u, \quad 0<x<1,\ t>0, \\ \text{BC} & u(0,t) &= 0, \\ & u(1,t) &= 0, \\ \text{IC} & u(x,0) &= \sin(\pi x) + 2\sin(3\pi x). \end{array} \tag{3.225}$$

Solution.

Using the transformation formula

$$u(x,t) = e^{-t}w(x,t), \tag{3.226}$$

carries (3.225) into

$$\begin{array}{lrl} \text{PDE} & w_t &= w_{xx}, \quad 0<x<1, t>0, \\ \text{BC} & w(0,t) &= 0, \\ & w(1,t) &= 0, \\ \text{IC} & w(x,0) &= \sin(\pi x) + 2\sin(3\pi x). \end{array} \tag{3.227}$$

Setting

$$w(x,t) = F(x)T(t), \tag{3.228}$$

and proceeding as before we obtain

$$w(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t} \sin(n\pi x), \tag{3.229}$$

obtained upon using the boundary conditions. To determine the arbitrary constants B_n, $n \geqslant 1$, we substitute $t = 0$ in (3.229) and use the initial condition to find

$$B_1 \sin(\pi x) + B_2 \sin(2\pi x) + B_3 \sin(3\pi x) + \cdots = \sin(\pi x) + 2\sin(3\pi x), \tag{3.230}$$

which gives

$$B_1 = 1, \quad B_3 = 2, \quad B_k = 0, \quad k \neq 1,3. \tag{3.231}$$

In view of (3.231), Eq. (3.229) becomes

$$w(x,t) = e^{-\pi^2 t}\sin(\pi x) + 2e^{-9\pi^2 t}\sin(3\pi x). \tag{3.232}$$

Substituting (3.232) into (3.226) gives

$$u(x,t) = e^{-t}\left(e^{-\pi^2 t}\sin(\pi x) + 2e^{-9\pi^2 t}\sin(3\pi x)\right). \tag{3.233}$$

For comparisons reasons, the Adomian decomposition method will be used to solve Example 6. Applying the operator L_t^{-1} to both sides of (3.225) and using the initial condition we obtain

$$u(x,t) = \sin(\pi x) + 2\sin(3\pi x) + L_t^{-1}(L_x u - u). \tag{3.234}$$

Using the decomposition series for $u(x,t)$ gives

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) &= \sin(\pi x) + 2\sin(3\pi x)\\ &\quad + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right) - \sum_{n=0}^{\infty} u_n(x,t)\right).\end{aligned} \tag{3.235}$$

Using the recursive algorithm as discussed before we obtain

$$\begin{aligned} u_0 &= \sin(\pi x) + 2\sin(3\pi x),\\ u_1 &= -(\pi^2+1)t\sin(\pi x) - 2(9\pi^2+1)t\sin(3\pi x),\\ u_2 &= (\pi^2+1)^2\frac{t^2}{2!}\sin(\pi x) + 2(9\pi^2+1)^2\frac{t^2}{2!}\sin(3\pi x),\end{aligned} \tag{3.236}$$

and so on. Based on this, the solution in a series for is given by

$$\begin{aligned} u(x,t) &= \sin(\pi x)\left(1 - (\pi^2+1)t + (\pi^2+1)^2\frac{t^2}{2!} - \cdots\right)\\ &\quad + 2\sin(3\pi x)\left(1 - (9\pi^2+1)t + (9\pi^2+1)^2\frac{t^2}{2!} + \cdots\right),\end{aligned} \tag{3.237}$$

and in a closed form by

$$u(x,t) = \sin(\pi x)e^{-(\pi^2+1)t} + 2\sin(3\pi x)e^{-(9\pi^2+1)t}. \tag{3.238}$$

Example 7. Solve the following initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_t = & u_{xx} - 3u + 3, \quad 0 < x < \pi,\ t > 0,\\ \text{BC} & u(0,t) = & 1,\\ & u(\pi,t) = & 1,\\ \text{IC} & u(x,0) = & 1 + \sin x. \end{array} \tag{3.239}$$

Notice here that the lateral heat source has the coefficient -3. This means that if the coefficient of u is $-a$, we use a transformation formula in the form

$$u(x,t) = e^{-at} w(x,t). \tag{3.240}$$

Moreover, because of the inhomogeneous part 3, we add 1 to the transformation formula as will be seen.

Solution.

In this case we use the transformation formula

$$u(x,t) = 1 + e^{-3t} w(x,t), \tag{3.241}$$

that carries (3.239) into

$$\begin{array}{lrl} \text{PDE} & w_t & = w_{xx}, \quad 0 < x < \pi, t > 0, \\ \text{BC} & w(0,t) & = 0, \\ & w(\pi,t) & = 0, \\ \text{IC} & w(x,0) & = \sin x. \end{array} \tag{3.242}$$

Setting

$$w(x,t) = F(x)T(t), \tag{3.243}$$

and proceeding as before we obtain

$$w(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin(nx), \tag{3.244}$$

obtained upon using the boundary conditions. To determine the arbitrary constants B_n, $n \geqslant 1$, we substitute $t = 0$ in (3.244) and use the initial condition to find

$$B_1 \sin x + B_2 \sin(2x) + \cdots = \sin x, \tag{3.245}$$

which gives

$$B_1 = 1, \quad B_k = 0,\ k \geqslant 2. \tag{3.246}$$

In view of (3.246), Eq. (3.229) becomes

$$w(x,t) = e^{-t} \sin x. \tag{3.247}$$

Substituting (3.247) into (3.241) gives

$$u(x,t) = 1 + \sin x e^{-4t}. \tag{3.248}$$

Exercises 3.4.3

Use the method of separation of variables to solve the following heat equations with lateral heat loss:

1. $u_t = u_{xx} - u,\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = 0$
   $u(x,0) = \sin x$
2. $u_t = u_{xx} - u,\ 0 < x < 1,\ t > 0$
   $u(0,t) = 0,\ u(1,t) = 0$
   $u(x,0) = \sin(2\pi x)$
3. $u_t = u_{xx} - 3u,\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = 0$
   $u(x,0) = \sin(x)$
4. $u_t = u_{xx} - 2\pi^2 u,\ 0 < x < 1,\ t > 0$
   $u(0,t) = 0,\ u(1,t) = 0$
   $u(x,0) = \sin(\pi x)$
5. $u_t = u_{xx} - u + 1,\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 1,\ u(\pi,t) = 1$
   $u(x,0) = 1 + \sin x$
6. $u_t = u_{xx} - 2\pi^2 u + 6\pi^2,\ 0 < x < 1,\ t > 0$
   $u(0,t) = 3,\ u(1,t) = 3$
   $u(x,0) = 3 + 3\sin(\pi x)$

References

1. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, (1994).
2. G. Adomian, A review of the decomposition method and some recent results for nonlinear equation, *Math. Comput. Modelling*, 13(7), 17–43, (1992).
3. S.J. Farlow, Partial Differential Equations for Scientists and Engineers, Dover, New York, (1993).
4. J.H. He, A variational iteration method-a kind of nonlinear analytical technique: Some examples, *Int. J. Nonlinear Mech.*, 34, 699–708, (1999).
5. J.H. He, Variational iteration method: some recent results and new interpretations, *J. Comput. and Appl. Math.*, 207(1), 3–17, (2007).
6. F. John, Partial Differential Equations, Springer-Verlag, New York, (1982).
7. A.M. Wazwaz, A reliable modification of Adomian's decomposition method, *Appl. Math. and Comput.*, 92(1), 1–7, (1998).
8. A.M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema Publishers, The Netherlands, (2002).

Chapter 4
Higher Dimensional Heat Flow

4.1 Introduction

This chapter is devoted to the study of the PDEs that control the heat flow in two and three dimensional spaces. The higher dimensional heat flow has been the subject of intensive analytical and numerical investigations. The work in this chapter will run in a parallel manner to the work used in Chapter 3. The study of higher dimensional heat equation will be carried out ONLY by using Adomian decomposition method [1–2] and the method of separation of variables [3,4,6–9]. The two methods have been outlined in Chapters 2 and 3 and were implemented for the heat equation in one dimension.

The decomposition method has been used to obtain analytic and approximate solutions to a wide class of linear and nonlinear, differential and integral equations. It was found by many researchers, that unlike other series solution methods, the decomposition method is easy to program in engineering problems, and provides immediate and visible solution terms without linearization or discretization. As stated in Chapter 2, the concept of rapid convergence of the method was addressed extensively. The main advantage of the decomposition method is that it can be applied directly to all types of differential equations with homogeneous or inhomogeneous boundary conditions [5]. Another important advantage is that the method is capable of reducing the size of computational work, especially for nonlinear models [10–12].

To examine the performance of Adomian's method compared to existing techniques, the method of separation of variables will be implemented. The method of separation of variables provides the solution of a partial differential equation through reducing the equation to a system of ordinary differential equations. In addition, the method requires that the problem and the boundary conditions be linear and homogeneous, hence transformation formulae are usually used to meet this need.

4.2 Adomian Decomposition Method

The Adomian decomposition method has been receiving much attention in recent years in the area of series solutions. A considerable research work has been invested recently in applying this method to a wide class of differential and integral equations. A useful attraction of this method is that it has proved to be a competitive alternative to the Taylor series method [5, 10–12] and other series techniques.

The Adomian method consists of decomposing the unknown function $u(x,t)$ into an infinite sum of components [1–2]. The zeroth component $u_0(x,t)$ is identified by the terms arising from integrating the inhomogeneous term and the initial/boundary conditions. The successive terms are determined in a recursive manner. The method attacks inhomogeneous problems and homogeneous problems in a like manner, thus providing an easily computable technique.

4.2.1 Two Dimensional Heat Flow

The distribution of heat flow in a two dimensional space is governed by the following initial boundary value problem [3,7]

$$\begin{array}{lrl} \text{PDE} & u_t &= \overline{k}(u_{xx}+u_{yy}), 0<x<a,\, 0<y<b, t>0, \\ \text{BC} & u(0,y,t) &= u(a,y,t)=0, \\ & u(x,0,t) &= u(x,b,t)=0, \\ \text{IC} & u(x,y,0) &= f(x,y), \end{array} \tag{4.1}$$

where $u \equiv u(x,y,t)$ is the temperature of any point located at the position (x,y) of a rectangular plate at any time t, and $\overline{k}$ is the thermal diffusivity.

As discussed before, the solution in the t space, the x space, or the y space will produce the same series solution. However, the solution in the t space reduces the size of calculations compared with the other space solutions. For this reason the solution in the t direction will be followed in this chapter.

We first rewrite (4.1) in an operator form by

$$L_t u(x,y,t) = \overline{k}\left(L_x u + L_y u\right), \tag{4.2}$$

where the differential operators L_t, L_x, and L_y are defined by

$$L_t = \frac{\partial}{\partial t}, \quad L_x = \frac{\partial^2}{\partial x^2}, \quad L_y = \frac{\partial^2}{\partial y^2}, \tag{4.3}$$

so that the integral operator L_t^{-1} exists and given by

$$L_t^{-1}(\cdot) = \int_0^t (\cdot)dt. \tag{4.4}$$

Applying L_t^{-1} to both sides of (4.2) and using the initial condition leads to

$$u(x,y,t) = f(x,y) + \overline{k}L_t^{-1}\left(L_x u + L_y u\right). \tag{4.5}$$

The decomposition method defines the solution $u(x,y,t)$ as a series given by

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t), \tag{4.6}$$

where the components $u_n(x,y,t), n \geqslant 0$ will be easily computed by using a recursive algorithm. Substituting (4.6) into both sides of (4.5) yields

$$\sum_{n=0}^{\infty} u_n = f(x,y) + \overline{k}L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{4.7}$$

The decomposition method suggests that the zeroth component $u_0(x,y,t)$ is identified as the terms arising from the initial/boundary conditions and from source terms. The remaining components of $u(x,y,t)$ can be determined in a recursive manner such that each component is determined by using the previous component. Accordingly, the components $u_n(x,y,t)$, $n \geqslant 0$ can be completely determined by following the recurrence relation

$$\begin{aligned} u_0(x,y,t) &= f(x,y), \\ u_{k+1}(x,y,t) &= \overline{k}L_t^{-1}\left(L_x u_k + L_y u_k\right), \quad k \geqslant 0. \end{aligned} \tag{4.8}$$

As a result, the successive components are completely determined, and hence the solution in a series form is thus obtained. Recall that the components can be determined recursively as far as we like. For numerical purposes, the accuracy level can be improved significantly by increasing the number of components determined. As discussed earlier, the closed form solution may also be obtained.

To give a clear overview of the implementation of the decomposition method, we have chosen several examples, homogeneous and inhomogeneous, to illustrate the discussion given above.

Homogeneous Heat Equations

The Adomian decomposition method will be used to solve the following homogeneous heat equation in two dimensions with homogeneous or inhomogeneous boundary conditions [4,6,9].

Example 1. Use the Adomian decomposition method to solve the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_t &= u_{xx}+u_{yy},\ 0<x,y<\pi,\, t>0, \\ \text{BC} & u(0,y,t) &= u(\pi,y,t)=0, \\ & u(x,0,t) &= u(x,\pi,t)=0, \\ \text{IC} & u(x,y,0) &= \sin x \sin y. \end{array} \tag{4.9}$$

Solution.

We first write (4.9) in an operator form by

$$L_t u = L_x u + L_y u. \tag{4.10}$$

Applying the inverse operator L_t^{-1} to (4.10) and using the initial condition we obtain

$$u(x,y,t) = \sin x \sin y + L_t^{-1}\left(L_x u + L_y u\right). \tag{4.11}$$

The decomposition method defines the solution $u(x,y,t)$ as a series given by

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t), \tag{4.12}$$

where the components $u_n(x,y,t), n \geqslant 0$ are to be determined by using a recursive algorithm. Substituting (4.12) into both sides of (4.11) yields

$$\sum_{n=0}^{\infty} u_n = \sin x \sin y + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{4.13}$$

The zeroth component $u_0(x,y,t)$ is identified by all terms that are not included under L_t^{-1}. The components $u_n(x,y,t)$, $n \geqslant 0$ can be completely determined by following the recursive algorithm

$$\begin{aligned} u_0(x,y,t) &= \sin x \sin y, \\ u_{k+1}(x,y,t) &= L_t^{-1}\left(L_x u_k + L_y u_k\right), \quad k \geqslant 0. \end{aligned} \tag{4.14}$$

With u_0 defined as shown above, the first few terms of the decomposition (4.12) are given by

$$\begin{aligned} u_0(x,y,t) &= \sin x \sin y, \\ u_1(x,y,t) &= L_t^{-1}\left(L_x u_0 + L_y u_0\right) = -2t \sin x \sin y, \\ u_2(x,y,t) &= L_t^{-1}\left(L_x u_1 + L_y u_1\right) = \frac{(2t)^2}{2!} \sin x \sin y, \\ u_3(x,y,t) &= L_t^{-1}\left(L_x u_2 + L_y u_2\right) = -\frac{(2t)^3}{3!} \sin x \sin y, \end{aligned} \tag{4.15}$$

and so on. Combining (4.12) and (4.15), the solution in a series form is given by

$$u(x,y,t) = \sin x \sin y \left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \cdots\right), \tag{4.16}$$

and in a closed form by

$$u(x,y,t)=e^{-2t}\sin x\sin y. \tag{4.17}$$

Example 2. Use the Adomian decomposition method to solve the initial-boundary value problem with lateral heat loss

$$\begin{array}{lrcl} \text{PDE} & u_t &=& u_{xx}+u_{yy}-u,\, 0<x,y<\pi,\, t>0, \\ \text{BC} & u(0,y,t) &=& u(\pi,y,t)=0, \\ & u(x,0,t) &=& -u(x,\pi,t)=e^{-3t}\sin x, \\ \text{IC} & u(x,y,0) &=& \sin x\cos y. \end{array} \tag{4.18}$$

Solution.

Applying the inverse operator L_t^{-1} to (4.18) gives

$$u(x,y,t)=\sin x\cos y+L_t^{-1}\left(L_xu+L_yu-u\right). \tag{4.19}$$

The decomposition method defines the solution $u(x,y,t)$ as a series given by

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n(x,y,t). \tag{4.20}$$

Substituting (4.20) into both sides of (4.19) yields

$$\sum_{n=0}^{\infty}u_n=\sin x\cos y+L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n\right)+L_y\left(\sum_{n=0}^{\infty}u_n\right)-\sum_{n=0}^{\infty}u_n\right). \tag{4.21}$$

Proceeding as before we find

$$\begin{aligned} u_0(x,y,t) &= \sin x\cos y, \\ u_{k+1}(x,y,t) &= L_t^{-1}\left(L_xu_k+L_yu_k-u_k\right), \quad k\geqslant 0. \end{aligned} \tag{4.22}$$

It follows that

$$\begin{aligned} u_0(x,y,t) &= \sin x\cos y, \\ u_1(x,y,t) &= L_t^{-1}\left(L_xu_0+L_yu_0-u_0\right)=-3t\sin x\cos y, \\ u_2(x,y,t) &= L_t^{-1}\left(L_xu_1+L_yu_1-u_1\right)=\frac{(3t)^2}{2!}\sin x\cos y, \\ u_3(x,y,t) &= L_t^{-1}\left(L_xu_2+L_yu_2-u_2\right)=-\frac{(3t)^3}{3!}\sin x\cos y, \end{aligned} \tag{4.23}$$

and so on. Combining (4.20) and (4.23), the solution in a series form is given by

$$u(x,y,t)=\sin x\cos y\left(1-3t+\frac{(3t)^2}{2!}-\frac{(3t)^3}{3!}+\cdots\right), \tag{4.24}$$

and in a closed form by

$$u(x,y,t)=e^{-3t}\sin x\cos y. \tag{4.25}$$

Example 3. Use the Adomian decomposition method to solve the initial-boundary value problem

$$
\begin{array}{lrcl}
\text{PDE} & u_t &=& u_{xx}+u_{yy},\ 0<x,y<\pi,\ t>0,\\
\text{BC} & u(0,y,t) &=& -u(\pi,y,t)=e^{-2t}\sin y,\\
 & u(x,0,t) &=& -u(x,\pi,t)=e^{-2t}\sin x,\\
\text{IC} & u(x,y,0) &=& \sin(x+y).
\end{array}
\tag{4.26}
$$

Solution.

It is obvious that the boundary conditions are inhomogeneous. One major advantage of the decomposition method is that it handles any problem in a direct way without any need to transform the inhomogeneous conditions to homogeneous conditions.

Applying the operator L_t^{-1} to the operator form of (4.26) yields

$$
u(x,y,t)=\sin(x+y)+L_t^{-1}\left(L_x u+L_y u\right). \tag{4.27}
$$

Substituting the decomposition series for $u(x,y,t)$ into (4.27) gives

$$
\sum_{n=0}^{\infty} u_n=\sin(x+y)+L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right)+L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{4.28}
$$

Proceeding as before, we set the recurrence relation

$$
\begin{aligned}
u_0(x,y,t) &= \sin(x+y),\\
u_{k+1}(x,y,t) &= L_t^{-1}\left(L_x u_k+L_y u_k\right),\quad k\geqslant 0.
\end{aligned}
\tag{4.29}
$$

Using few terms of the decomposition gives

$$
\begin{aligned}
u_0(x,y,t) &= \sin(x+y),\\
u_1(x,y,t) &= L_t^{-1}\left(L_x u_0+L_y u_0\right)=-2t\sin(x+y),\\
u_2(x,y,t) &= L_t^{-1}\left(L_x u_1+L_y u_1\right)=\frac{(2t)^2}{2!}\sin(x+y),\\
u_3(x,y,t) &= L_t^{-1}\left(L_x u_2+L_y u_2\right)=-\frac{(2t)^3}{3!}\sin(x+y),
\end{aligned}
\tag{4.30}
$$

and so on. The solution in a series form is given by

$$
u(x,y,t)=\sin(x+y)\left(1-2t+\frac{(2t)^2}{2!}-\frac{(2t)^3}{3!}+\cdots\right), \tag{4.31}
$$

and in a closed form by

$$
u(x,y,t)=e^{-2t}\sin(x+y). \tag{4.32}
$$

Inhomogeneous Heat Equations

It was defined before that inhomogeneous heat equation contains one or more terms that do not contain the dependent variable $u(x,y,t)$. A useful advantage of Adomian's method is that it handles homogeneous and inhomogeneous problems in a like manner [10]. The decomposition method, that has been outlined before, will be applied to solve inhomogeneous heat flow equations given by the following illustrative examples.

Example 4. Use the Adomian decomposition method to solve the initial-boundary value problem

$$\begin{array}{lrcl} \text{PDE} & u_t &=& u_{xx}+u_{yy}+\sin y,\ 0<x,y<\pi,\ t>0,\\ \text{BC} & u(0,y,t) &=& u(\pi,y,t)=\sin y,\\ & u(x,0,t) &=& u(x,\pi,t)=0,\\ \text{IC} & u(x,y,0) &=& \sin x\sin y+\sin y. \end{array} \tag{4.33}$$

Solution.

It is obvious that the given equation is an inhomogeneous equation. Unlike the method of separation of variables, the decomposition method handles any problem in a direct way without any need to transform the inhomogeneous equation to a related homogeneous equation.

Operating with L_t^{-1} to the operator form of (4.33) gives

$$u(x,y,t)=\sin x\sin y+\sin y+t\sin y+L_t^{-1}\left(L_x u+L_y u\right). \tag{4.34}$$

Using the decomposition series for $u(x,y,t)$ into (4.34) leads to

$$\sum_{n=0}^{\infty} u_n=\sin x\sin y+\sin y+t\sin y+L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right)+L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{4.35}$$

It follows that the recursive relationship is given by

$$\begin{aligned} u_0(x,y,t) &= \sin x\sin y+\sin y+t\sin y,\\ u_{k+1}(x,y,t) &= L_t^{-1}\left(L_x u_k+L_y u_k\right),\quad k\geqslant 0. \end{aligned} \tag{4.36}$$

This gives

$$\begin{aligned} u_0(x,y,t) &= \sin x\sin y+\sin y+t\sin y,\\ u_1(x,y,t) &= L_t^{-1}\left(L_x u_0+L_y u_0\right)=-2t\sin x\sin y-t\sin y-\frac{t^2}{2!}\sin y,\\ u_2(x,y,t) &= L_t^{-1}\left(L_x u_1+L_y u_1\right)=\frac{(2t)^2}{2!}\sin x\sin y+\frac{t^2}{2!}\sin y+\frac{t^3}{3!}\sin y, \end{aligned} \tag{4.37}$$

and so on. The solution in a series form is given by

$$\begin{aligned} u(x,y,t) &= \sin y + \sin x \sin y\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \cdots\right) \\ &\quad + \left(t \sin y - t \sin y - \frac{t^2}{2!}\sin y + \frac{t^2}{2!}\sin y - \cdots\right), \end{aligned} \tag{4.38}$$

and in a closed form by

$$u(x,y,t) = \sin y + e^{-2t}\sin x \sin y, \tag{4.39}$$

where other terms vanish in the limit.

Example 5. Use the Adomian decomposition method to solve the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_t &= u_{xx} + u_{yy} + 2\cos x \cos y,\ 0 < x, y < \pi,\ t > 0, \\ \text{BC} & u(0,y,t) &= -u(\pi,y,t) = (1 - e^{-2t})\cos y, \\ & u(x,0,t) &= -u(x,\pi,t) = (1 - e^{-2t})\cos x, \\ \text{IC} & u(x,y,0) &= 0. \end{array} \tag{4.40}$$

Solution.

The given partial differential equation and the boundary conditions are inhomogeneous. Our approach will be analogous to that employed in the previous examples.

Applying the inverse operator L_t^{-1} we obtain

$$u(x,y,t) = 2t\cos x \cos y + L_t^{-1}\left(L_x u + L_y u\right). \tag{4.41}$$

Using the decomposition series for $u(x,y,t)$ gives

$$\sum_{n=0}^{\infty} u_n = 2t\cos x \cos y + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{4.42}$$

Proceeding as before, we set

$$\begin{aligned} u_0(x,y,t) &= 2t\cos x \cos y, \\ u_{k+1}(x,y,t) &= L_t^{-1}\left(L_x u_k + L_y u_k\right), \quad k \geqslant 0. \end{aligned} \tag{4.43}$$

Using few terms of the decomposition gives

$$\begin{aligned} u_0(x,y,t) &= 2t\cos x \cos y, \\ u_1(x,y,t) &= L_t^{-1}\left(L_x u_0 + L_y u_0\right) = -\frac{(2t)^2}{2!}\cos x \cos y, \\ u_2(x,y,t) &= L_t^{-1}\left(L_x u_1 + L_y u_1\right) = \frac{(2t)^3}{3!}\cos x \cos y, \end{aligned} \tag{4.44}$$

and so on. The solution in a series form is given by

$$u(x,y,t) = \cos x \cos y \left(2t - \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots\right), \tag{4.45}$$

and in a closed form by

$$u(x,y,t) = (1 - e^{-2t}) \cos x \cos y. \tag{4.46}$$

Exercises 4.2.1

In Exercises 1–6, use the decomposition method to solve the homogeneous initial-boundary value problems:

1. $u_t = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = \sin x \sin y$

2. $u_t = u_{xx} + u_{yy},\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = 2\sin x \sin y$

3. $u_t = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = -u(\pi,y,t) = e^{-4t} \cos y$
   $u(x,0,t) = -u(x,\pi,t) = e^{-4t} \cos x$
   $u(x,y,0) = \cos(x+y)$

4. $u_t = 3(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = -u(\pi,y,t) = -e^{-6t} \sin y$
   $u(x,0,t) = -u(x,\pi,t) = e^{-6t} \sin x$
   $u(x,y,0) = \sin(x-y)$

5. $u_t = 2(u_{xx} + u_{yy}) - u,\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = \sin x \sin y$

6. $u_t = 3(u_{xx} + u_{yy}) - 2u,\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = -u(\pi,y,t) = e^{-8t} \sin y$
   $u(x,0,t) = -u(x,\pi,t) = e^{-8t} \sin x$
   $u(x,y,0) = \sin(x+y)$

In Exercises 7–12, use the decomposition method to solve the inhomogeneous initial-boundary value problems:

7. $u_t = 2(u_{xx} + u_{yy}) + 2\sin x,\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 0$

$u(x,0,t) = u(x,\pi,t) = \sin x$
$u(x,y,0) = \sin x \sin y + \sin x$

8. $u_t = 3(u_{xx} + u_{yy}) + 3\cos x,\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = -u(\pi,y,t) = 1$
   $u(x,0,t) = u(x,\pi,t) = \cos x$
   $u(x,y,0) = \sin x \sin y + \cos x$

9. $u_t = u_{xx} + u_{yy} + 2\cos(x+y),\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = (e^{-2t}\sin y + \cos y)$
   $u(\pi,y,t) = -(e^{-2t}\sin y + \cos y)$
   $u(x,0,t) = (e^{-2t}\sin x + \cos x)$
   $u(x,\pi,t) = -(e^{-2t}\sin x + \cos x)$
   $u(x,y,0) = \sin(x+y) + \cos(x+y)$

10. $u_t = u_{xx} + u_{yy} = \sin x + \sin y,\ 0 < x,y < \pi,\ t > 0$
    $u(0,y,t) = u(\pi,y,t) = \sin y$
    $u(x,0,t) = u(x,\pi,t) = \sin x$
    $u(x,y,0) = \sin x(1 + \sin y) + \sin y$

11. $u_t = u_{xx} + u_{yy} - 2,\ 0 < x,y < \pi,\ t > 0$
    $u(0,y,t) = 0,\ u(\pi,y,t) = \pi^2$
    $u(x,0,t) = u(x,\pi,t) = x^2$
    $u(x,y,0) = x^2 + \sin x \sin y$

12. $u_t = u_{xx} + u_{yy} - 2,\ 0 < x,y < \pi,\ t > 0$
    $u(0,y,t) = u(\pi,y,t) = y^2$
    $u(x,0,t) = 0,\ u(x,\pi,t) = \pi^2$
    $u(x,y,0) = \sin x \sin y + y^2$

4.2.2 Three Dimensional Heat Flow

The distribution of heat flow in a three dimensional space [8, 10] is governed by the following initial boundary value problem

$$
\begin{array}{lrl}
\text{PDE} & u_t = & \overline{k}(u_{xx} + u_{yy} + u_{zz}),\ t > 0, \\
 & & 0 < x < a, 0 < y < b, 0 < z < c, \\
\text{BC} & u(0,y,z,t) = & u(a,y,z,t) = 0, \\
 & u(x,0,z,t) = & u(x,b,z,t) = 0, \\
 & u(x,y,0,t) = & u(x,y,c,t) = 0, \\
\text{IC} & u(x,y,z,0) = & f(x,y,z),
\end{array}
\tag{4.47}
$$

where $u \equiv u(x,y,z,t)$ is the temperature of any point located at the position (x,y,z) of a rectangular volume at any time t, and $\overline{k}$ is the thermal diffusivity.

We first rewrite (4.47) in an operator form by

$$L_t u = \overline{k}(L_x u + L_y u + L_z u), \tag{4.48}$$

where the differential operators L_x, L_y, and L_z are defined by

$$L_t = \frac{\partial}{\partial t}, \quad L_x = \frac{\partial^2}{\partial x^2}, \quad L_y = \frac{\partial^2}{\partial y^2}, \quad L_z = \frac{\partial^2}{\partial z^2}, \tag{4.49}$$

so that the integral operator L_t^{-1} exists and given by

$$L_t^{-1}(\cdot) = \int_0^t (\cdot)dt. \tag{4.50}$$

Applying L_t^{-1} to both sides of (4.48) and using the initial condition leads to

$$u(x,y,t) = f(x,y,z) + \overline{k}L_t^{-1}\left(L_x u + L_y u + L_z u\right). \tag{4.51}$$

The decomposition method defines the solution $u(x,y,z,t)$ as a series given by

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t). \tag{4.52}$$

Substituting (4.52) into both sides of (4.51) yields

$$\begin{aligned}\sum_{n=0}^{\infty} u_n &= f(x,y,z)\\ &\quad + \overline{k}L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right).\end{aligned} \tag{4.53}$$

The components $u_n(x,y,z,t)$, $n \geqslant 0$ can be completely determined by using the recursive relationship

$$\begin{aligned} u_0(x,y,z,t) &= f(x,y,z), \\ u_{k+1}(x,y,z,t) &= \overline{k}L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k\right), \quad k \geqslant 0. \end{aligned} \tag{4.54}$$

The components can be determined recursively as far as we like. Consequently, the components $u_n, n \geqslant 0$, are completely determined and the solution in a series form follows immediately.

Homogeneous Heat Equations

The decomposition method will be used to discuss the following homogeneous heat equations.

Example 6. Solve the following initial-boundary value problem

$$\begin{array}{lrcl} \text{PDE} & u_t & = & u_{xx}+u_{yy}+u_{zz},\ 0<x,y,z<\pi,\ t>0, \\ \text{BC} & u(0,y,z,t) & = & u(\pi,y,z,t)=0, \\ & u(x,0,z,t) & = & u(x,\pi,z,t)=0, \\ & u(x,y,0,t) & = & u(x,y,\pi,t)=0, \\ \text{IC} & u(x,y,z,0) & = & 2\sin x\sin y\sin z. \end{array} \quad (4.55)$$

Solution.

Applying the inverse operator L_t^{-1} to the operator form of (4.55) gives

$$u(x,y,z,t)=2\sin x\sin y\sin z+L_t^{-1}\left(L_x u+L_y u+L_z u\right). \quad (4.56)$$

Using the decomposition series

$$u(x,y,z,t)=\sum_{n=0}^{\infty} u_n(x,y,z,t) \quad (4.57)$$

into (4.56) yields

$$\begin{aligned} \sum_{n=0}^{\infty} u_n &= 2\sin x\sin y\sin z \\ &+L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right)+L_y\left(\sum_{n=0}^{\infty} u_n\right)+L_z\left(\sum_{n=0}^{\infty} u_n\right)\right). \end{aligned} \quad (4.58)$$

The components $u_n(x,y,z,t)$, $n\geqslant 0$ can be determined by using the recurrence relation

$$\begin{aligned} u_0(x,y,z,t) &= 2\sin x\sin y\sin z, \\ u_{k+1}(x,y,z,t) &= L_t^{-1}\left(L_x u_k+L_y u_k+L_z u_k\right), \quad k\geqslant 0. \end{aligned} \quad (4.59)$$

It follows that the first few terms of the decomposition series of $u(x,y,z,t)$ are given by

$$\begin{aligned} u_0(x,y,z,t) &= 2\sin x\sin y\sin z, \\ u_1(x,y,z,t) &= L_t^{-1}\left(L_x u_0+L_y u_0+L_z u_0\right)=-2(3t)\sin x\sin y\sin z, \\ u_2(x,y,z,t) &= L_t^{-1}\left(L_x u_1+L_y u_1+L_z u_1\right)=\frac{2(3t)^2}{2!}\sin x\sin y\sin z, \\ u_3(x,y,z,t) &= L_t^{-1}\left(L_x u_2+L_y u_2+L_z u_2\right)=-\frac{2(3t)^3}{3!}\sin x\sin y\sin z, \end{aligned} \quad (4.60)$$

and so on. As indicated before, further components can be easily computed to increase the level of accuracy.

Combining (4.57) and (4.60), the solution in a series form is given by

$$u(x,y,z,t)=2\sin x\sin y\sin z\left(1-3t+\frac{(3t)^2}{2!}-\frac{(3t)^3}{3!}+\cdots\right), \quad (4.61)$$

and in a closed form by

$$u(x,y,z,t) = 2e^{-3t} \sin x \sin y \sin z. \tag{4.62}$$

Example 7. Solve the following initial-boundary value problem with lateral heat loss

$$\begin{array}{lrcl} \text{PDE} & u_t &=& u_{xx} + u_{yy} + u_{zz} - 2u, \quad 0 < x,y,z < \pi,\ t > 0, \\ \text{BC} & u(0,y,z,t) &=& u(\pi,y,z,t) = 0, \\ & u(x,0,z,t) &=& u(x,\pi,z,t) = 0, \\ & u(x,y,0,t) &=& u(x,y,\pi,t) = 0, \\ \text{IC} & u(x,y,z,0) &=& \sin x \sin y \sin z. \end{array} \tag{4.63}$$

Solution.

Operating with L_t^{-1} on (4.63) we obtain

$$u(x,y,z,t) = \sin x \sin y \sin z + L_t^{-1}\left(L_x u + L_y u + L_z u - 2u\right). \tag{4.64}$$

Proceeding as before we find

$$\begin{aligned} \sum_{n=0}^{\infty} u_n &= \sin x \sin y \sin z \\ &+ L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right) - 2\left(\sum_{n=0}^{\infty} u_n\right)\right). \end{aligned} \tag{4.65}$$

Using the assumptions of the decomposition method yields

$$\begin{aligned} u_0(x,y,z,t) &= \sin x \sin y \sin z, \\ u_{k+1}(x,y,z,t) &= L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k - 2u_k\right), \quad k \geqslant 0. \end{aligned} \tag{4.66}$$

Consequently, we obtain

$$\begin{aligned} u_0 &= \sin x \sin y \sin z, \\ u_1 &= L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0 - 2u_0\right) = -5t \sin x \sin y \sin z, \\ u_2 &= L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1 - 2u_1\right) = \frac{(5t)^2}{2!} \sin x \cos y \cos z, \\ u_3 &= L_t^{-1}\left(L_x u_2 + L_y u_2 + L_z u_2 - 2u_2\right) = -\frac{(5t)^3}{3!} \sin x \sin y \sin z. \end{aligned} \tag{4.67}$$

The solution in a series form is given by

$$u(x,y,z,t) = \sin x \sin y \sin z \left(1 - 5t + \frac{(5t)^2}{2!} - \frac{(5t)^3}{3!} + \cdots\right), \tag{4.68}$$

and in a closed form by

$$u(x,y,z,t) = e^{-5t}\sin x \sin y \sin z, \tag{4.69}$$

obtained upon using the Taylor series of e^{-5t}.

Example 8. Solve the following initial-boundary value problem

$$\begin{array}{lrl}
\text{PDE} & u_t &= 2(u_{xx}+u_{yy}+u_{zz}), \quad 0<x,y,z<\pi, t>0,\\
\text{BC} & u(0,y,z,t) &= u(\pi,y,z,t) = 0,\\
 & u(x,0,z,t) &= -u(x,\pi,z,t) = e^{-6t}\sin x \cos z,\\
 & u(x,y,0,t) &= -u(x,y,\pi,t) = e^{-6t}\sin x \cos y,\\
\text{IC} & u(x,y,z,0) &= \sin x \cos y \cos z.
\end{array} \tag{4.70}$$

Solution.

We first note that the boundary conditions are inhomogeneous. Following the previous discussion we obtain

$$u(x,y,z,t) = \sin x \cos y \cos z + 2L_t^{-1}\left(L_x u + L_y u + L_z u\right), \tag{4.71}$$

and hence we find

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n &= \sin x \cos y \cos z\\
&+ 2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right).
\end{aligned} \tag{4.72}$$

With u_0 defined as shown above, we set the relation

$$\begin{aligned}
u_0(x,y,z,t) &= \sin x \cos y \cos z,\\
u_{k+1}(x,y,z,t) &= 2L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k\right), \quad k \geqslant 0.
\end{aligned} \tag{4.73}$$

Consequently, the first few components

$$\begin{aligned}
u_0(x,y,z,t) &= \sin x \cos y \cos z,\\
u_1(x,y,z,t) &= 2L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0\right) = -6t \sin x \cos y \cos z,\\
u_2(x,y,z,t) &= 2L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1\right) = \frac{(6t)^2}{2!}\sin x \cos y \cos z,\\
u_3(x,y,z,t) &= 2L_t^{-1}\left(L_x u_2 + L_y u_2 + L_z u_2\right) = -\frac{(6t)^3}{3!}\sin x \cos y \cos z,
\end{aligned} \tag{4.74}$$

are obtained. The solution in a series form

$$u(x,y,z,t) = \sin x \cos y \cos z\left(1 - 6t + \frac{(6t)^2}{2!} - \frac{(6t)^3}{3!} + \cdots\right), \tag{4.75}$$

is readily obtained, and hence the exact solution

$$u(x,y,z,t) = e^{-6t}\sin x\cos y\cos z, \tag{4.76}$$

follows immediately.

Inhomogeneous Heat Equations

In the following, the Adomian decomposition method will be applied to inhomogeneous heat equations. The method will be implemented in a like manner to that used in homogeneous cases.

Example 9. Solve the following initial-boundary value problem

$$\begin{array}{lrcl} \text{PDE} & u_t &=& (u_{xx}+u_{yy}+u_{zz})-2, \quad 0<x,y,z<\pi,\, t>0,\\ \text{BC} & u(0,y,z,t) &=& u(\pi,y,z,t)=z^2,\\ & u(x,0,z,t) &=& u(x,\pi,z,t)=z^2,\\ & u(x,y,0,t) &=& 0,\, u(x,y,\pi,t)=\pi^2,\\ \text{IC} & u(x,y,z,0) &=& z^2+\sin x\sin y\sin z. \end{array} \tag{4.77}$$

Solution.

We first note that the PDE and the boundary conditions are inhomogeneous. Applying the inverse operator L_t^{-1} to (4.77) and using the initial condition we obtain

$$u(x,y,z,t) = -2t+z^2+\sin x\sin y\sin z+L_t^{-1}\left(L_xu+L_yu+L_zu\right), \tag{4.78}$$

and proceeding as before we find

$$\begin{aligned} \sum_{n=0}^{\infty} u_n &= -2t+z^2+\sin x\sin y\sin z\\ &+L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right)+L_y\left(\sum_{n=0}^{\infty} u_n\right)+L_z\left(\sum_{n=0}^{\infty} u_n\right)\right). \end{aligned} \tag{4.79}$$

We next set the recurrence relation

$$\begin{aligned} u_0(x,y,z,t) &= -2t+z^2+\sin x\sin y\sin z,\\ u_{k+1}(x,y,z,t) &= L_t^{-1}\left(L_xu_k+L_yu_k+L_zu_k\right), \quad k\geqslant 0. \end{aligned} \tag{4.80}$$

The first few terms of the decomposition series are

$$\begin{aligned} u_0(x,y,z,t) &= -2t+z^2+\sin x\sin y\sin z,\\ u_1(x,y,z,t) &= -3t\sin x\sin y\sin z+2t,\\ u_2(x,y,z,t) &= \frac{(3t)^2}{2!}\sin x\sin y\sin z,\\ u_3(x,y,z,t) &= -\frac{(3t)^3}{3!}\sin x\sin y\sin z. \end{aligned} \tag{4.81}$$

The solution in a series form is given by

$$u(x,y,z,t) = z^2 + \sin x \sin y \sin z \left(1 - 3t + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \cdots\right), \tag{4.82}$$

and in a closed form by

$$u(x,y,z,t) = z^2 + e^{-3t} \sin x \sin y \sin z. \tag{4.83}$$

Example 10. Solve the following initial-boundary value problem

$$\begin{array}{lrcl} \text{PDE} & u_t &=& (u_{xx} + u_{yy} + u_{zz}) + \sin z, \quad 0 < x,y,z < \pi,\, t > 0, \\ \text{BC} & u(0,y,z,t) &=& \sin z + e^{-2t} \sin y, \\ & u(\pi,y,z,t) &=& \sin z - e^{-2t} \sin y, \\ & u(x,0,z,t) &=& \sin z + e^{-2t} \sin x, \\ & u(x,\pi,z,t) &=& \sin z - e^{-2t} \sin x, \\ & u(x,y,0,t) &=& u(x,y,\pi,t) = e^{-2t} \sin(x+y), \\ \text{IC} & u(x,y,z,0) &=& \sin(x+y) + \sin z. \end{array} \tag{4.84}$$

Solution.

It is clear that the PDE and the boundary conditions are inhomogeneous. Applying the inverse operator L_t^{-1} to (4.84) gives

$$u(x,y,z,t) = \sin(x+y) + \sin z + t \sin z + L_t^{-1} \left(L_x u + L_y u + L_z u\right), \tag{4.85}$$

and this in turn gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n &= \sin(x+y) + \sin z + t \sin z \\ &+ L_t^{-1} \left(L_x \left(\sum_{n=0}^{\infty} u_n \right) + L_y \left(\sum_{n=0}^{\infty} u_n \right) + L_z \left(\sum_{n=0}^{\infty} u_n \right) \right). \end{aligned} \tag{4.86}$$

Accordingly, we set the recursive relationship

$$\begin{aligned} u_0(x,y,z,t) &= \sin(x+y) + \sin z + t \sin z, \\ u_{k+1}(x,y,z,t) &= L_t^{-1} \left(L_x u_k + L_y u_k + L_z u_k\right), \quad k \geqslant 0. \end{aligned} \tag{4.87}$$

The first few terms of the decomposition are

$$\begin{aligned} u_0(x,y,z,t) &= \sin(x+y) + \sin z + t \sin z, \\ u_1(x,y,z,t) &= -2t \sin(x+y) - t \sin z - \frac{t^2}{2!} \sin z, \\ u_2(x,y,z,t) &= \frac{(2t)^2}{2!} \sin(x+y) + \frac{t^2}{2!} \sin z + \frac{t^3}{3!} \sin z. \end{aligned} \tag{4.88}$$

The solution in a series form is given by

$$\begin{aligned} u(x,y,z,t) &= \sin z + \sin(x+y)\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \cdots\right) \\ &\quad + \left(t\sin z - t\sin z - \frac{t^2}{2!}\sin z + \frac{t^2}{2!}\sin z + \cdots\right), \end{aligned} \tag{4.89}$$

and in a closed form by

$$u(x,y,z,t) = \sin z + e^{-2t}\sin(x+y). \tag{4.90}$$

Exercises 4.2.2

In Exercises 1–4, use the decomposition method to solve the homogeneous initial-boundary value problems:

1. $u_t = 2(u_{xx} + u_{yy} + u_{zz}),\ 0 < x,y,z < \pi,\ t > 0$
   $u(0,y,z,t) = u(\pi,y,z,t) = 0$
   $u(x,0,z,t) = u(x,\pi,z,t) = 0$
   $u(x,y,0,t) = u(x,y,\pi,t) = 0$
   $u(x,y,z,0) = \sin x \sin y \sin z$

2. $u_t = u_{xx} + u_{yy} + u_{zz},\ 0 < x,y,z < \pi,\ t > 0$
   $u(0,y,z,t) = u(\pi,y,z,t) = 0$
   $u(x,0,z,t) = u(x,\pi,z,t) = 0$
   $u(x,y,0,t) = u(x,y,\pi,t) = 0$
   $u(x,y,z,0) = 2\sin x \sin y \sin z$

3. $u_t = u_{xx} + u_{yy} + u_{zz},\ 0 < x,y,z < \pi,\ t > 0$
   $u(0,y,z,t) = -u(\pi,y,z,t) = e^{-3t}\sin(y+z)$
   $u(x,0,z,t) = -u(x,\pi,z,t) = e^{-3t}\sin(x+z)$
   $u(x,y,0,t) = -u(x,y,\pi,t) = e^{-3t}\sin(x+y)$
   $u(x,y,z,0) = \sin(x+y+z)$

4. $u_t = u_{xx} + u_{yy} + u_{zz} - u,\ 0 < x,y,z < \pi,\ t > 0$
   $u(0,y,z,t) = u(\pi,y,z,t) = 0$
   $u(x,0,z,t) = u(x,\pi,z,t) = 0$
   $u(x,y,0,t) = u(x,y,\pi,t) = 0$
   $u(x,y,z,0) = \sin x \sin y \sin z$

In Exercises 5–8, solve the inhomogeneous initial-boundary value problems:

5. $u_t = u_{xx} + u_{yy} + u_{zz} - 4,\ 0 < x,y,z < \pi,\ t > 0$
   $u(0,y,z,t) = 0,\ u(\pi,y,z,t) = 2\pi^2$
   $u(x,0,z,t) = u(x,\pi,z,t) = 2x^2$

$u(x,y,0,t) = u(x,y,\pi,t) = 2x^2$
$u(x,y,z,0) = 2x^2 + \sin x \sin y \sin z$

6. $u_t = u_{xx} + u_{yy} + u_{zz} - 2,\ 0 < x,y,z < \pi,\ t > 0$
$u(0,y,z,t) = u(\pi,y,z,t) = y^2$
$u(x,0,z,t) = 0,\ u(x,\pi,z,t) = \pi^2$
$u(x,y,0,t) = u(x,y,\pi,t) = y^2$
$u(x,y,z,0) = y^2 + \sin x \sin y \sin z$

7. $u_t = u_{xx} + u_{yy} + u_{zz} + \sin x,\ 0 < x,y,z < \pi,\ t > 0$
$u(0,y,z,t) = u(\pi,y,z,t) = e^{-2t}\sin(y+z)$
$u(x,0,z,t) = \sin x + e^{-2t}\sin z,\ u(x,\pi,z,t) = \sin x - e^{-2t}\sin z$
$u(x,y,0,t) = \sin x + e^{-2t}\sin y,\ u(x,y,\pi,t) = \sin x - e^{-2t}\sin y$
$u(x,y,z,0) = \sin x + \sin(y+z)$

8. $u_t = u_{xx} + u_{yy} + u_{zz} - 2,\ 0 < x,y,z < \pi,\ t > 0$
$u(0,y,z,t) = e^{-3t}(\sin y + \sin z)$
$u(\pi,y,z,t) = \pi^2 + e^{-3t}(\sin y + \sin z)$
$u(x,0,z,t) = u(x,\pi,z,t) = x^2 + e^{-3t}(\sin x + \sin z)$
$u(x,y,0,t) = u(x,y,\pi,t) = x^2 + e^{-3t}(\sin x + \sin y)$
$u(x,y,z,0) = x^2 + (\sin x + \sin y + \sin z)$

4.3 Method of Separation of Variables

In this section, the heat flow in a two dimensional space and a three dimensional space will be discussed by using the classical method of the separation of variables. As discussed in Chapter 3, this method converts the partial differential equation by a system of ordinary differential equations that are usually easy to handle. The resulting ODEs are then solved independently. We then proceed as discussed in Chapter 3 and apply the boundary and the initial equations to determine the constants of integration. Unlike the Adomian decomposition method, it is well known that the method of separation of variables [7, 10] is commonly used for the case where the PDE and the boundary conditions are linear and homogeneous. For inhomogeneous equations, transformation formulas are used to convert the inhomogeneous equations to homogeneous equations.

4.3.1 Two Dimensional Heat Flow

The distribution of heat flow in a two dimensional space is governed by the following initial boundary value problem

$$
\begin{array}{lrl}
\text{PDE} & u_t = & \overline{k}(u_{xx}+u_{yy}), \quad 0<x<a, 0<y<b, t>0, \\
\text{BC} & u(0,y,t) = & u(a,y,t)=0, \\
 & u(x,0,t) = & u(x,b,t)=0, \\
\text{IC} & u(x,y,0) = & f(x,y).
\end{array}
\tag{4.91}
$$

where $u \equiv u(x,y,t)$ defines the temperature of any point at the position (x,y) of a rectangular plate at any time t, and $\overline{k}$ is the thermal diffusivity.

The method of separation of variables is based on an assumption that the solution $u(x,y,t)$ can be expressed as the product of distinct functions $F(x), G(y)$, and $T(t)$, such that each function depends on one variable only. Based on this assumption, we first set

$$
u(x,y,t) = F(x)G(y)T(t). \tag{4.92}
$$

Differentiating both sides of (4.92) with respect to t and twice with respect to x and y respectively, we obtain

$$
\begin{aligned}
u_t &= F(x)G(y)T'(t), \\
u_{xx} &= F''(x)G(y)T(t), \\
u_{yy} &= F(x)G''(y)T(t).
\end{aligned}
\tag{4.93}
$$

Substituting (4.93) into (4.91) leads to

$$
F(x)G(y)T'(t) = \overline{k}\left(F''(x)G(y)T(t) + F(x)G''(y)T(t)\right). \tag{4.94}
$$

Dividing both sides of (4.94) by $\overline{k}F(x)G(y)T(t)$ yields

$$
\frac{T'(t)}{\overline{k}T(t)} = \frac{F''(x)}{F(x)} + \frac{G''(y)}{G(y)}. \tag{4.95}
$$

It is obvious that the left hand side depends only on t and the right hand side depends only on x and y. This means that the equality holds only if both sides are equal to the same constant. Assuming that the right hand side is a constant, it is valid to assume that it is the sum of two constants. Therefore, we set

$$
\frac{F''(x)}{F(x)} = -\lambda^2, \tag{4.96}
$$

and

$$
\frac{G''(y)}{G(y)} = -\mu^2. \tag{4.97}
$$

Consequently, we find

$$
F''(x) + \lambda^2 F(x) = 0, \tag{4.98}
$$

and

$$
G''(y) + \mu^2 G(y) = 0. \tag{4.99}
$$

This means that the left hand side of (4.95) is equal to $-(\lambda^2+\mu^2)$. Accordingly, we obtain

$$\frac{T'(t)}{\overline{k}T(t)} = -(\lambda^2 + \mu^2), \tag{4.100}$$

or equivalently

$$T'(t) + \overline{k}(\lambda^2 + \mu^2)T(t) = 0. \tag{4.101}$$

The selection of $-(\lambda^2 + \mu^2)$ is the only selection that will provide nontrivial solutions. Besides, this selection is made in accordance with the natural fact that the factor $T(t)$, and hence the temperature $u(x,y,t)$, must vanish as $t \to \infty$. From physics, we know that the temperature component $T(t)$ follows the exponential decay phenomena.

It is interesting to note that the partial differential equation (4.91) has been transformed to three ordinary differential equations, two second order ODEs given by (4.98) and (4.99), and a first order ODE given by (4.101).

The solution of (4.101) is given by

$$T(t) = Ce^{-\overline{k}(\lambda^2+\mu^2)t}, \tag{4.102}$$

where C is a constant. The result (4.102) explains the fact that $T(t)$ must follow the exponential decay of heat flow. If accidently we selected the constant to equal $(\lambda^2 + \mu^2)$, this will result in an exponential growth of the Temperature factor $T(t)$. In this case, $T(t) \to \infty$ and consequently $u(x,y,t) \to \infty$ as $t \to \infty$. This contradicts the natural behavior of the heat flow [3, 10].

The second order differential equations (4.98) and (4.99) give the solutions

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{4.103}$$

and

$$G(y) = \alpha\cos(\mu y) + \beta\sin(\mu y), \tag{4.104}$$

where A, B, α, and β are constants that will be determined.

To determine the constants A and B, we use the boundary conditions at $x = 0$ and at $x = a$ to find that

$$\begin{aligned} F(0)G(y)T(t) &= 0, \\ F(a)G(y)T(t) &= 0, \end{aligned} \tag{4.105}$$

which gives

$$\begin{aligned} F(0) &= 0, \\ F(a) &= 0. \end{aligned} \tag{4.106}$$

Substituting (4.106) into (4.103) gives

$$A = 0, \tag{4.107}$$

and

$$\lambda_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \cdots. \tag{4.108}$$

It is important to note here that we exclude $n = 0$ and $B = 0$ because each will lead to the trivial solution $u(x,y,t) = 0$. Using the results obtained for the constants A

and λ_n, we therefore write the functions

$$F_n(x) = B_n \sin(\frac{n\pi}{a}x), \quad n = 1,2,3,\cdots. \tag{4.109}$$

In a parallel manner, we use the second boundary condition at $y=0$ and at $y=b$ into (4.104) to find that

$$\alpha = 0, \tag{4.110}$$

and

$$\lambda_m = \frac{m\pi}{b}, \quad m = 1,2,3,\cdots. \tag{4.111}$$

We exclude $m=0$ and $\beta=0$, because each will lead to the trivial solution as indicated before. Consequently, $G(y)$ can be the functions

$$G_m(y) = \beta_m \sin(\frac{m\pi}{b}y), \quad m = 1,2,3,\cdots. \tag{4.112}$$

Based on the infinite number of values for λ_n and μ_m, then $T(t)$ of (4.102) takes the functions

$$T_{nm} = \overline{C}_{nm} e^{-\overline{k}(\frac{n^2}{a^2}+\frac{m^2}{b^2})\pi^2 t} \tag{4.113}$$

Ignoring the constants B_n, β_m, and $\overline{C}_{nm}$, we conclude that the functions, that form the set of fundamental solutions,

$$\begin{aligned} u_{nm} &= F_n(x)G_m(y)T_{nm}(t) \\ &= \sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)e^{-\overline{k}(\frac{n^2}{a^2}+\frac{m^2}{b^2})\pi^2 t}, \quad n,m = 1,2,\cdots, \end{aligned} \tag{4.114}$$

satisfy (4.91) and the boundary conditions.

Using the superposition principle, we obtain

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} e^{-\overline{k}(\frac{n^2}{a^2}+\frac{m^2}{b^2})\pi^2 t} \sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y), \tag{4.115}$$

where the arbitrary constants C_{nm} are as yet undetermined.

To determine the constants C_{nm}, we use the given initial condition to find

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y) = f(x,y). \tag{4.116}$$

The constants C_{nm} are completely determined by using a double Fourier coefficients where we find

$$C_{nm} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)dx\,dy. \tag{4.117}$$

Having determined the constants C_{nm}, the solution given by (4.115) is completely determined.

It is interesting to point out that the constants C_{nm} can also be determined by equating the coefficients of both sides if the initial condition is defined explicitly in terms of $\sin(\gamma x)\sin(\delta y)$, where γ and δ are constants. This will reduce the massive size of calculations required by the computational work of the double Fourier series.

The method will be illustrated by discussing the following examples.

Example 1. Use the method of separation of variables to solve the initial-boundary value problem:

$$\begin{array}{llll} \text{PDE} & u_t &= u_{xx}+u_{yy}, & 0<x,y<\pi, t>0,\\ \text{BC} & u(0,y,t) &= u(\pi,y,t)=0, & \\ & u(x,0,t) &= u(x,\pi,t)=0, & \\ \text{IC} & u(x,y,0) &= 2\sin x\sin y. & \end{array} \tag{4.118}$$

Solution.

As discussed before, we first set

$$u(x,y,t)=F(x)G(y)T(t). \tag{4.119}$$

Proceeding as before, we obtain

$$F''(x)+\lambda^2F(x)=0, \tag{4.120}$$

$$G''(y)+\mu^2G(y)=0, \tag{4.121}$$

and

$$T'(t)+(\lambda^2+\mu^2)T(t)=0, \tag{4.122}$$

where λ and μ are constants.

The second order ordinary differential equations (4.120) and (4.121) give the solutions

$$F(x)=A\cos(\lambda x)+B\sin(\lambda x), \tag{4.123}$$

and

$$G(y)=\alpha\cos(\mu y)+\beta\sin(\mu y), \tag{4.124}$$

respectively, where A,B,α and β are constants. Using the boundary conditions of (4.118) into (4.123) gives

$$A=0, \tag{4.125}$$

and

$$\lambda_n=n, \quad n=1,2,3,\cdots, \tag{4.126}$$

so that

$$F_n(x)=B_n\sin(nx), \quad n=1,2,3,\cdots. \tag{4.127}$$

Similarly, using the boundary conditions of (4.118) into (4.124) gives

$$\alpha=0, \tag{4.128}$$

and

$$\mu_m = m, \quad m = 1,2,3,\cdots, \tag{4.129}$$

so that

$$G_m(y) = \beta_m \sin(my), \quad m = 1,2,3,\cdots. \tag{4.130}$$

The solution of the first order differential equation (4.122) is given by

$$T_{nm}(t) = \overline{C}_{nm} e^{-(\lambda^2+\mu^2)t}, \tag{4.131}$$

and by substituting λ and μ we obtain

$$T_{nm}(t) = \overline{C}_{nm} e^{-(n^2+m^2)t}. \tag{4.132}$$

Combining (4.127), (4.130) and (4.132) and using the superposition principle, the general solution of the problem is given by the double series

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} e^{-(n^2+m^2)t} \sin(nx)\sin(my). \tag{4.133}$$

To determine the constants C_{nm}, we use the given initial condition and expand the double series to find

$$C_{11} \sin x \sin y + C_{12} \sin x \sin(2y) + \cdots = 2 \sin x \sin y. \tag{4.134}$$

Equating the coefficients on both sides yields

$$C_{11} = 2, C_{ij} = 0, i \neq 1, j \neq 1. \tag{4.135}$$

Accordingly, the particular solution is given by

$$u(x,y,t) = 2e^{-2t} \sin x \sin y. \tag{4.136}$$

Example 2. Use the method of separation of variables to solve the initial-boundary value problem:

$$\begin{array}{lrcl} \text{PDE} & u_t &=& u_{xx} + u_{yy}, \quad 0 < x,y < \pi, t > 0, \\ \text{BC} & u(0,y,t) &=& u(\pi,y,t) = 0, \\ & u(x,0,t) &=& u(x,\pi,t) = 0, \\ \text{IC} & u(x,y,0) &=& \sin x \sin y + 2 \sin x \sin(2y). \end{array} \tag{4.137}$$

Solution.

As discussed before, we set

$$u(x,y,t) = F(x)G(y)T(t). \tag{4.138}$$

Proceeding as before, we obtain

$$\frac{T'(t)}{T(t)} = \frac{F''(x)}{F(x)} + \frac{G''(y)}{G(y)}. \tag{4.139}$$

Proceeding as before, we obtain

$$F''(x) + \lambda^2 F(x) = 0, \tag{4.140}$$

$$G''(y) + \mu^2 G(y) = 0, \tag{4.141}$$

and

$$T'(t) + (\lambda^2 + \mu^2)T(t) = 0. \tag{4.142}$$

Solving (4.140) and (4.141) and using the boundary conditions leads to

$$F_n(x) = B_n \sin(nx), \quad \lambda_n = n, \quad n = 1,2,3,\cdots, \tag{4.143}$$

and

$$G_m(y) = \beta_m \sin(my), \quad \mu_m = m, \quad m = 1,2,3,\cdots, \tag{4.144}$$

respectively. The solution of the first order differential equation (4.142) is given by

$$T(t) = Ce^{-(\lambda^2+\mu^2)t}, \tag{4.145}$$

and by substituting λ and μ we obtain

$$T_{nm}(t) = \overline{C}_{nm} e^{-(n^2+m^2)t}. \tag{4.146}$$

Combining (4.143), (4.144), and (4.146) and using the superposition principle, the general solution of the problem is given by the double series

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} e^{-(n^2+m^2)t} \sin(nx)\sin(my). \tag{4.147}$$

To determine the constants C_{nm}, we use the given initial condition and expand the double series to obtain

$$C_{11}\sin x \sin y + C_{21}\sin x \sin(2y) + \cdots = \sin x \sin y + 2\sin x \sin(2y). \tag{4.148}$$

Equating the coefficients on both sides yields

$$C_{11} = 1, \tag{4.149}$$

and

$$C_{12} = 2, \tag{4.150}$$

and other coefficients are zeros. Accordingly, the particular solution is given by

$$u(x,y,t) = e^{-2t}\sin x \sin y + 2e^{-5t}\sin x \sin(2y), \tag{4.151}$$

obtained by substituting (4.149) and (4.150) into (4.147).

Example 3. Use the method of separation of variables to solve the initial-boundary value problem with mixed boundary conditions

$$\begin{array}{lrcl} \text{PDE} & u_t & = & 3(u_{xx}+u_{yy}), \quad 0<x,y<\pi, t>0, \\ \text{BC} & u_x(0,y,t) & = & u_x(\pi,y,t)=0, \\ & u(x,0,t) & = & u(x,\pi,t)=0, \\ \text{IC} & u(x,y,0) & = & \sin y+\cos x\sin y. \end{array} \tag{4.152}$$

Solution.

Proceeding as before we obtain

$$F''(x)+\lambda^2 F(x)=0, \tag{4.153}$$

$$G''(y)+\mu^2 G(y)=0, \tag{4.154}$$

and

$$T'(t)+3(\lambda^2+\mu^2)T(t)=0. \tag{4.155}$$

Solving (4.153) we find

$$F(x)=A\cos(\lambda x)+B\sin(\lambda x). \tag{4.156}$$

It is important to note that the boundary conditions

$$u_x(0,y,t)=u_x(\pi,y,t)=0, \tag{4.157}$$

implies that

$$F'(0)=0, F'(\pi)=0. \tag{4.158}$$

Using (4.158) into (4.156) gives

$$B=0, \quad \lambda=n,\ n=0,1,2,\cdots, \tag{4.159}$$

so that $\lambda=0$ is included because it does not provide the trivial solution. Consequently, we find

$$F_n(x)=A_n\cos(nx), \quad n=0,1,2,\cdots. \tag{4.160}$$

Solving (4.154) and using the proper boundary conditions we obtain

$$G_m(y)=\beta_m\sin(my), \quad m=1,2,3,\cdots. \tag{4.161}$$

The solution of the first order differential equation (4.155) is given by

$$T(t)=Ce^{-3(\lambda^2+\mu^2)t}, \tag{4.162}$$

and by substituting λ and μ we obtain

$$T_{nm}(t)=\overline{C}_{nm}e^{-3(n^2+m^2)t}. \tag{4.163}$$

Combining (4.160), (4.161), and (4.163) and using the superposition principle, the general solution of the problem is given by the double series

$$u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{nm}e^{-3(n^2+m^2)t}\cos(nx)\sin(my). \tag{4.164}$$

We then expand the double series (4.164) and use the given initial condition to find

$$C_{01}\sin y + C_{11}\cos x\sin y + \cdots = \sin y + \cos x\sin y. \tag{4.165}$$

Equating the coefficients on both sides yields

$$C_{01} = 1, \tag{4.166}$$

and

$$C_{11} = 1. \tag{4.167}$$

In addition, other coefficients are zeros. Accordingly, the particular solution is given by

$$u(x,y,t) = e^{-3t}\sin y + e^{-6t}\cos x\sin y. \tag{4.168}$$

Example 4. Use the method of separation of variables to solve the initial-boundary value problem with Neumann boundary conditions

$$\begin{array}{lrcl} \text{PDE} & u_t &=& 2(u_{xx}+u_{yy}), \quad 0<x,y<\pi, t>0, \\ \text{BC} & u_x(0,y,t) &=& u_x(\pi,y,t)=0, \\ & u_y(x,0,t) &=& u_y(x,\pi,t)=0, \\ \text{IC} & u(x,y,0) &=& 1+\cos x\cos y. \end{array} \tag{4.169}$$

Solution.

We first set

$$u(x,y,t) = F(x)G(y)T(t), \tag{4.170}$$

to obtain

$$F''(x) + \lambda^2 F(x) = 0, \tag{4.171}$$

$$G''(y) + \mu^2 G(y) = 0, \tag{4.172}$$

and

$$T'(t) + 2(\lambda^2+\mu^2)T(t) = 0. \tag{4.173}$$

Solving (4.171) and (4.172) and using the boundary conditions we obtain

$$F_n(x) = A_n\cos(nx), \quad \lambda_n = n, n = 0,1,2,\cdots, \tag{4.174}$$

and

$$G_m(y) = \beta_m\cos(my), \quad \mu_m = m, m = 0,1,2,\cdots, \tag{4.175}$$

respectively. The solution of the equation (4.173) is therefore given by

$$T_{nm}(t) = \overline{C}_{nm} e^{-2(n^2+m^2)t}. \tag{4.176}$$

Combining the results obtained above and using the superposition principle lead to the general solution of the problem given by

$$u(x,y,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm} e^{-2(n^2+m^2)t} \cos(nx) \cos(my). \tag{4.177}$$

To determine the constants C_{nm}, we expand the double series (4.177) and we use the given initial condition to obtain

$$C_{00} + C_{11} \cos x \cos y + \cdots = 1 + \cos x \cos y. \tag{4.178}$$

Equating the coefficients of like terms on both sides yields

$$C_{00} = 1, \tag{4.179}$$

and

$$C_{11} = 1, \tag{4.180}$$

and other coefficients are zeros. Accordingly, the particular solution is given by

$$u(x,y,t) = 1 + e^{-4t} \cos x \cos y. \tag{4.181}$$

Exercises 4.3.1

Use the method of separation of variables in the following initial-boundary value problems:

1. $u_t = u_{xx} + u_{yy},\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = \sin(2x)\sin(3y)$

2. $u_t = 3(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = \sin x \sin y + \sin(2x)\sin(2y)$

3. $u_t = 4(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = \sin x \sin y + \sin x \sin(2y) + \sin(2x)\sin y$

4. $u_t = u_{xx} + u_{yy},\ 0 < x,y < \pi,\ t > 0$
   $u_x(0,y,t) = u_x(\pi,y,t) = 0$

$u(x,0,t) = u(x,\pi,t) = 0$
$u(x,y,0) = \cos x \sin y$

5. $u_t = u_{xx} + u_{yy},\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = u(\pi,y,t) = 0$
$u_y(x,0,t) = u_y(x,\pi,t) = 0$
$u(x,y,0) = \sin x \cos y$

6. $u_t = u_{xx} + u_{yy},\ 0 < x,y < \pi,\ t > 0$
$u_x(0,y,t) = u_x(\pi,y,t) = 0$
$u(x,0,t) = u(x,\pi,t) = 0$
$u(x,y,0) = \cos x \sin y + \cos(2x)\sin(2y)$

7. $u_t = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = u(\pi,y,t) = 0$
$u_y(x,0,t) = u_y(x,\pi,t) = 0$
$u(x,y,0) = \sin x \cos y + \sin(2x)\cos(2y)$

8. $u_t = u_{xx} + u_{yy},\ 0 < x,y < \pi,\ t > 0$
$u_x(0,y,t) = u_x(\pi,y,t) = 0$
$u_y(x,0,t) = u_y(x,\pi,t) = 0$
$u(x,y,0) = \cos(2x)\cos(3y)$

9. $u_t = u_{xx} + u_{yy},\ 0 < x,y < \pi,\ t > 0$
$u_x(0,y,t) = u_x(\pi,y,t) = 0$
$u_y(x,0,t) = u_y(x,\pi,t) = 0$
$u(x,y,0) = 1 + \cos x \cos(2y)$

10. $u_t = 4(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
$u_x(0,y,t) = u_x(\pi,y,t) = 0$
$u_y(x,0,t) = u_y(x,\pi,t) = 0$
$u(x,y,0) = 4 + \cos(2x)\cos(2y)$

4.3.2 Three Dimensional Heat Flow

The distribution of heat flow in a three dimensional space is governed by the initial boundary value problem:

$$\begin{aligned}
&\text{PDE} && u_t = \overline{k}(u_{xx} + u_{yy} + u_{zz}),\\
& && 0 < x < a,\ 0 < y < b, 0 < z < c,\\
&\text{BC} && u(0,y,z,t) = u(a,y,z,t) = 0,\\
& && u(x,0,z,t) = u(x,b,z,t) = 0,\\
& && u(x,y,0,t) = u(x,y,c,t) = 0,\\
&\text{IC} && u(x,y,z,0) = f(x,y,z),
\end{aligned} \tag{4.182}$$

where $u \equiv u(x,y,z,t)$ defines the temperature of any point at the position (x,y,z) of a rectangular volume at any time t, $\overline{k}$ is the thermal diffusivity. In a parallel manner to the previous discussion, the separation of variables method assumes that the solution $u(x,y,z,t)$ consists of the product of four distinct functions each depends on one variable only. Accordingly, we set

$$u(x,y,z,t) = F(x)G(y)H(z)T(t). \tag{4.183}$$

As discussed before, differentiating (4.183) once with respect to t, and twice with respect to x,y, and z, substituting into (4.182), and by dividing both sides by $\overline{k}F(x)G(y)H(z)T(t)$ we obtain

$$\frac{T'(t)}{\overline{k}T(t)} = \frac{F''(x)}{F(x)} + \frac{G''(y)}{G(y)} + \frac{H''(z)}{H(z)}. \tag{4.184}$$

It is obvious that the equality in (4.184) holds only if both sides are equal to the same constant. This allows us to set

$$F''(x) + \lambda^2 F(x) = 0, \tag{4.185}$$
$$G''(y) + \mu^2 G(y) = 0, \tag{4.186}$$
$$H''(z) + \nu^2 H(z) = 0, \tag{4.187}$$
$$T'(t) + \overline{k}(\lambda^2 + \mu^2 + \nu^2)T(t) = 0, \tag{4.188}$$

where λ, μ, and ν are constants. By solving the second order normal forms (4.185)—(4.187), we obtain the following solutions

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{4.189}$$
$$G(y) = \alpha\cos(\mu y) + \beta\sin(\mu y), \tag{4.190}$$
$$H(z) = \gamma\cos(\nu z) + \delta\sin(\nu z), \tag{4.191}$$

respectively, where $A, B, \alpha, \beta, \gamma$, and δ are constants. Using the boundary conditions in a similar way as discussed before we find

$$A = 0,\ \ \lambda = \frac{n\pi}{a},\ n = 1,2,3,\cdots, \tag{4.192}$$

$$\alpha = 0\ \mu = \frac{m\pi}{b},\ m = 1,2,3,\cdots, \tag{4.193}$$

$$\gamma = 0,\ \ \nu = \frac{r\pi}{c},\ r = 1,2,3,\cdots, \tag{4.194}$$

so that

$$F_n(x) = B_n\sin(\frac{n\pi}{a}x),\ n = 1,2,3,\cdots, \tag{4.195}$$

$$G_m(y) = \beta_m\sin(\frac{m\pi}{b}y),\ m = 1,2,3,\cdots, \tag{4.196}$$

$$H_r(z) = \delta_r \sin(\frac{r\pi}{c}z),\ r = 1,2,3,\cdots. \tag{4.197}$$

The solution of (4.188) is therefore given by

$$T_{nmr}(t) = \overline{C}_{nmr} e^{-\overline{k}(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{r^2}{c^2})\pi^2 t}. \tag{4.198}$$

Consequently, we can formulate the general solution of (4.182) by using the superposition principle, therefore we find

$$u = \sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nmr}\, e^{-\overline{k}(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{r^2}{c^2})\pi^2 t} \sin(\frac{n\pi}{a}x)\ \sin(\frac{m\pi}{b}y)\ \sin(\frac{r\pi}{c}z). \tag{4.199}$$

It remains now to determine the constants C_{nmr}. Using the initial condition given in (4.182), the coefficients C_{nmr} are given by

$$C_{nmr} = \frac{8}{abc}\int_0^c\int_0^b\int_0^a f(x,y,z)\sin(\frac{n\pi}{a}x)\ \sin(\frac{m\pi}{b}y)\ \sin(\frac{r\pi}{c}z)\,dx\,dy\,dz. \tag{4.200}$$

The constants C_{nmr} can also be determined by equating the coefficients on both sides if the initial condition is given in terms of trigonometric functions identical to those included in $u(x,y,z,t)$. This technique reduces the massive size of calculations usually required by using the triple Fourier coefficients.

The following examples will illustrate the method presented above.

Example 5. Solve the initial-boundary value problem

$$\begin{array}{lrcl} \text{PDE} & u_t &=& u_{xx}+u_{yy}+u_{zz}, \quad 0<x,y,z<\pi,\ t>0,\\ \text{BC} & u(0,y,z,t) &=& u(\pi,y,z,t)=0,\\ & u(x,0,z,t) &=& u(x,\pi,z,t)=0,\\ & u(x,y,0,t) &=& u(x,y,\pi,t)=0,\\ \text{IC} & u(x,y,z,0) &=& 3\sin x\sin y\sin z. \end{array} \tag{4.201}$$

Solution.

Proceeding as before, we set

$$u(x,y,z,t) = F(x)G(y)H(z)T(t). \tag{4.202}$$

Substituting (4.202) into (4.201) and following the discussions above we find

$$F_n(x) = B_n\sin(nx), \quad \lambda_n = n,\ n = 1,2,3,\cdots, \tag{4.203}$$

$$G_m(y) = \beta_m\sin(my), \quad \mu_m = m,\ m = 1,2,3,\cdots, \tag{4.204}$$

$$H_r(z) = \delta_r\sin(rz), \quad \upsilon_r = r,\ r = 1,2,3,\cdots, \tag{4.205}$$

$$T(t) = Ce^{-(\lambda^2+\mu^2+\upsilon^2)t}. \tag{4.206}$$

Consequently, we can formulate the general solution is given by

$$u=\sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{nmr}\,e^{-(n^2+m^2+r^2)t}\sin(nx)\,\sin(my)\,\sin(rz). \tag{4.207}$$

To determine the constants C_{nmr}, we use the initial condition and expand (4.207) to find

$$C_{111}\,\sin x\,\sin y\,\sin z+\cdots=3\,\sin x\,\sin y\,\sin z \tag{4.208}$$

Equating the coefficients of like terms in both sides we obtain

$$\begin{aligned} C_{111}&=3,\\ C_{ijk}&=0,\ \text{for}\ i\neq 1,\ j\neq 1,\ k\neq 1. \end{aligned} \tag{4.209}$$

Consequently, the particular solution is given by

$$u(x,y,z,t)=3e^{-3t}\sin x\,\sin y\,\sin z, \tag{4.210}$$

obtained by inserting (4.209) into (4.207).

Example 6. Solve the initial-boundary value problem

$$\begin{array}{lrcl} \text{PDE} & u_t &=& u_{xx}+u_{yy}+u_{zz}, \quad 0<x,y,z<\pi,\ t>0,\\ \text{BC} & u(0,y,z,t) &=& u(\pi,y,z,t)=0,\\ & u(x,0,z,t) &=& u(x,\pi,z,t)=0,\\ & u(x,y,0,t) &=& u(x,y,\pi,t)=0,\\ \text{IC} & u(x,y,z,0) &=& \sin x\sin 2y\sin 3z. \end{array} \tag{4.211}$$

Solution.

Proceeding as before, we set

$$u(x,y,z,t)=F(x)G(y)H(z)T(t). \tag{4.212}$$

Substituting (4.212) into (4.211) and following the discussions above we find

$$F_n(x)=B_n\sin(nx),\quad \lambda_n=n,\ n=1,2,3,\cdots, \tag{4.213}$$

$$G_m(y)=\beta_m\sin(my),\quad \mu_m=m,\ m=1,2,3,\cdots, \tag{4.214}$$

$$H_r(z)=\delta_r\sin(rz),\quad \nu_r=r,\ r=1,2,3,\cdots, \tag{4.215}$$

$$T(t)=Ce^{-(\lambda^2+\mu^2+\nu^2)t}. \tag{4.216}$$

Consequently, we can formulate the general solution is given by

$$u=\sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{nmr}\,e^{-(n^2+m^2+r^2)t}\sin(nx)\,\sin(my)\,\sin(rz). \tag{4.217}$$

To determine the constants C_{nmr}, we use the initial condition and expand (4.217) to find

$$C_{123}\,\sin x\,\sin 2y\,\sin 3z+\cdots=\sin x\,\sin 2y\,\sin 3z \tag{4.218}$$

Equating the coefficients of like terms in both sides we obtain

$$\begin{aligned} C_{123} &= 1, \\ C_{ijk} &= 0, \text{ for } i \neq 1, j \neq 2, k \neq 3. \end{aligned} \tag{4.219}$$

Consequently, the particular solution is given by

$$u(x,y,z,t) = e^{-14t} \sin x \sin 2y \sin 3z. \tag{4.220}$$

In the next example, the Neumann boundary conditions in the spatial domain are used.

Example 7. Solve the initial-boundary value problem with Neumann boundary conditions

$$\begin{array}{lrl} \text{PDE} & u_t &= u_{xx} + u_{yy} + u_{zz}, \quad 0 < x,y,z < \pi, t > 0, \\ \text{BC} & u_x(0,y,z,t) &= u_x(\pi,y,z,t) = 0, \\ & u_y(x,0,z,t) &= u_y(x,\pi,z,t) = 0, \\ & u_z(x,y,0,t) &= u_z(x,y,\pi,t) = 0, \\ \text{IC} & u(x,y,z,0) &= 4 + \cos x \cos(2y) \cos(3z). \end{array} \tag{4.221}$$

Solution.

Proceeding as before, we set

$$u(x,y,z,t) = F(x)G(y)H(z)T(t). \tag{4.222}$$

Substituting in (4.221), using the boundary conditions, and following the discussions above we find

$$F_n(x) = A_n \cos(nx), \quad \lambda_n = n, \; n = 0,1,2,3,\cdots, \tag{4.223}$$

$$G_m(y) = \alpha_m \cos(my), \quad \mu_m = m, \; m = 0,1,2,3,\cdots, \tag{4.224}$$

$$H_r(z) = \gamma_r \cos(rz), \quad \nu_r = r, \; r = 0,1,2,3,\cdots, \tag{4.225}$$

$$T_{nmr}(t) = \overline{C}_{nmr} e^{-\overline{k}(n^2+m^2+r^2)t}. \tag{4.226}$$

Consequently, we can formulate the general solution expressed by

$$u = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{nmr}\, e^{-(n^2+m^2+r^2)t} \cos(nx) \cos(my) \cos(rz). \tag{4.227}$$

To determine the constants C_{nmr}, we using the initial condition and expand (4.227) to find

$$C_{000} + C_{123} \cos x \; \cos(2y) \cos(3z) + \cdots = 4 + \cos x \cos(2y) \cos(3z). \tag{4.228}$$

Equating the coefficients of like terms in both sides we obtain

$$\begin{aligned} C_{000} &= 4, \\ C_{123} &= 1, \text{ for } n = 1, m = 2, r = 3, \end{aligned} \tag{4.229}$$

where other coefficients vanish. Consequently, the particular solution is given by

$$u(x,y,z,t) = 4 + e^{-14t}\cos(x)\,\cos(2y)\,\cos(3z), \tag{4.230}$$

obtained upon combining (4.229) and (4.227).

Exercises 4.3.2

Use the method of separation of variables in the following initial-boundary value problems:

1. $u_t = u_{xx} + u_{yy} + u_{zz},\ 0 < x,y,z < \pi$
   $u(0,y,z,t) = u(\pi,y,z,t) = 0$
   $u(x,0,z,t) = u(x,\pi,z,t) = 0$
   $u(x,y,0,t) = u(x,y,\pi,0) = 0$
   $u(x,y,z,0) = \sin(2x)\sin(3y)\sin(4z)$

2. $u_t = u_{xx} + u_{yy} + u_{zz},\ 0 < x,y,z < \pi$
   $u(0,y,z,t) = u(\pi,y,z,t) = 0$
   $u(x,0,z,t) = u(x,\pi,z,t) = 0$
   $u(x,y,0,t) = u(x,y,\pi,0) = 0$
   $u(x,y,z,0) = \sin x \sin y \sin z + \sin(2x)\sin(2y)\sin(2z)$

3. $u_t = u_{xx} + u_{yy} + u_{zz},\ 0 < x,y,z < \pi$
   $u(0,y,z,t) = u(\pi,y,z,t) = 0$
   $u(x,0,z,t) = u(x,\pi,z,t) = 0$
   $u(x,y,0,t) = u(x,y,\pi,0) = 0$
   $u(x,y,z,0) = \sin x \sin y \sin(2z) + \sin x \sin(2y)\sin(3z)$

4. $u_t = u_{xx} + u_{yy} + u_{zz},\ 0 < x,y,z < \pi$
   $u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0$
   $u(x,0,z,t) = u(x,\pi,z,t) = 0$
   $u(x,y,0,t) = u(x,y,\pi,0) = 0$
   $u(x,y,z,0) = \cos x \sin y \sin z$

5. $u_t = u_{xx} + u_{yy} + u_{zz},\ 0 < x,y,z < \pi$
   $u(0,y,z,t) = u(\pi,y,z,t) = 0$
   $u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0$
   $u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$
   $u(x,y,z,0) = \sin x \cos y \cos z$

6. $u_t = u_{xx} + u_{yy} + u_{zz},\ 0 < x,y,z < \pi$
   $u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0$

$$u(x,0,z,t) = u(x,\pi,z,t) = 0$$
$$u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$$
$$u(x,y,z,0) = \cos x \sin y \cos z$$

7. $u_t = u_{xx} + u_{yy} + u_{zz},\ 0 < x,y,z < \pi$
$$u(0,y,z,t) = u(\pi,y,z,t) = 0$$
$$u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0$$
$$u(x,y,0,t) = u(x,y,\pi,0) = 0$$
$$u(x,y,z,0) = \sin x \cos y \sin z$$

8. $u_t = u_{xx} + u_{yy} + u_{zz},\ 0 < x,y,z < \pi$
$$u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0$$
$$u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0$$
$$u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$$
$$u(x,y,z,0) = 2 + 3\cos x \cos(2y)\cos z$$

9. $u_t = u_{xx} + u_{yy} + u_{zz},\ 0 < x,y,z < \pi$
$$u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0$$
$$u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0$$
$$u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$$
$$u(x,y,z,0) = 1 + \cos x \cos y \cos z + \cos(2x)\cos(2y)\cos(2z)$$

10. $u_t = u_{xx} + u_{yy} + u_{zz},\ 0 < x,y,z < \pi$
$$u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0$$
$$u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0$$
$$u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$$
$$u(x,y,z,0) = 1 + 2\cos x \cos y \cos z + 3\cos(2x)\cos(3y)\cos(4z)$$

References

1. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, (1994).
2. G. Adomian, A review of the decomposition method and some recent results for nonlinear equation, *Math. Comput. Modelling*, 13(7), 17 – 43, (1992).
3. P. Berg and J. McGregor, Elementary Partial Differential Equations, Holden Day, New York, (1966).
4. J.M. Cooper, Introduction to Partial Differential Equations with MATLAB, Birkhauser, Boston, (1998).
5. B.K. Datta, Introduction to Partial Differential Equations, New Central Book Agency, Calcutta, (1993).
6. R. Dennemeyer, Partial Differential Equations and Boundary Value Problems, McGraw Hill, New York, (1968).
7. S. J. Farlow, Partial Differential Equations for Scientists and Engineers, Dover, New York, (1993).
8. F. John, Partial Differential Equations, Springer-Verlag, New York, (1982).
9. R.C. McOwen, Partial Differential Equations, Prentice Hall, New Jersey, (1996).
10. A.M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema Publishers, The Netherlands, (2002).

11. A.M. Wazwaz, A First Course in Integral Equations, WSPC, New Jersey, (1997).
12. A.M. Wazwaz, Necessary conditions for the appearance of noise terms in decomposition solution series, *Appl. Math. Comput.*, 81, 199–204, (1997).

Chapter 5
One Dimensional Wave Equation

5.1 Introduction

In this chapter we will study the physical problem of the wave propagation. The wave equation usually describes water waves, the vibrations of a string or a membrane, the propagation of electromagnetic and sound waves, or the transmission of electric signals in a cable. The function $u(x,t)$ defines a small displacement of any point of a vibrating string at position x at time t. Unlike the heat equation, the wave equation contains the term u_{tt} that represents the vertical acceleration of a vibrating string at point x, which is due to the tension in the string [2–5].

The wave equation plays a significant role in various physical problems [7]. The study of wave equation is needed in diverse areas of science and engineering.

The typical model that describes the wave equation, as will be discussed later, is an initial-boundary value problem valid in a bounded domain or an initial value problem valid in an unbounded domain. It is interesting to note here that two initial conditions should be prescribed, namely the initial displacement $u(x,0) = f(x)$ and the initial velocity $u_t(x,0) = g(x)$ that describe the initial displacement and the initial velocity at the starting time $t = 0$ respectively [8–10].

In a parallel manner to our approach applied to the heat equation in Chapters 3 and 4, our concern will be focused on solving the PDE in conjunction with the given conditions. The approach will be identical to that applied before, therefore the mathematical derivation of the wave equation will not be examined in this text.

In this chapter, we will apply the newly developed *Adomian decomposition method* [1] and the *variational iteration method* [6] to handle the wave equation. Moreover, the traditional methods of separation of variables and D'Alembert method [2–5] will be used as well. Further, a particular solution of the wave equation will be established recalling that a general solution is of little use.

5.2 Adomian Decomposition Method

The Adomian decomposition method has been widely used with promising results in linear and nonlinear partial differential equations that describe wave propagations [7, 8]. The method has been presented in details in Chapters 2, 3, and 4 and the formal steps have been outlined and supported by several illustrative examples. The method introduces the solution of any equation in a series form with elegantly computed components. The method identifies the zeroth component u_0 by the terms that arise from the initial/boundary conditions and from integrating the source term if exists. The remaining components $u_n, n \geqslant 1$, are determined recursively as far as we like. The method will be illustrated by discussing the following typical wave model.

Without loss of generality, as a simple wave equation, we consider the following initial-boundary value problem:

$$\begin{array}{lrl} \text{PDE} & u_{tt} = c^2 u_{xx}, & 0 < x < L,\ t > 0, \\ \text{BC} & u(0,t) = 0,\ u(L,t) = 0, & t \geqslant 0, \\ \text{IC} & u(x,0) = f(x), & u_t(x,0) = g(x). \end{array} \tag{5.1}$$

where $u = u(x,t)$ is the displacement of any point of the string at the position x and at time t, and c is a constant related to the elasticity of the material of the string. The given boundary conditions indicate that the end points of the vibrating string are fixed. It is obvious the IBVP (5.1), that governs the wave displacement, contains the term u_{tt}. Consequently, two initial conditions should be given. The initial conditions describe the initial displacement and the initial velocity of any point at the starting time $t = 0$.

We begin our analysis by rewriting (5.1) in an operator form by

$$L_t u(x,t) = c^2 L_x u(x,t), \tag{5.2}$$

where the differential operators L_t and L_x are defined by

$$L_t = \frac{\partial^2}{\partial t^2}, \quad L_x = \frac{\partial^2}{\partial x^2}. \tag{5.3}$$

We assume that the integral operators L_t^{-1} and L_x^{-1} exist and may be regarded as two-fold indefinite integrals defined by

$$L_t^{-1}(\cdot) = \int_0^t \int_0^t (\cdot)\,dt\,dt, \tag{5.4}$$

and

$$L_x^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)\,dx\,dx. \tag{5.5}$$

This means that

$$L_t^{-1} L_t u(x,t) = u(x,t) - t u_t(x,0) - u(x,0), \tag{5.6}$$

and

$$L_x^{-1}L_xu(x,t) = u(x,t) - xu_x(0,t) - u(0,t). \tag{5.7}$$

Recall that the solution can be obtained by using the inverse operator L_t^{-1} or the inverse operator L_x^{-1}. However, using the inverse operator L_t^{-1} requires the use of the initial conditions only, whereas operating wit L_x^{-1} imposes the use of initial and boundary conditions. For this reason, and to reduce the size of calculations, we will apply the decomposition method in the t direction. Applying L_t^{-1} to both sides of (5.2) and using the initial conditions we obtain

$$u(x,t) = f(x) + tg(x) + c^2L_t^{-1}(L_xu(x,t)). \tag{5.8}$$

The Adomian's method decomposes the displacement function $u(x,t)$ into a sum of an infinite components defined by the infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{5.9}$$

where the components $u_n(x,t), n \geqslant 0$ will be easily calculated. Substituting (5.9) into both sides of (5.8) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + tg(x) + c^2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right), \tag{5.10}$$

or by using few components

$$u_0 + u_1 + u_2 + \cdots = f(x) + tg(x) + c^2L_t^{-1}\left(L_x\left(u_0 + u_1 + u_2 + \cdots\right)\right). \tag{5.11}$$

The method suggests that the zeroth component $u_0(x,t)$ is identified by the terms that are not included under L_t^{-1} in (5.10). The other components are determined by using the recursive relation

$$\begin{aligned} u_0(x,t) &= f(x) + tg(x), \\ u_{k+1}(x,t) &= c^2L_t^{-1}\left(L_x\left(u_k(x,t)\right)\right), \quad k \geqslant 0. \end{aligned} \tag{5.12}$$

In view of (5.12), the components $u_0(x,t), u_1(x,t), u_2(x,t), \cdots$ can be determined individually by

$$\begin{aligned} u_0(x,t) &= f(x) + tg(x), \\ u_1(x,t) &= c^2L_t^{-1}L_x(u_0) = c^2\left(\frac{t^2}{2!}f''(x) + \frac{t^3}{3!}g''(x)\right), \\ u_2(x,t) &= c^2L_t^{-1}L_x(u_1) = c^4\left(\frac{t^4}{4!}f^{(4)}(x) + \frac{t^5}{5!}g^{(4)}(x)\right), \\ u_3(x,t) &= c^2L_t^{-1}L_x(u_2) = c^6\left(\frac{t^6}{6!}f^{(6)}(x) + \frac{t^7}{7!}g^{(6)}(x)\right), \end{aligned} \tag{5.13}$$

and so on. It is obvious that the PDE (5.1) is reduced to solving simple integrals given in (5.13), where components can be determined easily as far as we like. The accuracy level can be enhanced significantly by determining more terms if a closed form solution is not obtained, where a truncated number of components is usually used for numerical purposes.

Having determined the components in (5.13), the solution of the partial differential equation (5.1) is obtained in a series form given by

$$u(x,t)=\sum_{n=0}^{\infty} c^{2n}\left(\frac{t^{2n}}{(2n)!}f^{(2n)}(x)+\frac{t^{2n+1}}{(2n+1)!}g^{(2n)}(x)\right), \tag{5.14}$$

obtained by substituting (5.13) into (5.9). It is important to note that the solution (5.14) can also be obtained by using the inverse operator L_x^{-1}. However, the solution in this way requires more work because the boundary condition $u_x(0,t)$ is not always available.

To give a clear overview of the decomposition method, we have selected homogeneous and inhomogeneous equations to illustrate the procedure discussed above.

5.2.1 Homogeneous Wave Equations

The Adomian decomposition method will be used to solve the following homogeneous equations.

Example 1. Use the Adomian decomposition method to solve the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt}=u_{xx}, & 0<x<\pi,\ t>0,\\ \text{BC} & u(0,t)=0, & u(\pi,t)=0,\ t\geqslant 0,\\ \text{IC} & u(x,0)=\sin x, & u_t(x,0)=0. \end{array} \tag{5.15}$$

Solution.

In an operator form, Equation (5.15) can be written as

$$L_t u(x,t)=L_x u(x,t), \tag{5.16}$$

where the differential operators L_t and L_x are defined by

$$L_t=\frac{\partial^2}{\partial t^2},\quad L_x=\frac{\partial^2}{\partial x^2}. \tag{5.17}$$

Accordingly, the inverse operator L_t^{-1} is a two-fold integral operator defined by

$$L_t^{-1}(\cdot)=\int_0^t\int_0^t(\cdot)\,dt\,dt, \tag{5.18}$$

so that

$$L_t^{-1} L_t\, u(x,t) = u(x,t) - t u_t(x,0) - u(x,0). \tag{5.19}$$

Applying L_t^{-1} to both sides of (5.16), noting (5.19), and using the initial conditions we find

$$u(x,t) = \sin x + L_t^{-1}\left(L_x u(x,t)\right). \tag{5.20}$$

The decomposition method defines the unknown function $u(x,t)$ by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{5.21}$$

that carries (5.20) into

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right), \tag{5.22}$$

or equivalently

$$u_0 + u_1 + u_2 + \cdots = \sin x + L_t^{-1}\left(L_x\left(u_0 + u_1 + u_2 + \cdots\right)\right). \tag{5.23}$$

Following the discussions presented above we set the recursive relation

$$\begin{aligned} u_0(x,t) &= \sin x, \\ u_{k+1}(x,t) &= L_t^{-1}\left(L_x\left(u_k\right)\right), \quad k \geqslant 0, \end{aligned} \tag{5.24}$$

and this in turn gives

$$\begin{aligned} u_0(x,t) &= \sin x, \\ u_1(x,t) &= L_t^{-1}\left(L_x\left(u_0\right)\right) = L_t^{-1}\left(-\sin x\right) = -\frac{1}{2!}t^2 \sin x, \\ u_2(x,t) &= L_t^{-1}\left(L_x\left(u_1\right)\right) = L_t^{-1}\left(\frac{t^2}{2!}\sin x\right) = \frac{1}{4!}t^4 \sin x, \\ u_3(x,t) &= L_t^{-1}\left(L_x\left(u_2\right)\right) = L_t^{-1}\left(-\frac{t^4}{4!}\sin x\right) = -\frac{1}{6!}t^6 \sin x, \end{aligned} \tag{5.25}$$

and so on. Consequently, the solution $u(x,t)$ in a series form is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\ &= \sin x\left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \cdots\right), \end{aligned} \tag{5.26}$$

and in a closed form by

$$u(x,t) = \sin x \cos t, \tag{5.27}$$

obtained upon using the Taylor expansion of $\cos t$. It is clear that the particular solution (5.27) satisfies the PDE, the boundary conditions and the initial conditions.

Example 2. Use the Adomian decomposition method to solve the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = u_{xx}, & 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = 0, & u(\pi,t) = 0, \\ \text{IC} & u(x,0) = 0, & u_t(x,0) = \sin x. \end{array} \tag{5.28}$$

Solution.

Applying L_t^{-1} to both sides of the operator form (5.28) gives

$$u(x,t) = t\sin x + L_t^{-1}\left(L_x(u(x,t))\right). \tag{5.29}$$

Substituting the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{5.30}$$

into both sides of (5.29) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = t\sin x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{5.31}$$

Proceeding as before we set

$$\begin{aligned} u_0(x,0) &= t\sin x, \\ u_{k+1}(x,t) &= L_t^{-1}\left(L_x(u_k(x,t))\right), \quad k \geqslant 0, \end{aligned} \tag{5.32}$$

hence we find

$$\begin{aligned} u_0(x,t) &= t\sin x, \\ u_1(x,t) &= L_t^{-1}\left(L_x\left(u_0\right)\right) = -\frac{1}{3!}t^3\sin x, \\ u_2(x,t) &= L_t^{-1}\left(L_x\left(u_1\right)\right) = \frac{1}{5!}t^5\sin x, \end{aligned} \tag{5.33}$$

and so on. Consequently, the solution $u(x,t)$ in a series form is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\ &= \sin x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots\right), \end{aligned} \tag{5.34}$$

and in a closed form by

$$u(x,t) = \sin x\sin t. \tag{5.35}$$

Example 3. Use the Adomian decomposition method to solve the wave equation problem

$$
\begin{array}{lll}
\text{PDE} & u_{tt} = u_{xx}, & 0 < x < \pi,\ t > 0, \\
\text{BC} & u(0,t) = 1 + \sin t, & u(\pi,t) = 1 - \sin t, \\
\text{IC} & u(x,0) = 1, & u_t(x,0) = \cos x.
\end{array}
\tag{5.36}
$$

Solution.

It is important to note that the boundary conditions are inhomogeneous. The method will be applied in a straightforward manner for all types of differential equations.

Applying L_t^{-1} to the operator form of (5.36) and using the initial conditions we find

$$u(x,t) = 1 + t\cos x + L_t^{-1}\left(L_x u(x,t)\right). \tag{5.37}$$

Substituting the decomposition series for $u(x,t)$ into both sides of (5.37) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 + t\cos x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{5.38}$$

Proceeding as before we find

$$
\begin{aligned}
u_0(x,t) &= 1 + t\cos x, \\
u_1(x,t) &= L_t^{-1}\left(L_x\left(u_0\right)\right) = -\frac{1}{3!}t^3\cos x, \\
u_2(x,t) &= L_t^{-1}\left(L_x\left(u_1\right)\right) = \frac{1}{5!}t^5\cos x,
\end{aligned}
\tag{5.39}
$$

and so on. It is clear that we can easily determine other components as far as we like.

In view of (5.39), the solution $u(x,t)$ in a series form is given by

$$
\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\
&= 1 + \cos x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots\right),
\end{aligned}
\tag{5.40}
$$

and in a closed form by

$$u(x,t) = 1 + \cos x \sin t. \tag{5.41}$$

Example 4. Use the Adomian decomposition method to solve the initial-boundary value problem

$$
\begin{array}{lll}
\text{PDE} & u_{tt} = u_{xx} - 3u, & 0 < x < \pi,\ t > 0, \\
\text{BC} & u(0,t) = \sin(2t), & u(\pi,t) = -\sin(2t), \\
\text{IC} & u(x,0) = 0, & u_t(x,0) = 2\cos x.
\end{array}
\tag{5.42}
$$

Solution.

In this example, an additional term $-3u$ is involved. This term arises when each element of the string is subject to an additional force which is proportional to its displacement. Applying L_t^{-1} to the operator form of (5.42) and using the initial conditions we find

$$u(x,t) = 2t\cos x + L_t^{-1}\left(L_x u(x,t) - 3u(x,t)\right). \tag{5.43}$$

Substituting the decomposition series for $u(x,t)$ into both sides of (5.43) yields

$$\sum_{n=0}^{\infty} u_n = 2t\cos x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) - 3\sum_{n=0}^{\infty} u_n\right). \tag{5.44}$$

The components $u_n(x,t), n \geqslant 0$ can be recursively determined as follows

$$\begin{aligned}
u_0(x,t) &= 2t\cos x,\\
u_1(x,t) &= -\frac{1}{3!}(2t)^3\cos x,\\
u_2(x,t) &= \frac{1}{5!}(2t)^5\cos x,\\
u_3(x,t) &= -\frac{1}{7!}(2t)^7\cos x,
\end{aligned} \tag{5.45}$$

and so on. Other components can be determined to improve the accuracy level if numerical approximations are required.

In view of (5.45), the solution $u(x,t)$ in a series form is given by

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots\\
&= \cos x\left(2t - \frac{1}{3!}(2t)^3 + \frac{1}{5!}(2t)^5 - \cdots\right),
\end{aligned} \tag{5.46}$$

and in a closed form by

$$u(x,t) = \cos x\sin(2t), \tag{5.47}$$

obtained upon using the Taylor series of $\sin(2t)$.

Example 5. Use the Adomian decomposition method to solve the initial-boundary value problem

$$\begin{array}{lll}
\text{PDE} & u_{tt} = u_{xx}, & 0 < x < \pi,\ t > 0,\\
\text{BC} & u_x(0,t) = 1, & u_x(\pi,t) = 1,\\
\text{IC} & u(x,0) = x, & u_t(x,0) = \cos x.
\end{array} \tag{5.48}$$

Solution.

The given Neumann boundary conditions $u_x(0,t)$ and $u_x(\pi,t)$ are inhomogeneous. Operating with L_t^{-1} gives

$$u(x,t) = x + t\cos x + L_t^{-1}\left(L_x u(x,t)\right), \tag{5.49}$$

and proceeding as before we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = x + t\cos x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{5.50}$$

The components $u_n(x,t), n \geqslant 0$ can be recursively determined as follows

$$\begin{aligned} u_0(x,t) &= x + t\cos x, \\ u_1(x,t) &= L_t^{-1}\left(L_x\left(u_0\right)\right) = -\frac{1}{3!}t^3\cos x, \\ u_2(x,t) &= L_t^{-1}\left(L_x\left(u_1\right)\right) = \frac{1}{5!}t^5\cos x. \end{aligned} \tag{5.51}$$

In view of (5.51), the series form for $u(x,t)$ is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\ &= x + \cos x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots\right), \end{aligned} \tag{5.52}$$

and in a closed form by

$$u(x,t) = x + \cos x \sin t. \tag{5.53}$$

In the following example, we will discuss equations where the coefficient of u_{xx} is a function rather than a constant.

Example 6. Use the Adomian decomposition method to solve the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = & \frac{x^2}{2}u_{xx}, \quad 0 < x < 1,\ t > 0, \\ \text{BC} & u(0,t) = 0, & u(1,t) = \sinh t, \\ \text{IC} & u(x,0) = 0, & u_t(x,0) = x^2. \end{array} \tag{5.54}$$

Solution.

Applying L_t^{-1} to both sides of (5.54) gives

$$u(x,t) = tx^2 + L_t^{-1}\left(\frac{x^2}{2}L_x u(x,t)\right), \tag{5.55}$$

so that

$$\sum_{n=0}^{\infty} u_n(x,t) = tx^2 + L_t^{-1}\left(\frac{x^2}{2}L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{5.56}$$

The components $u_n(x,t), n \geqslant 0$ can be recursively determined as follows

$$\begin{aligned}
u_0(x,t) &= tx^2,\\
u_1(x,t) &= L_t^{-1}\left(\frac{x^2}{2}L_x\left(u_0\right)\right) = \frac{x^2t^3}{3!},\\
u_2(x,t) &= L_t^{-1}\left(\frac{x^2}{2}L_x\left(u_1\right)\right) = \frac{x^2t^5}{5!},
\end{aligned} \tag{5.57}$$

and so on. Other components can be determined to improve the accuracy level for numerical purposes.

In view of (5.57), the solution $u(x,t)$ in a series form is given by

$$\begin{aligned}
u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots\\
&= x^2\left(t+\frac{1}{3!}t^3+\frac{1}{5!}t^5+\cdots\right),
\end{aligned} \tag{5.58}$$

and in a closed form by

$$u(x,t) = x^2\sinh t. \tag{5.59}$$

5.2.2 *Inhomogeneous Wave Equations*

Adomian's method will be used to handle the following inhomogeneous examples.

Example 7. Use the Adomian decomposition method to solve the inhomogeneous PDE

$$\begin{array}{lll}
\text{PDE} & u_{tt} = u_{xx}-2, & 0<x<\pi,\ t>0,\\
\text{BC} & u(0,t)=0, & u(\pi,t)=\pi^2,\ t\geqslant 0,\\
\text{IC} & u(x,0)=x^2, & u_t(x,0)=\sin x.
\end{array} \tag{5.60}$$

Solution.

In an operator form, Eq.(5.60) becomes

$$L_t u(x,t) = L_x u(x,t) - 2. \tag{5.61}$$

Operating with L_t^{-1} on both sides of (5.61) leads to

$$u(x,t) = x^2 + t\,\sin x - t^2 + L_t^{-1}\left(L_x u(x,t)\right), \tag{5.62}$$

and consequently we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = x^2 + t\sin x - t^2 + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{5.63}$$

It should be noted here that the zeroth component u_0 is assigned the terms that arise from integrating -2 and from using the initial conditions. The following recursive relation

$$\begin{aligned} u_0(x,t) &= x^2 + t\sin x - t^2, \\ u_{k+1}(x,t) &= L_t^{-1}\left(L_x\left(u_k(x,t)\right)\right), \quad k \geqslant 0, \end{aligned} \tag{5.64}$$

should be used to determine the components of $u(x,t)$. Proceeding as before, we set

$$\begin{aligned} u_0(x,t) &= x^2 + t\sin x - t^2, \\ u_1(x,t) &= L_t^{-1}\left(L_x\left(u_0\right)\right) = t^2 - \frac{1}{3!}t^3\sin x, \\ u_2(x,t) &= L_t^{-1}\left(L_x\left(u_1\right)\right) = \frac{1}{5!}t^5\sin x, \end{aligned} \tag{5.65}$$

and so on. Consequently, the solution $u(x,t)$ in a series form is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\ &= x^2 + \sin x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots\right), \end{aligned} \tag{5.66}$$

and in a closed form by

$$u(x,t) = x^2 + \sin x \sin t. \tag{5.67}$$

Example 8. Use the Adomian decomposition method to solve the inhomogeneous PDE

$$\begin{array}{lrl} \text{PDE} & u_{tt} = u_{xx} + \sin x, & 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = 0, & u(\pi,t) = 0,\ t \geqslant 0, \\ \text{IC} & u(x,0) = \sin x, & u_t(x,0) = \sin x. \end{array} \tag{5.68}$$

Solution.

Following the discussion presented above we obtain

$$u(x,t) = \sin x + t\sin x + \frac{1}{2!}t^2\sin x + L_t^{-1}\left(L_x u(x,t)\right). \tag{5.69}$$

Using the decomposition series for $u(x,t)$ we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin x + t\sin x + \frac{1}{2!}t^2\sin x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{5.70}$$

Proceeding as before, we use the recursive algorithm

$$\begin{aligned} u_0(x,t) &= \sin x + t\sin x + \frac{1}{2!}t^2\sin x, \\ u_1(x,t) &= L_t^{-1}\left(L_x\left(u_0\right)\right) = -\frac{1}{2!}t^2\sin x - \frac{1}{3!}t^3\sin x - \frac{1}{4!}t^4\sin x, \\ u_2(x,t) &= L_t^{-1}\left(L_x\left(u_1\right)\right) = \frac{1}{4!}t^4\sin x + \frac{1}{5!}t^5\sin x + \frac{1}{6!}t^6\sin x, \end{aligned} \tag{5.71}$$

and so on. In view of (5.71), the solution $u(x,t)$ in a series form is given by

$$
\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\
&= \sin x + \sin x \left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots \right),
\end{aligned} \tag{5.72}
$$

and in a closed form by

$$
u(x,t) = \sin x + \sin x \sin t. \tag{5.73}
$$

Example 9. Use the decomposition method to solve the initial-boundary value problem

$$
\begin{array}{lrl}
\text{PDE} & u_{tt} &= u_{xx} + 6t + 2x, \quad 0 < x < \pi,\, t > 0, \\
\text{BC} & u_x(0,t) &= t^2 + \sin t, \quad u_x(\pi,t) = t^2 - \sin t, \\
\text{IC} & u(x,0) &= 0, \quad u_t(x,0) = \sin x.
\end{array} \tag{5.74}
$$

Solution.

Operating with L_t^{-1} on both sides of (5.74) yields

$$
u(x,t) = t^3 + t^2 x + t \sin x + L_t^{-1}\left(L_x u(x,t)\right), \tag{5.75}
$$

so that

$$
\sum_{n=0}^{\infty} u_n(x,t) = t^3 + t^2 x + t \sin x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{5.76}
$$

Following our discussion above we find

$$
\begin{aligned}
u_0(x,t) &= t^3 + t^2 x + t \sin x, \\
u_1(x,t) &= L_t^{-1}\left(L_x\left(u_0\right)\right) = -\frac{1}{3!}t^3 \sin x, \\
u_2(x,t) &= L_t^{-1}\left(L_x\left(u_1\right)\right) = \frac{1}{5!}t^5 \sin x,
\end{aligned} \tag{5.77}
$$

and so on. In view of (5.77), the solution $u(x,t)$ in a series form is given by

$$
\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\
&= t^3 + t^2 x + \sin x \left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots \right),
\end{aligned} \tag{5.78}
$$

and in a closed form by

$$
u(x,t) = t^3 + t^2 x + \sin x \sin t. \tag{5.79}
$$

Exercises 5.2.2

In Exercises 1–8, use the decomposition method to solve the following homogeneous partial differential equations:

1. $u_{tt} = 4u_{xx},\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = 0,\ t \geqslant 0$
   $u(x,0) = \sin(2x),\ u_t(x,0) = 0$

2. $u_{tt} = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = 0,\ t \geqslant 0$
   $u(x,0) = \sin x,\ u_t(x,0) = \sin x$

3. $u_{tt} = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 2 + \cos t,\ u(\pi,t) = 2 - \cos t,\ t \geqslant 0$
   $u(x,0) = 2 + \cos x,\ u_t(x,0) = 0$

4. $u_{tt} = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 1,\ u(\pi,t) = 1 + \pi,\ t \geqslant 0$
   $u(x,0) = 1 + x,\ u_t(x,0) = \sin x$

5. $u_{tt} = u_{xx} - 8u,\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = 0,\ t \geqslant 0$
   $u(x,0) = \sin x,\ u_t(x,0) = 0$

6. $u_{tt} = u_{xx} - 3u,\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 0,\ u(\pi,t) = 0,\ t \geqslant 0$
   $u(x,0) = 0,\ u_t(x,0) = 2\sin x$

7. $u_{tt} = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u_x(0,t) = 0,\ u_x(\pi,t) = 0,\ t \geqslant 0$
   $u(x,0) = \cos x,\ u_t(x,0) = 0$

8. $u_{tt} = u_{xx},\ 0 < x < \pi,\ t > 0$
   $u_x(0,t) = 1,\ u_x(\pi,t) = 1,\ t \geqslant 0$
   $u(x,0) = x + \cos x,\ u_t(x,0) = 0$

In Exercises 9–14, solve the inhomogeneous initial-boundary value problems:

9. $u_{tt} = u_{xx} + \cos x,\ 0 < x < \pi,\ t > 0$
   $u(0,t) = 1,\ u(\pi,t) = -1,\ t \geqslant 0$
   $u(x,0) = \cos x,\ u_t(x,0) = \sin x$

10. $u_{tt} = u_{xx} + \sin x,\ 0 < x < \pi,\ t > 0$
    $u(0,t) = 0,\ u(\pi,t) = 0,\ t \geqslant 0$
    $u(x,0) = 2\sin x,\ u_t(x,0) = 0$

11. $u_{tt} = u_{xx} - 3u + 3,\ 0 < x < \pi,\ t > 0$
    $u(0,t) = 1,\ u(\pi,t) = 1,\ t \geqslant 0$

$u(x,0) = 1,\ u_t(x,0) = 2\sin x$

12. $u_{tt} = u_{xx} - 12x^2,\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = \pi^4,\ t \geqslant 0$
$u(x,0) = x^4 + \sin x,\ u_t(x,0) = 0$

13. $u_{tt} = u_{xx} - 6x,\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = \pi^3,\ t \geqslant 0$
$u(x,0) = x^3,\ u_t(x,0) = \sin x$

14. $u_{tt} = u_{xx} + \cos x,\ 0 < x < \pi,\ t > 0$
$u(0,t) = 2,\ u(\pi,t) = 0,\ t \geqslant 0$
$u(x,0) = 1 + \cos x,\ u_t(x,0) = \sin x$

In Exercises 15–20, solve the initial-boundary value problems:

15. $u_{tt} = u_{xx} - 4,\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = 2\pi^2,\ t \geqslant 0$
$u(x,0) = 2x^2 + \sin x,\ u_t(x,0) = 0$

16. $u_{tt} = u_{xx} - 2,\ 0 < x < \pi,\ t > 0$
$u(0,t) = \sin t,\ u(\pi,t) = \pi^2 - \sin t,\ t \geqslant 0$
$u(x,0) = x^2,\ u_t(x,0) = \cos x$

17. $u_{tt} = u_{xx} + \sin x,\ 0 < x < \pi,\ t > 0$
$u_x(0,t) = 1,\ u_x(\pi,t) = -1,\ t \geqslant 0$
$u(x,0) = \sin x,\ u_t(x,0) = \cos x$

18. $u_{tt} = u_{xx} - 2,\ 0 < x < \pi,\ t > 0$
$u_x(0,t) = 0,\ u_x(\pi,t) = 2\pi,\ t \geqslant 0$
$u(x,0) = x^2 + \cos x,\ u_t(x,0) = 0$

19. $u_{tt} = u_{xx} + 12t^2 + 6xt,\ 0 < x < \pi,\ t > 0$
$u_x(0,t) = t^3,\ u_x(\pi,t) = t^3,\ t \geqslant 0$
$u(x,0) = 0,\ u_t(x,0) = \cos x$

20. $u_{tt} = u_{xx} - 6x + 2,\ 0 < x < \pi,\ t > 0$
$u(0,t) = t^2,\ u(\pi,t) = t^2 + \pi^3,\ t \geqslant 0$
$u(x,0) = x^3,\ u_t(x,0) = \sin x$

In Exercise 21–24, solve the following equations where coefficient of u_{xx} is a function:

21. $u_{tt} = \dfrac{x^2}{2} u_{xx},\ 0 < x < 1,\ t > 0$
$u(0,t) = 0,\ u(1,t) = \cosh t,\ t \geqslant 0$
$u(x,0) = x^2,\ u_t(x,0) = 0$

22. $u_{tt} = \dfrac{x^2}{2} u_{xx},\ 0 < x < 1,\ t > 0$
$u(0,t) = 0,\ u_x(1,t) = 2e^t,\ t \geqslant 0$

$u(x,0) = x^2,\ u_t(x,0) = x^2$

23. $u_{tt} = \dfrac{x^2}{12} u_{xx},\ 0 < x < 1,\ t > 0$
$u(0,t) = 0,\ u_x(1,t) = 4\sinh t,\ t \geqslant 0$
$u(x,0) = 0,\ u_t(x,0) = x^4$

24. $u_{tt} = \dfrac{x^2}{6} u_{xx},\ 0 < x < 1,\ t > 0$
$u(0,t) = 0,\ u_x(1,t) = 3\cosh t,\ t \geqslant 0$
$u(x,0) = x^3,\ u_t(x,0) = 0$

5.2.3 Wave Equation in an Infinite Domain

The **initial value problem** of the one dimensional wave equation, where the domain of the space variable x is unbounded [10], will be discussed by using Adomian decomposition method. This type of equations describes the motion of a very long string that is considered not to have boundaries. Based on this, the wave motion is described by a PDE and initial conditions only, therefore, it is called initial value problem. It was discovered before that solutions of the wave equation behave quite differently than solutions of the heat equation. The solution $u(x,t)$ of the wave equation represents the displacement of the point x at time $t \geqslant 0$.

It is interesting to note that the classical method of separation of variables is not applicable for this type of problems because of the lack of boundary conditions. However, a classical method called D'Alembert solution is usually used [2, 5]. The D'Alembert solution will be discussed later.

In this section, the Adomian decomposition method will be used to handle the wave equation where the space of the variable x is unbounded. Recall that Adomian's method can easily handle problems with initial conditions only.

To achieve our goal, we consider the initial value problem:

$$\begin{array}{lrl} \text{PDE} & u_{tt} = c^2 u_{xx}, & -\infty < x < \infty,\ t > 0, \\ \text{IC} & u(x,0) = f(x), & u_t(x,0) = g(x). \end{array} \tag{5.80}$$

The attention will be focused upon the disturbance occurred at the center of the very long string. The initial displacement $u(x,0)$ and the initial displacement $u_t(x,0)$ are prescribed by $f(x)$ and $g(x)$ respectively.

Applying the inverse operator L_t^{-1} to both sides of the operator form of (5.80) and using the initial conditions we find

$$u(x,t) = f(x) + tg(x) + c^2 L_t^{-1}(L_x u(x,t)). \tag{5.81}$$

Identifying the zeroth component $u_0(x,t)$ and proceeding as before we find

$$
\begin{aligned}
u_0(x,t) &= f(x) + tg(x), \\
u_1(x,t) &= c^2 L_t^{-1}(L_x u_0), \\
&= f''(x)\frac{(ct)^2}{2!} + c^2 g''(x)\frac{t^3}{3!}, \\
u_2(x,t) &= c^2 L_t^{-1}(L_x u_1), \\
&= f^{(4)}(x)\frac{(ct)^4}{4!} + c^4 g^{(4)}(x)\frac{t^5}{5!},
\end{aligned}
\tag{5.82}
$$

and so on. In view of (5.82), the solution $u(x,t)$ of (5.80) in a series form is given by

$$
\begin{aligned}
u(x,t) = & \left(f(x) + f''(x)\frac{(ct)^2}{2!} + f^{(4)}(x)\frac{(ct)^4}{4!} + \cdots\right) \\
& + \left(g(x)t + c^2 g''(x)\frac{t^3}{3!} + c^4 g^{(4)}(x)\frac{t^5}{5!} + \cdots\right),
\end{aligned}
\tag{5.83}
$$

or equivalently

$$
u(x,t) = \sum_{n=0}^{\infty} \left(\frac{(ct)^{2n}}{(2n)!} f^{(2n)}(x) + c^{2n} \frac{t^{2n+1}}{(2n+1)!} g^{(2n)}(x) \right), \tag{5.84}
$$

The series solution (5.84) is easily obtained because it relies completely on differentiating the initial conditions $f(x)$ and $g(x)$ which is mostly an easy task. The approach we followed will be illustrated by discussing the following examples.

Example 10. Use the Adomian decomposition method to solve the initial value problem

$$
\begin{array}{lrl}
\text{PDE} & u_{tt} = 16u_{xx}, & -\infty < x < \infty,\ t > 0, \\
\text{IC} & u(x,0) = \sin x, & u_t(x,0) = 2.
\end{array}
\tag{5.85}
$$

Solution.

Note that $c = 4, f(x) = \sin x$ and $g(x) = 2$. We can easily apply the inverse operator as used in other examples. However, for simplicity reasons, we will use the result (5.84) hence we set

$$
f^{(2n)}(x) = (-1)^n \sin x, \quad n = 0, 1, 2, \cdots, \tag{5.86}
$$

and

$$
g^{(2n)}(x) = \begin{cases} 2, \text{ for } n = 0 \\ 0, \text{ for } n = 1, 2, \cdots \end{cases} \tag{5.87}
$$

The solution in a series form is readily obtained by substituting (5.86) and (5.87) into (5.83) and given by

$$
u(x,t) = \sin x \left(1 - \frac{(4t)^2}{2!} + \frac{(4t)^4}{4!} + \cdots\right) + 2t, \tag{5.88}
$$

and

$$u(x,t) = \sin x \cos(4t) + 2t. \tag{5.89}$$

Example 11. Use the Adomian decomposition method to solve the initial value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = 4u_{xx}, & -\infty < x < \infty,\ t > 0, \\ \text{IC} & u(x,0) = \sin x, & u_t(x,0) = 2\cos x. \end{array} \tag{5.90}$$

Solution.

Note that $c = 2, f(x) = \sin x$ and $g(x) = 2\cos x$. Proceeding as before, we set

$$f^{(2n)}(x) = (-1)^n \sin x,\ n = 0,1,2,\cdots, \tag{5.91}$$

and

$$g^{(2n)}(x) = 2(-1)^n \cos x,\ n = 0,1,2,\cdots. \tag{5.92}$$

The solution in a series form is readily obtained by substituting (5.92) and (5.91) into (5.83) and given by

$$\begin{aligned} u(x,t) = & \sin x \left(1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} + \cdots\right) \\ & + \cos x \left((2t) - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \cdots\right), \end{aligned} \tag{5.93}$$

and in a closed form by

$$\begin{aligned} u(x,t) &= \sin x \cos(2t) + \cos x \sin(2t), \\ &= \sin(x + 2t). \end{aligned} \tag{5.94}$$

Example 12. Use the decomposition method to solve the initial value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = u_{xx} + 2x + 6t, & -\infty < x < \infty, t > 0, \\ \text{IC} & u(x,0) = 0, & u_t(x,0) = \sin x. \end{array} \tag{5.95}$$

Solution.

Note that the initial value problem is inhomogeneous. Operating with L_t^{-1} on both sides of (5.95) and using the initial conditions we obtain

$$u(x,t) = xt^2 + t^3 + t\sin x + L_t^{-1}\left(L_x u(x,t)\right). \tag{5.96}$$

Using the decomposition series for $u(x,t)$ we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = xt^2 + t^3 + t\sin x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{5.97}$$

Proceeding as before, we use the recursive algorithm

$$\begin{aligned} u_0(x,t) &= xt^2 + t^3 + t\sin x, \\ u_1(x,t) &= L_t^{-1}\left(L_x\left(u_0\right)\right) = -\frac{1}{3!}t^3 \sin x, \\ u_2(x,t) &= L_t^{-1}\left(L_x\left(u_1\right)\right) = \frac{1}{5!}t^5 \sin x, \end{aligned} \tag{5.98}$$

and so on. In view of (5.98), the solution $u(x,t)$ in a series form is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\ &= xt^2 + t^3 + \sin x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots\right), \end{aligned} \tag{5.99}$$

and in a closed form by

$$u(x,t) = xt^2 + t^3 + \sin x \sin t. \tag{5.100}$$

Example 13. Use the decomposition method to solve the initial value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = u_{xx} + e^{-t}, & -\infty < x < \infty,\ t > 0, \\ \text{IC} & u(x,0) = 1, & u_t(x,0) = -1 + \sin x. \end{array} \tag{5.101}$$

Solution.

Note that the initial value problem is inhomogeneous. Operating with L_t^{-1} on both sides of (5.101) gives

$$u(x,t) = t\sin x + e^{-t} + L_t^{-1}\left(L_x u(x,t)\right), \tag{5.102}$$

so that

$$\sum_{n=0}^{\infty} u_n(x,t) = t\sin x + e^{-t} + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{5.103}$$

Proceeding as before we find

$$\begin{aligned} u_0(x,t) &= t\sin x + e^{-t}, \\ u_1(x,t) &= -\frac{1}{3!}t^3 \sin x, \\ u_2(x,t) &= \frac{1}{5!}t^5 \sin x, \end{aligned} \tag{5.104}$$

and so on. In view of (5.104), the solution $u(x,t)$ in a series form is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

$$= e^{-t} + \sin x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots\right), \tag{5.105}$$

and in a closed form by

$$u(x,t) = e^{-t} + \sin x \sin t. \tag{5.106}$$

Exercises 5.2.3

In Exercises 1–8, use the decomposition method to solve the following initial value problems:

1. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = 0,\ u_t(x,0) = 4 + \sin x$

2. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = \sin x,\ u_t(x,0) = \cos x$

3. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = \cos x,\ u_t(x,0) = -\sin x$

4. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = \sin x,\ u_t(x,0) = -\cos x$

5. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = \sin x,\ u_t(x,0) = 6$

6. $u_{tt} = u_{xx} + 4x,\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = 0,\ u_t(x,0) = 6$

7. $u_{tt} = u_{xx} + 4t,\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = 0,\ u_t(x,0) = x^2 + e^x$

8. $u_{tt} = u_{xx} + xe^t,\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = x,\ u_t(x,0) = x + \cos x$

9. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = x^2,\ u_t(x,0) = \sin x$

10. $u_{tt} = u_{xx} - \cos x,\ -\infty < x < \infty,\ t > 0$
    $u(x,0) = 0,\ u_t(x,0) = 1 + 2x$

11. $u_{tt} = u_{xx} - \sin x,\ -\infty < x < \infty,\ t > 0$
    $u(x,0) = x^2 - \sin x,\ u_t(x,0) = \sin x$

12. $u_{tt} = u_{xx} + \cos x,\ -\infty < x < \infty,\ t > 0$
    $u(x,0) = 2\cos x,\ u_t(x,0) = 0$

5.3 The Variational Iteration Method

As stated in Chapters 2 and 3, the variational iteration method (VIM) gives rapidly convergent successive approximations [6] of the exact solution if an exact solution exists. Otherwise, the method provides an approximation of high accuracy level by using only few iterations. In what follows, we only summarize the main steps of this method. For the differential equation

$$Lu + Nu = g(x,t), \tag{5.107}$$

where L and N are linear and nonlinear operators respectively, and $g(x,t)$ is the source inhomogeneous term, the variational iteration method admits the use of the correction functional for equation (5.107) which can be written as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\right) d\xi, \quad n \geqslant 0. \tag{5.108}$$

The Lagrange multiplier $\lambda(\xi)$ should be determined first. The successive approximations $u_{n+1}(x,t), n \geqslant 0$ of the solution $u(x,t)$ will be obtained readily upon using the obtained Lagrange multiplier and by using any selective function u_0. The initial values $u(x,0)$ and $u_t(x,0)$ should be used for selecting the zeroth approximation u_0. The exact solution may be obtained by using

$$u = \lim_{n\to\infty} u_n. \tag{5.109}$$

It is worth noting that

$$\int \lambda(\xi) u_n''(\xi) d\xi = \lambda(\xi) u_n'(\xi) - \lambda'(\xi) u_n(\xi) + \int \lambda''(\xi) u_n(\xi) d\xi. \tag{5.110}$$

5.3.1 Homogeneous Wave Equations

The variational iteration method will be used in the following wave equations. We will examine the same examples presented in the previous section.

Example 1. Use the variational iteration method to solve the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = u_{xx}, & 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = 0, & u(\pi,t) = 0,\ t \geqslant 0, \\ \text{IC} & u(x,0) = \sin x, & u_t(x,0) = 0. \end{array} \tag{5.111}$$

Solution.

The correction functional for this equation reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2}\right) d\xi. \tag{5.112}$$

This yields the stationary conditions

$$\begin{aligned} 1-\lambda'|_{\xi=t} &= 0, \\ \lambda|_{\xi=t} &= 0, \\ \lambda''|_{\xi=t} &= 0. \end{aligned} \tag{5.113}$$

This in turn gives

$$\lambda = \xi - t. \tag{5.114}$$

Substituting this value of the Lagrange multiplier into the functional (5.112) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right) d\xi, \quad n \geqslant 0. \tag{5.115}$$

Considering the given initial values, we can select $u_0(x,t) = \sin x$. Using this selection into (5.115) we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= \sin x, \\ u_1(x,t) &= \sin x - \frac{1}{2!}t^2 \sin x, \\ u_2(x,t) &= \sin x - \frac{1}{2!}t^2 \sin x + \frac{1}{4!}t^4 \sin x, \\ u_3(x,t) &= \sin x - \frac{1}{2!}t^2 \sin x + \frac{1}{4!}t^4 \sin x - \frac{1}{6!}t^6 \sin x, \\ &\vdots \\ u_n(x,t) &= \sin x\left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \cdots\right). \end{aligned} \tag{5.116}$$

This gives the exact solution by

$$u(x,t) = \sin x \cos t, \tag{5.117}$$

by noting that $u(x,t) = \lim_{n\to\infty} u_n$.

Example 2. Use the variational iteration method to solve the initial-boundary value problem

$$\begin{array}{lll} \text{PDE} & u_{tt} = u_{xx}, & 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = 0, & u(\pi,t) = 0, \\ \text{IC} & u(x,0) = 0, & u_t(x,0) = \sin x. \end{array} \tag{5.118}$$

Solution.

The correction functional for this equation reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2}\right) d\xi. \tag{5.119}$$

This yields the stationary conditions

$$1 - \lambda'|_{\xi=t} = 0, \quad \lambda|_{\xi=t} = 0, \quad \lambda''|_{\xi=t} = 0. \tag{5.120}$$

This in turn gives

$$\lambda = \xi - t. \tag{5.121}$$

Using Lagrange multiplier $\lambda = \xi - t$ into the functional (5.119) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right) d\xi, \quad n \geqslant 0. \tag{5.122}$$

Considering the given initial values, we can select $u_0(x,t) = t\sin x$. Using this selection into (5.122) we obtain the following successive approximations

$$\begin{aligned}
u_0(x,t) &= t\sin x,\\
u_1(x,t) &= t\sin x - \frac{1}{3!}t^3\sin x,\\
u_2(x,t) &= t\sin x - \frac{1}{3!}t^3\sin x + \frac{1}{5!}t^5\sin x,\\
u_3(x,t) &= t\sin x - \frac{1}{3!}t^3\sin x + \frac{1}{5!}t^5\sin x - \frac{1}{7!}t^7\sin x,\\
&\vdots\\
u_n(x,t) &= \sin x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots\right).
\end{aligned} \tag{5.123}$$

The exact solution is thus given by

$$u(x,t) = \sin x \sin t, \tag{5.124}$$

by using Taylor series for $\sin t$ and by noting that $u(x,t) = \lim_{n\to\infty} u_n$.

Example 3. Use the variational iteration method to solve the wave equation problem

$$\begin{array}{lrl}
\text{PDE} & u_{tt} = u_{xx}, & 0 < x < \pi,\ t > 0,\\
\text{BC} & u(0,t) = 1 + \sin t, & u(\pi,t) = 1 - \sin t,\\
\text{IC} & u(x,0) = 1, & u_t(x,0) = \cos x.
\end{array} \tag{5.125}$$

Solution.

Although the boundary conditions are inhomogeneous, the VIM will be applied directly for all types of differential equations.

Proceeding as before, and noting that

$$\lambda = \xi - t. \tag{5.126}$$

we obtain the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right) d\xi, \quad n \geqslant 0. \tag{5.127}$$

Considering the given initial values, we can select $u_0(x,t) = 1 + t\cos x$. Using this selection into (5.127) we obtain the successive approximations

$$\begin{aligned}
u_0(x,t) &= 1 + t\cos x,\\
u_1(x,t) &= 1 + t\cos x - \frac{1}{3!}t^3\cos x,\\
u_2(x,t) &= 1 + t\cos x - \frac{1}{3!}t^3\cos x + \frac{1}{5!}t^5\cos x,\\
u_3(x,t) &= 1 + t\cos x - \frac{1}{3!}t^3\cos x + \frac{1}{5!}t^5\cos x - \frac{1}{7!}t^7\cos x,\\
&\vdots\\
u_n(x,t) &= 1 + \cos x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots\right).
\end{aligned} \tag{5.128}$$

The exact solution is given by

$$u(x,t) = 1 + \cos x \sin t, \tag{5.129}$$

by using Taylor series for $\sin t$ and by noting that $u(x,t) = \lim_{n\to\infty} u_n$.

Example 4. Use the VIM to solve the initial-boundary value problem

$$\begin{array}{lrl}
\text{PDE} & u_{tt} &= u_{xx} - 3u, \quad 0 < x < \pi,\ t > 0,\\
\text{BC} & u(0,t) &= \sin(2t), \quad u(\pi,t) = -\sin(2t),\\
\text{IC} & u(x,0) &= 0, \quad u_t(x,0) = 2\cos x.
\end{array} \tag{5.130}$$

Solution.

The term $-3u$ arises when each element of the string is subject to an additional force which is proportional to its displacement.

The correction functional for this equation reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} + 3\tilde{u}_n(x,\xi)\right) d\xi. \tag{5.131}$$

This yields the stationary conditions

$$\begin{aligned}
1 - \lambda'|_{\xi=t} &= 0,\\
\lambda|_{\xi=t} &= 0,\\
\lambda''|_{\xi=t} &= 0.
\end{aligned} \tag{5.132}$$

This in turn gives

$$\lambda = \xi - t. \tag{5.133}$$

Substituting this value of the Lagrangian multiplier into the functional (5.131) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + 3u_n(x,\xi)\right) d\xi, \quad n \geqslant 0. \tag{5.134}$$

Considering the given initial values, we can select $u_0(x,t) = 2t\cos x$. Using this selection into (5.134) we obtain the following successive approximations

$$\begin{aligned}
u_0(x,t) &= 2t\cos x,\\
u_1(x,t) &= 2t\cos x - \frac{(2t)^3}{3!}\cos x,\\
u_2(x,t) &= 2t\cos x - \frac{(2t)^3}{3!}\cos x + \frac{(2t)^5}{5!}\cos x,\\
u_3(x,t) &= 2t\cos x - \frac{(2t)^3}{3!}\cos x + \frac{(2t)^5}{5!}\cos x - \frac{(2t)^7}{7!}\cos x,\\
&\ \vdots\\
u_n(x,t) &= \cos x\left((2t) - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \frac{(2t)^7}{7!} + \cdots\right).
\end{aligned} \tag{5.135}$$

This gives the exact solution by

$$u(x,t) = \cos x \sin 2t. \tag{5.136}$$

Example 5. Use the VIM to solve the initial-boundary value problem

$$\begin{array}{lrl}
\text{PDE} & u_{tt} = u_{xx}, & 0 < x < \pi,\ t > 0,\\
\text{BC} & u_x(0,t) = 1, & u_x(\pi,t) = 1,\\
\text{IC} & u(x,0) = x, & u_t(x,0) = \cos x.
\end{array} \tag{5.137}$$

Solution.

Proceeding as before, we obtain the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right) d\xi, \quad n \geqslant 0. \tag{5.138}$$

Considering the given initial values, we can select $u_0(x,t) = x + t\cos x$. Using this selection into (5.138) we obtain the following successive approximations

$$
\begin{aligned}
u_0(x,t) &= x + t\cos x, \\
u_1(x,t) &= x + t\cos x - \frac{1}{3!}t^3\cos x, \\
u_2(x,t) &= x + t\cos x - \frac{1}{3!}t^3\cos x + \frac{1}{5!}t^5\cos x, \\
u_3(x,t) &= x + t\cos x - \frac{1}{3!}t^3\cos x + \frac{1}{5!}t^5\cos x - \frac{1}{7!}t^7\cos x, \\
&\vdots \\
u_n(x,t) &= x + \cos x\left(1 - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}7^6 + \cdots\right).
\end{aligned}
\tag{5.139}
$$

This gives the exact solution by

$$u(x,t) = x + \cos x \sin t. \tag{5.140}$$

Example 6. Use the VIM to solve the wave-like equation

$$
\begin{array}{lll}
\text{PDE} & u_{tt} = \dfrac{x^2}{2}u_{xx}, & 0 < x < 1, t > 0, \\
\text{BC} & u(0,t) = 0, & u(1,t) = \sinh t, \\
\text{IC} & u(x,0) = 0, & u_t(x,0) = x^2.
\end{array}
\tag{5.141}
$$

Solution.

Proceeding as before, and using $\lambda = \xi - t$ we obtain the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{x^2}{2}\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right) d\xi, \quad n \geqslant 0. \tag{5.142}$$

Using the zeroth selection $u_0(x,t) = tx^2$ into (5.142) yields the following successive approximations

$$
\begin{aligned}
u_0(x,t) &= x^2 t, \\
u_1(x,t) &= x^2 t + x^2\frac{t^3}{3!}, \\
u_2(x,t) &= x^2 t + x^2\frac{t^3}{3!} + x^2\frac{t^5}{5!}, \\
u_3(x,t) &= x^2 t + x^2\frac{t^3}{3!} + x^2\frac{t^5}{5!} + x^2\frac{t^7}{7!}, \\
&\vdots \\
u_n(x,t) &= x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots\right).
\end{aligned}
\tag{5.143}
$$

This in turn gives the exact solution

$$u(x,t) = x^2\sinh t. \tag{5.144}$$

5.3.2 Inhomogeneous Wave Equations

The variational iteration method can be effectively used to handle the inhomogeneous wave equations. This can be illustrated by studying the following examples.

Example 7. Use the VIM to solve the inhomogeneous PDE

$$\begin{array}{lrl} \text{PDE} & u_{tt} = u_{xx} - 2, & 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = 0, & u(\pi,t) = \pi^2,\ t \geqslant 0, \\ \text{IC} & u(x,0) = x^2, & u_t(x,0) = \sin x. \end{array} \tag{5.145}$$

Solution.

The correction functional for this equation reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} + 2\right) d\xi. \tag{5.146}$$

Substituting the Lagrange multiplier $\lambda = \xi - t$ into the functional (5.146) leads to the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + 2\right) d\xi, \quad n \geqslant 0. \tag{5.147}$$

Considering the given initial values, we can select $u_0(x,t) = x^2 + t\sin x$. Using this selection into (5.147) we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= x^2 + t\sin x, \\ u_1(x,t) &= x^2 + t\sin x - \frac{1}{3!}t^3\sin x, \\ u_2(x,t) &= x^2 + t\sin x - \frac{1}{3!}t^3\sin x + \frac{1}{5!}t^5\sin x, \\ u_3(x,t) &= x^2 + t\sin x - \frac{1}{3!}t^3\sin x + \frac{1}{5!}t^5\sin x - \frac{1}{7!}t^7\sin x, \\ &\vdots \\ u_n(x,t) &= x^2 + \sin x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots\right), \end{aligned} \tag{5.148}$$

so that the exact solution is given by

$$u(x,t) = x^2 + \sin x \sin t, \tag{5.149}$$

by using Taylor series for $\sin t$ and by noting that $u(x,t) = \lim_{n\to\infty} u_n$.

Example 8. Use the variational iteration method to solve the inhomogeneous PDE

$$\begin{array}{lrl} \text{PDE} & u_{tt} = & u_{xx} + \sin x, \quad 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) = & 0, \quad u(\pi,t) = 0,\ t \geqslant 0, \\ \text{IC} & u(x,0) = & \sin x, \quad u_t(x,0) = \sin x. \end{array} \tag{5.150}$$

Solution.

Proceeding as before we obtain the Lagrange multiplier by

$$\lambda = \xi - t. \tag{5.151}$$

Consequently, we find the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - \sin x\right) d\xi, \quad n \geqslant 0. \tag{5.152}$$

We can select $u_0(x,t) = \sin x + t\sin x$. Using this selection into (5.152), the successive approximations

$$\begin{aligned} u_0(x,t) &= \sin x + t\sin x, \\ u_1(x,t) &= \sin x + t\sin x - \frac{1}{3!}t^3\sin x, \\ u_2(x,t) &= \sin x + t\sin x - \frac{1}{3!}t^3\sin x + \frac{1}{5!}t^5\sin x, \\ u_3(x,t) &= \sin x + t\sin x - \frac{1}{3!}t^3\sin x + \frac{1}{5!}t^5\sin x - \frac{1}{7!}t^7\sin x, \\ &\vdots \\ u_n(x,t) &= \sin x + \sin x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 - \cdots\right), \end{aligned} \tag{5.153}$$

follow immediately. The exact solution is given by

$$u(x,t) = \sin x + \sin x \sin t. \tag{5.154}$$

Example 9. Use the decomposition method to solve the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = & u_{xx} + 6t + 2x, \quad 0 < x < \pi, t > 0, \\ \text{BC} & u_x(0,t) = & t^2 + \sin t, \quad u_x(\pi,t) = t^2 - \sin t, \\ \text{IC} & u(x,0) = & 0, \quad u_t(x,0) = \sin x. \end{array} \tag{5.155}$$

Solution.

The correction functional for this equation is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} - 6\xi - 2x\right) d\xi. \tag{5.156}$$

The stationary conditions give

$$\lambda = \xi - t. \tag{5.157}$$

Substituting this value of the Lagrangian multiplier into the functional (5.156) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - 6\xi - 2x\right) d\xi. \tag{5.158}$$

Considering the given initial values, we can select $u_0(x,t) = t\sin x$. Using this selection into (5.158) we obtain the following successive approximations

$$\begin{aligned}
u_0(x,t) &= t\sin x,\\
u_1(x,t) &= t^3 + t^2x + t\sin x - \frac{1}{3!}t^3\sin x,\\
u_2(x,t) &= t^3 + t^2x + t\sin x - \frac{1}{3!}t^3\sin x - \frac{1}{5!}t^5\sin x,\\
u_3(x,t) &= t^3 + t^2x + t\sin x - \frac{1}{3!}t^3\sin x - \frac{1}{5!}t^5\sin x + \frac{1}{7!}t^7\sin x,\\
&\ \ \vdots\\
u_n(x,t) &= t^3 + t^2x + \sin x(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots).
\end{aligned} \tag{5.159}$$

Consequently, the exact solution is given by

$$u(x,t) = t^3 + t^2x + \sin x\sin t. \tag{5.160}$$

Exercises 5.3.2

Use the variational iteration method to solve the problems in Exercises 5.2.2.

5.3.3 Wave Equation in an Infinite Domain

In what follows we will examine the wave equation in an infinite domain. As stated before, this equation is usually solved by D'Alembert method [5, 8]. Because of the unbounded domain, boundary conditions are not given. The same examples presented before will be examined here.

Example 10. Use the variational iteration method to solve the initial value problem

$$\begin{array}{lll}
\text{PDE} & u_{tt} = 16u_{xx}, & -\infty < x < \infty,\ t > 0,\\
\text{IC} & u(x,0) = \sin x, & u_t(x,0) = 2.
\end{array} \tag{5.161}$$

Solution.

The correction functional for this PDE is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - 16\frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2}\right) d\xi. \tag{5.162}$$

The stationary conditions give

$$\lambda = \xi - t. \tag{5.163}$$

Substituting this value of the Lagrange multiplier into the functional (5.162) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - 16\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right) d\xi, \quad n \geqslant 0. \tag{5.164}$$

Considering the given initial values, we can select $u_0(x,t) = 2t + \sin x$. Using this selection into (5.164) we obtain the following successive approximations

$$\begin{aligned}
u_0(x,t) &= 2t + \sin x,\\
u_1(x,t) &= 2t + \sin x - \frac{(4t)^2}{2!}\sin x,\\
u_2(x,t) &= 2t + \sin x - \frac{(4t)^2}{2!}\sin x + \frac{(4t)^4}{4!}\sin x,\\
u_3(x,t) &= 2t + \sin x - \frac{(4t)^2}{2!}\sin x + \frac{(4t)^4}{4!}\sin x - \frac{(4t)^6}{6!}\sin x,\\
&\vdots\\
u_n(x,t) &= 2t + \sin x\left(1 - \frac{(4t)^2}{2!} + \frac{(4t)^4}{4!} - \frac{(4t)^6}{6!} + \cdots\right),
\end{aligned} \tag{5.165}$$

and this leads to the exact solution

$$u(x,t) = 2t + \sin x \cos(4t). \tag{5.166}$$

Example 11. Use the variational iteration method to solve the initial value problem

$$\begin{array}{lrl}
\text{PDE} & u_{tt} = 4u_{xx}, & -\infty < x < \infty,\ t > 0,\\
\text{IC} & u(x,0) = \sin x, & u_t(x,0) = 2\cos x.
\end{array} \tag{5.167}$$

Solution.

Proceeding as before, we find

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - 4\frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2}\right) d\xi. \tag{5.168}$$

Substituting the Lagrange multiplier $\lambda = \xi - t$ into the functional (5.168) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - 4\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right) d\xi, \quad n \geqslant 0. \tag{5.169}$$

Substituting $u_0(x,t) = \sin x + 2t\cos x$ into (5.169) gives the successive approximations

$$\begin{aligned}
u_0(x,t) &= \sin x + 2t\cos x,\\
u_1(x,t) &= \sin x + 2t\cos x - \frac{(2t)^2}{2!}\sin x - \frac{(2t)^3}{3!}\cos x,\\
u_2(x,t) &= \sin x + 2t\cos x - \frac{(2t)^2}{2!}\sin x - \frac{(2t)^3}{3!}\cos x + \frac{(2t)^4}{4!}\sin x + \frac{(2t)^5}{5!}\cos x,\\
&\vdots\\
u_n(x,t) &= \sin x\left(1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} - \cdots\right) + \cos x\left(2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \cdots\right),
\end{aligned} \tag{5.170}$$

that leads to the exact solution

$$u(x,t) = \sin x\cos(2t) + \cos x\sin(2t) = \sin(x+2t). \tag{5.171}$$

Example 12. Use the variational iteration method to solve the initial value problem

$$\begin{array}{lll}
\text{PDE} & u_{tt} = u_{xx} + 2x + 6t, & -\infty < x < \infty,\ t > 0,\\
\text{IC} & u(x,0) = 0, \quad u_t(x,0) = \sin x. &
\end{array} \tag{5.172}$$

Solution.

Proceeding as before, we find

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} - 2x - 6\xi\right) d\xi. \tag{5.173}$$

Substituting the Lagrange multiplier $\lambda = \xi - t$ into the functional (5.173) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - 2x - 6\xi\right) d\xi, \quad n \geqslant 0. \tag{5.174}$$

Substituting $u_0(x,t) = t\sin x$ into (5.174) gives the successive approximations

$$\begin{aligned}
u_0(x,t) &= t\sin x,\\
u_1(x,t) &= xt^2 + t^3 + t\sin x - \frac{t^3}{3!}\sin x,
\end{aligned}$$

$$
\begin{aligned}
u_2(x,t) &= xt^2+t^3+t\sin x-\frac{t^3}{3!}\sin x+\frac{t^5}{5!}\sin x,\\
&\vdots\\
u_n(x,t) &= xt^2+t^3+\sin x\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\right),
\end{aligned}
\tag{5.175}
$$

that leads to the exact solution

$$
u(x,t)=xt^2+t^3+\sin x\sin t. \tag{5.176}
$$

Example 13. Use the variational iteration method to solve the initial value problem

$$
\begin{array}{ll}
\text{PDE} & u_{tt}=u_{xx}+e^{-t}, \quad -\infty<x<\infty, \quad t>0,\\
\text{IC} & u(x,0)=1, \quad u_t(x,0)=-1+\sin x.
\end{array}
\tag{5.177}
$$

Solution.

Note that the initial value problem is inhomogeneous. Proceeding as before, we find

$$
u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}-\frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2}-e^{-\xi}\right)d\xi. \tag{5.178}
$$

Substituting the Lagrange multiplier $\lambda=\xi-t$ into the functional (5.178) gives the iteration formula

$$
u_{n+1}(x,t)=u_n(x,t)+\int_0^t (\xi-t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}-\frac{\partial^2 u_n(x,\xi)}{\partial x^2}-e^{-\xi}\right)d\xi, \quad n\geqslant 0. \tag{5.179}
$$

Substituting $u_0(x,t)=1-t+t\sin x$ into (5.179) gives the successive approximations

$$
\begin{aligned}
u_0(x,t) &= 1-t+t\sin x,\\
u_1(x,t) &= e^{-t}+t\sin x-\frac{1}{3!}t^3\sin x,\\
u_2(x,t) &= e^{-t}+t\sin x-\frac{1}{3!}t^3\sin x+\frac{1}{5!}t^5\sin x,\\
u_n(x,t) &= e^{-t}+\sin x\left(t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\cdots\right),
\end{aligned}
\tag{5.180}
$$

that gives the exact solution

$$
u(x,t)=e^{-t}+\sin x\sin t. \tag{5.181}
$$

Exercises 5.3.3

Use the variational iteration method to solve the problems in Exercises 5.2.3.

5.4 Method of Separation of Variables

In this section the homogeneous partial differential equation that describes the vibrations of a vibrating string will be discussed by using a well-known method called the *method of separation of variables*. The most important feature of the method of separation of variables [4] is that it reduces the partial differential equation into a system of ordinary differential equations that can be easily handled.

5.4.1 Analysis of the Method

As discussed before in the heat equation, the method of separation of variables requires that the PDE and the boundary conditions be linear and homogeneous. For this reason, we begin our analysis by discussing the vibrations of a freely vibrating string with fixed ends at $x = 0$ and $x = L$, initial position $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = g(x)$. The initial-boundary value problem that controls the vibrations of a string is given by

$$\begin{array}{lll} \text{PDE} & u_{tt} = c^2 u_{xx}, & 0 < x < L,\ t > 0, \\ \text{BC} & u(0,t) = 0, & u(L,t) = 0, \\ \text{IC} & u(x,0) = f(x), & u_t(x,0) = g(x). \end{array} \tag{5.182}$$

The wave function $u(x,t)$ is the displacement of any point of a vibrating string at position x at time t. The method of separation of variables consists of assuming that the displacement $u(x,t)$ is identified as the product of two distinct functions $F(x)$ and $T(t)$, where $F(x)$ depends on the space variable x and $T(t)$ depends on the time variable t. This assumption allows us to set

$$u(x,t) = F(x)T(t), \tag{5.183}$$

assuming that $F(x)$ and $T(t)$ are twice continuously differentiable. Differentiating both sides of (5.183) twice with respect to t and twice with respect to x we obtain

$$\begin{aligned} u_{tt}(x,t) &= F(x)T''(t), \\ u_{xx}(x,t) &= F''(x)T(t). \end{aligned} \tag{5.184}$$

Substituting (5.184) into (5.182) yields

$$F(x)T''(t) = c^2 F''(x)T(t). \tag{5.185}$$

Dividing both sides of (5.185) by $c^2F(x)T(t)$ gives

$$\frac{T''(t)}{c^2T(t)} = \frac{F''(x)}{F(x)}. \tag{5.186}$$

The left hand side of (5.186) depends only on t and the right hand side depends only on x. This means that the equality holds only if both sides are equal to the same constant. Therefore, we set

$$\frac{T''(t)}{c^2T(t)} = \frac{F''(x)}{F(x)} = -\lambda^2. \tag{5.187}$$

The selection of $-\lambda^2$ in (5.187) is essential to obtain nontrivial solutions. However, we can easily show that selecting the constant to be zero or λ^2 will produce the trivial solution $u(x,t) = 0$.

The result (5.187) gives two distinct ordinary differential equations given by

$$\begin{aligned} F''(x) + \lambda^2 F(x) &= 0, \\ T''(t) + c^2\lambda^2 T(t) &= 0. \end{aligned} \tag{5.188}$$

This means that the partial differential equation of (5.182) is reduced to the more familiar second order ordinary differential equations ODEs (5.188) where each equation relies only on one distinct variable.

To determine the function $F(x)$, we solve the second order linear ODE

$$F''(x) + \lambda^2 F(x) = 0, \tag{5.189}$$

to find that

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{5.190}$$

where A and B are constants. To determine the constants A, B, and λ, we use the homogeneous boundary conditions

$$u(0,t) = 0,\ u(L,t) = 0, \tag{5.191}$$

as given above by (5.182). Substituting (5.191) into the assumption (5.183) gives

$$F(0) = 0,\ F(L) = 0. \tag{5.192}$$

Using (5.192) into (5.190) leads to

$$A = 0, \tag{5.193}$$

and

$$\sin(\lambda L) = 0. \tag{5.194}$$

We exclude $B = 0$ since it gives the trivial solution $u(x,t) = 0$. Accordingly, we find

$$\lambda_n = \frac{n\pi}{L}, \quad n = 1,2,3,\cdots. \tag{5.195}$$

It is important to note that $n = 0$ is excluded since it gives the trivial solution $u(x,t) = 0$. The function $F_n(x)$ associated with λ_n is

$$F_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n = 1,2,3,\cdots. \tag{5.196}$$

Consequently, the solution $T_n(t)$ associated with λ_n must satisfy

$$T_n''(t) + c^2\lambda_n^2 T_n(t) = 0. \tag{5.197}$$

The general solution of (5.197) is given by

$$\begin{aligned} T_n(t) &= C_n\cos(\lambda_n ct) + D_n\sin(\lambda_n ct), \\ &= C_n\cos\left(\frac{n\pi c}{L}t\right) + D_n\sin\left(\frac{n\pi c}{L}t\right), \quad n = 1,2,3,\cdots, \end{aligned} \tag{5.198}$$

where C_n and D_n are constants.

Combining the results (5.196) and (5.198) we obtain the infinite sequence of product functions

$$\begin{aligned} u_n(x,t) &= F_n(x)T_n(t), \\ &= \sin\left(\frac{n\pi}{L}x\right)\left(C_n\cos\left(\frac{n\pi c}{L}t\right) + D_n\sin\left(\frac{n\pi c}{L}t\right)\right), \quad n = 1,2,\cdots. \end{aligned} \tag{5.199}$$

Recall that the superposition principle admits that a linear combination of the functions $u_n(x,t)$ also satisfies the given equation and the boundary conditions. Therefore, using this principle gives the general solution by

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(C_n\cos\left(\frac{n\pi c}{L}t\right) + D_n\sin\left(\frac{n\pi c}{L}t\right)\right), \tag{5.200}$$

where the arbitrary constants $C_n, D_n, n \geqslant 1$, are as yet undetermined. The derivative of (5.200) with respect to t is

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(-\frac{n\pi c}{L}C_n\sin\left(\frac{n\pi c}{L}t\right) + \frac{n\pi c}{L}D_n\cos\left(\frac{n\pi c}{L}t\right)\right). \tag{5.201}$$

To determine $C_n, n \geqslant 1$, we substitute $t = 0$ in (5.200) and by using the initial condition $u(x,0) = f(x)$, we obtain

$$\sum_{n=1}^{\infty} C_n\sin\left(\frac{n\pi}{L}x\right) = f(x), \tag{5.202}$$

so that the constants C_n can be determined in this case by using Fourier coefficients given by the formula

$$C_n = \frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{n\pi}{L}x\right)\,dx. \tag{5.203}$$

To determine $D_n, n \geqslant 1$, we substitute $t = 0$ in (5.201) and by using the initial condition $u_t(x,0) = g(x)$ we obtain

$$\sum_{n=1}^{\infty} \frac{n\pi c}{L} D_n \sin\left(\frac{n\pi}{L}x\right) = g(x), \tag{5.204}$$

so that

$$D_n = \frac{2}{n\pi c}\int_0^L g(x)\,\sin\left(\frac{n\pi}{L}x\right)\,dx. \tag{5.205}$$

Having determined the constants C_n and D_n, the particular solution $u(x,t)$ follows immediately upon substituting (5.203) and (5.205) into (5.200).

It is to be noted that the use of the Fourier coefficients requires a considerable size of calculations. However, if the initial conditions $f(x)$ and $g(x)$ are given in terms of $\sin(nx)$ and $\cos(mx)$, it seems reasonable to expand the solution (5.200) and then equate the coefficients of like terms in both sides to determine the constants C_n and D_n.

To give a clear overview of the method of separation of variables, we have selected several examples of homogeneous PDEs with homogeneous boundary conditions to illustrate the discussion presented above.

Example 1. Use the method of separation of variables to solve the following initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= u_{xx}, \quad 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) &= 0, \quad u(\pi,t) = 0, \\ \text{IC} & u(x,0) &= \sin(2x), \quad u_t(x,0) = 0. \end{array} \tag{5.206}$$

Solution.

We first set

$$u(x,t) = F(x)T(t). \tag{5.207}$$

Using (5.207) into (5.206) and proceeding as discussed before we find

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x). \tag{5.208}$$

$$T(t) = C\cos(\lambda t) + B\sin(\lambda t). \tag{5.209}$$

Using the boundary conditions of (5.206) into (5.208) gives

$$\begin{aligned} A &= 0, \\ \lambda_n &= n, \quad n = 1,2,3,\cdots. \end{aligned} \tag{5.210}$$

Based on this, equations (5.208) and (5.209) become

$$\begin{aligned} F_n(x) &= B_n \sin(nx), \quad n = 1, 2, \cdots, \\ T_n(t) &= C_n \cos(nt) + D_n \sin(nt). \end{aligned} \tag{5.211}$$

This gives the infinite sequence of product functions

$$u_n(x,t) = \sin(nx)\left(C_n \cos(nt) + D_n \sin(nt)\right). \tag{5.212}$$

Using the superposition principle we obtain

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(C_n \cos(nt) + D_n \sin(nt)\right), \tag{5.213}$$

and its derivative with respect to t is

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(-nC_n \sin(nt) + nD_n \cos(nt)\right). \tag{5.214}$$

To determine C_n, we use the initial condition $u(x,0) = \sin(2x)$ and substitute $t = 0$ in (5.213) to find

$$C_1 \sin x + C_2 \sin(2x) + C_3 \sin(3x) + \cdots = \sin(2x). \tag{5.215}$$

Equating the coefficients of like terms of both sides gives

$$C_2 = 1, \quad C_j = 0,\ j \neq 2. \tag{5.216}$$

To determine D_n, substitute $t = 0$ in (5.214), and use the initial condition $u_t(x,0) = 0$ to find

$$D_1 \sin x + 2D_2 \sin(2x) + 3D_3 \sin(3x) + \cdots = 0, \tag{5.217}$$

so that

$$D_j = 0, \quad j \geqslant 1. \tag{5.218}$$

Combining (5.213), (5.216), and (5.218), the particular solution is given by

$$u(x,t) = \sin(2x)\cos(2t). \tag{5.219}$$

Example 2. Use the method of separation of variables to solve the following initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= u_{xx}, \quad 0 < x < \pi,\ t > 0, \\ \text{BC} & u(0,t) &= 0, \quad u(\pi,t) = 0, \\ \text{IC} & u(x,0) &= \sin x, \quad u_t(x,0) = 2\sin x. \end{array} \tag{5.220}$$

Solution.

We first set

$$u(x,t) = F(x)T(t). \tag{5.221}$$

Proceeding as before and substituting (5.221) into (5.220), the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(C_n \cos(nt) + D_n \sin(nt)\right). \tag{5.222}$$

To determine C_n, we use the initial condition $u(x,0) = \sin x$ and replace t by zero in (5.222) to find

$$C_1 \sin x + C_2 \sin(2x) + \cdots = \sin x. \tag{5.223}$$

Equating coefficients of like terms of both sides of (5.223) gives

$$C_1 = 1, C_j = 0, j \neq 1. \tag{5.224}$$

To determine D_n, we use the initial condition $u_t(x,0) = 2\sin x$ and replace t by zero in the derivative of (5.222) to find

$$D_1 \sin x + 2D_2 \sin(2x) + \cdots = 2\sin x. \tag{5.225}$$

Equating coefficients of like terms of both sides of (5.225) gives

$$D_1 = 2, \quad D_j = 0, \ j \neq 1. \tag{5.226}$$

Combining the results (5.222), (5.224) and (5.226), the particular solution is given by

$$u(x,t) = \sin x \cos t + 2\sin x \sin t. \tag{5.227}$$

Example 3. Use the method of separation of variables to solve the following initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = u_{xx}, & 0 < x < \pi, \ t > 0, \\ \text{BC} & u_x(0,t) = 0, & u_x(\pi,t) = 0, \\ \text{IC} & u(x,0) = 0, & u_t(x,0) = \cos x. \end{array} \tag{5.228}$$

Solution.

It is interesting to note that the boundary conditions are of the second kind defined by the derivatives $u_x(0,t) = 0$ and $u_x(\pi,t) = 0$. This means that the ends of the string are free and not fixed. We first set

$$u(x,t) = F(x)T(t). \tag{5.229}$$

Proceeding as before and substituting (5.229) into (5.228) and solving the resulting equations we obtain

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{5.230}$$

so that by using the boundary conditions we find

$$\begin{aligned} B &= 0, \quad A \neq 0, \\ \lambda_n &= n, \quad n = 0,1,2,3,\cdots. \end{aligned} \tag{5.231}$$

Note that $\lambda_n = 0$ is considered because it did not give the trivial solution. Accordingly, we find an infinite number of solutions for $F_n(x)$ defined by

$$F_n(x) = A_n \cos(nx), \quad n = 0, 1, 2, 3, \cdots. \tag{5.232}$$

Using the values obtained for λ_n, we obtain

$$T_n(t) = C_n \cos(nt) + D_n \sin(nt), \tag{5.233}$$

where C_n and D_n are as yet undetermined constants.

Using the superposition principle, the general solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} \cos(nx)\left(C_n \cos(nt) + D_n \sin(nt)\right). \tag{5.234}$$

To determine C_n, we use the initial condition $u(x,0) = 0$ and replace t by zero in (5.234) to find

$$C_0 + C_1 \cos x + C_2 \cos(2x) + \cdots = 0, \tag{5.235}$$

and this gives

$$C_j = 0, \quad j \geqslant 0. \tag{5.236}$$

To determine D_n, we use the initial condition $u_t(x,0) = \cos x$ and replace t by zero in the derivative of (5.234) with respect to t to find

$$D_1 \cos x + 2D_2 \cos(2x) + \cdots = \cos x, \tag{5.237}$$

so that

$$D_1 = 1, \quad D_j = 0, \quad j \neq 1. \tag{5.238}$$

Combining the results (5.234), (5.236) and (5.238), the particular solution is given by

$$u(x,t) = \cos x \sin t, \tag{5.239}$$

obtained by substituting (5.236) and (5.238) into (5.234).

Example 4. Use the method of separation of variables to solve the following initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = & u_{xx}, \quad 0 < x < \pi,\ t > 0, \\ \text{BC} & u_x(0,t) = & 0, \quad u_x(\pi,t) = 0, \\ \text{IC} & u(x,0) = & 1 + \cos x, \quad u_t(x,0) = 0. \end{array} \tag{5.240}$$

Solution.

Proceeding as before we obtain

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{5.241}$$

so that by using the boundary conditions we find

$$\begin{aligned} B &= 0, \quad A \neq 0, \\ \lambda_n &= n, \quad n = 0, 1, 2, 3, \cdots. \end{aligned} \tag{5.242}$$

Consequently we find

$$F_n(x) = A_n \cos(nx), \quad n = 0, 1, 2, 3, \cdots. \tag{5.243}$$

Using the values obtained for λ_n, we obtain

$$T_n(t) = C_n \cos(nt) + D_n \sin(nt). \tag{5.244}$$

Using the superposition principle gives the general solution by

$$u(x,t) = \sum_{n=0}^{\infty} \cos(nx) \left(C_n \cos(nt) + D_n \sin(nt)\right), \tag{5.245}$$

so that

$$u_t(x,t) = \sum_{n=0}^{\infty} \cos(nx) \left(-nC_n \sin(nt) + nD_n \cos(nt)\right). \tag{5.246}$$

To determine C_n, we replace t by zero in (5.245) to find

$$C_0 + C_1 \cos x + C_2 \cos(2x) + \cdots = 1 + \cos x, \tag{5.247}$$

and this gives

$$C_0 = 1, \quad C_1 = 1, \quad C_j = 0, \ j \geqslant 2. \tag{5.248}$$

We next use the initial condition $u_t(x,0) = 0$ and replace t by zero in (5.246) to find

$$D_1 \cos x + 2D_2 \cos(2x) + \cdots = 0, \tag{5.249}$$

so that

$$D_j = 0, \quad j \geqslant 1. \tag{5.250}$$

Combining the results obtained above gives

$$u(x,t) = 1 + \cos x \cos t. \tag{5.251}$$

Example 5. Use the method of separation of variables to solve the following initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = u_{xx}, & 0 < x < \pi, \ t > 0, \\ \text{BC} & u(0,t) = 0, & u(\pi,t) = 0, \\ \text{IC} & u(x,0) = 1, & u_t(x,0) = 0. \end{array} \tag{5.252}$$

Solution.

Following the analysis introduced before leads to

$$F_n(x) = A_n \sin(nx), \quad n = 1, 2, 3, \cdots. \tag{5.253}$$

and

$$T_n(t) = C_n \cos(nt) + D_n \sin(nt), \tag{5.254}$$

which in turn gives

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(C_n \cos(nt) + D_n \sin(nt)\right). \tag{5.255}$$

To determine C_n, we substitute $t = 0$ into (5.255), and use the initial condition $u(x,0) = 1$ to find

$$\sum_{n=1}^{\infty} C_n \sin(nx) = 1. \tag{5.256}$$

The arbitrary constants C_n are determined by using the Fourier coefficients method, therefore we find

$$\begin{aligned} C_n &= \frac{2}{\pi}\int_0^{\pi} \sin(nx)dx, \\ &= \frac{2}{n\pi}(1 - \cos(n\pi)), \end{aligned} \tag{5.257}$$

so that

$$C_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{4}{n\pi}, & \text{if } n \text{ is odd.} \end{cases} \tag{5.258}$$

This means that we can express C_n by

$$\begin{aligned} C_{2m} &= 0, \\ C_{2m+1} &= \frac{4}{(2m+1)\pi}, \quad m = 0,1,2,\cdots. \end{aligned} \tag{5.259}$$

To determine D_n, we substitute $t = 0$ into the derivative of (5.255) to find

$$D_n = 0, \quad n = 1,2,3,\cdots. \tag{5.260}$$

Combining the results obtained above, the particular solution is given by

$$u(x,t) = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{1}{2m+1} \sin(2m+1)x \cos(2m+1)t. \tag{5.261}$$

The initial condition $u(x,0) = 1$ can be justified by using Appendix F.

Exercises 5.4.1

In Exercises 1–6, where the ends of the string are fixed, solve the initial-boundary value problems by the method of separation of variables:

1. $u_{tt} = u_{xx}$, $0 < x < \pi$, $t > 0$

$u(0,t) = 0,\ u(\pi,t) = 0$
$u(x,0) = 0, u_t(x,0) = 3\sin(3x)$

2. $u_{tt} = u_{xx},\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = 0$
$u(x,0) = \sin x, u_t(x,0) = 0$

3. $u_{tt} = u_{xx},\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = 0$
$u(x,0) = 0, u_t(x,0) = 4\sin(2x)$

4. $u_{tt} = 4u_{xx},\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = 0$
$u(x,0) = \sin x, u_t(x,0) = 0$

5. $u_{tt} = 4u_{xx},\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = 0$
$u(x,0) = \sin(2x), u_t(x,0) = 0$

6. $u_{tt} = 9u_{xx},\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = 0$
$u(x,0) = 0, u_t(x,0) = 3\sin x$

In Exercises 7–10, where the ends of the string are free, solve the initial-boundary value problems by the method of separation of variables:

7. $u_{tt} = 9u_{xx},\ 0 < x < \pi,\ t > 0$
$u_x(0,t) = 0,\ u_x(\pi,t) = 0$
$u(x,0) = 1, u_t(x,0) = 3\cos x$

8. $u_{tt} = 4u_{xx},\ 0 < x < \pi,\ t > 0$
$u_x(0,t) = 0,\ u_x(\pi,t) = 0$
$u(x,0) = 2 + \cos x, u_t(x,0) = 0$

9. $u_{tt} = 9u_{xx},\ 0 < x < \pi,\ t > 0$
$u_x(0,t) = 0,\ u_x(\pi,t) = 0$
$u(x,0) = 0, u_t(x,0) = 3\cos x$

10. $u_{tt} = u_{xx},\ 0 < x < \pi,\ t > 0$
$u_x(0,t) = 0,\ u_x(\pi,t) = 0$
$u(x,0) = \cos x, u_t(x,0) = \cos x$

In Exercises 11–12, use the Fourier coefficients to solve the following initial-boundary value problems:

11. $u_{tt} = u_{xx},\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = 0$
$u(x,0) = 0, u_t(x,0) = x$

12. $u_{tt} = 4u_{xx},\ 0 < x < \pi,\ t > 0$
$u(0,t) = 0,\ u(\pi,t) = 0$

$$u(x,0) = 0, u_t(x,0) = x(1-x)$$

5.4.2 Inhomogeneous Boundary Conditions

In this section we will consider the case where the boundary conditions of the vibrating string are inhomogeneous. It is well known that the method of separation of variables requires that the equation and the boundary conditions are linear and homogeneous. Therefore, transformation formulas should be used to convert the inhomogeneous boundary conditions to homogeneous boundary conditions.

In this section we will discuss wave equations where Dirichlet boundary conditions and Neumann boundary conditions are not homogeneous. It is normal to seek transformation formulas to convert these inhomogeneous conditions to homogeneous conditions.

Dirichlet Boundary Conditions

In this first type of boundary conditions, the displacements $u(0,t) = \alpha$ and $u(L,t) = \beta$ of a vibrating string of length L are given. We begin our analysis by considering the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= c^2 u_{xx}, \quad 0 < x < L,\ t > 0, \\ \text{BC} & u(0,t) &= \alpha, \quad u(L,t) = \beta,\ t \geqslant 0, \\ \text{IC} & u(x,0) &= f(x), \quad u_t(x,0) = g(x). \end{array} \tag{5.262}$$

To convert the inhomogeneous boundary conditions of (5.262) to homogeneous boundary conditions, we simply use the conversion formula that we used before in Section 3.4.2. In other words, the following transformation formula

$$u(x,t) = \left(\alpha + \frac{x}{L}(\beta - \alpha)\right) + v(x,t), \tag{5.263}$$

should be used to achieve this goal.

Substituting (5.263) into (5.262) shows that $v(x,t)$ is governed by the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & v_{tt} &= c^2 v_{xx}, \quad 0 < x < L,\ t > 0, \\ \text{BC} & v(0,t) &= 0, \quad v(L,t) = 0, \\ \text{IC} & v(x,0) &= f(x) - \left(\alpha + \frac{x}{L}(\beta - \alpha)\right), \quad v_t(L,t) = g(x). \end{array} \tag{5.264}$$

In view of (5.264), the method of separation of variables can be easily used in (5.264) as discussed before. Having determined $v(x,t)$ of (5.264), the wave function $u(x,t)$ of (5.262) follows immediately upon substituting $v(x,t)$ into (5.263).

To get a better understanding of the implementation of the transformation formula (5.263), we will discuss the following illustrative examples.

Example 6. Solve the following initial-boundary value problem

$$\begin{array}{lll} \text{PDE} & u_{tt} = u_{xx}, & 0 < x < 1,\ t > 0, \\ \text{BC} & u(0,t) = 1, & u(1,t) = 2,\ t \geqslant 0, \\ \text{IC} & u(x,0) = 1+x, & u_t(x,0) = \pi\sin(\pi x). \end{array} \tag{5.265}$$

Solution.

Using the transformation formula (5.263) we obtain

$$u(x,t) = (1+x) + v(x,t), \tag{5.266}$$

that carries (5.265) into

$$\begin{array}{lll} \text{PDE} & v_{tt} = v_{xx}, & 0 < x < 1,\ t > 0, \\ \text{BC} & v(0,t) = 0, & v(1,t) = 0, \\ \text{IC} & v(x,0) = 0, & v_t(x,0) = \pi\sin(\pi x). \end{array} \tag{5.267}$$

Assuming that

$$v(x,t) = F(x)T(t), \tag{5.268}$$

and proceeding as before we obtain

$$F_n(x) = B_n\sin(n\pi x), \quad n = 1,2,\cdots \tag{5.269}$$

and

$$T_n(t) = C_n\cos(n\pi t) + D_n\sin(n\pi t), \tag{5.270}$$

so that

$$v(x,t) = \sum_{n=1}^{\infty}\sin(n\pi x)\left(C_n\cos(n\pi t) + D_n\sin(n\pi t)\right), \tag{5.271}$$

where C_n and D_n are as yet undetermined constants. Using the initial condition $v(x,0) = 0$ in (5.271) gives

$$C_n = 0, \quad n \geqslant 1. \tag{5.272}$$

Using the initial condition $v_t(x,0) = \pi\sin(\pi x)$ into the derivative of (5.271) we obtain

$$D_1 = 1, \quad D_k = 0,\ k \neq 1. \tag{5.273}$$

This gives the solution for $v(x,t)$ by

$$v(x,t) = \sin(\pi x)\sin(\pi t), \tag{5.274}$$

so that the particular solution $u(x,t)$ of (5.265) is given by

$$u(x,t) = 1 + x + \sin(\pi x)\sin(\pi t). \tag{5.275}$$

At this point, it seems reasonable to use the Adomian decomposition method to solve the initial-boundary value problem of this example. This will enable us to compare the performance of the decomposition method and the classical method of separation of variables.

Applying the inverse operator L_t^{-1} to the operator form of (5.265) and using the initial conditions we obtain

$$u(x,t) = 1 + x + (\pi t)\sin(\pi x) + L_t^{-1}\left(L_x u(x,t)\right). \tag{5.276}$$

Using the decomposition series of $u(x,t)$ into both sides of equation (5.276) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 + x + (\pi t)\sin(\pi x) + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{5.277}$$

Using the recursive algorithm we obtain

$$\begin{aligned} u_0 &= 1 + x + (\pi t)\sin(\pi x),\\ u_1 &= L_t^{-1}(L_x u_0) = -\frac{(\pi t)^3}{3!}\sin(\pi x),\\ u_2 &= L_t^{-1}(L_x u_1) = \frac{(\pi t)^5}{5!}\sin(\pi x), \end{aligned} \tag{5.278}$$

and so on. Consequently, the solution in a series form is given by

$$u(x,t) = 1 + x + \sin(\pi x)\left(\pi t - \frac{(\pi t)^3}{3!} + \frac{(\pi t)^5}{5!} - \cdots\right), \tag{5.279}$$

and in a closed form

$$u(x,t) = 1 + x + \sin(\pi x)\sin(\pi t). \tag{5.280}$$

It is obvious that we obtained the solution (5.280) by employing less computational work if compared with the method of separation of variables. The power of the decomposition method for solving differential equations is thus emphasized.

Example 7. Solve the following initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= u_{xx}, \quad 0 < x < 1,\ t > 0,\\ \text{BC} & u(0,t) &= 2, \quad u(1,t) = 3,\ t \geqslant 0,\\ \text{IC} & u(x,0) &= 2 + x + \sin(\pi x), \quad u_t(x,0) = 0. \end{array} \tag{5.281}$$

Solution.

Using the transformation formula (5.263) where $\alpha = 2$ and $\beta = 3$, we obtain

$$u(x,t) = (2 + x) + v(x,t). \tag{5.282}$$

In view of (5.282), we find that

$$\begin{array}{lll} \text{PDE} & v_{tt} = v_{xx}, & 0 < x < 1,\ t > 0 \\ \text{BC} & v(0,t) = 0, & v(1,t) = 0 \\ \text{IC} & v(x,0) = \sin(\pi x), & v_t(x,0) = 0 \end{array} \tag{5.283}$$

Following the analysis presented before gives

$$F_n(x) = B_n \sin(n\pi x), \quad n = 1,2,\cdots \tag{5.284}$$

and

$$T_n(t) = C_n \cos(n\pi t) + D_n \sin(n\pi t), \tag{5.285}$$

so that

$$v(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(C_n \cos(n\pi t) + D_n \sin(n\pi t)\right), \tag{5.286}$$

where C_n and D_n are as yet undetermined constants Using the initial condition $v(x,0) = \sin(\pi x)$ in (5.286) gives

$$C_1 = 1, \quad C_k = 0, k \neq 1. \tag{5.287}$$

Differentiating (5.286) and using the initial condition $v_t(x,0) = 0$ we obtain

$$D_n = 0, \quad n \geqslant 1. \tag{5.288}$$

This gives the solution for $v(x,t)$ by

$$v(x,t) = \sin(\pi x)\cos(\pi t), \tag{5.289}$$

so that the particular solution $u(x,t)$ of (5.281) is given by

$$u(x,t) = 2 + x + \sin(\pi x)\cos(\pi t). \tag{5.290}$$

Neumann Boundary Conditions

We next consider the second kind of boundary conditions, where $u_x(0,t) = \alpha$ and $u_x(L,t) = \beta$ are given. We point out that the transformation formula (5.263) works effectively for the first kind of boundary conditions, but cannot be used for the second kind of boundary conditions. To study the proper formula in this case, we consider the initial-boundary value problem

$$\begin{array}{lll} \text{PDE} & u_{tt} = c^2 u_{xx}, & 0 < x < L,\ t > 0, \\ \text{BC} & u_x(0,t) = \alpha, & u_x(L,t) = \beta,\ t \geqslant 0, \\ \text{IC} & u(x,0) = f(x), & u_t(x,0) = g(x). \end{array} \tag{5.291}$$

It is interesting to note that an alternative formula should be used to convert the inhomogeneous boundary conditions of (5.291) to homogeneous boundary conditions. We can easily prove that the transformation formula

$$u(x,t) = \alpha x + \left(\frac{\beta-\alpha}{2L}\right)x^2 + c^2\left(\frac{\beta-\alpha}{2L}\right)t^2 + v(x,t) \tag{5.292}$$

is an appropriate formula that can be used to achieve our goal of conversion. Differentiating (5.292) twice with respect to t and to x we obtain

$$\begin{aligned} u_{tt} &= c^2\left(\frac{\beta-\alpha}{L}\right) + v_{tt}, \\ u_{xx} &= \frac{\beta-\alpha}{L} + v_{xx}. \end{aligned} \tag{5.293}$$

Using (5.292) and (5.293), it can be easily shown that $v(x,t)$ is governed by the initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & v_{tt} &= c^2 v_{xx}, \quad 0 < x < L,\ t > 0, \\ \text{BC} & v_x(0,t) &= 0, \quad v_x(L,t) = 0, \\ \text{IC} & v(x,0) &= f(x) - \left(\alpha x + \dfrac{\beta-\alpha}{2L}x^2\right), \quad v_t(x,0) = g(x). \end{array} \tag{5.294}$$

It is clear that an initial-boundary value problem (5.294) with homogeneous boundary conditions is obtained. The use of the transformation formula (5.292) will be explained by the following illustrative examples.

Example 8. Solve the following initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= u_{xx}, \quad 0 < x < 1,\ t > 0, \\ \text{BC} & u_x(0,t) &= 1, \quad u_x(1,t) = 3,\ t \geqslant 0, \\ \text{IC} & u(x,0) &= x^2 + x,\ u_t(x,0) = \pi\cos(\pi x). \end{array} \tag{5.295}$$

Solution.

Using the transformation formula (5.292), where $\alpha = 1, \beta = 3$, and $c = 1$, we find

$$u(x,t) = x + x^2 + t^2 + v(x,t) \tag{5.296}$$

In view of (5.296), the initial-boundary value problem for $v(x,t)$ is given by

$$\begin{array}{lrl} \text{PDE} & v_{tt} &= v_{xx}, \quad 0 < x < L,\ t > 0, \\ \text{BC} & v_x(0,t) &= 0, \quad v_x(1,t) = 0,\ t \geqslant 0, \\ \text{IC} & v(x,0) &= 0, \quad v_t(x,0) = \pi\cos(\pi x). \end{array} \tag{5.297}$$

Proceeding as discussed before we find

$$v(x,t) = \sum_{n=0}^{\infty} \cos(n\pi x)\left(C_n\cos(n\pi t) + D_n\sin(n\pi t)\right). \tag{5.298}$$

Using the initial conditions gives

$$C_n = 0, n \geqslant 0. \tag{5.299}$$

and

$$D_1 = 1, \quad D_k = 0,\ k \neq 1. \tag{5.300}$$

This leads to

$$v(x,t) = \cos(\pi x)\sin(\pi t), \tag{5.301}$$

which gives

$$u(x,t) = x + x^2 + t^2 + \cos(\pi x)\sin(\pi t). \tag{5.302}$$

Exercises 5.4.2

Use the method of separation of variables to solve Exercises 1–6, where the first kind of boundary conditions are given:

1. $u_{tt} = u_{xx},\ 0 < x < 1,\ t > 0$
   $u(0,t) = 1,\ u(1,t) = 1$
   $u(x,0) = 1,\ u_t(x,0) = \pi\sin(\pi x)$

2. $u_{tt} = u_{xx},\ 0 < x < 1,\ t > 0$
   $u(0,t) = 2,\ u(1,t) = 3$
   $u(x,0) = 2 + x + 2\sin(\pi x),\ u_t(x,0) = 0$

3. $u_{tt} = u_{xx},\ 0 < x < 1,\ t > 0$
   $u(0,t) = 0,\ u(1,t) = 3$
   $u(x,0) = 3x,\ u_t(x,0) = 4\pi\sin(\pi x)$

4. $u_{tt} = u_{xx},\ 0 < x < 1,\ t > 0$
   $u(0,t) = 4,\ u(1,t) = 1$
   $u(x,0) = 4 - 3x,\ u_t(x,0) = \pi\sin(\pi x)$

5. $u_{tt} = 4u_{xx},\ 0 < x < 1,\ t > 0$
   $u(0,t) = 3,\ u(1,t) = 7$
   $u(x,0) = 3 + 4x,\ u_t(x,0) = 2\pi\sin(\pi x)$

6. $u_{tt} = 4u_{xx},\ 0 < x < 1,\ t > 0$
   $u(0,t) = 1,\ u(1,t) = 2$
   $u(x,0) = 1 + x,\ u_t(x,0) = 4\pi\sin(2\pi x)$

In Exercises 7–12, where the boundary conditions $u_x(0,t)$ and $u_x(1,t)$ are given, use the method of separation of variables to solve the initial-boundary value problems:

7. $u_{tt} = u_{xx},\ 0 < x < 1,\ t > 0$
   $u_x(0,t) = 3,\ u_x(1,t) = 5$
   $u(x,0) = 3x + x^2 + \cos(\pi x),\ u_t(x,0) = 0$

8. $u_{tt} = u_{xx},\ 0 < x < 1,\ t > 0$
   $u_x(0,t) = 4,\ u_x(1,t) = 4$
   $u(x,0) = 4x,\ u_t(x,0) = \pi\cos(\pi x)$

9. $u_{tt} = u_{xx}$, $0 < x < 1$, $t > 0$
   $u_x(0,t) = 2$, $u_x(1,t) = 6$
   $u(x,0) = 2x + 2x^2 + \cos(\pi x)$, $u_t(x,0) = 0$

10. $u_{tt} = 4u_{xx}$, $0 < x < 1$, $t > 0$
    $u_x(0,t) = 1$, $u_x(1,t) = 1$
    $u(x,0) = x$, $u_t(x,0) = 2\pi\cos(\pi x)$

11. $u_{tt} = 4u_{xx}$, $0 < x < 1$, $t > 0$
    $u_x(0,t) = 2$, $u_x(1,t) = 2$
    $u(x,0) = 2x + \cos(\pi x)$, $u_t(x,0) = 0$

12. $u_{tt} = 9u_{xx}$, $0 < x < 1$, $t > 0$
    $u_x(0,t) = 1$, $u_x(1,t) = 3$
    $u(x,0) = x + x^2$, $u_t(x,0) = 3\pi\cos(\pi x)$

5.5 Wave Equation in an Infinite Domain: D'Alembert Solution

In Section 5.2.3, the motion of a very long string, that is considered not to have boundaries, has been handled by using the decomposition method. The physical model that controls the wave motion of a very long string is governed by a PDE and initial conditions only. As mentioned before, the method of separation of variables is not applicable in this case.

However, a standard method, known as D'Alembert solution, allows us to solve the initial value problem on an infinite domain.

To derive D'Alembert formula, we consider a typical wave equation in an infinite domain given by

$$\begin{array}{lrl} \text{PDE} & u_{tt} = c^2 u_{xx}, & -\infty < x < \infty,\ t > 0, \\ \text{IC} & u(x,0) = f(x), & u_t(x,0) = g(x). \end{array} \tag{5.303}$$

As stated in a previous section, the attention will be focused upon the disturbance occurred at the center of the very long string. The initial displacement $u(x,0)$ and the initial velocity $u_t(x,0)$ are prescribed by $f(x)$ and $g(x)$ respectively.

To derive D'Alembert solution, we consider two new variables ξ and η defined by

$$\begin{aligned} \xi &= x + ct, \\ \eta &= x - ct. \end{aligned} \tag{5.304}$$

Using the chain rule we obtain

$$\begin{aligned} u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\ u_{tt} &= c^2\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right). \end{aligned} \tag{5.305}$$

Substituting (5.305) into (5.303) gives

$$u_{\xi\eta} = 0. \tag{5.306}$$

Integrating (5.306) first with respect ξ then with respect η we obtain the general solution given by

$$u(\xi,\eta) = F(\xi) + G(\eta), \tag{5.307}$$

where F and G are arbitrary functions. Using (5.304), equation (5.307) can be rewritten as

$$u(x,t) = F(x+ct) + G(x-ct), \tag{5.308}$$

Using the initial condition $u(x,0) = f(x)$ into (5.308) yields

$$F(x) + G(x) = f(x). \tag{5.309}$$

Substituting the initial condition $u_t(x,0) = g(x)$ into (5.308) gives

$$cF'(x) - cG'(x) = g(x). \tag{5.310}$$

Integrating both sides of (5.310) from 0 to x gives

$$F(x) - G(x) = \frac{1}{c}\int_0^x g(r)\,dr + K, \tag{5.311}$$

where K is the constant of integration. Solving (5.309) and (5.311) we find

$$\begin{aligned} F(x) &= \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(r)\,dr + \frac{1}{2}K, \\ G(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(r)\,dr - \frac{1}{2}K. \end{aligned} \tag{5.312}$$

This means that

$$\begin{aligned} F(x+ct) &= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(r)\,dr + \frac{1}{2}K, \\ G(x-ct) &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(r)\,dr - \frac{1}{2}K, \end{aligned} \tag{5.313}$$

so that by using (5.308) we obtain the D'Alembert formula given by

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(r)\,dr. \tag{5.314}$$

This completes the formal derivation of D'Alembert solution. To explain D'Alembert's formula, we consider the following examples.

Example 1. Use the D'Alembert formula to solve the initial value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = u_{xx}, & -\infty < x < \infty,\ t > 0, \\ \text{IC} & u(x,0) = \sin x, & u_t(x,0) = 0. \end{array} \tag{5.315}$$

Solution.

Substituting $c = 1, f(x) = \sin x$ and $g(x) = 0$ into (5.314) gives the particular solution by

$$u(x,t) = \frac{\sin(x+t)+\sin(x-t)}{2}, \tag{5.316}$$

which gives

$$u(x,t) = \sin x \cos t. \tag{5.317}$$

Example 2. Use the D'Alembert formula to solve the initial value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = 4u_{xx}, & -\infty < x < \infty,\ t > 0, \\ \text{IC} & u(x,0) = \sin x, & u_t(x,0) = 4. \end{array} \tag{5.318}$$

Solution.

Substituting $c = 2, f(x) = \sin x$ and $g(x) = 4$ into (5.314) gives the particular solution

$$u(x,t) = \frac{\sin(x+2t)+\sin(x-2t)}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} 4\,dr, \tag{5.319}$$

which gives

$$u(x,t) = \sin x \cos(2t) + 4t. \tag{5.320}$$

Example 3. Use the D'Alembert formula to solve the initial value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = 9u_{xx}, & -\infty < x < \infty,\ t > 0, \\ \text{IC} & u(x,0) = \sin x, & u_t(x,0) = 3\cos x. \end{array} \tag{5.321}$$

Solution.

Note that $c = 3, f(x) = \sin x$ and $g(x) = 3\cos x$. Substituting into (5.314) gives

$$\begin{aligned} u(x,t) &= \frac{\sin(x+3t)+\sin(x-3t)}{2} + \frac{1}{6}\int_{x-3t}^{x+3t} 3\cos r\,dr, \\ &= \frac{\sin(x+3t)+\sin(x-3t)}{2} + \frac{\sin(x+3t)-\sin(x-3t)}{2} \end{aligned} \tag{5.322}$$

which gives

$$u(x,t) = \sin(x+3t). \tag{5.323}$$

Example 4. Use the D'Alembert formula to solve the initial value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} = u_{xx}, & -\infty < x < \infty,\ t > 0, \\ \text{IC} & u(x,0) = e^{-x}, & u_t(x,0) = \dfrac{2}{1+x^2}. \end{array} \tag{5.324}$$

Solution.

Substituting $c = 1, f(x) = e^{-x}$ and $g(x) = \dfrac{2}{1+x^2}$ into (5.314) gives

$$\begin{aligned} u(x,t) &= \frac{e^{-(x+t)} + e^{-(x-t)}}{2} + \frac{1}{2}\int_{x-t}^{x+t} \frac{2}{1+r^2}\,dr, \\ &= \frac{e^{-(x+t)} + e^{-(x-t)}}{2} + [\arctan r]_{x-t}^{x+t} \\ &= \frac{e^{-x}(e^t + e^{-t})}{2} + \arctan(x+t) - \arctan(x-t), \end{aligned} \tag{5.325}$$

which gives the particular solution

$$u(x,t) = e^{-x}\cosh t + \arctan(x+t) - \arctan(x-t). \tag{5.326}$$

Example 5. If $u(x,t) = \sin x \cos t + 2xt$ is a solution of the initial value problem

$$\begin{array}{lll} \text{PDE} & u_{tt} = u_{xx}, & -\infty < x < \infty,\ t > 0, \\ \text{IC} & u(x,0) = f(x), & u_t(x,0) = 2x. \end{array} \tag{5.327}$$

Use the D'Alembert formula to find $f(x)$.

Solution.

Substituting $c = 1$ and $g(x) = 2x$ into (5.314) gives

$$\begin{aligned} \sin x \cos t + 2xt &= \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} 2r\,dr, \\ &= \frac{f(x+t) + f(x-t)}{2} + 2xt. \end{aligned} \tag{5.328}$$

This in turn gives

$$f(x+t) + f(x-t) = 2\sin x \cos t = \sin(x+t) + \sin(x-t). \tag{5.329}$$

Based on this result we obtain

$$f(x) = \sin x. \tag{5.330}$$

Exercises 5.5

Use the D'Alembert formula to solve the following initial value problems:

1. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = 0,\ u_t(x,0) = 2 + \sin x$

2. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = \sin x,\ u_t(x,0) = \cos x$

3. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$

$u(x,0) = \cos x,\ u_t(x,0) = -\sin x$

4. $u_{tt} = 16u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = \sin x,\ u_t(x,0) = 4\cos x$

5. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = \sin x,\ u_t(x,0) = 2$

6. $u_{tt} = 4u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = \cos x,\ u_t(x,0) = -2\sin x$

7. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = \sinh x,\ u_t(x,0) = \cosh x$

8. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = x,\ u_t(x,0) = e^{-x}$

9. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
   $u(x,0) = \cosh x,\ u_t(x,0) = 0$

10. $u_{tt} = 4u_{xx},\ -\infty < x < \infty,\ t > 0$
    $u(x,0) = 0,\ u_t(x,0) = 2\sinh x$

11. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
    $u(x,0) = \cos x,\ u_t(x,0) = 1 + 2x$

12. $u_{tt} = u_{xx},\ -\infty < x < \infty,\ t > 0$
    $u(x,0) = \sin x,\ u_t(x,0) = 4 + 4x$

References

1. G. Adomian, Nonlinear Stochastic Operator Equations, Academic Press, San Diego, (1986).
2. N. Asmar, Partial Differential Equations, Prentice Hall, New Jersey, (2005).
3. J.M. Cooper, Introduction to Partial Differential Equations with MATLAB, Birkhauser, Boston, (1998).
4. S.J. Farlow, Partial Differential Equations for Scientists and Engineers, Dover, New York, (1993).
5. R. Haberman, Applied Partial Differential Equations, Pearson, New York, (2003).
6. J.H. He, A variational iteration method–a kind of nonlinear analytical technique: Some examples, *Int. J. Nonlinear Mech.*, 34, 699–708, (1999).
7. R.C. McOwen, Partial Differential Equations, Prentice Hall, New Jersey, (1996).
8. A.M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema, Leiden, (2002).
9. A.M. Wazwaz, A reliable technique for solving the wave equation in an infinite one-dimensional medium, *Appl. Math. Comput.*, 79, 37–44, (1998).
10. A.M.Wazwaz, Blow-up for solutions of some linear wave equations with mixed nonlinear boundary conditions, *Appl. Math. Comput.*, 123, 133–140, (2001).

Chapter 6
Higher Dimensional Wave Equation

6.1 Introduction

In this chapter we will discuss the initial-boundary value problems that control the wave propagation in two and three dimensional spaces. The methods that will be applied are the Adomian decomposition method [1] and the method of separation of variables [2–5]. The two methods have been outlined before and were applied to the one dimensional wave equation in Chapter 5.

The decomposition method decomposes the solution u of any equation into an infinite series of components $u_0, u_1, u_2, \cdots$ where these components are elegantly computed. The determination of these components can be achieved in an easy way through a recursive relation that involves simple integrals [6].

The method of separation of variables provides the solution of a partial differential equation through converting the partial differential equation into several easily solvable ordinary differential equations. In addition, the method requires that the problem and the boundary conditions be linear and homogeneous, hence transformation formulas are usually used to justify this need. The method of separation of variables cannot handle initial value problems because boundary conditions are not prescribed. However, the decomposition method can easily handle these problems.

6.2 Adomian Decomposition Method

In previous chapters we have discussed Adomian decomposition method and have applied it to partial differential equations of any order, homogeneous and inhomogeneous. The decomposition method consists of decomposing the unknown function u into an infinite sum of components $u_0, u_1, u_2, \cdots$, and concerns itself with determining these components recurrently. The zeroth component u_0 is usually identified by the terms arising from integrating inhomogeneous terms and from initial/boundary conditions. The successive components $u_1, u_2, \cdots$ are determined in a recursive manner. It was found that few components can give an insight into the character and behavior of the solution. For numerical purposes, accuracy can be easily enhanced by determining as many components as we like [1,6].

Throughout this section, the decomposition method will be applied to two dimensional and three dimensional wave equations.

6.2.1 Two Dimensional Wave Equation

The propagation of waves in a two dimensional vibrating membrane of length a and width b is governed by the following initial-boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} & = c^2(u_{xx}+u_{yy}), \quad 0<x<a,\, 0<y<b, t>0, \\ \text{BC} & u(0,y,t) & = u(a,y,t)=0, \\ & u(x,0,t) & = u(x,b,t)=0, \\ \text{IC} & u(x,y,0) & = f(x,y), \quad u_t(x,y,0)=g(x,y). \end{array} \tag{6.1}$$

where $u=u(x,y,t)$ is the displacement function of any point located at the position (x,y) of a vibrating membrane at any time t, and c is related to the elasticity of the material of the rectangular plate.

As discussed before, the solution in the t direction, in the x space, or in the y space will lead to identical results. However, the solution in the t direction reduces the size of calculations compared with the other space solutions because it uses the initial conditions only. For this reason the solution in the t direction will be followed in this chapter.

We first rewrite (6.1) in an operator form by

$$L_t u(x,y,t) = c^2\left(L_x u(x,y,t) + L_y u(x,y,t)\right), \tag{6.2}$$

where the differential operators L_t, L_x, and L_y are defined by

$$L_t = \frac{\partial^2}{\partial t^2}, \quad L_x = \frac{\partial^2}{\partial x^2}, \quad L_y = \frac{\partial^2}{\partial y^2}, \tag{6.3}$$

so that the integral operator L_t^{-1} exists and given by

$$L_t^{-1}(\cdot) = \int_0^t \int_0^t (\cdot)\,dt\,dt. \tag{6.4}$$

This means that

$$L_t^{-1} L_t\, u(x,y,t) = u(x,y,t) - u(x,y,0) - t u_t(x,y,0). \tag{6.5}$$

Applying L_t^{-1} to both sides of (6.2) and using the initial conditions leads to

$$u(x,y,t) = f(x,y) + t g(x,y) + c^2 L_t^{-1}\left(L_x u + L_y u\right). \tag{6.6}$$

The decomposition method defines the solution $u(x,y,t)$ as an infinite series given by

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t), \tag{6.7}$$

where the components $u_n(x,y,t), n \geqslant 0$ will be easily computed by using a recursive relation. Substituting (6.7) into both sides of (6.6) yields

$$\sum_{n=0}^{\infty} u_n = f(x,y) + tg(x,y) + c^2 L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{6.8}$$

To construct the recursive scheme, the decomposition method suggests that the zeroth component $u_0(x,y,t)$ is identified as the terms arising from the initial/boundary conditions and from integrating inhomogeneous terms if exist. The components $u_n(x,y,t)$, $n \geqslant 0$ can be completely determined by using the recursive relation

$$\begin{aligned} u_0(x,y,t) &= f(x,y) + tg(x,y), \\ u_{k+1}(x,y,t) &= c^2 L_t^{-1}\left(L_x u_k + L_y u_k\right), \quad k \geqslant 0. \end{aligned} \tag{6.9}$$

Consequently, the successive components can be completely computed, hence the solution in a series form follows immediately.

To give a clear overview of the implementation of the decomposition method, we have chosen several examples, homogeneous and inhomogeneous, to illustrate the discussion given above.

Homogeneous Wave Equations

The Adomian decomposition method will be used to solve the following homogeneous wave equations in two dimensional vibrating membrane with homogeneous boundary conditions.

Example 1. Use the Adomian decomposition method to solve the initial-boundary value problem.

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= 2(u_{xx} + u_{yy}), \quad 0 < x,y < \pi, t > 0, \\ \text{BC} & u(0,y,t) &= u(\pi,y,t) = 0, \\ & u(x,0,t) &= u(x,\pi,t) = 0, \\ \text{IC} & u(x,y,0) &= \sin x \sin y, \quad u_t(x,y,0) = 0. \end{array} \tag{6.10}$$

Solution.

In an operator form, Equation (6.10) becomes

$$L_t u(x,y,t) = 2(L_x u(x,y,t) + L_y u(x,y,t)). \tag{6.11}$$

Applying the inverse operator L_t^{-1} to (6.11)gives

$$u(x,y,t) = \sin x \sin y + 2L_t^{-1}(L_x u + L_y u). \tag{6.12}$$

The decomposition method decomposes the solution $u(x,y,t)$ by the decomposition series

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t). \tag{6.13}$$

Substituting (6.13) into both sides of (6.12) yields

$$\sum_{n=0}^{\infty} u_n = \sin x \sin y + 2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{6.14}$$

The zeroth component $u_0(x,y,t)$ is usually identified by all terms that are not included under the inverse operator L_t^{-1} in (6.14). Consequently, we set the recursive relation

$$\begin{aligned} u_0(x,y,t) &= \sin x \sin y, \\ u_{k+1}(x,y,t) &= 2L_t^{-1}(L_x u_k + L_y u_k), \quad k \geqslant 0. \end{aligned} \tag{6.15}$$

This in turn gives

$$\begin{aligned} u_0(x,y,t) &= \sin x \sin y, \\ u_1(x,y,t) &= 2L_t^{-1}(L_x u_0 + L_y u_0) = -\frac{(2t)^2}{2!}\sin x \sin y, \\ u_2(x,y,t) &= 2L_t^{-1}(L_x u_1 + L_y u_1) = \frac{(2t)^4}{4!}\sin x \sin y, \end{aligned} \tag{6.16}$$

and so on. The solution in a series form is given by

$$u(x,y,t) = \sin x \sin y\left(1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} - \cdots\right), \tag{6.17}$$

and in a closed form by

$$u(x,y,t) = \sin x \sin y \cos(2t), \tag{6.18}$$

obtained upon using the Taylor expansion of $\cos(2t)$.

Example 2. Use the Adomian decomposition method to solve the initial-boundary value problem.

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= 8(u_{xx} + u_{yy}), \quad 0 < x,y < \pi,\ t > 0, \\ \text{BC} & u(0,y,t) &= u(\pi,y,t) = 0, \\ & u(x,0,t) &= u(x,\pi,t) = 0, \\ \text{IC} & u(x,y,0) &= 0, \quad u_t(x,y,0) = 4\sin x \sin y. \end{array} \tag{6.19}$$

Solution.

Proceeding as in Example 1 we find

$$u(x,y,t) = 4t\sin x\sin y + 8L_t^{-1}\left(L_x u + L_y u\right). \tag{6.20}$$

We next define $u(x,y,t)$ by an infinite series

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t), \tag{6.21}$$

that carries (6.20) into

$$\sum_{n=0}^{\infty} u_n = 4t\sin x\sin y + 8L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{6.22}$$

Following Adomian's assumptions we find

$$\begin{aligned} u_0(x,y,t) &= 4t\sin x\sin y, \\ u_{k+1}(x,y,t) &= 8L_t^{-1}\left(L_x u_k + L_y u_k\right), \quad k \geqslant 0. \end{aligned} \tag{6.23}$$

It follows that

$$\begin{aligned} u_0(x,y,t) &= 4t\sin x\sin y, \\ u_1(x,y,t) &= 8L_t^{-1}\left(L_x u_0 + L_y u_0\right) = -\frac{(4t)^3}{3!}\sin x\sin y, \\ u_2(x,y,t) &= 8L_t^{-1}\left(L_x u_1 + L_y u_1\right) = \frac{(4t)^5}{5!}\sin x\sin y, \end{aligned} \tag{6.24}$$

and so on. Combining (6.24) and (6.21), the solution in a series form is given by

$$u(x,y,t) = \sin x\sin y\left(4t - \frac{(4t)^3}{3!} + \frac{(4t)^5}{5!} - \cdots\right), \tag{6.25}$$

and the exact solution

$$u(x,y,t) = \sin x\sin y\sin(4t), \tag{6.26}$$

follows immediately.

Example 3. Use the Adomian decomposition method to solve the initial-boundary value problem.

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= \frac{1}{2}(u_{xx} + u_{yy}), \quad 0 < x,y < \pi,\ t > 0, \\ \text{BC} & u(0,y,t) &= u(\pi,y,t) = 1, \\ & u(x,0,t) &= 1 + \sin x\sin t, \quad u(x,\pi,t) = 1 - \sin x\sin t, \\ \text{IC} & u(x,y,0) &= 1, \quad u_t(x,y,0) = \sin x\cos y. \end{array} \tag{6.27}$$

Solution.

We note that the wave equation is homogeneous and the boundary conditions are inhomogeneous. The decomposition method will be applied in a direct way as used before.

Applying the inverse operator L_t^{-1} to the operator form of (6.27) leads to

$$u(x,y,t) = 1 + t\sin x\cos y + \frac{1}{2}L_t^{-1}\left(L_x u + L_y u\right), \tag{6.28}$$

where by using the decomposition series for $u(x,y,t)$ we obtain

$$\sum_{n=0}^{\infty} u_n = 1 + t\sin x\cos y + \frac{1}{2}L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{6.29}$$

The components of $u(x,y,t)$ can be easily determined in a recursive manner by

$$\begin{aligned} u_0(x,y,t) &= 1 + t\sin x\cos y,\\ u_{k+1}(x,y,t) &= \frac{1}{2}L_t^{-1}\left(L_x u_k + L_y u_k\right), \quad k \geqslant 0. \end{aligned} \tag{6.30}$$

The first few components of the solution $u(x,y,t)$ are given by

$$\begin{aligned} u_0(x,y,t) &= 1 + t\sin x\cos y,\\ u_1(x,y,t) &= \frac{1}{2}L_t^{-1}\left(L_x u_0 + L_y u_0\right) = -\frac{1}{3!}t^3\sin x\cos y,\\ u_2(x,y,t) &= \frac{1}{2}L_t^{-1}\left(L_x u_1 + L_y u_1\right) = \frac{1}{5!}t^5\sin x\cos y, \end{aligned} \tag{6.31}$$

and so on. The solution in a series form is given by

$$u(x,y,t) = 1 + \sin x\cos y\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots\right), \tag{6.32}$$

and in a closed form by

$$u(x,y,t) = 1 + \sin x\cos y\sin t. \tag{6.33}$$

Example 4. Use the Adomian decomposition method to solve the initial-boundary value problem.

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= u_{xx} + u_{yy} - 2u, \quad 0 < x,y < \pi,\ t > 0,\\ \text{BC} & u(0,y,t) &= -u(\pi,y,t) = \cos y\sin(2t),\\ & u(x,0,t) &= -u(x,\pi,t) = \cos x\sin(2t),\\ \text{IC} & u(x,y,0) &= 0, \quad u_t(x,y,0) = 2\cos x\cos y. \end{array} \tag{6.34}$$

Solution.

We note that an additional term $-2u$ is included in the standard wave equation. This arises when each element of the membrane is subjected to an additional force which is proportional [5,6] to its displacement $u(x,y,t)$.

Applying the inverse operator L_t^{-1} to the operator form of (6.34) and using the initial conditions we obtain

$$u(x,y,t) = 2t\cos x\cos y + L_t^{-1}\left(L_x u + L_y u - 2u\right). \tag{6.35}$$

Using the decomposition series of $u(x,y,t)$ into both sides of (6.35) gives

$$\begin{aligned}\sum_{n=0}^{\infty} u_n &= 2t\cos x\cos y \\ &+L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) - 2\left(\sum_{n=0}^{\infty} u_n\right)\right).\end{aligned} \tag{6.36}$$

The components of $u(x,y,t)$ can be recursively determined by

$$\begin{aligned} u_0(x,y,t) &= 2t\cos x\cos y, \\ u_{k+1}(x,y,t) &= L_t^{-1}\left(L_x u_k + L_y u_k - 2u_k\right), \quad k \geqslant 0, \end{aligned} \tag{6.37}$$

so that

$$\begin{aligned} u_0(x,y,t) &= 2t\cos x\cos y, \\ u_1(x,y,t) &= -\frac{1}{3!}(2t)^3\cos x\cos y, \\ u_2(x,y,t) &= \frac{1}{5!}(2t)^5\cos x\cos y. \end{aligned} \tag{6.38}$$

In view of (6.38), the solution in a series form is given by

$$u(x,y,t) = \cos x\cos y\left(2t - \frac{1}{3!}(2t)^3 + \frac{1}{5!}(2t)^5 - \cdots\right), \tag{6.39}$$

and in a closed form by

$$u(x,y,t) = \cos x\cos y\sin(2t). \tag{6.40}$$

Example 5. Use the Adomian decomposition method to solve the initial-boundary value problem.

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= 2(u_{xx}+u_{yy}), \quad 0<x,y<\pi,\ t>0, \\ \text{BC} & u(0,y,t) &= y, \quad u(\pi,y,t) = \pi + y, \\ & u(x,0,t) &= x, \quad u(x,\pi,t) = \pi + x, \\ \text{IC} & u(x,y,0) &= x+y+\sin x\sin y, \quad u_t(x,y,0) = 0. \end{array} \tag{6.41}$$

Solution.

Note that the boundary conditions are inhomogeneous and given by functions and not constants. The decomposition method will attack the problem directly without any need to convert the inhomogeneous conditions to homogeneous conditions.

Operating with L_t^{-1} on (6.41) gives

$$u(x,y,t) = x + y + \sin x \sin y + 2L_t^{-1}\left(L_x u + L_y u\right). \tag{6.42}$$

It then follows

$$\sum_{n=0}^{\infty} u_n = x + y + \sin x \sin y + 2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{6.43}$$

Consequently, we set the relation

$$\begin{aligned} u_0(x,y,t) &= x + y + \sin x \sin y, \\ u_{k+1}(x,y,t) &= 2L_t^{-1}\left(L_x u_k + L_y u_k\right), \quad k \geqslant 0. \end{aligned} \tag{6.44}$$

This gives the first few components of the solution $u(x,y,t)$ by

$$\begin{aligned} u_0(x,y,t) &= x + y + \sin x \sin y, \\ u_1(x,y,t) &= 2L_t^{-1}\left(L_x u_0 + L_y u_0\right) = -\frac{1}{2!}(2t)^2 \sin x \sin y, \\ u_2(x,y,t) &= 2L_t^{-1}\left(L_x u_1 + L_y u_1\right) = \frac{1}{4!}(2t)^4 \sin x \sin y. \end{aligned} \tag{6.45}$$

The series solution is given by

$$u(x,y,t) = x + y + \sin x \sin y\left(1 - \frac{1}{2!}(2t)^2 + \frac{1}{4!}(2t)^4 - \cdots\right), \tag{6.46}$$

and in a closed form by

$$u(x,y,t) = x + y + \sin x \sin y \cos(2t). \tag{6.47}$$

Example 6. Use the Adomian decomposition method to solve the initial-boundary value problem.

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= \dfrac{x^2}{4}u_{xx} + \dfrac{y^2}{4}u_{yy}, \quad 0 < x,y < 1,\, t > 0, \\ \text{BC} & u(0,y,t) &= 0, \quad u(1,y,t) = y^2 \cosh t, \\ & u(x,0,t) &= 0, \quad u(x,1,t) = x^2 \cosh t, \\ \text{IC} & u(x,y,0) &= x^2 y^2, \quad u_t(x,y,0) = 0. \end{array} \tag{6.48}$$

Solution.

It is important to note that the coefficients of u_{xx} and u_{yy} are functions and not constants. Applying the inverse operator L_t^{-1} to the operator form of the PDE of (6.48) yields

$$u(x,y,t) = x^2y^2 + L_t^{-1}\left(\frac{x^2}{4}L_x u + \frac{y^2}{4}L_y u\right). \tag{6.49}$$

Using the decomposition series of $u(x,y,t)$ into both sides of (6.49) gives

$$\sum_{n=0}^{\infty} u_n = x^2y^2 + L_t^{-1}\left(\frac{x^2}{4}L_x\left(\sum_{n=0}^{\infty} u_n\right) + \frac{y^2}{4}L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{6.50}$$

The recursive relation

$$\begin{aligned} u_0(x,y,t) &= x^2y^2, \\ u_{k+1}(x,y,t) &= L_t^{-1}\left(\frac{x^2}{4}L_x u_k + \frac{y^2}{4}L_y u_k\right), \quad k \geqslant 0, \end{aligned} \tag{6.51}$$

follows immediately. It then follows that

$$\begin{aligned} u_0(x,y,t) &= x^2y^2, \\ u_1(x,y,t) &= L_t^{-1}\left(\frac{x^2}{4}L_x u_0 + \frac{y^2}{4}L_y u_0\right) = \frac{1}{2!}t^2x^2y^2, \\ u_2(x,y,t) &= L_t^{-1}\left(\frac{x^2}{4}L_x u_1 + \frac{y^2}{4}L_y u_1\right) = \frac{1}{4!}t^4x^2y^2, \end{aligned} \tag{6.52}$$

Consequently, the series solution

$$u(x,y,t) = x^2y^2\left(1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \cdots\right), \tag{6.53}$$

and the exact solution

$$u(x,y,t) = x^2y^2 \cosh t \tag{6.54}$$

are readily obtained.

Inhomogeneous Wave Equations

We now consider the inhomogeneous wave equation of the form

$$u_{tt} = c^2(u_{xx} + u_{yy}) + h(x,y,t), \tag{6.55}$$

where $h(x,y,t)$ is the inhomogeneous term. One significant advantage [1,6] of Adomian decomposition method is that it handles the inhomogeneous partial differential equations in an identical manner to that used before in handling homogeneous differential equations. The zeroth component u_0 is identified by all terms that arise from using initial conditions and from integrating inhomogeneous terms as well. In the following, the decomposition method will be illustrated by discussing the inhomogeneous equations.

Example 7. Use the Adomian decomposition method to solve the initial-boundary value problem.

$$
\begin{array}{lrl}
\text{PDE} & u_{tt} &= \frac{1}{2}(u_{xx}+u_{yy})-2, \quad 0<x,y<\pi, t>0,\\
\text{BC} & u(0,y,t) &= y^2, \quad u(\pi,y,t)=\pi^2+y^2,\\
 & u(x,0,t) &= x^2, \quad u(x,\pi,t)=\pi^2+x^2,\\
\text{IC} & u(x,y,0) &= x^2+y^2, \quad u_t(x,y,0)=\sin x\sin y.
\end{array}
\tag{6.56}
$$

Solution.

Note that the PDE (6.56) contains the term -2; hence it is inhomogeneous equation. In addition, the boundary conditions are inhomogeneous.

Applying the inverse operator L_t^{-1} to the operator form of (6.56) gives

$$
u(x,y,t)=x^2+y^2+t\sin x\sin y-t^2+\frac{1}{2}L_t^{-1}\left(L_x u+L_y u\right), \tag{6.57}
$$

obtained by using the initial conditions and by integrating the constant term. Using the series representation of $u(x,y,t)$ in both sides of (6.57) we obtain

$$
\sum_{n=0}^{\infty}u_n=x^2+y^2+t\sin x\sin y-t^2+\frac{1}{2}L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n\right)+L_y\left(\sum_{n=0}^{\infty}u_n\right)\right). \tag{6.58}
$$

To determine the components of u, we use the recursive relation

$$
\begin{aligned}
u_0(x,y,t) &= x^2+y^2+t\sin x\sin y-t^2,\\
u_{k+1}(x,y,t) &= \frac{1}{2}L_t^{-1}\left(L_x u_k+L_y u_k\right), \quad k\geqslant 0,
\end{aligned}
\tag{6.59}
$$

that gives

$$
\begin{aligned}
u_0(x,y,t) &= x^2+y^2+t\sin x\sin y-t^2,\\
u_1(x,y,t) &= \frac{1}{2}L_t^{-1}\left(L_x u_0+L_y u_0\right)=t^2-\frac{1}{3!}t^3\sin x\sin y,\\
u_2(x,y,t) &= \frac{1}{2}L_t^{-1}\left(L_x u_1+L_y u_1\right)=\frac{1}{5!}t^5\sin x\sin y.
\end{aligned}
\tag{6.60}
$$

It follows that the solution in a series form is

$$
u(x,y,t)=x^2+y^2+\sin x\sin y\left(t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\cdots\right), \tag{6.61}
$$

and in a closed form is

$$u(x,y,t) = x^2 + y^2 + \sin x \sin y \sin t. \tag{6.62}$$

Unlike Example 7 where the inhomogeneous term is a constant, the inhomogeneous in the following example is a function of y.

Example 8. Use the Adomian decomposition method to solve the initial-boundary value problem.

$$\begin{array}{lrcl} \text{PDE} & u_{tt} &=& (u_{xx}+u_{yy}) + \cos y, \quad 0 < x,y < \pi,\ t > 0, \\ \text{BC} & u(0,y,t) &=& u(\pi,y,t) = \cos y, \\ & u(x,0,t) &=& 1 + \sin x \sin t, \quad u(x,\pi,t) = -1 + \sin x \sin t, \\ \text{IC} & u(x,y,0) &=& \cos y, \quad u_t(x,y,0) = \sin x. \end{array} \tag{6.63}$$

Solution.

Proceeding as before we obtain

$$u(x,y,t) = \cos y + t \sin x + \frac{1}{2!}t^2 \cos y + L_t^{-1}\left(L_x u + L_y u\right). \tag{6.64}$$

It then follows that

$$\sum_{n=0}^{\infty} u_n = \cos y + t \sin x + \frac{1}{2!}t^2 \cos y + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{6.65}$$

Identifying the zeroth component $u_0(x,y,t)$ by all terms that arise from initial conditions and from applying L_t^{-1} to the inhomogeneous term $\cos y$, the recursive relationship is therefore defined by

$$\begin{aligned} u_0(x,y,t) &= \cos y + t \sin x + \frac{1}{2!}t^2 \cos y, \\ u_{k+1}(x,y,t) &= L_t^{-1}\left(L_x u_k + L_y u_k\right), \quad k \geqslant 0, \end{aligned} \tag{6.66}$$

hence we find

$$\begin{aligned} u_0(x,y,t) &= \cos y + t \sin x + \frac{1}{2!}t^2 \cos y, \\ u_1(x,y,t) &= L_t^{-1}\left(L_x u_0 + L_y u_0\right) = -\frac{1}{2!}t^2 \cos y - \frac{1}{3!}t^3 \sin x - \frac{1}{4!}t^4 \cos y, \\ u_2(x,y,t) &= L_t^{-1}\left(L_x u_1 + L_y u_1\right) = \frac{1}{4!}t^4 \cos y + \frac{1}{5!}t^5 \sin x + \frac{1}{6!}t^6 \cos y. \end{aligned} \tag{6.67}$$

This gives the solution in a series form by

$$u(x,y,t) = \cos y + \sin x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots\right) \tag{6.68}$$

and in a closed form by

$$u(x,y,t) = \cos y + \sin x \sin t. \tag{6.69}$$

Example 9. Use the Adomian decomposition method to solve the initial-boundary value problem.

$$\begin{array}{ll} \text{PDE} & u_{tt} = 2(u_{xx}+u_{yy}) + 6t + 2x + 4y, \quad 0 < x,y < \pi,\, t > 0, \\ \text{BC} & u(0,y,t) = t^3 + 2t^2 y, \quad u(\pi,y,t) = t^3 + \pi t^2 + 2t^2 y, \\ & u(x,0,t) = t^3 + t^2 x, \quad u(x,\pi,t) = t^3 + t^2 x + 2\pi t^2, \\ \text{IC} & u(x,y,0) = 0, \quad u_t(x,y,0) = 2\sin x \sin y. \end{array} \tag{6.70}$$

Solution.

We follow the discussion introduced before to obtain

$$u(x,y,t) = 2t\sin x\sin y + t^3 + t^2 x + 2t^2 y + 2L_t^{-1}\left(L_x u + L_y u\right), \tag{6.71}$$

where we find

$$\sum_{n=0}^{\infty} u_n = 2t\sin x\sin y + t^3 + t^2 x + 2t^2 y + 2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{6.72}$$

We next set the recursive relation

$$\begin{aligned} u_0(x,y,t) &= 2t\sin x\sin y + t^3 + t^2 x + 2t^2 y, \\ u_{k+1}(x,y,t) &= 2L_t^{-1}\left(L_x u_k + L_y u_k\right)), \quad k \geqslant 0, \end{aligned} \tag{6.73}$$

that gives the first few components

$$\begin{aligned} u_0(x,y,t) &= 2t\sin x\sin y + t^3 + t^2 x + 2t^2 y, \\ u_1(x,y,t) &= -\frac{(2t)^3}{3!}\sin x\sin y, \\ u_2(x,y,t) &= \frac{(2t)^5}{5!}\sin x\sin y, \\ u_3(x,y,t) &= -\frac{(2t)^7}{7!}\sin x\sin y, \end{aligned} \tag{6.74}$$

In view of (6.74), the solution in a series form is

$$u(x,y,t) = t^3 + t^2 x + 2t^2 y + \sin x\sin y\left(2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \frac{(2t)^7}{7!} + \cdots\right), \tag{6.75}$$

and in a closed form is

$$u(x,y,t) = t^3 + t^2 x + 2t^2 y + \sin x\sin y\sin(2t). \tag{6.76}$$

Exercises 6.2.1

In Exercises 1–8, use the decomposition method to solve the homogeneous initial-boundary value problems:

1. $u_{tt} = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = 0,\ u_t(x,y,0) = 2\sin x \sin y$

2. $u_{tt} = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = 0, u_t(x,y,0) = 4\sin(2x)\sin(2y)$

3. $u_{tt} = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = \sin(2x)\sin(2y), u_t(x,y,0) = 0$

4. $u_{tt} = \frac{1}{2}(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 2$
   $u(x,0,t) = u(x,\pi,t) = 2$
   $u(x,y,0) = 2, u_t(x,y,0) = \sin x \sin y$

5. $u_{tt} = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 1 + y$
   $u(x,0,t) = 1, u(x,\pi,t) = 1 + \pi$
   $u(x,y,0) = 1 + y, u_t(x,y,0) = 2\sin x \sin y$

6. $u_{tt} = u_{xx} + u_{yy},\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = 1 + \sin y \sin t, u(\pi,y,t) = 1 + \pi + \sin y \sin t$
   $u(x,0,t) = u(x,\pi,t) = 1 + x$
   $u(x,y,0) = 1 + x, u_t(x,y,0) = \sin y$

7. $u_{tt} = u_{xx} + u_{yy} - 2u,\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = 0, u_t(x,y,0) = 2\sin x \sin y$

8. $u_{tt} = u_{xx} + u_{yy} - 7u,\ 0 < x,y < \pi,\ t > 0$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = \sin x \sin y, u_t(x,y,0) = 0$

In Exercises 9–14, use the decomposition method to solve the inhomogeneous initial-boundary value problems:

9. $u_{tt} = \frac{1}{2}(u_{xx} + u_{yy}) + \frac{1}{2}\sin x,\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = u(\pi,y,t) = 0$
$u(x,0,t) = u(x,\pi,t) = \sin x$
$u(x,y,0) = \sin x, u_t(x,y,0) = \sin x \sin y$

10. $u_{tt} = u_{xx} + u_{yy} + \cos x,\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = 1 + \sin y \sin t, u(\pi,y,t) = -1 + \sin y \sin t$
$u(x,0,t) = u(x,\pi,t) = \cos x$
$u(x,y,0) = \cos x, u_t(x,y,0) = \sin y$

11. $u_{tt} = u_{xx} + u_{yy} - 4,\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = y^2, u(\pi,y,t) = \pi^2 + y^2$
$u(x,0,t) = x^2 + \sin x \sin t, u(x,\pi,t) = \pi^2 + x^2 + \sin x \sin t$
$u(x,y,0) = x^2 + y^2, u_t(x,y,0) = \sin x$

12. $u_{tt} = u_{xx} + u_{yy} - 8,\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = 2y^2, u(\pi,y,t) = 2\pi^2 + 2y^2$
$u(x,0,t) = 2x^2, u(x,\pi,t) = 2\pi^2 + 2x^2$
$u(x,y,0) = 2x^2 + 2y^2 + 2\sin x \sin y, u_t(x,y,0) = 0$

13. $u_{tt} = \frac{1}{2}(u_{xx} + u_{yy}) + 2,\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = t^2 + ty, u(\pi,y,t) = t^2 + \pi t + ty$
$u(x,0,t) = t^2 + tx, u(x,\pi,t) = t^2 + \pi t + tx$
$u(x,y,0) = 0, u_t(x,y,0) = x + y + \sin x \sin y$

14. $u_{tt} = 2(u_{xx} + u_{yy}) + 6t + 2x,\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = t^3 + ty, u(\pi,y,t) = t^3 + \pi t^2 + ty$
$u(x,0,t) = t^3 + t^2 x, u(x,\pi,t) = t^3 + t^2 x + \pi t$
$u(x,y,0) = \sin x \sin y, u_t(x,y,0) = y$

In Exercises 15–20, use the decomposition method to solve the initial-boundary value problems:

15. $u_{tt} = u_{xx} + u_{yy} - 2,\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = u(\pi,y,t) = y^2$
$u(x,0,t) = \sin x \cos t, u(x,\pi,t) = \pi^2 + \sin x \cos t$
$u(x,y,0) = y^2 + \sin x, u_t(x,y,0) = 0$

16. $u_{tt} = u_{xx} + u_{yy} - 2,\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = \sin y \sin t, u(\pi,y,t) = \pi^2 + \sin y \sin t$
$u(x,0,t) = u(x,\pi,t) = x^2$
$u(x,y,0) = x^2, u_t(x,y,0) = \sin y$

17. $u_{tt} = u_{xx} + u_{yy} + \sin x,\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = u(\pi,y,t) = \sin y \sin t$
$u(x,0,t) = u(x,\pi,t) = \sin x$
$u(x,y,0) = \sin x, u_t(x,y,0) = \sin y$

18. $u_{tt} = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = -u(\pi,y,t) = \cos y \sin(2t)$
$u(x,0,t) = -u(x,\pi,t) = \cos x \sin(2t)$
$u(x,y,0) = 0, u_t(x,y,0) = 2\cos x \cos y$

19. $u_{tt} = \frac{1}{2}(u_{xx} + u_{yy}) + 12t^2 + 2y,\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = u(\pi,y,t) = t^4 + t^2 y$
$u(x,0,t) = t^4, u(x,\pi,t) = t^4 + \pi t^2$
$u(x,y,0) = 0, u_t(x,y,0) = \sin x \sin y$

20. $u_{tt} = \frac{1}{2}(u_{xx} + u_{yy}),\ 0 < x,y < \pi,\ t > 0$
$u(0,y,t) = t^2 + y^2, u(\pi,y,t) = \pi^2 + t^2 + y^2$
$u(x,0,t) = t^2 + x^2, u(x,\pi,t) = \pi^2 + t^2 + x^2$
$u(x,y,0) = x^2 + y^2, u_t(x,y,0) = \sin x \sin y$

In Exercises 21–24, solve the partial differential equations where coefficients of u_{xx} and u_{yy} are functions and constants:

21. $u_{tt} = \frac{x^2}{4} u_{xx} + \frac{y^2}{4} u_{yy},\ 0 < x,y < 1,\ t > 0$
$u(0,y,t) = 0, u(1,y,t) = y^2 \sinh t$
$u(x,0,t) = 0, u(x,1,t) = x^2 \sinh t$
$u(x,y,0) = 0, u_t(x,y,0) = x^2 y^2$

22. $u_{tt} = \frac{x^2}{4} u_{xx} + \frac{y^2}{4} u_{yy}, 0 < x,y < 1,\ t > 0$
$u(0,y,t) = 0, u(1,y,t) = y^2 e^t$
$u(x,0,t) = 0, u(x,1,t) = x^2 e^t$
$u(x,y,0) = x^2 y^2, u_t(x,y,0) = x^2 y^2$

23. $u_{tt} = \frac{x^2}{2} u_{xx} + \frac{y^2}{2} u_{yy}, 0 < x,y < 1,\ t > 0$
$u(0,y,t) = y^2 \cosh t, u(1,y,t) = \sinh t + y^2 \cosh t$
$u(x,0,t) = x^2 \sinh t, u(x,1,t) = x^2 \sinh t + \cosh t$
$u(x,y,0) = y^2, u_t(x,y,0) = x^2$

24. $u_{tt} = \frac{x^2}{2} u_{xx} + \frac{y^2}{2} u_{yy}, 0 < x,y < 1,\ t > 0$
$u(0,y,t) = y^2 e^t, u(1,y,t) = e^{-t} + y^2 e^t$
$u(x,0,t) = x^2 e^{-t}, u(x,1,t) = e^t + x^2 e^{-t}$
$u(x,y,0) = x^2 + y^2, u_t(x,y,0) = y^2 - x^2$

6.2.2 Three Dimensional Wave Equation

The propagation of waves in a three dimensional volume of length a, width b, and height d is governed by the following initial boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= c^2(u_{xx}+u_{yy}+u_{zz}), \quad t>0, \\ \text{BC} & u(0,y,z,t) &= u(a,y,z,t)=0, \\ & u(x,0,z,t) &= u(x,b,z,t)=0, \\ & u(x,y,0,t) &= u(x,y,d,t)=0, \\ \text{IC} & u(x,y,z,0) &= f(x,y,z), \quad u_t(x,y,z,0)=g(x,y,z). \end{array} \tag{6.77}$$

where $0<x<a, 0<y<b, 0<z<d$, and $u=u(x,y,z,t)$ is the displacement of any point located at the position (x,y,z) of a rectangular volume at any time t, and c is the velocity of a propagation wave.

As discussed before, the solution in the t space minimizes the volume of calculations. Accordingly, the operator L_t^{-1} will be applied here. We first rewrite (6.77) in an operator form by

$$L_t u = c^2(L_x u + L_y u + L_z u), \tag{6.78}$$

where the differential operators L_x, L_y, and L_z are defined by

$$L_t = \frac{\partial^2}{\partial t^2}, \quad L_x = \frac{\partial^2}{\partial x^2}, \quad L_y = \frac{\partial^2}{\partial y^2}, \quad L_z = \frac{\partial^2}{\partial z^2}, \tag{6.79}$$

so that the integral operator L_t^{-1} represents a two-fold integration from 0 to t given by

$$L_t^{-1}(\cdot) = \int_0^t \int_0^t (\cdot)dt. \tag{6.80}$$

This means that

$$L_t^{-1}L_t u(x,y,z,t) = u(x,y,z,t) - u(x,y,z,0) - tu_t(x,y,z,0). \tag{6.81}$$

Applying L_t^{-1} to both sides of (6.78), noting (6.81) and using the initial conditions we find

$$u(x,y,z,t) = f(x,y,z) + tg(x,y,z) + c^2 L_t^{-1}\left(L_x u + L_y u + L_z u\right). \tag{6.82}$$

The decomposition method defines the solution $u(x,y,z,t)$ as a series given by

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t). \tag{6.83}$$

Substituting (6.83) into both sides of (6.82) yields

$$\sum_{n=0}^{\infty} u_n = f(x,y,z) + tg(x,y,z)$$

$$+c^2 L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right)+L_y\left(\sum_{n=0}^{\infty} u_n\right)+L_z\left(\sum_{n=0}^{\infty} u_n\right)\right). \quad (6.84)$$

The components $u_n(x,y,z,t)$, $n \geqslant 0$ can be completely determined by using the recursive relation

$$\begin{aligned} u_0(x,y,z,t) &= f(x,y,z)+tg(x,y,z), \\ u_{k+1}(x,y,z,t) &= c^2 L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k\right), \quad k \geqslant 0. \end{aligned} \quad (6.85)$$

Having determined the components $u_n, n \geqslant 0$ by applying the scheme (6.85), the solution in a series form follows immediately.

Homogeneous Wave Equations

The decomposition method will be used to discuss the following homogeneous wave equations in three dimensional space with homogeneous or inhomogeneous boundary conditions.

Example 10. Use the Adomian decomposition method to solve the initial boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= 3(u_{xx}+u_{yy}+u_{zz}), \quad 0<x,y,z<\pi,\ t>0 \\ \text{BC} & u(0,y,z,t) &= u(\pi,y,z,t)=0, \\ & u(x,0,z,t) &= u(x,\pi,z,t)=0, \\ & u(x,y,0,t) &= u(x,y,\pi,t)=0, \\ \text{IC} & u(x,y,z,0) &= 0, \quad u_t(x,y,z,0)=3\sin x\sin y\sin z. \end{array} \quad (6.86)$$

Solution.

The PDE of (6.86) can be rewritten by

$$L_t u = 3\left(L_x u + L_y u + L_z u\right). \quad (6.87)$$

Applying the inverse operator L_t^{-1} to (6.87), using (6.81) and substituting the initial conditions we obtain

$$u(x,y,z,t) = 3t\sin x\sin y\sin z + 3L_t^{-1}\left(L_x u + L_y u + L_z u\right). \quad (6.88)$$

Using the decomposition series

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t), \quad (6.89)$$

into both sides of (6.88) yields

$$\sum_{n=0}^{\infty} u_n = 3t\sin x\sin y\sin z + 3L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right). \quad (6.90)$$

Identifying the zeroth component as discussed before we then set the relation

$$\begin{aligned} u_0(x,y,z,t) &= 3t\sin x\sin y\sin z, \\ u_{k+1}(x,y,z,t) &= 3L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k\right), \quad k \geqslant 0. \end{aligned} \quad (6.91)$$

The first few components of the decomposition of u are given by

$$\begin{aligned} u_0(x,y,z,t) &= 3t\sin x\sin y\sin z, \\ u_1(x,y,z,t) &= 3L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0\right) = -\frac{(3t)^3}{3!}\sin x\sin y\sin z, \\ u_2(x,y,z,t) &= 3L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1\right) = \frac{(3t)^5}{5!}\sin x\sin y\sin z. \end{aligned} \quad (6.92)$$

For numerical purposes, further components can be computed to improve the accuracy level of the approximation.

Combining (6.89) and (6.92), the solution in a series form is given by

$$u(x,y,z,t) = \sin x\sin y\sin z\left(3t - \frac{(3t)^3}{3!} + \frac{(3t)^5}{5!} - \cdots\right), \quad (6.93)$$

and in a closed form by

$$u(x,y,z,t) = \sin x\sin y\sin z\sin(3t). \quad (6.94)$$

Example 11. Use the Adomian decomposition method to solve the initial boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= u_{xx} + u_{yy} + u_{zz} - u, \quad 0 < x,y,z < \pi,\ t > 0, \\ \text{BC} & u(0,y,z,t) &= u(\pi,y,z,t) = 0, \\ & u(x,0,z,t) &= u(x,\pi,z,t) = 0, \\ & u(x,y,0,t) &= u(x,y,\pi,t) = 0, \\ \text{IC} & u(x,y,z,0) &= 0, \quad u_t(x,y,z,0) = 2\sin x\sin y\sin z. \end{array} \quad (6.95)$$

Solution.

We note that an additional term $-u$ is contained in the standard wave equation. This term usually arises when each element of the rectangular volume is subjected to an additional force.

Applying L_t^{-1} to the operator form of the PDE of (6.95), using (6.81) and substituting the initial conditions we obtain

$$u = 2t\sin x\sin y\sin z + L_t^{-1}\left(L_x u + L_y u + L_z u - u\right). \tag{6.96}$$

Following the analysis made above we find

$$\begin{aligned}\sum_{n=0}^{\infty} u_n &= 2t\sin x\sin y\sin z\\ &+L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right) - \left(\sum_{n=0}^{\infty} u_n\right)\right).\end{aligned} \tag{6.97}$$

Therefore we set the relation

$$\begin{aligned}u_0(x,y,z,t) &= 2t\sin x\sin y\sin z,\\ u_{k+1}(x,y,z,t) &= L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k - u_k\right), \quad k \geqslant 0,\end{aligned} \tag{6.98}$$

that gives

$$\begin{aligned}u_0(x,y,z,t) &= 2t\sin x\sin y\sin z,\\ u_1(x,y,z,t) &= L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0 - u_0\right) = -\frac{(2t)^3}{3!}\sin x\sin y\sin z,\\ u_2(x,y,z,t) &= L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1 - u_1\right) = \frac{(2t)^5}{5!}\sin x\sin y\sin z.\end{aligned} \tag{6.99}$$

The series solution is given by

$$u(x,y,z,t) = \sin x\sin y\sin z\left(2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \cdots\right), \tag{6.100}$$

and the exact solution is

$$u(x,y,z,t) = \sin x\sin y\sin z\sin(2t). \tag{6.101}$$

Example 12. Use the Adomian decomposition method to solve the initial boundary value problem

$$\begin{array}{lrcl}\text{PDE} & u_{tt} &=& u_{xx} + u_{yy} + u_{zz} - u, \quad 0 < x,y,z < \pi,\ t > 0,\\ \text{BC} & u(0,y,z,t) &=& -u(\pi,y,z,t) = \sin y\sin(z+2t),\\ & u(x,0,z,t) &=& -u(x,\pi,z,t) = \sin x\sin(z+2t),\\ & u(x,y,0,t) &=& -u(x,y,\pi,t) = \sin(x+y)\sin(2t),\\ \text{IC} & u(x,y,z,0) &=& \sin(x+y)\sin z, \quad u_t(x,y,z,0) = 2\sin(x+y)\cos z.\end{array} \tag{6.102}$$

Solution.

It is interesting to note that the boundary conditions are inhomogeneous and given by functions and not constants.

Applying L_t^{-1} to (6.102), using (6.81) gives

$$\begin{aligned} u(x,y,z,t) &= \sin(x+y)\sin z + 2t\sin(x+y)\cos z \\ &\quad + L_t^{-1}\left(L_x u + L_y u + L_z u - u\right), \end{aligned} \tag{6.103}$$

therefore we find

$$\begin{aligned} \sum_{n=0}^{\infty} u_n &= \sin(x+y)\sin z + 2t\sin(x+y)\cos z \\ &\quad + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right) - \left(\sum_{n=0}^{\infty} u_n\right)\right). \end{aligned} \tag{6.104}$$

This means that

$$\begin{aligned} u_0(x,y,z,t) &= \sin(x+y)\sin z + 2t\sin(x+y)\cos z, \\ u_{k+1}(x,y,z,t) &= L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k - u_k\right), \quad k \geqslant 0, \end{aligned} \tag{6.105}$$

and therefore we obtain

$$\begin{aligned} u_0(x,y,z,t) &= \sin(x+y)\sin z + 2t\sin(x+y)\cos z, \\ u_1(x,y,z,t) &= L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0 - u_0\right) \\ &= -\frac{(2t)^2}{2!}\sin(x+y)\sin z - \frac{(2t)^3}{3!}\sin(x+y)\cos z, \\ u_2(x,y,z,t) &= L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1 - u_1\right), \\ &= \frac{(2t)^4}{4!}\sin(x+y)\sin z + \frac{(2t)^5}{5!}\sin(x+y)\cos z. \end{aligned} \tag{6.106}$$

In view of (6.106), the solution in a series form is given by

$$\begin{aligned} u(x,y,z,t) &= \sin(x+y)\sin z\left(1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} - \cdots\right) \\ &\quad + \sin(x+y)\cos z\left(2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \cdots\right), \end{aligned} \tag{6.107}$$

and consequently the exact solution is

$$u(x,y,z,t) = \sin(x+y)\left(\sin z\cos(2t) + \cos z\sin(2t)\right), \tag{6.108}$$

obtained upon using the Taylor series of $\cos(2t)$ and $\sin(2t)$ respectively. Using the trigonometric identities, the exact solution in (6.108) can be simplified to

$$u(x,y,z,t) = \sin(x+y)\sin(z+2t). \tag{6.109}$$

Example 13. Use the Adomian decomposition method to solve the initial boundary value problem

$$\begin{array}{lrll}
\text{PDE} & u_{tt} &= \frac{x^2}{18}u_{xx} + \frac{y^2}{18}u_{yy} + \frac{z^2}{18}u_{zz} - u, & 0 < x,y,z < 1,\, t > 0, \\
\text{BC} & u(0,y,z,t) &= 0, \quad u(1,y,z,t) = y^4z^4 \sinh t, & \\
& u(x,0,z,t) &= 0, \quad u(x,1,z,t) = x^4z^4 \sinh t, & \\
& u(x,y,0,t) &= 0, \quad u(x,y,1,t) = x^4y^4 \sinh t, & \\
\text{IC} & u(x,y,z,0) &= 0, \quad u_t(x,y,z,0) = x^4y^4z^4. &
\end{array} \tag{6.110}$$

Solution.

The coefficients of the derivatives of u are functions and not constants. Following the discussions presented above we find

$$u(x,y,z,t) = tx^4y^4z^4 + L_t^{-1}\left(\frac{x^2}{18}L_xu + \frac{y^2}{18}L_yu + \frac{z^2}{18}L_zu - u\right), \tag{6.111}$$

that leads to gives

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n &= x^4y^4z^4 \\
&+ L_t^{-1}\left(\frac{x^2}{18}L_x\left(\sum_{n=0}^{\infty} u_n\right) + \frac{y^2}{18}L_y\left(\sum_{n=0}^{\infty} u_n\right) + \frac{z^2}{18}L_z\left(\sum_{n=0}^{\infty} u_n\right) - \left(\sum_{n=0}^{\infty} u_n\right)\right).
\end{aligned} \tag{6.112}$$

The recursive relation

$$\begin{aligned}
u_0(x,y,z,t) &= tx^4y^4z^4, \\
u_{k+1}(x,y,z,t) &= L_t^{-1}\left(\frac{x^2}{18}L_xu_k + \frac{y^2}{18}L_yu_k + \frac{z^2}{18}L_zu_k - u_k\right), \quad k \geqslant 0.
\end{aligned} \tag{6.113}$$

gives the first few components by

$$\begin{aligned}
u_0(x,y,z,t) &= tx^4y^4z^4, \\
u_1(x,y,z,t) &= L_t^{-1}\left(\frac{x^2}{18}L_xu_0 + \frac{y^2}{18}L_yu_0 + \frac{z^2}{18}L_zu_0 - u_0\right) = \frac{t^3}{3!}x^4y^4z^4, \\
u_2(x,y,z,t) &= L_t^{-1}\left(\frac{x^2}{18}L_xu_1 + \frac{y^2}{18}L_yu_1 + \frac{z^2}{18}L_zu_1 - u_1\right) = \frac{t^5}{5!}x^4y^4z^4.
\end{aligned} \tag{6.114}$$

In view of (6.114), the solution in a series form is given by

$$u(x,y,z,t) = x^4y^4z^4\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right), \tag{6.115}$$

and in a closed form by

$$u(x,y,z,t) = x^4y^4z^4 \sinh t. \tag{6.116}$$

Inhomogeneous Wave Equations

We now consider the inhomogeneous wave equation in a three dimensional space of the form

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}) + h(x,y,z), \tag{6.117}$$

where $h(x,y,z)$ is an inhomogeneous term. The decomposition method can be applied without any need to transform this equation to a homogeneous equation. The following example will be used to explain the implementation of the method.

Example 14. Use the Adomian decomposition method to solve the initial boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= u_{xx} + u_{yy} + u_{zz} + \sin x + \sin y, \quad 0 < x,y,z < \pi, \\ \text{BC} & u(0,y,z,t) &= u(\pi,y,z,t) = \sin y + \sin z \sin t, \\ & u(x,0,z,t) &= u(x,\pi,z,t) = \sin x + \sin z \sin t, \\ & u(x,y,0,t) &= u(x,y,\pi,t) = \sin x + \sin y, \\ \text{IC} & u(x,y,z,0) &= \sin x + \sin y, \quad u_t(x,y,z,0) = \sin z. \end{array} \tag{6.118}$$

Solution.

Operating with L_t^{-1} on (6.118) gives

$$u = \sin x + \sin y + t\sin z + \frac{t^2}{2!}\sin x + \frac{t^2}{2!}\sin y + L_t^{-1}\left(L_x u + L_y u + L_z u\right). \tag{6.119}$$

Following the analysis made above, we substitute the decomposition series

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t), \tag{6.120}$$

into both sides of (6.119) to obtain

$$\begin{aligned} \sum_{n=0}^{\infty} u_n &= \sin x + \sin y + t\sin z + \frac{t^2}{2!}\sin x + \frac{t^2}{2!}\sin y \\ &\quad + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right). \end{aligned} \tag{6.121}$$

This leads to

$$\begin{aligned} u_0(x,y,z,t) &= \sin x + \sin y + t\sin z + \frac{t^2}{2!}\sin x + \frac{t^2}{2!}\sin y, \\ u_1(x,y,z,t) &= L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0\right) \\ &= -\frac{t^2}{2!}\sin x - \frac{t^2}{2!}\sin y - \frac{t^3}{3!}\sin z - \frac{t^4}{4!}\sin x - \frac{t^4}{4!}\sin y, \\ u_2(x,y,z,t) &= L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1\right), \end{aligned} \tag{6.122}$$

$$\begin{aligned}
&= \frac{t^4}{4!}\sin x + \frac{t^4}{4!}\sin y + \frac{t^5}{5!}\sin z + \frac{t^6}{6!}\sin x + \frac{t^6}{6!}\sin y, \\
u_3(x,y,z,t) &= L_t^{-1}\left(L_x u_2 + L_y u_2 + L_z u_2\right), \\
&= -\frac{t^6}{6!}\sin x - \frac{t^6}{6!}\sin y - \frac{t^7}{7!}\sin z - \frac{t^8}{8!}\sin x - \frac{t^8}{8!}\sin y.
\end{aligned}$$

The series solution and the exact solution are given by

$$u(x,y,z,t) = \sin x + \sin y + \sin z\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right), \tag{6.123}$$

and

$$u(x,y,z,t) = \sin x + \sin y + \sin z \sin t, \tag{6.124}$$

respectively.

Exercises 6.2.2

Use the decomposition method to solve the homogeneous problems in Exercises 1–8, and the inhomogeneous problems in Exercises 9–15:

1. $u_{tt} = 3(u_{xx}+u_{yy}+u_{zz}),\ 0<x,y,z<\pi,\ t>0$
$u(0,y,z,t) = u(\pi,y,z,t) = 0$
$u(x,0,z,t) = u(x,\pi,z,t) = 0$
$u(x,y,0,t) = u(x,y,\pi,t) = 0$
$u(x,y,z,0) = 0, u_t(x,y,z,0) = 6\sin(2x)\sin(2y)\sin(2z)$

2. $u_{tt} = \frac{1}{3}(u_{xx}+u_{yy}+u_{zz}),\ 0<x,y,z<\pi,\ t>0$
$u(0,y,z,t) = u(\pi,y,z,t) = 1$
$u(x,0,z,t) = u(x,\pi,z,t) = 1$
$u(x,y,0,t) = u(x,y,\pi,t) = 1$
$u(x,y,z,0) = 1, u_t(x,y,z,0) = \sin x\sin y\sin z$

3. $u_{tt} = 3(u_{xx}+u_{yy}+u_{zz}),\ 0<x,y,z<\pi,\ t>0$
$u(0,y,z,t) = u(\pi,y,z,t) = 3$
$u(x,0,z,t) = u(x,\pi,z,t) = 3$
$u(x,y,0,t) = u(x,y,\pi,t) = 3$
$u(x,y,z,0) = 3+\sin x\sin y\sin z, u_t(x,y,z,0) = 0$

4. $u_{tt} = u_{xx}+u_{yy}+u_{zz}-6u,\ 0<x,y,z<\pi,\ t>0$
$u(0,y,z,t) = u(\pi,y,z,t) = 0$
$u(x,0,z,t) = u(x,\pi,z,t) = 0$
$u(x,y,0,t) = u(x,y,\pi,t) = 0$
$u(x,y,z,0) = 0, u_t(x,y,z,0) = 3\sin x\sin y\sin z$

5. $u_{tt} = \frac{1}{2}(u_{xx} + u_{yy} + u_{zz}) - u,\ 0 < x,y,z < \pi,\ t > 0$
$u(0,y,z,t) = -u(\pi,y,z,t) = \sin(2y)\sin(z+2t)$
$u(x,0,z,t) = u(x,\pi,z,t) = \sin x \sin(z+2t)$
$u(x,y,0,t) = -u(x,y,\pi,t) = \sin(x+2y)\sin(2t)$
$u(x,y,z,0) = \sin(x+2y)\sin z, u_t(x,y,z,0) = 2\sin(x+2y)\cos z$

6. $u_{tt} = u_{xx} + u_{yy} + u_{zz} - 2u,\ 0 < x,y,z < \pi,\ t > 0$
$u(0,y,z,t) = u(\pi,y,z,t) = 0$
$u(x,0,z,t) = u(x,\pi,z,t) = 0$
$u(x,y,0,t) = u(x,y,\pi,t) = 0$
$u(x,y,z,0) = \sin x \sin(2y)\sin(3z), u_t(x,y,z,0) = 0$

7. $u_{tt} = \frac{1}{3}(u_{xx} + u_{yy} + u_{zz}),\ 0 < x,y,z < \pi,\ t > 0$
$u(0,y,z,t) = -u(\pi,y,z,t) = \cos y \sin(z+t)$
$u(x,0,z,t) = -u(x,\pi,z,t) = \cos x \sin(z+t)$
$u(x,y,0,t) = -u(x,y,\pi,t) = \cos(x+y)\sin t$
$u(x,y,z,0) = \cos(x+y)\sin z, u_t(x,y,z,0) = \cos(x+y)\cos z$

8. $u_{tt} = \frac{1}{3}(u_{xx} + u_{yy} + u_{zz}),\ 0 < x,y,z < \pi,\ t > 0$
$u(0,y,z,t) = u(\pi,y,z,t) = 1 + z$
$u(x,0,z,t) = u(x,\pi,z,t) = 1 + z$
$u(x,y,0,t) = 1, u(x,y,\pi,t) = 1 + \pi$
$u(x,y,z,0) = 1 + z, u_t(x,y,z,0) = \sin x \sin y \sin z$

9. $u_{tt} = u_{xx} + u_{yy} + u_{zz} + \sin x + \sin y,\ 0 < x,y,z < \pi,\ t > 0$
$u(0,y,z,t) = u(\pi,y,z,t) = \sin y + \sin z \sin t$
$u(x,0,z,t) = u(x,\pi,z,t) = \sin x + \sin z \sin t$
$u(x,y,0,t) = u(x,y,\pi,t) = \sin x + \sin y$
$u(x,y,z,0) = \sin x + \sin y, u_t(x,y,z,0) = \sin z$

10. $u_{tt} = u_{xx} + u_{yy} + u_{zz} + \cos x + \cos y,\ 0 < x,y,z < \pi,\ t > 0$
$u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0$
$u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0$
$u_z(x,y,0,t) = -u_z(x,y,\pi,t) = \sin t$
$u(x,y,z,0) = \cos x + \cos y, u_t(x,y,z,0) = \sin z$

11. $u_{tt} = u_{xx} + u_{yy} + u_{zz} - 6 - \sin t,\ 0 < x,y,z < \pi,\ t > 0$
$u_x(0,y,z,t) = 0, u_x(\pi,y,z,t) = 2\pi$
$u_y(x,0,z,t) = 0, u_y(x,\pi,z,t) = 2\pi$
$u_z(x,y,0,t) = 0, u_z(x,y,\pi,t) = 2\pi$
$u(x,y,z,0) = x^2 + y^2 + z^2, u_t(x,y,z,0) = 1$

12. $u_{tt} = u_{xx} + u_{yy} + u_{zz} + 2 - \sin t,\ 0 < x,y,z < \pi,\ t > 0$
$u_x(0,y,z,t) = u_x(\pi,y,z,t) = t$
$u_y(x,0,z,t) = u_y(x,\pi,z,t) = t$
$u_z(x,y,0,t) = u_z(x,y,\pi,t) = t$

$u(x,y,z,0) = 0, u_t(x,y,z,0) = 1+x+y+z$

13. $u_{tt} = u_{xx}+u_{yy}+u_{zz}+2(x+y+z), 0 < x,y,z < \pi, t > 0$
$u_x(0,y,z,t) = u_x(\pi,y,z,t) = t^2$
$u_y(x,0,z,t) = t^2+\sin t, u_y(x,\pi,z,t) = t^2-\sin t$
$u_z(x,y,0,t) = u_z(x,y,\pi,t) = t^2$
$u(x,y,z,0) = 0, u_t(x,y,z,0) = \sin y$

14. $u_{tt} = u_{xx}+u_{yy}+u_{zz}-6, 0 < x,y,z < \pi, t > 0$
$u_x(0,y,z,t) = 0, u_x(\pi,y,z,t) = 2\pi$
$u_y(x,0,z,t) = 0, u_y(x,\pi,z,t) = 2\pi$
$u_z(x,y,0,t) = 0, u_z(x,y,\pi,t) = 2\pi$
$u(x,y,z,0) = x^2+y^2+z^2+\cos y, u_t(x,y,z,0) = 0$

15. $u_{tt} = \frac{1}{2}(u_{xx}+u_{yy}+u_{zz})-1, 0 < x,y,z < \pi, t > 0$
$u_x(0,y,z,t) = 0, u_x(\pi,y,z,t) = 2\pi$
$u_y(x,0,z,t) = -u_y(x,\pi,z,t) = \sin z\cos t$
$u_z(x,y,0,t) = -u_z(x,y,\pi,t) = \sin y\cos t$
$u(x,y,z,0) = x^2+\sin x\sin z, u_t(x,y,z,0) = 0$

In Exercises 16–20, use the decomposition method to solve the initial-boundary value problems:

16. $u_{tt} = u_{xx}+u_{yy}+u_{zz}-u, 0 < x,y,z < \pi, t > 0$
$u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0$
$u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0$
$u_z(x,y,0,t) = u_z(x,y,\pi,t) = 0$
$u(x,y,z,0) = 0, u_t(x,y,z,0) = 2\cos x\cos y\cos z$

17. $u_{tt} = u_{xx}+u_{yy}+u_{zz}-u+1, 0 < x,y,z < \pi, t > 0$
$u(0,y,z,t) = u(\pi,y,z,t) = 1$
$u(x,0,z,t) = u(x,\pi,z,t) = 1$
$u(x,y,0,t) = u(x,y,\pi,t) = 1$
$u(x,y,z,0) = 1+\sin x\sin y\sin z, u_t(x,y,z,0) = 0$

18. $u_{tt} = u_{xx}+u_{yy}+u_{zz}+2\sin x\sin y, 0 < x,y,z < \pi, t > 0$
$u(0,y,z,t) = u(\pi,y,z,t) = 1+\sin z\sin t$
$u(x,0,z,t) = u(x,\pi,z,t) = 1+\sin z\sin t$
$u(x,y,0,t) = u(x,y,\pi,t) = 1+\sin x\sin y$
$u(x,y,z,0) = 1+\sin x\sin y, u_t(x,y,z,0) = \sin z$

19. $u_{tt} = \frac{1}{3}(u_{xx}+u_{yy}+u_{zz}), 0 < x,y,z < \pi, t > 0$
$u_x(0,y,z,t) = 0, u_x(\pi,y,z,t) = 2\pi$
$u_y(x,0,z,t) = 0, u_y(x,\pi,z,t) = 2\pi$
$u_z(x,y,0,t) = 0, u_z(x,y,\pi,t) = 2\pi$
$u(x,y,z,0) = x^2+y^2+z^2+\cos x\cos y\cos z, u_t(x,y,z,0) = 0$

20. $u_{tt} = u_{xx} + u_{yy} + u_{zz} - 12x^2 - 12y^2,\ 0 < x,y,z < \pi,\ t > 0$
$u_x(0,y,z,t) = 0, u_x(\pi,y,z,t) = 4\pi^3$
$u_y(x,0,z,t) = 0, u_y(x,\pi,z,t) = 4\pi^3$
$u_z(x,y,0,t) = u_z(x,y,\pi,t) = 0$
$u(x,y,z,0) = x^4 + y^4 + \cos z, u_t(x,y,z,0) = 0$

In Exercises 21–24, solve the inhomogeneous initial-boundary value problems where the coefficients of the derivatives are functions and not constants:

21. $u_{tt} = \frac{x^2}{6}u_{xx} + \frac{y^2}{6}u_{yy} + \frac{z^2}{6}u_{zz},\ 0 < x,y,z < 1,\ t > 0$
$u(0,y,z,t) = 0, u(1,y,z,t) = y^2z^2\cosh t$
$u(x,0,z,t) = 0, u(x,1,z,t) = x^2z^2\cosh t$
$u(x,y,0,t) = 0, u(x,y,1,t) = x^2y^2\cosh t$
$u(x,y,z,0) = x^2y^2z^2, u_t(x,y,z,0) = 0$

22. $u_{tt} = \frac{x^2}{6}u_{xx} + \frac{y^2}{6}u_{yy} + \frac{z^2}{6}u_{zz},\ 0 < x,y,z < 1,\ t > 0$
$u_x(0,y,z,t) = 0, u_x(1,y,z,t) = 3\sinh t$
$u_y(x,0,z,t) = 0, u_y(x,1,z,t) = 3\cosh t$
$u_z(x,y,0,t) = 0, u_z(x,y,1,t) = 3\cosh t$
$u(x,y,z,0) = (y^3 + z^3)\cosh t, u_t(x,y,z,0) = x^3$

23. $u_{tt} = \frac{x^2}{2}u_{xx} + \frac{y^2}{2}u_{yy} + \frac{z^2}{2}u_{zz},\ 0 < x,y,z < 1,\ t > 0$
$u_x(0,y,z,t) = 0, u_x(1,y,z,t) = 2e^t$
$u_y(x,0,z,t) = 0, u_y(x,1,z,t) = 2e^{-t}$
$u_z(x,y,0,t) = 0, u_z(x,y,1,t) = 2e^t$
$u(x,y,z,0) = x^2 + y^2 + z^2, u_t(x,y,z,0) = x^2 - y^2 + z^2$

24. $u_{tt} = \frac{x^2}{6}u_{xx} + \frac{y^2}{6}u_{yy} + \frac{z^2}{6}u_{zz},\ 0 < x,y,z < 1,\ t > 0$
$u_x(0,y,z,t) = 0, u_x(1,y,z,t) = 3\sinh t$
$u_y(x,0,z,t) = 0, u_y(x,1,z,t) = 3\sinh t$
$u_z(x,y,0,t) = 0, u_z(x,y,1,t) = 3\sinh t$
$u(x,y,z,0) = 0, u_t(x,y,z,0) = x^3 + y^3 + z^3$

6.3 Method of Separation of Variables

In this section, the homogeneous partial differential equations that describe the wave propagation in a two dimensional space and in a three dimensional space will be discussed by using the classical method of *separation of variables*. The most significant feature of this method is that it reduces the partial differential equation into a system of ordinary differential equations, where each ODE depends on one variable only, and can be solved independently [4]. The boundary conditions and the initial conditions are then used to determine the constants of integration.

The complete details of the method can be found in the preceding chapters, hence emphasis will be focused on applying the method.

6.3.1 Two Dimensional Wave Equation

The propagation of waves in a two dimensional vibrating membrane of length a and width b is governed by the following initial boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= c^2(u_{xx}+u_{yy}), \quad 0<x<a, 0<y<b, t>0, \\ \text{BC} & u(0,y,t) &= u(a,y,t)=0, \\ & u(x,0,t) &= u(x,b,t)=0, \\ \text{IC} & u(x,y,0) &= f(x,y), \quad u_t(x,y,0)=g(x,y). \end{array} \tag{6.125}$$

where $u=u(x,y,t)$ defines the displacement function of any point at the position (x,y) of a vibrating membrane at any time t, and c is related to the elasticity of the material of the membrane.

The method of separation of variables is based on an assumption that the solution $u(x,y,t)$ can be expressed as the product of distinct functions $F(x)$, $G(y)$, and $T(t)$, such that each function depends on one variable only. Based on this assumption, we first set

$$u(x,y,t)=F(x)G(y)T(t). \tag{6.126}$$

Differentiating both sides of (6.126) twice with respect to t, x and y respectively, we obtain

$$\begin{aligned} u_{tt} &= F(x)G(y)T''(t), \\ u_{xx} &= F''(x)G(y)T(t), \\ u_{yy} &= F(x)G''(y)T(t). \end{aligned} \tag{6.127}$$

Substituting (6.127) into the PDE of (6.125) gives

$$F(x)G(y)T''(t)=c^2\left(F''(x)G(y)T(t)+F(x)G''(y)T(t)\right). \tag{6.128}$$

Dividing both sides of (6.128) by $c^2F(x)G(y)T(t)$ yields

$$\frac{T''(t)}{c^2T(t)}=\frac{F''(x)}{F(x)}+\frac{G''(y)}{G(y)}. \tag{6.129}$$

It is easily observed from (6.129) that the left hand side depends only on t and the right hand side depends only on x and y. This means that the equality holds only if both sides are equal to the same constant. Assuming that the right hand side is a constant, it is valid to assume that it is the sum of two constants. This admits the use of

$$\frac{F''(x)}{F(x)}=-\nu^2, \tag{6.130}$$

and

$$\frac{G''(y)}{G(y)} = -\mu^2. \tag{6.131}$$

Consequently, we find

$$F''(x) + \nu^2 F(x) = 0, \tag{6.132}$$

and

$$G''(y) + \mu^2 G(y) = 0. \tag{6.133}$$

The left hand side of (6.129) is thus equal to the constant $-(\nu^2 + \mu^2)$, hence we set

$$\frac{T''(t)}{c^2 T(t)} = -(\nu^2 + \mu^2), \tag{6.134}$$

or equivalently

$$T''(t) + c^2(\nu^2 + \mu^2)T(t) = 0. \tag{6.135}$$

The selection of $-(\nu^2 + \mu^2)$ is the only selection that will provide nontrivial solutions.

It is interesting to note that the partial differential equation of (6.125) has been transformed to three second-order ordinary differential equations given by (6.132), (6.133), and (6.135).

The second-order differential equations (6.132) and (6.133) give the solutions

$$F(x) = A\cos(\nu x) + B\sin(\nu x), \tag{6.136}$$

and

$$G(y) = \alpha\cos(\mu y) + \beta\sin(\mu y), \tag{6.137}$$

where A, B, α, and β are constants.

To determine the constants A and B, we use the boundary conditions at $x = 0$ and at $x = a$ to find that

$$F(0) = 0, \quad F(a) = 0. \tag{6.138}$$

Substituting (6.138) into (6.136) gives

$$A = 0, \tag{6.139}$$

and

$$\nu_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \cdots. \tag{6.140}$$

It is important to note here that we exclude $n = 0$ and $B = 0$ because each will lead to the trivial solution $u(x, y, t) = 0$. Using the results obtained for the constants A and ν_n, we therefore find

$$F_n(x) = B_n \sin(\frac{n\pi}{a}x), \quad n = 1, 2, 3, \cdots. \tag{6.141}$$

In a parallel manner, we use the second boundary condition at $y = 0$ and at $y = b$ into (6.137) to find that

$$\alpha = 0, \tag{6.142}$$

and

$$\mu_m = \frac{m\pi}{b}, \quad m = 1, 2, 3, \cdots. \tag{6.143}$$

We exclude $m = 0$ and $\beta = 0$, because each will lead to the trivial solution as indicated before. Consequently, we obtain

$$G_m(y) = \beta_m \sin\frac{m\pi}{b}, \quad m = 1, 2, 3, \cdots. \tag{6.144}$$

The solution of (6.135) is therefore given by

$$T_{nm}(t) = \tilde{C}_{nm}\cos(c\lambda_{nm}t) + \tilde{D}_{nm}\sin(c\lambda_{nm}t), \tag{6.145}$$

where

$$\lambda_{mn}^2 = (\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2, \tag{6.146}$$

and $\tilde{C}_{nm}$ and $\tilde{D}_{nm}$ are constants.

Combining the results obtained for $F_n(x)$, $G_m(y)$, and $T_{nm}(t)$ we obtain the infinite sequence of product functions

$$\begin{aligned} u_{nm} &= F_n(x)G_m(y)T_{nm}(t) \\ &= \sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\left(\tilde{C}_{nm}\cos(c\lambda_{nm}t) + \tilde{D}_{nm}\sin(c\lambda_{nm}t)\right), \end{aligned} \tag{6.147}$$

that satisfies the PDE of (6.125) and the boundary conditions. Using the superposition principle gives the general solution
$u(x,y,t) =$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin(\frac{n\pi}{a}x)\,\sin(\frac{m\pi}{b}y)(C_{nm}\cos(c\lambda_{nm}t) + D_{nm}\sin(c\lambda_{nm}t)), \tag{6.148}$$

where the arbitrary constants C_{nm} and D_{nm} are as yet undetermined.

To determine the constants C_{nm} and D_{nm}, we use the given initial conditions to find

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin(\frac{n\pi}{a}x)\,\sin(\frac{m\pi}{b}y) = f(x,y), \tag{6.149}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\lambda_{nm}D_{nm}\sin(\frac{n\pi}{a}x)\,\sin(\frac{m\pi}{b}y) = g(x,y). \tag{6.150}$$

Consequently, the arbitrary constants C_{nm} and D_{nm} are completely determined by using double Fourier coefficients where we find

$$C_{nm} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin(\frac{n\pi}{a}x)\,\sin(\frac{m\pi}{b}y)dx\,dy, \tag{6.151}$$

and

$$D_{nm} = \frac{4}{\lambda_{nm}abc}\int_0^b\int_0^a g(x,y)\sin(\frac{n\pi}{a}x)\,\sin(\frac{m\pi}{b}y)dx\,dy. \tag{6.152}$$

Having determined the constants C_{nm} and D_{nm}, the particular solution $u(x,y,t)$ that satisfies the initial boundary value problem (6.125) is readily obtained.

It is interesting to point out that the constants C_{nm} and D_{nm} can also be determined by expanding the double Fourier series of (6.148), applying the initial conditions, and then by equating the coefficients of like terms on both sides. Clearly this works if the initial conditions are defined explicitly in terms of trigonometric functions of sines and cosines.

For illustration, several examples will be discussed to emphasize the use of the method.

Example 1. Use the method of separation of variables to solve the initial-boundary value problem:

$$\begin{array}{lrcl} \text{PDE} & u_{tt} &=& 2(u_{xx}+u_{yy}), \quad 0<x,y<\pi, t>0, \\ \text{BC} & u(0,y,t) &=& u(\pi,y,t)=0, \\ & u(x,0,t) &=& u(x,\pi,t)=0, \\ \text{IC} & u(x,y,0) &=& \sin x\sin y, \quad u_t(x,y,0)=0. \end{array} \tag{6.153}$$

Solution.

The method of separation of variables assumes that

$$u(x,y,t) = F(x)G(y)T(t). \tag{6.154}$$

Proceeding as before, we obtain

$$F''(x)+\nu^2F(x)=0, \tag{6.155}$$

$$G''(y)+\mu^2G(y)=0, \tag{6.156}$$

and

$$T''(t)+2\lambda^2T(t)=0, \tag{6.157}$$

where

$$\lambda^2=\nu^2+\mu^2, \tag{6.158}$$

and ν, μ and λ are constants.

From (6.155) and (6.156), we find

$$F(x) = A\cos(\nu x)+B\sin(\nu x), \tag{6.159}$$

and

$$G(y) = \alpha\cos(\mu y)+\beta\sin(\mu y), \tag{6.160}$$

respectively, where A, B, α and β are constants. Inserting the boundary conditions at $x=0$ and at $x=\pi$ into (6.159) gives

$$A = 0, \tag{6.161}$$

and

$$\nu_n = n, \quad n = 1, 2, 3, \cdots, \tag{6.162}$$

so that

$$F_n(x) = B_n \sin(nx),\ n = 1, 2, 3, \cdots. \tag{6.163}$$

Likewise, using the boundary conditions at $y = 0$ and at $y = \pi$ into (6.160) gives

$$\alpha = 0, \tag{6.164}$$

and

$$\mu_m = m, \quad m = 1, 2, 3, \cdots, \tag{6.165}$$

so that

$$G_m(y) = \beta_m \sin(my),\ m = 1, 2, 3, \cdots. \tag{6.166}$$

The solution of (6.157) is therefore given by

$$T_{nm}(t) = \tilde{C}_{nm} \cos(\sqrt{2}\lambda_{nm} t) + \tilde{D}_{nm} \sin(\sqrt{2}\lambda_{nm} t), \tag{6.167}$$

where

$$\lambda_{nm} = \sqrt{n^2 + m^2}. \tag{6.168}$$

Combining the results obtained above, we obtain

$$u_{nm}(x, y, t) = \sin(nx)\sin(my)(\tilde{C}_{nm} \cos(\sqrt{2}\lambda_{nm} t) + \tilde{D}_{nm} \sin(\sqrt{2}\lambda_{nm} t)). \tag{6.169}$$

Using the superposition principle, the general solution of the problem is given by the double series

$$u = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin(nx)\, \sin(my)(C_{nm} \cos(\sqrt{2}\lambda_{nm} t) + D_{nm} \sin(\sqrt{2}\lambda_{nm} t)). \tag{6.170}$$

To determine the constants C_{nm}, we use $u(x, y, 0) = \sin x \sin y$ and expand the double series (6.170) to find

$$C_{11} \sin x \sin y + C_{12} \sin x \sin 2y + \cdots = \sin x \sin y. \tag{6.171}$$

Equating the coefficients of like terms on both sides yields

$$C_{11} = 1, \quad C_{ij} = 0,\ i \neq 1, j \neq 1. \tag{6.172}$$

Likewise, inserting the second initial condition $u_t(x, y, 0) = 0$ into the derivative of (6.170) gives

$$D_{nm} = 0, \quad n \geqslant 1, m \geqslant 1. \tag{6.173}$$

The particular solution is therefore given by

$$u(x,y,t) = \sin x \sin y \cos(2t). \tag{6.174}$$

Example 2. Use the method of separation of variables to solve the initial-boundary value problem:

$$\begin{array}{lrcl} \text{PDE} & u_{tt} &=& 2(u_{xx}+u_{yy}), \quad 0<x,y<\pi,\, t>0, \\ \text{BC} & u_x(0,y,t) &=& u_x(\pi,y,t)=0, \\ & u_y(x,0,t) &=& u_y(x,\pi,t)=0, \\ \text{IC} & u(x,y,0) &=& 1+\cos x\cos y, \quad u_t(x,y,0)=0. \end{array} \tag{6.175}$$

Solution.

Note first that the boundary conditions are of the second kind where the derivatives of $u(x,y,t)$ are given instead of the initial displacements. As discussed before, we set

$$u(x,y,t) = F(x)G(y)T(t). \tag{6.176}$$

Substituting (6.176) into (6.175) and proceeding as before we obtain

$$F''(x) + \nu^2 F(x) = 0, \tag{6.177}$$

$$G''(y) + \mu^2 G(y) = 0, \tag{6.178}$$

and

$$T''(t) + 2(\nu^2+\mu^2)T(t) = 0. \tag{6.179}$$

Solving (6.177) we find

$$F(x) = A\cos(\nu x) + B\sin(\nu x). \tag{6.180}$$

It is important to note that the boundary conditions

$$u_x(0,y,t) = u_x(\pi,y,t) = 0, \tag{6.181}$$

imply that

$$F'(0) = 0, \quad F'(\pi) = 0. \tag{6.182}$$

Substituting (6.182) into (6.180) gives

$$B = 0, \quad \nu = n, n = 0,1,2,\cdots, \tag{6.183}$$

so that $\nu = 0$ is included because it does not provide the trivial solution. Consequently, we find

$$F_n(x) = A_n\cos(nx), \quad n = 0,1,2,\cdots. \tag{6.184}$$

Solving (6.178) and using the proper boundary conditions we obtain

$$G_m(y) = \alpha_m\cos(my), \quad m = 0,1,2,3,\cdots. \tag{6.185}$$

The solution of the second order differential equation (6.179) is therefore given by

$$T_{nm}(t) = \tilde{C}_{nm}\cos(\sqrt{2}\lambda_{nm}t) + \tilde{D}_{nm}\sin(\sqrt{2}\lambda_{nm}t), n \geqslant 0, m \geqslant 0, \tag{6.186}$$

where

$$\lambda_{nm} = \sqrt{n^2+m^2}. \tag{6.187}$$

The general solution is given by

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \cos(nx)\cos(my)\left(C_{nm}\cos(\sqrt{2}\lambda_{nm}t) + D_{nm}\sin(\sqrt{2}\lambda_{nm}t)\right). \tag{6.188}$$

To determine the constants C_{nm}, we first expand the series (6.188) and use the first initial condition to find

$$C_{00} + C_{11}\cos x\cos y + \cdots = 1 + \cos x\cos y. \tag{6.189}$$

Equating the coefficients of like terms on both sides gives

$$\begin{aligned} C_{00} &= 1, \\ C_{11} &= 1, \end{aligned} \tag{6.190}$$

where other coefficients are zeros.

To determine the constants D_{nm}, we use the second boundary condition in the derivative of (6.188) to find that

$$D_{nm} = 0, \quad n \geqslant 0, m \geqslant 0. \tag{6.191}$$

Accordingly, the particular solution is given by

$$u(x,y,t) = 1 + \cos x\cos y\cos(2t). \tag{6.192}$$

Example 3. Use the method of separation of variables to solve the initial-boundary value problem:

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= u_{xx} + u_{yy}, \quad 0 < x,y < \pi,\ t > 0, \\ \text{BC} & u(0,y,t) &= u(\pi,y,t) = 0, \\ & u(x,0,t) &= u(x,\pi,t) = 0, \\ \text{IC} & u(x,y,0) &= 1, \quad u_t(x,y,0) = 0. \end{array} \tag{6.193}$$

Solution.

We first set

$$u(x,y,t) = F(x)G(y)T(t), \tag{6.194}$$

Proceeding as before we obtain

$$F_n(x) = B_n\sin(nx), \quad \lambda_n = n,\ n = 1,2,\cdots, \tag{6.195}$$

$$G_m(y) = \beta_m\sin(my), \quad \mu_m = m,\ m = 1,2,\cdots, \tag{6.196}$$

$$T_{nm}(t) = \tilde{C}_{nm}\cos(\lambda_{nm}t) + \tilde{D}_{nm}\sin(\lambda_{nm}t), \tag{6.197}$$

where

$$\lambda_{nm} = \sqrt{n^2 + m^2}. \tag{6.198}$$

Combining (6.195), (6.196), and (6.197) and using the superposition principle, the general solution of the problem is given by

$$u = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin(nx) \sin(my)(C_{nm} \cos(\lambda_{nm} t) + D_{nm} \sin(\lambda_{nm} t)). \tag{6.199}$$

To determine the constants C_{nm}, we use the double series coefficients method to obtain

$$C_{nm} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} \sin(nx) \sin(my)\, dx\, dy, \tag{6.200}$$

which gives

$$C_{nm} = \begin{cases} 0, & \text{for } n \text{ or } m \text{ is even,} \\ \dfrac{16}{\pi^2 nm}, & \text{for } n \text{ and } m \text{ are odd.} \end{cases} \tag{6.201}$$

To determine D_{nm}, we use the second boundary condition to find

$$D_{nm} = 0, \quad n \geqslant 1, m \geqslant 1. \tag{6.202}$$

Accordingly, the particular solution in a series form is given by

$$u = \frac{16}{\pi^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(2n+1)(2m+1)} \sin(2n+1)x \sin(2m+1)y \cos(\lambda_{nm} t), \tag{6.203}$$

where

$$\lambda_{nm} = \sqrt{(2n+1)^2 + (2m+1)^2}, \quad n \geqslant 0, m \geqslant 0. \tag{6.204}$$

Example 4. Use the method of separation of variables to solve the initial-boundary value problem:

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= u_{xx} + u_{yy}, \quad 0 < x,y < \pi, t > 0, \\ \text{BC} & u(0,y,t) &= u(\pi,y,t) = 0, \\ & u(x,0,t) &= u(x,\pi,t) = 0, \\ \text{IC} & u(x,y,0) &= 0, u_t(x,y,0) = 1. \end{array} \tag{6.205}$$

Solution.

We first set

$$u(x,y,t) = F(x)G(y)T(t), \tag{6.206}$$

Following Example 3, we obtain

$$F_n(x) = B_n \sin(nx),\ \nu_n = n,\ n = 1,2,\cdots, \tag{6.207}$$

$$G_m(y) = \beta_m \sin(my),\ \mu_m = m,\ m = 1,2,\cdots, \tag{6.208}$$

and

$$T_{nm}(t) = \tilde{C}_{nm}\cos(\lambda_{nm}t) + \tilde{D}_{nm}\sin(\lambda_{nm}t), \tag{6.209}$$

where

$$\lambda_{nm} = \sqrt{n^2+m^2}. \tag{6.210}$$

Combining (6.207), (6.208), and (6.209) and using the superposition principle, the general solution of the problem is given by

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin(nx)\sin(my)\left(C_{nm}\cos(\lambda_{nm}t) + D_{nm}\sin(\lambda_{nm}t)\right). \tag{6.211}$$

To determine the constants C_{nm}, we use the double Fourier coefficients method to obtain

$$C_{nm} = 0, \quad n \geqslant 1, m \geqslant 1. \tag{6.212}$$

Using the second boundary condition we find

$$\lambda_{nm}D_{nm} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} \sin(nx)\sin(my)\,dx\,dy, \tag{6.213}$$

which gives

$$D_{nm} = \begin{cases} 0, & \text{when } n \text{ or } m \text{ is even,} \\ \dfrac{16}{\lambda_{nm}\pi^2 nm}, & \text{when } n \text{ and } m \text{ are odd.} \end{cases} \tag{6.214}$$

Accordingly, the particular solution of the initial-boundary value problem (6.205) is given by the double series form

$$\begin{aligned} u(x,y,t) = {} & \frac{16}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\lambda_{nm}(2n+1)(2m+1)} \\ & \times \sin(2n+1)x\sin(2m+1)y\sin(\lambda_{nm}t), \end{aligned} \tag{6.215}$$

where

$$\lambda_{nm} = \sqrt{(2n+1)^2+(2m+1)^2}. \tag{6.216}$$

Exercises 6.3.1

Use the method of separation of variables in the following initial-boundary value problems:

1. $u_{tt} = 2(u_{xx}+u_{yy}),\ 0 < x,y < \pi$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = \sin(2x)\sin(2y),\ u_t(x,y,0) = 0$

2. $u_{tt} = 5(u_{xx} + u_{yy}),\ 0 < x,y < \pi$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = \sin x \sin(2y), u_t(x,y,0) = 0$

3. $u_{tt} = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = 0, u_t(x,y,0) = 2\sin x \sin y$

4. $u_{tt} = 5(u_{xx} + u_{yy}),\ 0 < x,y < \pi$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = 0, u_t(x,y,0) = 5\sin x \sin(2y)$

5. $u_{tt} = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi$
   $u_x(0,y,t) = u_x(\pi,y,t) = 0$
   $u_y(x,0,t) = u_y(x,\pi,t) = 0$
   $u(x,y,0) = 2, u_t(x,y,0) = 2\cos x \cos y$

6. $u_{tt} = 8(u_{xx} + u_{yy}),\ 0 < x,y < \pi$
   $u_x(0,y,t) = u_x(\pi,y,t) = 0$
   $u_y(x,0,t) = u_y(x,\pi,t) = 0$
   $u(x,y,0) = 1 + \cos x \cos y, u_t(x,y,0) = 0$

7. $u_{tt} = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi$
   $u(0,y,t) = u(\pi,y,t) = 0$
   $u_y(x,0,t) = u_y(x,\pi,t) = 0$
   $u(x,y,0) = 0, u_t(x,y,0) = 2\sin x \cos y$

8. $u_{tt} = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi$
   $u_x(0,y,t) = u_x(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = \cos x \sin y, u_t(x,y,0) = 0$

9. $u_{tt} = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi$
   $u_x(0,y,t) = u_x(\pi,y,t) = 0$
   $u(x,0,t) = u(x,\pi,t) = 0$
   $u(x,y,0) = 0, u_t(x,y,0) = 2\cos x \sin y$

10. $u_{tt} = 5(u_{xx} + u_{yy}),\ 0 < x,y < \pi$
    $u_x(0,y,t) = u_x(\pi,y,t) = 0$
    $u_y(x,0,t) = u_y(x,\pi,t) = 0$
    $u(x,y,0) = 3, u_t(x,y,0) = 5\cos x \cos(2y)$

11. $u_{tt} = 2(u_{xx} + u_{yy}),\ 0 < x,y < \pi$
    $u(0,y,t) = u(\pi,y,t) = 0$
    $u(x,0,t) = u(x,\pi,t) = 0$
    $u(x,y,0) = 2, u_t(x,y,0) = 0$

12. $u_{tt} = 2(u_{xx} + u_{yy}),\ 0 < x, y < \pi$
$u(0,y,t) = u(\pi,y,t) = 0$
$u(x,0,t) = u(x,\pi,t) = 0$
$u(x,y,0) = 0, u_t(x,y,0) = 3$

6.3.2 Three Dimensional Wave Equation

The propagation of waves in a three dimensional space of length a, width b and of height d is governed by the initial boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{tt} &= c^2(u_{xx} + u_{yy} + u_{zz}), \quad 0 < x < a, 0 < y < b, 0 < z < d, \\ \text{BC} & u(0,y,z,t) &= u(a,y,z,t) = 0, \\ & u(x,0,z,t) &= u(x,b,z,t) = 0, \\ & u(x,y,0,t) &= u(x,y,d,t) = 0, \\ \text{IC} & u(x,y,z,0) &= f(x,y,z), \quad u_t(x,y,z,0) = g(x,y,z). \end{array} \tag{6.217}$$

where the unknown function $u = u(x,y,z,t)$ defines the displacement of any point at the position (x,y,z) of a rectangular volume at any time t, c is the velocity of a propagation wave.

The method of separation of variables assumes that $u(x,y,z,t)$ consists of the product of four distinct functions each depends on one variable only. This means that we can set

$$u(x,y,z,t) = F(x)G(y)H(z)T(t). \tag{6.218}$$

Substituting (6.218) into (6.217), and dividing both sides of the resulting equation by $c^2F(x)G(y)H(z)T(t)$ we obtain

$$\frac{T''(t)}{c^2T(t)} = \frac{F''(x)}{F(x)} + \frac{G''(y)}{G(y)} + \frac{H''(z)}{H(z)}. \tag{6.219}$$

The equality in (6.219) holds only if both sides are equal to the same constant. This allows us to set

$$F''(x) + \nu^2 F(x) = 0, \tag{6.220}$$

$$G''(y) + \mu^2 G(y) = 0, \tag{6.221}$$

$$H''(z) + \eta^2 H(z) = 0, \tag{6.222}$$

$$T''(t) + c^2\lambda^2 T(t) = 0, \tag{6.223}$$

where ν, μ, η, and λ are constants, and

$$\lambda^2 = (\nu^2 + \mu^2 + \eta^2). \tag{6.224}$$

Solving the second order normal forms (6.220) – (6.222), we obtain the following solutions

$$F(x) = A\cos(\nu x) + B\sin(\nu x), \tag{6.225}$$
$$G(y) = \alpha\cos(\mu y) + \beta\sin(\mu y), \tag{6.226}$$
$$H(z) = \gamma\cos(\eta z) + \delta\sin(\eta z), \tag{6.227}$$

respectively, where $A, B, \alpha, \beta, \gamma$, and δ are constants. Using the proper boundary conditions into (6.225) – (6.227) as applied before we find

$$A = 0, \quad \nu_n = \frac{n\pi}{a}, \; n = 1,2,3,\cdots, \tag{6.228}$$
$$\alpha = 0 \; \mu_m = \frac{m\pi}{b}, \; m = 1,2,3,\cdots, \tag{6.229}$$
$$\gamma = 0, \quad \eta_r = \frac{r\pi}{d}, \; r = 1,2,3,\cdots, \tag{6.230}$$

so that

$$F_n(x) = B_n \sin(\frac{n\pi}{a}x), \; n = 1,2,3,\cdots, \tag{6.231}$$
$$G_m(y) = \beta_m \sin(\frac{m\pi}{b}y), \; m = 1,2,3,\cdots, \tag{6.232}$$
$$H_r(z) = \delta_r \sin(\frac{r\pi}{d}z), \; r = 1,2,3,\cdots. \tag{6.233}$$

The solution of (6.223) is therefore given by

$$T_{nmr}(t) = \tilde{C}_{nmr}\cos(c\lambda_{nmr}t) + \tilde{D}_{nmr}\sin(c\lambda_{nmr}t), \tag{6.234}$$

where

$$\lambda_{nmr} = (\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 + (\frac{r\pi}{d})^2. \tag{6.235}$$

Combining the results obtained above for $F_n(x), G_m(y), H_r(z)$, and $T_{nmr}(t)$ and using the superposition principle we can formulate the general solution of (6.218) in the form

$$\begin{aligned} u(x,y,z,t) = {} & \sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin(\frac{n\pi}{a}x)\,\sin(\frac{m\pi}{b}y)\,\sin(\frac{r\pi}{d}z) \\ & \times \left(C_{nmr}\cos(c\lambda_{nmr}t) + D_{nmr}\sin(c\lambda_{nmr}t)\right). \end{aligned} \tag{6.236}$$

It remains now to determine the constants C_{nmr} and D_{nmr}. Using the initial condition $u(x,y,z,0) = f(x,y,z)$ into (6.236), the coefficients C_{nmr} are given by

$C_{nmr} =$

$$\frac{8}{abd}\int_0^d\int_0^b\int_0^a f(x,y,z)\sin(\frac{n\pi}{a}x)\,\sin(\frac{m\pi}{b}y)\,\sin(\frac{r\pi}{d}z)\,dx\,dy\,dz. \tag{6.237}$$

Using the initial condition $u_t(x,y,z,0) = g(x,y,z)$ into the derivative of (6.236), the coefficients D_{nmr} can be determined in the following form

$D_{nmr} =$

$$\frac{8}{\lambda_{nmr} cabd} \int_0^d \int_0^b \int_0^a g(x,y,z) \sin(\frac{n\pi}{a}x) \sin(\frac{m\pi}{b}y) \sin(\frac{r\pi}{d}z) dx\,dy\,dz. \tag{6.238}$$

Having determined the coefficients C_{nmr} and D_{nmr}, the particular solution of the initial-boundary value problem follows immediately upon substituting (6.237) and (6.238) into (6.236).

To explain the use of the method of separation of variables, several illustrative examples will now be given.

Example 5. Solve the initial-boundary value problem

$$\begin{array}{lrcl} \text{PDE} & u_{tt} & = & 3(u_{xx} + u_{yy} + u_{zz}), \quad 0 < x,y,z < \pi, t > 0, \\ \text{BC} & u(0,y,z,t) & = & u(\pi,y,z,t) = 0, \\ & u(x,0,z,t) & = & u(x,\pi,z,t) = 0, \\ & u(x,y,0,t) & = & u(x,y,\pi,t) = 0, \\ \text{IC} & u(x,y,z,0) & = & \sin x \sin y \sin z, \quad u_t(x,y,z,0) = 0. \end{array} \tag{6.239}$$

Solution.

Proceeding as before, we set

$$u(x,y,z,t) = F(x)G(y)H(z)T(t). \tag{6.240}$$

Following the discussions presented above we find

$$F_n(x) = B_n \sin(nx), \quad \nu_n = n,\ n = 1,2,3,\cdots, \tag{6.241}$$

$$G_m(y) = \beta_m \sin(my), \quad \mu_m = m,\ m = 1,2,3,\cdots, \tag{6.242}$$

$$H_r(z) = \delta_r \sin(rz), \quad \eta_r = r,\ r = 1,2,3,\cdots, \tag{6.243}$$

$$T_{nmr}(t) = \tilde{C}_{nmr} \cos(\sqrt{3}\lambda_{nmr}t) + \tilde{D}_{nmr} \sin(\sqrt{3}\lambda_{nmr}t), \tag{6.244}$$

where

$$\lambda_{nmr} = \sqrt{n^2 + m^2 + r^2}. \tag{6.245}$$

Recall that we exclude $n = 0, m = 0$, and $r = 0$.

Proceeding as before and using the superposition principle, we can formulate the general solution in the form

$$\begin{aligned} u(x,y,z,t) = & \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(nx) \sin(my) \sin(rz) \\ & \times \left(C_{nmr} \cos(\sqrt{3}\lambda_{nmr}t) + D_{nmr} \sin(\sqrt{3}\lambda_{nmr}t) \right). \end{aligned} \tag{6.246}$$

To determine the constants C_{nmr}, we use the first initial condition and expand (6.246) to find

$$C_{111} \sin x \sin y \sin z + \cdots = \sin x \sin y \sin z. \tag{6.247}$$

Equation the coefficients of like terms in both sides we obtain

$$
\begin{aligned}
C_{111} &= 1, \quad n=1, m=1, r=1, \\
C_{ijk} &= 0, \quad i \neq 1, j \neq 1, k \neq 1.
\end{aligned}
\tag{6.248}
$$

To determine the constants D_{nmr}, we use the second initial condition into the derivative of (6.246) to find

$$
D_{nmr} = 0, \quad n \geqslant 1, m \geqslant 1, r \geqslant 1. \tag{6.249}
$$

Substituting the results (6.248) and (6.249) into (6.246) gives the particular solution

$$
u(x,y,z,t) = \sin x \sin y \sin z \cos(3t). \tag{6.250}
$$

Example 6. Solve the initial-boundary value problem

$$
\begin{array}{lrl}
\text{PDE} & u_{tt} &= 3(u_{xx}+u_{yy}+u_{zz}), \quad 0<x,y,z<\pi, t>0, \\
\text{BC} & u(0,y,z,t) &= u(\pi,y,z,t) = 0, \\
 & u(x,0,z,t) &= u(x,\pi,z,t) = 0, \\
 & u(x,y,0,t) &= u(x,y,\pi,t) = 0, \\
\text{IC} & u(x,y,z,0) &= 1, \quad u_t(x,y,z,0) = 0.
\end{array}
\tag{6.251}
$$

Solution.

Proceeding as before, the general solution is expressed in the form

$$
\begin{aligned}
u(x,y,z,t) = \sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} & \sin(nx)\sin(my)\sin(rz) \\
& \times \left(C_{nmr}\cos(\lambda_{nmr}t) + D_{nmr}\sin(\lambda_{nmr}t)\right).
\end{aligned}
\tag{6.252}
$$

We next use the initial condition $u(x,y,z,0) = 1$ into (6.252) to find

$$
C_{nmr} = \frac{8}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \sin(nx)\,\sin(my)\,\sin(rz)\,dx\,dy\,dz, \tag{6.253}
$$

which gives

$$
C_{nmr} = \begin{cases} 0, & \text{when } n,m \text{ or } r \text{ is even,} \\ \dfrac{64}{\pi^3 nmr}, & \text{when } n,m \text{ and } r \text{ are odd.} \end{cases} \tag{6.254}
$$

It is clear that the coefficients D_{nmr} are given by

$$
D_{nmr} = 0, \quad n \geqslant 1, m \geqslant 1, r \geqslant 1. \tag{6.255}
$$

Consequently, the particular solution is given by

$$
\begin{aligned}
u(x,y,z,t) = \frac{64}{\pi^3}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} & \frac{1}{(2n+1)(2m+1)(2r+1)}\sin(2n+1)x \\
& \times \sin(2m+1)y\sin(2r+1)z\cos(\lambda_{nmr}t),
\end{aligned}
\tag{6.256}
$$

where

$$\lambda_{nmr} = \sqrt{(2n+1)^2 + (2m+1)^2 + (2r+1)^2}. \tag{6.257}$$

Exercises 6.3.2

Use the method of separation of variables in the following initial-boundary value problems:

1. $u_{tt} = 12(u_{xx} + u_{yy} + u_{zz}),\ 0 < x,y,z < \pi$
   $u(0,y,z,t) = u(\pi,y,z,t) = 0$
   $u(x,0,z,t) = u(x,\pi,z,t) = 0$
   $u(x,y,0,t) = u(x,y,\pi,0) = 0$
   $u(x,y,z,0) = 0, u_t(x,y,z,0) = 12\sin(2x)\sin(2y)\sin(2z)$

2. $u_{tt} = 14(u_{xx} + u_{yy} + u_{zz}),\ 0 < x,y,z < \pi$
   $u(0,y,z,t) = u(\pi,y,z,t) = 0$
   $u(x,0,z,t) = u(x,\pi,z,t) = 0$
   $u(x,y,0,t) = u(x,y,\pi,0) = 0$
   $u(x,y,z,0) = \sin x\sin(2y)\sin(3z), u_t(x,y,z,0) = 0$

3. $u_{tt} = 6(u_{xx} + u_{yy} + u_{zz}),\ 0 < x,y,z < \pi$
   $u(0,y,z,t) = u(\pi,y,z,t) = 0$
   $u(x,0,z,t) = u(x,\pi,z,t) = 0$
   $u(x,y,0,t) = u(x,y,\pi,0) = 0$
   $u(x,y,z,0) = 0, u_t(x,y,z,0) = 6\sin x\sin y\sin(2z)$

4. $u_{tt} = 4(u_{xx} + u_{yy} + u_{zz}),\ 0 < x,y,z < \pi$
   $u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0$
   $u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0$
   $u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$
   $u(x,y,z,0) = 0, u_t(x,y,z,0) = 6\cos x\cos(2y)\cos(2z)$

5. $u_{tt} = 12(u_{xx} + u_{yy} + u_{zz}),\ 0 < x,y,z < \pi$
   $u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0$
   $u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0$
   $u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$
   $u(x,y,z,0) = 3, u_t(x,y,z,0) = 6\cos x\cos y\cos z$

6. $u_{tt} = 12(u_{xx} + u_{yy} + u_{zz}),\ 0 < x,y,z < \pi$
   $u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0$
   $u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0$
   $u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$
   $u(x,y,z,0) = 4 + \cos x\cos y\cos z, u_t(x,y,z,0) = 0$

7. $u_{tt} = 3(u_{xx} + u_{yy} + u_{zz}),\ 0 < x,y,z < \pi$

$u(0,y,z,t) = u(\pi,y,z,t) = 0$
$u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0$
$u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$
$u(x,y,z,0) = 0, u_t(x,y,z,0) = 3\sin x \cos y \cos z$

8. $u_{tt} = 3(u_{xx} + u_{yy} + u_{zz}), 0 < x,y,z < \pi$
$u(0,y,z,t) = u(\pi,y,z,t) = 0$
$u(x,0,z,t) = u(x,\pi,z,t) = 0$
$u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$
$u(x,y,z,0) = 0, u_t(x,y,z,0) = 3\sin x \sin y \cos z$

9. $u_{tt} = 12(u_{xx} + u_{yy} + u_{zz}), 0 < x,y,z < \pi$
$u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0$
$u(x,0,z,t) = u(x,\pi,z,t) = 0$
$u(x,y,0,t) = u(x,y,\pi,0) = 0$
$u(x,y,z,0) = \cos x \sin y \sin z, u_t(x,y,z,0) = 0$

10. $u_{tt} = 6(u_{xx} + u_{yy} + u_{zz}), 0 < x,y,z < \pi$
$u(0,y,z,t) = u(\pi,y,z,t) = 0$
$u(x,0,z,t) = u(x,\pi,z,t) = 0$
$u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$
$u(x,y,z,0) = \sin x \sin y \cos(2z), u_t(x,y,z,0) = 0$

11. $u_{tt} = u_{xx} + u_{yy} + u_{zz}, 0 < x,y,z < \pi$
$u(0,y,z,t) = u(\pi,y,z,t) = 0$
$u(x,0,z,t) = u(x,\pi,z,t) = 0$
$u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$
$u(x,y,z,0) = 0, u_t(x,y,z,0) = \sqrt{3}\sin x \sin y \cos z$

12. $u_{tt} = 6(u_{xx} + u_{yy} + u_{zz}), 0 < x,y,z < \pi$
$u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0$
$u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0$
$u_z(x,y,0,t) = u_z(x,y,\pi,0) = 0$
$u(x,y,z,0) = 0, u_t(x,y,z,0) = 6\cos x \cos y \cos 2z$

References

1. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, (1994).
2. N. Asmar, Partial Differential Equations, Prentice Hall, New Jersey, (2005).
3. J.M. Cooper, Introduction to Partial Differential Equations with MATLAB, Birkhauser, Boston, (1998).
4. S.J. Farlow, Partial Differential Equations for Scientists and Engineers, Dover, New York, (1993).
5. R. Haberman, Applied Partial Differential Equations, Pearson, New York, (2003).
6. A.M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema, Leiden, (2002).

Chapter 7
Laplace's Equation

7.1 Introduction

In Chapter 4 we have discussed the PDEs that control the heat flow in two and three dimensional spaces given by

$$\begin{aligned} u_t &= \overline{k}(u_{xx} + u_{yy}), \\ u_t &= \overline{k}(u_{xx} + u_{yy} + u_{zz}), \end{aligned} \tag{7.1}$$

respectively, where $\overline{k}$ is the thermal diffusivity. If the temperature u reaches a steady state, that is, when u does not depend on time t and depends only on the space variables, then the time derivative u_t vanishes as $t \longrightarrow \infty$. In view of this, we substitute $u_t = 0$ into (7.1), hence we obtain the Laplace's equations in two and three dimensions given by

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ u_{xx} + u_{yy} + u_{zz} &= 0. \end{aligned} \tag{7.2}$$

The Laplace's equation [2,3,5] is used to describe gravitational potential in absence of mass, to define electrostatic potential in absence of charges [4,7], and to describe temperature in a steady-state heat flow. The Laplace's equation is often called the potential equation [10] because u defines the potential function.

Recall that the heat and the wave equations investigate the evolution of temperature and displacement respectively. However, it is worth noting that Laplace's equation describes physical phenomena at equilibrium. Moreover, since the solution of the Laplace's equation does not depend on time t, initial conditions are not specified and boundary conditions at the edges of a rectangle or at the faces of a rectangular volume are specified [8]. For this reason, Laplace's equation is best described as a Boundary Value Problem (BVP).

In this chapter we will discuss the Laplace's equation in two or three dimensional spaces and in polar coordinates. The methods that will be used are the Adomian decomposition method [1], the variational iteration method [6,9], and the method of separation of variables. The three methods have been outlined in previous chapters and have been applied in heat flow and wave equations.

7.2 Adomian Decomposition Method

The Adomian decomposition method is now well known and it has been used in details in the previous five chapters. The Adomian method, as discussed before, provides the solution in terms of a rapidly convergent series. In a manner parallel to that used in the preceding chapters, we will apply the Adomian decomposition method to Laplace's equation with specified boundary conditions.

7.2.1 Two Dimensional Laplace's Equation

The two dimensional Laplace's equation will be discussed using all types of boundary conditions. As stated above, initial conditions are irrelevant. The boundary conditions associated with Laplace's equation can be identified into three types of boundary conditions, namely:

1. Dirichlet boundary conditions:

In this type, the solution $u(x,y)$ of Laplace's equation is specified on the boundary. The Laplace's equation in this case is known as Dirichlet problem for a rectangle.

2. Neumann boundary conditions:

In this type, the normal derivative u_n is specified on the boundary. The Laplace's equation in this case is known as Neumann problem [7,8].

3. Robin boundary conditions:

In this type, the function u is specified on parts of the boundary and the directional derivative u_n is specified on other parts of the boundary.

Without loss of generality, we consider the two dimensional Laplace's equation is given by the boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{xx}+u_{yy}=0, & 0<x<a, 0<y<b, \\ \text{BC} & u(0,y)=0, & u(a,y)=f(y), \\ & u(x,0)=0, & u(x,b)=0, \end{array} \tag{7.3}$$

where $u=u(x,y)$ is the solution of the Laplace's equation at any point located at the position (x,y) of a rectangular plate.

We first write (7.3) in an operator form by

$$L_y u(x,y) = -L_x u(x,y), \tag{7.4}$$

where the differential operators L_x and L_y are defined by

$$L_x = \frac{\partial^2}{\partial x^2}, \quad L_y = \frac{\partial^2}{\partial y^2}, \tag{7.5}$$

so that the inverse operators L_x^{-1} and L_y^{-1} are two-fold integral operators defined by

$$\begin{aligned} L_x^{-1}(\cdot) &= \int_0^x \int_0^x (\cdot)\,dx\,dx, \\ L_y^{-1}(\cdot) &= \int_0^y \int_0^y (\cdot)\,dy\,dy. \end{aligned} \tag{7.6}$$

Applying the inverse operator L_y^{-1} to both sides of (7.4) and using the boundary conditions we obtain

$$u(x,y) = yg(x) - L_y^{-1}L_x u(x,y), \tag{7.7}$$

where

$$g(x) = u_y(x,0), \tag{7.8}$$

a boundary condition that is not given but will be determined.

Using the decomposition series

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y), \tag{7.9}$$

into both sides of (7.7) gives

$$\sum_{n=0}^{\infty} u_n(x,y) = yg(x) - L_y^{-1}L_x\left(\sum_{n=0}^{\infty} u_n(x,y)\right). \tag{7.10}$$

Adomian's analysis admits the use of the recursive relation

$$\begin{aligned} u_0(x,y) &= yg(x), \\ u_{k+1}(x,y) &= -L_y^{-1}L_x(u_k), \quad k \geqslant 0. \end{aligned} \tag{7.11}$$

This leads to

$$\begin{aligned} u_0(x,y) &= yg(x), \\ u_1(x,y) &= -L_y^{-1}L_x(u_0) = -\frac{1}{3!}y^3 g''(x) \\ u_2(x,y) &= -L_y^{-1}L_x(u_1) = \frac{1}{5!}y^5 g^{(4)}(x), \end{aligned} \tag{7.12}$$

and so on. We can determine as many components as we like to enhance the accuracy level.

In view of (7.12), we can write

$$u(x,y) = yg(x) - \frac{1}{3!}y^3 g''(x) + \frac{1}{5!}y^5 g^{(4)}(x) - \cdots. \tag{7.13}$$

To complete the determination of the series solution of $u(x,y)$, we should determine $g(x)$. This can be easily done by using the inhomogeneous boundary condition $u(a,y) = f(y)$. Substituting $x = a$ into (7.13), using the Taylor expansion for $f(y)$, and equating the coefficients of like terms in both sides leads to the complete determination of $g(x)$.

Having determined the function $g(x)$, the series solution (7.9) of $u(x,y)$ is thus established.

To give a clear overview of the use of the decomposition method in Laplace's equation, we discuss below the following illustrative boundary value problems.

Example 1. Use the Adomian decomposition method to solve the boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{xx}+u_{yy}=0, & 0<x,y<\pi, \\ \text{BC} & u(0,y)=0, & u(\pi,y)=\sinh\pi\sin y, \\ & u(x,0)=0, & u(x,\pi)=0. \end{array} \tag{7.14}$$

Solution.

Applying the inverse operator L_y^{-1} to the operator form of (7.14), and using the proper boundary conditions we find

$$u(x,y)=yg(x)-L_y^{-1}L_xu(x,y), \tag{7.15}$$

where

$$g(x)=u_y(x,0). \tag{7.16}$$

The decomposition method defines the solution $u(x,y)$ by an infinite series given by

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y). \tag{7.17}$$

Substituting (7.17) into both sides of (7.15) gives

$$\sum_{n=0}^{\infty}u_n(x,y)=yg(x)-L_y^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n(x,y)\right)\right). \tag{7.18}$$

This gives the recursive relation

$$\begin{aligned} u_0(x,y)&=yg(x),\\ u_{k+1}(x,y)&=-L_y^{-1}L_x(u_k(x,y)), \quad k\geqslant 0, \end{aligned} \tag{7.19}$$

that gives the first few components

$$\begin{aligned} u_0(x,y)&=yg(x),\\ u_1(x,y)&=-L_y^{-1}L_x(u_0(x,y))=-\frac{1}{3!}y^3g''(x),\\ u_2(x,y)&=-L_y^{-1}L_x(u_1(x,y))=\frac{1}{5!}y^5g^{(4)}(x),\\ u_3(x,y)&=-L_y^{-1}L_x(u_2(x,y))=-\frac{1}{7!}y^7g^{(6)}(x). \end{aligned} \tag{7.20}$$

Combining the above results obtained for the components yields

$$u(x,y)=yg(x)-\frac{1}{3!}y^3g''(x)+\frac{1}{5!}y^5g^{(4)}(x)-\frac{1}{7!}y^7g^{(6)}(x)+\cdots. \tag{7.21}$$

To determine the function $g(x)$, we use the inhomogeneous boundary condition $u(\pi,y)=\sinh\pi\sin y$, and by using the Taylor expansion of $\sin y$ we obtain

$$\begin{aligned} &yg(\pi)-\frac{1}{3!}y^3g''(\pi)+\frac{1}{5!}y^5g^{(4)}(\pi)-\frac{1}{7!}y^7g^{(6)}(\pi)+\cdots \\ &=\sinh\pi(y-\frac{1}{3!}y^3+\frac{1}{5!}y^5-\frac{1}{7!}y^7+\cdots). \end{aligned} \tag{7.22}$$

Equating the coefficients of like terms on both sides gives

$$g(\pi)=g''(\pi)=g^{(4)}(\pi)=\cdots=\sinh\pi. \tag{7.23}$$

This means that

$$g(x)=\sinh x, \tag{7.24}$$

the only function that when substituted in (7.21) will also satisfy the remaining boundary conditions. Consequently, the solution in a series form is given by

$$u(x,y)=\sinh x\left(y-\frac{1}{3!}y^3+\frac{1}{5!}y^5-\frac{1}{7!}y^7+\cdots\right), \tag{7.25}$$

and in a closed form by

$$u(x,y)=\sinh x\sin y, \tag{7.26}$$

obtained by using the Taylor expansion for $\sin y$.

Example 2. Use the Adomian decomposition method to solve the boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{xx}+u_{yy}=0, & 0<x,y<\pi, \\ \text{BC} & u(0,y)=0, & u(\pi,y)=0, \\ & u(x,0)=0, & u(x,\pi)=\sin x\sinh\pi. \end{array} \tag{7.27}$$

Solution.

We first rewrite (7.27) in an operator form by

$$L_xu(x,y)=-L_yu(x,y). \tag{7.28}$$

Applying the inverse operator L_x^{-1} to both sides of (7.28), and using the proper boundary conditions we find

$$u(x,y)=xh(y)-L_x^{-1}L_yu(x,y), \tag{7.29}$$

where

$$h(y)=u_x(0,y). \tag{7.30}$$

Using the series representation of $u(x,y)$ gives

$$\sum_{n=0}^{\infty} u_n(x,y) = xh(y) - L_x^{-1}\left(L_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right), \tag{7.31}$$

that admits the use of the recursive relation

$$\begin{aligned} u_0(x,y) &= xh(y), \\ u_{k+1}(x,y) &= -L_x^{-1}L_y(u_k(x,y)), \quad k \geqslant 0, \end{aligned} \tag{7.32}$$

that in turn gives

$$\begin{aligned} u_0(x,y) &= xh(y), \\ u_1(x,y) &= -L_x^{-1}L_y(u_0(x,y)) = -\frac{1}{3!}x^3h''(y), \\ u_2(x,y) &= -L_x^{-1}L_y(u_1(x,y)) = \frac{1}{5!}x^5h^{(4)}(y). \end{aligned} \tag{7.33}$$

Using the above results obtained for the components gives

$$u(x,y) = xh(y) - \frac{1}{3!}x^3h''(y) + \frac{1}{5!}x^5h^{(4)}(y) - \frac{1}{7!}x^7h^{(6)}(y) + \cdots. \tag{7.34}$$

The function $h(y)$ should be determined to complete the determination of the series solution. Using the boundary condition $u(x,\pi) = \sinh\pi\sin x$, and using the Taylor expansion of $\sin x$ we obtain

$$\begin{aligned} & xh(\pi) - \frac{1}{3!}x^3h''(\pi) + \frac{1}{5!}x^5h^{(4)}(\pi) - \frac{1}{7!}x^7h^{(6)}(\pi) + \cdots \\ &= \sinh\pi(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots). \end{aligned} \tag{7.35}$$

Equating the coefficients of like terms on both sides gives

$$h(\pi) = h''(\pi) = h^{(4)}(\pi) = \cdots = \sinh\pi. \tag{7.36}$$

This means, considering the remaining boundary condition, that

$$h(y) = \sinh y. \tag{7.37}$$

Combining the results obtained above in (7.34) and (7.36), the solution in a series form is given by

$$u(x,y) = \sinh y\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right), \tag{7.38}$$

and in a closed form by

$$u(x,y) = \sin x \sinh y. \tag{7.39}$$

Example 3. Use the Adomian decomposition method to solve the boundary value problem

$$
\begin{array}{lrl}
\text{PDE} & u_{xx}+u_{yy}=0, & 0<x,y<\pi, \\
\text{BC} & u_x(0,y)=0, & u_x(\pi,y)=0, \\
 & u_y(x,0)=\cos x, & u_y(x,\pi)=\cosh\pi\cos x.
\end{array}
\tag{7.40}
$$

Solution.

We point out that the boundary conditions are the Neumann boundary conditions where the directional derivatives are specified. The decomposition method can be applied in a direct way. Applying the inverse operator L_y^{-1} to both sides of the operator form of (7.40) we find

$$u(x,y)=g(x)+y\cos x-L_y^{-1}L_x u(x,y), \tag{7.41}$$

where

$$g(x)=u(x,0). \tag{7.42}$$

This in turn gives

$$\sum_{n=0}^{\infty}u_n(x,y)=g(x)+y\cos x-L_y^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n(x,y)\right)\right), \tag{7.43}$$

so that the recursive relation is given by

$$
\begin{aligned}
u_0(x,y) &= g(x)+y\cos x,\\
u_{k+1}(x,y) &= -L_y^{-1}L_x(u_k(x,y)), \quad k\geqslant 0.
\end{aligned}
\tag{7.44}
$$

It then follows that

$$
\begin{aligned}
u_0(x,y) &= g(x)+y\cos x,\\
u_1(x,y) &= -L_y^{-1}L_x(u_0(x,y))=-\frac{1}{2!}y^2g''(x)+\frac{1}{3!}y^3\cos x,\\
u_2(x,y) &= -L_y^{-1}L_x(u_1(x,y))=\frac{1}{4!}y^4g^{(4)}(x)+\frac{1}{5!}y^5\cos x,
\end{aligned}
\tag{7.45}
$$

and so on. This gives

$$
\begin{aligned}
u(x,y) &= \cos x\left(y+\frac{1}{3!}y^3+\frac{1}{5!}y^5+\cdots\right)\\
&\qquad +g(x)-\frac{1}{2!}y^2g''(x)+\frac{1}{4!}y^4g^{(4)}(x)+\cdots,
\end{aligned}
\tag{7.46}
$$

or equivalently

$$u(x,y)=\cos x\sinh y+g(x)-\frac{1}{2!}y^2g''(x)+\frac{1}{4!}y^4g^{(4)}(x)-\cdots. \tag{7.47}$$

To determine $g(x)$, we use the boundary condition $u_y(x,\pi)=\cosh\pi\cos x$ to obtain

$$\cosh\pi\cos x+\left(-\pi g''(x)+\frac{1}{3!}\pi^3 g^{(4)}(x)+\cdots\right)=\cosh\pi\cos x. \tag{7.48}$$

Equating the coefficients of like terms on both sides and following the discussion presented in Ex. 2, we find

$$g(x)=0. \tag{7.49}$$

Consequently, we obtain

$$u(x,y)=\cos x\sinh y. \tag{7.50}$$

It is important to note that Neumann problem has a property that the solution is determined up to an additive constant. An arbitrary constant C_0 cannot be determined by the decomposition method and by the classic method of separation of variables as will be seen later. Based on this, the solution should be given by

$$u(x,y)=C_0+\cos x\sinh y, \tag{7.51}$$

that satisfies the equation and the boundary conditions.

Example 4. Use the Adomian decomposition method to solve the boundary value problem

$$\begin{array}{lll} \text{PDE} & u_{xx}+u_{yy}=0, & 0<x,y<\pi, \\ \text{BC} & u_x(0,y)=0, & u_x(\pi,y)=0, \\ & u(x,0)=\cos x, & u(x,\pi)=\cosh\pi\cos x. \end{array} \tag{7.52}$$

Solution.

We point out that the boundary conditions are the mixed boundary conditions where the solution $u(x,y)$ is specified at two edges and the derivatives $u_x(x,y)$ are specified at the remaining two edges.

Applying the inverse operator L_y^{-1} to both sides of the operator form of (7.52), and using the proper boundary conditions we find

$$u(x,y)=\cos x+yg(x)-L_y^{-1}L_xu(x,y), \tag{7.53}$$

where

$$g(x)=u_y(x,0). \tag{7.54}$$

Using the series representation of $u(x,y)$ gives

$$\sum_{n=0}^{\infty}u_n(x,y)=\cos x+yg(x)-L_y^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n(x,y)\right)\right). \tag{7.55}$$

Following the analysis presented above we find

$$u_0(x,y)=\cos x+yg(x),$$

$$\begin{aligned}u_1(x,y) &= -L_y^{-1}L_x(u_0(x,y)) = \frac{1}{2!}y^2\cos x - \frac{1}{3!}y^3 g''(x),\\ u_2(x,y) &= -L_y^{-1}L_x(u_1(x,y)) = \frac{1}{4!}y^4\cos x + \frac{1}{5!}y^5 g^{(4)}(x).\end{aligned} \tag{7.56}$$

This gives

$$\begin{aligned}u(x,y) = \cos x\left(1+\frac{1}{2!}y^2+\frac{1}{4!}y^4+\cdots\right)\\ +yg(x)-\frac{1}{3!}y^3g''(x)+\frac{1}{5!}y^5g^{(4)}(x)-\cdots,\end{aligned} \tag{7.57}$$

or equivalently

$$u(x,y) = \cos x\cosh y + yg(x) - \frac{1}{3!}y^3g''(x)+\frac{1}{5!}y^5g^{(4)}(x)-\cdots. \tag{7.58}$$

To determine $g(x)$, we use the boundary condition $u(x,\pi)=\cosh\pi\cos x$ to find

$$g(x)=0. \tag{7.59}$$

The solution in a closed form is given by

$$u(x,y)=\cos x\cosh y, \tag{7.60}$$

obtained by substituting $g(x)=0$ into (7.58).

Exercises 7.2.1

Use the decomposition method to solve the following Laplace's equations:

1. $u_{xx}+u_{yy}=0,\ 0<x,y<\pi$
   $u(0,y)=0, u(\pi,y)=\sinh\pi\cos y$
   $u(x,0)=\sinh x, u(x,\pi)=-\sinh x$

2. $u_{xx}+u_{yy}=0,\ 0<x,y<\pi$
   $u(0,y)=\sin y, u(\pi,y)=\cosh\pi\sin y$
   $u(x,0)=0, u(x,\pi)=0$

3. $u_{xx}+u_{yy}=0,\ 0<x,y<\pi$
   $u(0,y)=\cos y, u(\pi,y)=\cosh\pi\cos y$
   $u(x,0)=\cosh x, u(x,\pi)=-\cosh x$

4. $u_{xx}+u_{yy}=0,\ 0<x,y<\pi$
   $u(0,y)=0, u(\pi,y)=0$
   $u(x,0)=0, u(x,\pi)=\sinh(2\pi)\sin(2x)$

5. $u_{xx}+u_{yy}=0,\, 0<x,y<\pi$
$u(0,y)=0, u(\pi,y)=0$
$u(x,0)=\sin(2x), u(x,\pi)=\cosh(2\pi)\sin(2x)$

6. $u_{xx}+u_{yy}=0,\, 0<x,y<\pi$
$u(0,y)=\cosh(3y), u(\pi,y)=-\cosh(3y)$
$u(x,0)=\cos(3x), u(x,\pi)=\cosh(3\pi)\cos(3x)$

7. $u_{xx}+u_{yy}=0,\, 0<x,y<\pi$
$u(0,y)=0, u(\pi,y)=\sinh(2\pi)\cos(2y)$
$u(x,0)=\sinh(2x), u(x,\pi)=\sinh(2x)$

8. $u_{xx}+u_{yy}=0,\, 0<x,y<\pi$
$u(0,y)=\cos(2y), u(\pi,y)=\cosh(2\pi)\cos(2y)$
$u(x,0)=\cosh(2x), u(x,\pi)=\cosh(2x)$

9. $u_{xx}+u_{yy}=0,\, 0<x,y<\pi$
$u_x(0,y)=0, u_x(\pi,y)=0$
$u_y(x,0)=0, u_y(x,\pi)=\sinh\pi\cos x$

10. $u_{xx}+u_{yy}=0,\, 0<x,y<\pi$
$u_x(0,y)=\cosh y, u_x(\pi,y)=-\cosh y$
$u_y(x,0)=0, u_y(x,\pi)=\sinh\pi\sin x$

11. $u_{xx}+u_{yy}=0,\, 0<x,y<\pi$
$u_x(0,y)=\cos y, u_x(\pi,y)=\cosh\pi\cos y$
$u(x,0)=\sinh x, u(x,\pi)=-\sinh x$

12. $u_{xx}+u_{yy}=0,\, 0<x,y<\pi$
$u_x(0,y)=\cosh y, u_x(\pi,y)=-\cosh y$
$u(x,0)=\sin x, u(x,\pi)=\cosh\pi\sin x$

13. $u_{xx}+u_{yy}=0,\, 0<x,y<\pi$
$u(0,y)=0, u(\pi,y)=\pi$
$u(x,0)=x, u(x,\pi)=x+\sinh\pi\sin x$

14. $u_{xx}+u_{yy}=0,\, 0<x,y<\pi$
$u(0,y)=y, u(\pi,y)=y$
$u(x,0)=\sin x, u(x,\pi)=\pi+\cosh\pi\sin x$

15. $u_{xx}+u_{yy}=0,\, 0<x,y<\pi$
$u(0,y)=1, u(\pi,y)=1$
$u(x,0)=1, u(x,\pi)=1+\sinh\pi\sin x$

16. $u_{xx}+u_{yy}=0,\, 0<x,y<\pi$
$u(0,y)=1+\sinh y, u(\pi,y)=1-\sinh y$
$u(x,0)=1, u(x,\pi)=1+\sinh\pi\cos x$

7.3 The Variational Iteration Method

As stated before, the variational iteration method (VIM) gives rapidly convergent successive approximations of the exact solution if an exact solution exists. Moreover, the method provides an approximation of high accuracy level by using only few iterations [6]. The variational iteration method uses the correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\right) d\xi, \quad n \geqslant 0. \tag{7.61}$$

for the differential equation

$$Lu + Nu = g(x,t). \tag{7.62}$$

The Lagrange multiplier $\lambda(\xi)$ should be determined first, and therefore, the successive approximations $u_{n+1}(x,t), n \geqslant 0$, of the solution $u(x,t)$ follow immediately by using any selective function $u_0(x,t)$. The exact solution may be obtained by using

$$u = \lim_{n\to\infty} u_n. \tag{7.63}$$

Recall that

$$\int \lambda(\xi)u_n''(\xi)d\xi = \lambda(\xi)u_n'(\xi) - \lambda'(\xi)u_n(\xi) + \int \lambda''(\xi)u_n(\xi)d\xi. \tag{7.64}$$

To give a clear overview of the use of the variational iteration method in Laplace's equation, we discuss the same examples investigated before by using Adomian method.

Example 1. Use the variational iteration method to solve the boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{xx} + u_{yy} = 0, & 0 < x,y < \pi, \\ \text{BC} & u(0,y) = 0, & u(\pi,y) = \sinh\pi\sin y, \\ & u(x,0) = 0, & u(x,\pi) = 0. \end{array} \tag{7.65}$$

Solution.

The correction functional for this equation reads

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \lambda(\xi)\left(\frac{\partial^2 u_n(\xi,y)}{\partial \xi^2} + \frac{\partial^2 \tilde{u}_n(\xi,y)}{\partial y^2}\right) d\xi. \tag{7.66}$$

This yields the stationary conditions

$$\begin{aligned} 1 - \lambda'|_{\xi=x} &= 0, \\ \lambda|_{\xi=x} &= 0, \\ \lambda''|_{\xi=x} &= 0. \end{aligned} \tag{7.67}$$

This in turn gives

$$\lambda = \xi - x. \tag{7.68}$$

Substituting this value of the Lagrange multiplier into the functional (7.66) gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x (\xi - x)\left(\frac{\partial^2 u_n(\xi,y)}{\partial \xi^2} + \frac{\partial^2 u_n(\xi,y)}{\partial y^2}\right) d\xi. \tag{7.69}$$

Considering the given boundary conditions, it is clear that the solution contains $\sin y$ in addition to other functions that depend on x. Therefore, we can select $u_0(x,t) = x\sin y$. Using this selection into (7.69) we obtain the following successive approximations

$$\begin{aligned}
u_0(x,y) &= x\sin y,\\
u_1(x,y) &= x\sin y + \frac{1}{3!}x^3 \sin y,\\
u_2(x,y) &= x\sin y + \frac{1}{3!}x^3 \sin y + \frac{1}{5!}x^5 \sin y,\\
u_3(x,y) &= x\sin y + \frac{1}{3!}x^3 \sin y + \frac{1}{5!}x^5 \sin y + \frac{1}{7!}x^7 \sin y,\\
&\vdots\\
u_n(x,y) &= \sin y\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots\right).
\end{aligned} \tag{7.70}$$

Recall that

$$u = \lim_{n\to\infty} u_n, \tag{7.71}$$

that gives the exact solution by

$$u(x,y) = \sinh x \sin y, \tag{7.72}$$

obtained upon using the Taylor expansion for $\sinh x$.

Example 2. Use the variational iteration method to solve the boundary value problem

$$\begin{array}{lrl}
\text{PDE} & u_{xx} + u_{yy} = 0, & 0 < x,y < \pi,\\
\text{BC} & u(0,y) = 0, & u(\pi,y) = 0,\\
 & u(x,0) = 0, & u(x,\pi) = \sin x \sinh \pi.
\end{array} \tag{7.73}$$

Solution.

Because the boundary conditions at $y = 0, \pi$ include $\sin x$ and 0, it is normal to consider $u_0(x,t) = y\sin x$. Based on this, the correction functional for this equation will be

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda(\xi)\left(\frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} + \frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}\right) d\xi. \tag{7.74}$$

This again gives

$$\lambda = \xi - y. \tag{7.75}$$

Substituting this value of the Lagrange multiplier into the functional (7.74) gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y (\xi - y)\left(\frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}\right) d\xi. \tag{7.76}$$

Selecting $u_0(x,t) = y\sin x$ gives the following successive approximations

$$\begin{aligned}
u_0(x,y) &= y\sin x,\\
u_1(x,y) &= y\sin x + \frac{1}{3!}y^3\sin x,\\
u_2(x,y) &= y\sin x + \frac{1}{3!}y^3\sin x + \frac{1}{5!}y^5\sin x,\\
u_3(x,y) &= y\sin x + \frac{1}{3!}y^3\sin x + \frac{1}{5!}y^5\sin x + \frac{1}{7!}y^7\sin x,\\
&\vdots\\
u_n(x,y) &= \sin x\left(y + \frac{1}{3!}y^3 + \frac{1}{5!}y^5 + \frac{1}{7!}y^7 + \cdots\right),
\end{aligned} \tag{7.77}$$

that gives the exact solution by

$$u(x,y) = \sin x \sinh y, \tag{7.78}$$

obtained upon using the Taylor expansion for $\sinh y$.

Example 3. Use the variational iteration method to solve the boundary value problem

$$\begin{array}{lll}
\text{PDE} & u_{xx} + u_{yy} = 0, & 0 < x,y < \pi,\\
\text{BC} & u_x(0,y) = 0, & u_x(\pi,y) = 0,\\
 & u_y(x,0) = \cos x, & u_y(x,\pi) = \cosh\pi\cos x.
\end{array} \tag{7.79}$$

Solution.

As stated before, the boundary conditions are the Neumann boundary conditions where the directional derivatives are specified. Because the boundary conditions include $\cos x$, it is normal to consider $u_0(x,t) = y\cos x$. Proceeding as in Example 2 we find

$$\lambda = \xi - y. \tag{7.80}$$

Using this value of λ gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y (\xi - y)\left(\frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}\right) d\xi. \tag{7.81}$$

Selecting $u_0(x,t) = y\cos x$ gives the following successive approximations

$$\begin{aligned}
u_0(x,y) &= y\cos x,\\
u_1(x,y) &= y\cos x + \frac{1}{3!}y^3\cos x,
\end{aligned}$$

$$\begin{aligned}
u_2(x,y) &= y\cos x + \frac{1}{3!}y^3\cos x + \frac{1}{5!}y^5\cos x,\\
u_3(x,y) &= y\cos x + \frac{1}{3!}y^3\cos x + \frac{1}{5!}y^5\cos x + \frac{1}{7!}y^7\cos x,\\
&\vdots\\
u_n(x,y) &= \cos x\left(y + \frac{1}{3!}y^3 + \frac{1}{5!}y^5 + \frac{1}{7!}y^7 + \cdots\right),
\end{aligned} \tag{7.82}$$

that gives the exact solution by

$$u(x,y) = \cos x \sinh y, \tag{7.83}$$

obtained upon using the Taylor expansion for $\sinh y$.

It is important to note that Neumann problem has a property that the solution is determined up to an additive constant. An arbitrary constant C_0 cannot be determined by using this method and even by the classic method of the separation of variables as will be seen later. Based on this, the solution should be given by

$$u(x,y) = C_0 + \cos x \sinh y, \tag{7.84}$$

that satisfies the equation and the boundary conditions.

Example 4. Use the variational iteration method to solve the boundary value problem

$$\begin{aligned}
&\text{PDE} \quad & u_{xx} + u_{yy} &= 0, \quad 0 < x,y < \pi,\\
&\text{BC} & u_x(0,y) &= 0, \quad u_x(\pi,y) = 0,\\
& & u(x,0) &= \cos x, \quad u(x,\pi) = \cosh\pi\cos x.
\end{aligned} \tag{7.85}$$

Solution.

The Robin boundary conditions are mixed boundary conditions where the solution $u(x,y)$ is specified at two edges and the derivatives $u_x(x,y)$ are specified at the remaining two edges. Moreover, because the boundary conditions include $\cos x$ at $u(x,0)$ and at $u(x,\pi)$, it is normal to consider $u_0(x,t) = \cos x + \frac{1}{2}y^2\cos x$. Using $\lambda = \xi - y$, obtain the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y (\xi - y)\left(\frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}\right) d\xi. \tag{7.86}$$

Selecting $u_0(x,t) = \cos x(1 + \frac{1}{2!}y^2)$ gives the following successive approximations

$$\begin{aligned}
u_0(x,y) &= \cos x(1 + \frac{1}{2!}y^2),\\
u_1(x,y) &= \cos x(1 + \frac{1}{2!}y^2) + \cos x(\frac{1}{4!}y^4 + \frac{1}{6!}y^6),
\end{aligned}$$

$$
\begin{aligned}
u_2(x,y) &= \cos x(1+\frac{1}{2!}y^2)+\cos x(\frac{1}{4!}y^4+\frac{1}{6!}y^6)+\cos x(\frac{1}{8!}y^8+\frac{1}{10!}y^{10}),\\
&\ \vdots \\
u_n(x,y) &= \cos x(1+\frac{1}{2!}y^2+\frac{1}{4!}y^4+\frac{1}{6!}y^6+\frac{1}{8!}y^8+\frac{1}{10!}y^{10}+\cdots),
\end{aligned} \tag{7.87}
$$

The exact solution is given by

$$u(x,y) = \cos x \cosh y. \tag{7.88}$$

Exercises 7.3

Use the variational iteration method to solve the following Laplace's equations in Exercises 7.2.1.

7.4 Method of Separation of Variables

In this section, the method of separation of variables will be used to solve the Laplace's equation in two and three dimensional spaces and in a circular disc. Recall that the method reduces the partial differential equation into a system of ordinary differential equations, where each ordinary differential equation depends on one variable only. We then solve each ordinary differential equation independently. The homogeneous boundary conditions are used to evaluate the constants of integration. The superposition principle will be used to establish a general solution. The remaining inhomogeneous boundary condition will be employed to determine the particular solution that will satisfy the equation and the boundary conditions.

Because the solution of the Laplace's equation does not depend on time, initial conditions are irrelevant and only boundary conditions are specified at the edges of a rectangle or faces of a rectangular volume [8].

The complete details of the method of separation of variables have been outlined before, therefore we will focus our discussion on the implementation of the method.

7.4.1 Laplace's Equation in Two Dimensions

The two dimensional Laplace's equation is governed by the following boundary value problem

$$
\begin{array}{lrl}
\text{PDE} & u_{xx}+u_{yy}=0, & 0<x<a, 0<y<b,\\
\text{BC} & u(0,y)=0, & u(a,y)=0,\\
 & u(x,0)=0, & u(x,b)=g(x),
\end{array} \tag{7.89}
$$

where $u = u(x,y)$ is the solution of the Laplace's equation at any point located at the position (x,y) of a rectangle.

As indicated before, the method of separation of variables suggests that the solution $u(x,y)$ can be assumed as the product of distinct functions $F(x)$ and $G(y)$ such that each function depends on one variable only. This means that we can set

$$u(x,y) = F(x)G(y). \tag{7.90}$$

Differentiating both sides of (7.90) twice with respect to x and y respectively and substituting in the PDE of (7.89) we find

$$F''(x)G(y) + F(x)G''(y) = 0. \tag{7.91}$$

Dividing both sides by $F(x)G(y)$ gives

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}. \tag{7.92}$$

It is obvious that the left hand side depends only on the variable x and the right hand side depends only on the variable y. The equality holds only if both sides are equal to the same constant. Accordingly, we set

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -\lambda^2. \tag{7.93}$$

In view of (7.93) we obtain the two second order ordinary differential equations

$$F''(x) + \lambda^2 F(x) = 0, \tag{7.94}$$

and

$$G''(y) - \lambda^2 G(y) = 0. \tag{7.95}$$

The second order differential equations (7.94) and (7.95) give the solutions

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{7.96}$$

and

$$G(y) = \alpha\cosh(\lambda y) + \beta\sinh(\lambda y), \tag{7.97}$$

where A, B, α and β are constants. To achieve this goal, we use (7.90) and the boundary conditions $0 = u(0,y) = F(0)$ and $0 = u(a,y) = F(a)$ into (7.96) gives

$$A = 0, \tag{7.98}$$

and

$$\lambda_n = \frac{n\pi}{a}, \quad n = 1,2,3,\cdots. \tag{7.99}$$

It is obvious that $n = 0$ and $B = 0$ are excluded because each will give the trivial solution. We therefore conclude that

$$F_n(x) = \sin(\frac{n\pi}{a}x), \quad n = 1,2,3,\cdots. \tag{7.100}$$

Using (7.90) and the boundary condition $0 = u(x,0) = G(0)$ in (7.97) yields

$$\alpha = 0, \tag{7.101}$$

and hence

$$G_n(y) = \sinh(\frac{n\pi}{a}y), \quad n = 1,2,3,\cdots. \tag{7.102}$$

Combining (7.100) and (7.102) we obtain the fundamental solutions

$$u_n(x,y) = \sin(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}y), \quad n = 1,2,3,\cdots, \tag{7.103}$$

that satisfy the partial differential equation in (7.89) and the three homogeneous boundary conditions for each value of n.

Using the superposition principle we obtain

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}y), \tag{7.104}$$

where the constants C_n are as yet undetermined. To determine C_n, we use the inhomogeneous boundary condition to find

$$\sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}b) = g(x). \tag{7.105}$$

The constants C_n are then determined by using Fourier series to find

$$C_n \sinh(\frac{n\pi}{a}b) = \frac{2}{a}\int_0^a \sin(\frac{n\pi}{a}x)g(x)\,dx. \tag{7.106}$$

Consequently, the solution of the Laplace's equation is given by (7.104) with C_n defined by (7.106).

The method of separation of variables will be illustrated by discussing the following examples.

Example 1. Use the method of separation of variables to solve the boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{xx} + u_{yy} = 0, & 0 < x,y < \pi, \\ \text{BC} & u(0,y) = 0, & u(\pi,y) = 0, \\ & u(x,0) = 0, & u(x,\pi) = \sinh\pi\sin x. \end{array} \tag{7.107}$$

Solution.

The method of separation of variables suggests

$$u(x,y) = F(x)G(y), \tag{7.108}$$

that gives the second order ordinary differential equations

$$F''(x) + \lambda^2 F(x) = 0, \tag{7.109}$$

and

$$G''(y) - \lambda^2 G(y) = 0. \tag{7.110}$$

The second order differential equations (7.109) and (7.110) give the solutions

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{7.111}$$

and

$$G(y) = \alpha\cosh(\lambda y) + \beta\sinh(\lambda y), \tag{7.112}$$

where A, B, α and β are constants. To determine these constants, we use (7.108) and the boundary conditions $0 = u(0,y) = F(0)$, and $0 = u(\pi,y) = F(\pi)$ into (7.111) gives

$$A = 0, \tag{7.113}$$

and

$$\lambda_n = n, \quad n = 1,2,3,\cdots. \tag{7.114}$$

This gives

$$F_n(x) = \sin(nx), \quad n = 1,2,3,\cdots. \tag{7.115}$$

Using (7.108) and the boundary condition $0 = u(x,0) = G(0)$ into (7.112) yields

$$\alpha = 0, \tag{7.116}$$

and hence

$$G_n(y) = \sinh(ny), \quad n = 1,2,3,\cdots. \tag{7.117}$$

Combining (7.115) and (7.117) we obtain the fundamental solutions

$$u_n(x,y) = \sin(nx)\sinh(ny), \quad n = 1,2,3,\cdots, \tag{7.118}$$

that satisfy the partial differential equation and the three homogeneous boundary conditions for each value of n.

Using the superposition principle, the solution $u(x,y)$ can be written in the form

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin(nx)\sinh(ny), \tag{7.119}$$

where the constants C_n are as yet undetermined. To determine C_n, we use the inhomogeneous boundary condition $u(x,\pi) = \sinh\pi\sin y$ to find

$$\sum_{n=1}^{\infty} C_n \sinh(n\pi)\sin(nx) = \sinh\pi\sin x. \tag{7.120}$$

The constants C_n are then determined by using Fourier series or by expanding the series and equating coefficients of like terms, hence we obtain

$$C_1 = 1, \quad C_k = 0, \ k \neq 1. \tag{7.121}$$

Consequently, the solution of the Laplace's equation is given by

$$u(x,y) = \sin x \sinh y. \tag{7.122}$$

Example 2. Use the method of separation of variables to solve the boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{xx} + u_{yy} = 0, & 0 < x,y < \pi, \\ \text{BC} & u(0,y) = 0, & u(\pi,y) = 0, \\ & u(x,0) = 0, & u(x,\pi) = 1. \end{array} \tag{7.123}$$

Solution.

We first set

$$u(x,y) = F(x)G(y). \tag{7.124}$$

Substituting (7.124) into (7.123) and proceeding as before we find

$$F_n(x) = \sin(nx), \quad n = 1,2,3,\cdots, \tag{7.125}$$

and

$$G_n(y) = \sinh(ny), \quad n = 1,2,3,\cdots. \tag{7.126}$$

Accordingly, we obtain the the fundamental solutions

$$u_n(x,y) = \sin(nx)\sinh(ny), \quad n = 1,2,3,\cdots. \tag{7.127}$$

Using the superposition principle we obtain

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin(nx)\sinh(ny), \tag{7.128}$$

where $C_n, n \geqslant 1$ are constants. To determine C_n, we use the inhomogeneous boundary condition $u(x,\pi) = 1$ in (7.128) to find

$$\sum_{n=1}^{\infty} C_n \sin(nx)\sinh(n\pi) = 1. \tag{7.129}$$

The constants C_n can be determined by using the Fourier series, hence we find

$$C_n \sinh(n\pi) = \frac{2}{\pi}\int_0^{\pi} \sin(nx)dx, \tag{7.130}$$

so that

$$C_n \sinh(n\pi) = \begin{cases} \dfrac{4}{n\pi}, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}. \end{cases} \tag{7.131}$$

Consequently, the solution is given by

$$u(x,y) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin((2m+1)x)\sinh((2m+1)y)}{(2m+1)\sinh((2m+1)\pi)}. \tag{7.132}$$

The boundary condition $u(x,\pi) = 1$ is justified by using Appendix F.

Example 3. Use the method of separation of variables to solve the boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{xx} + u_{yy} = 0, & 0 < x,y < \pi, \\ \text{BC} & u(0,y) = 0, & u(\pi,y) = 0, \\ & u(x,0) = \sinh\pi \sin x, & u(x,\pi) = 0. \end{array} \tag{7.133}$$

Solution.

We first set

$$u(x,y) = F(x)G(y). \tag{7.134}$$

Proceeding as before we find

$$F''(x) + \lambda^2 F(x) = 0, \tag{7.135}$$

and

$$G''(y) - \lambda^2 G(y) = 0. \tag{7.136}$$

Solving these equations and using the boundary conditions we find

$$F_n(x) = \sin(nx), \quad n = 1,2,3,\cdots. \tag{7.137}$$

Solving (7.136) we obtain

$$G_n(y) = \alpha_n \cosh(ny) + \beta_n \sinh(ny). \tag{7.138}$$

To properly use the boundary condition $u(x,\pi) = 0$, we first rewrite (7.138) in the form

$$G_n(y) = C_n \sinh n(K-y). \tag{7.139}$$

Using the boundary condition $u(x,\pi) = 0$ in (7.139) yields

$$K = \pi, \quad C_n \neq 0. \tag{7.140}$$

This gives

$$G_n(y) = C_n \sinh n(\pi - y), \quad n = 1,2,3,\cdots. \tag{7.141}$$

Using the superposition principle we obtain

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin(nx) \sinh n(\pi - y), \tag{7.142}$$

where the constants C_n are as yet undermined. Using the inhomogeneous boundary condition $u(x,0) = \sin x \sinh \pi$ in (7.142) to find

$$\sum_{n=1}^{\infty} C_n \sin(nx) \sinh(n\pi) = \sin x \sinh \pi. \tag{7.143}$$

The constants C_n can be determined by expanding the series and equating the coefficients of like terms on both sides where we obtain

$$C_1 = 1, \quad C_k = 0, \ k \neq 1. \tag{7.144}$$

Consequently, the solution is given by

$$u(x,y) = \sin x \sinh(\pi - y). \tag{7.145}$$

Example 4. Use the method of separation of variables to solve the boundary value problem

$$\begin{array}{lrl} \text{PDE} & u_{xx} + u_{yy} = 0, & 0 < x,y < \pi, \\ \text{BC} & u_x(0,y) = 0, & u_x(\pi,y) = 0, \\ & u_y(x,0) = 0, & u_y(x,\pi) = \sinh \pi \cos x. \end{array} \tag{7.146}$$

Solution.

It is important to note that this type of problems is well known as the Neumann problem where the directional derivatives of $u(x,y)$ are prescribed on the boundary. A necessary condition for this problem to be solvable is that the integral of the inhomogeneous boundary condition vanishes. This can be easily satisfied by noting that

$$\int_0^{\pi} \sinh \pi \cos x dx = 0. \tag{7.147}$$

In addition, we will show that the solution of the Neumann problem will be determined up to an additive constant.

We first set

$$u(x,y) = F(x)G(y). \tag{7.148}$$

Proceeding as before we find

$$F''(x) + \lambda^2 F(x) = 0, \tag{7.149}$$

and

$$G''(y) - \lambda^2 G(y) = 0. \tag{7.150}$$

Solving these equations and using the boundary conditions we find

$$F_n(x) = \cos(nx), \quad n = 0,1,2,3,\cdots, \tag{7.151}$$

and

$$G_n(y) = C_n \cosh(ny), \quad n = 0,1,2,3,\cdots. \tag{7.152}$$

Using the superposition principle we obtain

$$u(x,y) = \sum_{n=0}^{\infty} C_n \cos(nx)\cosh(ny), \tag{7.153}$$

or equivalently

$$u(x,y) = C_0 + \sum_{n=1}^{\infty} C_n \cos(nx)\cosh(ny), \tag{7.154}$$

where the constants C_n are as yet undetermined. Using the inhomogeneous boundary condition $u_y(x,\pi) = \sinh\pi\cos x$ in the derivative of (7.154) and expand the series to find

$$C_1 \sinh\pi\cos x + C_2 \sinh(2\pi)\cos(2x) + \cdots = \sinh\pi\cos x. \tag{7.155}$$

Equating the coefficients of like terms on both sides gives

$$C_1 = 1, \quad C_k = 0, \; k \neq 1. \tag{7.156}$$

It is important to note that the constant C_0 is eliminated when the boundary condition $u_y(x,\pi)$ is used, and there is no prescribed condition that will determine C_0. Accordingly, the constant C_0 remains arbitrary and the solution is therefore given by

$$u(x,y) = C_0 + \cos x \cosh y. \tag{7.157}$$

This confirms the well-known property that Neumann problem is solved up to an additive arbitrary constant.

Exercises 7.4.1

Use the method of separation of variables to solve the following Laplace's equations:

1. $u_{xx} + u_{yy} = 0, 0 < x < \pi, \; 0 < y < \pi$
   $u(0,y) = 0, u(\pi,y) = 0$
   $u(x,0) = 0, u(x,\pi) = \sinh(2\pi)\sin(2x)$

2. $u_{xx} + u_{yy} = 0, 0 < x < \pi, \; 0 < y < \pi$
   $u(0,y) = 0, u(\pi,y) = \sinh(3\pi)\sin(3y)$
   $u(x,0) = 0, u(x,\pi) = 0$

3. $u_{xx} + u_{yy} = 0, 0 < x < \pi, \; 0 < y < \pi$
   $u(0,y) = 0, u(\pi,y) = 4\sinh 2\pi \sin 2y$
   $u(x,0) = 0, u(x,\pi) = 0$

4. $u_{xx}+u_{yy}=0, 0<x<\pi,\ 0<y<\pi$
   $u(0,y)=\cosh y, u(\pi,y)=-\cosh y$
   $u(x,0)=\cos x, u(x,\pi)=\cosh\pi\cos x$

5. $u_{xx}+u_{yy}=0, 0<x<\pi,\ 0<y<\pi$
   $u(0,y)=0, u(\pi,y)=0$
   $u(x,0)=\sin x, u(x,\pi)=\cosh\pi\sin x$

6. $u_{xx}+u_{yy}=0, 0<x<\pi,\ 0<y<\pi$
   $u(0,y)=\cosh 2y, u(\pi,y)=\cosh 2y$
   $u(x,0)=\cos 2x, u(x,\pi)=\cosh 2\pi\cos 2x$

7. $u_{xx}+u_{yy}=0, 0<x<\pi,\ 0<y<\pi$
   $u(0,y)=0, u(\pi,y)=\sinh\pi\sin(\pi-y)$
   $u(x,0)=0, u(x,\pi)=0$

8. $u_{xx}+u_{yy}=0, 0<x<\pi,\ 0<y<\pi$
   $u(0,y)=0, u(\pi,y)=0$
   $u(x,0)=\sinh(2\pi)\sin(2x), u(x,\pi)=0$

9. $u_{xx}+u_{yy}=0, 0<x<\pi,\ 0<y<\pi$
   $u_x(0,y)=0, u_x(\pi,y)=0$
   $u_y(x,0)=0, u_y(x,\pi)=2\sinh(2\pi)\cos(2x)$

10. $u_{xx}+u_{yy}=0, 0<x<\pi,\ 0<y<\pi$
    $u_x(0,y)=0, u_x(\pi,y)=2\sinh(2\pi)\cos(2y)$
    $u_y(x,0)=0, u_y(x,\pi)=0$

7.4.2 Laplace's Equation in Three Dimensions

The Laplace's equation in three dimensional space is governed by the boundary value problem

$$\begin{array}{lll}\text{PDE} & u_{xx}+u_{yy}+u_{zz}=0, & \\ & \quad 0<x<a, 0<y<b, 0<z<c, & \\ \text{BC} & u(0,y,z)=0, \quad u(a,y,z)=0, & \\ & u(x,0,z)=0, \quad u(x,b,z)=0, & \\ & u(x,y,0)=0, \quad u(x,y,c)=f(x,y), & \end{array} \tag{7.158}$$

where $u=u(x,y,z)$ is the solution of Laplace's equation at any point located at the position (x,y,z) of a rectangular volume.

Following the steps used in the previous section, we first establish a set of fundamental solutions that will satisfy the partial differential equation and the homogeneous boundary conditions. Next we use the superposition principle to establish a general solution. The remaining constant of integration is determined by using the remaining inhomogeneous boundary condition.

The method of separation of variables assumes that $u(x,y,z)$ consists of the product of three distinct functions $F(x)$, $G(y)$, and $H(z)$ such that each function depends on one variable only. This means that we can set

$$u(x,y,z) = F(x)G(y)H(z). \tag{7.159}$$

Differentiating both sides of (7.159) twice with respect to x, y, and z and substituting into the PDE of (7.158) we find

$$F''(x)G(y)H(z) + F(x)G''(y)H(z) + F(x)G(y)H''(z) = 0. \tag{7.160}$$

Dividing both sides by $F(x)G(y)H(z)$ gives

$$\frac{F''(x)}{F(x)} = -\left(\frac{G''(y)}{G(y)} + \frac{H''(z)}{H(z)}\right). \tag{7.161}$$

It is obvious that the left hand side depends only on the variable x and the right hand side depends only on the variables y and z. The equality holds only if both sides are equal to the same constant. Accordingly, we set

$$\frac{F''(x)}{F(x)} = -\left(\frac{G''(y)}{G(y)} + \frac{H''(z)}{H(z)}\right) = -\lambda^2. \tag{7.162}$$

Equation (7.162) yields the second order ordinary differential equations

$$F''(x) + \lambda^2 F(x) = 0, \tag{7.163}$$
$$G''(y) + \mu^2 G(y) = 0, \tag{7.164}$$
$$H''(z) - (\lambda^2 + \mu^2)H(z) = 0, \tag{7.165}$$

where λ, and μ are constants.

Solving the second order differential equations (7.163) – (7.165) gives

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{7.166}$$
$$G(y) = \alpha\cos(\mu y) + \beta\sin(\mu y), \tag{7.167}$$
$$H(z) = \gamma\cosh(\nu z) + \delta\sinh(\nu z), \tag{7.168}$$

where

$$\nu = \sqrt{\lambda^2 + \mu^2}, \tag{7.169}$$

and A, B, α, β, γ, and δ are constants.

Using the homogeneous boundary conditions gives

$$A = 0, \quad \lambda_n = \frac{n\pi}{a},\ n = 1,2,3\cdots, \tag{7.170}$$
$$\alpha = 0, \quad \mu_m = \frac{m\pi}{b},\ m = 1,2,3,\cdots, \tag{7.171}$$

$$\gamma = 0, \quad \nu_{nm} = \sqrt{(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2}, \tag{7.172}$$

so that

$$F_n(x) = \sin(\frac{n\pi}{a}x), \quad n = 1,2,3\cdots, \tag{7.173}$$

$$G_m(y) = \sin(\frac{m\pi}{b}y), \quad m = 1,2,3\cdots, \tag{7.174}$$

$$H_{nm}(z) = \sinh\left(\sqrt{(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2}\, z\right). \tag{7.175}$$

Combining (7.173) – (7.175) we obtain the fundamental set of solutions

$$u_n = \sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sinh\left(\sqrt{(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2}\, z\right), \quad n,m = 1,2,\cdots. \tag{7.176}$$

Using the superposition principle we obtain

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nm}\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sinh\left(\sqrt{(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2}\, z\right), \tag{7.177}$$

where the constants C_{nm} are as yet undetermined. To determine the constants C_{nm}, we use the inhomogeneous boundary condition $u(x,y,c) = f(x,y)$ to find

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nm}\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sinh\left(\sqrt{(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2}\, c\right) = f(x,y). \tag{7.178}$$

The Fourier coefficients are then given by

$$\begin{aligned} C_{nm}\sinh&\left(\sqrt{(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2}\, c\right) \\ &= \frac{4}{ab}\int_0^a\int_0^b \sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{a}y) f(x,y)\, dx\, dy. \end{aligned} \tag{7.179}$$

Consequently, the solution of the Laplace's equation is given by (7.177) with C_{nm} defined by (7.179).

The method of separation of variables will be illustrated by discussing the following examples.

Example 5. Use the method of separation of variables to solve the boundary value problem

$$\begin{array}{ll} \text{PDE} & u_{xx} + u_{yy} + u_{zz} = 0, \quad 0 < x,y,z < \pi, \\ \text{BC} & u(0,y,z) = 0, \quad u(\pi,y,z) = 0, \\ & u(x,0,z) = 0, \quad u(x,\pi,z) = 0, \\ & u(x,y,0) = 0, \quad u(x,y,\pi) = \sinh(\sqrt{2}\pi)\sin x \sin y. \end{array} \tag{7.180}$$

Solution.

Proceeding as discussed above, we obtain (7.163)–(7.165), and

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{7.181}$$
$$G(y) = \alpha\cos(\mu y) + \beta\sin(\mu y), \tag{7.182}$$
$$H(z) = \gamma\cosh(\nu z) + \delta\sinh(\nu z), \tag{7.183}$$

where

$$\nu = \sqrt{\lambda^2 + \mu^2}, \tag{7.184}$$

and A, B, α, β, γ, and δ are constants.

Using the homogeneous boundary conditions gives

$$A = 0, \quad \lambda_n = n, n = 1,2,3\cdots, \tag{7.185}$$
$$\alpha = 0, \quad \mu_m = m, m = 1,2,3,\cdots, \tag{7.186}$$
$$\gamma = 0, \quad \nu_{nm} = \sqrt{n^2 + m^2}, \tag{7.187}$$

so that

$$F_n(x) = \sin(nx), \quad n = 1,2,3\cdots, \tag{7.188}$$
$$G_m(y) = \sin(my), \quad m = 1,2,3\cdots, \tag{7.189}$$
$$H_{nm}(z) = \sinh\left(\sqrt{n^2 + m^2}\,z\right). \tag{7.190}$$

Consequently, we obtain the fundamental solutions

$$u_n(x,y,z) = \sin(nx)\sin(my)\sinh\left(\sqrt{n^2 + m^2}\,z\right), \quad n,m = 1,2,\cdots, \tag{7.191}$$

so that the general solution is

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nm}\sin(nx)\sin(my)\sinh\left(\sqrt{n^2 + m^2}\,z\right). \tag{7.192}$$

To determine the constants C_{nm}, we use the inhomogeneous boundary condition $u(x,y,\pi) = \sin x \sin y \sinh(\sqrt{2}\pi)$ to find

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nm}\sin(nx)\sin(my)\sinh\left(\sqrt{n^2 + m^2}\,\pi\right) = \sin x \sin y \sinh(\sqrt{2}\pi). \tag{7.193}$$

Expanding the double series and equating the coefficients of like terms on both sides we find

$$C_{11} = 1, \text{ for } n = 1, m = 1, \quad C_{nm} = 0, \text{ for } n \neq 1, m \neq 1. \tag{7.194}$$

This gives the exact solution

$$u(x,y,z) = \sin x \sin y \sinh \sqrt{2}z, \tag{7.195}$$

obtained upon substituting (7.194) into (7.192).

Example 6. Use the method of separation of variables to solve the boundary value problem

$$\begin{array}{ll} \text{PDE} & u_{xx} + u_{yy} + u_{zz} = 0,\, 0 < x,y,z < \pi, \\ \text{BC} & u(0,y,z) = 0,\, u(\pi,y,z) = 0, \\ & u(x,0,z) = 0,\, u(x,\pi,z) = 0, \\ & u(x,y,0) = \sinh\sqrt{2}\pi \sin x \sin y, \quad u(x,y,\pi) = 0. \end{array} \tag{7.196}$$

Solution.

Proceeding as before, we obtain

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{7.197}$$
$$G(y) = \alpha\cos(\mu y) + \beta\sin(\mu y), \tag{7.198}$$
$$H(z) = \gamma\cosh(\nu z) + \delta\sinh(\nu z), \tag{7.199}$$

where

$$\nu = \sqrt{\lambda^2 + \mu^2}, \tag{7.200}$$

and A, B, α, β, γ, and δ are constants.

Because $H(\pi) = 0$, it is useful to rewrite (7.199) into the equivalent form

$$H(z) = C\sinh\nu(K - z), \tag{7.201}$$

where K and C are constants. Using the homogeneous boundary conditions lead to

$$A = 0, \quad \lambda_n = n,\ n = 1,2,3\cdots, \tag{7.202}$$
$$\alpha = 0, \quad \mu_m = m,\ m = 1,2,3,\cdots \tag{7.203}$$
$$K = \pi, \quad \nu_{nm} = \sqrt{n^2 + m^2}, \quad C \neq 0, \tag{7.204}$$

so that

$$F_n(x) = \sin(nx), n = 1,2,3\cdots, \tag{7.205}$$
$$G_m(y) = \sin(my), m = 1,2,3\cdots, \tag{7.206}$$
$$H_{nm}(z) = \sinh\left(\sqrt{n^2 + m^2}(\pi - z)\right). \tag{7.207}$$

This gives

$$u_n = \sin(nx)\sin(my)\sinh\left(\sqrt{n^2 + m^2}(\pi - z)\right), \quad n,m = 1,2,\cdots. \tag{7.208}$$

Using the superposition principle gives

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nm}\sin(nx)\sin(my)\sinh\left(\sqrt{n^2+m^2}(\pi - z)\right). \tag{7.209}$$

To determine the constants C_{nm}, we substitute the inhomogeneous boundary condition $u(x,y,0) = \sinh(\sqrt{2}\pi)\sin x \sin y$ into (7.209), and equate the coefficients of like terms on both sides, we find

$$C_{11} = 1, \quad C_{nm} = 0,\ n \neq 1, m \neq 1. \tag{7.210}$$

This gives the exact solution

$$u(x,y,z) = \sin x \sin y \sinh\sqrt{2}(\pi - z), \tag{7.211}$$

obtained by combining (7.210) and (7.209).

Example 7. Use the method of separation of variables to solve the boundary value problem

$$\begin{array}{lll} \text{PDE} & u_{xx} + u_{yy} + u_{zz} = 0, \quad 0 < x,y,z < \pi, & \\ \text{BC} & u(0,y,z) = 0, \quad u(\pi,y,z) = 0, & \\ & u(x,0,z) = 0, \quad u(x,\pi,z) = 0, & \\ & u(x,y,0) = 1, \quad u(x,y,\pi) = 0. & \end{array} \tag{7.212}$$

Solution.

Proceeding as in Ex. 6, using the boundary conditions and following the last example we obtain

$$A = 0, \quad \lambda_n = n,\ n = 1,2,3\cdots, \tag{7.213}$$

$$\alpha = 0, \quad \mu_m = m,\ m = 1,2,3,\cdots \tag{7.214}$$

$$K = \pi, \quad \nu_{nm} = \sqrt{n^2+m^2}, \quad C \neq 0, \tag{7.215}$$

so that

$$F_n(x) = \sin(nx), \quad n = 1,2,3\cdots, \tag{7.216}$$

$$G_m(y) = \sin(my), \quad m = 1,2,3\cdots, \tag{7.217}$$

$$H_{nm}(z) = \sinh\left(\sqrt{n^2+m^2}(\pi - z)\right). \tag{7.218}$$

Using the superposition principle gives

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nm}\sin(nx)\sin(my)\sinh\left(\sqrt{n^2+m^2}(\pi - z)\right). \tag{7.219}$$

To determine the constants C_{nm}, we use the inhomogeneous boundary condition $u(x,y,0) = 1$ in (7.219) to find

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nm}\sin(nx)\sin(my)\sinh\left(\sqrt{n^2+m^2}\,\pi\right) = 1. \tag{7.220}$$

This gives

$$C_{nm}\sinh(\sqrt{n^2+m^2}\,\pi) = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi}\sin(nx)\sin(my)\,dx\,dy,$$
$$= \begin{cases} \dfrac{16}{\pi^2 nm} & \text{for } n,m \text{ odd} \\ 0 & \text{for } n \text{ or } m \text{ even} \end{cases} \tag{7.221}$$

The solution is given by

$$u(x,y,z) = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(2n+1)(2m+1)}\sin((2n+1)x)\sin((2m+1)y)$$
$$\times\sinh(\sqrt{(2n+1)^2+(2m+1)^2}(\pi-z)). \tag{7.222}$$

Example 8. Use the method of separation of variables to solve the boundary value problem

$$\begin{array}{ll} \text{PDE} & u_{xx}+u_{yy}+u_{zz}=0, \quad 0<x,y,z<\pi, \\ \text{BC} & u_x(0,y,z)=0, \quad u_x(\pi,y,z)=0, \\ & u_y(x,0,z)=0, \quad u_y(x,\pi,z)=0, \\ & u_z(x,y,0)=0, \quad u_z(x,y,\pi)=\sqrt{2}\sinh(\sqrt{2}\pi)\cos x\cos y. \end{array} \tag{7.223}$$

Solution.

This equation is a Neumann problem in three dimensions. Following the techniques used before gives

$$F_n(x) = \cos(nx), \quad n=0,1,2,3\cdots, \tag{7.224}$$
$$G_m(y) = \cos(my), \quad m=0,1,2,3\cdots, \tag{7.225}$$
$$H_{nm}(z) = \cosh\left(\sqrt{n^2+m^2}\,z\right). \tag{7.226}$$

Using the superposition principle gives

$$u(x,y,z) = C_0+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{nm}\cos(nx)\cos(my)\cosh\left(\sqrt{n^2+m^2}\,z\right). \tag{7.227}$$

Using the inhomogeneous boundary condition $u(x,y,\pi)=\sqrt{2}\sinh(\sqrt{2}\pi)\cos x\cos y$ gives

$$C_{11}=1, \quad C_{nm}=0,\ n\neq 1, m\neq 1. \tag{7.228}$$

Accordingly, the solution is given in the form

$$u(x,y,z) = C_0+\cos x\cos y\cosh(\sqrt{2}z). \tag{7.229}$$

Exercises 7.4.2

Use the method of separation of variables to solve the following three dimensional Laplace's equations:

1. $u_{xx}+u_{yy}+u_{zz}=0,\ 0<x,y,z<\pi$
   $u(0,y,z)=u(\pi,y,z)=0$
   $u(x,0,z)=u(x,\pi,z)=0$
   $u(x,y,0)=0,\ u(x,y,\pi)=\sinh(\sqrt{5}\pi)\sin x\sin(2y)$

2. $u_{xx}+u_{yy}+u_{zz}=0,\ 0<x,y,z<\pi$
   $u(0,y,z)=u(\pi,y,z)=0$
   $u(x,0,z)=u(x,\pi,z)=0$
   $u(x,y,0)=0,\ u(x,y,\pi)=\sinh(10\pi)\sin(6x)\sin(8y)$

3. $u_{xx}+u_{yy}+u_{zz}=0,\ 0<x,y,z<\pi$
   $u(0,y,z)=u(\pi,y,z)=0$
   $u(x,0,z)=u(x,\pi,z)=0$
   $u(x,y,0)=0,\ u(x,y,\pi)=\sinh(\sqrt{8}\pi)\sin(2x)\sin(2y)$

4. $u_{xx}+u_{yy}+u_{zz}=0,\ 0<x,y,z<\pi$
   $u(0,y,z)=u(\pi,y,z)=0$
   $u(x,0,z)=u(x,\pi,z)=0$
   $u(x,y,0)=\sinh(\sqrt{5}\pi)\sin x\sin(2y),\ u(x,y,\pi)=0$

5. $u_{xx}+u_{yy}+u_{zz}=0,\ 0<x,y,z<\pi$
   $u(0,y,z)=u(\pi,y,z)=0$
   $u(x,0,z)=u(x,\pi,z)=0$
   $u(x,y,0)=\sinh(5\pi)\sin(3x)\sin(4y),\ u(x,y,\pi)=0$

6. $u_{xx}+u_{yy}+u_{zz}=0,\ 0<x,y,z<\pi$
   $u(0,y,z)=u(\pi,y,z)=0$
   $u(x,0,z)=u(x,\pi,z)=0$
   $u(x,y,0)=\sinh(13\pi)\sin(5x)\sin(12y),\ u(x,y,\pi)=0$

7. $u_{xx}+u_{yy}+u_{zz}=0,\ 0<x,y,z<\pi$
   $u_x(0,y,z)=u_x(\pi,y,z)=0$
   $u_y(x,0,z)=u_y(x,\pi,z)=0$
   $u_z(x,y,0)=0,\ u_z(x,y,\pi)=\sqrt{5}\sinh(\sqrt{5}\pi)\cos x\cos(2y)$

8. $u_{xx}+u_{yy}+u_{zz}=0,\ 0<x,y,z<\pi$
   $u_x(0,y,z)=u_x(\pi,y,z)=0$
   $u_y(x,0,z)=u_y(x,\pi,z)=0$
   $u_z(x,y,0)=0,\ u_z(x,y,\pi)=13\sinh(13\pi)\cos(5x)\cos(12y)$

9. $u_{xx}+u_{yy}+u_{zz}=0,\ 0<x,y,z<\pi$
   $u_x(0,y,z)=u_x(\pi,y,z)=0$
   $u_y(x,0,z)=u_y(x,\pi,z)=0$

$u_z(x,y,0) = -5\sinh(5\pi)\cos(3x)\cos(4y)$, $u_z(x,y,\pi) = 0$

10. $u_{xx} + u_{yy} + u_{zz} = 0$, $0 < x,y,z < \pi$
    $u_x(0,y,z) = u_x(\pi,y,z) = 0$
    $u_y(x,0,z) = u_y(x,\pi,z) = 0$
    $u_z(x,y,0) = -\sqrt{8}\sinh(\sqrt{8}\pi)\cos(2x)\cos(2y)$, $u_z(x,y,\pi) = 0$

11. $u_{xx} + u_{yy} + u_{zz} = 0$, $0 < x,y,z < \pi$
    $u(0,y,z) = u(\pi,y,z) = 0$
    $u(x,0,z) = u(x,\pi,z) = 0$
    $u(x,y,0) = \sin 8x \sin 15y$, $u(x,y,\pi) = \cosh 17\pi \sin 8x \sin 15y$

12. $u_{xx} + u_{yy} + u_{zz} = 0$, $0 < x,y,z < \pi$
    $u(0,y,z) = u(\pi,y,z) = 0$
    $u(x,0,z) = u(x,\pi,z) = \sin 3x \sinh 5z$
    $u(x,y,0) = 0$, $u(x,y,\pi) = \sinh 5\pi \sin 3x \cos 4y$

7.5 Laplace's Equation in Polar Coordinates

In the last two sections we have studied the two dimensional Dirichlet problem for a rectangle governed by the equation

$$u_{xx} + u_{yy} = 0. \tag{7.230}$$

The boundary conditions are specified on the boundary of a rectangle. However, if the domain of the solution $u(x,y)$ is a disc or a circular annulus, it is useful to study the two dimensional Laplace's equation in polar coordinates. It is well known that the polar coordinates (r,θ) of any point are related to its Cartesian coordinates (x,y) by the familiar formulas

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{7.231}$$

Using these formulas along with the chain rule, the Laplace's equation for a circular domain becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < a, 0 \leqslant \theta \leqslant 2\pi. \tag{7.232}$$

Recall that Laplace's equation is a boundary value problem. Accordingly, the boundary condition that describes the solution $u(r,\theta)$ at the circumference of a circular domain should be specified. Therefore, we set the boundary condition for a disc by

$$u(a,\theta) = f(\theta), \quad 0 < r < a. \tag{7.233}$$

7.5.1 Laplace's Equation for a Disc

In this part, we will study Laplace's equation for a circular disc of radius a where the top and the bottom faces of the disc are insulated. The boundary condition at the circular edge is specified. The phenomenon that the temperature reaches a steady state inside the disc is governed by the Laplace's equation in polar coordinates [8], and expressed by the boundary value problem

$$\begin{array}{ll} \text{PDE} & \dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0, \quad 0<r<a, 0\leqslant \theta \leqslant 2\pi, \\ \text{BC} & u(a,\theta)=f(\theta). \end{array} \tag{7.234}$$

Two important facts should be taken into consideration, namely:
1. the solution $u(r,\theta)$ should be **bounded** at $r=0$.
2. the solution $u(r,\theta)$ should be **periodic** with period 2π. This means that $u(r,\theta+2\pi)=u(r,\theta)$.

It is interesting to point out that these facts are not really boundary conditions. The solution $u(r,\theta)$ being bounded at $r=0$ and the periodicity of the solution $u(r,\theta)$ play a major role in determining the solution. In addition, the boundary condition $f(\theta)$ must be periodic with period 2π. Also note that the coefficients $\frac{1}{r}$ and $\frac{1}{r^2}$ become infinite at $r=0$, hence $r=0$ is excluded from the domain of the solution.

To solve (7.234), we use the method of separation of variables, hence we set $u(r,\theta)$ in the form

$$u(r,\theta)=F(r)G(\theta). \tag{7.235}$$

This gives

$$\begin{aligned} u_r(r,\theta) &= F'(r)G(\theta), \\ u_{rr}(r,\theta) &= F''(r)G(\theta), \\ u_{\theta\theta}(r,\theta) &= F(r)G''(\theta). \end{aligned} \tag{7.236}$$

Substituting (7.236) into the PDE of (7.234) gives

$$F''(r)G(\theta)+\frac{1}{r}F'(r)G(\theta)+\frac{1}{r^2}F(r)G''(\theta)=0. \tag{7.237}$$

Dividing both sides of (7.237) by $F(r)G(\theta)$ yields

$$\frac{G''(\theta)}{G(\theta)}=-\left(r^2\frac{F''(r)}{F(r)}+r\frac{F'(r)}{F(r)}\right). \tag{7.238}$$

It is well known now that the equality holds only if each side is equal to the same constant. Therefore, we set

$$\frac{G''(\theta)}{G(\theta)}=-\lambda^2, \tag{7.239}$$

so that

$$-\left(r^2\frac{F''(r)}{F(r)}+r\frac{F'(r)}{F(r)}\right)=-\lambda^2. \tag{7.240}$$

This gives the second order differential equations

$$G''(\theta) + \lambda^2 G(\theta) = 0, \tag{7.241}$$

and

$$r^2 F''(r) + rF'(r) - \lambda^2 F(r) = 0. \tag{7.242}$$

Equation (7.241) gives the solution

$$G(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta). \tag{7.243}$$

As mentioned earlier, $u(r,\theta)$ is periodic, and hence, so is $G(\theta)$. Accordingly, the periodicity implies that

$$\lambda_n = n, \quad n = 0, 1, 2, \cdots. \tag{7.244}$$

Notice that $n = 0$ is included in our values for n since it results in a constant, which is also periodic. In view of (7.244), equation (7.243) becomes

$$G_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta), \quad n = 0, 1, 2, \cdots. \tag{7.245}$$

Substituting $\lambda_n = n$ into (7.242) gives

$$r^2 F''(r) + rF'(r) - n^2 F(r) = 0. \tag{7.246}$$

Equation (7.246) is the well-known second order Euler ordinary differential equation with general solution given by

$$F_0(r) = C_0 + D_0 \ln r, \tag{7.247}$$

and

$$F_n(r) = C_n r^n + D_n r^{-n}, \quad n = 1, 2, 3, \cdots. \tag{7.248}$$

It is important to recall that $u(r,\theta)$, and hence $F(r)$ should be bounded at $r = 0$. However, each of the components $\ln r$ in (7.247) and r^{-n} in (7.248) approaches infinity at $r = 0$. This means that we must set

$$D_0 = D_n = 0, \tag{7.249}$$

so that $u(r,\theta)$ becomes bounded at $r = 0$. Combining (7.247) – (7.249) gives the general solutions of Euler equation (7.246) by

$$F_n(r) = C_n r^n, \quad n = 0, 1, 2, \cdots. \tag{7.250}$$

Using the superposition principle, and combining the results obtained above, we find

$$u(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n \left(A_n\cos(n\theta) + B_n\sin(n\theta)\right), \tag{7.251}$$

which is usually written in a more convenient equivalent form by

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (\frac{r}{a})^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right), \tag{7.252}$$

simply by modifying the constants.

To determine the constants $a_n, n \geqslant 0$ and $b_n, n \geqslant 1$, we use the boundary condition $u(a,\theta) = f(\theta)$. Setting $r = a$ in (7.252) gives

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) = f(\theta). \tag{7.253}$$

It is obvious that a_n and b_n are Fourier coefficients, and therefore can be determined by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta, \quad n = 0,1,2,\cdots, \tag{7.254}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta, \quad n = 1,2,\cdots. \tag{7.255}$$

To determine the constants a_n and b_n, Appendix A can be used to evaluate the integrals in (7.254) and in (7.255).

However, the constants a_n and b_n can also be determined by equating the coefficients of like terms if the boundary condition is given in terms of sines and cosines as discussed in previous chapters.

The technique discussed above will be illustrated by discussing the following examples.

Example 1. Use the method of separation of variables to solve the Dirichlet problem

$$\begin{array}{ll} \text{PDE} & \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < 1, 0 \leqslant \theta \leqslant 2\pi, \\ \text{BC} & u(1,\theta) = \cos^2\theta. \end{array} \tag{7.256}$$

Solution.

Using the method of separation of variables gives

$$G''(\theta) + \lambda^2 G(\theta) = 0, \tag{7.257}$$

and

$$r^2 F''(r) + rF'(r) - \lambda^2 F(r) = 0. \tag{7.258}$$

The solution of (7.257) is given by

$$G(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta). \tag{7.259}$$

The periodicity of $u(r,\theta)$ implies that

$$\lambda_n = n, \quad n = 0,1,2,\cdots. \tag{7.260}$$

Equation (7.259) becomes

$$G_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), \quad n = 0, 1, 2, \cdots. \tag{7.261}$$

Substituting $\lambda_n = n$ into (7.258) gives

$$r^2 F''(r) + rF'(r) - n^2 F(r) = 0, \tag{7.262}$$

a second order Euler differential equation with the general solution given by

$$F_0(r) = C_0 + D_0 \ln r, \tag{7.263}$$

and

$$F_n(r) = C_n r^n + D_n r^{-n}, \quad n = 1, 2, 3, \cdots. \tag{7.264}$$

The essential fact that $u(r, \theta)$, and hence $F(r)$ should be bounded at $r = 0$ means that we must set

$$D_0 = D_n = 0, \quad n = 1, 2, 3 \tag{7.265}$$

This gives the general solutions of (7.262) by

$$F_n(r) = C_n r^n, \quad n = 0, 1, 2, \cdots. \tag{7.266}$$

Using the superposition principle and proceeding as before, we find

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right). \tag{7.267}$$

To determine the constants $a_n, n \geqslant 0$ and $b_n, n \geqslant 1$, set $r = 1$ in (7.267) and using the boundary condition, we obtain

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) = \frac{1}{2} + \frac{1}{2}\cos(2\theta), \tag{7.268}$$

where the identity

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta) \tag{7.269}$$

was used. Expanding the series in (7.268) and equating the coefficients of like terms in both sides we find

$$a_0 = 1, \tag{7.270}$$

$$a_2 = \frac{1}{2} \tag{7.271}$$

and

$$\begin{aligned} a_n &= 0, \quad n \neq 0, 2, \\ b_n &= 0, \quad n = 1, 2, 3, \cdots. \end{aligned} \tag{7.272}$$

This gives the particular solution by

$$u(r,\theta)=\frac{1}{2}+\frac{1}{2}r^2\cos(2\theta). \tag{7.273}$$

Example 2. Use the method of separation of variables to solve the Dirichlet problem

$$\begin{array}{ll}\text{PDE} & \dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0, \quad 0<r<1, 0\leqslant\theta\leqslant 2\pi,\\ \text{BC} & u(1,\theta)=2\sin^2\theta+\sin\theta.\end{array} \tag{7.274}$$

Solution.

We first set $u(r,\theta)$ in the form

$$u(r,\theta)=F(r)G(\theta). \tag{7.275}$$

Substituting (7.275) into the PDE of (7.274) and proceeding as before we obtain

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty} r^n\left(a_n\cos(n\theta)+b_n\sin(n\theta)\right). \tag{7.276}$$

To determine the constants $a_n, n\geqslant 0$ and $b_n, n\geqslant 1$, we substitute $r=1$ in (7.276) and we use the boundary condition, we obtain

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos(n\theta)+b_n\sin(n\theta)\right)=1-\cos(2\theta)+\sin\theta, \tag{7.277}$$

where the identity

$$\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos(2\theta), \tag{7.278}$$

was used. Expanding the series in (7.277) and equating the coefficients of like terms in both sides we find

$$a_0=2, \quad a_2=-1, \quad b_1=1, \tag{7.279}$$

where each of the remaining coefficients is zero. Hence, the particular solution is given by

$$u(r,\theta)=1-r^2\cos(2\theta)+r\sin\theta. \tag{7.280}$$

Example 3. Use the method of separation of variables to solve the Dirichlet problem

$$\begin{array}{ll}\text{PDE} & \dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0, \quad 0<r<1, 0\leqslant\theta\leqslant 2\pi,\\ \text{BC} & u(1,\theta)=\mid 2\theta\mid.\end{array} \tag{7.281}$$

Solution.

We first set $u(r,\theta)$ in the form

$$u(r,\theta)=F(r)G(\theta). \tag{7.282}$$

Substituting (7.282) into the PDE of (7.281) and proceeding as before we obtain

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right). \tag{7.283}$$

To determine the constants $a_n, n \geqslant 0$ and $b_n, n \geqslant 1$, we substitute $r = 1$ in (7.283) and by using the boundary condition, we obtain

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) = \mid 2\theta \mid . \tag{7.284}$$

The Fourier coefficients can be determined by using the formulas (7.254) and (7.255). For a_0, we find

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \mid 2\theta \mid d\theta, \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0} -2\theta d\theta + \int_{0}^{\pi} 2\theta d\theta\right) \\ &= 2\pi, \end{aligned} \tag{7.285}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \mid 2\theta \mid \cos(n\theta)\, d\theta, \\ &= -\frac{8}{\pi n^2}, \text{ for } n \text{ odd, 0 otherwise,} \end{aligned} \tag{7.286}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \mid 2\theta) \mid \sin(n\theta)\, d\theta, \\ &= 0. \end{aligned} \tag{7.287}$$

Therefore, the solution is given by

$$u(r,\theta) = \pi - \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} r^{2k+1} \cos(2k+1)\theta. \tag{7.288}$$

The boundary condition $u(1,\theta) = |2\theta|$ can be justified by using Appendix F.

Example 4. Use the method of separation of variables to solve the Neumann problem

$$\begin{aligned} &\text{PDE} \qquad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < 1, 0 \leqslant \theta \leqslant 2\pi, \\ &\text{BC} \qquad u_r(1,\theta) = 2\cos(2\theta). \end{aligned} \tag{7.289}$$

Solution.

Note that a Neumann boundary condition is given. Recall that the solution will be determined up to an arbitrary constant, hence the solution is not unique. We first set $u(r,\theta)$ in the form

$$u(r,\theta) = F(r)G(\theta). \tag{7.290}$$

Substituting (7.290) into the PDE of (7.289) and proceeding as before we obtain

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right). \tag{7.291}$$

To determine the constants $a_n, n \geqslant 0$ and $b_n, n \geqslant 1$, we set $r = 1$ in the derivative of (7.291) and by using the boundary condition, we obtain

$$\sum_{n=1}^{\infty} n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) = 2\cos(2\theta). \tag{7.292}$$

Expanding the series in (7.292) and equating the coefficients of like terms in both sides we find

$$a_2 = 1 \tag{7.293}$$

where each of the remaining coefficients is zero. Hence, the particular solution is given by

$$u(r,\theta) = C_0 + r^2 \cos(2\theta), \tag{7.294}$$

where C_0 is an arbitrary constant.

Exercises 7.5.1

Use the method of separation of variables to solve the following Laplace's equations:

1. $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, 0 < r < 1, 0 \leqslant \theta \leqslant 2\pi$

   $u(1,\theta) = 2 + 3\sin\theta + 4\cos\theta$

2. $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, 0 < r < 1, 0 \leqslant \theta \leqslant 2\pi$

   $u(1,\theta) = 2\cos^2(2\theta)$

3. $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, 0 < r < 1, 0 \leqslant \theta \leqslant 2\pi$

   $u(1,\theta) = 2\sin^2(3\theta)$

4. $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, 0 < r < 1, 0 \leqslant \theta \leqslant 2\pi$

   $u(1,\theta) = \sin(2\theta) + \cos(2\theta)$

5. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0, 0<r<1, 0\leqslant\theta\leqslant 2\pi$

   $u_r(1,\theta)=8\sin 4\theta$

6. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0, 0<r<1, 0\leqslant\theta\leqslant 2\pi$

   $u_r(1,\theta)=\sin\theta-\cos\theta$

7. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0, 0<r<1, 0\leqslant\theta\leqslant 2\pi$

   $u_r(1,\theta)=2\sin(2\theta)$

8. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0, 0<r<1, 0\leqslant\theta\leqslant 2\pi$

   $u_r(1,\theta)=3\cos(3\theta)$

9. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0, 0<r<1, 0\leqslant\theta\leqslant 2\pi$

   $u_r(1,\theta)=2\sin(2\theta)+3\cos(3\theta)$

10. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0, 0<r<1, 0\leqslant\theta\leqslant 2\pi$

    $u_r(1,\theta)=2\cos(2\theta)$

7.5.2 Laplace's Equation for an Annulus

In this part, we will study the Laplace's equation in the domain lying between the concentric circles K_1 and K_2 of radii a and b where $0<a<b$. In other words, the domain of the solution includes two circular edges, an interior boundary with radius $r_1=a$ and an exterior boundary with radius $r_2=b$. As a result, two boundary conditions should be specified in this case.

The Laplace's equation for an annulus is therefore defined by the boundary value problem

$$
\begin{array}{ll}
\text{PDE} & \dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0, \quad 0<a<r<b, 0\leqslant\theta\leqslant 2\pi, \\
\text{BC} & u(a,\theta)=f(\theta), \quad u(b,\theta)=g(\theta).
\end{array}
\tag{7.295}
$$

It is essential to note that $u(r,\theta)$ must be periodic with period 2π. This means that $u(r,\theta+2\pi)=u(r,\theta)$. Accordingly, the boundary conditions $f(\theta)$ and $g(\theta)$ must be periodic with period 2π. Moreover, we do not claim that $u(r,\theta)$ is bounded at $r=0$. This is due to the fact that $a<r<b$, and r will never be 0.

We begin our analysis by seeking a solution expressed as a product of two distinct functions, where each function depends on one variable only. The method of separation of variables suggests that

$$u(r,\theta) = F(r)G(\theta). \tag{7.296}$$

Substituting in (7.295) gives

$$\frac{G''(\theta)}{G(\theta)} = -\left(r^2\frac{F''(r)}{F(r)} + r\frac{F'(r)}{F(r)}\right) = -\lambda^2. \tag{7.297}$$

This gives the differential equations

$$G''(\theta) + \lambda^2 G(\theta) = 0, \tag{7.298}$$

and

$$r^2F''(r) + rF'(r) - \lambda^2 F(r) = 0. \tag{7.299}$$

Equation (7.298) gives the solution

$$G(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta). \tag{7.300}$$

As mentioned earlier, $u(r,\theta)$ is periodic, and hence, so is $G(\theta)$. Accordingly, the periodicity implies that

$$\lambda_n = n, \quad n = 0, 1, 2, \cdots. \tag{7.301}$$

Notice that $n = 0$ is included in our values for n since it results in a constant, which is also periodic. In view of (7.301), Eq. (7.300) becomes

$$G_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta), \quad n = 0, 1, 2, \cdots. \tag{7.302}$$

Substituting $\lambda_n = n$ into (7.299) gives

$$r^2F''(r) + rF'(r) - n^2F(r) = 0. \tag{7.303}$$

Equation (7.303) is the well-known second order Euler ordinary differential equation with general solution given by

$$F_0(r) = \frac{1}{2}(C_0 + D_0\ln r), \tag{7.304}$$

and

$$F_n(r) = C_n r^n + D_n r^{-n}, \quad n = 1, 2, 3, \cdots. \tag{7.305}$$

Using the superposition principle, the general solution is given by

$$u(r,\theta) = \frac{1}{2}(a_0 + b_0\ln r)$$

$$+\sum_{n=1}^{\infty}\left((a_n r^n + b_n r^{-n})\cos(n\theta) + (c_n r^n + d_n r^{-n})\sin(n\theta)\right), \tag{7.306}$$

where a_n, b_n, c_n and d_n are constants.

Using the boundary conditions, we first set $r = b$ into (7.306) to obtain

$$\frac{1}{2}(a_0 + b_0 \ln b)$$
$$+\sum_{n=1}^{\infty}\left((a_n b^n + b_n b^{-n})\cos(n\theta) + (c_n b^n + d_n b^{-n})\sin(n\theta)\right) = g(\theta). \tag{7.307}$$

The Fourier coefficients are thus given by

$$a_0 + b_0 \ln b = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta, \tag{7.308}$$
$$a_n b^n + b_n b^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\,d\theta, \tag{7.309}$$
$$c_n b^n + d_n b^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta. \tag{7.310}$$

Substituting $r = a$ into (7.306) gives

$$\frac{1}{2}(a_0 + b_0 \ln a)$$
$$+\sum_{n=1}^{\infty}\left((a_n a^n + b_n a^{-n})\cos(n\theta) + (c_n a^n + d_n a^{-n})\sin(n\theta)\right) = f(\theta), \tag{7.311}$$

so that the Fourier coefficients are given by

$$a_0 + b_0 \ln a = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta, \tag{7.312}$$
$$a_n a^n + b_n a^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta, \tag{7.313}$$
$$c_n a^n + d_n a^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta. \tag{7.314}$$

Solving (7.308) and (7.312) for a_0 and b_0, (7.309) and (7.313) for a_n and b_n, and (7.310) and (7.314) for c_n and d_n completes the determination of the constants a_0, b_0, a_n, b_n, c_n, and d_n. This gives the formal solution of Laplace's equation for a circular annulus.

For simplicity reasons, the following illustrative examples will include the boundary conditions in terms of sines and cosines so as to equate the coefficients of both sides as applied before.

Example 5. Use the method of separation of variables to solve the following Dirichlet problem for an annulus:

$$
\begin{array}{ll}
\text{PDE} & \dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0, \quad 1<r<2, 0\leqslant \theta\leqslant 2\pi, \\
\text{BC} & u(1,\theta)=\dfrac{1}{2}+\sin\theta, \quad u(2,\theta)=\dfrac{1}{2}+\dfrac{1}{2}\ln 2+\cos\theta.
\end{array}
\tag{7.315}
$$

Solution.

Following the procedure outlined above, we set

$$u(r,\theta)=F(r)G(\theta). \tag{7.316}$$

Substituting in the PDE of (7.315) and using the periodicity condition give

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2}(a_0+b_0\ln r) \\
&\quad +\sum_{n=1}^{\infty}\left((a_n r^n+b_n r^{-n})\cos(n\theta)+(c_n r^n+d_n r^{-n})\sin(n\theta)\right).
\end{aligned}
\tag{7.317}
$$

We first set $r=1$ into (7.317) and use the related boundary condition give

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}((a_n+b_n)\cos(n\theta)+(c_n+d_n)\sin(n\theta))=\frac{1}{2}+\sin\theta. \tag{7.318}$$

Expanding the series at the left side, and equating the coefficients of like terms on both sides give

$$a_0=1, \tag{7.319}$$

$$a_n+b_n=0, \quad n\geqslant 1, \tag{7.320}$$

$$c_1+d_1=1, \quad c_n+d_n=0, \quad n>1. \tag{7.321}$$

We next set $r=2$ into (7.317), using the related boundary condition give

$$
\begin{aligned}
\frac{1}{2}a_0+\frac{1}{2}b_0\ln 2+\sum_{n=1}^{\infty}\left((a_n 2^n+b_n 2^{-n})\cos(n\theta)+(c_n 2^n+d_n 2^{-n})\sin(n\theta)\right) \\
=\frac{1}{2}+\frac{1}{2}\ln 2+\cos\theta
\end{aligned}
\tag{7.322}
$$

Expanding the series at the left side, and equating the coefficients of like terms on both sides we obtain

$$b_0=1, \tag{7.323}$$

$$2a_1+\frac{1}{2}b_1=1, \quad 2^n a_n+2^{-n}b_n=0, \ n>1, \tag{7.324}$$

$$2^n c_n+2^{-n}d_n=0, \quad n\geqslant 1. \tag{7.325}$$

Equations (7.319) and (7.324) give

$$a_0 = 1, \quad b_0 = 1. \tag{7.326}$$

Equations (7.320) and (7.324) give

$$a_1 = \frac{2}{3}, \quad b_1 = -\frac{2}{3}. \tag{7.327}$$

Equations (7.321) and (7.325) give

$$c_1 = -\frac{1}{3}, \quad d_1 = \frac{4}{3}, \tag{7.328}$$

where all other coefficients vanish. Accordingly, the solution is given by

$$u(r,\theta) = \frac{1}{2} + \frac{1}{2}\ln r + (\frac{2}{3}r - \frac{2}{3}r^{-1})\cos\theta + (-\frac{1}{3}r + \frac{4}{3}r^{-1})\sin\theta, \tag{7.329}$$

obtained upon substituting (7.326) – (7.328) into (7.317).

Example 6. Use the method of separation of variables to solve the following Dirichlet problem for an annulus:

$$\begin{array}{ll} \text{PDE} & \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, \quad 1 < r < e, 0 \leqslant \theta \leqslant 2\pi, \\ \text{BC} & u(1,\theta) = \dfrac{1}{2} + 4\cos\theta, \\ & u(e,\theta) = 1 + 4\cosh 1 \cos\theta + 4\sinh 1 \sin\theta. \end{array} \tag{7.330}$$

Solution.

Following the procedure outlined above, we find

$$u(r,\theta) = \frac{1}{2}(a_0 + b_0 \ln r) + \sum_{n=1}^{\infty} \left((a_n r^n + b_n r^{-n})\cos(n\theta) + (c_n r^n + d_n r^{-n})\sin(n\theta)\right). \tag{7.331}$$

Substituting $r = 1$ into (7.331) and using the related boundary condition give

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left((a_n + b_n)\cos(n\theta) + (c_n + d_n)\sin(n\theta)\right) = \frac{1}{2} + 4\cos\theta. \tag{7.332}$$

Expanding the series at the left side, and equating the coefficients of like terms on both sides we obtain

$$a_0 = 1, \tag{7.333}$$
$$a_1 + b_1 = 4, \quad a_n + b_n = 0, \quad n > 1, \tag{7.334}$$
$$c_n + d_n = 0, \quad n \geqslant 1, \tag{7.335}$$

Substituting $r = e$ into (7.331) and using the related boundary condition give

$$\frac{1}{2}a_0 + \frac{1}{2}b_0 + \sum_{n=1}^{\infty}\left((a_n e^n + b_n e^{-n})\cos(n\theta) + (c_n e^n + d_n e^{-n})\sin(n\theta)\right)$$
$$= 1 + 4\cosh 1 \cos\theta + 4\sinh 1 \sin\theta. \tag{7.336}$$

Expanding the series at the left side, and equating the coefficients of like terms on both sides we obtain

$$\frac{1}{2}a_0 + \frac{1}{2}b_0 = 1, \tag{7.337}$$
$$ea_1 + e^{-1}b_1 = 2(e + e^{-1}), \quad a_n e^n + b_n e^{-n} = 0, n > 1, \tag{7.338}$$
$$ec_1 + e^{-1}d_1 = 2(e - e^{-1}), \quad c_n e^n + d_n e^{-n} = 0, n > 1. \tag{7.339}$$

Equations (7.333) and (7.337) give

$$a_0 = 1, \quad b_0 = 1. \tag{7.340}$$

Equations (7.334) and (7.338) give

$$a_1 = 2, \quad b_1 = 2. \tag{7.341}$$

Equations (7.335) and (7.339) give

$$c_1 = 2, \quad d_1 = -2, \tag{7.342}$$

and all other coefficients vanish. Accordingly, the solution is given by

$$u(r,\theta) = \frac{1}{2} + \frac{1}{2}\ln r + 2(r + r^{-1})\cos\theta + 2(r - r^{-1})\sin\theta. \tag{7.343}$$

Example 7. Use the method of separation of variables to solve the following Neumann problem for an annulus:

$$\begin{array}{ll} \text{PDE} & \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, \quad 1 < r < 2, 0 \leqslant \theta \leqslant 2\pi, \\ \text{BC} & u_r(1,\theta) = 1, \quad u_r(2,\theta) = \dfrac{1}{2} + \dfrac{3}{4}\cos\theta. \end{array} \tag{7.344}$$

Solution.

Note that this is a Neumann problem for an annulus. Following the procedure outlined above, we find

$$u(r,\theta) = \frac{1}{2}(a_0 + b_0 \ln r)$$
$$+ \sum_{n=1}^{\infty}\left((a_n r^n + b_n r^{-n})\cos(n\theta) + (c_n r^n + d_n r^{-n})\sin(n\theta)\right). \tag{7.345}$$

We first set $r = 1$ into the derivative of (7.345) and use the related boundary condition to obtain

$$\frac{1}{2}b_0 + \sum_{n=1}^{\infty}\left((na_n - nb_n)\cos(n\theta) + (nc_n - nd_n)\sin(n\theta)\right) = 1. \tag{7.346}$$

Expanding the series at the left side, and equating the coefficients of like terms on both sides we obtain

$$b_0 = 2, \tag{7.347}$$
$$a_n - b_n = 0, \quad n \geqslant 1, \tag{7.348}$$
$$c_n - d_n = 0, \quad n \geqslant 1. \tag{7.349}$$

We next set $r = 2$ into the derivative of (7.345), using the related boundary condition, and equating the coefficients of like terms on both sides we obtain

$$b_0 = 2, \tag{7.350}$$
$$a_1 - \frac{1}{4}b_1 = \frac{3}{4}, \quad n > 1, \tag{7.351}$$
$$n(2^{n-1}c_n - 2^{-n-1}d_n) = 0, \quad n \geqslant 1, \tag{7.352}$$

Proceeding as before we find

$$\begin{aligned} b_0 &= 2, \\ a_1 &= 1, \quad b_1 = 1. \end{aligned} \tag{7.353}$$

Note that other coefficients vanish. Accordingly, the solution is given by

$$u(r,\theta) = C_0 + \ln r + (r + r^{-1})\cos\theta. \tag{7.354}$$

Example 8. Use the method of separation of variables to solve the following Neumann problem for an annulus:

$$\begin{array}{ll} \text{PDE} & \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, \quad 1 < r < 2, 0 \leqslant \theta \leqslant 2\pi, \\ \text{BC} & u_r(1,\theta) = 0, \quad u_r(2,\theta) = \dfrac{3}{4}\cos\theta + \dfrac{3}{4}\sin\theta. \end{array} \tag{7.355}$$

Solution.

Following the procedure outlined above, we find

$$u(r,\theta) = \frac{1}{2}(a_0 + b_0 \ln r)$$
$$+ \sum_{n=1}^{\infty}\left((a_n r^n + b_n r^{-n})\cos(n\theta) + (c_n r^n + d_n r^{-n})\sin(n\theta)\right), \tag{7.356}$$

Using the boundary condition at $r = 1$ gives condition give

$$\frac{1}{2}b_0 + \sum_{n=1}^{\infty}(n(a_n - b_n)\cos(n\theta) + n(c_n - d_n)\sin(n\theta)) = 0, \tag{7.357}$$

that gives

$$b_0 = 0, \quad a_n - b_n = 0, \quad c_n - d_n = 0, \quad n \geqslant 1, \tag{7.358}$$

Using the boundary condition at $r = 2$ and proceeding as before we obtain

$$b_0 = 0, \quad a_1 - \frac{1}{4}b_1 = \frac{3}{4}, \quad c_1 - \frac{1}{4}d_1 = \frac{3}{4}. \tag{7.359}$$

Solving the last equations gives

$$b_0 = 0, \quad a_1 = 1, \quad b_1 = 1, \quad c_1 = 1, \quad d_1 = 1, \tag{7.360}$$

and all other coefficients vanish. Accordingly, the solution is given by

$$u(r,\theta) = C_0 + (r + r^{-1})\cos\theta + (r + r^{-1})\sin\theta. \tag{7.361}$$

Exercises 7.5.2

Use the method of separation of variables to solve the following Laplace's equations:

1. $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, 1 < r < e, 0 \leqslant \theta \leqslant 2\pi$

   $u(1,\theta) = 1,\ u(e,\theta) = 2 + 2\sinh 1(\cos\theta + \sin\theta)$

2. $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, 1 < r < e, 0 \leqslant \theta \leqslant 2\pi$

   $u(1,\theta) = 1 + \cos\theta + \sin\theta,\ u(e,\theta) = 2 + e(\cos\theta + \sin\theta)$

3. $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, 1 < r < 2, 0 \leqslant \theta \leqslant 2\pi$

   $u(1,\theta) = 1,\ \ u(2,\theta) = 1 + 1.5\sin\theta$

4. $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, 1 < r < 2, 0 \leqslant \theta \leqslant 2\pi$

   $u(1,\theta) = 1 - \cos\theta - \sin\theta,\ \ u(2,\theta) = 1 + \cos\theta + \sin\theta$

5. $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, 1 < r < e^2, 0 \leqslant \theta \leqslant 2\pi$

$$u(1,\theta)=1,\ \ u(e^2,\theta)=3+2\sinh 2\cos\theta$$

6. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}=0,\ 1<r<e^2,\ 0\leqslant\theta\leqslant 2\pi$

$$u(1,\theta)=1,\ \ u(e^2,\theta)=3+2\sinh 2\sin\theta$$

7. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}=0,\ \dfrac{1}{2}<r<1,\ 0\leqslant\theta\leqslant 2\pi$

$$u_r(\frac{1}{2},\theta)=2-3\sin\theta,\ \ u_r(1,\theta)=1$$

8. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}=0,\ \dfrac{1}{2}<r<1,\ 0\leqslant\theta\leqslant 2\pi$

$$u_r(\frac{1}{2},\theta)=5\cos\theta,\ \ u_r(1,\theta)=2\cos\theta$$

9. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}=0,\ 1<r<2,\ 0\leqslant\theta\leqslant 2\pi$

$$u_r(1,\theta)=\sin\theta,\ \ u_r(2,\theta)=2.5\sin\theta$$

10. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}=0,\ 1<r<2,\ 0\leqslant\theta\leqslant 2\pi$

$$u_r(1,\theta)=1,\ \ u_r(2,\theta)=\frac{1}{2}+\frac{3}{4}\cos\theta+\frac{3}{4}\sin\theta$$

11. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}=0,\ 1<r<2,\ 0\leqslant\theta\leqslant 2\pi$

$$u_r(1,\theta)=4\cos\theta,\ \ u_r(2,\theta)=2.5\cos\theta$$

12. $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}=0,\ 1<r<2,\ 0\leqslant\theta\leqslant 2\pi$

$$u_r(1,\theta)=\sin\theta,\ \ u_r(2,\theta)=2.5\sin\theta$$

References

1. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, (1994).
2. N. Asmar, Partial Differential Equations, Prentice Hall, New Jersey, (2005).
3. L.C. Evans, Partial Differential Equations, AMS, Providence, (1998).
4. M.S. Gockenbach, Partial Differential Equations, SIAM, Philadelphia, (2002).
5. R. Haberman, Applied Partial Differential Equations, Pearson, New York, (2003).
6. J.H.He, A variational iteration method—a kind of nonlinear analytical technique: Some examples, *Int. J. Nonlinear Mech.*, 34, 699–708, (1999).
7. A.P. Selvaduri, Partial Differential Equations in Mechanics 1 & 2, Springer, Berlin (2000).

8. A.M. Wazwaz, Partial Differential Equations: Methods and Applications, Bal-kema Publishers, The Netherlands, (2002).
9. A.M. Wazwaz, The variational iteration method for exact solutions of Laplace equation, *Phys. Lett. A*, 363, 260–262, (2007).
10. G.B. Whitham, Linear and Nonlinear Waves , John Wiley, New York, (1976).

Chapter 8
Nonlinear Partial Differential Equations

8.1 Introduction

So far in this text we have been mainly concerned in applying classic methods, the Adomian decomposition method [3–5], and the variational iteration method [8–10] in studying first order and second order linear partial differential equations. In this chapter, we will focus our study on the nonlinear partial differential equations. The nonlinear partial differential equations arise in a wide variety of physical problems such as fluid dynamics, plasma physics, solid mechanics and quantum field theory. Systems of nonlinear partial differential equations have been also noticed to arise in chemical and biological applications. The nonlinear wave equations and the solitons concept have introduced remarkable achievements in the field of applied sciences. The solutions obtained from nonlinear wave equations are different from the solutions of the linear wave equations [1–2].

Recently, a special type of KdV equation has been under a thorough investigation and new phenomenon was observed. It was discovered that when the wave dispersion is purely nonlinear, some features may be observed which is the existence of the so-called compactons: solitons with finite wave length [12]. As will be discussed in Chapter 9, solitons appear as a result of balance between weak nonlinearity and dispersion. The characteristics of the solitons and the compactons concepts will be addressed in Chapter 11.

It is important to note that several traditional methods, such as the method of characteristics and the variational principle, are among the methods that are used to handle the nonlinear partial differential equations. Moreover, nonlinear partial differential equations are not easy to handle especially if the questions of uniqueness and stability of solutions are to be discussed. It is interesting to point out that the superposition principle, that we used for the linear partial differential equations, cannot be applied to nonlinear partial differential equations. For this reason numerical solutions are usually established for nonlinear partial differential equations.

It is well known that a general method for determining analytical solutions for partial differential equations has not been found among traditional methods.

However, we believe that the Adomian decomposition method, the noise terms phenomenon [13], and the related modification [14] presented in previous chapters provide an effective, reliable, and powerful tool for handling nonlinear partial differential equations. Moreover, the variational iteration method gives an effective tool to handle nonlinear partial differential equations without any need to the so-called Adomian polynomials.

In Chapter 2, a detailed outline about the works conducted on Adomian's method, the implementation of this method to many scientific models and frontier physics problems, and the comparisons of this method with existing techniques were introduced. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. Cherruault examined the convergence of Adomian's method in [6]. In addition, Cherruault and Adomian presented a new proof of convergence of the method in [7].

However, Adomian decomposition method does not assure on its own existence and uniqueness of the solution. In fact, it can be safely applied when a fixed point theorem holds. A theorem developed by Re'paci [11] indicates that the decomposition method can be used as an algorithm for the approximation of the dynamical response in a sequence of time intervals $[0,t_1),[t_1,t_2),\cdots,[t_{n-1},T)$ such that the condition at t_p is taken as initial condition in the interval $[t_p,t_{p+1})$ which follows.

Unlike the preceding chapters, we will apply only the Adomian decomposition method, all related phenomena, and the variational iteration method in discussing the topic of nonlinear partial differential equations. This is due to the fact that the methods are efficient in that these two methods provide the solution in a rapidly convergent series and reduce the volume of computational work.

The nonlinear partial differential equation was defined in Chapter 1. The first order nonlinear partial differential equation in two independent variables x and y can be generally expressed in the form

$$F(x,y,u,u_x,u_y)=f, \tag{8.1}$$

where f is a function of one or two of the independent variables x and y. Similarly, the second order nonlinear partial differential equation in two independent variables x and y can be expressed by

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})=f. \tag{8.2}$$

The nonlinear partial differential equation is called *homogeneous* if $f=0$, and *inhomogeneous* if $f\neq 0$. Examples of the first order nonlinear partial differential equations are given by

$$u_t+2uu_x=0, \tag{8.3}$$
$$u_x-u^2u_y=0, \tag{8.4}$$
$$u_x+uu_y=6x, \tag{8.5}$$
$$u_t+uu_x=\sin x, \tag{8.6}$$

Note that Eqs. (8.3) and (8.4) are homogeneous equations. On the other hand, equations (8.5) and (8.6) are inhomogeneous equations. Examples of second order nonlinear partial differential equations are given by

$$u_t + uu_x - \nu u_{xx} = 0, \tag{8.7}$$

$$u_{tt} - c^2 u_{xx} + \sin u = 0. \tag{8.8}$$

On the other hand, the modified Korteweg-de Vries equation

$$u_t - 6u^2 u_x + u_{xxx} = 0, \tag{8.9}$$

is an example of a third order nonlinear homogeneous partial differential equation.

In the following section, the Adomian decomposition method will be presented for finding analytical solutions of nonlinear partial differential equations, homogeneous or inhomogeneous.

8.2 Adomian Decomposition Method

The Adomian decomposition method has been outlined before in previous chapters and has been applied to a wide class of linear partial differential equations. The method has been applied directly and in a straightforward manner to homogeneous and inhomogeneous problems without any restrictive assumptions or linearization. The method usually decomposes the unknown function u into an infinite sum of components that will be determined recursively through iterations as discussed before.

The Adomian decomposition method will be applied in this chapter and in the coming chapters to handle nonlinear partial differential equations. An important remark should be made here concerning the representation of the nonlinear terms that appear in the equation. Although the linear term u is expressed as an infinite series of components, the Adomian decomposition method requires a special representation for the nonlinear terms such as $u^2, u^3, u^4, \sin u, e^u, uu_x, u_x^2$, etc. that appear in the equation. The method introduces a formal algorithm to establish a proper representation for all forms of nonlinear terms. The representation of the nonlinear terms is necessary to handle the nonlinear equations in an effective and successful way.

In the following, the Adomian scheme for calculating representation of nonlinear terms will be introduced in details. The discussion will be supported by several illustrative examples that will cover a wide variety of forms of nonlinearity. In a like manner, an alternative algorithm for calculating Adomian polynomials will be outlined in details supported by illustrative examples.

8.2.1 Calculation of Adomian Polynomials

It is well known now that Adomian decomposition method suggests that the unknown linear function u may be represented by the decomposition series

$$u = \sum_{n=0}^{\infty} u_n, \tag{8.10}$$

where the components $u_n, n \geqslant 0$ can be elegantly computed in a recursive way. However, the nonlinear term $F(u)$, such as u^2, u^3, u^4, $\sin u, e^u, uu_x, u_x^2$, etc. can be expressed by an infinite series of the so-called Adomian polynomials A_n given in the form

$$F(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \cdots, u_n), \tag{8.11}$$

where the so-called Adomian polynomials A_n can be evaluated for all forms of nonlinearity. Several schemes have been introduced in the literature by researchers to calculate Adomian polynomials. Adomian introduced a scheme for the calculation of Adomian polynomials that was formally justified. An alternative reliable method that is based on algebraic and trigonometric identities and on Taylor series has been developed and will be examined later. The alternative method employs only elementary operations and does not require specific formulas.

The Adomian polynomials A_n for the nonlinear term $F(u)$ can be evaluated by using the following expression

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \cdots. \tag{8.12}$$

The general formula (8.12) can be simplified as follows. Assuming that the nonlinear function is $F(u)$, therefore by using (8.12), Adomian polynomials [3] are given by

$$\begin{aligned}
A_0 &= F(u_0), \\
A_1 &= u_1 F'(u_0), \\
A_2 &= u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0), \\
A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0), \\
A_4 &= u_4 F'(u_0) + (\frac{1}{2!} u_2^2 + u_1 u_3) F''(u_0) + \frac{1}{2!} u_1^2 u_2 F'''(u_0) + \frac{1}{4!} u_1^4 F^{(4)}(u_0).
\end{aligned} \tag{8.13}$$

Other polynomials can be generated in a similar manner.

Two important observations can be made here. First, A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends only on u_0, u_1 and u_2, and so on. Second, substituting (8.13) into (8.11) gives

$$\begin{aligned}
F(u) &= A_0 + A_1 + A_2 + A_3 + \cdots \\
&= F(u_0) + (u_1 + u_2 + u_3 + \cdots) F'(u_0)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2!}(u_1^2+2u_1u_2+2u_1u_3+u_2^2+\cdots)F''(u_0)+\cdots\\
&+\frac{1}{3!}(u_1^3+3u_1^2u_2+3u_1^2u_3+6u_1u_2u_3+\cdots)F'''(u_0)+\cdots\\
=\;& F(u_0)+(u-u_0)F'(u_0)+\frac{1}{2!}(u-u_0)^2F''(u_0)+\cdots.
\end{aligned}
$$

The last expansion confirms a fact that the series in A_n polynomials is a Taylor series about a function u_0 and not about a point as is usually used. The few Adomian polynomials given above in (8.13) clearly show that the sum of the subscripts of the components of u of each term of A_n is equal to n. As stated before, it is clear that A_0 depends only on u_0, A_1 depends only u_0 and u_1, A_2 depends only on u_0, u_1 and u_2. The same conclusion holds for other polynomials.

In the following, we will calculate Adomian polynomials for several forms of nonlinearity that may arise in nonlinear ordinary or partial differential equations.

Calculation of Adomian Polynomials A_n

I. Nonlinear Polynomials

Case 1. $F(u)=u^2$

The polynomials can be obtained as follows:

$$
\begin{aligned}
A_0 &= F(u_0)=u_0^2,\\
A_1 &= u_1F'(u_0)=2u_0u_1,\\
A_2 &= u_2F'(u_0)+\frac{1}{2!}u_1^2F''(u_0)=2u_0u_2+u_1^2,\\
A_3 &= u_3F'(u_0)+u_1u_2F''(u_0)+\frac{1}{3!}u_1^3F'''(u_0)=2u_0u_3+2u_1u_2.
\end{aligned}
$$

Case 2. $F(u)=u^3$

The polynomials are given by

$$
\begin{aligned}
A_0 &= F(u_0)=u_0^3,\\
A_1 &= u_1F'(u_0)=3u_0^2u_1,\\
A_2 &= u_2F'(u_0)+\frac{1}{2!}u_1^2F''(u_0)=3u_0^2u_2+3u_0u_1^2,\\
A_3 &= u_3F'(u_0)+u_1u_2F''(u_0)+\frac{1}{3!}u_1^3F'''(u_0)=3u_0^2u_3+6u_0u_1u_2+u_1^3.
\end{aligned}
$$

Case 3. $F(u)=u^4$

Proceeding as before we find

$$
A_0 = u_0^4,
$$

$$A_1 = 4u_0^3 u_1,$$
$$A_2 = 4u_0^3 u_2 + 6u_0^2 u_1^2,$$
$$A_3 = 4u_0^3 u_3 + 4u_1^3 u_0 + 12u_0^2 u_1 u_2.$$

In a parallel manner, Adomian polynomials can be calculated for nonlinear polynomials of higher degrees.

II. Nonlinear Derivatives

Case 1. $F(u) = (u_x)^2$

$$A_0 = u_{0_x}^2,$$
$$A_1 = 2u_{0_x} u_{1_x},$$
$$A_2 = 2u_{0_x} u_{2_x} + u_{1_x}^2,$$
$$A_3 = 2u_{0_x} u_{3_x} + 2u_{1_x} u_{2_x}.$$

Case 2. $F(u) = u_x^3$

The Adomian polynomials are given by

$$A_0 = u_{0_x}^3,$$
$$A_1 = 3u_{0_x}^2 u_{1_x},$$
$$A_2 = 3u_{0_x}^2 u_{2_x} + 3u_{0_x} u_{1_x}^2,$$
$$A_3 = 3u_{0_x}^2 u_{3_x} + 6u_{0_x} u_{1_x} u_{2_x} + u_{1_x}^3.$$

Case 3. $F(u) = uu_x = \frac{1}{2} L_x(u^2)$

The Adomian polynomials for this nonlinearity are given by

$$A_0 = F(u_0) = u_0 u_{0_x},$$
$$A_1 = \frac{1}{2} L_x(2u_0 u_1) = u_{0_x} u_1 + u_0 u_{1_x},$$
$$A_2 = \frac{1}{2} L_x(2u_0 u_2 + u_1^2) = u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0,$$
$$A_3 = \frac{1}{2} L_x(2u_0 u_3 + 2u_1 u_2) = u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0.$$

III. Trigonometric Nonlinearity

Case 1. $F(u) = \sin u$

The Adomian polynomials for this form of nonlinearity are given by

$$A_0 = \sin u_0,$$
$$A_1 = u_1 \cos u_0,$$

$$A_2 = u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0,$$

$$A_3 = u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0.$$

Case 2. $F(u) = \cos u$

Proceeding as before gives

$$A_0 = \cos u_0,$$

$$A_1 = -u_1 \sin u_0,$$

$$A_2 = -u_2 \sin u_0 - \frac{1}{2!} u_1^2 \cos u_0,$$

$$A_3 = -u_3 \sin u_0 - u_1 u_2 \cos u_0 + \frac{1}{3!} u_1^3 \sin u_0.$$

IV. Hyperbolic Nonlinearity

Case 1. $F(u) = \sinh u$

The A_n polynomials for this form of nonlinearity are given by

$$A_0 = \sinh u_0,$$

$$A_1 = u_1 \cosh u_0,$$

$$A_2 = u_2 \cosh u_0 + \frac{1}{2!} u_1^2 \sinh u_0,$$

$$A_3 = u_3 \cosh u_0 + u_1 u_2 \sinh u_0 + \frac{1}{3!} u_1^3 \cosh u_0.$$

Case 2. $F(u) = \cosh u$

The Adomian polynomials are given by

$$A_0 = \cosh u_0,$$

$$A_1 = u_1 \sinh u_0,$$

$$A_2 = u_2 \sinh u_0 + \frac{1}{2!} u_1^2 \cosh u_0,$$

$$A_3 = u_3 \sinh u_0 + u_1 u_2 \cosh u_0 + \frac{1}{3!} u_1^3 \sinh u_0.$$

V. Exponential Nonlinearity

Case 1. $F(u) = e^u$

The Adomian polynomials for this form of nonlinearity are given by

$$A_0 = e^{u_0},$$

$$A_1 = u_1 e^{u_0},$$

$$A_2 = (u_2 + \frac{1}{2!}u_1^2)e^{u_0},$$
$$A_3 = (u_3 + u_1u_2 + \frac{1}{3!}u_1^3)e^{u_0}.$$

Case 2. $F(u) = e^{-u}$

Proceeding as before gives

$$A_0 = e^{-u_0},$$
$$A_1 = -u_1e^{-u_0},$$
$$A_2 = (-u_2 + \frac{1}{2!}u_1^2)e^{-u_0},$$
$$A_3 = (-u_3 + u_1u_2 - \frac{1}{3!}u_1^3)e^{-u_0}.$$

VI. Logarithmic Nonlinearity

Case 1. $F(u) = \ln u, u > 0$

The A_n polynomials for logarithmic nonlinearity are give by

$$A_0 = \ln u_0,$$
$$A_1 = \frac{u_1}{u_0},$$
$$A_2 = \frac{u_2}{u_0} - \frac{1}{2}\frac{u_1^2}{u_0^2},$$
$$A_3 = \frac{u_3}{u_0} - \frac{u_1u_2}{u_0^2} + \frac{1}{3}\frac{u_1^3}{u_0^3}.$$

Case 2. $F(u) = \ln(1+u), -1 < u \leqslant 1$

The A_n polynomials are give by

$$A_0 = \ln(1+u_0),$$
$$A_1 = \frac{u_1}{1+u_0},$$
$$A_2 = \frac{u_2}{1+u_0} - \frac{1}{2}\frac{u_1^2}{(1+u_0)^2},$$
$$A_3 = \frac{u_3}{1+u_0} - \frac{u_1u_2}{(1+u_0)^2} + \frac{1}{3}\frac{u_1^3}{(1+u_0)^3}.$$

8.2.2 Alternative Algorithm for Calculating Adomian Polynomials

It is worth noting that a considerable amount of research work has been invested to develop an alternative method to Adomian algorithm for calculating Adomian polynomials A_n. The aim was to develop a practical technique that will calculate Adomian polynomials in a practical way without any need to the formulae introduced before. However, the methods developed so far in this regard are identical to that used by Adomian.

We believe that a simple and reliable technique can be established to make the calculations less dependable on the formulae presented before.

In the following, we will introduce an alternative algorithm that can be used to calculate Adomian polynomials for nonlinear terms in an easy way. The newly developed method in [15–16] depends mainly on algebraic and trigonometric identities, and on Taylor expansions as well. Moreover, we should use the fact that the sum of subscripts of the components of u in each term of the polynomial A_n is equal to n.

The alternative algorithm suggests that we substitute u as a sum of components $u_n, n \geqslant 0$ as defined by the decomposition method. It is clear that A_0 is always determined independent of the other polynomials $A_n, n \geqslant 1$, where A_0 is defined by

$$A_0 = F(u_0). \tag{8.14}$$

The alternative method assumes that we first separate $A_0 = F(u_0)$ for every nonlinear term $F(u)$. With this separation done, the remaining components of $F(u)$ can be expanded by using algebraic operations, trigonometric identities, and Taylor series as well. We next collect all terms of the expansion obtained such that the sum of the subscripts of the components of u in each term is the same. Having collected these terms, the calculation of the Adomian polynomials is thus completed. Several examples have been tested, and the obtained results have shown that Adomian polynomials can be elegantly computed without any need to the formulas established by Adomian. The technique will be explained by discussing the following illustrative examples.

Adomian Polynomials by Using the Alternative Method

I. Nonlinear Polynomials

Case 1. $F(u) = u^2$

We first set

$$u = \sum_{n=0}^{\infty} u_n. \tag{8.15}$$

Substituting (8.15) into $F(u) = u^2$ gives

$$F(u) = (u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \cdots)^2. \tag{8.16}$$

Expanding the expression at the right hand side gives

$$F(u) = u_0^2 + 2u_0u_1 + 2u_0u_2 + u_1^2 + 2u_0u_3 + 2u_1u_2 + \cdots. \tag{8.17}$$

The expansion in (8.17) can be rearranged by grouping all terms with the sum of the subscripts is the same. This means that we can rewrite (8.17) as

$$\begin{aligned} F(u) = {} & \underbrace{u_0^2}_{A_0} + \underbrace{2u_0u_1}_{A_1} + \underbrace{2u_0u_2 + u_1^2}_{A_2} + \underbrace{2u_0u_3 + 2u_1u_2}_{A_3} \\ & + \underbrace{2u_0u_4 + 2u_1u_3 + u_2^2}_{A_4} + \underbrace{2u_0u_5 + 2u_1u_4 + 2u_2u_3}_{A_5} + \cdots. \end{aligned} \tag{8.18}$$

This completes the determination of Adomian polynomials given by

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0u_1, \\ A_2 &= 2u_0u_2 + u_1^2, \\ A_3 &= 2u_0u_3 + 2u_1u_2, \\ A_4 &= 2u_0u_4 + 2u_1u_3 + u_2^2, \\ A_5 &= 2u_0u_5 + 2u_1u_4 + 2u_2u_3. \end{aligned}$$

Case 2. $F(u) = u^3$

Proceeding as before, we set

$$u = \sum_{n=0}^{\infty} u_n. \tag{8.19}$$

Substituting (8.19) into $F(u) = u^3$ gives

$$F(u) = (u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \cdots)^3. \tag{8.20}$$

Expanding the right hand side yields

$$\begin{aligned} F(u) = {} & u_0^3 + 3u_0^2u_1 + 3u_0^2u_2 + 3u_0u_1^2 + 3u_0^2u_3 + 6u_0u_1u_2 + u_1^3 \\ & + 3u_0^2u_4 + 3u_1^2u_2 + 3u_2^2u_0 + 6u_0u_1u_3 \cdots. \end{aligned} \tag{8.21}$$

The expansion in (8.21) can be rearranged by grouping all terms with the sum of the subscripts is the same. This means that we can rewrite (8.21) as

$$\begin{aligned} F(u) = {} & \underbrace{u_0^3}_{A_0} + \underbrace{3u_0^2u_1}_{A_1} + \underbrace{3u_0^2u_2 + 3u_0u_1^2}_{A_2} + \underbrace{3u_0^2u_3 + 6u_0u_1u_2 + u_1^3}_{A_3} \\ & + \underbrace{3u_0^2u_4 + 3u_1^2u_2 + 3u_2^2u_0 + 6u_0u_1u_3}_{A_4} + \cdots. \end{aligned} \tag{8.22}$$

Consequently, Adomian polynomials can be written by

$$\begin{aligned} A_0 &= u_0^3, \\ A_1 &= 3u_0^2u_1, \\ A_2 &= 3u_0^2u_2 + 3u_0u_1^2, \end{aligned}$$

$$A_3 = 3u_0^2u_3 + 6u_0u_1u_2 + u_1^3,$$
$$A_4 = 3u_0^2u_4 + 3u_1^2u_2 + 3u_2^2u_0 + 6u_0u_1u_3.$$

II. Nonlinear Derivatives

Case 1. $F(u) = u_x^2$

We first set

$$u_x = \sum_{n=0}^{\infty} u_{n_x}. \tag{8.23}$$

Substituting (8.23) into $F(u) = u_x^2$ gives

$$F(u) = (u_{0_x} + u_{1_x} + u_{2_x} + u_{3_x} + u_{4_x} + \cdots)^2. \tag{8.24}$$

Squaring the right side gives

$$F(u) = u_{0_x}^2 + 2u_{0_x}u_{1_x} + 2u_{0_x}u_{2_x} + u_{1_x}^2 + 2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x} + \cdots. \tag{8.25}$$

Grouping the terms as discussed above we find

$$\begin{aligned} F(u) = & \underbrace{u_{0_x}^2}_{A_0} + \underbrace{2u_{0_x}u_{1_x}}_{A_1} + \underbrace{2u_{0_x}u_{2_x} + u_{1_x}^2}_{A_2} \\ & + \underbrace{2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x}}_{A_3} + \underbrace{u_{2_x}^2 + 2u_{0_x}u_{4_x} + 2u_{1_x}u_{3_x}}_{A_4} + \cdots. \end{aligned} \tag{8.26}$$

Adomian polynomials are given by

$$A_0 = u_{0_x}^2,$$
$$A_1 = 2u_{0_x}u_{1_x},$$
$$A_2 = 2u_{0_x}u_{2_x} + u_{1_x}^2,$$
$$A_3 = 2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x},$$
$$A_4 = 2u_{0_x}u_{4_x} + 2u_{1_x}u_{3_x} + u_{2_x}^2.$$

Case 2. $F(u) = uu_x$

We first set

$$\begin{aligned} u &= \sum_{n=0}^{\infty} u_n, \\ u_x &= \sum_{n=0}^{\infty} u_{n_x}. \end{aligned} \tag{8.27}$$

Substituting (8.27) into $F(u) = uu_x$ yields

$$\begin{aligned} F(u) = & (u_0 + u_1 + u_2 + u_3 + u_4 + \cdots)\times \\ & (u_{0_x} + u_{1_x} + u_{2_x} + u_{3_x} + u_{4_x} + \cdots). \end{aligned} \tag{8.28}$$

Multiplying the two factors gives

$$\begin{aligned} F(u) = {} & u_0 u_{0_x} + u_{0_x} u_1 + u_0 u_{1_x} + u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0 + u_{0_x} u_3 \\ & + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0 + u_{0_x} u_4 + u_0 u_{4_x} + u_{1_x} u_3 \\ & + u_1 u_{3_x} + u_2 u_{2_x} + \cdots . \end{aligned} \tag{8.29}$$

Proceeding with grouping the terms we obtain

$$\begin{aligned} F(u) = {} & \underbrace{u_{0_x} u_0}_{A_0} + \underbrace{u_{0_x} u_1 + u_{1_x} u_0}_{A_1} + \underbrace{u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0}_{A_2} \\ & + \underbrace{u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0}_{A_3} \\ & + \underbrace{u_{0_x} u_4 + u_{1_x} u_3 + u_{2_x} u_2 + u_{3_x} u_1 + u_{4_x} u_0}_{A_4} + \cdots . \end{aligned} \tag{8.30}$$

It then follows that Adomian polynomials are given by

$$\begin{aligned} A_0 &= u_{0_x} u_0, \\ A_1 &= u_{0_x} u_1 + u_{1_x} u_0, \\ A_2 &= u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0, \\ A_3 &= u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0, \\ A_4 &= u_{0_x} u_4 + u_{1_x} u_3 + u_{2_x} u_2 + u_{3_x} u_1 + u_{4_x} u_0. \end{aligned}$$

III. Trigonometric Nonlinearity

Case 1. $F(u) = \sin u$

Note that algebraic operations cannot be applied here. Therefore, our main aim is to separate $A_0 = F(u_0)$ from other terms. To achieve this goal, we first substitute

$$u = \sum_{n=0}^{\infty} u_n, \tag{8.31}$$

into $F(u) = \sin u$ to obtain

$$F(u) = \sin[u_0 + (u_1 + u_2 + u_3 + u_4 + \cdots)]. \tag{8.32}$$

To calculate A_0, recall the trigonometric identity

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi. \tag{8.33}$$

Accordingly, Equation (8.32) becomes

$$\begin{aligned} F(u) = {} & \sin u_0 \cos(u_1 + u_2 + u_3 + u_4 + \cdots) \\ & + \cos u_0 \sin(u_1 + u_2 + u_3 + u_4 + \cdots). \end{aligned} \tag{8.34}$$

Separating $F(u_0) = \sin u_0$ from other factors and using Taylor expansions for $\cos(u_1 + u_2 \cdots)$ and $\sin(u_1 + u_2 + \cdots)$ give

$$\begin{aligned} F(u) = \sin u_0 & \left(1 - \frac{1}{2!}(u_1 + u_2 + \cdots)^2 + \frac{1}{4!}(u_1 + u_2 + \cdots)^4 - \cdots\right) \\ & + \cos u_0 \left((u_1 + u_2 + \cdots) - \frac{1}{3!}(u_1 + u_2 + \cdots)^3 + \cdots\right), \end{aligned} \tag{8.35}$$

so that

$$\begin{aligned} F(u) = \sin u_0 & \left(1 - \frac{1}{2!}(u_1^2 + 2u_1u_2 + \cdots)\right) \\ & + \cos u_0 \left((u_1 + u_2 + \cdots) - \frac{1}{3!}u_1^3 + \cdots\right). \end{aligned} \tag{8.36}$$

Note that we expanded the algebraic terms; then few terms of each expansion are listed. The last expansion can be rearranged by grouping all terms with the same sum of subscripts. This means that Eq. (8.36) can be rewritten in the form

$$\begin{aligned} F(u) = & \underbrace{\sin u_0}_{A_0} + \underbrace{u_1 \cos u_0}_{A_1} + \underbrace{u_2 \cos u_0 - \frac{1}{2!}u_1^2 \sin u_0}_{A_2} \\ & + \underbrace{u_3 \cos u_0 - u_1u_2 \sin u_0 - \frac{1}{3!}u_1^3 \cos u_0}_{A_3} + \cdots \end{aligned} \tag{8.37}$$

Case 2. $F(u) = \cos u$

Proceeding as before we obtain

$$\begin{aligned} F(u) = & \underbrace{\cos u_0}_{A_0} - \underbrace{u_1 \sin u_0}_{A_1} + \underbrace{(-u_2 \sin u_0 - \frac{1}{2!}u_1^2 \cos u_0)}_{A_2} \\ & + \underbrace{(-u_3 \sin u_0 - u_1u_2 \cos u_0 + \frac{1}{3!}u_1^3 \sin u_0)}_{A_3} + \cdots \end{aligned} \tag{8.38}$$

IV. Hyperbolic Nonlinearity

Case 1. $F(u) = \sinh u$

To calculate the A_n polynomials for $F(u) = \sinh u$, we first substitute

$$u = \sum_{n=0}^{\infty} u_n, \tag{8.39}$$

into $F(u) = \sinh u$ to obtain

$$F(u) = \sinh(u_0 + (u_1 + u_2 + u_3 + u_4 + \cdots)). \tag{8.40}$$

To calculate A_0, recall the hyperbolic identity

$$\sinh(\theta + \phi) = \sinh\theta \cosh\phi + \cosh\theta \sinh\phi. \tag{8.41}$$

Accordingly, Eq. (8.40) becomes

$$\begin{aligned} F(u) &= \sinh u_0 \cosh(u_1 + u_2 + u_3 + u_4 + \cdots) \\ &\quad + \cosh u_0 \sinh(u_1 + u_2 + u_3 + u_4 + \cdots). \end{aligned} \tag{8.42}$$

Separating $F(u_0) = \sinh u_0$ from other factors and using Taylor expansions for $\cosh(u_1 + u_2 + \cdots)$ and $\sinh(u_1 + u_2 + \cdots)$ give

$$\begin{aligned} F(u) &= \sinh u_0 \\ &\quad \times \left(1 + \frac{1}{2!}(u_1 + u_2 + \cdots)^2 + \frac{1}{4!}(u_1 + u_2 + \cdots)^4 + \cdots\right) \\ &\quad + \cosh u_0 \left((u_1 + u_2 + \cdots) + \frac{1}{3!}(u_1 + u_2 + \cdots)^3 + \cdots\right) \\ &= \sinh u_0 \left(1 + \frac{1}{2!}(u_1^2 + 2u_1u_2 + \cdots)\right) \\ &\quad + \cosh u_0 \left((u_1 + u_2 + \cdots) + \frac{1}{3!}u_1^3 + \cdots\right). \end{aligned}$$

By grouping all terms with the same sum of subscripts we find

$$\begin{aligned} F(u) &= \underbrace{\sinh u_0}_{A_0} + \underbrace{u_1 \cosh u_0}_{A_1} + \underbrace{u_2 \cosh u_0 + \frac{1}{2!}u_1^2 \sinh u_0}_{A_2} \\ &\quad + \underbrace{u_3 \cosh u_0 + u_1u_2 \sinh u_0 + \frac{1}{3!}u_1^3 \cosh u_0}_{A_3} + \cdots. \end{aligned} \tag{8.43}$$

Case 2. $F(u) = \cosh u$

Proceeding as in $\sinh x$ we find

$$\begin{aligned} F(u) &= \underbrace{\cosh u_0}_{A_0} + \underbrace{u_1 \sinh u_0}_{A_1} + \underbrace{u_2 \sinh u_0 + \frac{1}{2!}u_1^2 \cosh u_0}_{A_2} \\ &\quad + \underbrace{u_3 \sinh u_0 + u_1u_2 \cosh u_0 + \frac{1}{3!}u_1^3 \sinh u_0}_{A_3} + \cdots. \end{aligned} \tag{8.44}$$

V. Exponential Nonlinearity

Case 1. $F(u) = e^u$

Substituting

$$u = \sum_{n=0}^{\infty} u_n, \tag{8.45}$$

into $F(u) = e^u$ gives

$$F(u) = e^{(u_0+u_1+u_2+u_3+\cdots)}, \tag{8.46}$$

or equivalently

$$F(u) = e^{u_0} e^{(u_1+u_2+u_3+\cdots)}. \tag{8.47}$$

Keeping the term e^{u_0} and using the Taylor expansion for the other factor we obtain

$$F(u) = e^{u_0} \times \left(1 + (u_1 + u_2 + u_3 + \cdots) + \frac{1}{2!}(u_1 + u_2 + u_3 + \cdots)^2 + \cdots\right). \tag{8.48}$$

By grouping all terms with identical sum of subscripts we find

$$\begin{aligned} F(u) = {} & \underbrace{e^{u_0}}_{A_0} + \underbrace{u_1 e^{u_0}}_{A_1} + \underbrace{(u_2 + \frac{1}{2!}u_1^2)e^{u_0}}_{A_2} + \underbrace{(u_3 + u_1 u_2 + \frac{1}{3!}u_1^3)e^{u_0}}_{A_3} \\ & + \underbrace{(u_4 + u_1 u_3 + \frac{1}{2!}u_2^2 + \frac{1}{2!}u_1^2 u_2 + \frac{1}{4!}u_1^4)e^{u_0}}_{A_4} + \cdots. \end{aligned} \tag{8.49}$$

Case 2. $F(u) = e^{-u}$

Proceeding as before we find

$$\begin{aligned} F(u) = {} & \underbrace{e^{-u_0}}_{A_0} + \underbrace{(-u_1)e^{-u_0}}_{A_1} + \underbrace{(-u_2 + \frac{1}{2!}u_1^2)e^{-u_0}}_{A_2} \\ & + \underbrace{(-u_3 + u_1 u_2 - \frac{1}{3!}u_1^3)e^{-u_0}}_{A_3} \\ & + \underbrace{(-u_4 + u_1 u_3 + \frac{1}{2!}u_2^2 - \frac{1}{2!}u_1^2 u_2 + \frac{1}{4!}u_1^4)e^{-u_0}}_{A_4} + \cdots. \end{aligned} \tag{8.50}$$

VI. Logarithmic Nonlinearity

Case 1. $F(u) = \ln u, u > 0$

Substituting

$$u = \sum_{n=0}^{\infty} u_n, \tag{8.51}$$

into $F(u) = \ln u$ gives

$$F(u) = \ln(u_0 + u_1 + u_2 + u_3 + \cdots). \tag{8.52}$$

Equation (8.52) can be written as

$$F(u) = \ln\left(u_0\left(1 + \frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right)\right). \tag{8.53}$$

Using the fact that $\ln(\alpha\beta) = \ln\alpha + \ln\beta$, Equation (8.53) becomes

$$F(u) = \ln u_0 + \ln\left(1 + \frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right). \tag{8.54}$$

Separating $F(u_0) = \ln u_0$ and using the Taylor expansion for the remaining term we obtain

$$\begin{aligned} F(u) = \ln u_0 + \Bigg\{ & \left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right) - \frac{1}{2}\left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right)^2 \\ & + \frac{1}{3}\left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right)^3 - \frac{1}{4}\left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right)^4 + \cdots \Bigg\}. \end{aligned} \tag{8.55}$$

Proceeding as before, Equation (8.55) can be written as

$$F(u) = \underbrace{\ln u_0}_{A_0} + \underbrace{\frac{u_1}{u_0}}_{A_1} + \underbrace{\frac{u_2}{u_0} - \frac{1}{2}\frac{u_1^2}{u_0^2}}_{A_2} + \underbrace{\frac{u_3}{u_0} - \frac{u_1 u_2}{u_0^2} + \frac{1}{3}\frac{u_1^3}{u_0^3}}_{A_3} + \cdots. \tag{8.56}$$

Case 2. $F(u) = \ln(1+u), -1 < u \leqslant 1$

In a like manner we obtain

$$\begin{aligned} F(u) = & \underbrace{\ln(1+u_0)}_{A_0} + \underbrace{\frac{u_1}{1+u_0}}_{A_1} + \underbrace{\frac{u_2}{1+u_0} - \frac{1}{2}\frac{u_1^2}{(1+u_0)^2}}_{A_2} \\ & + \underbrace{\frac{u_3}{1+u_0} - \frac{u_1 u_2}{(1+u_0)^2} + \frac{1}{3}\frac{u_1^3}{(1+u_0)^3}}_{A_3} + \cdots. \end{aligned} \tag{8.57}$$

As stated before, there are other methods that can be used to evaluate Adomian polynomials. However, these methods suffer from the huge size of calculations. For this reason, the most commonly used methods are presented in this text.

Exercises 8.2

Use Adomian algorithm or the alternative method to calculate the first four Adomian polynomials of the following nonlinear terms:

1. $F(u) = u^4$
2. $F(u) = u^2 + u^3$
3. $F(u) = \cos 2u$
4. $F(u) = \sinh 2u$
5. $F(u) = e^{2u}$
6. $F(u) = u^2 u_x$
7. $F(u) = u u_x^2$
8. $F(u) = u e^u$
9. $F(u) = u \sin u$
10. $F(u) = u \cosh u$
11. $F(u) = u^2 + \sin u$
12. $F(u) = u + \cos u$
13. $F(u) = u + \ln u, u > 0$
14. $F(u) = u \ln u, u > 0$
15. $F(u) = u^{\frac{1}{2}}, u > 0$
16. $F(u) = u^{-1}, u > 0$

8.3 Nonlinear ODEs by Adomian Method

Although this book is devoted to investigate partial differential equations, it seems useful to employ the Adomian decomposition method first to nonlinear ordinary differential equations. It is well known that nonlinear ordinary differential equations are, in general, difficult to handle. The Adomian decomposition method will be applied in a direct manner as discussed in previous chapters except that nonlinear terms should be represented by the so called Adomian polynomials. It is interesting to point out that the modified decomposition method and the noise terms phenomenon that were introduced in Chapter 2 will be used here at proper places.

Recall that in solving differential or integral equations, solutions are usually obtained as exact solutions defined in closed form expressions, or as series solutions normally obtained from concrete problems.

To apply the Adomian decomposition method for solving nonlinear ordinary differential equations, we consider the equation

$$Ly + R(y) + F(y) = g(x), \tag{8.58}$$

where the differential operator L may be considered as the highest order derivative in the equation, R is the remainder of the differential operator, $F(y)$ expresses the nonlinear terms, and $g(x)$ is an inhomogeneous term. If L is a first order operator defined by

$$L = \frac{d}{dx}, \tag{8.59}$$

then, we assume that L is invertible and the inverse operator L^{-1} is given by

$$L^{-1}(\cdot) = \int_0^x (\cdot)dx, \tag{8.60}$$

so that

$$L^{-1}Ly = y(x) - y(0). \tag{8.61}$$

However, if L is a second order differential operator given by

$$L = \frac{d^2}{dx^2}, \tag{8.62}$$

so that the inverse operator L^{-1} is regarded a two-fold integration operator defined by

$$L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)dx\,dx, \tag{8.63}$$

which means that

$$L^{-1}Ly = y(x) - y(0) - xy'(0). \tag{8.64}$$

In a parallel manner, if L is a third order differential operator, we can easily show that

$$L^{-1}Ly = y(x) - y(0) - xy'(0) - \frac{1}{2!}x^2y''(0). \tag{8.65}$$

For higher order operators we can easily define the related inverse operators in a similar way.

Applying L^{-1} to both sides of (8.58) gives

$$y(x) = \psi_0 - L^{-1}g(x) - L^{-1}Ry - L^{-1}F(y), \tag{8.66}$$

where

$$\psi_0 = \begin{cases} y(0), & \text{for } L = \dfrac{d}{dx}, \\ y(0) + xy'(0), & \text{for } L = \dfrac{d^2}{dx^2}, \\ y(0) + xy'(0) + \dfrac{1}{2!}x^2 y''(0), & \text{for } L = \dfrac{d^3}{dx^3}, \\ y(0) + xy'(0) + \dfrac{1}{2!}x^2 y''(0) + \dfrac{1}{3!}x^3 y'''(0), & \text{for } L = \dfrac{d^4}{dx^4}, \\ y(0) + xy'(0) + \dfrac{1}{2!}x^2 y''(0) + \dfrac{1}{3!}x^3 y'''(0) + \dfrac{1}{4!}x^4 y^{(4)}(0), & \text{for } L = \dfrac{d^5}{dx^5}, \end{cases} \tag{8.67}$$

and so on. The Adomian decomposition method admits the decomposition of y into an infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n, \tag{8.68}$$

and the nonlinear term $F(y)$ be equated to an infinite series of polynomials

$$F(y) = \sum_{n=0}^{\infty} A_n, \tag{8.69}$$

where A_n are the Adomian polynomials. Substituting (8.68) and (8.69) into (8.66) gives

$$\sum_{n=0}^{\infty} y_n = \psi_0 - L^{-1}g(x) - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{8.70}$$

The various components y_n of the solution y can be easily determined by using the recursive relation

$$\begin{aligned} y_0 &= \psi_0 - L^{-1}(g(x)), \\ y_{k+1} &= -L^{-1}(Ry_k) - L^{-1}(A_k), k \geqslant 0. \end{aligned} \tag{8.71}$$

Consequently, the first few components can be written as

$$\begin{aligned} y_0 &= \psi_0 - L^{-1}g(x), \\ y_1 &= -L^{-1}(Ry_0) - L^{-1}(A_0), \\ y_2 &= -L^{-1}(Ry_1) - L^{-1}(A_1), \\ y_3 &= -L^{-1}(Ry_2) - L^{-1}(A_2), \\ y_4 &= -L^{-1}(Ry_3) - L^{-1}(A_3). \end{aligned} \tag{8.72}$$

Having determined the components y_n, $n \geqslant 0$, the solution y in a series form follows immediately. As stated before, the series may be summed to provide the solution in a closed form. However, for concrete problems, the $n-$term partial sum

$$\phi_n = \sum_{k=0}^{n-1} y_k, \tag{8.73}$$

may be used to give the approximate solution.

In the following, several examples will be discussed for illustration.

Example 1. Solve the first order nonlinear ordinary differential equation

$$y' - y^2 = 1, y(0) = 0. \tag{8.74}$$

Solution.

In an operator form, Eq. (8.74) can be written as

$$Ly = 1 + y^2, y(0) = 0, \tag{8.75}$$

where L is a first order differential operator. It is clear that L^{-1} is invertible and given by

$$L^{-1}(\cdot) = \int_0^x (\cdot)\, dx. \tag{8.76}$$

Applying L^{-1} to both sides of (8.75) and using the initial condition give

$$y = x + L^{-1}(y^2). \tag{8.77}$$

The decomposition method suggests that the solution $y(x)$ be expressed by the decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{8.78}$$

and the nonlinear terms y^2 be equated to

$$y^2 = \sum_{n=0}^{\infty} A_n, \tag{8.79}$$

where $y_n(x), n \geqslant 0$ are the components of $y(x)$ that will be determined recursively, and A_n, $n \geqslant 0$ are the Adomian polynomials that represent the nonlinear term y^2.

Inserting (8.78) and (8.79) into (8.77) yields

$$\sum_{n=0}^{\infty} y_n(x) = x + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{8.80}$$

The zeroth component y_0 is usually defined by all terms that are not included under the operator L^{-1}. The remaining components can be determined recurrently such that each term is determined by using the previous component. Consequently, the components of $y(x)$ can be elegantly determined by using the recursive relation

$$\begin{aligned} y_0(x) &= x, \\ y_{k+1}(x) &= L^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{8.81}$$

Note that the Adomian polynomials A_n for the nonlinear term y^2 were determined before by using Adomian algorithm and by using the alternative method where we found

$$\begin{aligned}
A_0 &= y_0^2, \\
A_1 &= 2y_0y_1, \\
A_2 &= 2y_0y_2 + y_1^2, \\
A_3 &= 2y_0y_3 + 2y_1y_2, \\
A_4 &= 2y_0y_4 + 2y_1y_3 + y_2^2.
\end{aligned}$$

and so on. Using these polynomials into (8.81), the first few components can be determined recursively by

$$\begin{aligned}
y_0(x) &= x, \\
y_1(x) &= L^{-1}A_0 = L^{-1}(y_0^2) = \frac{1}{3}x^3, \\
y_2(x) &= L^{-1}A_1 = L^{-1}(2y_0y_1) = \frac{2}{15}x^5, \\
y_3(x) &= L^{-1}A_2 = L^{-1}(2y_0y_2 + y_1^2) = \frac{17}{315}x^7,
\end{aligned} \tag{8.82}$$

Consequently, the solution in a series form is given by

$$y(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots, \tag{8.83}$$

and in a closed form by

$$y(x) = \tan x. \tag{8.84}$$

We point out that the ordinary differential equation (8.74) can be solved as a separable differential equation.

Example 2. Solve the first order nonlinear ordinary differential equation

$$y' + y^2 = 1, \quad y(0) = 0. \tag{8.85}$$

Solution.

Operating with L^{-1} we obtain

$$y = x - L^{-1}(y^2). \tag{8.86}$$

Using the decomposition series for the solution $y(x)$ and the polynomial representation for y^2 give

$$\sum_{n=0}^{\infty} y_n(x) = x - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{8.87}$$

This leads to the recursive relation

$$\begin{aligned} y_0 &= x, \\ y_{k+1} &= -L^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{8.88}$$

The Adomian polynomials A_n for the nonlinear term y^2 were used in the previous example, hence, the first few components can be determined recursively as

$$\begin{aligned} y_0 &= x, \\ y_1 &= -L^{-1}A_0 = -L^{-1}(y_0^2) = -\frac{1}{3}x^3, \\ y_2 &= -L^{-1}A_1 = -L^{-1}(2y_0y_1) = \frac{2}{15}x^5, \\ y_3 &= -L^{-1}A_2 = -L^{-1}(2y_0y_2 + y_1^2) = -\frac{17}{315}x^7, \end{aligned} \tag{8.89}$$

and so on. Consequently, the solution in a series form is given by

$$y(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots, \tag{8.90}$$

and in a closed form by

$$y(x) = \tanh x. \tag{8.91}$$

We point out that the ordinary differential equation (8.85) can be solved as a separable differential equation where partial fractions should be used.

Example 3. Use the modified decomposition method to solve the Riccati differential equation

$$y' = 1 - x^2 + y^2, \quad y(0) = 0. \tag{8.92}$$

Solution.

Applying the inverse operator L^{-1} we obtain

$$y = x - \frac{1}{3}x^3 + L^{-1}(y^2). \tag{8.93}$$

Using the decomposition series $y(x)$ and the polynomial representation for y^2 give

$$\sum_{n=0}^{\infty} y_n(x) = x - \frac{1}{3}x^3 + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{8.94}$$

It is important to point out that the modified decomposition method is recommended here. In this approach we split the polynomial $x - \frac{1}{3}x^3$ into two parts, namely, x will be assigned to the zeroth component y_0, and $-\frac{1}{3}x^3$ that will be assigned to the component y_1 among other terms. In this case, we use a modified recursive relation to accelerate the convergence of the solution. The modified recursive relation is defined by

$$\begin{aligned} y_0 &= x, \\ y_1 &= -\frac{1}{3}x^3 + L^{-1}(A_0), \\ y_{k+2} &= L^{-1}(A_{k+1}), \quad k \geqslant 0. \end{aligned} \tag{8.95}$$

Consequently, the first few components are given by

$$\begin{aligned} y_0 &= x, \\ y_1 &= -\frac{1}{3}x^3 + L^{-1}A_0 = -\frac{1}{3}x^3 + L^{-1}(y_0^2) = 0, \\ y_{k+2} &= 0, \quad k \geqslant 0. \end{aligned} \tag{8.96}$$

The exact solution is given by

$$y(x) = x. \tag{8.97}$$

We next consider a first order nonlinear differential equation where a closed form solution is not easily observed.

Example 4. Solve the first order nonlinear differential equation

$$y' = -y + y^2, \quad y(0) = 2. \tag{8.98}$$

Solution.

Applying the inverse operator L^{-1} and using the initial condition give

$$y(x) = 2 - L^{-1}(y) + L^{-1}(y^2). \tag{8.99}$$

We next represent the linear term $y(x)$ by the decomposition series of components y_n, $n \geqslant 0$, and equate the nonlinear term y^2 by the series of Adomian polynomials A_n, $n \geqslant 0$, to find

$$\sum_{n=0}^{\infty} y_n(x) = 2 - L^{-1}\left(\sum_{n=0}^{\infty} y_n(x)\right) + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{8.100}$$

The Adomian polynomials A_n for y^2 have been derived and used before. Following the decomposition method we set the recursive relation

$$\begin{aligned} y_0 &= 2, \\ y_{k+1} &= -L^{-1}(y_k) + L^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{8.101}$$

Consequently, the first few components of the solution are given by

$$
\begin{aligned}
y_0 &= 2,\\
y_1 &= -L^{-1}(y_0)+L^{-1}A_0 = -L^{-1}(2)+L^{-1}(4) = 2x,\\
y_2 &= -L^{-1}(y_1)+L^{-1}A_1 = -L^{-1}(2x)+L^{-1}(8x) = 3x^2,\\
y_3 &= -L^{-1}(y_2)+L^{-1}A_2 = -L^{-1}(3x^2)+L^{-1}(16x^2) = \frac{13}{3}x^3,
\end{aligned} \tag{8.102}
$$

and so on. Based on these calculations, the solution in a series form is given by

$$
y(x) = 2+2x+3x^2+\frac{13}{3}x^3+\cdots, \quad x<\ln 2. \tag{8.103}
$$

It is clear that a closed form solution is not easily observed. However, the closed form solution is given by

$$
y(x) = \frac{2}{2-e^x}. \tag{8.104}
$$

For numerical purposes, additional terms can be easily evaluated to enhance the accuracy of the approximate solution in (8.103). As will be discussed in Chapter 10, it will be shown that few terms only can lead to a high accuracy level with minimum error.

Example 5. Solve the first order nonlinear differential equation

$$
y' = \frac{y^2}{1-xy}, \quad y(0)=1. \tag{8.105}
$$

Solution.

We first rewrite the equation by

$$
y' = xyy'+y^2, \quad y(0)=1. \tag{8.106}
$$

It is useful to note that the differential equation (8.106) contains two nonlinear terms yy' and y^2. The Adomian polynomials for both terms have been derived before; hence will be implemented directly. Applying the inverse operator L^{-1} and using the initial condition give

$$
y(x) = 1+L^{-1}(xyy')+L^{-1}(y^2). \tag{8.107}
$$

We next represent the linear term $y(x)$ by the decomposition series of components $y_n, n \geqslant 0$, equate the nonlinear term yy' by the Adomian polynomials $A_n, n \geqslant 0$, and equate the nonlinear term y^2 by the series of Adomian polynomials $B_n, n \geqslant 0$, to find

$$
\sum_{n=0}^{\infty} y_n(x) = 1+L^{-1}\left(\sum_{n=0}^{\infty} xA_n\right)+L^{-1}\left(\sum_{n=0}^{\infty} B_n\right). \tag{8.108}
$$

Identifying the zeroth component y_0, and following the decomposition method we set the recursive relation

$$\begin{aligned} y_0(x) &= 1, \\ y_{k+1}(x) &= L^{-1}(xA_k) + L^{-1}(B_k), \quad k \geqslant 0. \end{aligned} \tag{8.109}$$

This relation leads to the component-by-component identification

$$\begin{aligned} y_0 &= 1, \\ y_1 &= L^{-1}(xA_0) + L^{-1}(B_0) = L^{-1}(0) + L^{-1}(1) = x, \\ y_2 &= L^{-1}(xA_1) + L^{-1}(B_1) = L^{-1}(x) + L^{-1}(2x) = \frac{3}{2}x^2, \\ y_3 &= L^{-1}(xA_2) + L^{-1}(B_2) = L^{-1}(4x^2) + L^{-1}(4x^2) = \frac{8}{3}x^3, \end{aligned} \tag{8.110}$$

and so on. Based on these calculations, the solution in a series form is given by

$$y(x) = 1 + x + \frac{3}{2}x^2 + \frac{8}{3}x^3 + \frac{125}{24}x^4 + \cdots. \tag{8.111}$$

It is clear that a closed form solution where y is expressed explicitly in terms of x cannot be found. However, the exact solution can be expressed in the implicit expression

$$y = e^{xy}. \tag{8.112}$$

In the following example, the ordinary differential equation contains an exponential nonlinearity. The Adomian polynomials for this form of nonlinearity have been calculated before.

Example 6. Solve the first order nonlinear differential equation

$$y' - e^y = 0, \quad y(0) = 1. \tag{8.113}$$

Solution.

Applying the inverse operator L^{-1} and using the initial condition give

$$y(x) = 1 + L^{-1}(e^y). \tag{8.114}$$

Equating the linear term $y(x)$ by an infinite series of components $y_n, n \geqslant 0$, and representing the nonlinear term e^y by an infinite series of Adomian polynomials $A_n, n \geqslant 0$, we obtain

$$\sum_{n=0}^{\infty} y_n(x) = 1 + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{8.115}$$

The Adomian polynomials A_n for e^y have been calculated before and given by

$$\begin{aligned} A_0 &= e^{y_0}, \\ A_1 &= y_1 e^{y_0}, \\ A_2 &= \left(y_2 + \frac{1}{2!}y_1^2\right) e^{y_0}, \end{aligned}$$

$$\begin{aligned} A_3 &= \left(\frac{1}{3!}y_1^3 + y_1y_2 + y_3\right)e^{y_0}, \\ A_4 &= \left(y_4 + y_1y_3 + \frac{1}{2!}y_2^2 + \frac{1}{2!}y_1^2y_2 + \frac{1}{4!}y_1^4\right)e^{y_0}. \end{aligned} \tag{8.116}$$

Identifying $y_0 = 1$ and applying the decomposition method give the recursive relation

$$\begin{aligned} y_0 &= 1, \\ y_{k+1} &= L^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{8.117}$$

This gives

$$\begin{aligned} y_0 &= 1, \\ y_1 &= L^{-1}A_0 = L^{-1}(e) = ex, \\ y_2 &= L^{-1}A_1 = L^{-1}(e^2x) = \frac{e^2}{2}x^2, \\ y_3 &= L^{-1}A_2 = L^{-1}(e^3x^2) = \frac{e^3}{3}x^3, \end{aligned} \tag{8.118}$$

and so on. The solution in a series form is given by

$$y(x) = 1 + ex + \frac{1}{2}(ex)^2 + \frac{1}{3}(ex)^3 + \cdots, \; -1 \leqslant ex < 1, \tag{8.119}$$

and in a closed form by

$$y(x) = 1 - \ln(1 - ex), \; -1 \leqslant ex < 1. \tag{8.120}$$

Example 7. Use the noise terms phenomenon to solve the second order nonlinear differential equation

$$y'' + (y')^2 + y^2 = 1 - \sin x, \quad y(0) = 0, y'(0) = 1. \tag{8.121}$$

Solution.

Applying the two-fold integral operator L^{-1} to both sides of Eq. (8.121) gives

$$y(x) = \sin x + \frac{1}{2}x^2 - L^{-1}\left((y')^2 + y^2\right). \tag{8.122}$$

Using the assumptions of the decomposition method yields

$$\sum_{n=0}^{\infty} y_n(x) = \sin x + \frac{1}{2}x^2 - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{8.123}$$

This leads to the recursive relation

$$\begin{aligned} y_0 &= \sin x + \frac{1}{2}x^2, \\ y_{k+1} &= -L^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{8.124}$$

This relation leads to the identification

$$\begin{aligned} y_0 &= \sin x + \frac{1}{2}x^2, \\ y_1 &= -L^{-1}((y_0')^2 + y_0^2) = -\frac{1}{2!}x^2 + \cdots. \end{aligned} \tag{8.125}$$

The zeroth component contains the trigonometric function $\sin x$, therefore it is recommended that the noise terms phenomenon be used here. By canceling the noise terms $\frac{1}{2}x^2$ and $-\frac{1}{2}x^2$ between y_0 and y_1, and justifying that the remaining non-canceled term of y_0 satisfies the differential equation leads to the exact solution given by

$$y(x) = \sin x. \tag{8.126}$$

This example shows that the noise terms phenomenon, if exists, works effectively for inhomogeneous ordinary differential equations and for inhomogeneous partial differential equations as well.

Exercises 8.3

In Exercises 1–4, use the Adomian scheme to find the exact solution of each of the following nonlinear ordinary differential equations:

1. $y' - 3y^2 = 3, y(0) = 0$
2. $y' + 4y^2 = 4, y(0) = 0$
3. $y' - y^2 = -2x - x^2, y(0) = 1$
4. $y' + e^y = 0, y(0) = 1$

In Exercises 5–8, use the Adomian decomposition method to find the exact solution of the following Riccati differential equations:

5. $y' = (2x - 3) - xy + y^2, y(0) = -2$
6. $y' = 1 - 2y + y^2, y(0) = 2$
7. $y' = 1 - x^2 + y^2, y(0) = 0$
8. $y' = -1 + xy + y^2, y(0) = 0$

In Exercises 9–12, use the Adomian decomposition method to find the first four terms of the series solution of the following first order nonlinear ordinary differential equations:

9. $y' + y = \sin y, y(0) = \frac{\pi}{2}$

10. $y' = x^2 + y^2, y(0) = 1$

11. $y' + y = y^2, y(0) = 2$

12. $y' = -\dfrac{y^2}{1+xy}, y(0) = 1$

In Exercises 13–16, use the Adomian decomposition method to find the exact solution of the following second order nonlinear ordinary differential equations:

13. $y'' + (y')^2 + y^2 = 1 - \cos x, y(0) = 1, y'(0) = 0$

14. $y'' - 2yy' = 0, y(0) = 0, y'(0) = 1$

15. $y'' + 2yy' = 0, y(0) = 0, y'(0) = 1$

16. $y'' + yy' + (y')^2 = 0, y(0) = 0, y'(0) = 1$

In Exercises 17–20, use the Adomian decomposition method to find the first four terms of the series solution of the following second order nonlinear ordinary differential equations:

17. $y'' + y^2 = 0, y(0) = 1, y'(0) = 0$

18. $y'' - y^3 = 0, y(0) = 1, y'(0) = 0$

19. $y'' - \sin y = 0, y(0) = \dfrac{\pi}{2}, y'(0) = 0$

20. $y'' - ye^y = 0, y(0) = 1, y'(0) = 0$

In Exercises 21–24, use the Adomian decomposition method to find the exact solution of the following third order nonlinear ordinary differential equations:

21. $y''' + (y')^2 - 12y' = 6, y(0) = y'(0) = y''(0) = 0$

22. $y''' + (y'')^2 + (y')^2 = 2 + \cos x, y(0) = 0, y'(0) = 2, y''(0) = 0$

23. $y''' + (y'')^2 + (y')^2 = 1 - \sin x, y(0) = 0, y'(0) = 0, y''(0) = 1$

24. $y''' - (y'')^2 + (y')^2 = 1 + \cosh x, y(0) = 0, y'(0) = 1, y''(0) = 0$

8.4 Nonlinear ODEs by VIM

The variational iteration method (VIM) was presented before. The method gives rapidly convergent successive approximations of the exact solution if an exact solution exists. The obtained approximations by this method are of high accuracy level even if few iterations are used. As introduced before, the method employs the correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(Lu_n(x,\xi) + N\tilde{u}_n(x,\xi) - g(x,\xi)\right) d\xi, \quad n \geqslant 0. \tag{8.127}$$

for the differential equation

$$Lu + Nu = g(x,t). \tag{8.128}$$

The Lagrange multiplier $\lambda(\xi)$ should be determined first. This value of λ allows us to determine the successive approximations $u_{n+1}(x,t), n \geqslant 0$, of the solution $u(x,t)$ by using any zeroth approximation $u_0(x,t)$. The exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{8.129}$$

To find λ we sometimes use

$$\begin{aligned}\int \lambda(\xi)u_n'(\xi)d\xi &= \lambda(\xi)u_n(\xi) - \int \lambda'(\xi)u_n(\xi)d\xi,\\ \int \lambda(\xi)u_n''(\xi)d\xi &= \lambda(\xi)u_n'(\xi) - \lambda'(\xi)u_n(\xi) + \int \lambda''(\xi)u_n(\xi)d\xi,\end{aligned} \tag{8.130}$$

and so on for derivatives of u_n of higher orders. The method has been used so far for handling linear problems only.

The variational iteration method will be used to handle nonlinear problems in a manner similar to that used before for linear problem [17–18]. The method facilitates the computational work for nonlinear problems compared to Adomian method. Unlike Adomian decomposition method, the variational iteration method does not require specific treatment for nonlinear operators. There is no need for Adomian polynomials that require a huge size of computational work. Moreover, the variational iteration method does not require specific assumptions or restrictive conditions as required by other methods such as perturbation techniques. The effectiveness and the efficiency of the method can be confirmed by discussing the following nonlinear ordinary differential equations. The same examples that were studied in the previous section will be examined here.

Example 1. Solve the first order nonlinear ordinary differential equation

$$y' - y^2 = 1, \quad y(0) = 0. \tag{8.131}$$

Solution.

The correction functional for equation (8.131) is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left(y_n'(\xi) - \tilde{y}_n^2(\xi) - 1\right) d\xi. \tag{8.132}$$

The stationary conditions

$$\begin{aligned}1 + \lambda|_{\xi=x} &= 0,\\ \lambda'|_{\xi=x} &= 0,\end{aligned} \tag{8.133}$$

follow immediately. This in turn gives

$$\lambda = -1. \tag{8.134}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (8.132) gives the iteration formula

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(\xi) - y_n^2(\xi) - 1\right) d\xi, \quad n \geqslant 0. \tag{8.135}$$

We can select $y_0(x) = y(0) = 0$ from the given condition. Using this selection into (8.135) we obtain the following successive approximations

$$\begin{aligned}
y_0(x) &= 0,\\
y_1(x) &= 0 - \int_0^x \left(y_0'(\xi) - y_0^2(\xi) - 1\right) d\xi = x,\\
y_2(x) &= x - \int_0^x \left(y_1'(\xi) - y_1^2(\xi) - 1\right) d\xi = x + \frac{1}{3}x^3,\\
y_3(x) &= x + \frac{1}{3}x^3 - \int_0^x \left(y_2'(\xi) - y_2^2(\xi) - 1\right) d\xi = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{1}{63}x^7,\\
y_4(x) &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{1}{63}x^7 - \int_0^x \left(y_3'(\xi) - y_3^2(\xi) - 1\right) d\xi\\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots,\\
&\vdots\\
y_n(x) &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots.
\end{aligned} \tag{8.136}$$

The VIM admits the use of

$$y(x) = \lim_{n\to\infty} y_n(x), \tag{8.137}$$

that gives the exact solution by

$$y(x) = \tan x. \tag{8.138}$$

The last result is consistent with the result obtained before by Adomian method. We point out here that the separable nonlinear ODE (8.131) is solved without any need for the so-called Adomian polynomials.

Example 2. Solve the first order nonlinear ordinary differential equation

$$y' + y^2 = 1, \quad y(0) = 0. \tag{8.139}$$

Solution.

The correction functional for equation (8.139) is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi) \left(y_n'(\xi) + \tilde{y}_n^2(\xi) - 1\right) d\xi. \tag{8.140}$$

Following the discussion in Example 1, the stationary conditions give

$$\lambda = -1. \tag{8.141}$$

Substituting $\lambda = -1$ into the functional (8.140) gives the iteration formula

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(\xi) + y_n^2(\xi) - 1\right) d\xi, n \geqslant 0. \tag{8.142}$$

We can select $y_0(x) = y(0) = 0$ from the given condition. Using this selection into (8.142) we obtain the following successive approximations

$$\begin{aligned}
y_0(x) &= 0,\\
y_1(x) &= 0 - \int_0^x \left(y_0'(\xi) + y_0^2(\xi) - 1\right) d\xi = x,\\
y_2(x) &= x - \int_0^x \left(y_1'(\xi) + y_1^2(\xi) - 1\right) d\xi = x - \frac{1}{3}x^3,\\
y_3(x) &= x - \frac{1}{3}x^3 - \int_0^x \left(y_2'(\xi) + y_2^2(\xi) - 1\right) d\xi = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{1}{63}x^7,\\
y_4(x) &= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{1}{63}x^7 - \int_0^x \left(y_3'(\xi) + y_3^2(\xi) - 1\right) d\xi\\
&= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots,\\
&\vdots\\
y_n(x) &= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots.
\end{aligned} \tag{8.143}$$

Consequently, the exact solution is

$$y(x) = \tanh x. \tag{8.144}$$

Example 3. Use the modified decomposition method to solve the Riccati differential equation

$$y' = 1 - x^2 + y^2, \quad y(0) = 0. \tag{8.145}$$

Solution.

The correction functional for equation (8.145) is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left(y_n'(\xi) - \tilde{y}_n^2(\xi) + \xi^2 - 1\right) d\xi. \tag{8.146}$$

The stationary conditions give

$$\lambda = -1. \tag{8.147}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (8.146) gives the iteration formula

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(\xi) - y_n^2(\xi) + \xi^2 - 1\right) d\xi, \quad n \geqslant 0. \tag{8.148}$$

We can select $y_0(x) = y(0) = 0$ from the given condition. Proceeding as before we obtain the successive approximations

$$\begin{aligned}
y_0(x) &= 0, \\
y_1(x) &= 0 - \int_0^x \left(y_0'(\xi) - y_0^2(\xi) + \xi^2 - 1\right) d\xi = x - \frac{1}{3}x^3, \\
y_2(x) &= x - \frac{1}{3}x^3 - \int_0^x \left(y_1'(\xi) - y_1^2(\xi) + \xi^2 - 1\right) d\xi \\
&= x - \frac{1}{3}x^3 + \frac{1}{3}x^3 - \frac{2}{15}x^5 + \frac{1}{63}x^7, \\
y_3(x) &= x - \frac{1}{3}x^3 + \frac{1}{3}x^3 - \frac{2}{15}x^5 + \frac{1}{63}x^7 - \int_0^x \left(y_2'(\xi) - y_2^2(\xi) + \xi^2 - 1\right) d\xi \\
&= x - \frac{1}{3}x^3 + \frac{1}{3}x^3 - \frac{2}{15}x^5 + \frac{2}{15}x^5 + \frac{1}{63}x^7 + \cdots, \\
&\vdots \\
y_n(x) &= x - \frac{1}{3}x^3 + \frac{1}{3}x^3 - \frac{2}{15}x^5 + \frac{2}{15}x^5 + \frac{1}{63}x^7 - \frac{1}{63}x^7 + \cdots.
\end{aligned} \tag{8.149}$$

It is clear that the noise terms vanish in the limit, and the exact solution is

$$y(x) = x. \tag{8.150}$$

Example 4. Solve the first order nonlinear differential equation

$$y' = -y + y^2, \quad y(0) = 2. \tag{8.151}$$

Solution.

The correction functional for this equation is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left(y_n'(\xi) - \tilde{y}_n^2(\xi) + \tilde{y}_n(\xi)\right) d\xi. \tag{8.152}$$

It is clear that

$$\lambda = -1. \tag{8.153}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the correction functional gives the iteration formula

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(\xi) - y_n^2(\xi) + y_n(\xi)\right) d\xi, \quad n \geqslant 0. \tag{8.154}$$

We can select $y_0(x) = y(0) = 2$ from the given condition. Using this selection into (8.154) we obtain the following successive approximations

$$
\begin{aligned}
y_0(x) &= 2, \\
y_1(x) &= 2 - \int_0^x \left(y_0'(\xi) - y_0^2(\xi) + y_0\right) d\xi = 2 + 2x, \\
y_2(x) &= 2 + 2x - \int_0^x \left(y_1'(\xi) - y_1^2(\xi) + y_1\right) d\xi = 2 + 2x + 3x^2 + \frac{4}{3}x^3, \\
y_3(x) &= 2 + 2x + 3x^2 + \frac{4}{3}x^3 - \int_0^x \left(y_2'(\xi) - y_2^2(\xi) + y_2\right) d\xi \\
&= 2 + 2x + 3x^2 + \frac{13}{3}x^3 + 4x^4 + \cdots, \\
&\ \vdots \\
y_n(x) &= 2 + 2x + 3x^2 + \frac{13}{3}x^3 + \frac{25}{4}x^4 + \frac{541}{60}x^5 + \cdots.
\end{aligned} \tag{8.155}
$$

As stated before, it is clear that a closed form solution is not easily observed. However, the closed form solution is given by

$$
y(x) = \frac{2}{2 - e^x}. \tag{8.156}
$$

Example 5. Solve the first order nonlinear differential equation by the VIM

$$
y' = \frac{y^2}{1 - xy}, \quad y(0) = 1. \tag{8.157}
$$

Solution.

We first rewrite the equation by

$$
y' = xyy' + y^2, \quad y(0) = 1. \tag{8.158}
$$

Proceeding as before, we use $\lambda = -1$. This gives the iteration formula

$$
y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(\xi) - \xi y_n y_n' - y_n^2(\xi)\right) d\xi, \quad n \geqslant 0. \tag{8.159}
$$

Selecting $y_0(x) = y(0) = 1$ gives the successive approximations

$$
\begin{aligned}
y_0(x) &= 1, \\
y_1(x) &= 1 - \int_0^x \left(y_0'(\xi) - \xi y_0 y_0' - y_0^2(\xi)\right) d\xi = 1 + x, \\
y_2(x) &= 1 + x - \int_0^x \left(y_1'(\xi) - \xi y_1 y_1' - y_1^2(\xi)\right) d\xi = 1 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3, \\
&\ \vdots \\
y_n(x) &= 1 + x + \frac{3}{2}x^2 + \frac{8}{3}x^3 + \frac{125}{24}x^4 + \cdots.
\end{aligned} \tag{8.160}
$$

The solution is given implicitly by

$$y(x) = e^{xy}. \tag{8.161}$$

In the following example, the ordinary differential equation contains an exponential nonlinearity. The Adomian polynomials for this form of nonlinearity have been calculated before.

Example 6. Solve the first order nonlinear differential equation by the VIM

$$y' - e^{y} = 0, \quad y(0) = 1. \tag{8.162}$$

Solution.

Using the Lagrange multiplier $\lambda = -1$ gives the iteration formula

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(\xi) - e^{y_n(\xi)}\right) d\xi, \quad n \geqslant 0. \tag{8.163}$$

We can select $y_0(x) = y(0) = 1$ to obtain the following successive approximations

$$\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= 1 - \int_0^x \left(y_0'(\xi) - e^{y_0}\right) d\xi = 1 + ex,\\
y_2(x) &= 1 + ex - \int_0^x \left(y_1'(\xi) - e^{y_1(\xi)}\right) d\xi = 1 + ex + \frac{1}{2}e^2x^2 + \frac{1}{6}e^3x^3 + \cdots,\\
y_3(x) &= 1 + ex + \frac{1}{2}e^2x^2 + \frac{1}{6}e^3x^3 - \int_0^x \left(y_2'(\xi) - e^{y_2(\xi)}\right) d\xi,\\
&= 1 + ex + \frac{1}{2}e^2x^2 + \frac{1}{3}e^3x^3 + \frac{1}{4}e^4x^4 + \cdots,\\
&\vdots\\
y_n(x) &= 1 + ex + \frac{1}{2}e^2x^2 + \frac{1}{3}e^3x^3 + \frac{1}{4}e^4x^4 + \frac{1}{5}e^5x^5 + \cdots, -1 \leqslant ex < 1,
\end{aligned} \tag{8.164}$$

where the Taylor series for $e^{y_i(\xi)}$ is used for integration. The exact solution is given by

$$y(x) = 1 - \ln(1 - ex), \quad -1 \leqslant ex < 1. \tag{8.165}$$

Example 7. Use the VIM to solve the second order nonlinear differential equation

$$y'' + (y')^2 + y^2 = 1 - \sin x, \quad y(0) = 0, \quad y'(0) = 1. \tag{8.166}$$

Solution.

The correction functional for this equation is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left(y_n''(\xi) + (\tilde{y}_n')^2(\xi) + \tilde{y}_n^2(\xi) + \sin\xi - 1\right) d\xi. \tag{8.167}$$

Notice that this is a second order differential equation. It was discussed in Chapters 5 and 7 that

$$\lambda = \xi - x. \tag{8.168}$$

Substituting this value of the Lagrange multiplier into the correction functional gives the iteration formula

$$y_{n+1}(x) = y_n(x) + \int_0^x (\xi - x)\left(y_n''(\xi) + (y_n')^2(\xi) + y_n^2(\xi) + \sin\xi - 1\right) d\xi, \quad n \geqslant 0. \tag{8.169}$$

We select $y_0(x) = x$ is any selective function. As a result, we obtain the following successive approximations

$$\begin{aligned}
y_0(x) &= x,\\
y_1(x) &= -\frac{1}{12}x^4 + \sin x,\\
y_2(x) &= x - \frac{1}{3!}x^3 + \frac{1}{4!}x^5 - \frac{1}{240}x^7 + \cdots,\\
y_3(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,\\
&\vdots\\
y_n(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots.
\end{aligned} \tag{8.170}$$

The exact solution is given by

$$y(x) = \sin x. \tag{8.171}$$

Notice that in evaluating the approximations, we used the Taylor series for $\sin x$.

Exercises 8.4

Use the variational iteration method to solve the problems in Exercises 8.3.

8.5 Nonlinear PDEs by Adomian Method

As stated before, nonlinear partial differential equations arise in different areas of physics, engineering, and applied mathematics such as fluid mechanics, condensed matter physics, soliton physics and quantum field theory. A considerable amount of research work has been invested in the study of numerous problems modeled by nonlinear partial differential equations. In this section, a straightforward implementation of the decomposition method to nonlinear partial differential equations will be carried out in general. However, a wide variety of physically significant problems modeled by nonlinear partial differential equations, such as the advection problem, the KdV equation, the modified KdV equation, the KP equation, Boussinesq equation, will be investigated in details in Chapter 9. Moreover, a comparative study will be conducted in Chapter 11 to show the physical behavior of the solitons concept and the recently discovered compactons: solitons with the absence of infinite wings.

An important note worth mentioning is that there is no general method that can be employed for obtaining analytical solutions for nonlinear partial differential equations. Several methods are usually used and numerical solutions are often obtained. Further, transformation methods are sometimes used to convert a nonlinear equation to an ordinary equation or to a system of ordinary differential equations. Furthermore, perturbation techniques and discretization methods, that require a massive size of computational work, are also used for some types of equations.

However, we have stated before that Adomian decomposition method can be used generally for all types of differential and integral equations. The method can be applied in a straightforward manner and it provides a rapidly convergent series solution.

The description of the decomposition method has been presented in details in the preceding chapters. However, in the following we will discuss a general description of the method that will be used for nonlinear partial differential equations. We first consider the nonlinear partial differential equation given in an operator form

$$L_x u(x,y) + L_y u(x,y) + R(u(x,y)) + F(u(x,y)) = g(x,y), \tag{8.172}$$

where L_x is the highest order differential in x, L_y is the highest order differential in y, R contains the remaining linear terms of lower derivatives, $F(u(x,y))$ is an analytic nonlinear term, and $g(x,y)$ is an inhomogeneous or forcing term.

The solutions for $u(x,y)$ obtained from the operator equations $L_x u$ and $L_y u$ are called partial solutions. It has been shown before that these partial solutions are equivalent and each converges to the exact solution. However, the decision as to which operator L_x or L_y should be used to solve the problem depends mainly on two bases:

(i) The operator of lowest order should be selected to minimize the size of computational work.

(ii) The selected operator of lowest order should be of best known conditions to accelerate the evaluation of the components of the solution.

Assuming that the operator L_x meets the two bases of selection, therefore we set

$$L_x u(x,y) = g(x,y) - L_y u(x,y) - R(u(x,y)) - F(u(x,y)). \tag{8.173}$$

Applying L_x^{-1} to both sides of (8.173) gives

$$\begin{aligned} u(x,y) = \Phi_0 &- L_x^{-1} g(x,y) - L_x^{-1} L_y u(x,y) - L_x^{-1} R(u(x,y)) \\ &- L_x^{-1} F(u(x,y)), \end{aligned} \tag{8.174}$$

where

$$\Phi_0 = \begin{cases} u(0,y) & \text{for } L = \dfrac{\partial}{\partial x}, \\ u(0,y) + xu_x(0,y) & \text{for } L = \dfrac{\partial^2}{\partial x^2}, \\ u(0,y) + xu_x(0,y) + \dfrac{1}{2!}x^2 u_{xx}(0,y) & \text{for } L = \dfrac{\partial^3}{\partial x^3}, \\ u(0,y) + xu_x(0,y) + \dfrac{1}{2!}x^2 u_{xx}(0,y) + \dfrac{1}{3!}x^3 u_{xxx}(0,y) & \text{for } L = \dfrac{\partial^4}{\partial x^4}, \end{cases}$$

We proceed in exactly the same manner by calculating the solution $u(x,y)$ in a series form

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y), \tag{8.175}$$

and the nonlinear term $F(u(x,y))$ by

$$F(u(x,y)) = \sum_{n=0}^{\infty} A_n, \tag{8.176}$$

where A_n are Adomian polynomials that can be generated for all forms of nonlinearity. Based on these assumptions, Eq. (8.174) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,y) = \Phi_0 - L_x^{-1}g(x,y) - L_x^{-1}L_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right) \\ - L_x^{-1}R\left(\sum_{n=0}^{\infty} u_n(x,y)\right) - L_x^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \end{aligned} \tag{8.177}$$

The components $u_n(x,y), n \geqslant 0$ of the solution $u(x,y)$ can be recursively determined by using the relation

$$\begin{aligned} u_0(x,y) &= \Phi_0 - L_x^{-1}g(x,y), \\ u_{k+1}(x,y) &= -L_x^{-1}L_y u_k - L_x^{-1}R(u_k) - L_x^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{8.178}$$

Using the algorithms described before for calculating A_n for the nonlinear term $F(u)$, the first few components can be identified by

$$\begin{aligned} u_0(x,y) &= \Phi_0 - L_x^{-1}g(x,y), \\ u_1(x,y) &= -L_x^{-1}L_y u_0(x,y) - L_x^{-1}R(u_0(x,y)) - L_x^{-1}A_0, \\ u_2(x,y) &= -L_x^{-1}L_y u_1(x,y) - L_x^{-1}R(u_1(x,y)) - L_x^{-1}A_1, \\ u_3(x,y) &= -L_x^{-1}L_y u_2(x,y) - L_x^{-1}R(u_2(x,y)) - L_x^{-1}A_2, \\ u_4(x,y) &= -L_x^{-1}L_y u_3(x,y) - L_x^{-1}R(u_3(x,y)) - L_x^{-1}A_3, \end{aligned}$$

where each component can be determined by using the preceding component. Having calculated the components $u_n(x,y), n \geqslant 0$, the solution in a series form is readily obtained.

In the following, several distinct nonlinear partial differential equations will be discussed to illustrate the procedure outlined above.

Example 1. Solve the nonlinear partial differential equation

$$u_t + uu_x = 0, \quad u(x,0) = x, \quad t > 0, \tag{8.179}$$

where $u = u(x,t)$.

Solution.

In an operator form, Eq. (8.179) becomes

$$L_t u(x,t) = -uu_x, \tag{8.180}$$

where L_t is defined by

$$L_t = \frac{\partial}{\partial t}. \tag{8.181}$$

The inverse operator L_t^{-1} is identified by

$$L_t^{-1}(\cdot) = \int_0^t (\cdot)dt. \tag{8.182}$$

Applying L_t^{-1} to both sides of (8.180) and using the initial condition we obtain

$$u(x,t) = x - L_t^{-1} uu_x. \tag{8.183}$$

Substituting

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{8.184}$$

and the nonlinear term by

$$uu_x = \sum_{n=0}^{\infty} A_n, \tag{8.185}$$

into (8.183) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = x - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{8.186}$$

This gives the recursive relation

$$\begin{aligned} u_0(x,t) &= x, \\ u_{k+1}(x,t) &= -L_t^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{8.187}$$

The first few components are given by

$$\begin{aligned} u_0(x,t) &= x, \\ u_1(x,t) &= -L_t^{-1}A_0 = -L_t^{-1}(x) = -xt, \\ u_2(x,t) &= -L_t^{-1}A_1 = -L_t^{-1}(-2xt) = xt^2, \\ u_3(x,t) &= -L_t^{-1}A_2 = -L_t^{-1}(3xt^2) = -xt^3, \end{aligned} \tag{8.188}$$

where additional terms can be easily computed. Combining the results obtained above, the solution in a series form is given by

$$u(x,t) = x(1 - t + t^2 - t^3 + \cdots), \tag{8.189}$$

and in a closed form by

$$u(x,t) = \frac{x}{1+t}, \quad |t| < 1. \tag{8.190}$$

In the next example we will use the modified decomposition method presented in Chapter 2 to minimize the size of calculations.

Example 2. Use the modified decomposition method to solve the nonlinear partial differential equation

$$u_t + uu_x = x + xt^2, \quad u(x,0) = 0, \; t > 0, \tag{8.191}$$

where $u = u(x,t)$.

Solution.

Note that the equation is an inhomogeneous equation. Proceeding as in Example 1, Equation (8.191) becomes

$$L_t u(x,t) = x + xt^2 - uu_x. \tag{8.192}$$

Applying the inverse operator L_t^{-1} to both sides of (8.192) and using the initial condition we find

$$u(x,t) = xt + \frac{1}{3}xt^3 - L_t^{-1}uu_x. \tag{8.193}$$

Using the decomposition assumptions for the linear term $u(x,t)$ and for the nonlinear term uu_x defined by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{8.194}$$

and

$$uu_x = \sum_{n=0}^{\infty} A_n, \tag{8.195}$$

into (8.193) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = xt + \frac{1}{3}xt^3 - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{8.196}$$

The modified decomposition method admits the use of a modified recursive relation given by

$$\begin{aligned} u_0(x,t) &= xt, \\ u_1(x,t) &= \frac{1}{3}xt^3 - L_t^{-1}(A_0), \\ u_{k+2}(x,t) &= -L_t^{-1}A_{k+1}, \quad k \geqslant 0. \end{aligned} \tag{8.197}$$

Consequently, we obtain

$$\begin{aligned} u_0(x,t) &= xt, \\ u_1(x,t) &= \frac{1}{3}xt^3 - L_t^{-1}(xt^2) = 0, \\ u_{k+2}(x,t) &= 0, \quad k \geqslant 0. \end{aligned} \tag{8.198}$$

In view of (8.198), the exact solution is given by

$$u(x,t) = xt. \tag{8.199}$$

Example 3. Solve the nonlinear partial differential equation

$$u_t = x^2 + \frac{1}{4}u_x^2, \quad u(x,0) = 0, \tag{8.200}$$

where $u = u(x,t)$.

Solution.

Operating with L_t^{-1} we find

$$u(x,t) = x^2 t + \frac{1}{4}L_t^{-1}u_x^2. \tag{8.201}$$

The decomposition method suggests that $u(x,t)$ can be defined by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{8.202}$$

and the nonlinear term u_x^2 by

$$u_x^2 = \sum_{n=0}^{\infty} A_n, \tag{8.203}$$

where $A_n, n \geqslant 0$, are Adomian polynomials. Using these assumptions gives

$$\sum_{n=0}^{\infty} u_n(x,t) = x^2 t + \frac{1}{4}L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{8.204}$$

This gives the recursive relation

$$\begin{aligned} u_0(x,t) &= x^2 t, \\ u_{k+1}(x,t) &= \frac{1}{4} L_t^{-1} A_k, \quad k \geqslant 0. \end{aligned} \tag{8.205}$$

The Adomian polynomials A_n for this form of nonlinearity are given by

$$\begin{aligned} A_0 &= u_{0_x}^2, \\ A_1 &= 2u_{0_x} u_{1_x}, \\ A_2 &= 2u_{0_x} u_{2_x} + u_{1_x}^2, \\ A_3 &= 2u_{0_x} u_{3_x} + 2u_{1_x} u_{2_x}, \end{aligned}$$

and so on. The first few components are determined as follows:

$$\begin{aligned} u_0(x,t) &= x^2 t, \\ u_1(x,t) &= \frac{1}{4} L_t^{-1} A_0 = \frac{1}{4} L_t^{-1} (4x^2 t^2) = \frac{1}{3} x^2 t^3, \\ u_2(x,t) &= \frac{1}{4} L_t^{-1} A_1 = \frac{1}{4} L_t^{-1} (\frac{8}{3} x^2 t^4) = \frac{2}{15} x^2 t^5, \\ u_3(x,t) &= \frac{1}{4} L_t^{-1} A_2 = \frac{1}{4} L_t^{-1} (\frac{68}{45} x^2 t^6) = \frac{17}{315} x^2 t^7, \end{aligned} \tag{8.206}$$

and so on. Combining the results obtained for the components, the solution in a series form is given by

$$u(x,t) = x^2 \left(t + \frac{1}{3} t^3 + \frac{2}{15} t^5 + \frac{17}{315} t^7 + \cdots \right), \tag{8.207}$$

and in a closed form by

$$u(x,t) = x^2 \tan t. \tag{8.208}$$

Example 4. Solve the nonlinear partial differential equation by the modified decomposition method

$$u_{xx} - u_x u_{yy} = -x + u, \quad u(0,y) = \sin y, \quad u_x(0,y) = 1, \tag{8.209}$$

where $u = u(x,y)$.

Solution.

Note that the equation is an inhomogeneous equation. We first write Eq. (8.209) in an operator form

$$L_x u = -x + u + u_x u_{yy}, \tag{8.210}$$

where L_x is a second order partial differential operator given by

$$L_x = \frac{\partial^2}{\partial x^2}. \tag{8.211}$$

Assuming that L_x^{-1} is invertible, and the inverse operator L_x^{-1} is a two-fold integral operator defined by

$$L_x^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx\, dx, \tag{8.212}$$

so that

$$\begin{aligned} L_x^{-1} L_x u &= \int_0^x \int_0^x u_{xx} dx\, dx, \\ &= u(x,y) - u(0,y) - x u_x(0,y). \end{aligned} \tag{8.213}$$

Proceeding as before we find

$$u(x,y) = \sin y + x - \frac{1}{3!}x^3 + L_x^{-1}(u + u_x u_{yy}). \tag{8.214}$$

Following Adomian method we obtain

$$\sum_{n=0}^{\infty} u_n(x,y) = \sin y + x - \frac{1}{3!}x^3 + L_x^{-1}\left(\sum_{n=0}^{\infty} u_n(x,y) + \sum_{n=0}^{\infty} A_n\right), \tag{8.215}$$

where A_n are the Adomian polynomials that represent the nonlinear term $u_x u_{yy}$.

To use the modified decomposition method, we identify the component u_0 by $u_0(x,y) = \sin y + x$, and the remaining term $-\frac{1}{3!}x^3$ will be assigned to $u_1(x,y)$ among other terms. Consequently, we obtain the recursive relation

$$\begin{aligned} u_0(x,y) &= \sin y + x, \\ u_1(x,y) &= -\frac{1}{3!}x^3 + L_x^{-1}(u_0 + A_0), \\ u_{k+1}(x,y) &= L_x^{-1}(u_k + A_k), \quad k \geqslant 1. \end{aligned} \tag{8.216}$$

Consequently, we obtain

$$\begin{aligned} u_0(x,y) &= \sin y + x, \\ u_1(x,y) &= -\frac{1}{3!}x^3 + L_x^{-1}(u_0 + A_0) = -\frac{1}{3!}x^3 + L_x^{-1}(x) = 0, \end{aligned} \tag{8.217}$$

The exact solution is therefore given by

$$u(x,y) = x + \sin y. \tag{8.218}$$

Example 5. Solve the nonlinear partial differential equation

$$u_{xx} + \frac{1}{4}u_y^2 = u, \quad u(0,y) = 1 + y^2, \quad u_x(0,y) = 1, \tag{8.219}$$

where $u = u(x,y)$.

Solution.

Operating with L_x^{-1} on (8.219) and using the given conditions we find

$$u(x,y) = y^2 + 1 + x + L_x^{-1}\left(u(x,y) - \frac{1}{4}u_y^2\right). \tag{8.220}$$

Proceeding as before we obtain

$$\sum_{n=0}^{\infty} u_n(x,y) = y^2 + 1 + x + L_x^{-1}\left(\sum_{n=0}^{\infty} u_n(x,y) - \frac{1}{4}\sum_{n=0}^{\infty} A_n\right), \tag{8.221}$$

where A_n are the Adomian polynomials that represent the nonlinear term u_y^2. The decomposition method admits the use of the recursive relation

$$\begin{aligned} u_0(x,y) &= y^2 + 1 + x \\ u_{k+1}(x,y) &= L_x^{-1}(u_k - \frac{1}{4}A_k), \quad k \geqslant 0. \end{aligned} \tag{8.222}$$

The Adomian polynomials are given by

$$\begin{aligned} A_0 &= u_{0_y}^2, \\ A_1 &= 2u_{0_y}u_{1_y}, \\ A_2 &= 2u_{0_y}u_{2_y} + u_{1_y}^2, \\ A_3 &= 2u_{0_y}u_{3_y} + 2u_{1_y}u_{2_y}, \end{aligned}$$

and so on. The first few components of the solution $u(x,y)$ are given by

$$\begin{aligned} u_0(x,y) &= y^2 + 1 + x, \\ u_1(x,y) &= L_x^{-1}(u_0 - \frac{1}{4}A_0) = L_x^{-1}(1+x) = \frac{1}{2!}x^2 + \frac{1}{3!}x^3, \\ u_2(x,y) &= L_x^{-1}(u_1 - \frac{1}{4}A_1) = L_x^{-1}(\frac{1}{2!}x^2 + \frac{1}{3!}x^3) = \frac{1}{4!}x^4 + \frac{1}{5!}x^5, \end{aligned} \tag{8.223}$$

and so on for other components. Consequently, the solution in a series form is given by

$$u(x,y) = y^2 + (1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots), \tag{8.224}$$

which gives the solution in a closed form by

$$u(x,y) = y^2 + e^x. \tag{8.225}$$

Example 6. Solve the nonlinear partial differential equation

$$u_{xx} + u^2 - u_y^2 = 0, \quad u(0,y) = 0, \quad u_x(0,y) = e^y, \tag{8.226}$$

where $u = u(x,y)$.

Solution.

We first write Eq. (8.226) in an operator form by

$$L_x u = u_y^2 - u^2, \tag{8.227}$$

where L_x is a second order partial differential operator. Operating with L_x^{-1} gives

$$u(x,y) = xe^y + L_x^{-1}\left(u_y^2 - u^2\right), \tag{8.228}$$

so that

$$\sum_{n=0}^{\infty} u_n(x,y) = xe^y + L_x^{-1}\left(\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} B_n\right), \tag{8.229}$$

where A_n and B_n are the Adomian polynomials that represent the nonlinear terms u_y^2 and u^2 respectively. We next set the recursive relation

$$\begin{aligned} u_0(x,y) &= xe^y \\ u_{k+1}(x,y) &= L_x^{-1}(A_k - B_k), \quad k \geqslant 0. \end{aligned} \tag{8.230}$$

The first few components of the solution $u(x,y)$ are given by

$$\begin{aligned} u_0(x,y) &= xe^y, \\ u_1(x,y) &= L_x^{-1}(A_0 - B_0) = 0, \end{aligned} \tag{8.231}$$

and therefore other components vanish. Consequently, the exact solution is given by

$$u(x,y) = xe^y. \tag{8.232}$$

Example 7. Solve the nonlinear partial differential equation

$$u_{xx} + u^2 - u_{yy}^2 = 0, \quad u(0,y) = 0, \quad u_x(0,y) = \cos y, \tag{8.233}$$

where $u = u(x,y)$.

Solution.

Operating with the two-fold integral operator L_x^{-1} on (8.233) leads to

$$u(x,y) = x\cos y + L_x^{-1}\left(u_{yy}^2 - u^2\right). \tag{8.234}$$

Following Adomian decomposition method we obtain

$$\sum_{n=0}^{\infty} u_n(x,y) = x\cos y + L_x^{-1}\left(\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} B_n\right), \tag{8.235}$$

where A_n and B_n are the Adomian polynomials that represent the nonlinear terms $(u_{yy})^2$ and u^2 respectively. This gives the recursive relation

$$\begin{aligned} u_0(x,y) &= x\cos y \\ u_{k+1}(x,y) &= L_x^{-1}(A_k - B_k), \quad k \geqslant 0. \end{aligned} \tag{8.236}$$

The first few components of the solution $u(x,y)$ are given by

$$\begin{aligned} u_0(x,y) &= x\cos y, \\ u_1(x,y) &= L_x^{-1}(A_0 - B_0) = 0, \end{aligned} \tag{8.237}$$

and other components vanish as well. Consequently, the exact solution is given by

$$u(x,y) = x\cos y. \tag{8.238}$$

Example 8. Solve the nonlinear partial differential equation

$$u_t + \frac{1}{36}xu_{xx}^2 = x^3, \quad u(x,0) = 0, \tag{8.239}$$

where $u = u(x,t)$.

Solution.

Using the integral operator L_t^{-1} on (8.239) and using the given condition we obtain

$$u(x,t) = x^3 t - \frac{1}{36}L_t^{-1}(xu_{xx}^2). \tag{8.240}$$

Following the analysis presented before we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = x^3 t - \frac{1}{36}L_t^{-1}\left(x\sum_{n=0}^{\infty} A_n\right), \tag{8.241}$$

where A_n are the Adomian polynomials that represent the nonlinear terms u_{xx}^2. This gives the recursive relation

$$\begin{aligned} u_0(x,t) &= x^3 t \\ u_{k+1}(x,t) &= -\frac{1}{36}L_t^{-1}(xA_k), \quad k \geqslant 0. \end{aligned} \tag{8.242}$$

The Adomian polynomials are given by

$$\begin{aligned} A_0 &= u_{0_{xx}}^2, \\ A_1 &= 2u_{0_{xx}}u_{1_{xx}}, \\ A_2 &= 2u_{0_{xx}}u_{2_{xx}} + u_{1_{xx}}^2, \end{aligned}$$

and so on. The first few components of the solution $u(x,y)$ are given by

$$u_0(x,t) = x^3 t,$$

$$\begin{aligned}
u_1(x,t) &= -\frac{1}{36}L_t^{-1}(36x^3t^2) = -\frac{1}{3}x^3t^3,\\
u_2(x,t) &= -\frac{1}{36}L_t^{-1}(-24x^3t^4) = \frac{2}{15}x^3t^5,\\
u_3(x,t) &= -\frac{1}{36}L_t^{-1}(\frac{68}{5}x^3t^6) = -\frac{17}{315}x^3t^7,
\end{aligned} \tag{8.243}$$

and so on. Consequently, the solution in a series form is given by

$$u(x,t) = x^3(t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \cdots), \tag{8.244}$$

and in a closed form by

$$u(x,t) = x^3 \tanh t. \tag{8.245}$$

Example 9. Solve the nonlinear partial differential equation

$$u_t + u^2u_x = 0, \quad u(x,0) = 2x,\ t > 0 \tag{8.246}$$

where $u = u(x,t)$.

Solution.

Applying L_t^{-1} on (8.246) and using the given condition we obtain

$$u(x,t) = 2x - L_t^{-1}(u^2u_x). \tag{8.247}$$

It follows that

$$\sum_{n=0}^{\infty} u_n(x,t) = 2x - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{8.248}$$

where A_n are the Adomian polynomials that represent the nonlinear terms u^2u_x. This gives the recursive relation

$$\begin{aligned}
u_0(x,t) &= 2x\\
u_{k+1}(x,t) &= -L_t^{-1}(A_k), \quad k \geqslant 0.
\end{aligned} \tag{8.249}$$

This gives the first few components of $u(x,y)$ by

$$\begin{aligned}
u_0(x,t) &= 2x,\\
u_1(x,t) &= -L_t^{-1}(A_0) = -L_t^{-1}(8x^2) = -8x^2t,\\
u_2(x,t) &= -L_t^{-1}(A_1) = -L_t^{-1}(-128x^3t) = 64x^3t^2\\
u_3(x,t) &= -L_t^{-1}(A_2) = -L_t^{-1}(1920x^4t^2) = -640x^4t^3,
\end{aligned} \tag{8.250}$$

and so on. It follows that the solution in a series form is given by

$$u(x,t) = 2x - 8x^2t + 64x^3t^2 - 640x^4t^3 + \cdots. \tag{8.251}$$

Two observations can be made here. First, we can easily observe that

$$u(x,t) = 2x, \quad t = 0, \tag{8.252}$$

that satisfies the initial condition. We next observe that for $t > 0$, the series solution in (8.251) can be formally expressed in a closed form by

$$u(x,t) = \frac{1}{4t}\left(\sqrt{1+16xt} - 1\right). \tag{8.253}$$

Combining (8.252) and (8.253) gives the solution in the form

$$u(x,t) = \begin{cases} 2x, & t = 0, \\ \frac{1}{4t}\left(\sqrt{1+16xt} - 1\right), & t > 0. \end{cases} \tag{8.254}$$

Example 10. Solve the nonlinear partial differential equation

$$u_t + uu_x = 0, \quad u(x,0) = \sin x, \ t > 0 \tag{8.255}$$

where $u = u(x,t)$.

Solution.

Equation (8.255) can be written in the form

$$L_t u = -uu_x. \tag{8.256}$$

Operating with L_t^{-1} on (8.256) gives

$$u(x,t) = \sin x - L_t^{-1}(uu_x). \tag{8.257}$$

Using the decomposition assumptions for the linear and the nonlinear terms we find

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin x - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{8.258}$$

where A_n are the Adomian polynomials that represent the nonlinear terms uu_x. The following recursive relation

$$\begin{aligned} u_0(x,t) &= \sin x \\ u_{k+1}(x,t) &= -L_t^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{8.259}$$

follows immediately. This gives the first few components of $u(x,y)$ by

$$\begin{aligned} u_0(x,t) &= \sin x, \\ u_1(x,t) &= -L_t^{-1}(A_0) = -t\sin x\cos x, \\ u_2(x,t) &= -L_t^{-1}(A_1) = (\sin x\cos^2 x - \frac{1}{2}\sin^3 x)t^2, \end{aligned} \tag{8.260}$$

and so on. It is clear that the solution obtained will be in terms of a series of functions rather than a power series. It then follows that

$$u(x,t) = \sin x - t\sin x\cos x + (\sin x\cos^2 x - \frac{1}{2}\sin^3 x)t^2 + \cdots. \tag{8.261}$$

However, by using the traditional method of characteristics, we can show that the solution can be expressed in the parametric form

$$\begin{aligned} u(x,t) &= \sin\xi, \\ \xi &= x - t\sin\xi. \end{aligned} \tag{8.262}$$

For numerical approximations, the series solution obtained above is more effective and practical compared to the parametric form solution given in (8.262).

In closing this section, it is important to note that the well-known nonlinear models that characterize physical models will be examined in details in Chapter 9. The aim of Chapter 8 is to introduce the algorithm in a general way so that it can be applied in scientific applications as will be seen in the coming chapters. For numerical purposes, the decomposition series solution will be combined with the powerful Padé approximants to handle the boundary condition at infinity in particular.

Exercises 8.5

In Exercises 1–12, use the Adomian decomposition method to find the exact solution of the following nonlinear partial differential equations:

1. $u_{xx} + u_y u_{yy} = 2$, $u(0,y) = 0$, $u_x(0,y) = y$
2. $u_{yy} + u_x u_{xx} = 2$, $u(x,0) = 0$, $u_y(x,0) = x$
3. $u_t + uu_x = 1 + x + t$, $u(x,0) = x$
4. $u_t + uu_x = x + t + xt^2$, $u(x,0) = 1$
5. $u_t - uu_x = 0$, $u(x,0) = x$
6. $u_t + uu_x = 1 + t\cos x + \sin x\cos x$, $u(x,0) = \sin x$
7. $u_t = 2x^2 - \frac{1}{8}u_x^2$, $u(x,0) = 0$
8. $u_t + \frac{1}{36}xu_{xx}^2 = x^3$, $u(x,0) = 0$
9. $u_t + u^2 u_x = 0$, $u(x,0) = 3x$
10. $u_x + u_y u_{yy} = \frac{1}{1+x^2}$, $u(0,y) = 2y$

11. $u_{xx} + 2u_x(u - t) = 0$, $u(0,t) = t$, $u_x(0,t) = 1$

12. $u_{xx} - 2u_x(u - t) = 0$, $u(0,t) = t$, $u_x(0,t) = 1$

In Exercises 13–24, use the modified decomposition method to find the exact solution of the following nonlinear partial differential equations:

13. $u_{xx} + uu_y = 2y^2 + 2x^4y^3$, $u(0,y) = 0$, $u_x(0,y) = 0$

14. $u_{xx} + uu_y = 2u - y$, $u(0,y) = 1 + y$, $u_x(0,y) = 1$

15. $u_{xx} + uu_x = x + \ln y$, $u(0,y) = \ln y$, $u_x(0,y) = 1$, $y > 0$

16. $u_{yy} + uu_y = y + \ln x$, $u(x,0) = \ln x$, $u_y(x,0) = 1$, $x > 0$

17. $u_{yy} + u_x^2 - u^2 = 0$, $u(x,0) = 0$, $u_y(x,0) = e^{-x}$

18. $u_{yy} - u_{xx}^2 + u^2 = 0$, $u(x,0) = 0$, $u_y(x,0) = \cos x$

19. $u_{xx} + uu_x = x + \ln(1 + y)$, $u(0,y) = \ln(1 + y)$,
 $u_x(0,y) = 1$, $y > -1$

20. $u_{xx} + yuu_x = 1 + xy$, $u(0,y) = \dfrac{1}{y}$, $u_x(0,y) = 1, y > 0$

21. $u_{yy} + u_x u_y = \dfrac{1}{1 + x^2}$, $u(x,0) = \arctan x$, $u_y(x,0) = 1$

22. $u_{xx} - uu_t = -t$, $u(0,t) = t$, $u_x(0,t) = 1$

23. $u_{xx} + uu_t = t$, $u(0,t) = t$, $u_x(0,t) = 1$

24. $u_{xx} + uu_t = t$, $u(0,t) = 1 + t$, $u_x(0,t) = 0$

In Exercises 25–30, use the Adomian decomposition method to find the series solution of the following nonlinear partial differential equations:

25. $u_t + uu_x = 0$, $u(x,0) = \sinh x$

26. $u_t + uu_x = 0$, $u(x,0) = \cos x$

27. $u_t + uu_{xx} = x^2$, $u(x,0) = 0$

28. $u_t + u_x^2 = 0$, $u(x,0) = x$

29. $u_t + u^2 u_x = 0$, $u(x,0) = x$

30. $u_t + uu_x^2 = 0$, $u(x,0) = x$

8.6 Nonlinear PDEs by VIM

As stated before, the variational iteration method handles nonlinear problems in a parallel manner to that used for linear problems. There is no need for Adomian polynomials [8,17]. The main step is to determine the Lagrange multiplier $\lambda(\xi)$, then the successive approximations can be obtained in a recursive manner. In the following, we will examine the same examples discussed before to illustrate the power of the VIM.

Example 1. Solve the nonlinear partial differential equation by the VIM

$$u_t + uu_x = 0, \quad u(x,0) = x,\ t > 0, \tag{8.263}$$

where $u = u(x,t)$.

Solution.

The correction functional for this equation reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{u}_n(x,\xi)\frac{\partial \tilde{u}_n(x,\xi)}{\partial x}\right) d\xi. \tag{8.264}$$

The stationary conditions

$$\begin{aligned} 1 + \lambda|_{\xi=t} &= 0, \\ \lambda'|_{\xi=t} &= 0, \end{aligned} \tag{8.265}$$

give

$$\lambda = -1. \tag{8.266}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (8.264) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + u_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x}\right) d\xi, \quad n \geqslant 0. \tag{8.267}$$

Selecting $u_0(x,t) = x$ from the given initial condition yields the successive approximations

$$\begin{aligned} u_0(x,t) &= x, \\ u_1(x,t) &= x - xt, \\ u_2(x,t) &= x - xt + xt^2 - \frac{1}{3}xt^3, \\ u_3(x,t) &= x - xt + xt^2 - xt^3 + \frac{2}{3}xt^4 + \cdots, \\ &\ \vdots \\ u_n(x,t) &= x\left(1 - t + t^2 - t^3 + t^4 + \cdots\right). \end{aligned} \tag{8.268}$$

The solution in a closed form is given by

$$u(x,t) = \frac{x}{1+t}, \quad |t| < 1. \tag{8.269}$$

Example 2. Use the VIM to solve the nonlinear partial differential equation

$$u_t + uu_x = x + xt^2, \quad u(x,0) = 0,\ t > 0, \tag{8.270}$$

where $u = u(x,t)$.

Solution.

Note that the equation is inhomogeneous. Proceeding as in Example 1, the correction functional for this equation reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{u}_n(x,\xi)\frac{\partial \tilde{u}_n(x,\xi)}{\partial x} - x - x\xi^2\right)d\xi. \tag{8.271}$$

The stationary conditions give $\lambda = -1$. Based on this, we obtain the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + u_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} - x - x\xi^2\right)d\xi, \quad n \geqslant 0. \tag{8.272}$$

Selecting $u_0(x,t) = 0$ from the given initial condition yields the successive approximations

$$\begin{aligned}
u_0(x,t) &= 0,\\
u_1(x,t) &= xt + \frac{1}{3}xt^3,\\
u_2(x,t) &= xt + (\frac{1}{3}xt^3 - \frac{1}{3}xt^3) - \frac{2}{15}xt^5 + \cdots,\\
u_3(x,t) &= xt + (\frac{1}{3}xt^3 - \frac{1}{3}xt^3) + (\frac{2}{15}xt^5 - \frac{2}{15}xt^5) + \cdots,\\
&\ \vdots\\
u_n(x,t) &= xt,
\end{aligned} \tag{8.273}$$

which is the exact solution obtained upon canceling the noise terms.

Example 3. Solve the nonlinear partial differential equation by the VIM

$$u_t = x^2 + \frac{1}{4}u_x^2, \quad u(x,0) = 0, \tag{8.274}$$

where $u = u(x,t)$.

Solution.

Proceeding as before we obtain the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{1}{4}u_{n_x}^2(x,\xi) - x^2\right)d\xi, \quad n \geqslant 0. \tag{8.275}$$

Selecting $u_0(x,t) = 0$ from the given initial condition yields the successive approximations

$$\begin{aligned} u_0(x,t) &= 0,\\ u_1(x,t) &= x^2t,\\ u_2(x,t) &= x^2t+\frac{1}{3}x^2t^3,\\ u_3(x,t) &= x^2t+\frac{1}{3}x^2t^3+\frac{2}{15}x^2t^5+\frac{1}{63}x^2t^7,\\ &\vdots\\ u_n(x,t) &= x^2\left(t+\frac{1}{3}t^3+\frac{2}{15}t^5+\frac{17}{315}t^7+\cdots\right), \end{aligned} \tag{8.276}$$

so that the solution in a closed form is

$$u(x,t) = x^2 \tan t. \tag{8.277}$$

Example 4. Solve the nonlinear partial differential equation by the VIM

$$u_{xx} - u_x u_{yy} = -x+u, \quad u(0,y) = \sin y, \quad u_x(0,y) = 1, \tag{8.278}$$

where $u = u(x,y)$.

Solution.

The correction functional for this equation reads

$$\begin{aligned} u_{n+1}(x,y) &= u_n(x,y)\\ &\quad +\int_0^x \lambda(\xi)\left(\frac{\partial^2 u_n(\xi,y)}{\partial \xi^2} - \tilde{u}_{n_\xi}(\xi,y)\frac{\partial^2 \tilde{u}_n(\xi,y)}{\partial y^2} - \tilde{u}_n(\xi,y)+\xi\right)d\xi. \end{aligned} \tag{8.279}$$

This yields the stationary conditions

$$\begin{aligned} 1-\lambda'|_{\xi=x} &= 0,\\ \lambda|_{\xi=x} &= 0,\\ \lambda''|_{\xi=x} &= 0. \end{aligned} \tag{8.280}$$

This in turn gives

$$\lambda = \xi - x. \tag{8.281}$$

Substituting this value of the Lagrangian multiplier into the functional (8.279) gives the iteration formula

$$\begin{aligned} u_{n+1}(x,y) &= u_n(x,y)\\ &\quad +\int_0^x (\xi - x)\left(\frac{\partial^2 u_n(\xi,y)}{\partial \xi^2} - u_{n_\xi}(\xi,y)\frac{\partial^2 u_n(\xi,y)}{\partial y^2} - u_n(\xi,y)+\xi\right)d\xi. \end{aligned} \tag{8.282}$$

Considering the given initial values, we can select $u_0(x,y) = x + \sin y$. Using this selection into (8.282) we obtain the following successive approximations

$$\begin{aligned} u_0(x,y) &= x + \sin y, \\ u_1(x,y) &= x + \sin y, \\ u_2(x,y) &= x + \sin y, \\ &\vdots \\ u_n(x,y) &= x + \sin y. \end{aligned} \tag{8.283}$$

This gives the exact solution by

$$u(x,y) = x + \sin y. \tag{8.284}$$

Example 5. Solve the nonlinear partial differential equation

$$u_{xx} + \frac{1}{4}u_y^2 = u, \quad u(0,y) = 1 + y^2, \quad u_x(0,y) = 1, \tag{8.285}$$

where $u = u(x,y)$.

Solution.

Proceeding as before gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x (\xi - x)\left(\frac{\partial^2 u_n(\xi,y)}{\partial \xi^2} + \frac{1}{4}(\frac{\partial u_n(\xi,y)}{\partial y})^2 - u_n(\xi,y)\right) d\xi. \tag{8.286}$$

Selecting $u_0(x,y) = y^2 + 1 + x$ gives the following successive approximations

$$\begin{aligned} u_0(x,y) &= y^2 + 1 + x, \\ u_1(x,y) &= y^2 + 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3, \\ u_2(x,y) &= y^2 + 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5, \\ &\vdots \\ u_n(x,y) &= y^2 + (1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots). \end{aligned} \tag{8.287}$$

This gives the exact solution by

$$u(x,y) = y^2 + e^x. \tag{8.288}$$

Example 6. Solve the nonlinear partial differential equation by the VIM

$$u_{xx} + u^2 - u_y^2 = 0, \quad u(0,y) = 0, \quad u_x(0,y) = e^y, \tag{8.289}$$

where $u = u(x,y)$.

Solution.

Proceeding as before gives

$$\lambda = \xi - x. \tag{8.290}$$

This in turn gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x (\xi - x)\left(\frac{\partial^2 u_n(\xi,y)}{\partial \xi^2} - (\frac{\partial u_n(\xi,y)}{\partial y})^2 + u_n^2(\xi,y)\right) d\xi. \tag{8.291}$$

Selecting $u_0(x,y) = xe^y$ gives the following successive approximations

$$\begin{aligned} u_0(x,y) &= xe^y, \\ u_1(x,y) &= xe^y, \\ u_2(x,y) &= xe^y, \\ &\vdots \\ u_n(x,y) &= xe^y. \end{aligned} \tag{8.292}$$

This gives the exact solution by

$$u(x,y) = xe^y. \tag{8.293}$$

Example 7. Solve the nonlinear partial differential equation

$$u_{xx} + u^2 - u_{yy}^2 = 0, \quad u(0,y) = 0, \quad u_x(0,y) = \cos y, \tag{8.294}$$

where $u = u(x,y)$.

Solution.

The Lagrange multiplier is

$$\lambda = \xi - x. \tag{8.295}$$

This in turn gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x (\xi - x)\left(\frac{\partial^2 u_n(\xi,y)}{\partial \xi^2} - (\frac{\partial^2 u_n(\xi,y)}{\partial y^2})^2 + u_n^2(\xi,y)\right) d\xi. \tag{8.296}$$

Selecting $u_0(x,y) = x\cos y$ gives the following successive approximations

$$\begin{aligned} u_0(x,y) &= x\cos y, \\ u_1(x,y) &= x\cos y, \\ u_2(x,y) &= x\cos y, \\ &\vdots \\ u_n(x,y) &= x\cos y. \end{aligned} \tag{8.297}$$

This gives the exact solution by

$$u(x,y) = x\cos y. \tag{8.298}$$

Example 8. Solve the nonlinear partial differential equation

$$u_t + \frac{1}{36}xu_{xx}^2 = x^3, \quad u(x,0) = 0. \tag{8.299}$$

Solution.

As discussed in Example 1, $\lambda = -1$. This gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{1}{36}x\left(\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right)^2 - x^3 \right) d\xi, \quad n \geqslant 0. \tag{8.300}$$

Selecting $u_0(x,t) = 0$ from the given initial condition yields the successive approximations

$$\begin{aligned}
u_0(x,t) &= 0,\\
u_1(x,t) &= x^3 t,\\
u_2(x,t) &= x^3 t - \frac{1}{3}x^3t^3,\\
u_3(x,t) &= x^3 t - \frac{1}{3}x^3t^3 + \frac{2}{15}x^3t^5 - \frac{1}{63}x^3t^7,\\
&\ \vdots\\
u_n(x,t) &= x^3\left(t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \cdots\right).
\end{aligned} \tag{8.301}$$

The solution in a closed form is given by

$$u(x,t) = x^3 \tanh t. \tag{8.302}$$

Example 9. Solve the nonlinear partial differential equation

$$u_t + u^2 u_x = 0, \quad u(x,0) = 2x,\ t > 0 \tag{8.303}$$

where $u = u(x,t)$.

Solution.

Proceeding as in the previous example we find

$$\lambda = -1. \tag{8.304}$$

This gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + u_n^2(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} \right) d\xi, \quad n \geqslant 0. \tag{8.305}$$

Selecting $u_0(x,t) = 2x$ from the given initial condition yields the successive approximations

$$\begin{aligned}
u_0(x,t) &= 2x, \\
u_1(x,t) &= 2x - 8x^2t, \\
u_2(x,t) &= 2x - 8x^2t + 64x^3t^2 - \frac{640}{3}x^4t^3 + \cdots, \\
u_3(x,t) &= 2x - 8x^2t + 64x^3t^2 - 640x^4t^3 + \cdots, \\
&\vdots \\
u_n(x,t) &= 2x - 8x^2t + 64x^3t^2 - 640x^4t^3 + 7168x^5t^4 + \cdots.
\end{aligned} \tag{8.306}$$

As concluded before, we can easily observe that

$$u(x,t) = 2x, \quad t = 0, \tag{8.307}$$

and for $t > 0$, the series solution in (8.307) can be formally expressed in a closed form by

$$u(x,t) = \frac{1}{4t}\left(\sqrt{1+16xt} - 1\right). \tag{8.308}$$

Combining (8.307) and (8.308) gives the solution in the form

$$u(x,t) = \begin{cases} 2x, & t = 0, \\ \dfrac{1}{4t}\left(\sqrt{1+16xt} - 1\right), & t > 0. \end{cases} \tag{8.309}$$

Example 10. Solve the nonlinear partial differential equation

$$u_t + uu_x = 0, \quad u(x,0) = \sin x, \ t > 0 \tag{8.310}$$

where $u = u(x,t)$.

Solution.

As presented above, the stationary conditions gives

$$\lambda = -1, \tag{8.311}$$

and a s a result we find the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + u_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x}\right) d\xi, \quad n \geqslant 0. \tag{8.312}$$

Selecting $u_0(x,t) = \sin x$ from the given initial condition yields the successive approximations

$$
\begin{aligned}
u_0(x,t) &= \sin x,\\
u_1(x,t) &= \sin x - t\sin x\cos x,\\
u_2(x,t) &= \sin x - t\sin x\cos x + \sin x(\cos^2 x - \frac{1}{2}\sin^2 x)t^2 - \frac{1}{6}t^3\sin 2x\cos 2x,\\
&\vdots\\
u_n(x,t) &= \sin x - t\sin x\cos x + t^2\sin x(\cos^2 x - \frac{1}{2}\sin^2 x)\\
&\quad -\frac{1}{6}t^3\sin 2x\cos 2x + \cdots.
\end{aligned} \tag{8.313}
$$

As stated in the previous section, using the traditional method of characteristics, we can show that the solution can be expressed in the parametric form

$$
\begin{aligned}
u(x,t) &= \sin\xi,\\
\xi &= x - t\sin\xi.
\end{aligned} \tag{8.314}
$$

For numerical approximations, the series solution obtained above is more effective and practical compared to the parametric form solution given in (8.314).

In closing this section, it is easily observed that the variational iteration method can be effectively used in handling nonlinear problems. There is no need for Adomian polynomials, perturbation techniques, or any restrictive assumptions that may change the physical behavior of the problem.

Exercises 8.6

Use the variational iteration method to solve the problems in Exercises 8.5.

8.7 Nonlinear PDEs Systems by Adomian Method

In this section, systems of nonlinear partial differential equations will be examined by using Adomian decomposition method. Systems of nonlinear partial differential equations arise in many scientific models such as the propagation of shallow water waves and the Brusselator model of chemical reaction-diffusion model. To achieve our goal in handling systems of nonlinear partial differential equations, we write a system in an operator form by

$$
\begin{aligned}
L_t u + L_x v + N_1(u,v) &= g_1,\\
L_t v + L_x u + N_2(u,v) &= g_2,
\end{aligned} \tag{8.315}
$$

with initial data

$$
\begin{aligned}
u(x,0) &= f_1(x),\\
v(x,0) &= f_2(x),
\end{aligned} \tag{8.316}
$$

where L_t and L_x are considered, without loss of generality, first order partial differential operators, N_1 and N_2 are nonlinear operators, and g_1 and g_2 are source terms. Operating with the integral operator L_t^{-1} to the system (8.315) and using the initial data (8.316) yields

$$\begin{aligned} u(x,t) &= f_1(x) + L_t^{-1} g_1 - L_t^{-1} L_x v - L_t^{-1} N_1(u,v), \\ v(x,t) &= f_2(x) + L_t^{-1} g_2 - L_t^{-1} L_x u - L_t^{-1} N_2(u,v). \end{aligned} \tag{8.317}$$

The linear unknown functions $u(x,t)$ and $v(x,t)$ can be decomposed by infinite series of components

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t), \\ v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t). \end{aligned} \tag{8.318}$$

However, the nonlinear operators $N_1(u,v)$ and $N_2(u,v)$ should be represented by using the infinite series of the so-called Adomian polynomials A_n and B_n as follows:

$$\begin{aligned} N_1(u,v) &= \sum_{n=0}^{\infty} A_n, \\ N_2(u,v) &= \sum_{n=0}^{\infty} B_n, \end{aligned} \tag{8.319}$$

where $u_n(x,t)$ and $v_n(x,t), n \geqslant 0$ are the components of $u(x,t)$ and $v(x,t)$ respectively that will be recurrently determined, and A_n and $B_n, n \geqslant 0$ are Adomian polynomials that can be generated for all forms of nonlinearity. The algorithms for calculating Adomian polynomials were introduced in Sections 8.2 and 8.3. Substituting (8.318) and (8.319) into (8.317) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) &= f_1(x) + L_t^{-1} g_1 - L_t^{-1} L_x (\sum_{n=0}^{\infty} v_n) - L_t^{-1} \left(\sum_{n=0}^{\infty} A_n \right), \\ \sum_{n=0}^{\infty} v_n(x,t) &= f_2(x) + L_t^{-1} g_2 - L_t^{-1} L_x (\sum_{n=0}^{\infty} u_n) - L_t^{-1} \left(\sum_{n=0}^{\infty} B_n \right). \end{aligned} \tag{8.320}$$

Two recursive relations can be constructed from (8.320) given by

$$\begin{aligned} u_0(x,t) &= f_1(x) + L_t^{-1} g_1, \\ u_{k+1}(x,t) &= -L_t^{-1} (L_x v_k) - L_t^{-1} (A_k), \quad k \geqslant 0, \end{aligned} \tag{8.321}$$

and

$$\begin{aligned} v_0(x,t) &= f_2(x) + L_t^{-1} g_2, \\ v_{k+1}(x,t) &= -L_t^{-1} (L_x u_k) - L_t^{-1} (B_k), \quad k \geqslant 0. \end{aligned} \tag{8.322}$$

It is an essential feature of the decomposition method that the zeroth components $u_0(x,t)$ and $v_0(x,t)$ are defined always by all terms that arise from initial data and from integrating the source terms. Having defined the zeroth pair (u_0, v_0), the remaining pair $(u_k, v_k), k \geqslant 1$ can be obtained in a recurrent manner by using (8.321) and (8.322). Additional pairs for the decomposition series solutions normally account for higher accuracy. Having determined the components of $u(x,t)$ and $v(x,t)$, the solution (u,v) of the system follows immediately in the form of a power series expansion upon using (8.318).

To give a clear overview of the analysis introduced above, two illustrative systems of nonlinear partial differential equations have been selected to demonstrate the efficiency of the method.

Example 1. Consider the nonlinear system:

$$\begin{aligned} u_t + v u_x + u &= 1, \\ v_t - u v_x - v &= 1, \end{aligned} \tag{8.323}$$

with the conditions

$$u(x,0) = e^x, \quad v(x,0) = e^{-x}. \tag{8.324}$$

Solution.

Operating with L_t^{-1} on (8.323) we obtain

$$\begin{aligned} u(x,t) &= e^x + t - L_t^{-1}(v u_x + u), \\ v(x,t) &= e^{-x} + t + L_t^{-1}(u v_x + v). \end{aligned} \tag{8.325}$$

The linear terms $u(x,t)$ and $v(x,t)$ can be represented by the decomposition series

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t), \\ v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t), \end{aligned} \tag{8.326}$$

and the nonlinear terms vu_x and uv_x by an infinite series of polynomials

$$\begin{aligned} v u_x &= \sum_{n=0}^{\infty} A_n, \\ u v_x &= \sum_{n=0}^{\infty} B_n, \end{aligned} \tag{8.327}$$

where A_n and B_n are the Adomian polynomials that can be generated for any form of nonlinearity. Substituting (8.326) and (8.327) into (8.325) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) &= e^x + t - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} u_n\right), \\ \sum_{n=0}^{\infty} v_n(x,t) &= e^{-x} + t + L_t^{-1}\left(\sum_{n=0}^{\infty} B_n + \sum_{n=0}^{\infty} v_n\right). \end{aligned} \tag{8.328}$$

To accelerate the convergence of the solution, the modified decomposition method will be applied here. The modified decomposition method defines the recursive relations in the form

$$\begin{aligned} u_0(x,t) &= e^x, \\ u_1(x,t) &= t - L_t^{-1}\left(A_0 + u_0\right), \\ u_{k+1}(x,t) &= -L_t^{-1}\left(A_k + u_k\right), \quad k \geqslant 1, \end{aligned} \tag{8.329}$$

and

$$\begin{aligned} v_0(x,t) &= e^{-x}, \\ v_1(x,t) &= t + L_t^{-1}\left(B_0 + v_0\right), \\ v_{k+1}(x,t) &= L_t^{-1}\left(B_k + v_k\right), \quad k \geqslant 1. \end{aligned} \tag{8.330}$$

The Adomian polynomials for the nonlinear term vu_x are given by

$$\begin{aligned} A_0 &= v_0 u_{0_x}, \\ A_1 &= v_1 u_{0_x} + v_0 u_{1_x}, \\ A_2 &= v_2 u_{0_x} + v_1 u_{1_x} + v_0 u_{2_x}, \\ A_3 &= v_3 u_{0_x} + v_2 u_{1_x} + v_1 u_{2_x} + v_0 u_{3_x}, \end{aligned}$$

and for the nonlinear term uv_x by

$$\begin{aligned} B_0 &= u_0 v_{0_x}, \\ B_1 &= u_1 v_{0_x} + u_0 v_{1_x}, \\ B_2 &= u_2 v_{0_x} + u_1 v_{1_x} + u_0 v_{2_x}, \\ B_3 &= u_3 v_{0_x} + u_2 v_{1_x} + u_1 v_{2_x} + u_0 v_{3_x}. \end{aligned}$$

Using the derived Adomian polynomials into (8.329) and (8.330), we obtain the following pairs of components

$$\begin{aligned} (u_0, v_0) &= (e^x, e^{-x}), \\ (u_1, v_1) &= (-te^x, te^{-x}), \\ (u_2, v_2) &= (\frac{t^2}{2!}e^x, \frac{t^2}{2!}e^{-x}), \\ (u_3, v_3) &= (-\frac{t^3}{3!}e^x, \frac{t^3}{3!}e^{-x}). \end{aligned} \tag{8.331}$$

Accordingly, the solution of the system in a series form is given by

$$(u,v) = \left(e^x(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots), e^{-x}(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots)\right), \tag{8.332}$$

and in a closed form by

$$(u,v) = \left(e^{x-t}, e^{-x+t}\right). \tag{8.333}$$

In what follows, a system of three nonlinear partial differential equations in three unknown functions $u(x,y,t), v(x,y,t)$ and $w(x,y,t)$ will be studied. It is worth noting that handling this system by traditional methods is quiet complicated.

Example 2. Consider the following nonlinear system:

$$\begin{aligned} u_t - v_x w_y &= 1, \\ v_t - w_x u_y &= 5, \\ w_t - u_x v_y &= 5, \end{aligned} \tag{8.334}$$

with the initial conditions

$$u(x,y,0) = x + 2y, \quad v(x,y,0) = x - 2y, \quad w(x,y,0) = -x + 2y. \tag{8.335}$$

Solution.

Following the analysis presented above we obtain

$$\begin{aligned} u(x,y,t) &= (x + 2y + t) + L_t^{-1}(v_x w_y), \\ v(x,y,t) &= (x - 2y + 5t) + L_t^{-1}(w_x u_y), \\ w(x,y,t) &= (-x + 2y + 5t) + L_t^{-1}(u_x v_y). \end{aligned} \tag{8.336}$$

Substituting the decomposition representations for linear and nonlinear terms into (8.336) yields

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,y,t) &= (x + 2y + t) + L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \\ \sum_{n=0}^{\infty} v_n(x,y,t) &= (x - 2y + 5t) + L_t^{-1}\left(\sum_{n=0}^{\infty} B_n\right), \\ \sum_{n=0}^{\infty} w_n(x,y,t) &= (-x + 2y + 5t) + L_t^{-1}\left(\sum_{n=0}^{\infty} C_n\right), \end{aligned} \tag{8.337}$$

where A_n, B_n, and C_n, are Adomian polynomials for the nonlinear terms $v_x w_y, w_x u_y$, and $u_x v_y$ respectively. For brevity, we list the first three Adomian polynomials for A_n, B_n, and C_n as follows:

For $v_x w_y$, we find

$$\begin{aligned} A_0 &= v_{0_x} w_{0_y}, \\ A_1 &= v_{1_x} w_{0_y} + v_{0_x} w_{1_y}, \\ A_2 &= v_{2_x} w_{0_y} + v_{1_x} w_{1_y} + v_{0_x} w_{2_y}, \end{aligned}$$

and for $w_x u_y$ we find

$$\begin{aligned} B_0 &= w_{0_x} u_{0_y}, \\ B_1 &= w_{1_x} u_{0_y} + w_{0_x} u_{1_y}, \end{aligned}$$

$$B_2 = w_{2_x}u_{0_y} + w_{1_x}u_{1_y} + w_{0_x}u_{2_y},$$

and for $u_x v_y$ we find

$$\begin{aligned}
C_0 &= u_{0_x}v_{0_y},\\
C_1 &= u_{1_x}v_{0_y} + u_{0_x}v_{1_y},\\
C_2 &= u_{2_x}v_{0_y} + u_{1_x}v_{1_y} + u_{0_x}v_{2_y}.
\end{aligned}$$

Substituting these polynomials into the appropriate recursive relations we find

$$\begin{aligned}
(u_0,v_0,w_0) &= (x+2y+t, x-2y+5t, -x+2y+5t),\\
(u_1,v_1,w_1) &= (2t, -2t, -2t),\\
(u_k,v_k,w_k) &= (0,0,0), \quad k \geqslant 2.
\end{aligned} \tag{8.338}$$

Consequently, the exact solution of the system of nonlinear partial differential equations is given by

$$(u,v,w) = (x+2y+3t, x-2y+3t, -x+2y+3t). \tag{8.339}$$

Exercises 8.7

Use Adomian decomposition method to solve the following systems of nonlinear partial differential equations:

1. $u_t + u_x v_x = 2,\ v_t + u_x v_x = 0$
   $u(x,0) = x,\ v(x,0) = x$

2. $u_t - v u_x - u = 1,\ v_t + u v_x + v = 1$
   $u(x,0) = e^{-x},\ v(x,0) = e^{x}$

3. $u_t + 2v u_x - u = 2,\ v_t - 3u v_x + v = 3$
   $u(x,0) = e^{x},\ v(x,0) = e^{-x}$

4. $u_t + v u_x - 3u = 2,\ v_t - u v_x + 3v = 2$
   $u(x,0) = e^{2x},\ v(x,0) = e^{-2x}$

5. $u_t + u_x v_x - w_y = 1,\ v_t + v_x w_x + u_y = 1,\ w_t + w_x u_x - v_y = 1$
   $u(x,y,0) = x+y,\ v(x,y,0) = x-y,\ w(x,y,0) = -x+y$

6. $u_t + v_x w_y - v_y w_x = -u,\ v_t + w_x u_y + w_y u_x = v$
   $w_t + u_x v_y + u_y v_x = w$
   $u(x,y,0) = e^{x+y},\ v(x,y,0) = e^{x-y},\ w(x,y,0) = e^{-x+y}$

7. $u_t + u_x v_x - u_y v_y + u = 0,\ v_t + v_x w_x - v_y w_y - v = 0$
   $w_t + w_x u_x + w_y u_y - w = 0$
   $u(x,y,0) = e^{x+y},\ v(x,y,0) = e^{x-y},\ w(x,y,0) = e^{-x+y}$

8. $u_t + u_y v_x = 1 + e^t,\ v_t + v_y w_x = 1 - e^{-t}$
$w_t + w_y u_y = 1 - e^{-t}$
$u(x,y,0) = 1 + x + y,\ v(x,y,0) = 1 + x - y,\ w(x,y,0) = 1 - x + y$

8.8 Systems of Nonlinear PDEs by VIM

Systems of nonlinear partial differential equations arise in many scientific models such as the propagation of shallow water waves and the Brusselator model of chemical reaction-diffusion model. To use the VIM, we write a system in an operator form by

$$\begin{aligned} L_t u + R_1(u,v,w) + N_1(u,v,w) &= g_1, \\ L_t v + R_2(u,v,w) + N_2(u,v,w) &= g_2, \\ L_t w + R_3(u,v,w) + N_3(u,v,w) &= g_3, \end{aligned} \tag{8.340}$$

with initial data

$$\begin{aligned} u(x,0) &= f_1(x), \\ v(x,0) &= f_2(x), \\ w(x,0) &= f_3(x), \end{aligned} \tag{8.341}$$

where L_t is considered a first order partial differential operator, $R_j, 1 \leqslant j \leqslant 3$ and $N_j, 1 \leqslant j \leqslant 3$ are linear and nonlinear operators respectively, and g_1, g_2 and g_3 are source terms. The correction functionals for equations of the system (8.340) can be written as

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) \\ &\quad + \int_0^t \lambda_1 \left(Lu_n(x,\xi) + R_1(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) + N_1(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) - g_1(\xi) \right) d\xi, \\ v_{n+1}(x,t) &= v_n(x,t) \\ &\quad + \int_0^t \lambda_2 \left(Lv_n(x,\xi) + R_2(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) + N_2(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) - g_2(\xi) \right) d\xi, \\ w_{n+1}(x,t) &= w_n(x,t) \\ &\quad + \int_0^t \lambda_3 \left(Lw_n(x,\xi) + R_3(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) + N_3(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) - g_3(\xi) \right) d\xi, \end{aligned} \tag{8.342}$$

where $\lambda_j, 1 \leqslant j \leqslant 3$ are general Lagrange's multipliers, which can be identified optimally via the variational theory, and $\tilde{u}_n, \tilde{v}_n$, and $\tilde{w}_n$ as restricted variations which means $\delta\tilde{u}_n = 0, \delta\tilde{v}_n = 0$ and $\delta\tilde{w}_n = 0$. It is required first to determine the Lagrange multipliers λ_j that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t), v_{n+1}(x,t), w_{n+1}(x,t), n \geqslant 0$ of the solutions $u(x,t), v(x,t)$ and $w(x,t)$ will follow immediately upon using the obtained Lagrange multipliers and by using selective functions u_0, v_0, and w_0. The initial values are usually used for the selective zeroth approximations. With the Lagrange multipliers λ_j determined, then several approximations $u_j(x,t), v_j(x,t), w_j(x,t), j \geqslant 0$ can be determined [18]. Consequently, the solutions are given by

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t), \\ v(x,t) &= \lim_{n\to\infty} v_n(x,t), \\ w(x,t) &= \lim_{n\to\infty} w_n(x,t). \end{aligned} \tag{8.343}$$

To give a clear overview of the analysis introduced above, we will apply the VIM to the same two illustrative systems of partial differential equations that were studied in the previous section.

Example 1. Consider the inhomogeneous nonlinear system

$$\begin{array}{ll} \text{PDE} & u_t + vu_x + u = 1, \\ & v_t - uv_x - v = 1, \\ \text{IC} & u(x,0) = e^x, \quad v(x,0) = e^{-x}. \end{array} \tag{8.344}$$

Solution.

The correction functionals for (8.344) read

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda_1(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{v}_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} + \tilde{u}_n(x,\xi) - 1\right) d\xi, \\ v_{n+1}(x,t) &= v_n(x,t) + \int_0^t \lambda_2(\xi)\left(\frac{\partial v_n(x,\xi)}{\partial \xi} - \tilde{u}_n(x,\xi)\frac{\partial v_n(x,\xi)}{\partial x} - \tilde{v}_n(x,\xi) - 1\right) d\xi. \end{aligned} \tag{8.345}$$

The stationary conditions are given by

$$\begin{aligned} 1+\lambda_1 &= 0, \quad \lambda_1'(\xi = t) = 0, \\ 1+\lambda_2 &= 0, \quad \lambda_2'(\xi = t) = 0, \end{aligned} \tag{8.346}$$

so that

$$\lambda_1 = \lambda_2 = -1. \tag{8.347}$$

Substituting these values of the Lagrange multipliers into the functionals (8.345) gives the iteration formulas

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + v_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} + u_n(x,\xi) - 1\right) d\xi, \\ v_{n+1}(x,t) &= v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} - u_n(x,\xi)\frac{\partial v_n(x,\xi)}{\partial x} - v_n(x,\xi) - 1\right) d\xi. \end{aligned} \tag{8.348}$$

The zeroth approximations $u_0(x,t) = e^x$, and $v_0(x,t) = e^{-x}$ are selected by using the given initial conditions. Therefore, we obtain the following successive approximations

$$
\begin{aligned}
u_0(x,t) &= e^x, \quad v_0(x,t) = e^{-x},\\
u_1(x,t) &= e^x - te^x, \quad v_1(x,t) = e^{-x} + te^{-x},\\
u_2(x,t) &= e^x - te^x + \frac{t^2}{2!}e^x + \text{noise terms},\\
v_2(x,t) &= e^{-x} + te^{-x} + \frac{t^2}{2!}e^{-x} + \text{noise terms},\\
&\vdots
\end{aligned}
\tag{8.349}
$$

By canceling the noise terms between $u_2, u_3, \cdots$ and between $v_2, v_3, \cdots$, we find

$$
u_n(x,t) = e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right), \quad v_n(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right),
\tag{8.350}
$$

and as a result, the exact solutions are given by

$$
\begin{aligned}
u(x,t) &= e^{x-t},\\
v(x,t) &= e^{-x+t},
\end{aligned}
\tag{8.351}
$$

obtained upon using the Taylor expansion for e^{-t} and e^t. It is obvious that we did not use any transformation formulas or linearization assumptions for handling the nonlinear terms.

In what follows, a system of three nonlinear partial differential equations in three unknown functions $u(x,y,t), v(x,y,t)$ and $w(x,y,t)$ will be studied. It is worth noting that handling this system by traditional methods is quiet complicated.

Example 2. Consider the following nonlinear system:

$$
\begin{aligned}
u_t - v_x w_y &= 1,\\
v_t - w_x u_y &= 5,\\
w_t - u_x v_y &= 5,
\end{aligned}
\tag{8.352}
$$

with the initial conditions

$$
u(x,y,0) = x + 2y, \quad v(x,y,0) = x - 2y, \quad w(x,y,0) = -x + 2y.
\tag{8.353}
$$

Solution.

Proceeding as before we find

$$
\lambda_1 = \lambda_2 = \lambda_3 = -1.
\tag{8.354}
$$

Substituting these values of the Lagrange multipliers gives the iteration formulas

$$
\begin{aligned}
u_{n+1}(x,y,t) &= u_n(x,y,t) - \int_0^t \left(\frac{\partial u_n(x,y,\xi)}{\partial \xi} - \frac{\partial v_n(x,y,\xi)}{\partial x} \times \frac{\partial w_n(x,y,\xi)}{\partial y} - 1 \right) d\xi, \\
v_{n+1}(x,y,t) &= v_n(x,y,t) - \int_0^t \left(\frac{\partial v_n(x,y,\xi)}{\partial \xi} - \frac{\partial w_n(x,y,\xi)}{\partial x} \times \frac{\partial u_n(x,y,\xi)}{\partial y} - 5 \right) d\xi, \\
w_{n+1}(x,y,t) &= w_n(x,y,t) - \int_0^t \left(\frac{\partial w_n(x,y,\xi)}{\partial \xi} - \frac{\partial u_n(x,y,\xi)}{\partial x} \times \frac{\partial v_n(x,y,\xi)}{\partial y} - 5 \right) d\xi.
\end{aligned}
\tag{8.355}
$$

The zeroth approximations

$$
\begin{aligned}
u_0(x,y,t) &= x+2y, \\
v_0(x,y,t) &= x-2y, \\
w_0(x,y,t) &= -x+2y,
\end{aligned}
\tag{8.356}
$$

are selected by using the given initial conditions. Consequently, the following successive approximations

$$
\begin{aligned}
&\begin{cases} u_0(x,y,t) = x+2y, \\ v_0(x,y,t) = x-2y, \\ w_0(x,y,t) = -x+2y. \end{cases} \\
&\begin{cases} u_1(x,y,t) = x+2y+3t, \\ v_1(x,y,t) = x-2y+3t, \\ w_1(x,y,t) = -x+2y+3t, \end{cases} \\
&\begin{cases} \vdots \end{cases} \\
&\begin{cases} u_n(x,y,t) = x+2y+3t, \\ v_n(x,y,t) = x-2y+3t, \\ w_n(x,y,t) = -x+2y+3t, \end{cases}
\end{aligned}
\tag{8.357}
$$

are readily obtained. Notice that the successive approximations became the same for u after obtaining the first approximation. The same conclusion can be made for v and w. Based on this, the exact solutions are given by

$$
\begin{aligned}
u(x,y,t) &= x+2y+3t, \\
v(x,y,t) &= x-2y+3t, \\
w(x,y,t) &= -x+2y+3t.
\end{aligned}
\tag{8.358}
$$

Exercises 8.8

Use the variational iteration method to solve the problems in Exercises 8.7.

References

1. M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, (1991).
2. M.J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia (1981).
3. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, (1994).
4. G. Adomian, A new approach to nonlinear partial differential equations, J. *Math. Anal. Appl.*, 102, 420–434,(1984).
5. G. Adomian, Nonlinear Stochastic Operator Equations, Academic Press, San Diego, (1986).
6. Y. Cherruault, Convergence of Adomian's method, *Kybernotes*, 18(2), 31–38, (1989).
7. Y. Cherruault and G. Adomian, Decomposition methods: a new proof of convergence, *Math. Comput. Modelling*, 18, 103–106, (1993).
8. J.H.He, Some asymptotic methods for strongly nonlinearly equations, *Int J. of Modern Math.*, 20(10), 1141–1199, (2006).
9. J.H.He, Variational iteration method for autonomous ordinary differential systems, *Appl. math. Comput.*, 114(2/3), 115–123, (2000).
10. J.H.He, Variational iteration method—a kind of nonlinear analytical technique: some examples, *Int. J. Nonlinear Mech*, 34, 799–708, (1999).
11. A. Répaci, Nonlinear dynamical systems: on the accuracy of Adomian's decomposition method, *Appl. Math. Lett.*, 3, 35–39, (1990).
12. P. Rosenau and J.M. Hyman, Compactons: Solitons with finite wavelengths, *Phys. Rev. Lett.*, 70(5),564–567, (1993).
13. A.M.Wazwaz, Necessary conditions for the appearance of noise terms in decomposition solution series, *Appl. Math. Comput.*, 81, 199–204, (1997).
14. A.M.Wazwaz, A reliable modification of Adomian's decomposition method, *Appl. Math. and Comput.*, 92, 1–7, (1998).
15. A.M.Wazwaz, A new technique for calculating Adomian polynomials for nonlinear operators, *Appl. Math. Comput.*, 111(1), 33–51, (2000).
16. A.M.Wazwaz, Partial Differential Equations: Methods and Applications, Balkema, Leiden, (2002).
17. A.M. Wazwaz, The variational iteration method: A reliable analytic tool for solving linear and nonlinear wave equations, *Comput. Math. Applic*. 54, 926–932, (2007).
18. A.M.Wazwaz, The variational iteration method for solving linear and nonlinear systems of PDEs, *Comput. Math. Applic*. 54, 895–902, (2007).

Chapter 9
Linear and Nonlinear Physical Models

9.1 Introduction

This chapter is devoted to treatments of linear and nonlinear particular applications that appear in applied sciences. A wide variety of physically significant problems modeled by linear and nonlinear partial differential equations has been the focus of extensive studies for the last decades. A huge size of research and investigation has been invested in these scientific applications. Several approaches have been used such as the characteristics method, spectral methods and perturbation techniques to examine these problems.

Nonlinear PDEs have undergone remarkable developments. Nonlinear problems arise in different areas including gravitation, chemical reaction, fluid dynamics, dispersion, nonlinear optics, plasma physics, acoustics, inviscid fluids and others. Nonlinear wave propagation problems have provided solutions of different physical structures than solutions of linear wave equations.

It is well known that many physical, chemical and biological problems are characterized by the interaction of convection and diffusion and by the interaction of diffusion and reaction processes. Burgers equation is considered as a model equation that describes the interaction of convection and diffusion, whereas Fisher equation is an appropriate model that describes the process of interaction between diffusion and reaction.

In this chapter, Adomian decomposition method, the modified decomposition method, and the self-canceling noise-terms phenomenon will be employed in the treatments of these models. Moreover, the variational iteration method will be used as well. Some of the examples will be solved by Adomian method and by the variational iteration method. The linear and nonlinear models will be approached directly and in a like manner. The series representation of the linear term u, and the representation of the nonlinear term $F(u)$ by a series of Adomian polynomials will be used in a like manner to that used in Chapter 8; hence details will be skipped.

9.2 The Nonlinear Advection Problem

The nonlinear partial differential equation of the advection problem is of the form

$$u_t + uu_x = f(x,t), \quad u(x,0) = g(x). \tag{9.1}$$

The problem has been handled by using the characteristic method, and recently by applying numerical methods such as Fourier series and Runge-Kutta method.

In this section, we approach the advection problem [13] by utilizing the decomposition method to find a rapidly convergent power series solution. The phenomenon of self-canceling noise terms will be used where appropriate.

In an operator form, Eq. (9.1) can be rewritten as

$$L_t u + \frac{1}{2} L_x(u^2) = f(x,t), \quad u(x,0) = g(x). \tag{9.2}$$

Operating with L_t^{-1} yields

$$u(x,t) = g(x) + L_t^{-1}(f(x,t)) - \frac{1}{2} L_t^{-1} L_x(u^2). \tag{9.3}$$

Substituting the linear term $u(x,t)$ by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{9.4}$$

and the nonlinear term u^2 by a series of Adomian polynomials

$$u^2(x,t) = \sum_{n=0}^{\infty} A_n, \tag{9.5}$$

where A_n are derived in Section 8.2.1, into (9.3) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = g(x) + L_t^{-1}(f(x,t)) - \frac{1}{2} L_t^{-1} L_x \left(\sum_{n=0}^{\infty} A_n \right). \tag{9.6}$$

Following Adomian approach, we obtain the recursive relation

$$\begin{aligned} u_0(x,t) &= g(x) + L_t^{-1}(f(x,t)), \\ u_{k+1}(x,t) &= -\frac{1}{2} L_t^{-1} L_x(A_k), \quad k \geqslant 0. \end{aligned} \tag{9.7}$$

In view of (9.7), the components u_n, $n \geqslant 0$ can be easily computed, and the series solution can be formally constructed.

As stated in Chapter 2, the phenomenon of the self-canceling noise terms may appear for inhomogeneous problems, whereas homogeneous problems do not produce the noise terms in $u_0(x,t)$ and $u_1(x,t)$. The self-canceling noise terms [14] will play a major role in accelerating the convergence. Moreover, the variational iteration method will be used for some of the examples for comparison reasons.

To give a clear overview of this analysis, the following illustrative examples will be discussed by using Adomian method [3]. Some of these examples will be also be handled by the variational iteration method

Example 1. Solve the inhomogeneous advection problem

$$u_t + \frac{1}{2}(u^2)_x = e^x + t^2 e^{2x}, \quad u(x,0) = 0. \tag{9.8}$$

Solution.

The Decomposition Method

Operating with L_t^{-1}, Eq. (9.8) becomes

$$u(x,t) = te^x + \frac{1}{3}t^3 e^{2x} - \frac{1}{2}L_t^{-1}L_x(u^2). \tag{9.9}$$

Substituting the decomposition representation for the linear term $u(x,t)$ and for the nonlinear term $u^2(x,t)$ gives

$$\sum_{n=0}^{\infty} u_n(x,t) = te^x + \frac{1}{3}t^3 e^{2x} - \frac{1}{2}L_t^{-1}L_x\left(\sum_{n=0}^{\infty} A_n\right), \tag{9.10}$$

where A_n are the Adomian polynomials for the nonlinear term u^2. This gives the recursive relation

$$\begin{aligned} u_0(x,t) &= te^x + \frac{1}{3}t^3 e^{2x}, \\ u_{k+1}(x,t) &= -\frac{1}{2}L_t^{-1}L_x\left(A_k\right), \quad k \geqslant 0, \end{aligned} \tag{9.11}$$

so that the first two components are given by

$$\begin{aligned} u_0(x,t) &= te^x + \frac{1}{3}t^3 e^{2x}, \\ u_1(x,t) &= -\frac{1}{2}L_t^{-1}L_x\left(A_0\right), \\ &= -\frac{1}{3}t^3 e^{2x} - \frac{1}{5}t^5 e^{3x} - \frac{2}{63}t^7 e^{4x}. \end{aligned} \tag{9.12}$$

The noise terms phenomenon suggests that if terms in u_0 are canceled by terms in u_1, even though u_1 contains further terms, then the remaining non-canceled terms of u_0 may provide the exact solution of the problem. This should be verified through substitution. Thus by canceling the term $-\frac{1}{3}t^3 e^{2x}$ in u_0, and by justifying that the remaining non-canceled term of u_0 satisfies the equation, it then follows that the exact solution is given by

$$u(x,t) = te^x. \tag{9.13}$$

The Variational Iteration Method

We now apply the variational iteration method [6] and set the correction functional for this equation by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{1}{2}\frac{\partial \tilde{u}_n^2(x,\xi)}{\partial x} - e^x - \xi^2 e^{2x}\right) d\xi. \tag{9.14}$$

The stationary conditions are given by

$$\begin{aligned} 1+\lambda|_{\xi=t} &= 0,\\ \lambda'|_{\xi=t} &= 0. \end{aligned} \tag{9.15}$$

Solving this system gives

$$\lambda = -1. \tag{9.16}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the correction functional gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{1}{2}\frac{\partial u_n^2(x,\xi)}{\partial x} - e^x - \xi^2 e^{2x}\right) d\xi, \quad n \geqslant 0. \tag{9.17}$$

Selecting $u_0(x,t) = 0$ from the given initial condition yields the successive approximations

$$\begin{aligned} u_0(x,t) &= 0,\\ u_1(x,t) &= te^x + \frac{t^3}{3}e^{2x},\\ u_2(x,t) &= te^x + (\frac{t^3}{3}e^{2x} - \frac{t^3}{3}e^{2x}) - \frac{t^5}{5}e^{3x} + \cdots,\\ u_3(x,t) &= te^x + (\frac{t^3}{3}e^{2x} - \frac{t^3}{3}e^{2x}) - (\frac{t^5}{5}e^{3x} - \frac{t^5}{5}e^{3x}) + \cdots,\\ &\vdots\\ u_n(x,t) &= te^x. \end{aligned} \tag{9.18}$$

The exact solution is therefore $u(x,t) = te^x$ obtained by canceling the noise terms.

Example 2. Solve the inhomogeneous advection problem

$$u_t + \frac{1}{2}(u^2)_x = -\sin(x+t) - \frac{1}{2}\sin 2(x+t), \quad u(x,0) = \cos x. \tag{9.19}$$

Solution.

Applying L_t^{-1} to both sides of (9.19) gives

$$u(x,t) = \cos(x+t) + \frac{1}{4}\cos 2(x+t) - \frac{1}{4}\cos 2x - \frac{1}{2}L_t^{-1}L_x(u^2). \tag{9.20}$$

Using the decomposition representation for the linear and nonlinear terms yields

$$\sum_{n=0}^{\infty} u_n(x,t) = \cos(x+t) + \frac{1}{4}\cos 2(x+t) - \frac{1}{4}\cos 2x - \frac{1}{2}L_t^{-1}L_x\left(\sum_{n=0}^{\infty} A_n\right), \quad (9.21)$$

where A_n are the Adomian polynomials for the nonlinear term u^2. Following Adomian analysis, the recursive relation

$$\begin{aligned} u_0(x,t) &= \cos(x+t) + \frac{1}{4}\cos 2(x+t) - \frac{1}{4}\cos 2x, \\ u_{k+1}(x,t) &= -\frac{1}{2}L_t^{-1}L_x\left(A_k\right), \quad k \geqslant 0, \end{aligned} \quad (9.22)$$

follows immediately. The first two components are given by

$$\begin{aligned} u_0(x,t) &= \cos(x+t) + \frac{1}{4}\cos 2(x+t) - \frac{1}{4}\cos 2x, \\ u_1(x,t) &= -\frac{1}{2}L_t^{-1}L_x\left(A_0\right) = -\frac{1}{4}\cos 2(x+t) + \frac{1}{4}\cos 2x - \cdots. \end{aligned} \quad (9.23)$$

It is easily observed that two noise terms appear in the components $u_0(x,t)$ and $u_1(x,t)$. By canceling these terms from u_0, the remaining non-canceled term of u_0 may provide the exact solution. It follows that the exact solution is given by

$$u(x,t) = \cos(x+t), \quad (9.24)$$

that can be justified by substitution.

Example 3. Solve the inhomogeneous advection problem

$$u_t + \frac{1}{2}(u^2)_x = x, \quad u(x,0) = 2. \quad (9.25)$$

Solution.

The Decomposition Method

Operating with L_t^{-1} gives

$$u(x,t) = 2 + xt - \frac{1}{2}L_t^{-1}L_x(u^2). \quad (9.26)$$

The decomposition method admits the use of

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (9.27)$$

and

$$u^2(x,t) = \sum_{n=0}^{\infty} A_n, \quad (9.28)$$

into (9.26) to obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = 2 + xt - \frac{1}{2} L_t^{-1} L_x \left(\sum_{n=0}^{\infty} A_n \right). \tag{9.29}$$

This gives the recursive relation

$$\begin{aligned} u_0(x,t) &= 2 + xt, \\ u_{k+1}(x,t) &= -\frac{1}{2} L_t^{-1} L_x (A_k), \quad k \geqslant 0, \end{aligned} \tag{9.30}$$

that gives

$$\begin{aligned} u_0(x,t) &= 2 + xt, \\ u_1(x,t) &= -\frac{1}{2} L_t^{-1} L_x (A_0) = -t^2 - \frac{1}{3} xt^3, \\ u_2(x,t) &= -\frac{1}{2} L_t^{-1} L_x (A_1) = \frac{5}{12} t^4 + \frac{2}{15} xt^5, \\ u_3(x,t) &= -\frac{1}{2} L_t^{-1} L_x (A_2) = -\frac{61}{360} t^6 - \frac{17}{315} xt^7. \end{aligned} \tag{9.31}$$

Although the advection problem (9.25) is inhomogeneous, it is clear that the noise terms do not appear in u_0 and u_1. Consequently, the series solution is given by

$$\begin{aligned} u(x,t) = 2 &\left(1 - \frac{1}{2!} t^2 + \frac{5}{4!} t^4 - \frac{61}{6!} t^6 + \cdots \right) \\ &+ x \left(t - \frac{1}{3} t^3 + \frac{2}{15} t^5 - \frac{17}{315} t^7 + \cdots \right), \end{aligned} \tag{9.32}$$

and as a result, the exact solution is given by

$$u(x,t) = 2 \operatorname{sech} t + x \tanh t. \tag{9.33}$$

The Variational Iteration Method

The variational iteration method gives the correction functional for this equation by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi) \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{1}{2} \frac{\partial \tilde{u}_n^2(x,\xi)}{\partial x} - x \right) d\xi. \tag{9.34}$$

Proceeding as in Example 1 we find

$$\lambda = -1. \tag{9.35}$$

Consequently, we obtain the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{1}{2} \frac{u_n^2(x,\xi)}{\partial x} - x \right) d\xi, \quad n \geqslant 0. \tag{9.36}$$

Selecting $u_0(x,t) = 2$ from the given initial condition yields the successive approximations

$$
\begin{aligned}
u_0(x,t) &= 2,\\
u_1(x,t) &= 2+xt,\\
u_2(x,t) &= 2+xt-\frac{1}{3}xt^3-t^2,\\
u_3(x,t) &= 2+xt-\frac{1}{3}xt^3-t^2+\frac{5}{12}t^4+\frac{2}{15}xt^5-\frac{1}{18}t^6-\frac{1}{63}xt^7,\\
&\vdots\\
u_n(x,t) &= 2(1-\frac{1}{2!}t^2+\frac{5}{4!}t^4-\frac{61}{6!}t^6+\cdots)+x(t-\frac{1}{3}t^3+\frac{2}{15}t^5-\frac{17}{315}t^7+\cdots).
\end{aligned}
\tag{9.37}
$$

Consequently, the exact solution is given by

$$
u(x,t) = 2\operatorname{sech} t + x\tanh t. \tag{9.38}
$$

Example 4. Solve the homogeneous nonlinear problem

$$
u_t + u^2u_x = 0, \quad u(x,0) = 3x. \tag{9.39}
$$

Solution.

Proceeding as before we find

$$
u(x,t) = 3x - L_t^{-1}(u^2u_x), \tag{9.40}
$$

so that

$$
\sum_{n=0}^{\infty} u_n(x,t) = 3x - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{9.41}
$$

Adomian's method introduces the recursive relation

$$
\begin{aligned}
u_0(x,t) &= 3x,\\
u_{k+1}(x,t) &= -L_t^{-1}(A_k), \quad k \geqslant 0,
\end{aligned}
\tag{9.42}
$$

which gives

$$
\begin{aligned}
u_0(x,t) &= 3x,\\
u_1(x,t) &= -L_t^{-1}(A_0) = -27x^2t,\\
u_2(x,t) &= -L_t^{-1}(A_1) = 486x^3t^2,\\
u_3(x,t) &= -L_t^{-1}(A_2) = -10935x^4t^3,
\end{aligned}
\tag{9.43}
$$

and so on. Consequently, the series solution

$$
u(x,t) = 3x - 27x^2t + 486x^3t^2 - 10935x^4t^3 + \cdots. \tag{9.44}
$$

Based on this, the solution can be expressed in the form

$$u(x,t)=\begin{cases}3x, & t=0,\\ \dfrac{1}{6t}(\sqrt{1+36xt}-1), & t>0.\end{cases} \tag{9.45}$$

Exercises 9.2

Use the variational iteration method or the decomposition method, and the noise terms phenomena where appropriate, to solve the following nonlinear advection problems:

1. $u_t+uu_x=1-e^{-x}(t+e^{-x})$, $u(x,0)=e^{-x}$
2. $u_t+uu_x=2t+x+t^3+xt^2$, $u(x,0)=0$
3. $u_t+uu_x=2x^2t+2xt^2+2x^3t^4$, $u(x,0)=1$
4. $u_t+uu_x=1+t\cos x+\frac{1}{2}\sin 2x$, $u(x,0)=\sin x$
5. $u_t+uu_x=0$, $u(x,0)=-x$
6. $u_t+uu_x=x$, $u(x,0)=-1$
7. $u_t+uu_x=1+x$, $u(x,0)=0$
8. $u_t+uu_x-u=e^t$, $u(x,0)=1+x$
9. $u_t+uu_x=0$, $u(x,0)=4x$
10. $u_t+uu_x=0$, $u(x,0)=x^2$

9.3 The Goursat Problem

In this section we will study the Goursat problem [5,12] that arise in linear and nonlinear partial differential equations with mixed derivatives. Several numerical methods such as Runge-Kutta method, finite difference method, finite elements method, and geometric mean averaging of the functional values of $f(x,y,u,u_x,u_y)$ have been used to approach the problem. However, the linear and nonlinear Goursat models will be approached more effectively and rapidly by using the Adomian decomposition method. The linear examples will be handled by the variational iteration method as well.

The Goursat problem in its standard form is given by

$$u_{xy}=f(x,y,u,u_x,u_y),\quad 0\leqslant x\leqslant a,\ 0\leqslant y\leqslant b, \tag{9.46}$$

$$u(x,0)=g(x),\quad u(0,y)=h(y),\quad g(0)=h(0)=u(0,0). \tag{9.47}$$

In an operator form, Eq. (9.46) can be rewritten as

$$L_xL_y u = f(x,y,u,u_x,u_y), \tag{9.48}$$

where

$$L_x = \frac{\partial}{\partial x}, \quad L_y = \frac{\partial}{\partial y}. \tag{9.49}$$

The inverse operators L_x^{-1} and L_y^{-1} can be defined as

$$L_x^{-1}(\cdot) = \int_0^x (\cdot)\,dx, \quad L_y^{-1}(\cdot) = \int_0^y (\cdot)\,dy. \tag{9.50}$$

Because the Goursat problem (9.46) involves two distinct differential operators L_x and L_y, two inverse integral operator L_x^{-1} and L_y^{-1} will be used. Applying L_y^{-1} to both sides of (9.48) gives

$$L_x[L_y^{-1}L_y u(x,y)] = L_y^{-1} f(x,y,u,u_x,u_y). \tag{9.51}$$

It then follows that

$$L_x[u(x,y) - u(x,0)] = L_y^{-1} f(x,y,u,u_x,u_y), \tag{9.52}$$

or equivalently

$$L_x u(x,y) = L_x u(x,0) + L_y^{-1} f(x,y,u,u_x,u_y). \tag{9.53}$$

Operating with L_x^{-1} on (9.53) yields

$$L_x^{-1}L_x u(x,y) = L_x^{-1}L_x u(x,0) + L_x^{-1}L_y^{-1} f(x,y,u,u_x,u_y). \tag{9.54}$$

This gives

$$u(x,y) = u(x,0) + u(0,y) - u(0,0) + L_x^{-1}L_y^{-1} f(x,y,u,u_x,u_y), \tag{9.55}$$

or equivalently

$$u(x,y) = g(x) + h(y) - g(0) + L_x^{-1}L_y^{-1} f(x,y,u,u_x,u_y), \tag{9.56}$$

obtained upon using the conditions given in (9.47). Substituting

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y), \tag{9.57}$$

into (9.56) leads to

$$\sum_{n=0}^{\infty} u_n(x,y) = g(x) + h(y) - g(0) + L_x^{-1}L_y^{-1} f(x,y,u,u_x,u_y). \tag{9.58}$$

Adomian's method admits the use of the recursive relation

$$\begin{aligned} u_0(x,y) &= \eta(x,y), \\ u_{k+1}(x,y) &= L_x^{-1}L_y^{-1}\sigma(u_k,u_{k_x},u_{k_y}), \quad k \geqslant 0, \end{aligned} \tag{9.59}$$

where

$\eta(x,y) =$

$$\begin{cases} g(x)+h(y)-g(0), & f=\sigma(u,u_x,u_y), \\ g(x)+h(y)-g(0)+L_x^{-1}L_y^{-1}\tau(x,y), & f=\tau(x,y)+\sigma(u,u_x,u_y). \end{cases} \tag{9.60}$$

In view of (9.59), the solution in a series form follows immediately. The resulting series solution may provide the exact solution. Otherwise, the n-term approximation ϕ_n can be used for numerical purposes. It can be shown that the difference between the exact solution and the n-term approximation decreases monotonically for all values of x and y as additional components are evaluated.

In the following, four linear and nonlinear Goursat models will be discussed for illustrative purposes.

Example 1. Solve the following linear Goursat problem

$$u_{xy} = -x+u, \tag{9.61}$$

subject to the conditions

$$u(x,0) = x+e^x, \quad u(0,y) = e^y, \quad u(0,0) = 1. \tag{9.62}$$

Solution.

The Decomposition Method

Following the previous discussion and using (9.55) we find

$$u(x,y) = x+e^x+e^y-1-\frac{1}{2}x^2y+L_x^{-1}L_y^{-1}\,u(x,y), \tag{9.63}$$

and by using the series representation for $u(x,t)$ into (9.63) gives

$$\sum_{n=0}^{\infty} u_n(x,y) = x+e^x+e^y-1-\frac{1}{2}x^2y+L_x^{-1}L_y^{-1}\left(\sum_{n=0}^{\infty} u_n(x,y)\right). \tag{9.64}$$

The recursive relation

$$\begin{aligned} u_0(x,y) &= x+e^x+e^y-1-\frac{1}{2}x^2y, \\ u_{k+1}(x,y) &= L_x^{-1}L_y^{-1}\,u_k(x,y), \quad k \geqslant 0, \end{aligned} \tag{9.65}$$

follows immediately. Consequently, the first three components of the solution $u(x,y)$ are given by

$$
\begin{aligned}
u_0(x,y) &= x + e^x + e^y - 1 - \frac{1}{2}x^2 y,\\
u_1(x,y) &= L_x^{-1}L_y^{-1}\,u_0(x,y)\\
&= \frac{1}{2}x^2 y + y(e^x-1) + x(e^y-1) - xy - \frac{1}{12}x^3y^2,\\
u_2(x,y) &= L_x^{-1}L_y^{-1}\,u_1(x,y)\\
&= \frac{1}{12}x^3y^2 + \frac{1}{2}y^2(e^x-1-x) + \frac{1}{2}x^2(e^y-1-y)\\
&\quad -\frac{1}{4}x^2y^2 - \frac{1}{144}y^4x^3,
\end{aligned}
\tag{9.66}
$$

This gives

$$
\begin{aligned}
u(x,y) = x &+ e^x\left(1+y+\frac{1}{2!}y^2+\frac{1}{3!}y^3+\cdots\right)\\
&+e^y\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right)\\
&-\left(1+x+y+xy+\frac{1}{2!}x^2+\frac{1}{2!}y^2+\frac{1}{3!}x^3+\frac{1}{3!}y^3+\frac{1}{2!}x^2y+\cdots\right),
\end{aligned}
\tag{9.67}
$$

or equivalently

$$
\begin{aligned}
u(x,y) = x &+ e^x\left(1+y+\frac{1}{2!}y^2+\frac{1}{3!}y^3+\cdots\right)\\
&+e^y\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right)\\
&-(1+x+\frac{1}{2!}x^2+\cdots)(1+y+\frac{1}{2!}y^2+\cdots).
\end{aligned}
\tag{9.68}
$$

Accordingly, the solution in a closed form is given by

$$
u(x,y) = x + e^{x+y},
\tag{9.69}
$$

obtained upon using the Taylor expansions for e^x and e^y.

The Variational Iteration Method

The correction functional for this equation reads

$$
u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial x\partial \xi} - \tilde{u}_n(x,\xi) + x\right)d\xi.
\tag{9.70}
$$

The stationary conditions

$$
\begin{aligned}
1+\lambda &= 0,\\
\lambda' &= 0,
\end{aligned}
\tag{9.71}
$$

give

$$
\lambda = -1.
\tag{9.72}
$$

Substituting the Lagrange multiplier $\lambda = -1$ into the correction functional gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) - \int_0^y \left(\frac{\partial^2 u_n(x,\xi)}{\partial x \partial \xi} - u_n(x,\xi) + x \right) d\xi, \quad n \geqslant 0. \tag{9.73}$$

Selecting $u_0(x,y) = x + Ae^x + Be^y$ gives the following successive approximations

$$\begin{aligned}
u_0(x,y) &= x + Ae^x + Be^y, \\
u_1(x,y) &= x + Ae^x(1+y) + 2Be^y - B, \\
u_2(x,y) &= x + Ae^x(1 + y + \frac{1}{2!}y^2) + 4Be^y - 3B - By, \\
u_3(x,y) &= x + Ae^x(1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3) + 8Be^y - 7B - 4By - \frac{1}{2}By^2, \\
&\vdots \\
u_n(x,y) &= x + Ae^x(1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots) \\
&\quad + 8Be^y - 7B - 4By - \frac{1}{2}By^2 + \cdots.
\end{aligned} \tag{9.74}$$

Using the boundary conditions $u(0,0) = 1$ and $u(x,0) = x + e^x$ gives the system

$$\begin{aligned}
A + B &= 1, \\
x + Ae^x + B &= x + e^x.
\end{aligned} \tag{9.75}$$

Solving this system gives $A = 1, B = 0$. Substituting these values into $u_n(x,t)$ gives the exact solution

$$u(x,y) = x + e^{x+y}, \tag{9.76}$$

obtained upon using the Taylor expansions for e^y.

Example 2. Solve the following linear Goursat problem

$$u_{xy} = 4xy - x^2y^2 + u, \tag{9.77}$$

subject to the conditions

$$u(x,0) = e^x, \; u(0,y) = e^y, \; u(0,0) = 1. \tag{9.78}$$

Solution.

The Decomposition Method

Proceeding as before we find

$$u(x,y) = x^2y^2 - \frac{1}{9}x^3y^3 - 1 + e^x + e^y + L_x^{-1}L_y^{-1}u(x,y). \tag{9.79}$$

This also gives

$$\sum_{n=0}^{\infty} u_n(x,y) = x^2y^2 - \frac{1}{9}x^3y^3 - 1 + e^x + e^y + L_x^{-1}L_y^{-1}\left(\sum_{n=0}^{\infty} u_n(x,y)\right). \tag{9.80}$$

The decomposition method introduces the recursive relation

$$\begin{aligned} u_0(x,y) &= x^2y^2 - \frac{1}{9}x^3y^3 - 1 + e^x + e^y, \\ u_{k+1}(x,y) &= L_x^{-1}L_y^{-1}u_k(x,y), \quad k \geqslant 0, \end{aligned} \tag{9.81}$$

that leads to

$$\begin{aligned} u_0(x,y) &= x^2y^2 - \frac{1}{9}x^3y^3 - 1 + e^x + e^y, \\ u_1(x,y) &= L_x^{-1}L_y^{-1}u_0(x,y) \\ &= \frac{1}{9}x^3y^3 - \frac{1}{144}x^4y^4 - xy + y(e^x - 1) + x(e^y - 1), \\ u_2(x,y) &= L_x^{-1}L_y^{-1}u_1(x,y) \\ &= \frac{1}{144}x^4y^4 - \frac{1}{3600}x^5y^5 - \frac{1}{4}x^2y^2 + \frac{1}{2}y^2(e^x - 1 - x) \\ &\quad + \frac{1}{2}x^2(e^y - 1 - y). \end{aligned} \tag{9.82}$$

In view of (9.82), the solution in a series form is given by

$$\begin{aligned} u(x,y) &= x^2y^2 + e^x\left(1 + y + \frac{1}{2!}y^2 + \cdots\right) + e^y\left(1 + x + \frac{1}{2!}x^2 + \cdots\right) \\ &\quad - \left(1 + x + y + xy + \frac{1}{2!}x^2 + \frac{1}{2!}y^2 + \frac{1}{3!}x^3 + \frac{1}{3!}y^3 + \cdots\right), \end{aligned} \tag{9.83}$$

and in a closed form by

$$u(x,y) = x^2y^2 + e^{x+y}. \tag{9.84}$$

The Variational Iteration Method

Proceeding as in Example 1 and using the Lagrange multiplier $\lambda = -1$ we obtain the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) - \int_0^y \left(\frac{\partial^2 u_n(x,\xi)}{\partial x \partial \xi} - u_n(x,\xi) - 4x\xi + x^2\xi^2\right) d\xi, \quad n \geqslant 0. \tag{9.85}$$

As stated before, we can select $u_0(x,y) = y^2x^2 + Ae^x + Be^y$. Using this selection into the iteration formula we obtain the following successive approximations

$$\begin{aligned} u_0(x,y) &= x^2y^2 + Ae^x + Be^y, \\ u_1(x,y) &= x^2y^2 + Ae^x(1+y) + 2Be^y - B, \end{aligned}$$

$$
\begin{aligned}
u_2(x,y) &= x^2y^2 + Ae^x(1+y+\frac{1}{2!}y^2) + 4Be^y - 3B - By, \\
&\vdots \\
u_n(x,t) &= x^2y^2 + Ae^x(1+y+\frac{1}{2!}y^2+\frac{1}{3!}y^3+\cdots) \\
&\quad +8Be^y - 7B - 4By - \frac{1}{2}By^2 + \cdots.
\end{aligned}
\tag{9.86}
$$

Using the boundary conditions $u(0,0) = 1$ and $u(x,0) = e^x$ gives the system

$$
\begin{aligned}
A + B &= 1, \\
Ae^x + B &= e^x,
\end{aligned}
\tag{9.87}
$$

so that $A = 1, B = 0$. This gives the exact solution

$$u(x,y) = x^2y^2 + e^{x+y}. \tag{9.88}$$

Example 3. Solve the following nonlinear Goursat problem

$$u_{xy} = e^{x+y}e^u, \tag{9.89}$$

subject to the conditions

$$u(x,0) = \ln 2 - 2\ln(1+e^x), \quad u(0,y) = \ln 2 - 2\ln(1+e^y), \quad u(0,0) = -\ln 2. \tag{9.90}$$

Solution.

Following the discussions presented above yields

$$u(x,y) = 3\ln 2 - 2\ln(1+e^x) - 2\ln(1+e^y) + L_x^{-1}L_y^{-1}e^{x+y}e^u. \tag{9.91}$$

Proceeding as before we obtain

$$\sum_{n=0}^{\infty} u_n(x,y) = 3\ln 2 - 2\ln(1+e^x) - 2\ln(1+e^y) + L_x^{-1}L_y^{-1}\left(\sum_{n=0}^{\infty} e^{x+y}A_n\right), \tag{9.92}$$

where A_n are the Adomian polynomials for the nonlinear term e^u. The Adomian polynomials for the exponential nonlinearity e^u were calculated before and given by

$$
\begin{aligned}
A_0 &= e^{u_0}, \\
A_1 &= u_1e^{u_0}, \\
A_2 &= (\frac{1}{2!}u_1^2 + u_2)e^{u_0}, \\
A_3 &= (\frac{1}{3!}u_1^3 + u_1u_2 + u_3)e^{u_0}.
\end{aligned}
$$

The decomposition method introduces the recursive relation

$$\begin{aligned} u_0(x,y) &= 3\ln 2 - 2\ln(1+e^x) - 2\ln(1+e^y),\\ u_{k+1}(x,y) &= L_x^{-1}L_y^{-1}\,(e^{x+y}A_k), \quad k \geqslant 0. \end{aligned} \tag{9.93}$$

The first three components of the solution $u(x,y)$ are given by

$$\begin{aligned} u_0(x,y) &= 3\ln 2 - 2\ln(1+e^x) - 2\ln(1+e^y),\\ u_1(x,y) &= L_x^{-1}L_y^{-1}\,e^{x+y}e^{u_0}\\ &= 8L_x^{-1}L_y^{-1}\left[\frac{e^x}{(e^x+1)^2}\times\frac{e^y}{(e^y+1)^2}\right]\\ &= 2\left[\frac{(e^x-1)(e^y-1)}{(e^x+1)(e^y+1)}\right],\\ u_2(x,y) &= L_x^{-1}L_y^{-1}\,e^{x+y}u_1e^{u_0}\\ &= 16L_x^{-1}L_y^{-1}\left[\frac{e^x(e^x-1)}{(e^x+1)^3}\times\frac{e^y(e^y-1)}{(e^y+1)^3}\right],\\ &= \left[\frac{(e^x-1)(e^y-1)}{(e^x+1)(e^y+1)}\right]^2,\\ u_3(x,y) &= L_x^{-1}L_y^{-1}\,(\frac{1}{2!}u_1^2+u_2)e^{x+y}e^{u_0}\\ &= \frac{2}{3}\left[\frac{(e^x-1)(e^y-1)}{(e^x+1)(e^y+1)}\right]^3, \end{aligned} \tag{9.94}$$

and so on. Note that the integrals involved above can be obtained by substituting $z = 1+e^t, dz = e^t\,dt$. In view of (9.94), the solution in a series form is given by

$$\begin{aligned} u(x,y) &= 3\ln 2 - 2\ln(1+e^x) - 2\ln(1+e^y)\\ &\quad +2\left(\sum_{n=1}^{\infty}\frac{K^n(x,y)}{n}\right), \end{aligned} \tag{9.95}$$

where

$$K(x,y) = \frac{(e^x-1)(e^y-1)}{(e^x+1)(e^y+1)}. \tag{9.96}$$

Recall that the Taylor expansion for $\ln(1-t)$ is given by

$$\begin{aligned} \ln(1-t) &= -(t+\frac{1}{2}t^2+\frac{1}{3}t^3+\cdots),\\ &= -\sum_{n=1}^{\infty}\frac{t^n}{n}, \quad -1 \leqslant t < 1. \end{aligned} \tag{9.97}$$

This means that Eq. (9.95) becomes

$$\begin{aligned}
u(x,y) &= 3\ln 2 - 2\ln(1+e^x) - 2\ln(1+e^y) - 2\ln[1-K(x,y)] \\
&= 3\ln 2 - 2\ln(1+e^x) - 2\ln(1+e^y) \\
&\quad -2\ln[1-\frac{(e^x-1)(e^y-1)}{(e^x+1)(e^y+1)}], \\
&= 3\ln 2 - 2\ln(1+e^x) - 2\ln(1+e^y) \\
&\quad -2\ln 2\left(\frac{(e^x+e^y)}{(e^x+1)(e^y+1)}\right), \\
&= \ln 2 - 2\ln(e^x+e^y).
\end{aligned} \tag{9.98}$$

Example 4. Solve the following nonlinear Goursat problem

$$u_{xy} = \frac{2}{3}e^{3u}, \tag{9.99}$$

subject to the conditions

$$u(x,0) = \frac{1}{3}x - \frac{2}{3}\ln(1+e^x),\, u(0,y) = \frac{1}{3}y - \frac{2}{3}\ln(1+e^y), u(0,0) = -\frac{2}{3}\ln 2. \tag{9.100}$$

Solution.

Proceeding as before we find

$$u(x,y) = \frac{x+y}{3} - \frac{2}{3}\ln(1+e^x) - \frac{2}{3}\ln(1+e^y) + \frac{2}{3}\ln 2 + \frac{2}{3}L_x^{-1}L_y^{-1}e^{3u}. \tag{9.101}$$

Substituting the series representation for the linear and the nonlinear terms into (9.101) we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x,y) &= \frac{x+y}{3} - \frac{2}{3}\ln(1+e^x) - \frac{2}{3}\ln(1+e^y) + \frac{2}{3}\ln 2 \\
&\quad + \frac{2}{3}L_x^{-1}L_y^{-1}\left(\sum_{n=0}^{\infty} A_n\right),
\end{aligned} \tag{9.102}$$

where A_n are Adomian polynomials for the nonlinear term e^{3u} given by

$$\begin{aligned}
A_0 &= e^{3u_0}, \\
A_1 &= 3u_1 e^{3u_0}, \\
A_2 &= (\frac{9}{2!}u_1^2 + 3u_2)e^{3u_0}, \\
A_3 &= (\frac{27}{3!}u_1^3 + 9u_1u_2 + 3u_3)e^{3u_0}.
\end{aligned}$$

Following Adomian analysis, we set the recursive relation

$$\begin{aligned} u_0(x,y) &= \frac{x+y}{3} - \frac{2}{3}\ln(1+e^x) - \frac{2}{3}\ln(1+e^y) + \frac{2}{3}\ln 2, \\ u_{k+1}(x,y) &= \frac{2}{3}L_x^{-1}L_y^{-1}(A_k), \quad k \geqslant 0, \end{aligned} \tag{9.103}$$

so that

$$\begin{aligned} u_0(x,y) &= \frac{x+y}{3} - \frac{2}{3}\ln(1+e^x) - \frac{2}{3}\ln(1+e^y) + \frac{2}{3}\ln 2, \\ u_1(x,y) &= \frac{2}{3}L_x^{-1}L_y^{-1}\, e^{3u_0} \\ &= \frac{8}{3}L_x^{-1}L_y^{-1}\left[\frac{e^x}{(e^x+1)^2} \times \frac{e^y}{(1+e^y)^2}\right], \\ &= \frac{2}{3}\left[\frac{(e^x-1)(e^y-1)}{(e^x+1)(e^y+1)}\right], \\ u_2(x,y) &= \frac{2}{3}L_x^{-1}L_y^{-1}\, 3u_1 e^{3u_0} \\ &= \frac{16}{3}L_x^{-1}L_y^{-1}\left[\frac{e^x(e^x-1)}{(e^x+1)^3} \times \frac{e^y(e^y-1)}{(e^y+1)^3}\right], \\ &= \frac{1}{3}\left[\frac{(e^x-1)(e^y-1)}{(e^x+1)(e^y+1)}\right]^2, \end{aligned} \tag{9.104}$$

and so on. Proceeding as before we obtain

$$\begin{aligned} u(x,y) &= \frac{x+y}{3} - \frac{2}{3}\ln(1+e^x) - \frac{2}{3}\ln(1+e^y) + \frac{2}{3}\ln 2 \\ &\quad + \frac{2}{3}\left(\sum_{n=1}^{\infty}\frac{K^n(x,y)}{n}\right), \end{aligned} \tag{9.105}$$

where

$$K(x,y) = \frac{(e^x-1)(e^y-1)}{(e^x+1)(e^y+1)}. \tag{9.106}$$

Using the Taylor expansion for $\ln(1-t)$ gives

$$\begin{aligned} u(x,y) &= \frac{x+y}{3} - \frac{2}{3}\ln(1+e^x) - \frac{2}{3}\ln(1+e^y) + \frac{2}{3}\ln 2 \\ &\quad - \frac{2}{3}\ln[1-K(x,y)], \\ &= \frac{x+y}{3} - \frac{2}{3}\ln(e^x+e^y). \end{aligned} \tag{9.107}$$

Exercises 9.3

In Exercises 1–6, use the variational iteration method or Adomian decomposition method to solve the following linear Goursat problems:

1. $u_{xy} = -y + u, u(x,0) = e^x, u(0,y) = y + e^y$

2. $u_{xy} = 1 - xy + u, u(x,0) = e^x, u(0,y) = e^y$

3. $u_{xy} = x + y + u, u(x,0) = -x + e^x, u(0,y) = -y + e^y$

4. $u_{xy} = -x^2 + u, u(x,0) = x^2 + e^x, u(0,y) = e^y$

5. $u_{xy} = u + 2e^{x+y}, u(x,0) = xe^x, u(0,y) = ye^y$

6. $u_{xy} = u, u(x,0) = e^x, u(0,y) = e^y$

In Exercises 7–12, use Adomian decomposition method to solve the following nonlinear Goursat problems:

7. $u_{xy} = e^{2u}, u(x,0) = \frac{1}{2}x - \ln(1+e^x), u(0,y) = \frac{1}{2}y - \ln(1+e^y)$

8. $u_{xy} = -e^{2u}, u(x,0) = \frac{1}{2}x - \ln(1+e^x), u(0,y) = -\frac{1}{2}y - \ln(1+e^{-y})$

9. $u_{xy} = e^y e^{2u}, u(x,0) = \frac{1}{2}x - \ln(1+e^x), u(0,y) = -\ln(1+e^y)$

10. $u_{xy} = e^x e^{2u}, u(x,0) = -\ln(1+e^x), u(0,y) = \frac{1}{2}y - \ln(1+e^y)$

11. $u_{xy} = \frac{2}{5}e^y e^{5u}, u(x,0) = \frac{1}{5}x - \frac{2}{5}\ln(1+e^x), u(0,y) = -\frac{2}{5}\ln(1+e^y)$

12. $u_{xy} = e^{x+y}e^{2y}, u(x,0) = -\ln(1+e^x), u(0,y) = -\ln(1+e^y)$

9.4 The Klein-Gordon Equation

The Klein-Gordon equation [15,16] is considered one of the most important mathematical models in quantum field theory. The equation appears in relativistic physics and is used to describe dispersive wave phenomena in general. In addition, it also appears in nonlinear optics and plasma physics. The Klein-Gordon equation arise in physics in linear and nonlinear forms.

The Klein-Gordon equation has been extensively studied by using traditional methods such as finite difference method, finite element method, or collocation method. Bäcklund transformations and the inverse scattering method were also applied to handle Klein-Gordon equation. The methods investigated the concepts of existence, uniqueness of the solution and the weak solution as well. The objectives of these studies were mostly focused on the determination of numerical solutions where a considerable volume of calculations is usually needed.

In this section, the Adomian decomposition method will be applied to obtain exact solutions if exist, and approximate to solutions for concrete problems. Moreover, the variational iteration method will be used for some of the examples for comparison reasons.

9.4.1 Linear Klein-Gordon Equation

The linear Klein-Gordon equation in its standard form is given by

$$u_{tt}(x,t) - u_{xx}(x,t) + au(x,t) = h(x,t), \tag{9.108}$$

subject to the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \tag{9.109}$$

where a is a constant and $h(x,t)$ is the source term. It is interesting to point here that if $a = 0$, Eq. (9.108) becomes the inhomogeneous wave equation that was introduced before. The linear Klein-Gordon equation is important in quantum mechanics. It is derived from the relativistic energy formula.

In an operator form, Eq. (9.108) can be rewritten as

$$L_t u(x,t) = u_{xx}(x,t) - au(x,t) + h(x,t), \tag{9.110}$$

where L_t is a second order differential operator and the inverse operator L_t^{-1} is a two-fold integral operator defined by

$$L_t^{-1}(\cdot) = \int_0^t \int_0^t (\cdot) dt\, dt. \tag{9.111}$$

Applying L_t^{-1} to both sides of (9.110) and using the initial conditions we find

$$u(x,t) = f(x) + tg(x) + L_t^{-1}(h(x,t)) + L_t^{-1}\left(u_{xx}(x,t) - au(x,t)\right). \tag{9.112}$$

Using the decomposition representation for $u(x,t)$ into both sides of (9.112) gives

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) &= f(x) + tg(x) + L_t^{-1}(h(x,t)) \\ &\quad + L_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx} - a\sum_{n=0}^{\infty} u_n(x,t)\right).\end{aligned} \tag{9.113}$$

We can formally set the recursive relation

$$\begin{aligned} u_0(x,t) &= f(x) + tg(x) + L_t^{-1}(h(x,t)), \\ u_{k+1}(x,t) &= L_t^{-1}\left(u_{k_{xx}}(x,t) - au_k(x,t)\right), \quad k \geqslant 0. \end{aligned} \tag{9.114}$$

This completes the determination of the components of $u(x,t)$. The solution in a series form follows immediately. In many cases we can obtain inductively the exact solution. The algorithm discussed above will be explained through the following illustrative examples.

Example 1. Solve the following linear homogeneous Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0, \quad u(x,0) = 0, \quad u_t(x,0) = x. \tag{9.115}$$

Solution.

The Decomposition Method

Applying L_t^{-1} to both sides of (9.115) and using the decomposition series for $u(x,t)$ give

$$\sum_{n=0}^{\infty} u_n(x,t) = xt + L_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx} - \sum_{n=0}^{\infty} u_n(x,t)\right). \tag{9.116}$$

Close examination of (9.116) suggests that the recursive relation is

$$\begin{aligned} u_0(x,t) &= xt, \\ u_{k+1}(x,t) &= L_t^{-1}\left(u_{k_{xx}}(x,t) - u_k(x,t)\right), \quad k \geqslant 0, \end{aligned} \tag{9.117}$$

that in turn gives

$$\begin{aligned} u_0(x,t) &= xt, \\ u_1(x,t) &= L_t^{-1}(u_{0_{xx}}(x,t) - u_0(x,t)) = -\frac{1}{3!}xt^3, \\ u_2(x,t) &= L_t^{-1}(u_{1_{xx}}(x,t) - u_1(x,t)) = \frac{1}{5!}xt^5. \end{aligned} \tag{9.118}$$

In view of (9.118) the series solution is given by

$$u(x,t) = x(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots), \tag{9.119}$$

and the exact solution is given by

$$u(x,t) = x\sin t. \tag{9.120}$$

The Variational Iteration Method

The correction functional for this equation reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} + \tilde{u}_n(x,\xi)\right) d\xi. \tag{9.121}$$

This yields the stationary conditions

$$\begin{aligned} 1-\lambda'|_{\xi=t} &= 0, \\ \lambda|_{\xi=t} &= 0, \\ \lambda''|_{\xi=t} &= 0. \end{aligned} \tag{9.122}$$

This in turn gives

$$\lambda = \xi - t. \tag{9.123}$$

Substituting this value of the Lagrange multiplier into the correction functional gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + u_n(x,\xi)\right) d\xi. \tag{9.124}$$

Considering the given initial values, we can select $u_0(x,t) = xt$. Using this selection into (9.124) we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= xt, \\ u_1(x,t) &= xt - \frac{1}{3!}xt^3, \\ u_2(x,t) &= xt - \frac{1}{3!}xt^3 + \frac{1}{5!}xt^5, \\ u_3(x,t) &= xt - \frac{1}{3!}xt^3 + \frac{1}{5!}xt^5 - \frac{1}{7!}xt^7, \\ &\vdots \\ u_n(x,t) &= x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots\right). \end{aligned} \tag{9.125}$$

This gives the exact solution by

$$u(x,t) = x \sin t. \tag{9.126}$$

Example 2. Solve the following linear inhomogeneous Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 2\sin x, \quad u(x,0) = \sin x, \quad u_t(x,0) = 1. \tag{9.127}$$

Solution.

The Decomposition Method

Proceeding as in Example 1 we find

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin x + t + t^2 \sin x + L_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx} - \sum_{n=0}^{\infty} u_n(x,t)\right). \tag{9.128}$$

Consequently, we set the relation

$$\begin{aligned} u_0(x,t) &= \sin x + t + t^2 \sin x \\ u_{k+1}(x,t) &= L_t^{-1}(u_{k_{xx}}(x,t) - u_k(x,t)), \quad k \geqslant 0, \end{aligned} \tag{9.129}$$

that gives

$$\begin{aligned} u_0(x,t) &= \sin x + t + t^2 \sin x, \\ u_1(x,t) &= L_t^{-1}(u_{0_{xx}}(x,t) - u_0(x,t)) = -t^2 \sin x - \frac{1}{6}t^4 \sin x - \frac{1}{3!}t^3, \\ u_2(x,t) &= L_t^{-1}(u_{1_{xx}}(x,t) - u_1(x,t)) = \frac{1}{6}t^4 \sin x + \frac{1}{90}t^6 \sin x + \frac{1}{5!}t^5. \end{aligned} \tag{9.130}$$

In view of (9.130), the series solution is given by

$$u(x,t) = \sin x + (t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots), \tag{9.131}$$

where noise terms vanish in the limit. The solution in a closed form

$$u(x,t) = \sin x + \sin t, \tag{9.132}$$

follows immediately.

The Variational Iteration Method

The correction functional for this equation reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi) \left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} + \tilde{u}_n(x,\xi) - 2\sin x \right) d\xi. \tag{9.133}$$

Proceeding as before we find

$$\lambda = \xi - t. \tag{9.134}$$

Substituting this value of the Lagrange multiplier into the correction functional gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t) \left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + u_n(x,\xi) - 2\sin x \right) d\xi. \tag{9.135}$$

Considering the given initial values, we can select $u_0(x,t) = t + \sin x$. Using this selection into the iteration formula we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= t + \sin x, \\ u_1(x,t) &= t + \sin x - \frac{1}{3!}t^3, \\ u_2(x,t) &= t + \sin x - \frac{1}{3!}t^3 + \frac{1}{5!}t^5, \end{aligned}$$

$$
\begin{aligned}
u_3(x,t) &= t+\sin x-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\frac{1}{7!}t^7,\\
&\vdots \\
u_n(x,t) &= \sin x+\left(t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\frac{1}{7!}t^7+\cdots\right).
\end{aligned}
\tag{9.136}
$$

The solution in a closed form

$$
u(x,t)=\sin x+\sin t, \tag{9.137}
$$

follows immediately.

9.4.2 Nonlinear Klein-Gordon Equation

The nonlinear Klein-Gordon equation [15] comes from quantum field theory and describes nonlinear wave interaction. The nonlinear Klein-Gordon equation in its standard form is given by

$$
u_{tt}(x,t)-u_{xx}(x,t)+au(x,t)+F(u(x,t))=h(x,t), \tag{9.138}
$$

subject to the initial conditions

$$
u(x,0)=f(x),\quad u_t(x,0)=g(x), \tag{9.139}
$$

where a is a constant, $h(x,t)$ is a source term and $F(u(x,t))$ is a nonlinear function of $u(x,t)$. The equation has been investigated using numerical methods such as finite difference method and the averaging techniques.

In a manner parallel to that used before, the decomposition method will be employed. The nonlinear term $F(u(x,t))$ will be equated to the infinite series of Adomian polynomials. Applying L_t^{-1} to both sides of (9.138) and using the initial conditions give

$$
\begin{aligned}
u(x,t) &= f(x)+tg(x)+L_t^{-1}(h(x,t))\\
&\quad +L_t^{-1}(u_{xx}(x,t)-au(x,t))-L_t^{-1}(F(u(x,t))).
\end{aligned}
\tag{9.140}
$$

Using the decomposition series for the linear term $u(x,t)$, the infinite series of Adomian polynomials for the nonlinear term $F(u(x,t))$, and proceeding as before we obtain the recursive relation

$$
\begin{aligned}
u_0(x,t) &= f(x)+tg(x)+L_t^{-1}(h(x,t)),\\
u_{k+1}(x,t) &= L_t^{-1}(u_{k_{xx}}(x,t)-u_k(x,t))-L_t^{-1}(A_k),\quad k\geqslant 0,
\end{aligned}
\tag{9.141}
$$

that leads to

$$
u_0(x,t)=f(x)+tg(x)+L_t^{-1}(h(x,t)),
$$

$$\begin{aligned} u_1(x,t) &= L_t^{-1}(u_{0_{xx}}(x,t) - u_0(x,t)) - L_t^{-1}(A_0), \\ u_2(x,t) &= L_t^{-1}(u_{1_{xx}}(x,t) - u_1(x,t)) - L_t^{-1}(A_1). \end{aligned} \tag{9.142}$$

This completes the determination of the first few components of the solution.

Based on this determination, the solution in a series form is readily obtained. In many cases, a closed form solution can be obtained inductively.

The following examples will be used to illustrate the algorithm discussed above. The noise terms phenomenon and the modified decomposition method will be implemented in this illustration to accelerate the convergence.

Example 3. Solve the following nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + u^2 = x^2t^2, \quad u(x,0) = 0, \quad u_t(x,0) = x. \tag{9.143}$$

Solution.

Following the discussion presented above we find

$$\sum_{n=0}^{\infty} u_n(x,t) = xt + \frac{1}{12}x^2t^4 + L_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx}\right) - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{9.144}$$

We will approach the problem by using the noise terms phenomenon. Equation (9.144) gives the recursive relation

$$\begin{aligned} u_0(x,t) &= xt + \tfrac{1}{12}x^2t^4, \\ u_{k+1}(x,t) &= L_t^{-1}u_{k_{xx}}(x,t) - L_t^{-1}A_k, \quad k \geqslant 0, \end{aligned} \tag{9.145}$$

that yields

$$\begin{aligned} u_0(x,t) &= xt + \frac{1}{12}x^2t^4, \\ u_1(x,t) &= L_t^{-1}u_{0_{xx}}(x,t) - L_t^{-1}A_0 \\ &= \frac{1}{180}t^6 - \frac{1}{12}x^2t^4 - \frac{1}{252}x^3t^7 + \frac{1}{12960}x^4t^{10}. \end{aligned} \tag{9.146}$$

Canceling the noise term $\frac{1}{12}x^2t^4$ from the component u_0, and verifying that the remaining non-canceled term satisfies the equation, the exact solution

$$u(x,t) = xt, \tag{9.147}$$

is readily obtained.

In the following we will solve this example by using the modified decomposition method. As introduced before we split the terms assigned to the zeroth component $u_0(x,t)$ to the first two components $u_0(x,t)$ and $u_1(x,t)$. Thus the modified recursive relation can be rewritten in the scheme

$$\begin{aligned} u_0(x,t) &= xt, \\ u_1(x,t) &= \frac{1}{12}x^2t^4 + L_t^{-1}(u_{0_{xx}}(x,t)) - L_t^{-1}(A_0), \\ u_{k+1}(x,t) &= L_t^{-1}(u_{k_{xx}}(x,t)) - L_t^{-1}(A_k), \quad k \geqslant 1. \end{aligned} \tag{9.148}$$

This leads to

$$\begin{aligned} u_0(x,t) &= xt, \\ u_1(x,t) &= \frac{1}{12}x^2t^4 + L_t^{-1}(u_{0_{xx}}(x,t)) - L_t^{-1}(A_0) = 0, \\ u_{k+1}(x,t) &= 0, \quad k \geqslant 1. \end{aligned} \tag{9.149}$$

Therefore, the exact solution is given by

$$u(x,t) = xt. \tag{9.150}$$

Example 4. Solve the following nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + u^2 = 2x^2 - 2t^2 + x^4t^4, \quad u(x,0) = u_t(x,0) = 0. \tag{9.151}$$

Solution.

The noise terms phenomenon will be used in this example. Proceeding as before gives

$$\begin{aligned} u_0(x,t) &= x^2t^2 - \frac{1}{6}t^4 + \frac{1}{30}x^4t^6, \\ u_{k+1}(x,t) &= L_t^{-1}(u_{k_{xx}}(x,t)) - L_t^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{9.152}$$

Based on this relation the first two components are given by

$$\begin{aligned} u_0(x,t) &= x^2t^2 - \frac{1}{6}t^4 + \frac{1}{30}x^4t^6, \\ u_1(x,t) &= L_t^{-1}(u_{0_{xx}}(x,t)) - L_t^{-1}A_0 = \frac{1}{6}t^4 - \frac{1}{30}x^4t^6 + \cdots. \end{aligned} \tag{9.153}$$

Canceling the noise terms in $u_0(x,t)$ that appear in $u_1(x,t)$ and verifying that the remaining term satisfies the equation leads to the exact solution

$$u(x,t) = x^2t^2. \tag{9.154}$$

Next we formally show that the modified decomposition method accelerates the convergence of the solution and minimizes the size of calculations. The modified method introduces the relation

$$\begin{aligned} u_0(x,t) &= x^2t^2, \\ u_1(x,t) &= -\frac{1}{6}t^4 + \frac{1}{30}x^4t^6 + L_t^{-1}(u_{0_{xx}}(x,t) - A_0) = 0, \\ u_{k+1}(x,t) &= 0, \quad k \geqslant 1. \end{aligned} \tag{9.155}$$

This formally gives the exact solution

$$u(x,t) = x^2t^2. \tag{9.156}$$

9.4.3 The Sine-Gordon Equation

The sine-Gordon equation [8,9,10] appeared first in differential geometry. This equation became the focus of a lot of research work because it appears in many physical phenomena such as the propagation of magnetic flux and the stability of fluid motions. The equation is considered an important nonlinear evolution equation that plays a major role in nonlinear physics.

The standard form of the sine-Gordon equation is given by

$$u_{tt} - c^2u_{xx} + \alpha \sin u = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \tag{9.157}$$

where c and α are constants. It is clear that this equation adds the nonlinear term $\sin u$ to the standard wave equation.

Several classical methods have been employed to handle the sine-Gordon equation. The Bäcklund transformations, the similarity method, and the inverse scattering method are mostly used to investigate this equation.

However, the sine-Gordon equation will be handled by using the Adomian decomposition method. Applying L_t^{-1} to (9.157) and using the initial conditions leads to

$$u(x,t) = f(x) + tg(x) + c^2L_t^{-1}(u_{xx}(x,t)) - \alpha L_t^{-1}(\sin u(x,t)). \tag{9.158}$$

Noting that $\sin u$ is a nonlinear term where the relevant Adomian polynomials have been derived before. Substituting the series decomposition for $u(x,t)$ and the infinite series of Adomian polynomials for $\sin u$ gives

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + tg(x) + L_t^{-1}\left(c^2\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx} - \alpha\left(\sum_{n=0}^{\infty} A_n\right)\right). \tag{9.159}$$

This gives the recursive relation

$$\begin{aligned} u_0(x,t) &= f(x) + tg(x), \\ u_{k+1}(x,t) &= c^2L_t^{-1}(u_{k_{xx}}(x,t)) - \alpha L_t^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{9.160}$$

This will lead to the determination of the solution in a series form. This can be illustrated as follows.

Example 5. Solve the following sine-Gordon equation

$$u_{tt} - u_{xx} = \sin u,\ u(x,0) = \frac{\pi}{2}, \quad u_t(x,0) = 0. \tag{9.161}$$

Solution.

Using the recursive scheme (9.160) yields

$$\begin{aligned} u_0(x,t) &= \frac{\pi}{2}, \\ u_{k+1}(x,t) &= L_t^{-1}(u_{k_{xx}}(x,t)) + L_t^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{9.162}$$

The first few Adomian polynomials for $\sin u$ are given by

$$\begin{aligned} A_0 &= \sin u_0, \\ A_1 &= u_1 \cos u_0, \\ A_2 &= u_2 \cos u_0 - \frac{1}{2!}u_1^2 \sin u_0, \\ A_3 &= u_3 \cos u_0 - u_2 u_1 \sin u_0 - \frac{1}{3!}u_1^3 \cos u_0. \end{aligned} \tag{9.163}$$

Combining (9.162) and (9.163) leads to

$$\begin{aligned} u_0(x,t) &= \frac{\pi}{2}, \quad u_1(x,t) = \frac{1}{2}t^2, \quad u_2(x,t) = 0, \\ u_3(x,t) &= -\frac{1}{240}t^6, \quad u_4(x,t) = 0, \quad u_5(x,t) = \frac{1}{34560}t^{10}, \end{aligned} \tag{9.164}$$

The series solution

$$u(x,t) = \frac{\pi}{2} + \frac{1}{2}t^2 - \frac{1}{240}t^6 + \frac{1}{34560}t^{10} + \cdots \tag{9.165}$$

is readily obtained.

Example 6. Solve the following sine-Gordon equation

$$u_{tt} - u_{xx} = \sin u, \quad u(x,0) = \frac{\pi}{2}, \quad u_t(x,0) = 1. \tag{9.166}$$

Solution.

Using the relation (9.160) gives

$$\begin{aligned} u_0(x,t) &= \frac{\pi}{2} + t, \\ u_{k+1}(x,t) &= L_t^{-1}(u_{k_{xx}}(x,t)) + L_t^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{9.167}$$

Using Adomian polynomials for $\sin u$ as shown above leads to the results

$$\begin{aligned} u_0(x,t) &= \frac{\pi}{2} + t, \\ u_1(x,t) &= 1 - \cos t, \\ u_2(x,t) &= \sin t - \frac{3}{4}t - \frac{1}{8}\sin 2t. \end{aligned} \tag{9.168}$$

Summing these iterates yields

$$u(x,t) = \frac{\pi}{2} + t + 1 - \cos t + \sin t - \frac{3}{4}t - \frac{1}{8}\sin 2t + \cdots, \tag{9.169}$$

so that the series solution

$$u(x,t) = \frac{\pi}{2} + t + \frac{1}{2!}t^2 - \frac{1}{4!}t^4 + \cdots, \tag{9.170}$$

obtained upon using Taylor expansion for the trigonometric functions involved.

It is important to note that another form of the sine-Gordon equation is sometimes used and given in the form

$$u_{xt} = \sin u. \tag{9.171}$$

Recall that the initial value problem of (9.171) has been discussed before as a Goursat problem.

Exercises 9.4

In Exercises 1–5, use the variational iteration method or Adomian decomposition method to solve the linear equations:

1. $u_{tt} - u_{xx} - u = -\cos x \cos t,\ u(x,0) = \cos x,\ u_t(x,0) = 0$
2. $u_{tt} - u_{xx} - u = -\cos x \sin t,\ u(x,0) = 0,\ u_t(x,0) = \cos x$
3. $u_{tt} - u_{xx} - u = -\sin x \sin t,\ u(x,0) = 0,\ u_t(x,0) = \sin x$
4. $u_{tt} - u_{xx} - u = 0,\ u(x,0) = 0,\ u_t(x,0) = \sin x$
5. $u_{tt} - u_{xx} + u = 0,\ u(x,0) = 0,\ u_t(x,0) = \cosh x$

In Exercises 6–10, use the variational iteration method or the modified decomposition method to solve the nonlinear equations:

6. $u_{tt} - u_{xx} - u + u^2 = xt + x^2t^2,\ u(x,0) = 1,\ u_t(x,0) = x$
7. $u_{tt} - u_{xx} + u^2 = 1 + 2xt + x^2t^2,\ u(x,0) = 1,\ u_t(x,0) = x$
8. $u_{tt} - u_{xx} + u^2 = 6xt(x^2 - t^2) + x^6t^6,\ u(x,0) = 0,\ u_t(x,0) = 0$
9. $u_{tt} - u_{xx} + u^2 = (t^2 + x^2)^2,\ u(x,0) = x^2,\ u_t(x,0) = 0$
10. $u_{tt} - u_{xx} + u + u^2 = x^2\cos^2 t,\ u(x,0) = x,\ u_t(x,0) = 0$

In Exercises 11–15, find the ϕ_3 approximant of the solution of the following sine-Gordon equations:

11. $u_{tt} - u_{xx} = \sin u,\ u(x,0) = \frac{\pi}{6},\ u_t(x,0) = 0$
12. $u_{tt} - u_{xx} = \sin u,\ u(x,0) = \frac{\pi}{4},\ u_t(x,0) = 0$
13. $u_{tt} - u_{xx} = \sin u,\ u(x,0) = 0,\ u_t(x,0) = 1$

14. $u_{tt} - u_{xx} = \sin u$, $u(x,0) = \pi$, $u_t(x,0) = 1$

15. $u_{tt} - u_{xx} = \sin u$, $u(x,0) = \dfrac{3\pi}{2}$, $u_t(x,0) = 1$

9.5 The Burgers Equation

The Burgers equation [4] is considered one of the fundamental model equations in fluid mechanics. The equation demonstrates the coupling between diffusion and convection processes.

The standard form of Burgers' equation is given by

$$u_t + uu_x = \nu u_{xx}, \quad t > 0, \tag{9.172}$$

where ν is a constant that defines the kinematic viscosity. If the viscosity $\nu = 0$, the equation is called *inviscid* Burgers equation. The inviscid Burgers equation governs gas dynamics. The inviscid Burgers equation has been discussed before as a homogeneous case of the advection problem. The inviscid equation can be elegantly handled as discussed before in Section 9.2.

Nonlinear Burgers equation is considered by most as a simple nonlinear partial differential equation incorporating both convection and diffusion in fluid dynamics. Burgers introduced this equation in [4] to capture some of the features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion. It is also used to describe the structure of shock waves, traffic flow, and acoustic transmission.

A great potential of research work has been invested on Burgers equation. Several exact solutions have been derived by using distinct approaches. Appendix C contains many of these exact solutions. The *Cole-Hopf* transformation is the commonly used approach. The solution $u(x,t)$ was replaced by ψ_x in (9.172) to obtain

$$\psi_{xt} + \psi_x\psi_{xx} = \nu\psi_{xxx}, \tag{9.173}$$

where by integrating this equation with respect to x we find

$$\psi_t + \frac{1}{2}\psi_x^2 = \nu\psi_{xx}. \tag{9.174}$$

Using the Cole-Hopf transformation

$$\psi = -2\nu\ln\phi, \tag{9.175}$$

so that

$$u(x,t) = \psi_x = -2\nu\frac{\phi_x}{\phi}, \tag{9.176}$$

transforms the nonlinear equation into the heat flow equation

$$\phi_t = \nu \phi_{xx}. \tag{9.177}$$

It is obvious that the difficult nonlinear Burgers equation (9.172) has been converted to an easily solvable linear equation. This will lead to exact solutions, each solution depends on the given conditions.

Another technique for deriving solutions to Burgers equation is the method of *symmetry reduction* in which solutions of the nonlinear Burgers equation are found in terms of parabolic cylinder functions or Airy functions. The symmetry reduction method was applied in a modified way where the Burgers equation was transformed to an ordinary differential equation.

However, it is the intention of this text to effectively apply the reliable Adomian decomposition method. We consider the Burgers equation

$$u_t + uu_x = u_{xx}, \quad u(x,0) = f(x). \tag{9.178}$$

Applying the inverse operator L_t^{-1} to (9.178) leads to

$$u(x,t) = f(x) + L_t^{-1}(u_{xx}) - L_t^{-1}(uu_x). \tag{9.179}$$

Using the decomposition series for the linear term $u(x,t)$ and the series of Adomian polynomials for the nonlinear term uu_x give

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + L_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx}\right) - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{9.180}$$

Identifying the zeroth component $u_0(x,t)$ by the term that arise from the initial condition and following the decomposition method, we obtain the recursive relation

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_{k+1}(x,t) &= L_t^{-1}(u_{k_{xx}}) - L_t^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{9.181}$$

The Adomian polynomials for the nonlinear term uu_x have been derived in the form

$$\begin{aligned} A_0 &= u_{0_x}u_0, \\ A_1 &= u_{0_x}u_1 + u_{1_x}u_0, \\ A_2 &= u_{0_x}u_2 + u_{1_x}u_1 + u_{2_x}u_0, \\ A_3 &= u_{0_x}u_3 + u_{1_x}u_2 + u_{2_x}u_1 + u_{3_x}u_0, \\ A_4 &= u_{0_x}u_4 + u_{1_x}u_3 + u_{2_x}u_2 + u_{3_x}u_1 + u_{4_x}u_0. \end{aligned} \tag{9.182}$$

In view of (9.181) and (9.182), the first few components can be identified by

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_1(x,t) &= L_t^{-1}(u_{0_{xx}}) - L_t^{-1}A_0, \\ u_2(x,t) &= L_t^{-1}(u_{1_{xx}}) - L_t^{-1}A_1, \\ u_3(x,t) &= L_t^{-1}(u_{2_{xx}}) - L_t^{-1}A_2. \end{aligned} \tag{9.183}$$

Additional components can be elegantly computed to enhance the accuracy level. The solution in a series form follows immediately. However, the n-term approximant ϕ_n can be determined by

$$\phi_n = \sum_{k=0}^{n-1} u_k(x,t). \tag{9.184}$$

Moreover, the variational iteration method will also be applied to investigate some of the Burgers equations. In the following we list some of the derived exact solutions of Burgers equation derived by many researchers:

$$\begin{aligned} u(x,t) &= 2\tan x,\ -2\cot x,\ -2\tanh x,\\ u(x,t) &= \frac{x}{t},\ \frac{x}{t}+\frac{2}{x+t}+\frac{x+t}{2t^2-t},\\ u(x,t) &= \frac{-2e^{-t}\cos x}{1+e^{-t}\sin x},\ \frac{2e^{-t}\sin x}{1+e^{-t}\cos x}, \end{aligned} \tag{9.185}$$

A table of solutions of Burgers equation can be found in Appendix C. The following examples will be used to illustrate the discussion carried out above by using Adomian decomposition method. Some of the proposed examples will be examined by using the variational iteration method.

Example 1. Solve the following Burgers equation

$$u_t + uu_x = u_{xx}, \quad u(x,0) = x. \tag{9.186}$$

Solution.

The Decomposition Method

Operating with L_t^{-1} and using (9.180) we find

$$\sum_{n=0}^{\infty} u_n(x,t) = x + L_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx}\right) - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{9.187}$$

This gives the recursive relation

$$\begin{aligned} u_0(x,t) &= x,\\ u_{k+1}(x,t) &= L_t^{-1}\left(u_{k_{xx}}(x,t)\right) - L_t^{-1}\left(A_k\right), \quad k \geqslant 0. \end{aligned} \tag{9.188}$$

Using Adomian polynomials we obtain

$$\begin{aligned} u_0(x,t) &= x,\\ u_1(x,t) &= L_t^{-1}\left(u_{0_{xx}}(x,t)\right) - L_t^{-1}\left(A_0\right) = -xt,\\ u_2(x,t) &= L_t^{-1}\left(u_{1_{xx}}(x,t)\right) - L_t^{-1}\left(A_1\right) = xt^2,\\ u_3(x,t) &= L_t^{-1}\left(u_{2_{xx}}(x,t)\right) - L_t^{-1}\left(A_2\right) = -xt^3. \end{aligned} \tag{9.189}$$

Summing these iterates gives the series solution

$$u(x,t) = x(1 - t + t^2 - t^3 + \cdots). \tag{9.190}$$

Consequently, the exact solution is given by

$$u(x,t) = \frac{x}{1+t}, \quad |t| < 1. \tag{9.191}$$

The Variational Iteration Method

The variational iteration method gives the correction functional for this equation by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{u}_n(x,\xi)\frac{\partial \tilde{u}_n(x,\xi)}{\partial x} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2}\right) d\xi. \tag{9.192}$$

The stationary conditions give

$$\lambda = -1. \tag{9.193}$$

Substituting $\lambda = -1$ into the correction functional gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + u_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right) d\xi. \tag{9.194}$$

Selecting $u_0(x,t) = x$ to obtain the successive approximations

$$\begin{aligned} u_0(x,t) &= x, \\ u_1(x,t) &= x - xt, \\ u_2(x,t) &= x - xt + xt^2 - \frac{1}{3}xt^3, \\ &\vdots \\ u_n(x,t) &= x(1 - t + t^2 - t^3 + \cdots). \end{aligned} \tag{9.195}$$

so that the exact solution is given by

$$u(x,t) = \frac{x}{1+t}, \quad |t| < 1. \tag{9.196}$$

Example 2. Solve the following Burgers equation

$$u_t + uu_x = u_{xx}, \quad u(x,0) = 1 - \frac{2}{x}, \quad x > 0. \tag{9.197}$$

Solution.

The Decomposition Method

Proceeding as before gives

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 - \frac{2}{x} + L_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx}\right) - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{9.198}$$

Consequently, we set the recursive relation

$$\begin{aligned} u_0(x,t) &= 1 - \frac{2}{x}, \\ u_{k+1}(x,t) &= L_t^{-1}\left(u_{k_{xx}}(x,t)\right) - L_t^{-1}\left(A_k\right), \quad k \geqslant 0, \end{aligned} \tag{9.199}$$

that gives

$$\begin{aligned} u_0(x,t) &= 1 - \frac{2}{x}, \\ u_1(x,t) &= L_t^{-1}\left(u_{0_{xx}}(x,t)\right) - L_t^{-1}\left(A_0\right) = L_t^{-1}(-\frac{2}{x^2}) = -\frac{2}{x^2}t, \\ u_2(x,t) &= L_t^{-1}\left(u_{1_{xx}}(x,t)\right) - L_t^{-1}\left(A_1\right) = L_t^{-1}(-\frac{4}{x^3}t) = -\frac{2}{x^3}t^2, \\ u_3(x,t) &= L_t^{-1}\left(u_{2_{xx}}(x,t)\right) - L_t^{-1}\left(A_2\right) = L_t^{-1}(-\frac{6}{x^4}t^2) = -\frac{2}{x^4}t^3. \end{aligned} \tag{9.200}$$

The series solution

$$u(x,t) = 1 - \frac{2}{x} - \frac{2}{x^2}t - \frac{2}{x^3}t^2 - \frac{2}{x^4}t^3 + \cdots, \tag{9.201}$$

is readily obtained. To determine the exact solution, Eq. (9.201) can be rewritten as

$$u(x,t) = 1 - \frac{2}{x}\left(1 + \frac{t}{x} + \frac{t^2}{x^2} + \frac{t^3}{x^3} + \cdots\right) = 1 - \frac{2}{x}\left(\frac{1}{1-\frac{t}{x}}\right) = 1 - \frac{2}{x-t}. \tag{9.202}$$

The Variational Iteration Method

Proceeding as in Example 1 and using the Lagrange multiplier $\lambda = -1$ into the correction functional gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + u_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right) d\xi. \tag{9.203}$$

Selecting $u_0(x,t) = 1 - \dfrac{2}{x}$ to obtain the successive approximations

$$\begin{aligned} u_0(x,t) &= 1 - \frac{2}{x}, \\ u_1(x,t) &= 1 - \frac{2}{x} - \frac{2}{x^2}t, \end{aligned}$$

$$\begin{aligned} u_2(x,t) &= 1-\frac{2}{x}-\frac{2}{x^2}t-\frac{2}{x^3}t^2 \\ &\vdots \\ u_n(x,t) &= 1-\frac{2}{x}-\frac{2}{x^2}t-\frac{2}{x^3}t^2-\frac{2}{x^4}t^3+\cdots. \end{aligned} \tag{9.204}$$

The series solution can be written as

$$u_n(x,t) = 1-\frac{2}{x}\left(1+\frac{t}{x}+\frac{t^2}{x^2}+\frac{t^3}{x^3}+\cdots\right). \tag{9.205}$$

This gives the exact solution by

$$u(x,t) = 1-\frac{2}{x}\left(\frac{1}{1-\frac{t}{x}}\right) = 1-\frac{2}{x-t}. \tag{9.206}$$

Example 3. Solve the following Burgers equation

$$u_t+uu_x = u_{xx}, \quad u(x,0) = 2\tan x. \tag{9.207}$$

Solution.

Following the analysis presented above gives

$$\sum_{n=0}^{\infty} u_n(x,t) = 2\tan x + L_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx}\right) - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{9.208}$$

The recursive relation

$$\begin{aligned} u_0(x,t) &= 2\tan x, \\ u_{k+1}(x,t) &= L_t^{-1}\left(u_{k_{xx}}(x,t)\right) - L_t^{-1}\left(A_k\right), \quad k \geqslant 0, \end{aligned} \tag{9.209}$$

leads to the determination of the first few components:

$$\begin{aligned} u_0(x,t) &= 2\tan x, \\ u_1(x,t) &= L_t^{-1}\left(u_{0_{xx}}(x,t)\right) - L_t^{-1}\left(A_0\right) = 0, \\ u_{k+2}(x,t) &= 0, \quad k \geqslant 0. \end{aligned} \tag{9.210}$$

Thus, the exact solution is given by

$$u(x,t) = 2\tan x. \tag{9.211}$$

Example 4. Solve the following Burgers equation

$$u_t+uu_x = u_{xx}, \quad u(0,t) = -\frac{2}{t}, \quad u_x(0,t) = \frac{1}{t}+\frac{2}{t^2}. \tag{9.212}$$

Solution.

It is important to note that, unlike the initial value problems discussed in the previous examples, the boundary conditions are given in this example. Hence, it is appropriate in this case to solve in the x direction. For this reason we first rewrite (9.212) in an operator form by

$$L_x u(x,t) = u_t + uu_x, \tag{9.213}$$

where L_x is a second order differential operator and the inverse operator L_x^{-1} is a two-fold integral operator defined by

$$L_x^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx\, dx. \tag{9.214}$$

Operating with L_x^{-1} on both sides of (9.213) gives

$$u(x,t) = -\frac{2}{t} + \left(\frac{1}{t} + \frac{2}{t^2}\right)x + L_x^{-1}(u_t) + L_x^{-1}(uu_x). \tag{9.215}$$

Substituting the linear term $u(x,t)$ by a series of components, and the nonlinear term uu_x by a series of Adomian polynomials, we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = -\frac{2}{t} + \left(\frac{1}{t} + \frac{2}{t^2}\right)x + L_x^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_t\right) + L_x^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{9.216}$$

The recursive relation

$$\begin{aligned} u_0(x,t) &= -\frac{2}{t} + \left(\frac{1}{t} + \frac{2}{t^2}\right)x, \\ u_{k+1}(x,t) &= L_x^{-1}\left(u_{k_t}(x,t)\right) + L_x^{-1}\left(A_k\right), \quad k \geqslant 0, \end{aligned} \tag{9.217}$$

gives

$$\begin{aligned} u_0(x,t) &= -\frac{2}{t} + \left(\frac{1}{t} + \frac{2}{t^2}\right)x, \\ u_1(x,t) &= L_x^{-1}\left(u_{0_t}(x,t)\right) + L_x^{-1}\left(A_0\right) = -2\frac{x^2}{t^3} + \frac{2}{3}\frac{x^3}{t^4}, \\ u_2(x,t) &= L_x^{-1}\left(u_{1_t}(x,t)\right) + L_x^{-1}\left(A_1\right) = \frac{4}{3}\frac{x^3}{t^4} + \cdots. \end{aligned} \tag{9.218}$$

Summing the resulting components, the series solution

$$u(x,t) = \frac{x}{t} - \frac{2}{t}\left(1 - \frac{x}{t} + \frac{x^2}{t^2} - \frac{x^3}{t^3} + \cdots\right), \tag{9.219}$$

is readily obtained. The exact solution

$$u(x,t) = \frac{x}{t} - \frac{2}{x+t}, \tag{9.220}$$

follows immediately.

Exercises 9.5

In Exercises 1–5, use the variational iteration method or Adomian decomposition method to solve the inviscid Burgers equations:

1. $u_t + uu_x = 0$, $u(x,0) = x$
2. $u_t + uu_x = 0$, $u(x,0) = -x$
3. $u_t + uu_x = 0$, $u(x,0) = 2x$
4. $u_t + uu_x = 0$, $u(x,0) = -2x$
5. $u_t + uu_x = 0$, $u(x,0) = \dfrac{1}{1+x}$

In Exercises 6–10, use the variational iteration method or Adomian decomposition method to solve the following Burgers equations:

6. $u_t + uu_x = u_{xx}$, $u(x,0) = -x$
7. $u_t + uu_x = u_{xx}$, $u(x,0) = 2x$
8. $u_t + uu_x = u_{xx}$, $u(x,0) = 4\tan 2x$
9. $u_t + uu_x = u_{xx}$, $u(0,t) = \dfrac{1}{2t-1}$, $u_x(0,t) = \dfrac{2}{2t-1}$
10. $u_t + uu_x = u_{xx}$, $u(0,t) = -\dfrac{2}{3t}$, $u_x(0,t) = \dfrac{1}{t} + \dfrac{2}{9t^2}$

9.6 The Telegraph Equation

The standard form of the telegraph equation [15] is given by

$$u_{xx} = au_{tt} + bu_t + cu, \tag{9.221}$$

where $u = u(x,t)$ is the resistance, and a, b and c are constants related to the inductance, capacitance and conductance of the cable respectively. Note that the telegraph equation is a linear partial differential equation. The telegraph equation arises in the propagation of electrical signals along a telegraph line. If we set $a = 0$ and $c = 0$, because of electrical properties of the cable, we then obtain

$$u_{xx} = bu_t, \tag{9.222}$$

which is the standard linear heat equation discussed in Chapter 3. On the other hand, the electrical properties may lead to $b = 0$ and $c = 0$; hence we obtain

$$u_{xx} = au_{tt}, \tag{9.223}$$

which is the standard linear wave equation presented in Chapter 5.

We now proceed formally to apply the decomposition method and the variational iteration method in a parallel manner to the approach used for handling other physical models. Without loss of generality, consider the initial boundary value telegraph equation

$$u_{xx} = u_{tt} + u_t + u, \quad 0 < x < L, \tag{9.224}$$

with boundary and initial conditions

$$\begin{array}{rl} \text{BC} & u(0,t) = f(t),\ u_x(0,t) = g(t), \\ \text{IC} & u(x,0) = h(x),\ u_t(x,0) = v(x). \end{array} \tag{9.225}$$

In an operator form, Eq. (9.224) becomes

$$L_x u(x,t) = u_{tt} + u_t + u, \tag{9.226}$$

where L_x is a second order differential operator with respect to x. Consequently, the inverse operator L_x^{-1} is considered a two-fold integral operator so that

$$L_x^{-1} L_x u(x,t) = u(x,t) - u(0,t) - xu_x(0,t). \tag{9.227}$$

Operating with L_x^{-1} on both sides of (9.226), using the boundary conditions, and noting (9.227) we obtain

$$u(x,t) = f(t) + xg(t) + L_x^{-1}(u_{tt} + u_t + u). \tag{9.228}$$

It is normal to define the recursive relation by

$$\begin{aligned} u_0(x,t) &= f(t) + xg(t), \\ u_{k+1}(x,t) &= L_x^{-1}\left(u_{k_{tt}} + u_{k_t} + u_k\right), \quad k \geqslant 0, \end{aligned} \tag{9.229}$$

that in turn gives

$$\begin{aligned} u_0(x,t) &= f(t) + xg(t), \\ u_1(x,t) &= L_x^{-1}\left(u_{0_{tt}} + u_{0_t} + u_0\right), \\ u_2(x,t) &= L_x^{-1}\left(u_{1_{tt}} + u_{1_t} + u_1\right), \\ u_3(x,t) &= L_x^{-1}\left(u_{2_{tt}} + u_{2_t} + u_2\right). \end{aligned} \tag{9.230}$$

Having determined the components of $u(x,t)$, the solution in a series form can thus be established upon summing these iterates. As indicated before, the resulting series may give the exact solution in a closed form.

The analysis presented above will be illustrated by discussing the following examples.

Example 1. Solve the following homogeneous telegraph equation:

$$u_{xx} = u_{tt} + u_t - u, \tag{9.231}$$

subject to the conditions

$$\begin{array}{ll} \text{BC} & u(0,t) = e^{-2t},\ u_x(0,t) = e^{-2t}, \\ \text{IC} & u(x,0) = e^x,\ u_t(x,0) = -2e^x. \end{array} \tag{9.232}$$

Solution.

The Decomposition Method

Operating with L_x^{-1} on (9.231) and using the boundary conditions yields

$$u(x,t) = e^{-2t} + xe^{-2t} + L_x^{-1}(u_{tt} + u_t - u). \tag{9.233}$$

Following the discussions presented before gives

$$\sum_{n=0}^{\infty} u_n(x,t) = e^{-2t} + xe^{-2t} + L_x^{-1}\left(\left(\sum_{n=0}^{\infty} u_n\right)_{tt} + \left(\sum_{n=0}^{\infty} u_n\right)_t - \sum_{n=0}^{\infty} u_n\right). \tag{9.234}$$

The decomposition method suggests the relation

$$\begin{aligned} u_0(x,t) &= e^{-2t} + xe^{-2t} \\ u_{k+1}(x,t) &= L_x^{-1}\left(u_{k_{tt}} + u_{k_t} - u_k\right), \quad k \geqslant 0, \end{aligned} \tag{9.235}$$

where the components of the solution $u(x,t)$ given by

$$\begin{aligned} u_0(x,t) &= e^{-2t} + xe^{-2t} \\ u_1(x,t) &= L_x^{-1}\left(u_{0_{tt}} + u_{0_t} - u_0\right) = \frac{1}{2!}x^2e^{-2t} + \frac{1}{3!}x^3e^{-2t}, \\ u_2(x,t) &= L_x^{-1}\left(u_{1_{tt}} + u_{1_t} - u_1\right) = \frac{1}{4!}x^4e^{-2t} + \frac{1}{5!}x^5e^{-2t}, \\ u_3(x,t) &= L_x^{-1}\left(u_{2_{tt}} + u_{2_t} - u_2\right) = \frac{1}{6!}x^6e^{-2t} + \frac{1}{7!}x^7e^{-2t}, \end{aligned} \tag{9.236}$$

follow immediately. In view of (9.236), the solution in a series form is given by

$$u(x,t) = e^{-2t}\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right), \tag{9.237}$$

which gives the exact solution in the form

$$u(x,t) = e^{x-2t}. \tag{9.238}$$

The Variational Iteration Method

The correction functional for this equation reads

$$
\begin{aligned}
u_{n+1}(x,t) &= u_n(x,t) \\
&\quad + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} + \frac{\partial \tilde{u}_n(x,\xi)}{\partial \xi} - \tilde{u}_n(x,\xi)\right) d\xi.
\end{aligned} \tag{9.239}
$$

The stationary conditions give

$$\lambda = \xi - t. \tag{9.240}$$

Substituting this value of the Lagrange multiplier into the correction functional gives the iteration formula

$$
\begin{aligned}
u_{n+1}(x,t) &= u_n(x,t) \\
&\quad + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \frac{\partial u_n(x,\xi)}{\partial \xi} - u_n(x,\xi)\right) d\xi.
\end{aligned} \tag{9.241}
$$

Using the selection $u_0(x,t) = (1-2t)e^x$ gives the successive approximations

$$
\begin{aligned}
u_0(x,t) &= (1-2t)e^x, \\
u_1(x,t) &= (1-2t+2t^2-\frac{2}{3}t^3)e^x, \\
u_2(x,t) &= (1-2t+2t^2-\frac{4}{3}t^3+\frac{2}{3}t^4+\cdots)e^x, \\
&\vdots \\
u_n(x,t) &= (1-2t+2t^2-\frac{4}{3}t^3+\frac{2}{3}t^4-\frac{4}{15}t^5+\cdots)e^x.
\end{aligned} \tag{9.242}
$$

This gives the exact solution by

$$u(x,t) = e^{x-2t}, \tag{9.243}$$

obtained upon using the Taylor series for e^{-2t}.

Example 2. Solve the following homogeneous telegraph equation:

$$u_{xx} = u_{tt} + 4u_t + 4u, \tag{9.244}$$

subject to the conditions

$$
\begin{aligned}
\text{BC} \quad & u(0,t) = 1+e^{-2t},\ u_x(0,t) = 2, \\
\text{IC} \quad & u(x,0) = 1+e^{2x},\ u_t(x,0) = -2.
\end{aligned} \tag{9.245}
$$

Solution.

The Decomposition Method

Applying the two-fold integral operator L_x^{-1} on (9.244) gives

$$u(x,t) = 1+e^{-2t}+2x+L_x^{-1}(u_{tt}+4u_t+4u), \tag{9.246}$$

where using the decomposition series for $u(x,t)$ we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 + e^{-2t} + 2x + L_x^{-1}\left(\left(\sum_{n=0}^{\infty} u_n\right)_{tt} + 4\left(\sum_{n=0}^{\infty} u_n\right)_t + 4\sum_{n=0}^{\infty} u_n\right). \tag{9.247}$$

A close observation of (9.247) suggests the recursive relation

$$\begin{aligned} u_0(x,t) &= 1 + e^{-2t} + 2x \\ u_{k+1}(x,t) &= L_x^{-1}\left(u_{k_{tt}} + 4u_{k_t} + 4u_k\right), \quad k \geqslant 0. \end{aligned} \tag{9.248}$$

In view of (9.248) we obtain

$$\begin{aligned} u_0(x,t) &= 1 + e^{-2t} + 2x \\ u_1(x,t) &= L_x^{-1}\left(u_{0_{tt}} + 4u_{0_t} + 4u_0\right) = 2x^2 + \frac{4}{3}x^3, \\ u_2(x,t) &= L_x^{-1}\left(u_{1_{tt}} + 4u_{1_t} + 4u_1\right) = \frac{2}{3}x^4 + \frac{4}{15}x^5. \end{aligned} \tag{9.249}$$

Other components can be computed in a similar manner. Consequently, the solution in a series form is given by

$$u(x,t) = e^{-2t} + \left(1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \cdots\right), \tag{9.250}$$

so that the exact solution

$$u(x,t) = e^{2x} + e^{-2t} \tag{9.251}$$

is readily obtained.

The Variational Iteration Method

Proceeding as in Example 1 we obtain the iteration formula

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) \\ &\quad + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + 4\frac{\partial u_n(x,\xi)}{\partial \xi} + 4u_n(x,\xi)\right) d\xi. \end{aligned} \tag{9.252}$$

Using the selection $u_0(x,t) = e^{2x} + 1 - 2t$ gives the successive approximations

$$\begin{aligned} u_0(x,t) &= e^{2x} + 1 - 2t, \\ u_1(x,t) &= e^{2x} + (1 - 2t + 2t^2 + \frac{4}{3}t^3), \\ u_2(x,t) &= e^{2x} + (1 - 2t + 2t^2 + \frac{4}{3}t^3 - 2t^4), \\ &\vdots \\ u_n(x,t) &= e^{2x} + (1 - 2t + 2t^2 - \frac{4}{3}t^3 + \frac{2}{3}t^4 + \cdots), \\ &= e^{2x} + \left(1 - (2t) + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \cdots\right). \end{aligned} \tag{9.253}$$

This gives the exact solution by

$$u(x,t) = e^{2x} + e^{-2t}. \tag{9.254}$$

Exercises 9.6

Use the variational iteration method or decomposition method to solve the telegraph equations:

1. $u_{xx} = u_{tt} + u_t + u$
   $u(0,t) = e^{-t}$, $u_x(0,t) = e^{-t}$
   $u(x,0) = e^x$, $u_t(x,0) = -e^x$

2. $u_{xx} = \frac{1}{3}(u_{tt} + u_t + u)$
   $u(0,t) = e^t$, $u_x(0,t) = e^t$
   $u(x,0) = e^x$, $u_t(x,0) = e^x$

3. $u_{xx} = u_{tt} + 2u_t + u$
   $u(0,t) = 1 + e^{-t}$, $u_x(0,t) = 1$
   $u(x,0) = 1 + e^x$, $u_t(x,0) = -1$

4. $u_{xx} = u_{tt} + u_t + 4u$
   $u(0,t) = e^{-t}$, $u_x(0,t) = 2e^{-t}$
   $u(x,0) = e^{2x}$, $u_t(x,0) = -e^{2x}$

5. $u_{xx} = 2u_{tt} + 3u_t + u$
   $u(0,t) = 1 - e^{-t}$, $u_x(0,t) = 1$
   $u(x,0) = e^x - 1$, $u_t(x,0) = 1$

6. $u_{xx} = u_{tt} + 2u_t + u$
   $u(0,t) = e^{-t}$, $u_x(0,t) = 1$
   $u(x,0) = 1 + \sinh x$, $u_t(x,0) = -1$

7. $u_{xx} = u_{tt} + 2u_t + u$
   $u(0,t) = 1 - e^{-t}$, $u_x(0,t) = 0$
   $u(x,0) = \cosh x - 1$, $u_t(x,0) = 1$

8. $u_{xx} = \frac{1}{3}u_{tt} + \frac{4}{3}u_t + u$
   $u(0,t) = 1 + e^{-3t}$, $u_x(0,t) = 1$
   $u(x,0) = 1 + e^x$, $u_t(x,0) = -3$

9. $u_{xx} = u_{tt} + 4u_t + 4u$
   $u(0,t) = 1 + e^{-2t}$, $u_x(0,t) = 2$
   $u(x,0) = 1 + e^{2x}$, $u_t(x,0) = -2$

10. $u_{xx} = u_{tt} + 4u_t + 4u$
    $u(0,t) = e^{-2t}$, $u_x(0,t) = 2$

$u(x,0) = 1 + \sinh 2x,\, u_t(x,0) = -2$

9.7 Schrodinger Equation

In this section, the linear and nonlinear Schrodinger equations [1,2] will be investigated. It is well-known that this equation arises in the study of the time evolution of the wave function.

9.7.1 The Linear Schrodinger Equation

The initial value problem for the linear Schrodinger equation for a free particle with mass m is given by the following standard form

$$u_t = iu_{xx}, \quad u(x,0) = f(x), \quad i^2 = -1, t > 0, \tag{9.255}$$

where $f(x)$ is continuous and square integrable. It is to be noted that Schrodinger equation (9.255) discusses the time evolution of a free particle. Moreover, the function $u(x,t)$ is complex, and Eq. (9.255) is a first order differential equation in t. The linear Schrodinger equation (9.255) is usually handled by using the spectral transform technique among other methods.

The Adomian decomposition method and the variational iteration method will be applied here to handle the linear and the nonlinear Schrodinger equations. To achieve this goal, we apply L_t^{-1} to both sides of (9.255) to obtain

$$u(x,t) = f(x) + iL_t^{-1}(u_{xx}), \tag{9.256}$$

and using the series representation for $u(x,t)$ yields

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + iL_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx}\right). \tag{9.257}$$

Applying the decomposition method leads to the recursive scheme

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_{k+1}(x,t) &= iL_t^{-1}(u_{k_{xx}}), \quad k \geqslant 0. \end{aligned} \tag{9.258}$$

Using few iterations of (9.258) gives

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_1(x,t) &= iL_t^{-1}(u_{0_{xx}}), \\ u_2(x,t) &= iL_t^{-1}(u_{1_{xx}}), \\ u_3(x,t) &= iL_t^{-1}(u_{2_{xx}}). \end{aligned} \tag{9.259}$$

Other components can be evaluated in a parallel manner. Having determined the first few components of $u(x,t)$, the solution in a series form is readily obtained.

The following examples will be used to illustrate the analysis discussed above.

Example 1. Solve the linear Schrodinger equation

$$u_t = iu_{xx}, \quad u(x,0) = e^{ix} \tag{9.260}$$

Solution.

The Decomposition Method

Following the discussions presented above we obtain

$$\begin{aligned} u_0(x,t) &= e^{ix}, \\ u_1(x,t) &= iL_t^{-1}\left(-e^{ix}\right) = -ite^{ix}, \\ u_2(x,t) &= L_t^{-1}\left(-te^{ix}\right) = -\frac{1}{2!}t^2e^{ix}, \\ u_3(x,t) &= iL_t^{-1}\left(\tfrac{1}{2!}t^2e^{ix}\right) = \frac{1}{3!}it^3e^{ix}. \end{aligned} \tag{9.261}$$

Summing these iterations yields the series solution

$$u(x,t) = e^{ix}\left(1 - it + \frac{1}{2!}(it)^2 - \frac{1}{3!}(it)^3 + \cdots\right), \tag{9.262}$$

that leads to the exact solution

$$u(x,t) = e^{i(x-t)}, \tag{9.263}$$

obtained upon using the Taylor expansion for e^{-it}.

The Variational Iteration Method

The correction functional for this equation reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} - i\frac{\partial^2(\tilde{u}_n)(x,\xi)}{\partial x^2}\right)d\xi. \tag{9.264}$$

The stationary conditions

$$\begin{aligned} 1 + \lambda &= 0, \\ \lambda' &= 0, \end{aligned} \tag{9.265}$$

follow immediately. This in turn gives

$$\lambda = -1. \tag{9.266}$$

Substituting $\lambda = -1$ into the correction functional gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - i\frac{\partial^2 u_n(x,\xi)}{\partial x^2} \right) d\xi, \quad n \geqslant 0. \tag{9.267}$$

Selecting $u_0(x,t) = e^{ix}$ leads to the successive approximations

$$\begin{aligned}
u_0(x,t) &= e^{ix}, \\
u_1(x,t) &= e^{ix}(1 - it), \\
u_2(x,t) &= e^{ix}(1 - it + \frac{1}{2!}(it)^2), \\
u_3(x,t) &= e^{ix}(1 - it + \frac{1}{2!}(it)^2 - \frac{1}{3!}(it)^3), \\
&\vdots \\
u_n(x,t) &= e^{ix}(1 - it + \frac{1}{2!}(it)^2 - \frac{1}{3!}(it)^3 + \cdots).
\end{aligned} \tag{9.268}$$

This gives the exact solution by

$$u(x,t) = e^{i(x-t)}. \tag{9.269}$$

Example 2. Solve the linear Schrodinger equation

$$\begin{aligned}
u_t &= iu_{xx}, \\
u(x,0) &= \sinh x.
\end{aligned} \tag{9.270}$$

Solution.

Proceeding as in Example 1, we obtain

$$\begin{aligned}
u_0(x,t) &= \sinh x, \\
u_1(x,t) &= iL_t^{-1}(\sinh x) = it\sinh x, \\
u_2(x,t) &= -L_t^{-1}(t\sinh x) = -\frac{1}{2!}t^2\sinh x, \\
u_3(x,t) &= -iL_t^{-1}\left(\frac{1}{2!}t^2\sinh x\right) = -\frac{1}{3!}it^3\sinh x.
\end{aligned} \tag{9.271}$$

Summing these components gives the series solution

$$u(x,t) = \sinh x\left(1 + (it) + \frac{1}{2!}(it)^2 + \frac{1}{3!}(it)^3 + \cdots\right), \tag{9.272}$$

and hence the exact solution is

$$u(x,t) = e^{it}\sinh x. \tag{9.273}$$

9.7.2 The Nonlinear Schrodinger Equation

We now turn to study the nonlinear Schrodinger equation (NLS) defined by its standard form

$$iu_t + u_{xx} + \gamma|u|^2u = 0, \tag{9.274}$$

where γ is a constant and $u(x,t)$ is complex. The Schrodinger equation (9.274) generally exhibits solitary type solutions. A soliton, or solitary wave, is a wave where the speed of propagation is independent of the amplitude of the wave. Solitons usually occur in fluid mechanics.

The nonlinear Schrodinger equations that are commonly used are given by

$$iu_t + u_{xx} + 2|u|^2u = 0, \tag{9.275}$$

and

$$iu_t + u_{xx} - 2|u|^2u = 0. \tag{9.276}$$

Moreover, other forms of nonlinear Schrodinger equations are used as well depending on the constant γ. The inverse scattering method is usually used to handle the nonlinear Schrodinger equation where solitary type solutions were derived.

The nonlinear Schrodinger equation will be handled differently in this section by using the Adomian decomposition method and the variational iteration method. We start our analysis by considering the initial value problem

$$iu_t + u_{xx} + \gamma|u|^2u = 0, u(x,0) = f(x). \tag{9.277}$$

Multiplying Eq. (9.277) by i, we may express this equation in an operator form as follows

$$L_tu(x,t) = iu_{xx} + i\gamma|u|^2u. \tag{9.278}$$

Applying L_t^{-1} to both sides of (9.278) gives

$$u(x,t) = f(x) + iL_t^{-1}u_{xx} + i\gamma L_t^{-1}F(u(x,t)), \tag{9.279}$$

where the nonlinear term $F(u(x,t))$ is given by

$$F(u(x,t)) = |u|^2u. \tag{9.280}$$

Substituting the decomposition series for $u(x,t)$ and the series of Adomian polynomials for $F(u(x,t))$ into (9.279) to obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + iL_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx}\right) + i\gamma L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{9.281}$$

Adomian's analysis introduces the recursive relation

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_{k+1}(x,t) &= iL_t^{-1}\left(u_{k_{xx}}\right) + i\gamma L_t^{-1}\left(A_k\right), \quad k \geqslant 0. \end{aligned} \tag{9.282}$$

Recall from complex analysis that

$$|u|^2 = u\overline{u}, \tag{9.283}$$

where $\overline{u}$ is the conjugate of u. This means that (9.280) can be rewritten as

$$F(u) = u^2\overline{u}. \tag{9.284}$$

In view of (9.284), and following the formal techniques used before to derive the Adomian polynomials, we can easily derive that $F(u)$ has the following polynomials representation

$$\begin{aligned} A_0 &= u_0^2\overline{u}_0, \\ A_1 &= 2u_0u_1\overline{u}_0 + u_0^2\overline{u}_1, \\ A_2 &= 2u_0u_2\overline{u}_0 + u_1^2\overline{u}_0 + 2u_0u_1\overline{u}_1 + u_0^2\overline{u}_2, \\ A_3 &= 2u_0u_3\overline{u}_0 + 2u_1u_2\overline{u}_0 + 2u_0u_2\overline{u}_1 + u_1^2\overline{u}_1 + 2u_0u_1\overline{u}_2 + u_0^2\overline{u}_3. \end{aligned} \tag{9.285}$$

In conjunction with (9.282) and (9.285), we can easily determine the first few components by

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_1(x,t) &= iL_t^{-1}\left(u_{0_{xx}}\right) + i\gamma L_t^{-1}\left(A_0\right), \\ u_2(x,t) &= iL_t^{-1}\left(u_{1_{xx}}\right) + i\gamma L_t^{-1}\left(A_1\right), \\ u_3(x,t) &= iL_t^{-1}\left(u_{2_{xx}}\right) + i\gamma L_t^{-1}\left(A_2\right). \end{aligned} \tag{9.286}$$

Other components can be determined as well. This completes the determination of the series solution.

The analysis introduced above will be illustrated by discussing the following examples.

Example 3. Use the decomposition method to solve the following nonlinear Schrodinger equation.

$$iu_t + u_{xx} + 2|u|^2u = 0, \quad u(x,0) = e^{ix}. \tag{9.287}$$

Solution.

The Decomposition Method

Following the analysis presented above gives

$$\sum_{n=0}^{\infty} u_n(x,t) = e^{ix} + iL_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx}\right) + 2iL_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{9.288}$$

The decomposition method suggests the use of the recursive relation

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_{k+1}(x,t) &= iL_t^{-1}\left(u_{k_{xx}}\right) + 2iL_t^{-1}\left(A_k\right), \quad k \geqslant 0, \end{aligned} \tag{9.289}$$

that in turn gives the first few components by

$$\begin{aligned}
u_0(x,t) &= e^{ix},\\
u_1(x,t) &= iL_t^{-1}\left(u_{0_{xx}}\right) + 2iL_t^{-1}\left(A_0\right) = ite^{ix},\\
u_2(x,t) &= iL_t^{-1}\left(u_{1_{xx}}\right) + 2iL_t^{-1}\left(A_1\right) = -\frac{1}{2!}t^2e^{ix},\\
u_3(x,t) &= iL_t^{-1}\left(u_{2_{xx}}\right) + 2iL_t^{-1}\left(A_2\right) = -\frac{1}{3!}it^3e^{ix}.
\end{aligned} \tag{9.290}$$

Accordingly, the series solution is given by

$$u(x,t) = e^{ix}\left(1 + it + \frac{1}{2!}(it)^2 + \frac{1}{3!}(it)^3 + \cdots\right), \tag{9.291}$$

that gives the exact solution by

$$u(x,t) = e^{i(x+t)}. \tag{9.292}$$

The Variational Iteration Method

Following the analysis presented above we obtain the correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(i\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial^2(\tilde{u}_n)(x,\xi)}{\partial x^2} + 2u_n^2\overline{u}_n\right)d\xi, \tag{9.293}$$

where $|u|^2 = u\overline{u}$, and $\overline{u}$ is the conjugate of u. The stationary conditions

$$\begin{aligned}
1 + i\lambda &= 0,\\
\lambda' &= 0,
\end{aligned} \tag{9.294}$$

give

$$\lambda = i. \tag{9.295}$$

Substituting this value of the Lagrange multiplier into the correction functional gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + i\int_0^t \left(i\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + 2u_n^2\overline{u}_n\right)d\xi, \quad n \geqslant 0. \tag{9.296}$$

Using $u_0(x,t) = e^{ix}$ gives the successive approximations

$$\begin{aligned}
u_0(x,t) &= e^{ix},\\
u_1(x,t) &= e^{ix} + ite^{ix},\\
u_2(x,t) &= e^{ix} + ite^{ix} + \frac{(it)^2}{2!}e^{ix},\\
u_3(x,t) &= e^{ix} + ite^{ix} + \frac{(it)^2}{2!}e^{ix} + \frac{(it)^3}{3!}e^{ix},\\
&\ \ \vdots\\
u_n(x,t) &= e^{ix}\left(1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots\right).
\end{aligned} \tag{9.297}$$

Consequently the exact solution

$$u(x,t) = e^{i(x+t)} \tag{9.298}$$

is readily obtained.

Example 4. Use the decomposition method to solve the following nonlinear Schrodinger equation.

$$iu_t + u_{xx} - 2|u|^2 u = 0, \quad u(x,0) = e^{ix}. \tag{9.299}$$

Solution.

Using the analysis of Example 3 yields

$$\sum_{n=0}^{\infty} u_n(x,t) = e^{ix} + iL_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx}\right) - 2i\left(\sum_{n=0}^{\infty} A_n\right). \tag{9.300}$$

This gives the recursive relation

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_{k+1}(x,t) &= iL_t^{-1}(u_{k_{xx}}) - 2iL_t^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{9.301}$$

Using the Adomian polynomials A_n that were derived before, the first few components are given by

$$\begin{aligned} u_0(x,t) &= e^{ix}, \\ u_1(x,t) &= iL_t^{-1}(u_{0_{xx}}) - 2iL_t^{-1}(A_0) = -3ite^{ix}, \\ u_2(x,t) &= iL_t^{-1}(u_{1_{xx}}) + 2iL_t^{-1}(A_1) = \frac{1}{2!}(3it)^2 e^{ix}, \\ u_3(x,t) &= iL_t^{-1}(u_{2_{xx}}) + 2iL_t^{-1}(A_2) = -\frac{1}{3!}(3it)^3 e^{ix}. \end{aligned} \tag{9.302}$$

In view of (9.302), the series solution is given by

$$u(x,t) = e^{ix}\left(1 - (3it) + \frac{1}{2!}(3it)^2 - \frac{1}{3!}(3it)^3 + \cdots\right). \tag{9.303}$$

The exact solution is therefore given by

$$u(x,t) = e^{i(x-3t)}. \tag{9.304}$$

Exercises 9.7

In Exercises 1–5, use the variational iteration method or Adomian decomposition method to solve the following linear Schrodinger equations:

1. $u_t = iu_{xx}$, $u(x,0) = e^{2ix}$
2. $u_t = iu_{xx}$, $u(x,0) = \sin x$
3. $u_t = iu_{xx}$, $u(x,0) = \cosh x$
4. $u_t = iu_{xx}$, $u(x,0) = 1+\cos 3x$
5. $u_t = iu_{xx}$, $u(x,0) = \sin 2x$

In Exercises 6–10, use the variational iteration method or Adomian decomposition method to solve the following nonlinear Schrodinger equations NLS:

6. $iu_t + u_{xx} + |u|^2 u = 0$, $u(x,0) = e^{2ix}$
7. $iu_t + u_{xx} + 2|u|^2 u = 0$, $u(x,0) = e^{-ix}$
8. $iu_t + u_{xx} + 6|u|^2 u = 0$, $u(x,0) = e^{3ix}$
9. $iu_t + u_{xx} - 2|u|^2 u = 0$, $u(x,0) = e^{2ix}$
10. $iu_t + u_{xx} + 17|u|^2 u = 0$, $u(0,t) = e^{8it}, u_x(0,t) = 3ie^{8it}$

9.8 Korteweg-deVries Equation

The Korteweg-deVries (KdV) equation in its simplest form [7,9] is given by

$$u_t + auu_x + u_{xxx} = 0. \tag{9.305}$$

The KdV equation arises in the study of shallow water waves [11]. In particular, the KdV equation is used to describe long waves traveling in canals. It is formally proved that this equation has solitary waves as solutions, hence it can have any number of solitons [7]. The KdV equation has received a lot of attention and has been extensively studied. Several numerical and analytical techniques were employed to study the solitary waves that result from this equation. The solitary waves of this equation will be presented in Chapters 11, 12 and forthcoming chapters.

In this section, the decomposition method and the variational iteration method will be used to handle the KdV equation. We first consider the initial value problem

$$u_t + auu_x + bu_{xxx} = 0, \quad u(x,0) = f(x), \tag{9.306}$$

where a and b are constants. In an operator form, the KdV equation becomes

$$L_t u = -bu_{xxx} - auu_x. \tag{9.307}$$

Applying L_t^{-1} on both sides of (9.307) yields

$$u(x,t) = f(x) - bL_t^{-1} u_{xxx} - aL_t^{-1} F(u(x,t)), \tag{9.308}$$

where the nonlinear term $F(u(x,t))$ is

$$F(u(x,t)) = uu_x. \tag{9.309}$$

Using the decomposition identification for the linear and nonlinear terms yields

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) - bL_t^{-1}\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xxx} - aL_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{9.310}$$

The typical approach of Adomian's method is the introduction of the recursive relation

$$\begin{aligned} u_0(x,t) &= f(x),\\ u_{k+1}(x,t) &= -bL_t^{-1}(u_{k_{xxx}}) - aL_t^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{9.311}$$

The components $u_n, n \geqslant 0$ can be elegantly calculated by

$$\begin{aligned} u_0(x,t) &= f(x),\\ u_1(x,t) &= -bL_t^{-1}(u_{0_{xxx}}) - aL_t^{-1}(A_0),\\ u_2(x,t) &= -bL_t^{-1}(u_{1_{xxx}}) - aL_t^{-1}(A_1),\\ u_3(x,t) &= -bL_t^{-1}(u_{2_{xxx}}) - aL_t^{-1}(A_2), \end{aligned} \tag{9.312}$$

where Adomian polynomials A_n for the nonlinearity uu_x were derived before and used in advection and Burgers problems. Summing the computed components (9.312) gives the solution in a series form. The discussion presented above will be illustrated as follows.

Example 1. Solve the following homogeneous KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0, \quad u(x,0) = 6x. \tag{9.313}$$

Solution.

The Decomposition Method

Proceeding as before we find

$$\sum_{n=0}^{\infty} u_n(x,t) = 6x - L_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xxx}\right) + 6L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{9.314}$$

A close observation of (9.314) admits the recursive relation

$$\begin{aligned} u_0(x,t) &= 6x,\\ u_{k+1}(x,t) &= -L_t^{-1}(u_{k_{xxx}}) + 6L_t^{-1}(A_k), \quad k \geqslant 0, \end{aligned} \tag{9.315}$$

that gives the first few components by

$$u_0(x,t) = 6x,$$

$$\begin{aligned}u_1(x,t) &= -L_t^{-1}(u_{0_{xxx}})+6L_t^{-1}(A_0)=6^3xt,\\ u_2(x,t) &= -L_t^{-1}(u_{1_{xxx}})+6L_t^{-1}(A_1)=6^5xt^2,\\ u_3(x,t) &= -L_t^{-1}(u_{2_{xxx}})+6L_t^{-1}(A_2)=6^7xt^3.\end{aligned} \tag{9.316}$$

In view of (9.316), the solution in a series form is given by

$$u(x,t)=6x(1+36t+(36t)^2+(36t)^3+\cdots), \tag{9.317}$$

and in a closed form by

$$u(x,t)=\frac{6x}{1-36t},\quad |36t|<1. \tag{9.318}$$

The Variational Iteration Method

Proceeding as in other examples and using $\lambda=-1$ we get the iteration formula

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left(\frac{\partial u_n(x,\xi)}{\partial\xi}-6u_n\frac{\partial u_n(x,\xi)}{\partial x}+\frac{\partial^3u_n(x,\xi)}{\partial x^3}\right)d\xi. \tag{9.319}$$

Selecting $u_0(x,t)=6x$ from the given initial condition yields the successive approximations

$$\begin{aligned}u_0(x,t) &= 6x,\\ u_1(x,t) &= 6x+6^3xt,\\ u_2(x,t) &= 6x+6^3xt+6^5xt^2+93312xt^3,\\ u_3(x,t) &= 6x+6^3xt+6^5xt^2+6^7xt^3+\cdots,\\ &\ \vdots\\ u_n(x,t) &= 6x(1+36t+(36t)^2+(36t)^3+(36t)^4+\cdots).\end{aligned} \tag{9.320}$$

This gives exact solution by

$$u(x,t)=\frac{6x}{1-36t},\quad |36t|<1. \tag{9.321}$$

Example 2. Solve the following KdV equation:

$$u_t-6uu_x+u_{xxx}=0,\quad u(x,0)=\frac{1}{6}(x-1). \tag{9.322}$$

Solution.

Proceeding as in Example 1 gives

$$\sum_{n=0}^{\infty}u_n(x,t)=\frac{1}{6}(x-1)-L_t^{-1}\left(\left(\sum_{n=0}^{\infty}u_n(x,t)\right)_{xxx}\right)+6L_t^{-1}\left(\sum_{n=0}^{\infty}A_n\right). \tag{9.323}$$

This gives the relation

$$\begin{aligned} u_0(x,t) &= \frac{1}{6}(x-1), \\ u_{k+1}(x,t) &= -L_t^{-1}\left(u_{k_{xxx}}\right) + 6L_t^{-1}\left(A_k\right), \quad k \geqslant 0, \end{aligned} \tag{9.324}$$

and as a result we find

$$\begin{aligned} u_0(x,t) &= \frac{1}{6}(x-1), \\ u_1(x,t) &= -L_t^{-1}\left(u_{0_{xxx}}\right) + 6L_t^{-1}(A_0) = \frac{1}{6}(x-1)t, \\ u_2(x,t) &= -L_t^{-1}\left(u_{1_{xxx}}\right) + 6L_t^{-1}(A_1) = \frac{1}{6}(x-1)t^2, \\ u_3(x,t) &= -L_t^{-1}\left(u_{2_{xxx}}\right) + 6L_t^{-1}(A_2) = \frac{1}{6}(x-1)t^3. \end{aligned} \tag{9.325}$$

The solution in a series form is therefore given by

$$u(x,t) = \frac{1}{6}(x-1)\left(1+t+t^2+t^3+\cdots\right), \tag{9.326}$$

and in a closed form by

$$u(x,t) = \frac{1}{6}\left(\frac{x-1}{1-t}\right), \quad |t| < 1. \tag{9.327}$$

Exercises 9.8

1. Show that $u = -\dfrac{2}{x^2}$ is a solution of $u_t + 6uu_x + u_{xxx} = 0$.
2. Show that $u = \dfrac{1}{6}\dfrac{x-3}{t-3}$ is a solution of $u_t + 6uu_x + u_{xxx} = 0$.
3. Show that $u = \dfrac{2}{(x-2)^2}$ is a solution of $u_t - 6uu_x + u_{xxx} = 0$.
4. Show that $u = \dfrac{6x(x^3-24t)}{(x^3+12t)^2}$ is a solution of $u_t - 6uu_x + u_{xxx} = 0$.
5. Show that $u = \dfrac{4(x-6t)^2-3}{4(x-6t)^2+1}$ is a solution of $u_t + 6u^2u_x + u_{xxx} = 0$.

Use the variational iteration method or Adomian decomposition method to solve the following KdV equations:

6. $u_t + 6uu_x + u_{xxx} = 0,\ u(x,0) = x$
7. $u_t - 6uu_x + u_{xxx} = 0,\ u(x,0) = \dfrac{2}{x^2}$

8. $u_t - 6uu_x + u_{xxx} = 0,\ u(x,0) = \frac{1}{12}(x-2)$

9. $u_t - 6uu_x + u_{xxx} = 0,\ u(x,0) = \frac{2}{(x-3)^2}$

10. $u_t - 6uu_x + u_{xxx} = 0,\ u(x,0) = \frac{1}{18}(x-4)$

9.9 Fourth-order Parabolic Equation

We close this chapter by discussing the fourth order parabolic linear partial differential equation with constant and variable coefficients.

9.9.1 Equations with Constant Coefficients

In what follows we study the fourth order parabolic linear partial differential equation with constant coefficients of the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = f(x,t), \quad 0 \leqslant x \leqslant 1, t > 0, \tag{9.328}$$

with initial conditions

$$u(x,0) = g(x),\ u_t(x,0) = h(x). \tag{9.329}$$

It is worth mentioning that the fourth order parabolic equation (9.328) governs the transverse vibrations of a homogeneous beam. In addition, equation (9.328), subject to specific initial and boundary conditions, was handled numerically by the finite difference method and by the alternating group explicit method. It is our main goal in this chapter to employ the Adomian decomposition method and the variational iteration method to physical applications.

In an operator form, Eq. (9.328) can be rewritten as

$$L_t u(x,t) = f(x,t) - L_x u(x,t), \tag{9.330}$$

where L_t is a second order partial derivative with respect to t, and L_x is a fourth-order partial derivative with respect to x. Operating with the two-fold integral operator L_t^{-1} and using the decomposition series for $u(x,t)$ give

$$\sum_{n=0}^{\infty} u_n(x,t) = g(x) + th(x) + L_t^{-1} f(x,t) - L_t^{-1} L_x \left(\sum_{n=0}^{\infty} u_n(x,t) \right). \tag{9.331}$$

It follows that

$$\begin{aligned} u_0(x,t) &= g(x)+th(x)+L_t^{-1}f(x,t), \\ u_{k+1}(x,t) &= -L_t^{-1}L_x\left(u_k(x,t)\right), k \geqslant 0. \end{aligned} \tag{9.332}$$

Using few iterations we obtain

$$\begin{aligned} u_0(x,t) &= g(x)+th(x)+L_t^{-1}f(x,t), \\ u_1(x,t) &= -L_t^{-1}L_x\left(u_0\right), \\ u_2(x,t) &= -L_t^{-1}L_x\left(u_1\right), \\ u_3(x,t) &= -L_t^{-1}L_x\left(u_2\right). \end{aligned} \tag{9.333}$$

The series solution follows immediately upon summing the components obtained in (9.333).

It is important to point out that the homogeneous and the inhomogeneous cases will be illustrated by discussing the following examples. For the inhomogeneous case, the noise terms will play a major role in accelerating the convergence of the solution.

Example 1. Solve the following homogeneous fourth order equation:

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}=0, \tag{9.334}$$

with initial conditions

$$u(x,0)=\cos x, \quad u_t(x,0)=-\sin x. \tag{9.335}$$

Solution.

The Decomposition Method

Operating with the two-fold integral operator L_t^{-1}, and representing $u(x,t)$ by the decomposition series of components we obtain

$$\sum_{n=0}^{\infty} u_n(x,t)=\cos x-t\sin x-L_t^{-1}L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right). \tag{9.336}$$

The recursive scheme

$$\begin{aligned} u_0(x,t) &= \cos x-t\sin x, \\ u_{k+1}(x,t) &= -L_t^{-1}L_x\left(\sum_{n=0}^{\infty} u_k(x,t)\right), \quad k \geqslant 0, \end{aligned} \tag{9.337}$$

follows immediately. Using few iterations we obtain

$$\begin{aligned} u_0(x,t) &= \cos x-t\sin x, \\ u_1(x,t) &= -L_t^{-1}L_x\left(u_0(x,t)\right)=-\frac{1}{2!}t^2\cos x+\frac{1}{3!}t^3\sin x, \\ u_2(x,t) &= -L_t^{-1}L_x\left(u_1(x,t)\right)=\frac{1}{4!}t^4\cos x-\frac{1}{5!}t^5\sin x. \end{aligned} \tag{9.338}$$

Therefore, the series solution is given by

$$u(x,t)=\cos x\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\cdots\right)-\sin x\left(t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\cdots\right). \quad (9.339)$$

Consequently, the exact solution is given by

$$u(x,t)=\cos x\cos t-\sin x\sin t=\cos(x+t). \quad (9.340)$$

The Variational Iteration Method

The correction functional for this equation reads

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}+\frac{\partial^4 \tilde{u}_n(x,\xi)}{\partial x^4}\right)d\xi. \quad (9.341)$$

The stationary conditions give

$$\lambda=\xi-t. \quad (9.342)$$

Substituting this value of the Lagrange multiplier into the correction functional gives the iteration formula

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t (\xi-t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}+\frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right)d\xi. \quad (9.343)$$

We can select $u_0(x,t)=\cos x-t\sin x$. Using this selection into the iteration formula we obtain the following successive approximations

$$\begin{aligned}
u_0(x,t) &= \cos x-t\sin x,\\
u_1(x,t) &= \cos x(1-\frac{1}{2!}t^2)-\sin x(t-\frac{1}{3!}t^3),\\
u_2(x,t) &= \cos x(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4)-\sin x(t-\frac{1}{3!}t^3+\frac{1}{5!}t^5),\\
&\ \vdots\\
u_n(x,t) &= \cos x(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4+\cdots)-\sin x(t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\cdots).
\end{aligned} \quad (9.344)$$

This gives the exact solution by

$$u(x,t)=\cos x\cos t-\sin x\sin t=\cos(x+t). \quad (9.345)$$

Example 2. Solve the following inhomogeneous fourth order equation:

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}=(\pi^4-1)\sin \pi x\sin t, \quad (9.346)$$

with initial conditions

$$u(x,0)=0, \quad u_t(x,0)=\sin \pi x. \tag{9.347}$$

Solution.

Proceeding as in Example 1 we obtain

$$\sum_{n=0}^{\infty} u_n(x,t)=t\sin \pi x+(\pi^4-1)\sin \pi x(t-\sin t)-L_t^{-1}L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right). \tag{9.348}$$

This gives the relation

$$\begin{aligned} u_0(x,t) &= t\sin \pi x+(\pi^4-1)\sin \pi x(t-\sin t), \\ u_{k+1}(x,t) &= -L_t^{-1}L_x\left(u_k(x,t)\right), \quad k \geqslant 0. \end{aligned} \tag{9.349}$$

To use the noise terms phenomenon we determine the first two components, hence we find

$$\begin{aligned} u_0(x,t) &= \sin \pi x \sin t+\pi^4 \sin \pi x(t-\sin t), \\ u_1(x,t) &= -L_t^{-1}L_x\left(u_0(x,t)\right) \\ &= -\pi^4 \sin \pi x(t-\sin t)-\pi^8 \sin \pi x(\tfrac{1}{3!}t^3+\sin t-t). \end{aligned} \tag{9.350}$$

A close examination of the first two components shows the appearance of the noise term $\pi^4 \sin \pi x(t-\sin t)$ with opposite signs in $u_0(x,t)$ and $u_1(x,t)$. By canceling this term from $u_0(x,t)$ and checking that the remaining term justifies the equation give the exact solution

$$u(x,t)=\sin \pi x \sin t. \tag{9.351}$$

9.9.2 Equations with Variable Coefficients

In what follows we investigate the variable coefficient fourth-order parabolic partial differential equation of the form

$$\frac{\partial^2 u}{\partial t^2}+\mu(x)\frac{\partial^4 u}{\partial x^4}=0, \quad \mu(x)>0, a<x<b, t>0, \tag{9.352}$$

where $\mu(x)>0$ is the ratio of flexural rigidity of the beam to its mass per unit length.

The initial conditions associated with (9.352) are of the form

$$\begin{aligned} u(x,0) &= f(x), \quad a \leqslant x \leqslant b, \\ u_t(x,0) &= g(x), \quad a \leqslant x \leqslant b, \end{aligned} \tag{9.353}$$

and the boundary conditions are given by

$$\begin{aligned} u(a,t) &= h(t), \;\; u(b,t)=r(t), t>0, \\ u_{xx}(a,t) &= s(t), \; u_{xx}(b,t)=q(t), t>0, \end{aligned} \tag{9.354}$$

where the functions $f(x), g(x), h(t), r(t), s(t)$ and $q(t)$ are continuous functions.

In an operator form, Equation (9.352) becomes

$$L_t u = -\mu(x)\frac{\partial^4 u}{\partial x^4}, \quad \mu(x) > 0. \tag{9.355}$$

Operating with the two fold integral operator and using the initial conditions yields

$$u(x,t) = f(x) + tg(x) - L_t^{-1}\left(\mu(x)\frac{\partial^4 u}{\partial x^4}\right). \tag{9.356}$$

Using the series representation of $u(x,t)$ leads to

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + tg(x) - L_t^{-1}\left(\mu(x)\frac{\partial^4}{\partial x^4}\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{9.357}$$

This gives the recurrence relation

$$\begin{aligned} u_0(x,t) &= f(x) + tg(x), \\ u_{k+1}(x,t) &= -L_t^{-1}\left(\mu(x)\frac{\partial^4 u_k}{\partial x^4}\right), \quad k \geqslant 0, \end{aligned} \tag{9.358}$$

so that

$$\begin{aligned} u_0(x,t) &= f(x) + tg(x), \\ u_1(x,t) &= -L_t^{-1}\left(\mu(x)\frac{\partial^4 u_0}{\partial x^4}\right), \\ u_2(x,t) &= -L_t^{-1}\left(\mu(x)\frac{\partial^4 u_1}{\partial x^4}\right), \\ u_3(x,t) &= -L_t^{-1}\left(\mu(x)\frac{\partial^4 u_2}{\partial x^4}\right). \end{aligned} \tag{9.359}$$

In view of (9.359), the series solution follows immediately.

Example 3. Solve the fourth order parabolic equation

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4} = 0, \quad \frac{1}{2} < x < 1, t > 0, \tag{9.360}$$

subject to the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 1 + \frac{x^5}{120}, \quad \frac{1}{2} < x < 1, \tag{9.361}$$

and the boundary conditions

$$\begin{aligned}
u\left(\frac{1}{2},t\right) &= \left(1+\frac{(1/2)^5}{120}\right)\sin t, \quad u(1,t) &= \frac{121}{120}\sin t, \quad t>0,\\
\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},t\right) &= \frac{1}{6}\left(\frac{1}{2}\right)^3\sin t, \quad \frac{\partial^2 u}{\partial x^2}(1,t) &= \frac{1}{6}\sin t, \quad t>0.
\end{aligned} \tag{9.362}$$

Solution.

The Decomposition Method

Adomian's analysis gives the recurrence relation

$$\begin{aligned}
u_0(x,t) &= \left(1+\frac{x^5}{120}\right)t,\\
u_{k+1}(x,t) &= -L_t^{-1}\left(\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 u_k}{\partial x^4}\right), \quad k\geqslant 0,
\end{aligned} \tag{9.363}$$

that gives

$$\begin{aligned}
u_0(x,t) &= \left(1+\frac{x^5}{120}\right)t,\\
u_1(x,t) &= -\left(1+\frac{x^5}{120}\right)\frac{t^3}{3!},\\
u_2(x,t) &= \left(1+\frac{x^5}{120}\right)\frac{t^5}{5!}.
\end{aligned} \tag{9.364}$$

The solution in a series form is

$$u(x,t) = \left(1+\frac{x^5}{120}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\right), \tag{9.365}$$

and in a closed form by

$$u(x,t) = \left(1+\frac{x^5}{120}\right)\sin t. \tag{9.366}$$

It is important to confirm the fact that we obtained the solution by using the initial conditions only. The obtained solution satisfies the four prescribed boundary conditions that were not used in the determination of the solution.

The Variational Iteration Method

Proceeding as before we obtain the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} + \left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right)d\xi. \tag{9.367}$$

We can select $u_0(x,t) = \left(1+\frac{x^5}{5!}\right)t$. Using this selection into the iteration formula we obtain the following successive approximations

$$
\begin{aligned}
u_0(x,t) &= \left(1+\frac{x^5}{5!}\right)t,\\
u_1(x,t) &= \left(1+\frac{x^5}{5!}\right)(t-\frac{1}{3!}t^3),\\
u_2(x,t) &= \left(1+\frac{x^5}{5!}\right)(t-\frac{1}{3!}t^3+\frac{1}{5!}t^5),\\
u_3(x,t) &= \left(1+\frac{x^5}{5!}\right)(t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\frac{1}{7!}t^7),\\
&\vdots\\
u_n(x,t) &= \left(1+\frac{x^5}{5!}\right)(t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\frac{1}{7!}t^7+\cdots).
\end{aligned}
\tag{9.368}
$$

This gives the exact solution by

$$
u(x,t)=\left(1+\frac{x^5}{5!}\right)\sin t. \tag{9.369}
$$

We close our analysis by discussing the following nonhomogeneous equation. The decomposition method will be combined with the effect of the noise terms phenomenon. This will facilitate the convergence of the solution.

Example 4. We finally consider the nonhomogeneous parabolic equation

$$
\frac{\partial^2 u}{\partial t^2}+(1+x)\frac{\partial^4 u}{\partial x^4}=(x^4+x^3-\frac{6}{7!}x^7)\cos t,\quad 0<x<1, t>0, \tag{9.370}
$$

subject to the initial conditions

$$
u(x,0)=\frac{6}{7!}x^7,\quad u_t(x,0)=0,\quad 0<x<1, \tag{9.371}
$$

and the boundary conditions

$$
\begin{aligned}
u(0,t)=0,\ u(1,t)=\frac{6}{7!}\cos t,\quad t>0,\\
\frac{\partial^2 u}{\partial x^2}(0,t)=0,\ \frac{\partial^2 u}{\partial x^2}(1,t)=\frac{1}{20}\cos t,\quad t>0,
\end{aligned}
\tag{9.372}
$$

Solution.

Following our discussions above, we obtain

$$
\begin{aligned}
u_0(x,t) &= \frac{6}{7!}x^7+(x^4+x^3-\frac{6}{7!}x^7)(1-\cos t),\\
u_{k+1}(x,t) &= -L_t^{-1}\left((1+x)\frac{\partial^4 u_k}{\partial x^4}\right),\quad k\geqslant 0.
\end{aligned}
\tag{9.373}
$$

Proceeding as before, the noise terms appear in the components $u_0(x,t)$, and $u_1(x,t)$, and by canceling these noise terms we obtain the exact solution

$$u(x,t) = \frac{6}{7!}x^7 \cos t. \tag{9.374}$$

Exercises 9.9

In Exercises 1–5, use the variational iteration method or the Adomian decomposition method to solve the following homogeneous fourth order equations:

1. $u_{tt} + u_{xxxx} = 0$, $u(x,0) = \sin x$, $u_t(x,0) = \cos x$
2. $u_{tt} + u_{xxxx} = 0$, $u(x,0) = \sin x$, $u_t(x,0) = 0$
3. $u_{tt} + u_{xxxx} = 0$, $u(x,0) = \cos x$, $u_t(x,0) = 0$
4. $u_{tt} + u_{xxxx} = 0$, $u(x,0) = 1 + \cos x$, $u_t(x,0) = -\sin x$
5. $u_{tt} + u_{xxxx} = 0$, $u(x,0) = 2$, $u_t(x,0) = \sin x$

In Exercises 6–10, use the variational iteration method or the Adomian decomposition method to solve the following inhomogeneous fourth order equations:

6. $u_{tt} + u_{xxxx} = 15\sin 2x \cos t$, $u(x,0) = \sin 2x$, $u_t(x,0) = 0$
7. $u_{tt} + u_{xxxx} = 2e^{x+t}$, $u(x,0) = e^x$, $u_t(x,0) = e^x$
8. $u_{tt} + u_{xxxx} = 12\sin 2x \sin 2t$, $u(x,0) = 0$, $u_t(x,0) = 2\sin 2x$
9. $u_{tt} + u_{xxxx} = 2e^{x-t}$, $u(x,0) = e^x$, $u_t(x,0) = -e^x$
10. $u_{tt} + u_{xxxx} = -3\sin(x+2t)$, $u(x,0) = \sin x$, $u_t(x,0) = 2\cos x$

In Exercises 11–15, use the variational iteration method or Adomian decomposition method to solve the following fourth order equations with variable coefficients, $0 < x < 1$:

11. $u_{tt} + (\frac{x}{\sin x} - 1)u_{xxxx} = 0$, $u(x,0) = -u_t(x,0) = x - \sin x$
12. $u_{tt} + \frac{x^4}{360}u_{xxxx} = 0$, $u(x,0) = 0$, $u_t(x,0) = \frac{x^6}{720}$
13. $u_{tt} + (\frac{x}{\cos x} - 1)u_{xxxx} = 0$, $u(x,0) = -u_t(x,0) = x - \cos x$
14. $u_{tt} + (1 + \frac{x^4}{360})u_{xxxx} = \frac{5}{2}x^2 \sin t$, $u(x,0) = 0$, $u_t(x,0) = \frac{5}{6!}x^6$
15. $u_{tt} + (1 + \frac{3!}{7!}x^4)u_{xxxx} = x^3(\sin t + \cos t)$, $u(x,0) = u_t(x,0) = \frac{6}{7!}x^7$

References

1. M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, (1991).
2. M.J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia (1981).
3. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, (1994).
4. J.M. Burgers, A mathematical model illustrating the theory of turbulence, *Adv. Appl. Mech.* 1, 171 – 199, (1948).
5. D.J. Evans and B.B. Sanugi, Numerical solution of the Goursat problem by a nonlinear trapezoidal formula, *Appl. Math. Lett.*, 1(3), 221 – 223, (1988).
6. J.H. He, Some asymptotic methods for strongly nonlinearly equations, *Int. J. of Modern Math.*, 20(10), 1141–1199, (2006).
7. W. Hereman and A. Nuseir, Symbolic methods to construct exact solutions of nonlinear partial differential equations, *Math. Comput. Simulation*, 43, 13–27, (1997).
8. J. Hietarinta, A search for bilinear equations passing Hirota's three-soliton condition. I. KdV-type bilinear equations, *J. Math. Phys.*, 28(8), 1732–1742, (1987).
9. R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge, (2004).
10. R. Hirota, Exact solutions of the Sine-Gordon equation for multiple collisions of solitons, *J. Phys. Soc. Japan*, 33(5), 1459–1463, (1972).
11. R.M. Miura, The Korteweg de-Vries equation: a survey of results, *SIAM Rev.*, 18, 412–459, (1976).
12. A.M. Wazwaz, The decomposition method for approximate solution of the Goursat problem, *Appl. Math. Comput.*, 69, 299–311, (1995).
13. A.M.Wazwaz, A new approach to the nonlinear advection problem, an application of the decomposition method, *Appl. Math. Comput.*, 72, 175–181, (1995).
14. A.M. Wazwaz, Necessary conditions for the appearance of noise terms in decomposition solution series, *Appl. Math. Comput.*, 81, 199–204, (1997).
15. A.M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema, Leiden, (2002).
16. G.B. Whitham, Linear and Nonlinear Waves, John Wiley, New York, (1976).

Chapter 10
Numerical Applications and Padé Approximants

10.1 Introduction

In this chapter we will apply Adomian decomposition method, the variational iteration method, and other numerical methods to handle linear and nonlinear differential equations numerically. Because the decomposition method and the variational iteration method provide a rapidly convergent series and approximations and faster than existing numerical techniques, it is therefore the two methods are considered efficient, reliable and easy to use from a computational viewpoint. It is to be noted that few terms or few approximations are usually needed to supply a reliable result much closer to the exact value. The overall error can be significantly decreased by computing additional terms of the decomposition series or additional approximations.

The common numerical techniques that are usually used are the *finite differences* method, *finite element* method, and Galerkin method. The finite differences method handles the differential equation by replacing the derivatives in the equation with difference quotients. The finite elements method reduces any partial differential equation to a system of ordinary differential equations. However, the Galerkin method approximates the solution of a differential equation by a finite linear combinations of basic functions with specific properties. In addition, other techniques are used such as Crank-Nicolson method, perturbation methods and collocation method.

Recently, several useful comparative discussions have been conducted between the decomposition method and other numerical approaches. In addition, comparative discussions have been conducted between the variational iteration method and other numerical schemes. The studies have formally proved that the decomposition method and the variational iteration method are faster and more efficient to use in numerical applications as well as in analytical approaches. Moreover, it was shown that both methods give approximations of a high degree of accuracy.

Concerning the decomposition method, the n-term approximant ϕ_n

$$\phi_n = \sum_{n=0}^{n-1} u_n, \tag{10.1}$$

offers a very good approximation for quite low values of n. The accuracy level can be significantly enhanced by computing additional components of the solution.

The decomposition method [1] and the variational iteration method [6] have been outlined before in previous chapters and have been extensively employed in the text. The best way to describe the use of the decomposition method for numerical studies is to work on several examples, ordinary and partial differential equations. Our approach will begin first with ordinary differential equations. Partial differential equations will be investigated as well. Some of these examples will be examined by more than one method.

10.2 Ordinary Differential Equations

10.2.1 Perturbation Problems

It is useful to consider a comparative study between at least two of the decomposition method, the variational iteration method, Taylor series method, and the perturbation technique. Two illustrative perturbation problems will be examined.

Example 1. Consider the Duffing equation

$$\frac{d^2y}{dt^2}+y+\varepsilon y^3=0,\quad y(0)=1,\quad y'(0)=0. \tag{10.2}$$

Solution.

The comparative study will be carried out by applying three methods.

The Decomposition Method

Applying the two-fold integral operator L_t^{-1} to both sides of (10.2) gives

$$y(t)=1-L_t^{-1}y-\varepsilon L_t^{-1}y^3. \tag{10.3}$$

Using the series representation for y and y^3 into (10.3) yields

$$\sum_{n=0}^{\infty}y_n=1-L_t^{-1}\left(\sum_{n=0}^{\infty}y_n\right)-\varepsilon L_t^{-1}\left(\sum_{n=0}^{\infty}A_n\right), \tag{10.4}$$

where A_n are Adomian polynomials for y^3. The decomposition method suggests the recursive relation

$$\begin{aligned} y_0(t) &= 1,\\ y_{k+1} &= -L_t^{-1}\left(y_k\right)-\varepsilon L_t^{-1}\left(A_k\right),\quad k\geqslant 0. \end{aligned} \tag{10.5}$$

The components $y_n(t)$ can be elegantly determined by

$$
\begin{aligned}
y_0(t) &= 1, \\
y_1(t) &= -L_t^{-1}(y_0) - \varepsilon L_t^{-1}(A_0) = -\frac{1}{2!}(1+\varepsilon)t^2, \\
y_2(t) &= -L_t^{-1}(y_1) - \varepsilon L_t^{-1}(A_1) = \frac{1}{4!}(1+4\varepsilon+3\varepsilon^2)t^4, \\
y_3(t) &= -L_t^{-1}(y_2) - \varepsilon L_t^{-1}(A_2) = -\frac{1}{6!}(1+25\varepsilon+51\varepsilon^2+27\varepsilon^3)t^6.
\end{aligned}
$$

Consequently, the ϕ_4 approximant is given by

$$
\begin{aligned}
\phi_4 &= \sum_{n=0}^{3} y_n(t), \\
&= \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6\right) - \varepsilon\left(\frac{1}{2!}t^2 - \frac{1}{3!}t^4 + \frac{25}{6!}t^6\right) + O(\varepsilon^2).
\end{aligned}
\tag{10.6}
$$

The Variational Iteration Method

The correction functional for this equation reads

$$
y_{n+1}(t) = y_n(t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 y_n(\xi)}{\partial \xi^2} + \tilde{y}_n(\xi) + \varepsilon \tilde{y}_n^3(\xi)\right) d\xi. \tag{10.7}
$$

This yields the stationary conditions

$$
\begin{aligned}
1 - \lambda'|_{\xi=t} &= 0, \\
\lambda|_{\xi=t} &= 0, \\
\lambda''|_{\xi=t} &= 0.
\end{aligned}
\tag{10.8}
$$

This in turn gives

$$
\lambda = \xi - t. \tag{10.9}
$$

Using this value of the Lagrange multiplier into the correction functional gives the iteration formula

$$
y_{n+1}(t) = y_n(t) + \int_0^t (\xi - t)\left(\frac{\partial^2 y_n(\xi)}{\partial \xi^2} + y_n(\xi) + \varepsilon y_n^3(\xi)\right) d\xi. \tag{10.10}
$$

Considering the given initial values, we can select $y_0(t) = 1$. Using this selection, we obtain the following successive approximations

$$
\begin{aligned}
y_0(t) &= 1, \\
y_1(t) &= 1 - \frac{1}{2}t^2 - \varepsilon\frac{1}{2}t^2, \\
y_2(t) &= 1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 - \varepsilon\left(\frac{1}{2}t^2 - \frac{1}{3!}t^4 + \frac{1}{40}t^6\right) + O(\varepsilon^2),
\end{aligned}
$$

$$\vdots$$

$$y_n(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6\right) - \varepsilon\left(\frac{1}{2}t^2 - \frac{1}{3!}t^4 + \frac{25}{6!}t^6\right) + O(\varepsilon^2). \tag{10.11}$$

The answer is consistent with the results obtained above. However, there is no need for Adomian polynomials or any transformation for the nonlinear term y^3.

The Perturbation Method

We next approach the Duffing equation (10.2) by applying the perturbation technique. To obtain a perturbative solution for this problem, we first represent $y(t)$ by a power series in ε as

$$y(t) = \sum_{n=0}^{\infty} \varepsilon^n y_n(t). \tag{10.12}$$

This means that the initial condition can be reduced to a set of initial conditions defined by

$$y_0(0) = 1, \quad y_0'(0) = 0, \quad y_k(0) = y_k'(0) = 0, \quad k \geqslant 1. \tag{10.13}$$

Substituting (10.12) into (10.2) and equating coefficients of like powers of ε gives the differential equations

$$\begin{aligned} y_0'' + y_0 &= 0, \quad y_0(0) = 1, \quad y_0'(0) = 0, \\ y_1'' + y_1 &= -y_0^3, \quad y_1(0) = 0, \quad y_1'(0) = 0 \\ y_2'' + y_2 &= -3y_0^2 y_1, \quad y_2(0) = 0, \quad y_2'(0) = 0. \end{aligned} \tag{10.14}$$

Solving the first homogeneous equation and using the result in solving the inhomogeneous equation gives

$$\begin{aligned} y_0 &= \cos t, \\ y_1 &= -\frac{1}{32}\cos t - \frac{3}{8}t\sin t + \frac{1}{32}\cos 3t. \end{aligned} \tag{10.15}$$

In view of (10.15), the first-order perturbative solution to the Duffing equation [10] is given by

$$y(t) = \cos t + \varepsilon\left(-\frac{1}{32}\cos t - \frac{3}{8}t\sin t + \frac{1}{32}\cos 3t\right) + O(\varepsilon^2). \tag{10.16}$$

A close examination of the decomposition method and the perturbation method clearly shows that the decomposition method can calculate the components elegantly where we integrated simple terms of the form t^n. However, using the perturbation technique requires solving homogeneous and inhomogeneous differential equations with trigonometric functions and trigonometric identities, such as $\cos 3t$ in this example. This shows that the perturbation technique suffers from the cumbersome work especially if a higher order solution is sought. On the other hand, we can easily evaluate additional components of the decomposition series as much as we like.

Example 2. Solve the equation

$$y' = y^2 \sin(\varepsilon t), \quad y(0) = 1. \tag{10.17}$$

Solution.

It is to be noted that the nonlinear equation (10.17) can be solved by using the separation of variables method where the analytic solution is

$$y = \frac{\varepsilon}{(\varepsilon - 1) + \cos \varepsilon t}. \tag{10.18}$$

We next carry out the comparison between the decomposition method and the perturbation technique.

The Decomposition Method

In an operator form, Eq. (10.17) can be rewritten as

$$L_t y(t) = y^2 \sin(\varepsilon t). \tag{10.19}$$

Applying the inverse operator L_t^{-1} to both sides of (10.19) yields

$$y(t) = 1 + L_t^{-1} y^2 \sin(\varepsilon t). \tag{10.20}$$

Using the decomposition assumptions for $y(t)$ and for the nonlinear term y^2 gives

$$\sum_{n=0}^{\infty} y_n(t) = 1 + L_t^{-1} \left(\sin(\varepsilon t) \sum_{n=0}^{\infty} A_n \right). \tag{10.21}$$

This suggests the recursive relation

$$\begin{aligned} y_0(t) &= 1, \\ y_{k+1} &= L_t^{-1} \left(\sin(\varepsilon t) A_k \right), \quad k \geqslant 0. \end{aligned} \tag{10.22}$$

This gives the first four components by

$$\begin{aligned} y_0(t) &= 1, \\ y_1(t) &= L_t^{-1} \left(\sin(\varepsilon t) A_0 \right) = \frac{1}{\varepsilon}(1 - \cos(\varepsilon t)), \\ y_2(t) &= L_t^{-1} \left(\sin(\varepsilon t) A_1 \right) = \frac{1}{\varepsilon^2}(1 - \cos(\varepsilon t))^2, \\ y_3(t) &= L_t^{-1} \left(\sin(\varepsilon t) A_2 \right) = \frac{1}{\varepsilon^3}(1 - \cos(\varepsilon t))^3, \end{aligned}$$

$$\begin{aligned} y_4(t) &= L_t^{-1}\left(\sin(\varepsilon t)A_3\right) = \frac{1}{\varepsilon^4}(1-\cos(\varepsilon t))^4, \\ y_5(t) &= L_t^{-1}\left(\sin(\varepsilon t)A_4\right) = \frac{1}{\varepsilon^5}(1-\cos(\varepsilon t))^5. \end{aligned} \tag{10.23}$$

This gives the ϕ_6 approximant by

$$\begin{aligned} \phi_6 &= \sum_{n=0}^{5} y_n, \\ &= 1+\frac{1}{\varepsilon}(1-\cos(\varepsilon t))+\frac{1}{\varepsilon^2}(1-\cos(\varepsilon t))^2 \\ &\quad +\frac{1}{\varepsilon^3}(1-\cos(\varepsilon t))^3+\frac{1}{\varepsilon^4}(1-\cos(\varepsilon t))^4+\frac{1}{\varepsilon^5}(1-\cos(\varepsilon t))^5. \end{aligned} \tag{10.24}$$

To enhance the approximation level, we should determine more components $y_n, n \geqslant 6$. Based on this, the approximants $\phi_n, n \geqslant 7$ can be used to achieve higher accuracy level.

The Perturbation Method

To obtain a perturbative solution for (10.17), we first represent $y(t)$ by a power series in ε as

$$y(t) = \sum_{n=0}^{\infty} \varepsilon^n y_n(t). \tag{10.25}$$

This means that the initial condition can be reduced to a set of initial conditions defined by

$$y_0(0) = 1, \quad y_0'(0) = 0, \quad y_k(0) = y_k'(0) = 0,\ k \geqslant 1. \tag{10.26}$$

Substituting (10.25) into (10.17) gives

$$\begin{aligned} & y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \varepsilon^3 y_3' + \varepsilon^4 y_4' + \cdots = \\ & \qquad \left(y_0^2 + 2\varepsilon y_0 y_1 + \varepsilon^2(2y_0y_2 + y_1^2)\right) \times \left(\varepsilon t - \frac{1}{3!}\varepsilon^3 t^3\right). \end{aligned} \tag{10.27}$$

Equating coefficients of like powers of ε in (10.27) leads to the set of differential equations

$$\begin{aligned} y_0' &= 0, \quad y_0(0) = 1, \\ y_1' &= t y_0^2, \quad y_1(0) = 0, \\ y_2' &= 2t y_0 y_1, \quad y_2(0) = 0, \\ y_3' &= t(2y_0y_2 + y_1^2) - \frac{1}{3!}t^3 y_0^2, \quad y_3(0) = 0, \\ y_4' &= t(2y_0y_3 + 2y_1y_2) - \frac{2}{3!}t^3 y_0 y_1, \quad y_4(0) = 0. \end{aligned} \tag{10.28}$$

Solving the resulting equations and using the relevant initial conditions give

$$y_0(t) = 1, y_1(t) = \frac{1}{2}t^2, y_2(t) = \frac{1}{4}t^4, y_3(t) = \frac{1}{8}t^6 - \frac{1}{24}t^4, y_4(t) = \frac{1}{16}t^8 - \frac{1}{72}t^6. \tag{10.29}$$

The perturbative solution is therefore given by

$$y(t) = 1 + \frac{1}{2}\varepsilon t^2 + \frac{1}{4}\varepsilon^2 t^4 + \varepsilon^3\left(\frac{1}{8}t^6 - \frac{1}{24}t^4\right) + \varepsilon^4\left(\frac{1}{16}t^8 - \frac{1}{72}t^6\right) + O(\varepsilon^5). \tag{10.30}$$

Table 10.1

t	y_{analytic}	$y_{\text{perturbation}}$	$y_{\text{decomposition}}$
0.8	1.033 04	1.033 04	1.033 04
1.5	1.126 49	1.126 51	1.126 47
2.0	1.248 96	1.248 84	1.248 56
3.0	1.807 13	1.780 24	1.775 02

Table 10.1 shows the performance of the perturbation and the decomposition methods by considering $\varepsilon = 0.1$. Comparing the performance of the decomposition method and the perturbation method in this example clearly shows that the decomposition method encountered the difficulties that arise from $\sin\varepsilon t$, where components of the decomposition series have been computed directly. However, using the perturbation technique, the Taylor expansion of $\sin\varepsilon t$ has been used to control the powers of ε in both sides of (10.27).

In addition, it is to be noted that approximating $\sin(\varepsilon t)$ in using the perturbation technique by the first two terms of the Taylor expansion will affect the approximation numerically. However, the decomposition method has been applied directly without using any expansion. More importantly from the point of view of numerical purposes, the calculations in Table 10.1 show slight improvements of the decomposition method over perturbation method for small values of t.

10.2.2 Nonperturbed Problems

In this section, nonperturbed ordinary differential equations will be handled from numerical viewpoint. A comparison will be carried here between the decomposition method and the Taylor series method. To achieve our goal, we study the following examples:

Example 3. Consider the first order ordinary differential equation

$$y' + y = \frac{1}{1+x^2}, \quad y(0) = 0. \tag{10.31}$$

Solution.

It is to be noted that the solution of the first order linear equation (10.31) cannot be found in a closed form. We next carry out the comparison between the decomposition method and the Taylor series method.

The Decomposition Method

In an operator form, Eq. (10.31) can be rewritten as

$$L_x y = \frac{1}{1+x^2} - y, \quad y(0) = 0. \tag{10.32}$$

Applying the inverse operator L_x^{-1} to both sides of (10.32) yields

$$y(x) = \arctan x - L_x^{-1} y(x). \tag{10.33}$$

Substituting the series expression for $y(x)$ carries (10.33) into

$$\sum_{n=0}^{\infty} y_n(x) = \arctan x - L_x^{-1}\left(\sum_{n=0}^{\infty} y_n(x)\right). \tag{10.34}$$

This suggests the recursive relation

$$\begin{aligned} y_0(x) &= \arctan x, \\ y_{k+1}(x) &= -L_x^{-1} y_k(x), \quad k \geqslant 0. \end{aligned} \tag{10.35}$$

This gives the first three components by

$$\begin{aligned} y_0(x) &= \arctan x, \\ y_1(x) &= -L_x^{-1}\left(y_0(x)\right) = -x \arctan x + \frac{1}{2}\ln(1+x^2), \\ y_2(x) &= -L_x^{-1}\left(y_1(x)\right) = \frac{1}{2}\left((x^2-1)\arctan x + x - x\ln(1+x^2)\right), \end{aligned} \tag{10.36}$$

where tables of integrals in Appendix A are used. This gives the ϕ_3 approximant by

$$\begin{aligned} \phi_3 &= \sum_{n=0}^{2} y_n(x), \\ &= \arctan x - x\arctan x + \frac{1}{2}\ln(1+x^2) \\ &\quad + \frac{1}{2}\left[(x^2-1)\arctan x + x - x\ln(1+x^2)\right]. \end{aligned} \tag{10.37}$$

The Taylor Series Method

This equation can be handled by using the integrating factor μ given by

$$\mu = e^x. \tag{10.38}$$

The solution of Eq. (10.31) is therefore given by the expression

$$y(x) = e^{-x} \int_0^x \frac{e^t}{1+t^2}\,dt. \tag{10.39}$$

It is clear that a closed form solution is not obtainable in this problem. Accordingly, we use the Taylor series method where we introduce the solution in the form of an infinite series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{10.40}$$

Our goal is now to determine the coefficients $a_n, n \geqslant 0$. Substituting (10.40) into (10.31) gives

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)_x + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}. \tag{10.41}$$

Note that the given condition $y(0) = 0$ gives $a_0 = 0$. The other coefficients a_n, $n \geqslant 1$ can be determined by equating the coefficients of like powers of x where we find

$$\begin{aligned} a_1 = 1,\ a_2 = -\frac{1}{2!},\ a_3 = -\frac{1}{3!} \\ a_4 = \frac{1}{4!},\ \ a_5 = \frac{23}{5!},\ \ a_6 = -\frac{23}{6!}. \end{aligned} \tag{10.42}$$

In view of (10.42), the solution in a series form is given by

$$y(x) = x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{23}{5!}x^5 - \frac{23}{6!}x^6 + \cdots. \tag{10.43}$$

Example 4. Solve the second order linear equation

$$y'' + 2xy' = 0, \quad y(0) = 0, \quad y'(0) = \frac{2}{\sqrt{\pi}}. \tag{10.44}$$

Solution.

It is to be noted that the exact solution of this equation is given by

$$y(x) = \operatorname{erf}(x), \tag{10.45}$$

where the error function $\operatorname{erf}(x)$ is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\,du, \tag{10.46}$$

and its complementary function $\operatorname{erfc}(x)$ is defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2}\,du, \tag{10.47}$$

such that

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = 1. \tag{10.48}$$

The Decomposition Method

Applying the two-fold integral operator L_x^{-1} on (10.44) gives

$$y(x) = \frac{2}{\sqrt{\pi}}x - 2L_x^{-1}(xy'(x)). \tag{10.49}$$

Proceeding as in the previous examples, the first four components of the solution $y(x)$ can be determined as follows:

$$\begin{aligned}
y_0(x) &= \frac{2}{\sqrt{\pi}}x,\\
y_1(x) &= -2L_x^{-1}(xy_0'(x)) = -\frac{2}{3\sqrt{\pi}}x^3,\\
y_2(x) &= -2L_x^{-1}(xy_1'(x)) = \frac{1}{5\sqrt{\pi}}x^5,\\
y_3(x) &= -2L_x^{-1}(xy_2'(x)) = -\frac{1}{21\sqrt{\pi}}x^7,\\
y_4(x) &= -2L_x^{-1}(xy_3'(x)) = \frac{1}{108\sqrt{\pi}}x^9,\\
y_5(x) &= -2L_x^{-1}(xy_4'(x)) = -\frac{1}{660\sqrt{\pi}}x^{11}.
\end{aligned}$$

Combining the obtained components gives the ϕ_6 approximant

$$\begin{aligned}
\phi_6 &= y_0 + y_1 + y_2 + y_3 + y_4 + y_5,\\
&= \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \frac{x^{11}}{11\cdot 5!}\right].
\end{aligned} \tag{10.50}$$

The Variational Iteration Method

Proceeding as in Example 1 we find the iteration formula

$$y_{n+1}(x) = y_n(x) + \int_0^x (\xi - x)\left(\frac{\partial^2 y_n(\xi)}{\partial \xi^2} + 2\xi\frac{\partial y_n(\xi)}{\partial \xi}\right)d\xi. \tag{10.51}$$

Selecting $y_0(x) = \frac{2}{\sqrt{\pi}}x$ gives the following successive approximations

$$y_0(x) = \frac{2}{\sqrt{\pi}}x,$$

$$y_1(x) = \frac{2}{\sqrt{\pi}}x - \frac{2}{3\sqrt{\pi}}x^3,$$

$$
\begin{aligned}
y_2(x) &= \frac{2}{\sqrt{\pi}}x - \frac{2}{3\sqrt{\pi}}x^3 + \frac{1}{5\sqrt{\pi}}x^5, \\
y_3(x) &= \frac{2}{\sqrt{\pi}}x - \frac{2}{3\sqrt{\pi}}x^3 + \frac{1}{5\sqrt{\pi}}x^5 - \frac{1}{21\sqrt{\pi}}x^7, \\
&\vdots
\end{aligned} \tag{10.52}
$$

This gives the ϕ_4 approximant

$$
\begin{aligned}
\phi_4 &= y_0 + y_1 + y_2 + y_3, \\
&= \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!}\right].
\end{aligned} \tag{10.53}
$$

The Taylor Series Method

To determine the series solution, several derivatives of $y(x)$ will be evaluated in this problem. To achieve this goal, we simply differentiate Eq. (10.44) successively to obtain

$$
\begin{aligned}
y''(x) &= -2xy'(x), \\
y'''(x) &= -2xy''(x) - 2y'(x), \\
y^{(4)}(x) &= -2xy'''(x) - 4y''(x), \\
y^{(5)}(x) &= -2xy^{(4)}(x) - 6y'''(x), \\
y^{(6)}(x) &= -2xy^{(5)}(x) - 8y^{(4)}(x), \\
y^{(7)}(x) &= -2xy^{(6)}(x) - 10y^{(5)}(x).
\end{aligned} \tag{10.54}
$$

Substituting the initial conditions

$$
\begin{aligned}
y(0) &= 0, \\
y'(0) &= \frac{2}{\sqrt{\pi}},
\end{aligned} \tag{10.55}
$$

into (10.54) gives

$$
\begin{gathered}
y''(0) = 0,\ y'''(0) = -\frac{4}{\sqrt{\pi}},\ y^{(4)}(0) = 0 \\
y^{(5)}(0) = \frac{24}{\sqrt{\pi}}, \quad y^{(6)}(0) = 0, \quad y^{(7)}(0) = -\frac{240}{\sqrt{\pi}}.
\end{gathered} \tag{10.56}
$$

Recall that the Taylor expansion of $y(x)$ is given by

$$
y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)}(0)\, x^n. \tag{10.57}
$$

Combining (10.56) and (10.57) gives the truncated Taylor series solution

$$
y(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!}\right]. \tag{10.58}
$$

It is clear that all three methods provided the same approximant. However, the Taylor method required more work compared to the other methods.

Table 10.2 below shows that by adding more terms in the decomposition series, we can easily enhance the accuracy level.

Table 10.2

x	erf(x)	ϕ_3	ϕ_4	ϕ_5
0.2	0.222 70	0.222 70	0.222 70	0.222 70
0.4	0.428 39	0.428 44	0.428 30	0.428 39
0.6	0.603 86	0.604 56	0.603 80	0.603 86
0.8	0.742 10	0.747 10	0.741 47	0.742 25
1.0	0.842 70	0.865 09	0.838 22	0.844 10

Exercises 10.2

In Exercises 1–5, use Adomian decomposition method to find the perturbative approximation for each equation.

1. Find the ϕ_3 approximant for the equation:

   $y'' = (\varepsilon x - 1)y, y(0) = 1, y'(0) = 0$

2. Find the ϕ_3 approximant for the equation:

   $y' = (\varepsilon x + 1)y, y(0) = 1$

3. Find the ϕ_4 approximant for the linear damping oscillator equation:

   $u'' + 2\varepsilon u' + u = 0,\ u(0) = 1,\ u'(0) = 0$

4. Find the ϕ_4 approximant for the Van der Pol equation:

   $u'' + \varepsilon(u^2 - 1)u' + u = 0,\ u(0) = 1,\ u'(0) = 0$

5. Find the ϕ_3 approximant for the nonlinear equation:

   $y' = y^2 \cos(\varepsilon x), y(0) = 1$

In Exercises 6 –10, use Adomian decomposition method to find the ϕ_4 approximant for each equation:

6. $y'' + 2xy' = 0, y(0) = 0, y'(0) = 1$
7. $y'' + 2xy' = 0, y(0) = 0, y'(0) = \dfrac{4}{\sqrt{\pi}}$
8. $y'' - 2xy' = 0, y(0) = 0, y'(0) = 1$
9. $y'' = 12x^2 - y^2, y(0) = 0, y'(0) = 0$
10. $y'' = 2 + y' + y^2, y(0) = 0, y'(0) = 0$

10.3 Partial Differential Equations

In this section, the decomposition method will be applied to partial differential equations to study the numerical approximations of the solutions. The basic outlines of the method are well known from previous chapters.

Example 1. Find the ϕ_5 approximation of the solution of the heat equation

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < \pi, t > 0, \\ u(0,t) &= 0, \quad u(\pi,t) = 0, \\ u(x,0) &= \sin x. \end{aligned} \tag{10.59}$$

Solution.

Recall that the heat equation (10.59) has been solved in Chapter 3 by using the decomposition and the separation of variables methods where we can easily show that the exact solution is given by

$$u(x,t) = e^{-t} \sin x. \tag{10.60}$$

To determine the ϕ_5 approximation, we apply the inverse operator L_t^{-1} to both sides of (10.59) to obtain

$$u(x,t) = \sin x + L_t^{-1}(u_{xx}). \tag{10.61}$$

Substituting $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ and using the resulting recursive relation, we can determine the first five components recurrently by

$$\begin{aligned} u_0(x,t) &= \sin x, \quad u_1(x,t) = -t \sin x, \quad u_2(x,t) = \frac{1}{2!}t^2 \sin x, \\ u_3(x,t) &= -\frac{1}{3!}t^3 \sin x, \quad u_4(x,t) = \frac{1}{4!}t^4 \sin x. \end{aligned} \tag{10.62}$$

Consequently, the ϕ_5 approximation is given by

$$\phi_5 = \sin x \left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4\right). \tag{10.63}$$

Table 10.3 below shows the error that results from using the approximation (10.63) for $t = 0.5$ and for appropriate values of x, where *error* $=|$exact value $- \phi_5|$.

Table 10.3

x	exact value	ϕ_5	error
0.5	0.290 786	0.290 901	1.15E−4
1.0	0.510 378	0.510 580	2.02E−4
1.5	0.605 011	0.605 251	2.4E−4
2.0	0.551 517	0.551 735	2.18E−4
2.5	0.362 992	0.363 135	1.43E−4
3.0	0.085 594	0.085 628	3.44E−5

The table clearly shows that errors can be significantly decreased by evaluating additional components.

Example 2. Find the ϕ_5 approximation of the solution of the wave equation

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < \pi, t > 0, \\ u(0,t) &= 0, \quad u(\pi,t) = \pi, \\ u(x,0) &= x, \quad u_t(x,0) = \sin x. \end{aligned} \tag{10.64}$$

Solution.

The wave equation (10.64) has been investigated in Chapter 5 by using the decomposition and the separation of variables methods where we can easily show that the exact solution is given by

$$u(x,t) = x + \sin x \sin t. \tag{10.65}$$

To determine the ϕ_5 approximation, we apply the two-fold inverse operator L_t^{-1} to both sides of (10.64) to obtain

$$u(x,t) = x + t\sin x + L_t^{-1}(u_{xx}). \tag{10.66}$$

Substituting $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ and proceeding as before, we can determine the first five components recurrently by

$$\begin{aligned} u_0(x,t) &= x + t\sin x, \quad u_1(x,t) = -\frac{1}{3!}t^3 \sin x, \quad u_2(x,t) = \frac{1}{5!}t^5 \sin x, \\ u_3(x,t) &= -\frac{1}{7!}t^7 \sin x, \quad u_4(x,t) = \frac{1}{9!}t^9 \sin x. \end{aligned} \tag{10.67}$$

Consequently, the ϕ_5 approximation is given by

$$\phi_5 = x + \sin x \left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4\right). \tag{10.68}$$

Example 3. Find the ϕ_5 approximation of the solution of the nonlinear partial differential equation

$$u_t + u^2 u_x = 0, \quad u(x,0) = x, t > 0. \tag{10.69}$$

Solution.

Operating with L_t^{-1} on (10.69) gives

$$u(x,t) = x - L_t^{-1}(u^2 u_x). \tag{10.70}$$

Substituting $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$, representing the nonlinear term $u^2 u_x$ by a series of Adomian polynomials, and proceeding as before, we find

$$
\begin{aligned}
u_0(x,t) &= x, \\
u_1(x,t) &= -L_t^{-1}(A_0) = -x^2t, \\
u_2(x,t) &= -L_t^{-1}(A_1) = 2x^3t^2, \\
u_3(x,t) &= -L_t^{-1}(A_2) = -5x^4t^3, \\
u_4(x,t) &= -L_t^{-1}(A_3) = 14x^5t^4.
\end{aligned}
\tag{10.71}
$$

Consequently, the ϕ_5 approximation is given by

$$\phi_5 = x(1 - xt + 2x^2t^2 - 5x^3t^3 + 14x^4t^4). \tag{10.72}$$

Table 10.4 below shows the errors obtained if the approximation (10.72) is used.

Table 10.4

$t \backslash x$	0.1	0.2	0.3	0.4	0.5
0.1	1.01E−9	2.30E−8	2.82E−7	1.53E−6	5.67E−6
0.2	1.1E−8	7.65E−7	8.25E−6	4.40E−5	1.60E−4
0.3	9.4E−8	5.50E−6	5.81E−5	3.04E−4	1.09E−3
0.4	3.83E−7	2.20E−5	2.28E−4	1.17E−3	4.15E−3
0.5	1.14E−6	6.40E−5	6.53E−4	3.32E−3	1.16E−2

It is to be noted that the exact solution is given by

$$
u(x,t) = \begin{cases} x, & t = 0, \\ \dfrac{1}{2t}(\sqrt{1+4xt} - 1), & t > 0, 1 + 4xt > 0. \end{cases} \tag{10.73}
$$

Exercises 10.3

In Exercises 1–5, use Adomian decomposition method to find the ϕ_4 approximation for each equation.

1. $u_t = u_{xx}$, $u(x,0) = \cos x$, $u(0,t) = e^{-t}$
2. $u_{tt} = u_{xx}$, $u(x,0) = \cos x$, $u(0,t) = \cos t$
3. $u_t + \dfrac{1}{36} x u_{xx}^2 = x^3$, $u(x,0) = 0$
4. $u_t + u^2 u_x = 0$, $u(x,0) = 4x$
5. $u_t + uu_x = x$, $u(x,0) = 2$

In Exercises 6–10, use Adomian decomposition method to find the ϕ_3 approximation for each equation.

6. $u_t + uu_x^2 = 0$, $u(x,0) = x$

7. $u_t + uu_x = x^2$, $u(x,0) = 0$

8. $u_t + u_x + u^2 = 0$, $u(x,0) = x$

9. $u_t = iu_{xx}$, $u(x,0) = \cosh x$

10. $u_t + u_{xx} = t\sinh x$, $u(x,0) = 1$

10.4 The Padé Approximants

In this section, the powerful Padé approximants [3,4] will be investigated. Our main concern will be directed in two ways. First, we will discuss the construction of Padé approximants for functions and polynomials. Next, we will explore the implementation of Padé approximants in boundary value problems where the domain is unbounded.

Polynomials are frequently used to approximate power series. However, polynomials tend to exhibit oscillations that may produce an approximation error bounds. In addition, polynomials can never blow up in a finite plane; and this makes the singularities not apparent. To overcome these difficulties, the Taylor series is best manipulated by Padé approximants for numerical approximations.

Padé approximant represents a function by the ratio of two polynomials [3,4,5]. The coefficients of the polynomials in the numerator and in the denominator are determined by using the coefficients in the Taylor expansion of the function. Padé rational approximations are widely used in numerical analysis and fluid mechanics, because they are more efficient than polynomials.

To explore the need of Padé approximants, we consider the function

$$f(x) = \sqrt{\frac{1+3x}{1+x}}. \tag{10.74}$$

The Taylor series of $f(x)$ in (10.74) is given by

$$f(x) = 1 + x - \frac{3}{2}x^2 + \frac{5}{2}x^3 - \frac{37}{8}x^4 + \frac{75}{8}x^5 - \frac{327}{16}x^6 + \frac{753}{16}x^7 + O(x^8). \tag{10.75}$$

The Taylor series (10.75) is often used to approximate $f(x)$ for values of x within the radius of convergence. However, if the polynomial obtained from using a finite number of the Taylor series (10.75) is to be evaluated for large positive values of x, such as $x = \infty$, the series or any truncated number of terms of (10.75) will definitely fail to provide a converging expression. Padé introduced a powerful tool that should be combined with power series for calculations work. This is highly needed especially for boundary value problems where, for specific cases, the domain of validity is unbounded. Using power series, isolated from other concepts, is not always useful because the radius of convergence of the series may not contain the two boundaries [5].

Padé approximant, symbolized by $[m/n]$, is a rational function defined by

$$[m/n] = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m}{1 + b_1 x + b_2 x^2 + \cdots + b_n x^n}, \tag{10.76}$$

where we considered $b_0 = 1$, and numerator and denominator have no common factors. If we selected $m = n$, then the approximants $[n/n]$ are called diagonal approximants.

Notice that in (10.76) there are $m+1$ independent numerator coefficients and n independent denominator coefficients, making altogether $m+n+1$ unknowns [5]. This suggests that $[m/n]$ Padé approximant fits the power series [5] of $f(x)$ through the orders $1, x, x^2, \cdots, x^{m+n}$.

In addition, the Padé approximant will converge on the entire real axis if the function $f(x)$ has no singularities. It was discussed by many that the diagonal Padé approximants, where $m = n$, are more accurate and efficient. Based on this, our study will be focused only on diagonal approximants.

In the following we will introduce the simple and the straightforward method to construct Padé approximants. Suppose that $f(x)$ has a Taylor series given by

$$f(x) = \sum_{k=0}^{\infty} c_k x^k. \tag{10.77}$$

Assuming that $f(x)$ can be manipulated by the diagonal Padé approximant defined in (10.76), where $m = n$. This admits the use of

$$\frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{1 + b_1 x + b_2 x^2 + \cdots + b_n x^n} = c_0 + c_1 x + c_2 x^2 + \cdots + c_{2n} x^{2n}. \tag{10.78}$$

By using cross multiplication in (10.78) we find

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n &= c_0 + (c_1 + b_1 c_0)x + (c_2 + b_1 c_1 + b_2 c_0)x^2 \\ &\quad + (c_3 + b_1 c_2 + b_2 c_1 + b_3 c_0)x^3 + \cdots. \end{aligned} \tag{10.79}$$

Equating powers of x leads to

$$\begin{aligned} &\text{coefficient of } x^0\text{:} & a_0 &= c_0, \\ &\text{coefficient of } x^1\text{:} & a_1 &= c_1 + b_1 c_0, \\ &\text{coefficient of } x^2\text{:} & a_2 &= c_2 + b_1 c_1 + b_2 c_0, \\ &\text{coefficient of } x^3\text{:} & a_3 &= c_3 + b_1 c_2 + b_2 c_1 + b_3 c_0, \\ & & &\vdots \\ &\text{coefficient of } x^n\text{:} & a_n &= c_n + \textstyle\sum_{k=1}^{n} b_k c_{n-k}. \end{aligned}$$

Notice that coefficients of $x^{n+1}, x^{n+2}, \cdots, x^{2n}$ should be equated to zero. This completes the determination of the constants of the polynomials in the numerator and in the denominator. The simple procedure outlined above will be illustrated by discussing the following examples.

Example 1. Find the Padé approximants [2/2] and [3/3] for the function

$$f(x)=\sqrt{\frac{1+3x}{1+x}}. \tag{10.80}$$

Solution.

The Taylor series for $f(x)$ of (10.80) is given by

$$f(x)=1+x-\frac{3}{2}x^2+\frac{5}{2}x^3-\frac{37}{8}x^4+\frac{75}{8}x^5-\frac{327}{16}x^6+\frac{753}{16}x^7+O(x^8). \tag{10.81}$$

The [2/2] approximant is defined by

$$[2/2]=\frac{a_0+a_1x+a_2x^2}{1+b_1x+b_2x^2}. \tag{10.82}$$

To determine the five coefficients of the two polynomials, the [2/2] approximant must fit the Taylor series of $f(x)$ in (10.81) through the orders of $1,x,\cdots,x^4$, hence we set

$$\frac{a_0+a_1x+a_2x^2}{1+b_1x+b_2x^2}=1+x-\frac{3}{2}x^2+\frac{5}{2}x^3-\frac{37}{8}x^4+\cdots. \tag{10.83}$$

Cross multiplying yields

$$\begin{aligned}1+(b_1+1)x+\left(b_1+b_2-\frac{3}{2}\right)x^2+\left(-\frac{3}{2}b_1+b_2+\frac{5}{2}\right)x^3\\ +\left(\frac{5}{2}b_1-\frac{3}{2}b_2-\frac{37}{8}\right)x^4=a_0+a_1x+a_2x^2.\end{aligned} \tag{10.84}$$

Equating powers of x leads to

$$\begin{aligned}
&\text{coefficient of } x^4: && \frac{5}{2}b_1-\frac{3}{2}b_2-\frac{37}{8} &&=0,\\
&\text{coefficient of } x^3: && -\frac{3}{2}b_1+b_2+\frac{5}{2} &&=0,\\
&\text{coefficient of } x^2: && b_1+b_2-\frac{3}{2} &&=a_2,\\
&\text{coefficient of } x^1: && b_1+1 &&=a_1,\\
&\text{coefficient of } x^0: && 1 &&=a_0.
\end{aligned}$$

The solution of this system of equations is

$$\begin{aligned}a_0=1,\ a_1=\frac{9}{2},\ a_2=\frac{19}{4},\\ b_1=\frac{7}{2},\ b_2=\frac{11}{4}.\end{aligned} \tag{10.85}$$

Consequently, the [2/2] Padé approximant is

$$[2/2] = \frac{1+\frac{9}{2}x+\frac{19}{4}x^2}{1+\frac{7}{2}x+\frac{11}{4}x^2}. \tag{10.86}$$

Two conclusions can be made here. First, we note that the Taylor series for the [2/2] approximant is given by

$$[2/2]_{\text{Taylor}} = 1+x-\frac{3}{2}x^2+\frac{5}{2}x^3-\frac{37}{8}x^4+\frac{149}{16}x^5-\frac{159}{8}x^6+O(x^7). \tag{10.87}$$

A close examination of the Taylor series of the approximant [2/2] given by (10.87) and the Taylor series of $f(x)$ given by (10.81), one can easily conclude that the two series are consistent up to x^4 of each. This is normal because in order to determine the five coefficients a_0, a_1, a_2, b_1, b_2, it was necessary to use the terms of orders $1, x, \cdots, x^4$ in Taylor series (10.81) of $f(x)$. Second, it was difficult to use Taylor series (10.81) when x is large, say $x = \infty$. However, the limit of Padé approximant (10.86) as $x \to \infty$ is $\frac{a_2}{b_2}$. In other words, as $x \to \infty$ we obtain

$$\lim_{x\to\infty} f(x) = \sqrt{3} \approx 1.732\,05 \tag{10.88}$$

and

$$\lim_{x\to\infty} [2/2] = \frac{19}{11} \approx 1.727\,27. \tag{10.89}$$

To determine the Padé approximant [3/3], we first set

$$[3/3] = \frac{a_0+a_1x+a_2x^2+a_3x^3}{1+b_1x+b_2x^2+b_3x^3} = 1+x-\frac{3}{2}x^2+\frac{5}{2}x^3-\frac{37}{8}x^4. \tag{10.90}$$

To determine the seven coefficients of the two polynomials, we use the terms through orders $1, x, \cdots, x^6$ in Taylor series of $f(x)$ in (10.81), hence we set

$$\begin{aligned}\frac{a_0+a_1x+a_2x^2+a_3x^3}{1+b_1x+b_2x^2+b_3x^3} &= 1+x-\frac{3}{2}x^2+\frac{5}{2}x^3-\frac{37}{8}x^4\\ &\quad +\frac{75}{8}x^5-\frac{327}{16}x^6+\cdots.\end{aligned} \tag{10.91}$$

Cross multiplying, equating coefficients of like powers of x and solving the resulting system of equations lead to

$$\begin{aligned} a_0 = 1,\ a_1 = \frac{13}{2},\ a_2 = \frac{27}{2},\ a_3 = \frac{71}{8}\\ b_1 = \frac{11}{2},\ b_2 = \frac{19}{2},\ b_3 = \frac{41}{8}.\end{aligned} \tag{10.92}$$

This gives

$$[3/3] = \frac{1 + \frac{13}{2}x + \frac{27}{2}x^2 + \frac{71}{8}x^3}{1 + \frac{11}{2}x + \frac{19}{2}x^2 + \frac{41}{8}x^3}. \tag{10.93}$$

It is to be noted that the Taylor series of (10.93) and the Taylor series (10.81) are consistent up to term of order x^6. In addition, the limit of Padé approximant [3/3] as $x \to \infty$ is $\frac{a_3}{b_3}$. In other words, as $x \to \infty$ we find

$$\lim_{x\to\infty} f(x) = \sqrt{3} \approx 1.732\,05 \tag{10.94}$$

and

$$\lim_{x\to\infty} [3/3] = \frac{71}{41} \approx 1.731\,707. \tag{10.95}$$

a better approximation to $\sqrt{3}$ compared to that obtained from [2/2].

Example 2. Establish the Padé approximants [2/2] and [3/3] for

$$f(x) = e^{-x}. \tag{10.96}$$

Solution.

The Taylor expansion for the exponential function is

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} + O(x^7). \tag{10.97}$$

The [2/2] approximant is defined by

$$[2/2] = \frac{a_0 + a_1x + a_2x^2}{1 + b_1x + b_2x^2}. \tag{10.98}$$

To determine the five coefficients of the two polynomials in the numerator and the denominator, we use the Taylor series of $f(x)$ in (10.97) as discussed before, hence we set

$$\frac{a_0 + a_1x + a_2x^2}{1 + b_1x + b_2x^2} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \tag{10.99}$$

Cross multiplying yields

$$\begin{aligned} 1 + (b_1 - 1)x + \left(-b_1 + b_2 + \frac{1}{2}\right)x^2 + \left(\frac{1}{2}b_1 - b_2 - \frac{1}{6}\right)x^3 \\ + \left(-\frac{1}{6}b_1 + \frac{1}{2}b_2 + \frac{1}{24}\right)x^4 = a_0 + a_1x + a_2x^2. \end{aligned} \tag{10.100}$$

Equating powers of x leads to

$$\begin{aligned}
&\text{coefficient of } x^4: && -\frac{1}{6}b_1+\frac{1}{2}b_2+\frac{1}{24} &&= 0,\\
&\text{coefficient of } x^3: && \frac{1}{2}b_1-b_2-\frac{1}{6} &&= 0,\\
&\text{coefficient of } x^2: && -b_1+b_2+\frac{1}{2} &&= a_2,\\
&\text{coefficient of } x^1: && b_1-1 &&= a_1,\\
&\text{coefficient of } x^0: && 1 &&= a_0.
\end{aligned}$$

This system of equations gives

$$\begin{aligned} a_0 = 1,\ a_1 = -\frac{1}{2},\ a_2 = \frac{1}{12},\\ b_1 = \frac{1}{2},\ \ b_2 = \frac{1}{12}, \end{aligned} \tag{10.101}$$

so that the Padé approximant is

$$[2/2] = \frac{1-\frac{1}{2}x+\frac{1}{12}x^2}{1+\frac{1}{2}x+\frac{1}{12}x^2}. \tag{10.102}$$

In a similar way, we can derive the Padé approximant [3/3] by

$$[3/3] = \frac{1-\frac{1}{2}x+\frac{1}{10}x^2-\frac{1}{120}x^3}{1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{120}x^3}. \tag{10.103}$$

Note that as $x \longrightarrow \infty$, Padé approximants fluctuate between -1 and 1 as can be easily seen from (10.102) and (10.103). In fact, $e^{-x} \longrightarrow 0$ as $x \longrightarrow \infty$.

Table 10.5 below shows the numerical approximations for e^{-x} for several values of x. We can easily observe that the Padé approximant [3/3] provides better approximation than the Taylor expansion. In general, the Padé approximants give small error near $x = 0$, but the error increases as $|x|$ increases.

Table 10.5

x	e^{-x}	[3/3]	Taylor
0.0	1	1	1
0.2	0.818 731	0.818 731	0.818 731
0.4	0.670 320	0.670 320	0.670 320
0.6	0.548 812	0.548 811	0.548 817
0.8	0.449 329	0.449 328	0.449 367
1.0	0.367 879	0.367 876	0.368 056

Example 3. Establish the Padé approximants [2/2] and [4/4] for

$$f(x) = \cos x. \tag{10.104}$$

Solution.

The Taylor expansion for the exponential function is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + O(x^{10}). \tag{10.105}$$

It is useful to note that $\cos x$ and its Padé approximants are even functions. To minimize the size of calculations, we substitute

$$z = x^2, \tag{10.106}$$

into (10.105) to obtain

$$\cos z^{\frac{1}{2}} = 1 - \frac{z}{2!} + \frac{z^2}{4!} - \frac{z^3}{6!} + \frac{z^4}{8!} + O(z^5). \tag{10.107}$$

The [2/2] approximant in this case is defined by

$$[2/2] = \frac{a_0 + a_1 z}{1 + b_1 z}. \tag{10.108}$$

To determine the three coefficients of the two polynomials, we proceed as before to find

$$\begin{aligned} a_0 = 1,\ a_1 &= -\frac{5}{12}, \\ b_1 &= \frac{1}{12}, \end{aligned} \tag{10.109}$$

so that the Padé approximant is

$$[2/2] = \frac{1 - \frac{5}{12}z}{1 + \frac{1}{12}z}, \tag{10.110}$$

or equivalently

$$[2/2] = \frac{1 - \frac{5}{12}x^2}{1 + \frac{1}{12}x^2}, \tag{10.111}$$

To determine the Padé approximant [4/4], we set

$$[4/4] = \frac{a_0 + a_1 z + a_2 z^2}{1 + b_1 z + b_2 z^2}. \tag{10.112}$$

Proceeding as before gives

$$[4/4]=\frac{1-\dfrac{115}{252}z+\dfrac{313}{15\,120}z^2}{1+\dfrac{11}{252}z+\dfrac{13}{15\,120}z^2}, \tag{10.113}$$

or by

$$[4/4]=\frac{1-\dfrac{115}{252}x^2+\dfrac{313}{15\,120}x^4}{1+\dfrac{11}{252}x^2+\dfrac{13}{15\,120}x^4}. \tag{10.114}$$

Example 4. Establish the Padé approximants [3/3] and [4/4] for

$$f(x)=\frac{\ln(1+x)}{x}. \tag{10.115}$$

Solution.

The Taylor expansion for function in (10.115) is

$$\frac{\ln(1+x)}{x}=1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\frac{x^5}{6}+\frac{x^6}{7}-\frac{x^7}{8}+\frac{x^8}{9}+O(x^9). \tag{10.116}$$

To establish [3/3] approximant, we set

$$[3/3]=\frac{a_0+a_1x+a_2x^2+a_3x^3}{1+b_1x+b_2x^2+b_3x^3}. \tag{10.117}$$

To determine the unknowns, we proceed as before and therefore we set

$$\frac{a_0+a_1x+a_2x^2+a_3x^3}{1+b_1x+b_2x^2+b_3x^3}=1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\frac{x^5}{6}+\frac{x^6}{7}. \tag{10.118}$$

Cross multiplying and proceeding as before we find

$$\begin{aligned} a_0=1,\ a_1=\frac{17}{14},\ a_2=\frac{1}{3},\ a_3=\frac{1}{140},\\ b_1=\frac{12}{7},\ b_2=\frac{6}{7},\ b_3=\frac{4}{35}, \end{aligned} \tag{10.119}$$

so that the Padé approximant is

$$[3/3]=\frac{420+510x+140x^2+3x^3}{420+720x+360x^2+48x^3}. \tag{10.120}$$

To determine the Padé approximant [4/4], we set

$$[4/4] = \frac{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4}{1 + b_1x + b_2x^2 + b_3x^3 + b_4x^4}. \tag{10.121}$$

Proceeding as before we obtain

$$[4/4] = \frac{3\,780 + 6\,510x + 3\,360x^2 + 505x^3 + 6x^4}{3\,780 + 8\,400x + 6\,300x^2 + 1\,800x^3 + 150x^4}. \tag{10.122}$$

We close this section by pointing out that many symbolic computer languages, such as Maple and Mathematica have a built-in function that finds Padé approximants when a Taylor series is used. In Appendix D, we list Padé tables for several well-known functions.

Exercises 10.4

1. (a) Establish the Padé approximants [2/2] and [3/3] for

$$f(x) = \sqrt{\frac{1+5x}{1+x}}$$

   (b) Use the result in part (a) to approximate $\sqrt{5}$

2. (a) Establish the Padé approximants [2/2] and [3/3] for

$$f(x) = \sqrt{\frac{1+13x}{1+x}}$$

   (b) Use the result in part (a) to approximate $\sqrt{13}$

3. (a) Establish the Padé approximants [2/2] and [3/3] for

$$f(x) = \sin x$$

   (b) Use the result in part (a) to approximate $\sin 1$

4. Establish the Padé approximants [2/2] and [3/3] for

$$f(x) = e^x$$

5. Establish the Padé approximants [2/2] and [3/3] for

$$f(x) = \frac{\ln(1-x)}{x}$$

6. Establish the Padé approximants [2/2] and [3/3] for

$$f(x) = \frac{\tan x}{x}$$

7. Establish the Padé approximants [3/3] and [4/4] for

$$f(x) = \tanh^{-1} x$$

8. Establish the Padé approximants [3/3] and [4/4] for

$$f(x) = \sinh x$$

9. Establish the Padé approximants [3/3] and [4/4] for

$$f(x) = \frac{1}{1+x}$$

10. Establish the Padé approximants [3/3] and [4/4] for

$$f(x) = \frac{\arctan x}{x}$$

11. Establish the Padé approximants [3/3] and [4/4] for

$$f(x) = e^{\sin x}$$

12. Establish the Padé approximants [3/3] and [4/4] for

$$f(x) = e^{\tan x}$$

10.5 Padé Approximants and Boundary Value Problems

In the previous section we have discussed the Padé approximants which have the advantage of manipulating the polynomial approximation into a rational functions of polynomials. By this manipulation we gain more information about the mathematical behavior of the solution. In addition, we have studied that power series are not useful for large values of x, say $x = \infty$. Boyd [5] and others have formally shown that power series in isolation are not useful to handle boundary value problems. This can be attributed to the possibility that the radius of convergence may not be sufficiently large to contain the boundaries of the domain. Based on this, it is essential to combine the series solution, obtained by the decomposition method or any series solution method, with the Padé approximants to provide an effective tool to handle boundary value problems on an infinite or semi-infinite domains. Recall that the Padé approximants can be easily evaluated by using built-in function in manipulation languages such as Maple or Mathematica.

In this section, the boundary value problems on an infinite or semi-infinite intervals will be investigated. Our approach stems mainly from the combination of the decomposition method and the diagonal approximants. The decomposition method is well addressed in the text and can be assumed known.

In what follows, we outline the basic steps to be followed for handling the boundary value problems on an unbounded domain of validity. In the first step, we use the decomposition method or the modified decomposition method to derive a series solution. In the second step, we form the diagonal Padé approximants $[n/n]$, because it is the most accurate and efficient approximation. Recall that

$$[n/n] = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{1 + b_1 x + b_2 x^2 + \cdots + b_n x^n}. \tag{10.123}$$

In the last step, the most effective use of the diagonal approximant is that it can be used to evaluate the limit as $x \to \infty$. In this case

$$\lim_{x\to\infty} [n/n] = \frac{a_n}{b_n}. \tag{10.124}$$

However, if the boundary condition at $x = \infty$ is given by

$$y(\infty) = 0, \tag{10.125}$$

it follows immediately that

$$a_n = 0. \tag{10.126}$$

Notice here that a_n is an expression that contains values from the prescribed boundary conditions. Consequently, equation (10.126) can be solved to find the unknown parameters of the given boundary condition.

It is interesting to note that solving the resulting polynomial (10.126) frequently leads to a set of roots. It is normal to discard complex roots and other roots that do not satisfy physical properties [5]. To better approximate the root α in (10.126), several Padé approximants should be established where the obtained roots converge to the accurate approximation of α.

To give a clear overview of the steps introduced above, three physical and population growth models, described by ordinary differential equations, will be discussed.

Model I: The Blasius equation.

We first consider the Blasius equation [10]

$$y''' + \frac{1}{2} y y'' = 0, \tag{10.127}$$

subject to the conditions

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = \alpha,\ \alpha > 0, \tag{10.128}$$

where α can be determined by using

$$\lim_{x\to-\infty} y' = 0. \tag{10.129}$$

The Decomposition Method

Following Adomian's method we find

$$\begin{aligned}
y(x) &= x+\frac{1}{2}\alpha x^2-\frac{1}{48}\alpha x^4-\frac{1}{240}\alpha^2x^5+\frac{1}{960}\alpha x^6\\
&\quad+\frac{11}{20\,160}\alpha^2x^7+\left(\frac{11}{161\,280}\alpha^3-\frac{1}{21\,504}\alpha\right)x^8\\
&\quad-\frac{43}{967\,680}\alpha^2x^9+\left(-\frac{5}{387\,072}\alpha^3+\frac{1}{552\,960}\alpha\right)x^{10}\\
&\quad+\left(-\frac{5}{4\,257\,792}\alpha^4+\frac{587}{212\,889\,600}\alpha^2\right)x^{11}+O(x^{12}).
\end{aligned} \tag{10.130}$$

To determine the constant α, we should use the condition

$$y'(x)=0, \quad x\to-\infty. \tag{10.131}$$

It is clear that this condition cannot be applied directly to the series of $y'(x)$, where

$$\begin{aligned}
y'(x) &= 1+\alpha x-\frac{1}{12}\alpha x^3-\frac{1}{48}\alpha^2x^4+\frac{1}{160}\alpha x^5+\frac{11}{2\,880}\alpha^2x^6\\
&\quad+\left(\frac{11}{20\,160}\alpha^3-\frac{1}{2\,688}\alpha\right)x^7-\frac{43}{107\,520}\alpha^2x^8\\
&\quad+\left(-\frac{25}{193\,536}\alpha^3+\frac{1}{55\,296}\alpha\right)x^9\\
&\quad+\left(-\frac{5}{387\,072}\alpha^4+\frac{587}{19\,353\,600}\alpha^2\right)x^{10}+O(x^{11}).
\end{aligned} \tag{10.132}$$

As stated above, the constant α can be evaluated by establishing the Padé approximants to $y'(x)$ in (10.132). Using computer tools, we list the first two diagonal approximants by

$$[2/2]=\frac{12+9\alpha x+(1-3\alpha^2)x^2}{12-3\alpha x+x^2}, \tag{10.133}$$

and

$$[3/3]=\frac{(300\alpha^2-40)+(300\alpha^3-70\alpha)x-3x^2+\left(\frac{45}{4}\alpha^3-3\alpha\right)x^3}{(300\alpha^2-40)-30\alpha x+(-3+30\alpha^2)x^2+\left(\frac{25}{4}\alpha^3-\frac{10}{3}\alpha\right)x^3}, \tag{10.134}$$

where the approximants $[4/4]$ and $[5/5]$ are computed but not listed. The condition (10.131) means that we should set the coefficient of x of highest power in the numerator polynomial to 0. In view of this, we obtain the following equations:

$$\begin{aligned}
-3\alpha^2+1&=0,\\
45\alpha^3-12\alpha&=0,\\
\frac{135}{112}\alpha^6-\frac{189}{64}\alpha^4+\frac{51}{28}\alpha^2-\frac{169}{560}&=0,\\
-\frac{12\,555}{56}\alpha^9+\frac{308\,295}{224}\alpha^7-\frac{226\,701}{112}\alpha^5+\frac{300\,603}{448}\alpha^3-\frac{113\,681}{1\,680}\alpha&=0,
\end{aligned} \tag{10.135}$$

obtained from [2/2], [3/3], [4/4], and [5/5] respectively. Solving these equations independently, we list the results of the roots of α in Table 10.6. As stated before, complex and negative roots may be obtained for α, where $\alpha > 0$.

Table 10.6

Padé approximants	roots
[2/2]	0.577 350, −0.577 350
[3/3]	0, 0.516 398, −0.516 398
[4/4]	0.522 703, −0.522 703
[5/5]	complex roots

Discarding the complex roots, and noting that $\alpha > 0$, indicates that the approximations of the roots of α converge to

$$\alpha = 0.522\,703. \tag{10.136}$$

The Variational Iteration Method

The correction functional for this equation reads

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left(\frac{\partial^3 y_n(\xi)}{\partial\xi^3} + \frac{1}{2}\tilde{y}_n(\xi)\frac{\partial^2\tilde{y}_n(\xi)}{\partial\xi^2}\right)d\xi. \tag{10.137}$$

The stationary conditions are given by

$$\begin{aligned}
\lambda'''(\xi)&=0,\\
1+\lambda''|_{\xi=x}&=0,\\
\lambda'|_{\xi=x}&=0,\\
\lambda_{\xi=x}&=0.
\end{aligned} \tag{10.138}$$

This in turn gives

$$\lambda = -\frac{1}{2}(\xi - x)^2. \tag{10.139}$$

Substituting this value of the Lagrange multiplier into the functional gives the iteration formula

$$y_{n+1}(x) = y_n(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left(\frac{\partial^3 y_n(\xi)}{\partial \xi^3} + \frac{1}{2} y_n(\xi)\frac{\partial^2 y_n(\xi)}{\partial \xi^2}\right) d\xi, \quad n \geqslant 0. \tag{10.140}$$

We select the initial value $y_0 = x + \frac{1}{2}\alpha x^2$ for y_0 by using the boundary conditions, where $\alpha = y''(0)$. Using this into the iteration formula gives the following successive approximations

$$\begin{aligned}
y_0(x) &= x + \frac{1}{2}\alpha x^2,\\
y_1(x) &= x + \frac{1}{2}\alpha x^2 - \frac{1}{48}\alpha x^4 - \frac{1}{240}\alpha^2 x^5,\\
y_2(x) &= x + \frac{1}{2}\alpha x^2 - \frac{1}{48}\alpha x^4 - \frac{1}{240}\alpha^2 x^5 + \frac{1}{960}\alpha x^6\\
&\quad + \frac{11}{20\,160}\alpha^2 x^7 + \left(\frac{11}{161\,280}\alpha^3 - \frac{1}{21\,504}\alpha\right)x^8 - \frac{43}{967\,680}\alpha^2 x^9\\
&\quad + \left(-\frac{5}{387\,072}\alpha^3 + \frac{1}{552\,960}\alpha\right)x^{10}\\
&\quad + \left(-\frac{5}{4\,257\,792}\alpha^4 + \frac{587}{212\,889\,600}\alpha^2\right)x^{11}\\
&\quad + O(x^{12}).\\
&\ \vdots
\end{aligned} \tag{10.141}$$

The constant α can then be determined by using Padé approximants as presented before.

Model II: Volterra's Population Model.

Volterra introduced a model for population growth of a species in a closed system. The model is characterized by the nonlinear integro-differential equation [8]

$$\kappa \frac{du}{dx} = u - u^2 - u\int_0^t u(x)\,dx, \quad u(0) = 0.15, \tag{10.142}$$

where κ is a prescribed parameter. To study the mathematical behavior of the scaled population of identical individuals $u(t)$, we first set

$$y(t) = \int_0^t u(x)\,dx, \tag{10.143}$$

so that

$$y'(t) = u(t), \quad y''(t) = u'(t). \tag{10.144}$$

Substituting (10.143) and (10.144) into (10.142), and using $\kappa = 0.25$ for numerical purposes we find

$$y'' = 4y' - 4(y')^2 - 4yy',\ y(0) = 0, \quad y'(0) = 0.15. \tag{10.145}$$

It is interesting to point out that the population growth model (10.142) and the related nonlinear differential equation (10.145) have been investigated by using many analytic and numerical techniques such as phase-plane and Runge-Kutta methods.

However, the decomposition method will be implemented here. Applying the inverse operator L_t^{-1} to both sides of (10.145) and using the initial conditions we obtain

$$y(t) = 0.15t + L_t^{-1}(4y' - 4(y')^2 - 4yy'). \tag{10.146}$$

Following Adomian decomposition method and proceeding as before, the series solution

$$\begin{aligned} u(t) &= 0.15 + 0.51t + 0.669t^2 + 0.1\,246t^3 - 0.85\,901t^4 - 1.186\,918t^5 \\ &\quad +0.109\,061t^6 + 1.562\,226t^7 + 1.865\,912t^8 \\ &\quad -0.162\,759t^9 - 2.791\,820t^{10} + O(t^{11}), \end{aligned} \tag{10.147}$$

is readily obtained upon using $u(t) = y'(t)$.

It is interesting to point out here that the focus of studies performed on this population model was on the phenomenon of the rapid growth of $u(t)$ to a certain peak along the logistic curve followed by the exponential decay as $t \to \infty$. As indicated before, the series solution (10.147) is not useful in isolation of other concepts. We cannot conduct the analysis to study the behavior of the solution $u(t)$ at $t = \infty$ by using the series (10.147). Consequently, the series (10.147) should be manipulated to construct several Padé approximants where the performance of the approximants show superiority over series solutions. Using computer tools we obtain the following approximants

$$[4/4] = \frac{0.15 + 0.253\,224t + 0.169\,713t^2 + 0.051\,551t^3 + 0.005\,221t^4}{1 - 1.711\,841t + 2.491\,682t^2 - 1.323\,902t^3 + 0.571\,875t^4}, \tag{10.148}$$

and

$$[5/5] = \\ \frac{0.15 - 0.126\,408t - 0.468\,575t^2 - 0.371\,927t^3 - 0.119\,542t^4 - 0.010\,943t^5}{1 - 4.242\,723t + 6.841\,424t^2 - 7.648\,48t^3 + 3.946\,155t^4 - 1.444\,76t^5}, \tag{10.149}$$

Fig. 10.1 above shows the behavior of $u(t)$ and explore the rapid growth that will reach a peak followed by a slow exponential decay. This behavior cannot be obtained if we graph the converted polynomial of the series solution (10.147).

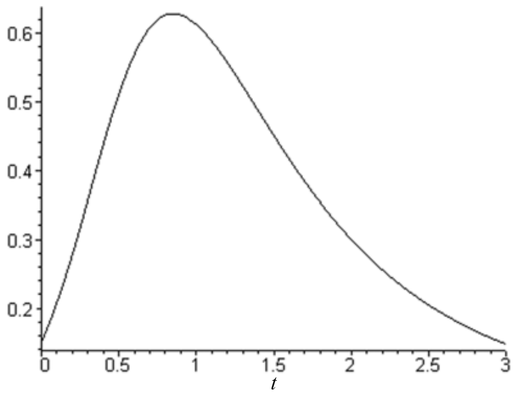

Fig. 10.1 The Padé approximant [4/4] shows the rapid growth followed by a slow exponential decay.

Model III: Thomas-Fermi Model.

The Thomas-Fermi model plays a major role in mathematical physics. The model was introduced to investigate the potentials [2,9] and charge densities of atoms having numerous electrons. The Thomas-Fermi model is characterized by the nonlinear equation

$$y'' = \frac{y^{\frac{3}{2}}}{x^{\frac{1}{2}}}, \tag{10.150}$$

subject to the boundary conditions

$$y(0) = 1, \quad \lim_{x\to\infty} y(x) = 0. \tag{10.151}$$

A considerable amount of research work has been invested in this important model. The focus of study was on obtaining an approximate solution to (10.150) and to determine a highly accurate value for the initial slope of the potential $y'(0)$.

To avoid the cumbersome work that will arise from the radical power in $y^{\frac{3}{2}}$, the modified decomposition method will be implemented. The method will facilitate our approach and will reduce the size of calculations.

The Decomposition Method

Applying L_x^{-1} to both sides of (10.150) gives

$$y(x) = 1 + Bx + L_x^{-1}\left(x^{-\frac{1}{2}} y^{\frac{3}{2}}\right), \tag{10.152}$$

where $B = y'(0)$. Using the decomposition assumption for $y(x)$ and $y^{\frac{3}{2}}$ yields

$$\sum_{n=0}^{\infty} y_n = 1 + Bx + L_x^{-1}\left(x^{-\frac{1}{2}} \sum_{n=0}^{\infty} A_n\right). \tag{10.153}$$

The modified decomposition method introduces the use of the recursive relation of the form

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= Bx + L_x^{-1}\left(x^{-\frac{1}{2}} A_0\right), \\ y_{k+1}(x) &= L_x^{-1}\left(x^{-\frac{1}{2}} A_k\right), \quad k \geqslant 1. \end{aligned} \tag{10.154}$$

This gives the first few components

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= Bx + L_x^{-1}\left(x^{-\frac{1}{2}} A_0\right) = Bx + \frac{4}{3}x^{\frac{3}{2}}, \\ y_2(x) &= L_x^{-1}\left(x^{-\frac{1}{2}} A_1\right) = \frac{2}{5}Bx^{\frac{5}{2}} + \frac{1}{3}x^3, \\ y_3(x) &= L_x^{-1}\left(x^{-\frac{1}{2}} A_2\right) = \frac{3}{70}B^2x^{\frac{7}{2}} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{\frac{9}{2}}. \end{aligned} \tag{10.155}$$

In view of (10.155), the series solution is given by

$$y(x) = 1 + Bx + \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}Bx^{\frac{5}{2}} + \frac{1}{3}x^3 + \frac{3}{70}B^2x^{\frac{7}{2}} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{\frac{9}{2}} + \cdots. \tag{10.156}$$

To achieve our goal of studying the mathematical behavior of the potential $y(x)$ and to determine the initial slope of the potential $y'(0)$, Padé approximants of different degrees should be established. To form Padé approximants it is useful to set

$$x^{\frac{1}{2}} = t, \tag{10.157}$$

into (10.156) to obtain

$$y(t) = 1 + Bt^2 + \frac{4}{3}t^3 + \frac{2}{5}Bt^5 + \frac{1}{3}t^6 + \frac{3}{70}B^2t^7 + \frac{2}{15}Bt^8 + \cdots. \tag{10.158}$$

Using any manipulation language such as Maple or Mathematica we find

$$\begin{aligned} [2/2] &= \frac{9B^2 - 12Bt + (9B^3 + 16)t^2}{9B^2 - 12Bt + 16t^2}, \\ [4/4] &= \frac{G(t)}{H(t)}, \end{aligned} \tag{10.159}$$

where

$$\begin{aligned} G(t) = {} & 27B^4+97B+\left(\frac{140}{9}+\frac{33}{10}B^3\right)t+\left(\frac{675}{28}B^5+\frac{437}{5}B^2\right)t^2 \\ & +\left(\frac{453}{14}B^4+\frac{1\,070}{9}B\right)t^3+\left(-\frac{81}{28}B^6-\frac{1\,096}{175}B^3+\frac{455}{27}\right)t^4, \end{aligned} \tag{10.160}$$

and

$$\begin{aligned} H(t) = {} & 27B^4+97B+\left(\frac{140}{9}+\frac{33}{10}B^3\right)t-\left(\frac{81}{28}B^5+\frac{48}{5}B^2\right)t^2 \\ & -\left(\frac{243}{35}B^4+26B\right)t^3-\left(\frac{186}{175}B^3+\frac{35}{9}\right)t^4. \end{aligned} \tag{10.161}$$

Other Padé approximants are also computed. To determine the initial slope $B = y'(0)$, we use the boundary condition at $t=\infty$ given by

$$\lim_{t\to\infty} y(t) = 0, \tag{10.162}$$

in the established Padé approximants. This means that we should equate the coefficient of x of highest power in the numerator of each approximant by zero. The resulting values of the initial slope $B = y'(0)$ are tabulated in Table 10.7 as shown below.

Table 10.7

Padé approximants	Initial slope B
[2/2]	−1.211 414
[4/4]	−1.550 526
[7/7]	−1.586 021

A better approximation has been obtained by evaluating more components of $y(x)$ and higher degree Padé approximants. Note that the accurate numerical solution of B is given by

$$B = y'(0) = -1.588\,071. \tag{10.163}$$

The Variational Iteration Method

The correction functional for this equation reads

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left(\frac{\partial^2 y_n(\xi)}{\partial \xi^2}+\frac{\tilde{y}_n^{\frac{3}{2}}(\xi)}{\sqrt{x}}\right)d\xi. \tag{10.164}$$

This gives

$$\lambda = \xi - x. \tag{10.165}$$

Using this value of the Lagrange multiplier into the correction functional gives the iteration formula

$$y_{n+1}(x) = y_n(x) + \int_0^x (\xi - x)\left(\frac{\partial^2 y_n(\xi)}{\partial \xi^2} + \frac{y_n^{\frac{3}{2}}(\xi)}{\sqrt{\xi}}\right) d\xi. \tag{10.166}$$

Considering the given initial values, we can select $y_0(x) = 1 + Bx$ where $B = y'(0)$. To avoid the cumbersome work that will arise from the radical power in $y^{\frac{3}{2}}$, we use the series of $y^{\frac{3}{2}}$ at every step of the calculations. This will facilitate our approach and will reduce the size of computation. Consequently, we obtain the following successive approximations

$$\begin{aligned}
y_0(x) &= 1 + Bx, \\
y_1(x) &= 1 + Bx + \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}Bx^{\frac{5}{2}} + \frac{3}{70}B^2x^{\frac{7}{2}} - \frac{1}{252}B^3x^{\frac{9}{2}} + \cdots, \\
y_2(x) &= 1 + Bx + \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}Bx^{\frac{5}{2}} + \frac{1}{3}x^3 + \frac{3}{70}B^2x^{\frac{7}{2}} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{\frac{9}{2}} + \cdots.
\end{aligned} \tag{10.167}$$

In view of this, the series solution is given by

$$y(x) = 1 + Bx + \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}Bx^{\frac{5}{2}} + \frac{1}{3}x^3 + \frac{3}{70}B^2x^{\frac{7}{2}} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{\frac{9}{2}} + \frac{4}{1\,485}Bx^{\frac{11}{2}} + \cdots. \tag{10.168}$$

To determine the potential $y'(0) = B$, we follow the same discussion introduced before where Padé approximants are used.

Model IV: Flierl-Petviashvili (FP) equation.

The Flierl-Petviashivili (FP) monopole $y(x)$ is the radially symmetric solution [5] to the equation

$$y'' + \frac{1}{x}y' - y^n - y^{n+1} = 0, \quad y(0) = \alpha, \quad y'(0) = 0,\ n \geqslant 1, \tag{10.169}$$

where the boundary condition is $y(\infty) = 0$. For $n = 1$ we obtain the standard Flierl-Petviashvili (FP) equation. The singularity behavior at $x = 0$ is a difficult element in this type of equations. A slight change in the implementation of Adomian method is necessary to overcome the singularity behavior at $x = 0$. The choice here is to define the differential operator L in terms of the two derivatives, $y'' + \frac{1}{x}y'$, contained in the problem. We first rewrite (10.169) in the form

$$Ly = g(y), \tag{10.170}$$

where $g(y) = y^n + y^{n+1}$, and the differential operator L employs the first two derivatives in the form

$$L = x^{-1}\frac{d}{dx}\left(x\frac{d}{dx}\right), \tag{10.171}$$

in order to overcome the singularity behavior the point at $x = 0$. In view of (10.171), the inverse operator L^{-1} is considered a two-fold integral operator defined by

$$L^{-1}(\cdot) = \int_0^x x^{-1} \int_0^x x^1(\cdot)\,dx\,dx. \tag{10.172}$$

Applying L^{-1} of (10.172) to the first two terms $y'' + \frac{1}{x}y'$ of Eq. (10.169) we find

$$\begin{aligned} L^{-1}\left(y'' + \frac{1}{x}y'\right) &= \int_0^x x^{-1} \int_0^x x\left(y'' + \frac{1}{x}y'\right) dx\,dx, \\ &= \int_0^x x^{-1}\left[xy' - \int_0^x y'\,dx + \int_0^x y'\,dx\right] dx, \\ &= \int_0^x y'\,dx, \\ &= y(x) - y(0), \end{aligned} \tag{10.173}$$

where integration by parts is used to integrate xy''.

Operating with L^{-1} on (10.170), it then follows

$$y(x) = \alpha + L^{-1}\left(y^n + y^{n+1}\right), \tag{10.174}$$

where

$$\alpha = y(0). \tag{10.175}$$

Recall that the FP equation

$$y'' + \frac{1}{x}y' - y^n - y^{n+1} = 0, \tag{10.176}$$

where the boundary conditions are given by

$$y(0) = \alpha, \quad y'(0) = 0, \quad y(\infty) = 0. \tag{10.177}$$

The general series solution for Eq. (10.176) is to be constructed for all possible values of $n \geqslant 1$. In an operator form, Eq. (10.176) becomes

$$Ly = y^n + y^{n+1}. \tag{10.178}$$

Recall that in this generalization $r = 1$. Applying L^{-1} to both sides of (10.178) and using the boundary condition we find

$$y(x) = \alpha + L^{-1}\left(y^n + y^{n+1}\right). \tag{10.179}$$

Using the decomposition suggestions for the linear term $y(x)$ and for the nonlinear term $y^n + y^{n+1}$ we obtain

$$\sum_{n=0}^{\infty} y_n(x) = \alpha + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{10.180}$$

The components $y_n(x)$ of the solution $y(x)$ can be elegantly computed recurrently by

$$\begin{aligned} y_0(x) &= \alpha, \\ y_{k+1}(x) &= L^{-1}(A_k), \quad k \geqslant 0. \end{aligned} \tag{10.181}$$

The first few Adomian polynomials are given by

$$\begin{aligned}
A_0 &= y_0^n + y_0^{n+1}, \\
A_1 &= n y_1 y_0^{n-1} + (n+1) y_1 y_0^n, \\
A_2 &= n y_2 y_0^{n-1} + n(n-1)\frac{y_1^2}{2!} y_0^{n-2} + (n+1) y_2 y_0^n + n(n+1)\frac{y_1^2}{2!} y_0^{n-1}, \\
A_3 &= n y_3 y_0^{n-1} + n(n-1) y_1 y_2 y_0^{n-2} + n(n-1)(n-2)\frac{y_1^3}{3!} y_0^{n-3} + n y_3 y_0^{n-1} \\
&\quad + n(n+1) y_1 y_2 y_0^{n-1} + n(n+1)(n-1)\frac{y_1^3}{3!} y_0^{n-2},
\end{aligned} \tag{10.182}$$

and so on. Inserting (10.182) into (10.181) gives

$$\begin{aligned}
y_0 &= \alpha, \\
y_1 &= \frac{(\alpha^n + \alpha^{n+1})}{4} x^2, \\
y_2 &= \frac{(\alpha^n + \alpha^{n+1})(n\alpha^n + (n+1)\alpha^{n+1})}{64\alpha} x^4, \\
y_3 &= \frac{(\alpha^n + \alpha^{n+1})\left(2n(3n-1)\alpha^{2n} + 2n(3n+1)\alpha^{2n+1} + (3n+1)(n+1)\alpha^{2n+2}\right)}{2\,304\alpha^2} x^6, \\
y_4 &= \frac{(\alpha^n + \alpha^{n+1})\left(\lambda_1 \alpha^{3n} + \lambda_2 \alpha^{3n+1} + \lambda_3 (3n\alpha^{3n+2} + (n+1)\alpha^{3n+3}\right)}{147\,456\alpha^3} x^8,
\end{aligned} \tag{10.183}$$

where

$$\begin{aligned} \lambda_1 &= n(18n^2 - 29n + 12), \\ \lambda_2 &= n(54n^2 - 33n + 7), \\ \lambda_3 &= 18n^2 + 7n + 1. \end{aligned} \tag{10.184}$$

Other components are also computed up to $O(x^{17})$, but not listed for brevity. In view of (10.183), the solution in a series form is thus given by

$$\begin{aligned} &y(x) \\ &= \alpha + \frac{(\alpha^n + \alpha^{n+1})}{4} x^2 + \frac{(\alpha^n + \alpha^{n+1})(n\alpha^n + (n+1)\alpha^{n+1})}{64\alpha} x^4 \end{aligned}$$

$$
\begin{aligned}
&+\frac{(\alpha^n+\alpha^{n+1})\left(2n(3n-1)\alpha^{2n}+2n(3n+1)\alpha^{2n+1}+(3n+1)(n+1)\alpha^{2n+2}\right)}{2\,304\alpha^2}x^6\\
&+\frac{(\alpha^n+\alpha^{n+1})\left(\lambda_1\alpha^{3n}+\lambda_2\alpha^{3n+1}+\lambda_3(3n\alpha^{3n+2}+(n+1)\alpha^{3n+3}\right)}{147\,456\alpha^3}x^8\\
&+\cdots.
\end{aligned}
\tag{10.185}
$$

The series solution (10.185) is used to obtain various Padé approximants $[m/m]$. Roots of the Padé approximants to the FP monopole α were obtained. The roots were obtained by using the limit of the Padé approximant $[m/m]$ as $x \to \infty$.

Table 10.8 Roots of the Padé approximants $[8/8]$ **monopole** α **for several values of** n.

n	$[8/8]$ Roots	n	$[8/8]$ Roots
1	$-2.392\,213\,866$	6	$-1.000\,861\,533$
2	–2.0	7	$-1.000\,708\,285$
3	$-1.848\,997\,181$	8	$-1.000\,601\,615$
4	$-1.286\,025\,892$	9	$-1.\,000\,523\,005$
5	$-1.001\,101\,141$	10	$-1.\,000\,462\,636$

Table 10.8 shows that the roots converge to -1 as n increases.

Model V: Third-order Boundary Layer Problem.

The third-order nonlinear boundary layer problem

$$
\begin{aligned}
&f''' + (n-1)ff'' - 2n(f')^2 = 0,\\
&f(0)=0,\quad f'(0)=1,\quad f''(0)=\alpha<0,\quad f'(\infty)=0,\quad n>0,
\end{aligned}
\tag{10.186}
$$

appears in boundary layers in fluid mechanics [7]. An analytic treatment will be approached to find the numerical values of $f''(0)=\alpha$ for several values of n. Equation (10.186) used by Kuiken [7] does not include the Blasius equation for a particular choice of n. Indeed, this equation is for backward boundary layers, that is, boundary layers originating at $-\infty$. It is interesting to point out that Kuiken [7] investigated this problem for three cases of n, namely for $0<n<1, n=1$ and for $n>1$.

The goal will be achieved by combining the series solution that will be obtained by the modified decomposition method with the diagonal Padé approximants as presented before. Recall that Padé approximants have the advantage of manipulating the polynomial approximation into a rational functions of polynomials. By this manipulation we gain more information about the mathematical behavior of the solution. In addition, power series are not useful for large values of x, say $x=\infty$.

To apply the modified decomposition method, we first rewrite (10.186) in an operator form

$$
Lf = -(n-1)ff'' + 2n(f')^2, \tag{10.187}
$$

where L is a third order differential operator, and hence L^{-1} is a three-fold integration operator defined by

$$L^{-1}(\cdot) = \int_0^t \int_0^t \int_0^t (\cdot)\,dt\,dt\,dt. \tag{10.188}$$

Operating with L^{-1} on both sides of (10.187) and using the initial conditions $f(0) = 0, f'(0) = 1, f''(0) = \alpha$ we obtain

$$f(x) = x + \frac{1}{2}\alpha x^2 - L^{-1}\left((n-1)ff'' - 2n(f')^2\right). \tag{10.189}$$

Recall that $f(x) = \sum_{n=0}^{\infty} f_n(x)$. The modified decomposition method suggests that the function $h(x)$ be divided into two parts

$$\begin{aligned} h(x) &= x + \frac{1}{2}\alpha x^2, \\ &= h_0(x) + h_1(x), \end{aligned} \tag{10.190}$$

where $h_0(x) = x$ is assigned to the component $f_0(x)$, and the other part $h_1(x) = \dfrac{1}{2}\alpha x^2$ is added to the definition of the component $f_1(x)$. Under this assumption, we set the modified recursive relation

$$\begin{aligned} f_0(x) &= x, \\ f_1(x) &= \frac{1}{2}\alpha x^2 - L^{-1}\left((n-1)A_0 - 2nB_0\right), \\ f_{k+2}(x) &= L^{-1}\left((n-1)A_{k+1} - 2nB_{k+1}\right), \quad k \geqslant 0. \end{aligned} \tag{10.191}$$

It is obvious that the slight variation is made on the components f_0 and f_1 only if compared with the standard Adomian method.

For convenience, we list below, few terms of the Adomian polynomials $A_n(x)$ for the nonlinear term ff'':

$$\begin{cases} A_0(x) = f_0(x)f_0''(x), \\ A_1(x) = f_0(x)f_1''(x) + f_1(x)f_0''(x), \\ A_2(x) = f_0(x)f_2''(x) + f_1(x)f_1''(x) + f_2(x)f_0''(x), \\ A_3(x) = f_0(x)f_3''(x) + f_1(x)f_2''(x) + f_2(x)f_1''(x) + f_3(x)f_0''(x), \\ A_4(x) = f_0(x)f_4''(x) + f_1(x)f_3''(x) + f_2(x)f_2''(x) + f_3(x)f_1''(x) + f_4(x)f_0''(x), \end{cases} \tag{10.192}$$

and for $B_n(x)$ for the nonlinear term $(f')^2$, we find:

$$\begin{cases} B_0(x) = (f_0')^2(x), \\ B_1(x) = 2f_0'(x)f_1'(x), \\ B_2(x) = (f_1')^2(x) + 2f_0'(x)f_2'(x), \\ B_3(x) = 2f_1'(x)f_2'(x) + 2f_0'(x)f_3'(x), \\ B_4(x) = (f_2')^2(x) + 2f_1'(x)f_3'(x) + 2f_0'(x)f_4'(x). \end{cases} \tag{10.193}$$

With f_0 defined in (10.191), this can be valuable in determining the other components. It then follows

$$\begin{aligned}
f_0(x) &= x,\\
f_1(x) &= \frac{1}{2}\alpha x^2 - L^{-1}\left((n-1)A_0 - 2nB_0\right),\\
f_2(x) &= L^{-1}\left((n-1)A_1 - 2nB_1\right),\\
f_3(x) &= L^{-1}\left((n-1)A_2 - 2nB_2\right),\\
f_4(x) &= L^{-1}\left((n-1)A_3 - 2nB_3\right),
\end{aligned} \tag{10.194}$$

and so on. This in turn gives

$$\begin{aligned}
f_0(x) &= x,\\
f_1(x) &= \frac{1}{2}\alpha x^2 + \frac{1}{3}nx^3,\\
f_2(x) &= \frac{1}{24}\alpha(3n+1)x^4 + \frac{1}{30}n(n+1)x^5,\\
f_3(x) &= \frac{1}{120}\alpha^2(3n+1)x^5 + \frac{1}{720}\alpha(19n^2+18n+3)x^6 + \frac{1}{315}n(2n^2+2n+1)x^7,\\
f_4(x) &= \frac{1}{5\,040}\alpha^2(27n^2+42n+11)x^7 + \frac{1}{40\,320}\alpha(167n^3+297n^2+161n+15)x^8\\
&\quad + \frac{1}{22\,680}n(13n^3+38n^2+23n+6)x^9.
\end{aligned} \tag{10.195}$$

Consequently we obtain

$$\begin{aligned}
f(x) &= x + \frac{\alpha x^2}{2} + \frac{nx^3}{3} + \left(\frac{1}{8}n\alpha + \frac{1}{24}\alpha\right)x^4\\
&+ \left(\frac{1}{30}n^2 + \frac{1}{40}n\alpha^2 + \frac{1}{120}\alpha^2 + \frac{1}{30}n\right)x^5 + \left(\frac{19}{720}n^2\alpha + \frac{1}{240}\alpha + \frac{1}{40}n\alpha\right)x^6\\
&+ \left(\frac{1}{120}n\alpha^2 + \frac{1}{315}n + \frac{2}{315}n^3 + \frac{11}{5\,040}\alpha^2 + \frac{3}{560}n^2\alpha^2 + \frac{2}{315}n^2\right)x^7\\
&+ \left(\frac{11}{40\,320}\alpha^3 + \frac{33}{4\,480}n^2\alpha + \frac{3}{4\,480}\alpha^3 n^2 + \frac{23}{5\,760}n\alpha + \frac{1}{2\,688}\alpha\right.\\
&\left.+ \frac{167}{40\,320}n^3\alpha + \frac{1}{960}\alpha^3 n\right)x^8\\
&+ \left(\frac{1}{3\,780}n + \frac{527}{362\,880}n^3\alpha^2 + \frac{19}{11\,340}n^3 + \frac{709}{362\,880}n\alpha^2 + \frac{23}{8\,064}n^2\alpha^2 + \frac{23}{22\,680}n^2\right.\\
&\left.+ \frac{13}{22\,680}n^4 + \frac{43}{120\,960}\alpha^2\right)x^9 + \cdots.
\end{aligned}$$

As stated before, Kuiken [7] studied the behavior of $f(x)$ for three specific cases, namely, for $0 < n < 1$, $n = 1$, and for $n > 1$. To study the mathematical behavior of the $f(x)$, it is normal to derive approximations for $f''(0) = \alpha < 0$ for the three

Table 10.9 Numerical values for $\alpha = f''(0)$ for $0 < n < 1$ by using Padé approximants.

n	[2/2]	[3/3]	[4/4]	[5/5]	[6/6]
0.2	−0.387 298 33	−0.382 153 38	−0.381 915 384	−0.381 914 808	−0.381 912 185
$\frac{1}{3}$	−0.577 350 27	−0.561 599 92	−0.561 406 658	−0.561 448 140	−0.561 449 193
0.4	−0.645 150 64	−0.639 100 06	−0.638 973 26	−0.638 989 268	−0.638 973 479
0.6	−0.840 796 16	−0.839 360 30	−0.839 606 047	−0.839 587 538	−0.839 605 677
0.8	−1.007 983 20	−1.007 796 98	−1.007 646 82	−1.007 646 83	−1.007 792 10

prescribed cases of n. This goal can be achieved by forming Padé approximants which will converge on the entire real axis if $f(x)$ is free of singularities on the real axis.

Using the boundary condition $f'(\infty) = 0$, the diagonal approximant [M/M] vanish if the coefficient of x with the highest power in the numerator vanishes. Using the Maple built-in utilities to solve the resulting polynomials gives the values of the initial slope $f''(0) = \alpha$ listed in Table 10.9.

Table 10.9 confirms the first two conclusions made in [7]. For $0 < n < 1$, the numerical value of α converges to $-1.007\ 792\ 1$. For $n = \frac{1}{3}$, the numerical value of α converges to $-0.561\ 449\ 193\ 4$. It is interesting to point out that Kuiken [7] examined the specific case where $n = \frac{1}{3}$. For this value of n, Eq. (10.186) can be reduced to

$$f' = \frac{1}{3}f^2 - \alpha x + 1, \tag{10.196}$$

upon substituting $n = \frac{1}{3}$ and integrating the resulting equation twice. In [1], the general solution for this case was derived as

$$f(x) = -3\left(\frac{\alpha^2}{9}\right)^{\frac{1}{6}} \frac{\mathrm{Ai}'}{\mathrm{Ai}}. \tag{10.197}$$

where Ai is Airy function. Using this exact solution, it was found that

$$\alpha = -0.561\ 449\ 193\ 46, \tag{10.198}$$

where our result obtained above in Table 10.9 is consistent with this result.

For $n = 1$, and using the average of all five diagonal Padé approximants, we find that

$$\alpha = -1.154\,948\,004. \tag{10.199}$$

For the third case where $n > 1$, and using the diagonal Padé approximants [2/2]–[6/6] we obtained the following results:
The results shown above confirms the exponential decay for $f(x)$, for $n > 1$ as formally derived in [7]. In closing, we confirm the fact that the function $f(x)$ decays

Table 10.10 Numerical values for $\alpha = f''(0)$ for $n > 1$ by using Padé approximants.

n	4	10	100	1 000	5 000
α	$-2.483\,954\,032$	$-4.026\,385\,103$	$-12.843\,343\,15$	$-40.655\,382\,18$	$-104.842\,067\,2$

algebraically for the case where $0 < n < 1$, and decays exponentially for the case where $n > 1$ when x tends to infinity.

Exercises 10.5

Use the Adomian decomposition method and the Padé approximants to study the following models:

1. $y'' = 10y' - 10(y')^2 - 10yy', y(0) = 0, y'(0) = 0.2$
2. $y'' = 5y' - 5(y')^2 - 5yy', y(0) = 0, y'(0) = 0.1$
3. Find the constant α in the Flierl-Petviashvili equation:

$$y_{xx} + \frac{1}{x}y_x - y - y^2 = 0, y(0) = \alpha, y'(0) = 0, \lim_{x\to\infty} y(x) = 0$$

References

1. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, (1994).
2. W.F. Ames, Nonlinear Partial Differential Equations in Engineering, Vol I, Academic Press, New York, (1965).
3. G.A. Baker, Essentials of Padé Approximants, Academic Press, London, (1975).
4. G.A. Baker and P. Graves-Morris, Essentials of Padé Approximants, Cambridge University Press, Cambridge, (1996).
5. J. Boyd, Padé approximant algorithm for solving nonlinear ordinary differential equation boundary value problems on an unbounded domain, *Computers in Physics*, 11(3), 299 – 303, (1997).
6. J.H. He, Some asymptotic methods for strongly nonlinearly equations, *Internat. J. of Modern Math.*, 20(10), 1141–1199, (2006).
7. H.K. Kuiken, On boundary layers in field mechanics that decay algebraically along stretches of wall that are not vanishing small, *IMA J. Applied Math.*, 27, 387–405, (1981).
8. A.M. Wazwaz, Analytical approximations and Padé approximants for Volterra's population model, *Appl. Math. and Comput.*, 100, 31–25, (1999).
9. A.M. Wazwaz, The modified decomposition method and Padé approximants for solving Thomas-Fermi equation, *Appl. Math. and Comput.*, 105, 11–19, (1999).
10. A.M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema, Leiden, (2002).

Chapter 11
Solitons and Compactons

11.1 Introduction

In 1834, John Scott Russell was the first to observe the solitary waves. He observed a large protrusion of water slowly traveling on the Edinburgh-Glasgow canal without change in shape. The bulge of water, that he observed and called "great wave of translation", was traveling along the channel of water for a long period of time while still retaining its shape. The remarkable discovery motivated Russell to conduct physical laboratory experiments to emphasize his observance and to study these solitary waves. He empirically derived the relation

$$c^2 = g(h+a), \tag{11.1}$$

that determines the speed c of the solitary wave, where a is the maximum amplitude above the water surface, h is the finite depth and g is the acceleration of gravity. The solitary waves are therfore called gravity waves.

The discovery of solitary waves inspired scientists to conduct a huge size of research work to study this concept. Two Dutchmen Korteweg and deVries derived a nonlinear partial differential equation, well known by the KdV equation, to model the height of the surface of shallow water in the presence of solitary waves [11]. The KdV equation also describes the propagation of plasma waves in a dispersive medium. The KdV equation was introduced before in Chapter 8 where it was handled in a traditional way. The KdV equation in its simplest form is given by

$$u_t + auu_x + u_{xxx} = 0, \tag{11.2}$$

where it indicates that dispersion and nonlinearity might occur. The solitary wave solutions are assumed to be of the form

$$u(x,t) = f(x-ct), \tag{11.3}$$

where c is the speed of the wave propagation, and $f(z), f'(z), f''(z) \to 0$ as $z \to \pm\infty$, $z = x - ct$.

In 1965, Zabusky and Kruskal [11] investigated the interaction of solitary waves and the recurrence of initial states. They discovered that solitary waves undergo nonlinear interaction following the KdV equation. Further, the waves emerge from this interaction retaining its shape and amplitude. The remarkable discovery, that solitary waves retain its identities and that its character resembles particle like behavior, motivated Zakusky and Kruskal [11] to call these solitary waves *solitons*. Zakusky and Kruskal marked the birth of the soliton, a name intended to signify particle like quantities. The interaction of two solitons emphasized the reality of the preservation of shapes and speeds and of the steady pulse like character of solitons.

A great deal of research work has been invested in recent years for the study of the soliton concept. Hirota [5,6] constructed the N-soliton solutions of the evolution equation by reducing it to the bilinear form. The bilinear formalism established by Hirota [5,6], and used by many such as in [3,4,7], was a very helpful tool in the study of the nonlinear equations. Nimmo and Freeman [9] introduced an alternative formulation of the N-soliton solutions in terms of some function of the Wronskian determinant of N functions.

Active research works have emerged worldwide in a diverse branches of scientific fields to study the soliton concept. It is now well known that solitons appear as a result of a balance between weak nonlinearity and dispersion. The soliton concept has attracted a huge size of studies due to its significant role in various scientific fields such as fluid dynamics, astrophysics, plasma physics, and magneto-acoustic waves and many others.

As will be seen in coming chapters, solitary waves appear in a variety of types, such as solitons, kinks, peakons, cuspons and other forms. each of these types has its own features.

Recently, in 1993, Rosenau and Hyman [10] discovered a class of solitary waves with compact support that are termed compactons. Compactons are defined by solitary waves with the remarkable soliton property that after colliding with other compactons, they reemerge with the same coherent shape. These particle like waves exhibit elastic collision that are similar to the soliton interaction associated with completely integrable PDEs supporting an infinite number of conservation laws.

It was found in [10] that when the wave dispersion is purely nonlinear, some novel features may be observed and the most remarkable one is the existence of the so called compactons. The definitions given so far for compactons are:
(i) compactons are solitons with finite wavelength;
(ii) compactons are solitary waves with compact support;
(iii) compactons are solitons free of exponential tails;
(iv) compactons are solitons characterized by the absence of infinite wings;
(v) compactons are robust soliton like solutions.

Two important features of compactons structure are observed, namely:
(i) unlike the standard KdV soliton where $f(z) \to 0$ as $z \to \infty$, the compacton is characterized by the absence of the exponential tails or wings, where $f(z)$ does not tend to 0 as $z \to \infty$;
(ii) unlike the standard KdV soliton where width narrows as the amplitude increases, the width of the compacton is independent of the amplitude.

The role of nonlinear dispersion in the formation of patterns in liquid drops was investigated by Rosenau and Hyman. The study in [10] was carried out by considering a genuinely nonlinear dispersive equation K(n,n), a special type of the KdV equation, that was subjected to experimental and analytical studies. The remarkable discovery by Rosenau and Hyman [10] is that solitary waves may compactify under the influence of nonlinear dispersion which is capable of causing deep qualitative changes in the nature of genuinely nonlinear phenomena. It was shown that certain solutions of the K(n,n) equation characterized by the absence of infinite wings can be constructed, and termed compactons. The derived results are new and of substantial interest.

The genuinely nonlinear dispersive K(n,n) equations, a family of nonlinear KdV like equations is of the form

$$u_t + a(u^n)_x + (u^n)_{xx} = 0, \quad a > 0, n > 1, \tag{11.4}$$

which supports compact solitary traveling structures for $a > 0$. The existence and stability of the compact entities was examined in [10].

It is important to note that Eq. (11.4) with $+a$ is called the focusing branch, whereas equation of the form

$$u_t - a(u^n)_x + (u^n)_{xx} = 0, \quad a > 0, n > 1, \tag{11.5}$$

is called the defocusing branch. The studies revealed that Eq. (11.5) supports solutions with solitary patterns having cusps or infinite slopes. Further, it was shown that while compactons are the essence of the focusing branch $(+a)$, spikes, peaks and cusps are the hallmark of the defocusing branch $(-a)$. This in turn means that the focusing branch (11.4) and the defocusing branch (11.5) represent two different models, each leading to a different physical structure. The remarkable discovery of compactons has led, in turn, to an intense study over the last few years. The study of compactons may give insight into many scientific processes [10] such as the super deformed nuclei, preformation of cluster in hydrodynamic models, the fission of liquid drops and inertial fusion.

For more details about compactons, see [10] and the references therein.

11.2 Solitons

In this section we will study the solitary wave solutions of some of the well known nonlinear equations that exhibit solitons. It is interesting to point out that there is no precise definition of a soliton. However, a soliton can be defined as a solution of a nonlinear partial differential equation that exhibits the following properties:
(i) the solution should demonstrate a wave of permanent form;
(ii) the solution is localized, which means that the solution either decays exponentially to zero such as the solitons provided by the KdV equation, or converges to a

constant at infinity such as the solitons given by the Sine-Gordon equation; (iii) the soliton interacts with other solitons preserving its character [3,4,7].

One basic expression of a solitary wave solution is of the form

$$u(x,t) = f(x - ct), \tag{11.6}$$

where c is the speed of wave propagation. For $c > 0$, the wave moves in the positive x direction, whereas the wave moves in the negative x direction for $c < 0$. More importantly, as will be seen later, the solutions of nonlinear equations may be a sech^2, sech, or $\arctan(e^{\alpha(x-ct)})$ function. Different methods were developed to obtain solitons. The inverse scattering transform method [1] and the bilinear formalism were developed and implemented in a huge size of research works. However, in this section we will use the direct substitution of the standard formula (11.6) and solve the obtained ordinary differential equation or by using Adomian decomposition method [2] if initial condition is given. In what follows, some of the well known nonlinear equations will be studied.

11.2.1 The KdV Equation

The nonlinear dispersive equation formulated by Korteweg and de Vries (KdV) in its simplest form [3] is given by

$$u_t - 6uu_x + u_{xxx} = 0, \tag{11.7}$$

with $u = u(x,t)$ is a differentiable function. We shall assume that the solution $u(x,t)$, along with its derivatives, tends to zero as $| x | \to \infty$.

Several different approaches, such as Bäcklund transformation, a bilinear form, and a Lax pair have been used independently by which soliton and multi-soliton solutions for nonlinear evolution equations are obtained.

As mentioned before, solitary wave solution can be written as

$$u(x,t) = f(x - ct), \tag{11.8}$$

where c is the soliton speed. Using (11.8) into (11.7) gives

$$-cf' - 6ff' + f''' = 0, \quad z = x - ct. \tag{11.9}$$

Integrating (11.9) gives

$$-cf - 3f^2 + f'' = 0, \tag{11.10}$$

where constant of integration is taken to be zero. Multiplying (11.10) by $2f'$ and integrating the resulting equation we find

$$(f')^2 = cf^2 + 2f^3, \tag{11.11}$$

an ordinary differential equation with explicit solution

$$f(z) = -\frac{c}{2}\,\mathrm{sech}^2 \frac{\sqrt{c}}{2} z. \tag{11.12}$$

Combining (11.12) and (11.8) gives

$$u(x,t) = -\frac{c}{2}\,\mathrm{sech}^2 \frac{\sqrt{c}}{2}(x-ct). \tag{11.13}$$

It is obvious that $f(x,t)$ in (11.12), along with its derivatives, tends to zero as $|\, x \,| \to \infty$.

Solving the KdV Equation by Adomian Method

In the following, Adomian decomposition method will be implemented to obtain a solitary wave solution for the KdV equation

$$\begin{aligned} u_t - 6uu_x + u_{xxx} &= 0, \\ u(x,0) &= -2\frac{k^2 e^{kx}}{(1+e^{kx})^2}, \end{aligned} \tag{11.14}$$

where $c = k^2$. Applying the inverse operator L_t^{-1} on both sides of (11.14) and using the decomposition series for $u(x,t)$ yields

$$\sum_{n=0}^{\infty} u_n(x,t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} + L_t^{-1}\left(6\,(\sum_{n=0}^{\infty} A_n) - (\sum_{n=0}^{\infty} u_n)_{xxx}\right). \tag{11.15}$$

Proceeding as before, Adomian decomposition method gives the recurrence relation

$$\begin{aligned} u_0(x,t) &= -2\frac{k^2 e^{kx}}{(1+e^{kx})^2}, \\ u_{k+1}(x,t) &= L_t^{-1}\left(6A_k - u_{k_{xxx}}\right), \quad k \geqslant 0. \end{aligned} \tag{11.16}$$

This in turn gives

$$\begin{aligned} u_0(x,t) &= -2\frac{k^2 e^{kx}}{(1+e^{kx})^2}, \\ u_1(x,t) &= L_t^{-1}\left(6A_0 - u_{0_{xxx}}\right) = -2\frac{k^5 e^{kx}(e^{kx}-1)}{(1+e^{kx})^3}\,t, \\ u_2(x,t) &= L_t^{-1}\left(6A_1 - u_{1_{xxx}}\right) = -\frac{k^8 e^{kx}(e^{2kx}-4e^{kx}+1)}{(1+e^{kx})^4}\,t^2. \end{aligned} \tag{11.17}$$

In view of (11.17), the solution in a series form is given by

$$u(x,t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} - 2\frac{k^5 e^{kx}(e^{kx}-1)}{(1+e^{kx})^3}t - \frac{k^8 e^{kx}(e^{2kx}-4e^{kx}+1)}{(1+e^{kx})^4}t^2 + \cdots, \tag{11.18}$$

so that the exact solution

$$u(x,t) = -\frac{c}{2}\operatorname{sech}^2 \frac{\sqrt{c}}{2}(x-ct) \tag{11.19}$$

is readily obtained.

It is worth noting that another form of the KdV equation given by

$$u_t + 6uu_x + u_{xxx} = 0 \tag{11.20}$$

can be proved to have the solitary wave solution

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2 \frac{\sqrt{c}}{2}(x-ct). \tag{11.21}$$

11.2.2 The Modified KdV Equation

We next consider the modified KdV (mKdV) equation [3,7] of the form

$$\begin{aligned} u_t + 6u^2 u_x + u_{xxx} &= 0, \\ u(x,0) &= g(x). \end{aligned} \tag{11.22}$$

We shall assume that the solution $u(x,t)$, along with its derivatives, tends to zero as $|\, x \,| \to \infty$.

Following the discussions made above, we look for a traveling wave solution in the form

$$u(x,t) = f(x-ct), \tag{11.23}$$

where c is the soliton speed, $z = x - ct, f(z), f'(z)$ and $f''(z)$ tend to 0 as $|\, x \,| \to \infty$.

Substituting (11.23) into (11.22) gives

$$-cf' + 6f^2 f' + f''' = 0. \tag{11.24}$$

Integrating (11.24) gives

$$-cf + 2f^3 + f'' = 0, \tag{11.25}$$

or equivalently

$$f'' = cf - 2f^3, \tag{11.26}$$

where we assumed that the constant of integration is zero. The exact solution of Eq. (11.22)

$$f(z) = \pm\sqrt{c}\operatorname{sech}\sqrt{c}z, \tag{11.27}$$

that gives

$$u(x,t) = \pm\sqrt{c}\,\mathrm{sech}\sqrt{c}\,(x-ct). \tag{11.28}$$

The intermediate calculations between (11.26) and (11.27) are left as an exercise. It is interesting to point out that the KdV equation has the soliton solution in terms of sech^2 function, whereas the solution of the mKdV equation is in terms of sech function [3,7].

Solving mKdV Equation by Adomian Method

In the following, the decomposition method will be applied for the modified KdV equation defined by

$$\begin{aligned} u_t + 6u^2u_x + u_{xxx} &= 0, \\ u(x,0) &= 2\frac{ke^{kx}}{1+e^{2kx}}. \end{aligned} \tag{11.29}$$

with $u = u(x,t)$ is a sufficiently-often differentiable function.

Operating with L_t^{-1} yields

$$\sum_{n=0}^{\infty} u_n(x,t) = 2\frac{ke^{kx}}{1+e^{2kx}} - L_t^{-1}\left(6\,(\sum_{n=0}^{\infty} A_n) + (\sum_{n=0}^{\infty} u_n)_{xxx}\right). \tag{11.30}$$

Adomian's method admits the use of the recurrence relation

$$\begin{aligned} u_0(x,t) &= 2\frac{ke^{kx}}{1+e^{2kx}}, \\ u_{k+1}(x,t) &= -L_t^{-1}\left(6A_k + u_{k_{xxx}}\right), \quad k \geqslant 0, \end{aligned} \tag{11.31}$$

that in turn gives

$$\begin{aligned} u_0(x,t) &= 2\frac{ke^{kx}}{1+e^{2kx}}, \\ u_1(x,t) &= -L_t^{-1}\left(6A_0 + u_{0_{xxx}}\right) = -2\frac{k^4e^{kx}(1-e^{2kx})}{(1+e^{2kx})^2}\,t, \\ u_2(x,t) &= -L_t^{-1}\left(6A_1 + u_{1_{xxx}}\right) = \frac{k^7e^{kx}(1-6e^{2kx}+e^{4kx})}{(1+e^{2kx})^3}\,t^2. \end{aligned} \tag{11.32}$$

The solution in a series form is given by

$$u(x,t) = 2\frac{ke^{kx}}{1+e^{2kx}} - 2\frac{k^4e^{kx}(1-e^{2kx})}{(1+e^{2kx})^2}\,t + \frac{k^7e^{kx}(1-6e^{2kx}+e^{4kx})}{(1+e^{2kx})^3}\,t^2 + \cdots, \tag{11.33}$$

so that the exact solution

$$u(x,t) = \pm\sqrt{c}\,\mathrm{sech}\sqrt{c}\,(x-ct), \tag{11.34}$$

is readily obtained noting that $c = k^2$.

11.2.3 The Generalized KdV Equation

A generalized form of the KdV equation [3] of the form

$$\begin{aligned} u_t + (n+1)(n+2)u^n u_x + u_{xxx} &= 0, \quad n = 1, 2, \cdots \\ u(x,0) &= g(x), \end{aligned} \tag{11.35}$$

will be investigated. Substituting (11.23) into (11.35) yields a differential equation for $f(z)$

$$-cf' + (n+1)(n+2)f^n f' + f''' = 0. \tag{11.36}$$

Integrating (11.36) gives

$$-cf + (n+2)f^{n+1} + f'' = 0, \tag{11.37}$$

so that

$$f''(z) = cf - (n+2)f^{n+1}, \tag{11.38}$$

with exact solution

$$f(z) = \left(\frac{1}{2}c\,\mathrm{sech}^2(\frac{1}{2}n\sqrt{c}\,z)\right)^{\frac{1}{n}}. \tag{11.39}$$

Combining (11.39) and (11.23) gives

$$u(x,t) = \left(\frac{1}{2}c\,\mathrm{sech}^2(\frac{1}{2}n\sqrt{c}\,(x-ct))\right)^{\frac{1}{n}}. \tag{11.40}$$

11.2.4 The Sine-Gordon Equation

The sine-Gordon equation [4,5] is given by

$$u_{tt} - u_{xx} + \sin u = 0. \tag{11.41}$$

The sine-Gordon equation arises in the study of superconductor transmission lines, crystals, geometry of surfaces, laser pulses, pendular motions, and in the propagation of magnetic flux.

To determine solitary wave solutions of Eq. (11.41) we let

$$u(x,t) = f(x-ct), \tag{11.42}$$

that carries the Sine-Gordon equation into

$$(c^2-1)f''f' + (\sin f)f' = 0, \tag{11.43}$$

obtained after multiplying it by f', noting that $z = x - ct$. Integrating (11.43) gives the first order equation

$$\frac{1}{2}(c^2-1)(f')^2-\cos f=C, \tag{11.44}$$

where C is a constant of integration. It may be shown that by choosing $C=-1$ will result that f will approach zero as z approaches infinity. This means that Eq. (11.44) becomes

$$(f')^2=\frac{4}{1-c^2}\sin^2(\frac{f}{2}), \quad |c|<1. \tag{11.45}$$

We can easily prove that one solution of Eq. (11.45) is given by

$$f(z)=4\arctan\left[\exp\left(-\frac{z}{\sqrt{1-c^2}}\right)\right], \tag{11.46}$$

so that the solitary wave solution is

$$u(x,t)=4\arctan\left[\exp\left(-\frac{x-ct}{\sqrt{1-c^2}}\right)\right]. \tag{11.47}$$

Recall that the solitary wave solutions of the KdV and the modified KdV equations are given by sech^2 and sech functions respectively. The solution obtained in (11.47) shows that the solitary wave solution in terms of $\arctan(e^{\alpha z})$. Moreover, we can easily observe that $u(x,t)\to 0$ as $x\to\infty$, and $u(x,t)\to 2\pi$ as $x\to-\infty$. A solution for which $u(x,t)$ increases by 2π is called a *kink*, and one which decreases by 2π an *antikink*. Another solution of the Sine-Gordon equation can be derived in the form

$$u(x,t)=4\arctan\left[\frac{\sinh(\frac{ct}{\sqrt{1-c^2}})}{c\cosh(\frac{x}{\sqrt{1-c^2}})}\right], \quad c^2<1. \tag{11.48}$$

11.2.5 The Boussinesq Equation

A well known model of nonlinear dispersive waves was proposed by Boussinesq in the form

$$u_{tt}=u_{xx}+3(u^2)_{xx}+u_{xxxx}, \quad a\leqslant x\leqslant b. \tag{11.49}$$

The Boussinesq equation (11.1) describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice. Using $u=f(z), z=(x-ct)$ into (11.49) gives

$$(c^2-1)f''=3(f^2)''+f^{(4)}, \tag{11.50}$$

where integrating twice yields

$$(c^2-1)f-3f^2=f''. \tag{11.51}$$

Multiplying both sides of (11.51) by $2f'$ and integrating gives

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}x + \frac{\sqrt{c}}{2}\sqrt{1+ct}\right]. \tag{11.52}$$

The exact solution can also be written as

$$u(x,t) = \frac{1}{2}(c^2-1)\operatorname{sech}^2\left[\frac{\sqrt{c^2-1}}{2}(x-ct)\right]. \tag{11.53}$$

The intermediate calculations between (11.51) and (11.53) are left an exercise.

Solving Boussinesq Equation by the Modified Adomian Method

In what follows we will use the modified decomposition method to determine the solitary wave solutions of a specific form of the Boussinesq equation, defined by

$$\begin{aligned} u_{tt} &= u_{xx} + 3(u^2)_{xx} + u_{xxxx}, \quad -80 \leqslant x \leqslant 80, \\ u(x,0) &= 2\frac{ak^2e^{kx}}{(1+ae^{kx})^2}, \quad u_t(x,0) = -2\frac{ak^3\sqrt{1+k^2}e^{kx}(ae^{kx}-1)}{(1+ae^{kx})^3}. \end{aligned} \tag{11.54}$$

Applying the inverse operator L_t^{-1} yields

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) = 2\frac{ak^2e^{kx}}{(1+ae^{kx})^2} - 2\frac{ak^3\sqrt{1+k^2}e^{kx}(ae^{kx}-1)}{(1+ae^{kx})^3}t \\ + L_t^{-1}\left(3\,(\sum_{n=0}^{\infty} A_n) + (\sum_{n=0}^{\infty} u_n)_{xx} + (\sum_{n=0}^{\infty} u_n)_{xxxx}\right). \end{aligned} \tag{11.55}$$

The modified decomposition method gives the recurrence relation

$$\begin{aligned} u_0(x,t) &= 2\frac{ak^2e^{kx}}{(1+ae^{kx})^2}, \\ u_1(x,t) &= -2\frac{ak^3\sqrt{1+k^2}e^{kx}(ae^{kx}-1)}{(1+ae^{kx})^3}t + L_t^{-1}\,(3A_0 + u_{0_{xx}} + u_{0_{xxxx}}), \\ u_{k+1}(x,t) &= L_t^{-1}\,(3A_k + u_{k_{xx}} + u_{k_{xxxx}}), \quad k \geqslant 1. \end{aligned} \tag{11.56}$$

Consequently, we obtain

$$\begin{aligned} u_0(x,t) &= 2\frac{ak^2e^{kx}}{(1+ae^{kx})^2}, \\ u_1(x,t) &= -2\frac{ak^3\sqrt{1+k^2}e^{kx}(ae^{kx}x-1)}{(1+ae^{kx})^3}t + L_t^{-1}\,(3A_0 + u_{0_{xx}} + u_{0_{xxxx}}) \end{aligned}$$

$$= -2\frac{ak^3\sqrt{1+k^2}e^{kx}(ae^{kx}-1)}{(1+ae^{kx})^3}t + \frac{ak^4e^{kx}(1+k^2)(a^2e^{2kx}-4ae^{kx}+1)}{(1+ae^{kx})^4}t^2, \tag{11.57}$$

where u_2 and u_3 are determined but not listed. In view of (11.57), the solution in a series form is given by

$$\begin{aligned} u(x,t) = {} & 2\frac{ak^2e^{kx}}{(1+ae^{kx})^2} - 2ak^3\sqrt{1+k^2}e^{kx}\frac{(ae^{kx}-1)}{(1+ae^{kx})^3}t \\ & +ak^4(1+k^2)e^{kx}\frac{(a^2e^{2kx}-4ae^{kx}+1)}{(1+ae^{kx})^4}t^2+\cdots, \end{aligned} \tag{11.58}$$

and consequently, we find that the exact solution is

$$u(x,t) = 2\frac{ak^2e^{kx+k\sqrt{1+k^2}\,t}}{(1+ae^{kx+k\sqrt{1+k^2}\,t})^2}, \tag{11.59}$$

or equivalently

$$u(x,t) = \frac{ak^2}{2}\operatorname{sech}^2\left[\frac{k\sqrt{a}x}{2}+\frac{k\sqrt{a}}{2}\sqrt{1+ak^2}\,t\right], \tag{11.60}$$

where $c = ak^2$.

11.2.6 The Kadomtsev-Petviashvili Equation

In 1970, Kadomtsev and Petviashvili generalized the KdV equation to two space variables and formulated the well-known Kadomtsev-Petviashvili equation [8] to provide an explanation of the general weakly dispersive waves. The KP equation is used to model shallow-water waves with weakly non-linear restoring forces. It is a natural generalization of the KdV equation and it gives multiple soliton solutions as will be discussed later.

The KP equation is of the form

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{11.61}$$

Adomian's method will be used to solve the specific KP equation

$$u_{xt} - 6u_x^2 - 6uu_{xx} + u_{xxxx} + 3u_{yy} = 0, \tag{11.62}$$

with the initial condition

$$u(x,y,0) = \frac{-8e^{2x+2y}}{(1+e^{2x+2y})^2}, \tag{11.63}$$

and the boundary conditions are zero at the boundary. Operating with the inverse operator L_{xt}^{-1}

$$L_{xt}^{-1}(\cdot) = \int_0^x\int_0^t(\cdot)dt\,dx, \tag{11.64}$$

gives the relation

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,y,t) &= \frac{-8e^{2x+2y}}{(1+e^{2x+2y})^2} \\ &+L_{xt}^{-1}\left(6\,(\sum_{n=0}^{\infty} A_n)+6\,(\sum_{n=0}^{\infty} B_n)-(\sum_{n=0}^{\infty} u_n)_{xxxx}-3(\sum_{n=0}^{\infty} u_n)_{yy}\right),\end{aligned} \tag{11.65}$$

where A_n and B_n are Adomian polynomials for u_x^2 and uu_{xx} respectively. This means that the first few components are derived as follows:

$$\begin{aligned}u_0(x,y,t) &= -\frac{8e^{2x+2y}}{(1+e^{2x+2y})^2}, \\ u_1(x,y,t) &= L_{xt}^{-1}\,(6A_0+6B_0-(u_0)_{xxxx}-3\,(u_0)_{yy}) \\ &= -112\left(\frac{(-1+e^{2x+2y})e^{2x+2y}}{(1+e^{2x+2y})^3}+112\frac{(-1+e^{-160+2y})e^{-160+2y}}{(1+e^{-160+2y})^3}\right)t.\end{aligned} \tag{11.66}$$

The solution in series form is given by

$$\begin{aligned}u(x,y,t) &= \frac{-8e^{2x+2y}}{(1+e^{2x+2y})^2} \\ &-112\left(\frac{(-1+e^{2x+2y})e^{2x+2y}}{(1+e^{2x+2y})^3}+112\frac{(-1+e^{-160+2y})e^{-160+2y}}{(1+e^{-160+2y})^3}\right)t+\cdots,\end{aligned} \tag{11.67}$$

so that the exact single soliton solution is given by

$$u(x,y,t) = -2\mathrm{sech}^2(x+y-7t), \tag{11.68}$$

or equivalently

$$u(x,y,t) = -\frac{8e^{2x+2y-14t}}{(1+e^{2x+2y-14t})^2}. \tag{11.69}$$

Exercises 11.2

Use the decomposition method or the substitution $u = f(x-ct)$ to find the solitary wave solutions of the following nonlinear problems:

1. $u_t - 6uu_x + u_{xxx} = 0,\, u(x,0) = -2\,\mathrm{sech}^2(x)$
2. $u_t + 6uu_x + u_{xxx} = 0,\, u(x,0) = 8\,\mathrm{sech}^2(2x)$
3. $u_t + 12u^2u_x + u_{xxx} = 0,\, u(x,0) = \sqrt{2}\,\mathrm{sech}(2x)$
4. $u_{tt} - u_{xx} + \sin u = 0,\, u(x,0) = 4\arctan(e^{-2x})$

5. $(u_t+6uu_x+u_{xxx})_x+3u_{yy}=0,\ u(x,y,0)=\frac{1}{2}\operatorname{sech}^2(\frac{1}{2}(x+y))$

11.3 Compactons

In 1993, Rosenau and Hyman [10] introduced a class of solitary waves with compact support that are termed *compactons*. Compactons can be defined as solitons with finite wave length or solitons free of exponential tails. In other words, compactons are solitons characterized by the absence of infinite wings and, unlike solitons, the width of the compacton is independent of the amplitude. Rosenau and Hyman discovered that solitary waves may compactify under the influence of nonlinear dispersion which is capable of causing deep qualitative changes in the nature of genuinely nonlinear phenomena. Compactons were proved to collide elastically and reemerge with the same coherent shape. Such solitary wave solutions, which vanish outside a finite core region, are solutions of a two parameter family of genuinely nonlinear dispersive equations $K(n,n)$:

$$u_t+(u^n)_x+(u^n)_{xxx}=0,\quad n>1. \tag{11.70}$$

As stated before, solitons appear as a result of a balance between dispersion and weak nonlinearity. However, when the wave dispersion is purely nonlinear, some novel features may be observed. The most interesting feature of the nonlinear dispersion is the existence of the so-called compactons: solitons with finite wavelength or solitons without exponential tails.

Unlike solitons, compactons are nonanalytic solutions. The points of non analyticity at the compacton edge are related to points of genuine nonlinearity of the equation. In addition, it was shown that in [10] that the inverse scattering tools are inapplicable. The pseudo spectral method was used to obtain the compactons solutions:

$$u(x,t)=\begin{cases}\{\sqrt{\dfrac{2cn}{n+1}}\cos[\dfrac{n-1}{2n}(x-ct)]\}^{\frac{2}{n-1}}, & |x-ct|\leqslant\dfrac{n\pi}{(n-1)},\quad n>1,\\ 0, & \text{otherwise}.\end{cases} \tag{11.71}$$

However, an additional general formula for the compactons solutions was derived in the form

$$u(x,t)=\begin{cases}\{\sqrt{\dfrac{2cn}{n+1}}\sin[\dfrac{n-1}{2n}(x-ct)]\}^{\frac{2}{n-1}}, & |x-ct|\leqslant\dfrac{2n\pi}{(n-1)},\ n>1,\\ 0, & \text{otherwise}.\end{cases} \tag{11.72}$$

The One Dimensional Focusing Branch

Consider the nonlinear dispersive equation

$$u_t + a(u^n)_x + b(u^n)_{xxx} = 0, \quad a,b > 0. \tag{11.73}$$

Following the discussions in [10], we assume that the general solution of Eq. (11.73) is of the form

$$u(x,t) = \rho \sin^{\frac{2}{n-1}}[\sigma(x-ct)], \tag{11.74}$$

or of the form

$$u(x,t) = \rho \cos^{\frac{2}{n-1}}[\sigma(x-ct)], \tag{11.75}$$

where ρ and σ are constants that will be determined. Substituting these assumptions into (11.73) and by solving the resulting equations for ρ and σ we find

$$\begin{aligned} \sigma &= \pm\frac{(n-1)}{2n}\sqrt{\frac{a}{b}}, \\ \rho &= \begin{cases} \left(\dfrac{2nc}{a(n+1)}\right)^{\frac{1}{n-1}}, & n \text{ is even}, \\ \pm\left(\dfrac{2nc}{a(n+1)}\right)^{\frac{1}{n-1}}, & n \text{ is odd}. \end{cases} \end{aligned} \tag{11.76}$$

Consequently, we find the following sets of general compactons solutions:

1. For n even, the general compactons solutions are given by:

$$u(x,t) = \begin{cases} \{\sqrt{\dfrac{2nc}{a(n+1)}}\sin[\dfrac{(n-1)}{2n}\sqrt{\dfrac{a}{b}}(x-ct)]\}^{\frac{2}{n-1}}, & |x-ct| \leqslant \dfrac{2n\pi}{\sigma}, \\ 0, & \text{otherwise}. \end{cases} \tag{11.77}$$

and

$$u(x,t) = \begin{cases} \{\sqrt{\dfrac{2nc}{a(n+1)}}\cos[\dfrac{(n-1)}{2n}\sqrt{\dfrac{a}{b}}(x-ct)]\}^{\frac{2}{n-1}}, & |x-ct| \leqslant \dfrac{n\pi}{\sigma}, \\ 0, & \text{otherwise}. \end{cases} \tag{11.78}$$

2. For n odd, the compactons and anticompactons solutions are defined by

$$u(x,t) = \begin{cases} \pm\{\sqrt{\dfrac{2nc}{a(n+1)}}\sin[\dfrac{(n-1)}{2n}\sqrt{\dfrac{a}{b}}(x-ct)]\}^{\frac{2}{n-1}}, & |x-ct| \leqslant \dfrac{2n\pi}{\sigma}, \\ 0, & \text{otherwise}. \end{cases} \tag{11.79}$$

and

$$u(x,t)=\begin{cases}\pm\{\sqrt{\dfrac{2nc}{a(n+1)}}\cos[\dfrac{(n-1)}{2n}\sqrt{\dfrac{a}{b}}(x-ct)]\}^{\frac{2}{n-1}}, & |x-ct|\leqslant\dfrac{n\pi}{\sigma},\\ 0, & \text{otherwise.}\end{cases}\tag{11.80}$$

We have chosen to examine two test problems, namely $K(2,2)$ and $K(3,3)$.

Example 1. We first consider the initial value problem $K(2,2)$

$$\begin{aligned} u_t+(u^2)_x+(u^2)_{xxx} &= 0,\\ u(x,0) &= \frac{4}{3}c\cos^2(\frac{1}{4}x).\end{aligned}\tag{11.81}$$

Solution.

Following Adomian analysis we find

$$u(x,t)=\frac{4}{3}c\cos^2(\frac{1}{4}x)-L_t^{-1}\left((u^2)_x+(u^2)_{xxx}\right).\tag{11.82}$$

Substituting the decomposition series for $u(x,t)$ into (11.82) gives

$$\sum_{n=0}^{\infty}u_n(x,t)=\frac{4}{3}c\cos^2(\frac{1}{4}x)-L_t^{-1}\left(\sum_{n=0}^{\infty}A_n+\sum_{n=0}^{\infty}B_n\right),\tag{11.83}$$

where A_n and B_n are Adomian polynomials that represent the nonlinear operators $(u^2)_x$ and $(u^2)_{xxx}$ respectively. In view of (11.83), the decomposition technique admits the use of the recursive relation

$$\begin{aligned} u_0(x,t) &= \frac{4}{3}c\cos^2(\frac{1}{4}x),\\ u_{k+1}(x,t) &= -L_t^{-1}\left(A_k+B_k\right),\quad k\geqslant 0.\end{aligned}\tag{11.84}$$

The Adomian polynomials A_n and B_n for $(u^2)_x$ and $(u^2)_{xxx}$ are given by

$$\begin{aligned} A_0 &= F(u_0)=(u_0^2)_x,\\ A_1 &= u_1F'(u_0)=(2u_1u_0)_x,\\ A_2 &= u_2F'(u_0)+\frac{1}{2}u_1^2F''(u_0)=(2u_2u_0+u_1^2)_x,\end{aligned}\tag{11.85}$$

and

$$\begin{aligned} B_0 &= G(u_0)=(u_0^2)_{xxx},\\ B_1 &= u_1G'(u_0)=(2u_1u_0)_{xxx},\\ B_2 &= u_2G'(u_0)+\frac{1}{2}u_1^2G''(u_0)=(2u_2u_0+u_1^2)_{xxx}.\end{aligned}\tag{11.86}$$

This in turn gives

$$u_0(x,t)=\frac{4}{3}c\cos^2(\frac{1}{4}x),$$

$$
\begin{aligned}
u_1(x,t) &= -L_t^{-1}(A_0+B_0) = \frac{1}{3}c^2 t\sin(\frac{1}{2}x),\\
u_2(x,t) &= -L_t^{-1}(A_1+B_1) = -\frac{1}{12}c^3t^2\cos(\frac{1}{2}x),\\
u_3(x,t) &= -L_t^{-1}(A_2+B_2) = -\frac{1}{72}c^4t^3\sin(\frac{1}{2}x).
\end{aligned} \tag{11.87}
$$

The solution in a series form

$$
u(x,t) = \frac{4}{3}c\cos^2(\frac{1}{4}x) + \frac{1}{3}c^2t\sin(\frac{1}{2}x) - \frac{1}{12}c^3t^2\cos(\frac{1}{2}x) - \frac{1}{72}c^4t^3\sin(\frac{1}{2}x) + \cdots, \tag{11.88}
$$

follows immediately, and as a result, the closed form solution

$$
u(x,t) = \begin{cases} \frac{4}{3}c\cos^2(\frac{1}{4}(x-ct)), & |\, x-ct\,| \leqslant 2\pi,\\ 0, & \text{otherwise}, \end{cases} \tag{11.89}
$$

is readily obtained.

Example 2. We now consider the initial value problem K$(3,3)$

$$
\begin{aligned}
u_t + (u^3)_x + (u^3)_{xxx} &= 0,\\
u(x,0) &= \sqrt{\frac{3c}{2}}\cos(\frac{1}{3}x).
\end{aligned} \tag{11.90}
$$

Solution.

Following the analysis presented above we obtain

$$
u(x,t) = \sqrt{\frac{3c}{2}}\cos(\frac{1}{3}x) - L_t^{-1}\left((u^3)_x + (u^3)_{xxx}\right). \tag{11.91}
$$

Using the decomposition series assumption for $u(x,t)$ gives

$$
\sum_{n=0}^{\infty} u_n(x,t) = \sqrt{\frac{3c}{2}}\cos(\frac{1}{3}x) - L_t^{-1}\left(\sum_{n=0}^{\infty}\tilde{A}_n + \sum_{n=0}^{\infty}\tilde{B}_n\right) \tag{11.92}
$$

where $\tilde{A}_n$ and $\tilde{B}_n$ are Adomian polynomials that represent the nonlinear operators $(u^3)_x$ and $(u^3)_{xxx}$ respectively. In view of (11.92), we use the recursive relation

$$
\begin{aligned}
u_0(x,t) &= \sqrt{\frac{3c}{2}}\cos(\frac{1}{3}x),\\
u_{k+1}(x,t) &= -L_t^{-1}\left(\tilde{A}_k + \tilde{B}_k\right), \quad k \geqslant 0.
\end{aligned} \tag{11.93}
$$

Adomian polynomials $\tilde{A}_n$ and $\tilde{B}_n$ can be calculated as before to find

$$
\tilde{A}_0 = (u_0^3)_x,\ \tilde{A}_1 = (3u_1u_0^2)_x, \quad \tilde{A}_2 = (3u_2u_0^2 + 3u_0u_1^2)_x, \tag{11.94}
$$

and

$$\tilde{B}_0 = (u_0^3)_{xxx},\ \tilde{B}_1 = (3u_1u_0^2)_{xxx}, \quad \tilde{B}_2 = (3u_2u_0^2 + 3u_0u_1^2)_{xxx}. \tag{11.95}$$

This gives

$$\begin{aligned} u_0(x,t) &= \frac{\sqrt{6c}}{2}\cos(\frac{1}{3}x),\\ u_1(x,t) &= -L_t^{-1}(\tilde{A}_0+\tilde{B}_0) = \frac{\sqrt{6c^3}}{6}t\sin(\frac{1}{3}x),\\ u_2(x,t) &= -L_t^{-1}(\tilde{A}_1+\tilde{B}_1) = -\frac{\sqrt{6c^5}}{36}t^2\cos(\frac{1}{3}x),\\ u_3(x,t) &= -L_t^{-1}(\tilde{A}_2+\tilde{B}_2) = -\frac{\sqrt{6c^7}}{324}t^3\sin(\frac{1}{3}x). \end{aligned} \tag{11.96}$$

The solution in a series form is given by

$$u(x,t) = \frac{\sqrt{6c}}{2}\cos(\frac{1}{3}x) + \frac{\sqrt{6c^3}}{6}t\sin(\frac{1}{3}x) - \frac{\sqrt{6c^5}}{36}t^2\cos(\frac{1}{3}x) - \cdots, \tag{11.97}$$

and in a closed form is given by

$$u(x,t) = \begin{cases} \dfrac{\sqrt{6c}}{2}\cos(\dfrac{1}{3}(x-ct), & |\, x-ct \,| \leqslant \dfrac{3\pi}{2},\\ 0, & \text{otherwise}. \end{cases} \tag{11.98}$$

Exercises 11.3

Use the decomposition method or any other method to find the compactons solutions for the following nonlinear dispersive equations:

1. $u_t + (u^3)_x + (u^3)_{xxx} = 0,\ u(x,0) = 3\cos(\frac{1}{3}x)$
2. $u_t + (u^4)_x + (u^4)_{xxx} = 0,\ u(x,0) = \left(2\sin(\frac{3}{8}x)\right)^{\frac{2}{3}}$
3. $u_t + (u^3)_x + (u^3)_{xxx} + (u^3)_{yyy} = 0,\ u(x,y,0) = 3\cos(\frac{1}{3\sqrt{2}}(x+y))$
4. $\frac{1}{2}(u^2)_t + (u^2)_x + (u^2)_{xxx} + (u^2)_{xxxxx} = 0,\ u(x,0) = \sqrt{\cos x}$
5. $\frac{1}{2}(u^2)_t + (u^2)_x + (u^2)_{xxx} + (u^2)_{xxxxx} + (u^2)_{yyyyy} = 0,\ u(x,y,0) = \sqrt{\cos(\frac{1}{\sqrt{2}}(x+y))}$

11.4 The Defocusing Branch of K(n,n)

As indicated before the defocusing branch

$$u_t - a(u^n)_x + (u^n)_{xxx} = 0, \quad n > 1, \tag{11.99}$$

where $a = -1$, was examined in the literature. It was revealed in these studies that solutions with solitary patterns having cusps or infinite slopes arise from the nonlinear dispersive equation of the form given in (11.99).

It is natural to seek a general solution of the dispersive Eq. (11.99) in the form

$$u(x,t) = \rho \sinh^{\frac{2}{n-1}}[\sigma(x-ct)], \tag{11.100}$$

or in the form

$$u(x,t) = \rho \cosh^{\frac{2}{n-1}}[\sigma(x-ct)], \tag{11.101}$$

where ρ and σ are constants that will be determined. Proceeding as before, it then follows that

$$\sigma = \pm\frac{(n-1)}{2n}\sqrt{\frac{a}{b}}, \quad \rho = \begin{cases} \left(\dfrac{2nc}{a(n+1)}\right)^{\frac{1}{n-1}}, & n \text{ is even}, \\ \pm\left(\dfrac{2nc}{a(n+1)}\right)^{\frac{1}{n-1}}, & n \text{ is odd}. \end{cases} \tag{11.102}$$

Substituting the last result into (11.100) and (11.101) gives the related solitary patterns solutions.

Exercises 11.4

Use the decomposition method or any other method to find the compactons solutions for the following nonlinear dispersive equations:

1. $u_t - (u^3)_x + (u^3)_{xxx} = 0,\ u(x,0) = 3\sinh(\frac{1}{3}x)$
2. $u_t - (u^4)_x + (u^4)_{xxx} = 0,\ u(x,0) = -\left(2\cosh(\frac{3}{8}x)\right)^{\frac{2}{3}}$
3. $u_t - (u^3)_x + (u^3)_{xxx} + (u^3)_{yyy} = 0,\ u(x,y,0) = 3\sinh\left(\frac{1}{3\sqrt{2}}(x+y)\right)$
4. $\frac{1}{2}(u^2)_t - (u^2)_x + (u^2)_{xxx} + (u^2)_{xxxxx} = 0,\ u(x,0) = \sqrt{\sinh x}$

5. $\frac{1}{2}(u^2)_t - (u^2)_x + (u^2)_{xxx} + (u^2)_{xxxxx} + (u^2)_{yyyyy} = 0, u(x,y,0) = \sqrt{\sinh(x+y)}$

References

1. M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, (1991).
2. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, (1994).
3. P.G. Drazin and R.S. Johnson, Solitons: an Introduction, Cambridge University Press, Cambridge, (1996).
4. J. Hietarinta, Introduction to the Bilinear Method, in "Integrability of Nonlinear Systems", eds. Y. Kosman-Schwarzbach, B. Grammaticos and K.M. Tamizhmani, Lecture Notes in Physics, Springer, Berlin, 638, 95–105 (2004).
5. R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge (2004).
6. R. Hirota, Exact solutions of the Korteweg-de Vries equation for multiple collisions of solitons, *Phys. Rev. Lett.*, 27(18), 1192–1194, (1971).
7. M. Ito, An extension of nonlinear evolution equations of the KdV (mKdV) type to higher orders, *J.Phys. Soc. Japan*, 49(2), 771–778, (1980).
8. B.B. Kadomtsev and V.I. Petviashvili, On the stability of solitary waves in weakly dispersive media, *Sov. Phys. Dokl.*, 15, 539–541, (1970).
9. J.J.C. Nimmo and N.C. Freeman, The use of Backlund transformations in obtaining the N-soliton solutions in terms of a Wronskian, *J. Phys. A:Math. General*, 17, 1415–1424, (1983).
10. P. Rosenau and J.M. Hyman, Compactons: Solitons with finite wavelengths, *Phys. Rev. Lett.*, 70(5), 564–567, (1993).
11. N.J. Zabusky and M.D. Kruskal, Interaction of solitons incollisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.*, 15, 240–243, (1965).

Part II
Solitray Waves Theory

Chapter 12
Solitary Waves Theory

12.1 Introduction

In 1844 the Scottish John Scott Russell was the first people to observe the solitary waves. As stated in Chapter 11, Russell called the bulge of water, that he observed, a “great wave of translation” [9]. The wave was traveling along the channel of water for a long period of time while still retaining its original identity.

In Russell’s own words: “I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped-not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called Wave of translation.”

This single humped wave of bulge of water is now called solitary waves or solitons. The solitons–localized, highly stable waves that retain its identity (shape and speed), upon interaction–was discovered *experimentally* by Russell.

In 1895, Diederik Johannes Korteweg (1848–1941) together with his Ph.D student, Gustav de Vries (1866–1934) derived *analytically* a nonlinear partial differential equation, well known now as the KdV equation. The KdV equation, that contains nonlinear and dispersive terms, describes the propagation of long waves of small but finite amplitude in dispersive media. The KdV equation is a generic model for the study of weakly nonlinear long waves, incorporating leading order nonlinearity and dispersion. The KdV equation which now bears the names of Korteweg and de Vries had already appeared in a work on water waves by Boussinesq in 1872.

The KdV equation in its simplest form is given by

$$u_t + auu_x + u_{xxx} = 0. \tag{12.1}$$

The term u_t in this equation describes the time evolution of the wave propagating in one direction. Moreover, this equation incorporates two competing effects: nonlinearity represented by uu_x that accounts for steepening of the wave, and linear dispersion represented by u_{xxx} that describes the spreading of the wave. Nonlinearity tends to localize the wave while dispersion spreads it out. In other words, in some nonlinear media, such as a layer of shallow water or an optical fiber, the widening of a wave packet due to dispersion could be balanced exactly by the narrowing effects due to nonlinearity of the medium. The balance between these weak nonlinear steepening and dispersion explains the formulation of solitons that consist of single humped waves. The stability of solitons stems from the delicate equilibrium between these two effects of nonlinearity and dispersion. As will be discussed later, this equation gives soliton solutions which characterize solitary waves with particle-like properties that decrease monotonically at infinity.

In 1965, Norman J. Zabusky (1929–) and Martin D. Kruskal (1925–2006) investigated *numerically* the nonlinear interaction of a large solitary-wave overtaking a smaller one, and the recurrence of initial states [11]. They discovered that solitary waves undergo nonlinear interaction following the KdV equation. Further, the waves emerge from this interaction retaining its original shape, amplitude and speed, and therefore conserved energy and mass. The only effect of the interaction was a phase shift. The remarkable discovery, that solitary waves retain their identities and that their character resembles particle like behavior, motivated Zabusky and Kruskal [11] to call these solitary waves *solitons*. Zabusky and Kruskal marked the birth of soliton, a name intended to signify particle like quantities. The interaction of two solitons emphasized the reality of the preservation of shapes and speeds and of the steady pulse like character of solitons, therefore the collision of KdV solitons is considered elastic. The name soliton have been coined by Zabusky and Kruskal after photon, phonon, proton, etc. However the name solitary wave is more general. Solitons are special kinds of solitary waves.

12.2 Definitions

It is interesting now to give some definitions of some concepts in the mathematical theory of waves. *Linear waves* like sinusoidal waves are different from solitons. The physical definition of a wave is a movement up and down or back and forth. Also wave is a disturbance that transmits energy from one place to another. The simplest wave propagation equation is given by

$$u_{tt} = c^2 u_{xx}, \tag{12.2}$$

where $u(x,t)$ represents the amplitude of the wave, and c is the speed of the wave. This equation has the general d'Alembert'solution

$$u(x,t) = f(x - ct) + g(x + ct), \tag{12.3}$$

where f and g are arbitrary functions which represent right and left propagating waves respectively. The two distinct waves f and g propagate without changing its identity. The functions f and g are usually determined by using the initial values $u(x,0)$ and $u_t(x,0)$ that are usually prescribed. Because the wave equation is linear, the two solutions can be added according to the superposition principle. Setting $g = 0$, the wave in this case is propagating in the right direction only as in the equation $u_t + u_x = 0$ with solution $u(x,t) = f(x - t)$ with speed $c = 1$.

On the other hand, *a travelling wave* is a wave in which the medium moves in the direction of prorogation of the wave. Travelling waves arise in the study of nonlinear differential equations where waves are represented by the form $u(x,t) = f(x - ct)$, where $u(x,t)$ represents a disturbance moving in the negative or positive x direction if $c < 0$ or $c > 0$ respectively. If the solution $u(x,t)$ depends only on the difference between the two coordinates of the partial differential equations, then the solution keeps its exact shape, and therefore called *solitary waves*. A solitary wave is a travelling wave whose transition from the asymptotic state at $\xi = -\infty$ to the other asymptotic state at $\xi = \infty$ is localized in ξ, where $\xi = x - ct$, and c is the wave speed. Hereman [3] defined solitary wave as a localized gravity wave that maintains its coherence, and has a finite amplitude and propagate with constant speed and constant shape.

Solitons are found in many physical phenomena. Solitons arise as the solutions of a widespread class of weakly nonlinear dispersive partial differential equations describing physical systems. Solitons are solitary waves with elastic scattering property. Solitons retain their shapes and speed after colliding with each other. As stated before, the KdV equation is the pioneer model that gives rise to solitons. The name soliton appears to have been coined by Zabusky and Kruskal. Solitons are caused by a delicate balance between nonlinear and dispersive effects in the medium. Solitons appear either in the sech^2 bell shape or in the form of a kink. Soliton possesses particle-like character and retains its identities in a collision. A precise definition of a soliton is not easy to find. However, Drazin et.al [1] defined a soliton as any solution of a nonlinear equation (or a system) which:
(i) is a solitary wave of permanent form;
(ii) is localized, so that it decays or approaches a constant at infinity;
(iii) can interact strongly with other solitons and retain its identity;
(iv) is caused by a delicate balance between nonlinear and dispersive effects.

In the physical literature, the difference between solitary waves and solitons has become blurred. Solitary waves may be defined as soliton-like solutions of nonlinear evolution equations describing wave processes in dispersive and dissipative media. It is usually referred to a single soliton solution as a solitary wave [1], but when more than one soliton appear in a solution they are called solitons. For equations other than the KdV equation, the solitary wave solution may not be a sech^2 function; but may be a sech or arctan($e^{\alpha x}$) [1].

12.2.1 Dispersion and Dissipation

It is useful to study some properties of wave phenomena. We first consider the equation

$$u_t + u_x = 0. \tag{12.4}$$

It can be easily seen that the solution of this equation is of the form

$$u(x,t) = f(x-t). \tag{12.5}$$

Examples of this solution are $\sin(x-t), \cos(x-t), e^{x-t}$, and many others. Also these solutions can be combined, hence the superposition principle is applicable here because the equation is linear. The shape of these waves (12.5) does not change as the wave propagate.

However, adding a *third order* spatial derivative, which is the dispersion term, to Eq. (12.4) gives the simplest dispersive equation

$$u_t + u_x + u_{xxx} = 0. \tag{12.6}$$

Assume that the wave solution is of the form

$$u(x,t) = e^{i(kx-\omega t)}, \tag{12.7}$$

where k is the wave number, and ω is the frequency. Substituting (12.7) into the dispersive equation (12.6) and using the real or imaginary part we obtain the dispersion relation

$$\omega = k - k^2, \tag{12.8}$$

and therefore the wave propagates at the velocity

$$c = \frac{\omega}{k} = 1 - k^2. \tag{12.9}$$

This indicates that dispersive waves are waves in which the velocity c varies with the wave number k as shown by (12.9). Dispersive effects usually gives a relationship between the frequency and the wave speed.

On the other hand, using an *even order* spatial derivative, which is the dissipative term, in (12.4) gives the dissipative equation

$$u_t + u_x - u_{xx} = 0. \tag{12.10}$$

Using the assumption (12.7) into (12.10) gives the relation

$$\omega = k(1 - ik), \tag{12.11}$$

and this in turn gives the solution

$$u(x,t) = e^{-k^2 t + ik(x-t)}. \tag{12.12}$$

It is obvious that the solution (12.12) indicates that the wave propagates at a unity speed. The dissipation, the exponential decay of (12.12), is also clear for $t \to \infty, k \neq 0$. A wave that loses amplitude, due to loss of energy over time, is called a dissipative wave.

We have discussed so far linear equations. However, if we replace u_x in (12.6) and (12.10) by a nonlinear term uu_x we obtain the nonlinear equations

$$u_t + uu_x + u_{xxx} = 0, \tag{12.13}$$

and

$$u_t + uu_x - u_{xx} = 0, \tag{12.14}$$

respectively. These equations are the well-known KdV and Burgers equations that will be studied in details in the forthcoming chapters. It is interesting to point out that the delicate balance between the nonlinearity effect of uu_x and the dispersion effect of u_{xxx} gives rise to *solitons*, that after a fully interaction with others, the solitons reemerge retaining their identities with the same speed and shape. However, the Burgers equation (12.14) combines the effects of nonlinearity and dissipation that gives rise to *kinks*. The KdV equation has solitary wave solutions characterized by analytic sech^2 functions that have exponentially decaying wings. The Burgers equation has kink solutions characterized by tanh function that approaches a constant at infinity.

It is important to note that the superposition principle, which works for linear equations, is not applicable for nonlinear wave equations. If two solitons of the KdV equation collide, the solitons simply pass through each other and emerge unchanged.

Moreover, Rosenau and Hyman [8] investigated the nonlinear dispersive equation $\mathrm{K}(n,n)$ given by

$$u_t + a(u^n)_x + (u^n)_{xxx} = 0, \quad n > 1. \tag{12.15}$$

This equation combines the nonlinear convection term $(u^n)_x$ and the genuinely nonlinear dispersive term $(u^n)_{xxx}$. The delicate interaction between the genuine nonlinearity and dispersion gives rise to *compacton*: soliton with compact support free of exponential wings. In [8], it was proved that solitary waves may compactify under the influence of purely nonlinear dispersion which is capable of causing deep changes in the nature of genuinely nonlinear phenomenon. One important feature of the compacton structure, in addition to the absence of infinite wings, is that the width of the compacton is independent of the amplitude. It is to be noted that while compactons are the essence of the focusing branch where $a > 0$, spikes, peakons, and cusps are the hallmark of the defocusing branch where $a < 0$. Furthermore, the defocusing branch was found to rise to solitary patterns having cusps or infinite slopes. This confirms the fact that the focusing branch and the defocusing branch represent two different sets of models each leading to a different physical structure. The $\mathrm{K}(n,n)$ equation cannot be derived from a first order Lagrangian except for $n=1$, and did not possess the usual conservation laws of energy that KdV equation possessed.

Solitons are analytic solutions whereas compactons are nonanalytic solutions. The points of non-analyticity at the edge of the compacton correspond to points of genuine nonlinearity for the differential equations. Compactons exhibit elastic collision where after colliding with other compactons they reemerge with the same coherent shape. The main difference between linear or weakly nonlinear equations such as the KdV and the Burgers equation versus the completely nonlinear equations such as the $\mathrm{K}(n,n)$ equations is that the fully nonlinear models admit nonanalytic solutions.

More definitions and explanations of these terms, solitons, compactons, kinks, peakon, as well as other terms will be addressed in the forthcoming sections and chapters. Several nonlinear evolution equations, which give travelling waves solutions, will be investigated in subsequent chapters by using basic methods.

It is also interesting to define the terminology *complete integrable* PDEs. A common feature of complete integrable PDEs is the existence of an infinite sequence of independent conservation laws, and hence give rise to N-soliton solutions. A conservation law for any equation is a divergence expression

$$\frac{\partial T}{\partial t}+\frac{\partial X}{\partial x}=0, \tag{12.16}$$

where T and X are named conserved density and conserved flux respectively and neither one involves derivatives with respect to t, is called a conservation law [1]. More about conservation laws will be presented at the end of this chapter. The KdV, Boussinesq, KP, sine-Gordon equations are examples of completely integrable equations that will be studied in the forthcoming chapters.

12.2.2 Types of Travelling Wave Solutions

The study of equations that model wave phenomena requires the study of travelling wave solutions. Travelling wave solution is a solution of permanent form moving with a constant velocity. The travelling wave solutions are usually obtained by reducing the nonlinear evolution equations to associated ordinary differential equations. This is mostly handled by using the ansatz $u(x,t)=u(\xi),\xi=x-ct$, c is the wave speed, that will transform the PDE in x,t to an ordinary differential equation in ξ which can be solved by several appropriate methods.

There are many types of travelling wave solutions that are of particular interest in solitary wave theory that is rapidly developing in many scientific fields from water waves in shallow water to plasma physics. As stated before, travelling waves appear in many types, and only some of these types will be addressed:

1. **Solitary Waves and Solitons**

Solitary waves are localized travelling waves travelling with constant speeds and shape, asymptotically zero at large distances. Solitons are special kinds of solitary waves. The soliton solution is spatially localized solution, hence $u'(\xi), u''(\xi)$, and $u'''(\xi) \to 0$ as $\xi \to \pm\infty$, $\xi = x - ct$. Solitons have a remarkable soliton property in that it keeps its identity upon interacting with other solitons.

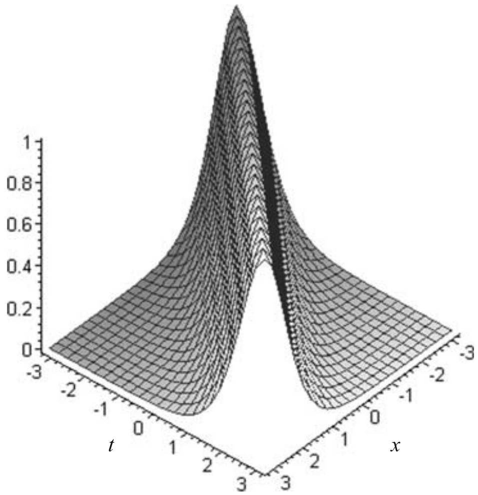

Fig. 12.1 Graph of a soliton solution $\operatorname{sech}^2(x-t), -\pi \leqslant x,t \leqslant \pi$, that has an infinite support or infinite tails.

The KdV equation is the pioneer model for analytic bell-shaped sech^2 solitary wave solutions. Fig. 12.1 above shows a graph of a bell-shaped sech^2 soliton solution characterized by infinite wings or infinite tails.

2. **Periodic Solutions**

Periodic solutions are travelling wave solutions that are periodic such as $\cos(x-t)$. The standard wave equation $u_{tt} = u_{xx}$ gives periodic solutions. As stated before, because this standard wave equation is linear, it admits d'Alembert solution, and components can be superposed. Fig. 12.2 above shows a periodic solution $u(x,t) = \cos(x-t), -\pi \leqslant x,t \leqslant \pi$ for a standard wave equation.

3. **Kink Waves**

Kink waves are travelling waves which rise or descend from one asymptotic state to another. The kink solution approaches a constant at infinity.

The standard dissipative Burgers equation

$$u_t + uu_x = \nu u_{xx}, \tag{12.17}$$

where ν is the viscosity coefficient, is a well-known equation that gives kink solutions. Other equations provide kinks solutions as well. Fig. 12.3 above shows a kink solution $u(x,t) = 1 - \tanh(x-t), -10 \leqslant x,t \leqslant 10$ for Burgers equation with $\nu = \frac{1}{2}$.

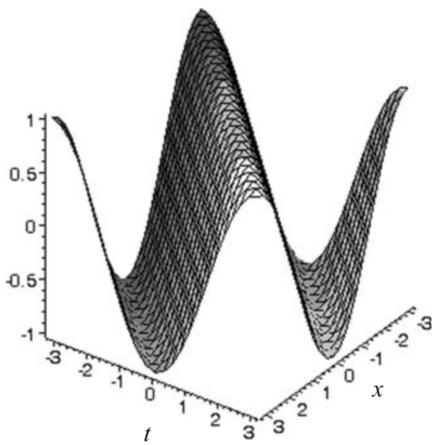

Fig. 12.2 Graph of a periodic solution $u(x,t) = \cos(x-t), -\pi \leqslant x,t \leqslant \pi$.

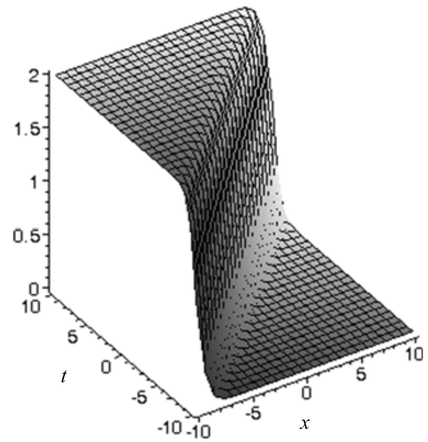

Fig. 12.3 Graph of a kink solution $u(x,t) = 1 - \tanh(x-t), -10 \leqslant x,t \leqslant 10$.

Fig. 12.4 shows a kink solution and a soliton solution.

4. **Peakons**

Peakons are peaked solitary wave solutions. In this case, the travelling wave solutions are smooth except for a peak at a corner of its crest. Peakons are the points at which spatial derivative changes sign so that peakons have a finite jump in first derivative of the solution $u(x,t)$. This means that peakons have discontinuities in the x-derivative but both one-sided derivatives exist and differ only by a sign [10]. The peakons are solitons retaining their shape and speed after interacting. In [7,10], peakons were investigated and classified as periodic peakons and peakons with exponential decay.

The integrable Camassa-Holm and the Degasperis-Procesi equations

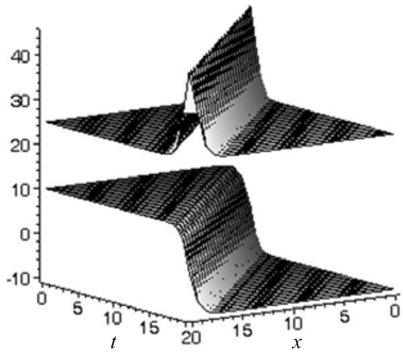

Fig. 12.4 Graphs of a kink solution (lower) and a soliton solution (upper).

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx} \tag{12.18}$$

for $b = 2$ and $b = 3$ respectively, admit peaked solitary wave solutions. The CH equation has peaked solitary wave solutions of the form

$$u(x,t) = ce^{-|x-ct|}, \tag{12.19}$$

where c is the wave speed. Fig.12.5 below shows a peakon solution $u(x,t) = e^{-|x-t|}, -2 \leqslant x,t \leqslant 2$ for CH equation with $c = 1$.

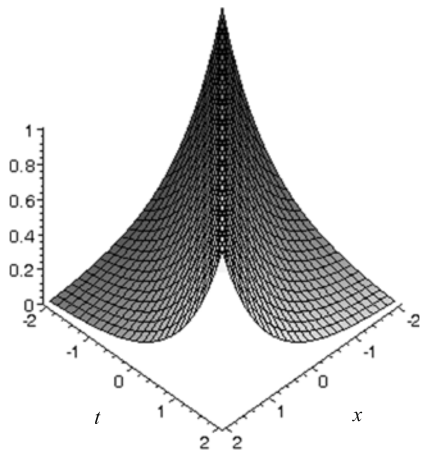

Fig. 12.5 Graph of a peakon solution $u(x,t) = e^{-|x-t|}, -2 \leqslant x,t \leqslant 2$ for CH equation with $c = 1$.

5. **Cuspons**

Cuspons are other forms of solitons where solution exhibits cusps at their crests. Unlike peakons where the derivatives at the peak differ only by a sign, the derivatives at the jump of a cuspon diverges. Fig. 12.6 below shows a virtual graph of a cuspon that is not derived from a well-known model.

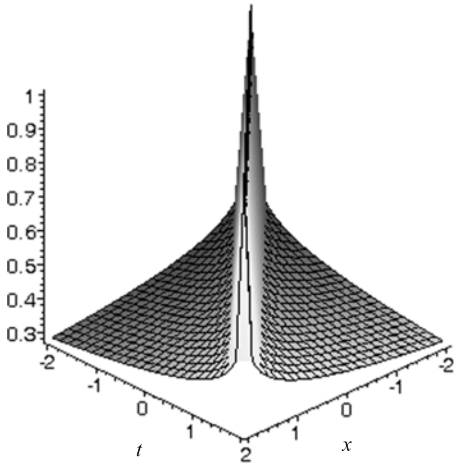

Fig. 12.6 Graph of a cuspon $u(x,t) = e^{-(|(x-ct)|)^{\frac{1}{6}}}, c = 1, -2 \leqslant x,t \leqslant 2.$

Fig. 12.6 above shows a cuspon with a cusp on its crest. The derivatives at the cusp diverges.

It is important to note that the soliton solution $u(x,t)$, along with its derivatives, tends to zero as $|\ x\ | \to \infty$. The Camassa-Holm equation and the DP equation are two equations that give cuspons for specific cases that are discussed in details in [7]. Unfortunately, we could not find an explicit expression for cuspons. Instead, we used a virtual expression to represent it graphically. The assumption is that cuspon can be represented as

$$u(x,t) = e^{-|x-ct|^{\frac{1}{n}}}, \quad n > 1. \tag{12.20}$$

We can easily show that $u_\xi = \infty$ at the cusp, and $u_\xi, u_{\xi\xi}, \ldots \to 0$ to characterize the soliton property.

In [7], and in the references therein, cuspons were investigated and classified as periodic cuspons and and cuspons with exponential decay.

6. **Compacton**

Compacton is a new class of solitons with compact spatial support such that each compacton is a soliton confined to a finite core. Compactons are defined by solitary waves with the remarkable soliton property that after colliding with other compactons, they reemerge with the same coherent shape [8]. These particle like waves exhibit elastic collision that are similar to the soliton collision. It was found that a compacton is a solitary wave with a compact support where the nonlinear dispersion confines it to a finite core, therefore the exponential wings vanish.

The genuinely nonlinear dispersive K(n,n) equations, a family of nonlinear KdV like equations is of the form

$$u_t + a(u^n)_x + (u^n)_{xx} = 0, \quad a > 0, n > 1, \tag{12.21}$$

which supports compact solitary traveling structures for $a > 0$.

The definitions given so far for compactons are:
(i) compactons are solitons with finite wavelength;
(ii) compactons are solitary waves with compact support;
(iii) compactons are solitons free of exponential tails;
(iv) compactons are solitons characterized by the absence of infinite wings;
(v) compactons are robust soliton-like solutions.

Two important features of compactons structures are observed, namely:
(i) unlike the standard KdV soliton where $u(\xi) \to 0$ as $\xi \to \infty$, the compacton is characterized by the absence of the exponential tails or wings, where $u(\xi)$ does not tend to 0 as $\xi \to \infty$;
(ii) unlike the standard KdV soliton where width narrows as the amplitude increases, the width of the compacton is independent of the amplitude.

It is important to note that Eq. (12.21) with $(+a)$ is called the focusing branch, and with $(-a)$, the equation is called the defocusing branch of the K(n,n) equations. Many studies revealed that the defocusing branch supports solutions with solitary patterns having cusps or infinite slopes. Further, it was shown that while compactons are the essence of the focusing branch $(+a)$, spikes, peaks and cusps are the hallmark of the defocusing branch $(-a)$. This in turn means that the focusing branch (12.21) and the defocusing branch represent two different models, each leading to a different physical structure. Fig. 12.7 below shows a graph of a compacton $u(x,t) = \cos^{\frac{1}{2}}(x-t), c = 1, 0 \leqslant x,t \leqslant 1$.

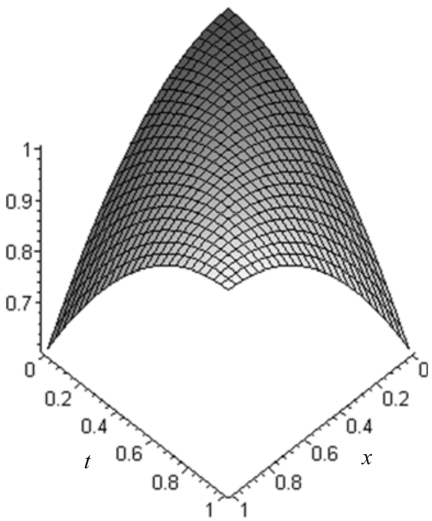

Fig. 12.7 Graph of a compacton $u(x,t) = \cos^{\frac{1}{2}}(x-t), c = 1, 0 \leqslant x,t \leqslant 1$.

The graph shows that a compacton is a solitary wave free of exponential wings.

The remarkable discovery of compactons has led to an intense study over the last few years. The study of compactons may give insight into many scientific processes such as the super deformed nuclei, preformation of cluster in hydrodynamic models, the fission of liquid drops and inertial fusion. The stability analysis has shown that compacton solutions are stable, where the stability condition is satisfied for arbitrary values of the nonlinearity parameter. The stability of the compactons solutions was investigated by means of both linear stability and by Lyapunov stability criteria. Moreover, the compactons are nonanalytic solutions whereas classical solitons are analytic solutions. *Solitons* and *compactons* with and without exponential wings respectively, are termed by using the suffix-*on* to indicate that it has the property of a particle, such as phonon, and photon. Fig. 12.8 below shows a compacton (left) and a soliton (right).

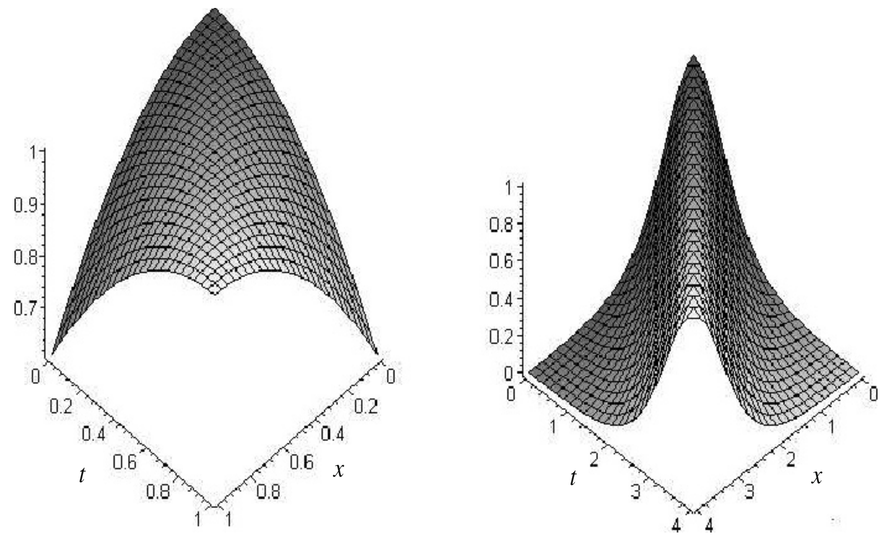

Fig. 12.8 The compacton graph (left) and the soliton graph without and with infinite wings respectively.

12.2.3 Nonanalytic Solitary Wave Solutions

It will be discussed later that some nonlinear dispersive equations give analytic solitary wave solutions, such as the KdV equation, whereas others give nonanalytic solutions such as the $K(n,n)$ equations. The appearance of nonanalytic solitary wave solutions, including compactons, peakons, and cuspons has increased the menagerie of solutions appearing in model equations, both completely integrable and nonintegrable [4,5,7].

The distinguishing feature of the systems admitting nonanalytic solitary wave solutions is that, in contrast to the classical nonlinear wave equations, they all include a genuinely nonlinear dispersion, such as $(u^n)_{xxx}$ of the K(n,n) equation, or the highest order derivatives (characterizing the dispersion relation) are typically multiplied by a function of the dependent variable as uu_{xxx} of the Camassa-Holm equation [4,5].

12.3 Analysis of the Methods

In the literature, researchers usually use a variety of distinct methods to analyze nonlinear evolution equations. The methods range from reasonable to difficult that require a huge size of work. There is no unified method that can be used for all types of nonlinear evolution equations. For single soliton solutions, several methods, such as the pseudo spectral method, the inverse scattering method [1], Hirota's bilinear method [4], the truncated Painlevé expansion, Bäcklund transformation method, homogeneous balance method, projective Riccati equation method, Jacobi elliptic functions method, and many others have been used. Lot of informations and details about these methods are presented in several texts. However, the tanh method [6], the tanh-coth method, and the sine-cosine method are proved to be powerful methods, and therefore are recently heavily used in several research works, but rarely in books. It is for this reason, our main methods in this text will be the implementation of these methods, namely the tanh-coth method and the sine-cosine method, to handle nonlinear dispersive and dissipative equations. However, for N-soliton solutions, the Hirota's bilinear form combined with the simplified version of Hirota's method established by Hereman [3] will be used as well.

The tanh-coth method and the sine-cosine method have been applied for a wide variety of nonlinear problems and will be used in this text for single travelling wave solution. These two methods were proved to be powerful, reliable and effective in handling a huge number of nonlinear dispersive and dissipative equations. Moreover, the Hirota bilinear formalism and a simplified version of this method will be used to address the concept of multiple soliton solutions. This does not mean in any way that other methods are not useful, but because these methods are used in most of the available tests, we prefered to implement these relatively new developed methods only. The main features of the tanh-coth method, sine-cosine method and the Hirota formalism will be presented.

12.3.1 The Tanh-coth Method

A wave variable $\xi = x - ct$ converts any PDE

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, \cdots) = 0, \tag{12.22}$$

to an ODE

$$Q(u, u', u'', u''', \cdots) = 0. \tag{12.23}$$

Equation (12.23) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.

The standard tanh method is developed by Malfliet [6] where the tanh is used as a new variable, since all derivatives of a tanh are represented by tanh itself. For example, if we set $T = \tanh(\xi)$, then we have

$$\begin{aligned} T &= \tanh(\xi), \\ T' &= 1 - T^2, \\ T'' &= -2T + 2T^3, \\ T''' &= -2 + 8T^2 - 6T^4, \\ T^{(4)} &= 16T - 40T^3 + 24T^5. \end{aligned} \tag{12.24}$$

In other words, introducing a new independent variable

$$Y = \tanh(\mu\xi),\ \xi = x - ct, \tag{12.25}$$

where μ is the wave number, leads to the change of derivatives:

$$\begin{aligned} \frac{d}{d\xi} &= \mu(1 - Y^2)\frac{d}{dY}, \\ \frac{d^2}{d\xi^2} &= -2\mu^2 Y(1 - Y^2)\frac{d}{dY} + \mu^2(1 - Y^2)^2\frac{d^2}{dY^2}, \\ \frac{d^3}{d\xi^3} &= 2\mu^3(1 - Y^2)(3Y^2 - 1)\frac{d}{dY} - 6\mu^3 Y(1 - Y^2)^2\frac{d^2}{dY^2} + \mu^3(1 - Y^2)^3\frac{d^3}{dY^3}, \\ \frac{d^4}{d\xi^4} &= -8\mu^4 Y(1 - Y^2)(3Y^2 - 2)\frac{d}{dY} + 4\mu^4(1 - Y^2)^2(9Y^2 - 2)\frac{d^2}{dY^2} \\ &\quad - 12\mu^4 Y(1 - Y^2)^3\frac{d^3}{dY^3} + \mu^4(1 - Y^2)^4\frac{d^4}{dY^4}, \end{aligned} \tag{12.26}$$

The tanh-coth method [10] admits the use of the finite expansion

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}, \tag{12.27}$$

where M is a positive integer, in most cases, that will be determined. For noninteger M, a transformation formula is used to overcome this difficulty. Expansion (12.27) reduces to the standard tanh method for $b_k = 0, 1 \leqslant k \leqslant M$. Substituting (12.27) into the reduced ODE results in an algebraic equation in powers of Y.

To carry out the balance method, we notice from (12.26) and (12.27) that the highest exponents for the function u and its derivatives are as follows

$$\begin{aligned} u &\to M, \\ u^n &\to nM, \end{aligned} \tag{12.28}$$

$$\begin{aligned} u' &\to M+1, \\ u'' &\to M+2, \\ u^{(r)} &\to M+r. \end{aligned} \tag{12.29}$$

To determine the parameter M, we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms by using the scheme given above. We then collect all coefficients of powers of Y in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters a_k, b_k, μ, and c. Having determined these parameters we obtain an analytic solution $u(x,t)$ in a closed form. The solutions we obtain may be solitons in terms of sech^2, or may be kinks in terms of tanh. However, this method may give periodic solutions as well.

12.3.2 The Sine-cosine Method

Proceeding as in the tanh-coth method, Equation (12.23) is integrated as long as all terms contain derivatives where integration constants are considered zeros. The sine-cosine method admits the use of the solutions in the forms

$$u(x,t) = \lambda \cos^{\beta}(\mu\xi), \quad |\xi| \leqslant \frac{\pi}{2\mu}, \tag{12.30}$$

and

$$u(x,t) = \lambda \sin^{\beta}(\mu\xi), \quad |\xi| \leqslant \frac{\pi}{\mu}, \tag{12.31}$$

where λ, μ, and β are parameters that will be determined, μ and c are the wave number and the wave speed respectively. Equations (12.30) and (12.31) give

$$\begin{aligned} (u^n)'(\mu\xi) &= -n\beta\mu\lambda^n \cos^{n\beta-1}(\mu\xi)\sin(\mu\xi), \\ (u^n)''(\mu\xi) &= -n^2\mu^2\beta^2\lambda^n \cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi), \end{aligned} \tag{12.32}$$

and

$$\begin{aligned} (u^n)'(\mu\xi) &= n\beta\mu\lambda^n \sin^{n\beta-1}(\mu\xi)\cos(\mu\xi), \\ (u^n)''(\mu\xi) &= -n^2\mu^2\beta^2\lambda^n \sin^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\sin^{n\beta-2}(\mu\xi). \end{aligned} \tag{12.33}$$

Substituting (12.32) or (12.33) into (12.23) gives a trigonometric equation of $\cos^R(\mu\xi)$ or $\sin^R(\mu\xi)$ terms. The parameters are then determined by first balancing the exponents of each pair of cosine or sine to determine R. We next collect all coefficients of the same power in $\cos^k(\mu\xi)$ or $\sin^k(\mu\xi)$, where these coefficients have to vanish. This gives a system of algebraic equations among the unknowns β, λ

and μ that will be determined. The solutions proposed in (12.30) and (12.31) follow immediately.

The algorithms described above certainly work well for a large class of very interesting nonlinear wave equations. The main advantage of the tanh-coth method and the sine-cosine method, presented above, is that the great capability of reducing the size of computational work compared to existing techniques such as the pseudo spectral method, the inverse scattering method, Hirota's bilinear method, and the truncated Painlevé expansion. The whole work will be changed from solving nonlinear differential equation to simply solving a system of algebraic equations that can be used by any manipulation computer program such as Mathematica or Maple. As will be seen later, the tanh-coth method and the sine-cosine method will give the same solutions for M is even. However, For M is odd, the two methods give distinct solutions.

Hirota [4] constructed the N-soliton solutions of the integrable evolution equations by reducing it to the bilinear form. As stated before, completely integrable PDEs are the equations that have infinitely many conservation laws and admit N-soliton solutions of any order. The bilinear formalism is a very helpful tool in the study of the nonlinear equations and it was the most suitable for computer algebra. In what follows we highlight the main steps of this method.

12.3.3 Hirota's Bilinear Method

A well-known third method, namely, the Hirota bilinear form, will be employed to handle specific integrable nonlinear equations. The method is widely used especially to handle the multi-soliton solutions of many evolution equations. Hirota introduced the customary definition of the Hirota's bilinear operators by

$$D_t^n D_x^m (a\cdot b) = \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m a(x,t)b(x',t')|x'=x,t'=t. \tag{12.34}$$

In what follows, we express some of the bilinear differentials operators:

$$\begin{aligned}
D_x(a\cdot b) &= a_x b - ab_x,\\
D_x^2(a\cdot b) &= a_{2x}b - 2a_x b_x + ab_{2x},\\
D_x D_t(a\cdot b) &= D_x(a_t b - ab_t) = a_{xt}b - a_t b_x - a_x b_t + ab_{xt},\\
D_x D_t(a\cdot a) &= 2(aa_{xt} - a_x a_t),\\
D_x^4(a\cdot b) &= a_{4x}b - 4a_{3x}b_x + 6a_{2x}b_{2x} - 4a_x b_{3x} + ab_{4x},\\
D^n(a\cdot a) &= 0, \text{for } n \text{ is odd.}
\end{aligned} \tag{12.35}$$

Moreover, more of the properties of the D-operators are as follows

$$
\begin{aligned}
\frac{D_t^2(f\cdot f)}{f^2} &= \iint u_{tt}\,dx\,dx,\\
\frac{D_tD_x^3(f\cdot f)}{f^2} &= u_{xt}+3u\int x u_t\,dx',\\
\frac{D_x^2(f\cdot f)}{f^2} &= u,\\
\frac{D_x^4(f\cdot f)}{f^2} &= u_{2x}+3u^2,\\
\frac{D_tD_x(f\cdot f)}{f^2} &= \ln(f^2)_{xt},\\
\frac{D_x^6(f\cdot f)}{f^2} &= u_{4x}+15uu_{2x}+15u^3,\\
\frac{D_t^2(f\cdot f)}{f^2} &= \iint u_{tt}\,dx\,dx,\\
\frac{D_tD_x^3(f\cdot f)}{f^2} &= u_{xt}+3u\int u_t\,dx',
\end{aligned}
\tag{12.36}
$$

where

$$u(x,t)=2(\ln f(x,t))_{xx}, \tag{12.37}$$

The solution of the canonical KdV equation

$$u_t+6uu_x+u_{xxx}=0, \tag{12.38}$$

can be expressed by

$$u(x,t)=2\frac{\partial^2}{\partial x^2}\log f, \tag{12.39}$$

where $f(x,t)$ is given by the perturbation expansion

$$f(x,t)=1+\sum_{n=1}^{\infty}\epsilon^n f_n(x,t), \tag{12.40}$$

where ϵ is a formal expansion parameter. For the one-soliton solution we set

$$f(x,t)=1+\epsilon f_1, \tag{12.41}$$

and for the two-soliton solution we set

$$f(x,t)=1+\epsilon f_1+\epsilon^2 f_2, \tag{12.42}$$

and so on. The functions $f_1, f_2, f_3, \cdots$ can be determined by using the Hirota's bilinear formalism or by direct substitution of (12.40) into the appropriate equation as will be seen later. The N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i), \tag{12.43}$$

where

$$\theta_i = k_i x - c_i t, \tag{12.44}$$

where k_i and c_i are arbitrary constants, k_i is called the wave number. The relation between k_i and c_i can be obtained by determining the dispersion relation.

In [3], a simplified form of the Hirota's bilinear formalism was introduced to minimize the cumbersome work of Hirota's method. The simplified approach in [3] will be examined in forthcoming chapters.

12.4 Conservation Laws

It is important to study the conservation laws of nonlinear evolution equations. The existence of a sequence of conserved densities (with gaps) predicts integrability [2]. The lower order conservation laws can be determined directly from the equation, whereas the higher order conservation laws need a huge size of long and tedious work. The first few conservation laws have a physical interpretation, and additional ones may facilitate the study of both quantitative and qualitative properties of the solution [2]. The non-existence of conserved quantities does not preclude integrability such as Burgers equation that has one conserved density [2].

A conservation law for any equation is a divergence expression

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0, \tag{12.45}$$

where T and X are named conserved density and conserved flux respectively and neither one involves derivatives with respect to t, is called a conservation law [1]. This means that T and X may depend on $x, t, u, u_x, \cdots$ but not on u_t. For most equations, the density-flux pairs are polynomials in u and derivatives of u with respect to x. For polynomial-type T and X, integration of (12.45) yields

$$P = \int_{-\infty}^{\infty} T\, dx = \text{constant}, \tag{12.46}$$

provided that X vanishes at infinity [2]. For example, we consider the canonical form of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0. \tag{12.47}$$

This equation is in conservation form [2] where

$$T = u, \quad X = u_{xx} - 3u^2. \tag{12.48}$$

This in turn gives the first conservation law

$$\int_{-\infty}^{\infty} u dx = \text{constant}. \tag{12.49}$$

Multiplying (12.47) by u yields

$$\frac{\partial}{\partial t}(\frac{1}{2}u^2) + \frac{\partial}{\partial x}(uu_{xx} - \frac{1}{2}(u_x)^2 - 2u^3) = 0, \tag{12.50}$$

that gives the second law of conservation laws

$$\int_{-\infty}^{\infty} u^2 dx = \text{constant}. \tag{12.51}$$

Multiplying (12.47) by $3u^2$ gives

$$3u^2(u_t - 6uu_x + u_{xxx}) = 0. \tag{12.52}$$

Multiplying the partial derivative of (12.47) with respect to x by u_x gives

$$u_x(u_{xt} - 6(u_x)^2 - 6uu_{xx} + u_{xxxx}) = 0. \tag{12.53}$$

Adding the last two quantities yields

$$\frac{\partial}{\partial t}\left(u^3 - \frac{1}{2}(u_x)^2\right) + \frac{\partial}{\partial x}\left(-\frac{9}{2}u^4 + 3u^2 u_{xx} - 6u(u_x)^2 + u_x u_{xxx} - \frac{1}{2}(u_{xx})^2\right) = 0. \tag{12.54}$$

This gives the third conservation law of the KdV equation

$$\int_{-\infty}^{\infty}\left(u^3 - \frac{1}{2}(u_x)^2\right) dx = \text{constant}. \tag{12.55}$$

It was formally proved that there is an infinite set of conservation laws for the KdV equation.

The existence of conservation laws has been considered as an indication of the integrability of the KdV. There is an infinite set of independent conservation laws for the KdV equation. The first five conservation laws of this set are

$$\begin{aligned}
&\int_{-\infty}^{\infty} u dx = \text{constant},\\
&\int_{-\infty}^{\infty} u^2 dx = \text{constant},\\
&\int_{-\infty}^{\infty} (u^3 - \frac{1}{2}(u_x)^2) dx = \text{constant},\\
&\int_{-\infty}^{\infty} (5u^4 + 10u(u_x)^2 + (u_{xx})^2) dx = \text{constant},\\
&\int_{-\infty}^{\infty} (21u^5 + 105u^2(u_x)^2 + 21u(u_{xx})^2 + (u_{xxx})^2) dx = \text{constant},
\end{aligned} \tag{12.56}$$

where each conservation law includes a higher power of u than the preceding law.

In what follows, we list some of the conservation laws for specific well-known nonlinear equations.
(i) The modified KdV (mKdV) equation is given by

$$u_t - 6u^2u_x + u_{xxx} = 0, \tag{12.57}$$

has many conservation laws. Some of the conservation laws are given by $u, u^2, u^4 + (u_x)^2$. This in turn gives the first three conservation laws:

$$\begin{aligned} &T_1 = u, \quad X_1 = 2u^3 + u_{xx}, \\ &T_2 = \frac{1}{2}u^2, \quad X_2 = \frac{3}{2}u^4 + uu_{xx} - \frac{1}{2}u_x^2, \\ &T_3 = \frac{1}{4}u^4 - \frac{1}{4}u_x^2, \quad X_3 = u^6 + u^3u_{xx} - 3u^2u_x^2 - \frac{1}{2}u_xu_{xxx} + \frac{1}{4}u_{xx}^2. \end{aligned} \tag{12.58}$$

(ii) The Lax fifth-order equation is given by

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{3x} + u_{5x} = 0. \tag{12.59}$$

The conserved densities T_i of the Lax equation are given by

$$\begin{aligned} T_1 &= u, \\ T_2 &= \frac{1}{2}u^2, \\ T_3 &= \frac{1}{3}u^3 - \frac{1}{6}u_x^2, \\ T_4 &= \frac{1}{4}u^4 - \frac{1}{2}uu_x^2 + \frac{1}{20}u_{2x}^2. \end{aligned} \tag{12.60}$$

(iii) The Sawada-Kotera fifth-order equation is given by

$$u_t + 5u^2u_x + 5u_xu_{xx} + 5uu_{3x} + u_{5x} = 0. \tag{12.61}$$

The conserved densities T_i of the Sawada-Kotera equation are given by

$$\begin{aligned} T_1 &= u, \\ T_2 &= -, \\ T_3 &= \frac{1}{3}u^3 - u_x^2, \\ T_4 &= \frac{1}{4}u^4 - \frac{9}{4}uu_x^2 + \frac{3}{4}u_{2x}^2. \end{aligned} \tag{12.62}$$

Notice that the Sawada-Kotera equation does not have a conserved density that includes u^2.
(iv) The Kaup-Kuperschmidt equation fifth-order equation is given by

$$u_t + 20u^2u_x + 25u_xu_{xx} + 10uu_{3x} + u_{5x} = 0. \tag{12.63}$$

The conserved densities T_i of the Kaup-Kuperschmidt equation are given by

$$\begin{aligned}
T_1 &= u, \\
T_2 &= -, \\
T_3 &= \frac{1}{3}u^3 - \frac{1}{8}u_x^2, \\
T_4 &= \frac{1}{4}u^4 - \frac{9}{16}uu_x^2 + \frac{3}{64}u_{2x}^2.
\end{aligned} \tag{12.64}$$

(v) The Ito fifth-order equation is given by

$$u_t + 2u^2u_x + 6u_xu_{xx} + 3uu_{3x} + u_{5x} = 0. \tag{12.65}$$

The Ito equation has a limited number of special conservation laws, hence it is not completely integrable. It was found that the Ito equation has following three conserved densities

$$\begin{aligned}
T_1 &= u, \\
T_2 &= \frac{1}{2}u^2, \\
T_3 &= -, \\
T_4 &= \frac{1}{4}u^4 - \frac{9}{2}uu_x^2 + \frac{3}{4}u_{2x}^2.
\end{aligned} \tag{12.66}$$

(vi)The seventh-order KdV equation (sKdV) is given by

$$u_t + 6uu_x + u_{3x} - u_{5x} + \alpha u_{7x} = 0, \tag{12.67}$$

where α is a nonzero constant, and $u = u(x,t)$ is a sufficiently often differentiable function. The sKdV equation has three polynomial type conserved quantities given by:

$$\begin{aligned}
T_1 &= u, \\
T_2 &= u^2, \\
T_3 &= -u^3 + \frac{1}{2}(u_x)^2 - \frac{1}{2}(u_{xx})^2 + \frac{1}{2}\alpha(u_{3x})^2.
\end{aligned} \tag{12.68}$$

(vii) The Sawada-Kotera-Ito seventh-order equation is given by

$$u_t + 252u^2u_x + 63u_x^3 + 378uu_xu_{2x} + 126u^2u_{3x} + 63u_{2x}u_{3x} + 42u_xu_{4x} + 21uu_{5x} + u_{7x} = 0. \tag{12.69}$$

This equation has an infinite number of conservation laws as proved in [2] and others. The first four conserved densities are given by

$$\begin{aligned}
T_1 &= u, \\
T_2 &= -, \\
T_3 &= \frac{1}{3}u^3 - \frac{1}{3}u_x^2,
\end{aligned}$$

$$T_4 = \frac{1}{4}u^4 - \frac{3}{4}uu_x^2 + \frac{1}{12}u_{2x}^2. \tag{12.70}$$

(viii) The Lax seventh-order equation is given by

$$u_t + 140u^2u_x + 70u_x^3 + 280uu_xu_{2x} + 70u^2u_{3x} + 70u_{2x}u_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x} = 0. \tag{12.71}$$

This equation has an infinite number of conservation laws as proved in [2] and others. The first four conserved densities are given by

$$\begin{aligned} T_1 &= u, \\ T_2 &= \frac{1}{2}u^2, \\ T_3 &= \frac{1}{3}u^3 - \frac{1}{6}u_x^2, \\ T_4 &= \frac{1}{4}u^4 - \frac{1}{2}uu_x^2 + \frac{1}{20}u_{2x}^2. \end{aligned} \tag{12.72}$$

(ix) The Kaup-Kuperschmidt seventh-order equation is given by

$$\begin{aligned} &u_t + 2016u^2u_x + 630u_x^3 + 2268uu_xu_{2x} + 504u^2u_{3x} + 252u_{2x}u_{3x} + \\ &147u_xu_{4x} + 42uu_{5x} + u_{7x} = 0. \end{aligned} \tag{12.73}$$

This equation has an infinite number of conservation laws as proved in [2] and others. The first four conserved densities are given by

$$\begin{aligned} T_1 &= u, \\ T_2 &= -, \\ T_3 &= u^3 - \frac{1}{8}u_x^2, \\ T_4 &= u^4 - \frac{3}{4}uu_x^2 + \frac{1}{48}u_{2x}^2. \end{aligned} \tag{12.74}$$

(x) The Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \tag{12.75}$$

has the following conservation densities:

$$\begin{aligned} T_1 &= u, \\ T_2 &= u^2 + u_x^2, \\ T_3 &= u^3 + uu_x^2. \end{aligned} \tag{12.76}$$

(xi) The sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0 \tag{12.77}$$

has the following conservation laws:

$$
\begin{aligned}
&(\frac{1}{2}u_t^2)_x-(1-\cos u)_t=0,\\
&(1-\cos u)_x-(\frac{1}{2}u_x^2)_t=0,\\
&(\frac{1}{4}u_x^4-u_{xx}^2)_t+(u_x^2\cos u)_x=0,\\
&(\frac{1}{6}u_x^6-\frac{2}{3}u_x^2u_{xx}^2+\frac{8}{9}u_x^3u_{xxx}+\frac{4}{3}u_{xxx}^2)_t+(\frac{1}{9}u_x^4\cos u-\frac{4}{3}u_{xx}^2\cos u)_x=0.
\end{aligned}
\tag{12.78}
$$

(xii) The sinh-Gordon equation

$$
u_{xt}=\alpha\sinh u \tag{12.79}
$$

has the following conservation laws [3]:

$$
\begin{aligned}
T_1&=u_x^2,\\
T_2&=u_x^4+4u_{xx}^2,\\
T_3&=u_x^6+20u_x^2u_{xx}^2+8u_{xxx}^2,\\
T_4&=5u_x^8+280u_x^4u_{xx}^2-112u_{xx}^4+224u_x^2u_{xxx}^2+64u_{xxxx}^2.
\end{aligned}
\tag{12.80}
$$

(xiii) The Schrodinger equation

$$
iu_t+u_{xx}+q|u|^2u=0,\quad -\infty<x<\infty \tag{12.81}
$$

has the following conserved densities [1]:

$$
\begin{aligned}
T_1&=|u|^2,\\
T_2&=u^*u_x-u_x^*,\\
T_3&=|u_x|^2-\frac{q}{2}|u|^4,
\end{aligned}
\tag{12.82}
$$

where u^* is the complex conjugate of u.
(xiv) The Benjamin-Ono equation

$$
u_t+4uu_x+H(u_{xx})=0, \tag{12.83}
$$

where H is the Hilbert transform defined by

$$
H[u(x,t)]=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{u(y,t)}{y-x}dy, \tag{12.84}
$$

where P refers to the principal value of the integral. The Benjamin-Ono equation has the following conserved densities [1]

$$
T_1=u,\ T_2=u^2,\ T_3=u^3-3u_xH(u). \tag{12.85}
$$

References

1. P.G. Drazin and R.S. Johnson, Solitons: an Introduction, Cambridge University Press, Cambridge, (1996).
2. U. Goktas and W. Hereman, Symbolic computation of conserved densities for systems of nonlinear evolution equations, *J. Symbolic Comput.*, 24(5), 591–621, (1997).
3. W. Hereman, Shallow Water Waves and Solitary Waves. In: Encyclopedia of Complexity and Systems Science, Springer-Verlag, Heibelberg, (2009), to appear.
4. R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge, (2004).
5. Y.A. Li and P.J. Olver, Convergence of solitary-wave solutions in a perturbed bi-Hamiltonian dynamical system. I. Compactons, and Peakons, *Discr. Contin. Dyn. Syste.*, 3(3), 419–432, (1997).
6. Malfliet W., Solitary wave solutions of nonlinear wave equations, *Am. J. Phys.*, 60 (7), 650–654, (1992).
7. E.J. Parkes and V.O. Vakhnenko, Explicit solutions of the Camassa-Holm equation, *Chaos Solitons Fractals*, 26(5), 1309–1316, (2005).
8. P. Rosenau and J.M. Hyman, Compactons: Solitons with finite wavelengths, *Phys. Rev. Lett.*, 70(5), 564–567, (1993).
9. J.S. Russell, Report on Waves: 14th Meeting of the British Association for the Advancement of Science, John Murray, London, 311–390, (1844).
10. A.M. Wazwaz, Peakons, kinks, compactons and solitary patterns solutions for a family of Camassa–Holm equations by using new hyperbolic schemes, *Appl. Math. Comput.*, 182(1), 412–424, (2006).
11. N.J. Zabusky and M.D. Kruskal, Interaction of solitons incollisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.*, 15(6), 240–243, (1965).

Chapter 13
The Family of the KdV Equations

13.1 Introduction

The ubiquitous Korteweg de Vries equation [14] in dimensionless variables reads

$$u_t + auu_x + u_{xxx} = 0, \tag{13.1}$$

where subscripts denote partial derivatives. The parameter a can be scaled to any real number, where the commonly used values are $a = \pm 1$ or $a = \pm 6$. The KdV equation models a variety of nonlinear phenomena, including ion acoustic waves in plasmas, and shallow water waves. The derivative u_t characterizes the time evolution of the wave propagating in one direction, the nonlinear term uu_x describes the steepening of the wave, and the linear term u_{xxx} accounts for the spreading or dispersion of the wave. The KdV equation was derived by Korteweg and de Vries to describe shallow water waves of long wavelength and small amplitude. The KdV equation is a nonlinear evolution equation that models a diversity of important finite amplitude dispersive wave phenomena. It has also been used to describe a number of important physical phenomena such as acoustic waves in a harmonic crystal and ion-acoustic waves in plasmas. As stated before, this equation is the simplest nonlinear equation embodying two effects: nonlinearity represented by uu_x, and linear dispersion represented by u_{xxx}. Nonlinearity of uu_x tends to localize the wave whereas dispersion spreads the wave out. The delicate balance between the weak nonlinearity of uu_x and the linear dispersion of u_{xxx} defines the formulation of solitons that consist of single humped waves. The stability of solitons is a result of the delicate equilibrium between the two effects of nonlinearity and dispersion. This equation is the pioneer of model equations that gives soliton solutions which characterize solitary waves that decrease monotonically at infinity [1–4].

The function $u(x,t)$ represents the water's free surface in non-dimensional variables. The nonlinear KdV equation gives a large variety of solutions. The solutions propagate at speed c while retaining its identity. We usually introduce the new wave variable $\xi = x - ct$, so that

$$u(x,t) = u(\xi). \tag{13.2}$$

The soliton solution is spatially localized solution, hence $u', u'', u''' \to 0$ as $\xi \to \pm\infty$, $\xi = x - ct$.

To give a preliminary approach for solving the KdV equation, we substitute (13.2) into (13.1) to obtain

$$-cu' + 6uu' + u''' = 0, \tag{13.3}$$

that gives

$$-cu + 3u^2 + u'' = 0, \tag{13.4}$$

upon integrating (13.3), where constant of integration is taken to be zero. Multiplying (13.4) by $2u'$ and integrating the resulting equation we find

$$(u')^2 = cu^2 - 2u^3, \tag{13.5}$$

or equivalently

$$\frac{du}{\sqrt{cu^2 - 2u^3}} = d\xi. \tag{13.6}$$

Using a change of variable gives the solution

$$u(\xi) = \frac{c}{2}\operatorname{sech}^2 \frac{\sqrt{c}}{2}\,\xi, \tag{13.7}$$

or equivalently

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2 \frac{\sqrt{c}}{2}(x - ct). \tag{13.8}$$

It is obvious that $u(x,t)$ in (13.8), along with its derivatives, approaches zero as $\xi \to \infty$. It is also clear from (13.8) that the amplitude of the wave is directly proportional to its speed c, and this in turn means that the taller the wave the faster it moves. It moves to the right for $(-c)$ and to the left if we replace $(-c)$ by $(+c)$. It is also clear that the wave has no dispersion because of the balance between the dispersion effect and the nonlinear effect. Consequently, the wave retains its identity and shape.

Based on (13.8), the following

$$u(x,t) = -\frac{c}{2}\operatorname{csch}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right), \tag{13.9}$$

is also a travelling wave solution of the KdV equation. Notice that the last solution is not a soliton, because $u(x,t)$ is unbounded at $\xi = 0$.

The KdV equation can also be approached by using the Bäcklund transformation. To achieve this, we introduce a function v such that $u = v_x$. This transformation will carry out the KdV equation (13.1) to

$$v_{xt} + 6v_x v_{xx} + v_{xxxx} = 0, \tag{13.10}$$

where by integrating this equation with respect to x we find

$$v_t + 3(v_x)^2 + v_{xxx} = 0. \tag{13.11}$$

The last equation is called the potential KdV equation that will be examined later in this chapter. Using the wave variable $\xi = x - ct$, and proceeding as before we can easily obtain the solutions

$$\begin{aligned} v &= \sqrt{c}\tanh(\frac{\sqrt{c}}{2}(x-ct)), \\ v &= \sqrt{c}\coth(\frac{\sqrt{c}}{2}(x-ct)). \end{aligned} \tag{13.12}$$

Recall that $u = v_x$, hence we find

$$\begin{aligned} u(x,t) &= \frac{c}{2}\text{sech}^2(\frac{\sqrt{c}}{2}(x-ct)), \\ u(x,t) &= -\frac{c}{2}\text{csch}^2(\frac{\sqrt{c}}{2}(x-ct)). \end{aligned} \tag{13.13}$$

13.2 The Family of the KdV Equations

This section focuses on the family of the KdV equations. The canonical KdV equation is a nonlinear dispersive equation of third order. However, the KdV equations appear in three, five, seven or more order forms. Other modified forms that include changes in the nonlinearity of uu_x will be examined as well. In what follows, a brief summary of these forms will be given. The complete analysis of each form will be addressed in the forthcoming sections.

13.2.1 Third-order KdV Equations

The family of third order Korteweg-de Vries is of the form

$$u_t + f(u)u_x + u_{xxx} = 0, \tag{13.14}$$

where $u(x,t)$ is a function of space x and time variable t. Constants can be used as coefficients of $f(u)u_x$ and u_{xxx}, but these constants can be usually scaled out. The nonlinear term $f(u)$ appears in the following forms

$$f(u) = \begin{cases} \alpha u, \\ \alpha u^2, \\ \alpha u^n, \\ \alpha u_x, \\ 2\alpha u - 3\beta u^2, \\ \alpha u^n - \beta u^{2n}. \end{cases} \tag{13.15}$$

(i) For $f(u) = \pm 6u$ we obtain one of the standard KdV equations

$$u_t \pm 6uu_x + u_{xxx} = 0, \tag{13.16}$$

where the factor ± 6 is appropriate for complete integrability. Also, $f(u) = \pm u$ is also used. The complete integrability means that the KdV equation has N-soliton solutions as will be presented later.

(ii) For $f(u) = 6u^2$, Eq. (13.14) is called the modified KdV (mKdV) equation given by

$$u_t + 6u^2 u_x + u_{xxx} = 0. \tag{13.17}$$

The mKdV equation is identical to the KdV equation in that both are completely integrable and each has infinitely many conserved quantities. The mKdV equation appears in electric circuits and multi-component plasmas. The mKdV equation gives algebraic solitons solutions in the form of a rational function. Stability and instability conditions of algebraic solitons of the mKdV equation have been investigated thoroughly in [1,2].

(iii) For $f(u) = \alpha u^n, n \geqslant 3$, Eq. (13.14) is called the generalized KdV (gKdV) equation given by

$$u_t + \alpha u^n u_x + u_{xxx} = 0, \quad n \geqslant 3. \tag{13.18}$$

Unlike the KdV equation and the mKdV equation, the generalized KdV equation (13.18) is not integrable for $n \geqslant 3$, and therefore does not give multiple-soliton solutions [8,9].

(iv) For $f(u) = \alpha u_x$, Eq. (13.14) is called the potential KdV equation given by

$$u_t + \alpha (u_x)^2 + u_{3x} = 0. \tag{13.19}$$

As stated before, the potential KdV equation [4] can be obtained from the standard KdV equation by setting $u = v_x$ and integrating the resulting equation with respect to x.

(v) For $f(u) = 2\alpha u - 3\beta u^2, \alpha, \beta > 0$, Eq. (13.14) is called the Gardner equation [5] or the combined KdV-mKdV equation, given by

$$u_t + (2\alpha u - 3\beta u^2) u_x + u_{xxx} = 0. \tag{13.20}$$

The Gardner equation is widely used in various branches of physics, such as plasma physics, fluid physics, quantum field theory. The equation plays a prominent role in ocean waves. The Gardner equation describes internal waves and admits quite interesting tanh type solutions. The Gardner equation has been investigated thoroughly in the literature because it is used to model a variety of nonlinear phenomena.

(vi) For $f(u) = \alpha u^n - \beta u^{2n}$ we obtain another generalized KdV equation with two power nonlinearities of the form

$$u_t + (\alpha u^n - \beta u^{2n}) u_x + u_{xxx} = 0. \tag{13.21}$$

This last equation models the propagation of nonlinear long acoustic-type waves [19]. The function f', where $f = (\frac{\alpha}{n+1}u^{n+1} - \frac{\beta}{2n+1}u^{2n+1})$ is regarded as a nonlinear correction to the limiting long-wave phase speed c. If the amplitude is not supposed to be small, Eq. (13.21) serves as an approximate model for the description of weak dispersive effects on the propagation of nonlinear waves along a characteristic direction. The well-known Gardner equation, that is also called the combined KdV-mKdV equation, can be obtained by setting $n = 1$ in Eq. (13.21).

Equation (13.21) appears in many scientific applications and gives rise to a variety of solitons. For this reason, this equation has been subjected to thorough studies in [19,21] and the references therein. The main focus of these studies was the solitary wave solutions, collapsing solitons, algebraic solitons, and solitary wave instability. Algebraic solitons are solitons that decay to zero at infinity or approach nonzero boundary values at an algebraic rate.

13.2.2 The **K**(n,n) *Equation*

It was stated before that the KdV equation is characterized by the presence of the weak nonlinearity term uu_x and the linear dispersion term u_{xxx}. The delicate balance between the two effects gives rise to solitons. However, a KdV-like equation was introduced by Rosenau *et. al* [18] and given by

$$u_t + a(u^n)_x + b(u^n)_{xxx} = 0. \tag{13.22}$$

The K(n,n) equation (13.22) is characterized by the genuinely nonlinear term $(u^n)_x$ and the genuinely nonlinear dispersion term $(u^n)_{xxx}$. The balance between the nonlinear convection term $(u^n)_x$ and the genuinely dispersion term $(u^n)_{xxx}$ gives rise to the so-called *compacton*, solitary wave with compact support and without tails or wings. Equation (13.22) was thoroughly studied in the literature, and it was found that other nonlinear evolution equations possess the property of introducing compacton solutions. The K(n,n) equation and the compactons phenomena were presented in the previous chapter and will be examined in this chapter as well.

13.3 The KdV Equation

The ubiquitous KdV equation in dimensionless variables reads

$$u_t + auu_x + u_{xxx} = 0. \tag{13.23}$$

This equation models a variety of nonlinear wave phenomena [10–13] such as shallow water waves, acoustic waves in a harmonic crystal, and ion-acoustic waves in plasmas. The KdV equation is completely integrable and gives rise to multiple-soliton solutions. The KdV equation has been studied by a variety of methods such as the inverse scattering method and the Bäcklund transformation method. Other methods were used as well.

As stated before, in this chapter and the forthcoming chapters, we will use the tanh-coth method, sine-cosine method, or both for analyzing the nonlinear equations under discussion. Moreover, the Hirota's direct method [10–13] will be used for completely integrable equations.

We first substitute the wave variable $\xi = x - ct$, c is the wave speed, into (13.23) and integrating once to obtain

$$-cu + \frac{a}{2}u^2 + u'' = 0. \tag{13.24}$$

13.3.1 Using the Tanh-coth Method

We first balance the terms u^2 with u''. This means that the highest power of u^2 is $2M$, and for u'' is $M+2$ obtained by using the scheme for the balance process presented in the previous chapter. Using the balance process leads to

$$2M = M + 2, \tag{13.25}$$

that gives

$$M = 2. \tag{13.26}$$

The tanh-coth method [23,24] allows us to use the substitution

$$u(x,t) = S(Y) = \sum_{j=0}^{2} a_j Y^j + \sum_{i=1}^{2} b_i Y^{-i}. \tag{13.27}$$

Substituting (13.27) into (13.24), collecting the coefficients of each power of $Y^r, 0 \leqslant r \leqslant 8$, setting each coefficient to zero, and solving the resulting system of algebraic equations we find the following sets of solutions

(i)

$$a_0 = \frac{3c}{a}, \quad a_1 = a_2 = b_1 = 0, \quad b_2 = -\frac{3c}{a}, \quad \mu = \frac{1}{2}\sqrt{c},\ c > 0. \tag{13.28}$$

(ii)

$$a_0 = -\frac{c}{a}, \quad a_1 = a_2 = b_1 = 0, \quad b_2 = \frac{3c}{a}, \quad \mu = \frac{1}{2}\sqrt{-c},\ c < 0. \tag{13.29}$$

(iii)

$$a_0 = \frac{3c}{a}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = -\frac{3c}{a}, \quad \mu = \frac{1}{2}\sqrt{c},\ c > 0. \tag{13.30}$$

(iv)

$$a_0 = -\frac{c}{a}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = \frac{3c}{a}, \quad \mu = \frac{1}{2}\sqrt{-c},\ c < 0. \tag{13.31}$$

Consequently, we obtain the following soliton solutions

$$\begin{aligned} u_1(x,t) &= \frac{3c}{a}\text{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct)\right], \quad c > 0, \\ u_2(x,t) &= -\frac{c}{a}\left(1 - 3\tanh^2\left[\frac{\sqrt{-c}}{2}(x-ct)\right]\right), \quad c < 0. \end{aligned} \tag{13.32}$$

Moreover, the travelling wave solutions

$$\begin{aligned} u_3(x,t) &= -\frac{3c}{a}\text{csch}^2\left[\frac{\sqrt{c}}{2}(x-ct)\right], \quad c > 0, \\ u_4(x,t) &= -\frac{c}{a}\left(1 - 3\coth^2\left[\frac{\sqrt{-c}}{2}(x-ct)\right]\right), \quad c < 0. \end{aligned} \tag{13.33}$$

follow immediately. Fig. 13.1 below shows a graph of a one-soliton solution $u_1(x,t)$ of (13.32) for $a = 3, c = 1$. The graph is characterized by infinite wings or infinite tails. This shows that $u \to 0$ as $\xi \to \pm\infty$, $\xi = x - ct$.

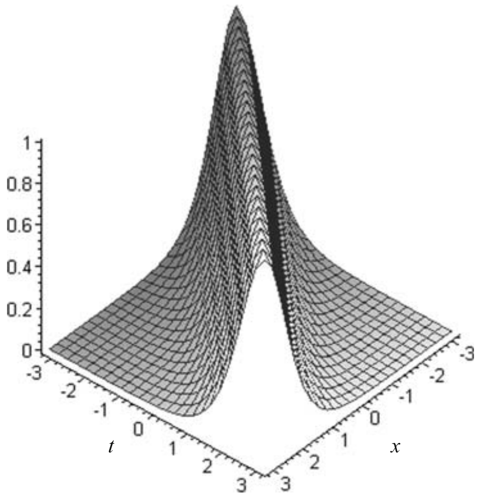

Fig. 13.1 Graph of the soliton solution $u_1, a = 3, c = 1$ characterized by an infinite wing.

The physical structures of the obtained solutions in (13.32) depend mainly on the sign of the wave speed c whether $c > 0$ or $c < 0$. Consequently, we obtain the following plane periodic solutions:

$$u_5(x,t) = \frac{3c}{a}\csc^2\left[\frac{\sqrt{-c}}{2}(x-ct)\right], \quad c < 0,$$

$$\begin{aligned}
u_6(x,t) &= -\frac{c}{a}\left(1+3\cot^2\left[\frac{\sqrt{c}}{2}(x-ct)\right]\right), \quad c>0,\\
u_7(x,t) &= \frac{3c}{a}\sec^2\left[\frac{\sqrt{-c}}{2}(x-ct)\right], \quad c<0,\\
u_8(x,t) &= -\frac{c}{a}\left(1+3\tan^2\left[\frac{\sqrt{c}}{2}(x-ct)\right]\right), \quad c>0.
\end{aligned} \tag{13.34}$$

13.3.2 Using the Sine-cosine Method

Substituting the cosine assumption into the reduced equation (13.24) gives

$$\begin{aligned}
&-c\lambda\cos^{\beta}(\mu\xi)+\frac{a}{2}\lambda^2\cos^{2\beta}(\mu\xi)\\
&-\lambda\mu^2\beta^2\cos^{\beta}(\mu\xi)+\lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi)=0.
\end{aligned} \tag{13.35}$$

We should also use the balance between the exponents of the cosine functions. This means that Eq. (13.35) is satisfied only if the following system of algebraic equations holds

$$\begin{aligned}
\beta-1 &\neq 0,\\
2\beta &= \beta-2,\\
\mu^2\beta^2\lambda &= -c\lambda,\\
\frac{a}{2}\lambda^2 &= -c\lambda\mu^2\beta(\beta-1).
\end{aligned} \tag{13.36}$$

This in turn gives

$$\begin{aligned}
\beta &= -2,\\
\mu &= \frac{1}{2}\sqrt{-c},\\
\lambda &= \frac{3c}{a}.
\end{aligned} \tag{13.37}$$

The results in (13.37) can be easily obtained if we also use the sine assumption. Moreover, the last results give the solutions $u_1(x,t), u_3(x,t)$, and $u_7(x,t)$ that were obtained before. It is easily observed that the tanh-coth method gives more solutions than the sine-cosine method.

13.3.3 Multiple-soliton Solutions of the KdV Equation

In this section, we will examine multiple-soliton solutions of the canonical KdV equation

$$u_t+6uu_x+u_{xxx}=0. \tag{13.38}$$

Hirota [10–12] established a method for the determination of exact solutions of nonlinear PDEs. The method is called the Hirota's direct method or the Hirota's bilinear formalism. A necessary condition for the direct method to be applicable is that the PDE can be brought into a bilinear form. Hirota proposed a bilinear form where it was shown that soliton solutions are just polynomials of exponentials. Finding bilinear forms for nonlinear PDEs, if they exist at all, is highly nontrivial [6,7]. Considering $u(x,t) = 2(\ln(f))_{xx}$, the *bilinear form* for the KdV equation is

$$B(f,f) = (D_x^4 + D_x D_t)(f \cdot f) = 0.$$

Hereman *et.al.* [6,7] introduced a simplified version of Hirota's method, where exact solitons can be obtained by solving a perturbation scheme using a symbolic manipulation package, and without any need to use bilinear forms. In what follows, we summarize the main steps of the simplified version of Hirota's method.

The simplified version of Hirota method introduces the change of dependent variable

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{f f_{2x} - (f_x)^2}{f^2}, \tag{13.39}$$

to carry out the KdV equation (13.38) into a quadratic equation of the form

$$[f(f_{xt} + f_{4x})] - \left[f_x f_t + 4 f_x f_{3x} - 3 f_{2x}^2\right] = 0. \tag{13.40}$$

Equation (13.40) can be decomposed into linear operator L and nonlinear operator N [7] defined by

$$\begin{aligned} L &= \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4}, \\ N(f,f) &= -f_x f_t - 4 f_x f_{3x} + 3 f_{2x} f_{2x}. \end{aligned} \tag{13.41}$$

The function $f(x,t)$ is assumed to have a perturbation expansion given by

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \tag{13.42}$$

where ε is a non small formal expansion parameter. Following Hirota's method [10–12] and the simplified version of Hereman *et. al.* established in [7], we substitute (13.42) into (13.41) and equate to zero the powers of ε to obtain

$$\begin{aligned} &O(\varepsilon^0) : B(1 \cdot 1) = 0, \\ &O(\varepsilon^1) : B(1 \cdot f_1 + f_1 \cdot 1) = 0, \\ &O(\varepsilon^2) : B(1 \cdot f_2 + f_1 \cdot f_1 + f_2 \cdot 1) = 0, \\ &O(\varepsilon^3) : B(1 \cdot f_3 + f_1 \cdot f_2 + f_2 \cdot f_1 + f_3 \cdot 1) = 0, \\ &O(\varepsilon^4) : B(1 \cdot f_4 + f_1 \cdot f_3 + f_2 \cdot f_2 + f_3 \cdot f_1 + f_4 \cdot 1) = 0, \\ &O(\varepsilon^5) : B(1 \cdot f_5 + f_1 \cdot f_4 + f_2 \cdot f_3 + f_3 \cdot f_2 + f_4 \cdot f_1 + f_5 \cdot 1) = 0, \\ &\qquad \vdots \\ &O(\varepsilon^n) : B(\textstyle\sum_{j=0}^{n} f_j \cdot f_{n-j}) = 0, \end{aligned}$$

where the bilinear form B is defined by

$$B(f,f) = (D_x^4 + D_x D_t)(f \cdot f) = 0.$$

It is interesting to note that the previous scheme is the same for every bilinear operator B. In other words we can set

$$
\begin{aligned}
O(\varepsilon^1) &: Lf_1 = 0, && (13.43)\\
O(\varepsilon^2) &: Lf_2 = -N(f_1,f_1), && (13.44)\\
O(\varepsilon^3) &: Lf_3 = -f_1Lf_2 - f_2Lf_1 - N(f_1,f_2) - N(f_2,f_1), && (13.45)\\
O(\varepsilon^4) &: Lf_4 = -f_1Lf_3 - f_2Lf_2 - f_3Lf_1 && (13.46)\\
&\qquad -N(f_1,f_3) - N(f_2,f_2) - N(f_3,f_1), &&\\
O(\varepsilon^5) &: Lf_5 = -f_1Lf_4 - f_2Lf_3 - f_3Lf_2 - f_4Lf_1 && (13.47)\\
&\qquad -N(f_1,f_4) - N(f_2,f_3) - N(f_3,f_2) - N(f_4,f_1), &&\\
&\vdots &&\\
O(\varepsilon^n) &: Lf_n = -\sum_{j=1}^{n-1} [f_jLf_{n-j} + N(f_j,f_{n-j})] = 0. && (13.48)
\end{aligned}
$$

The N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i), \tag{13.49}$$

where

$$\theta_i = k_ix - c_it, \tag{13.50}$$

where k_i and c_i are arbitrary constants, k_i is called the wave number. Substituting

$$u(x,t) = e^{k_ix - c_it} \tag{13.51}$$

into the linear terms of the KdV equation (13.38) gives the dispersion relation

$$c_i = k_i^3. \tag{13.52}$$

We therefore find

$$\theta_i = k_ix - k_i^3t. \tag{13.53}$$

This means that

$$f_1 = \exp(\theta_1) = \exp(k_1(x - k_1^2t)), \tag{13.54}$$

obtained by using $N = 1$ in (13.49).

For the one-soliton solution, we set

$$f = 1 + \exp(\theta_1) = 1 + \exp(k_1(x - k_1^2t)), \tag{13.55}$$

where we used $\varepsilon = 1$. Recall that $u(x,t) = 2(\ln f)_{xx}$. This means that the one soliton solution is given by

$$u(x,t) = \frac{2k_1^2\exp(k_1(x - k_1^2t))}{1 + \exp(k_1(x - k_1^2t))^2}, \tag{13.56}$$

or equivalently

$$u(x,t) = \frac{k_1^2}{2}\operatorname{sech}^2\left[\frac{k_1}{2}(x - k_1^2 t)\right]. \tag{13.57}$$

Setting $k_1 = \sqrt{c}$ in (13.57) gives the one-soliton solution obtained above by using the tanh-coth and the sine-cosine methods.

To determine the two-soliton solutions, we first use $N = 2$ in (13.49) to get

$$f_1 = \exp(\theta_1) + \exp(\theta_2). \tag{13.58}$$

To determine f_2, we substitute (13.58) into (13.45) to evaluate the right hand side and equate it with the left hand side to obtain

$$f_2 = \sum_{1\leqslant i<j\leqslant N} a_{ij}\exp(\theta_i + \theta_j), \tag{13.59}$$

where the phase factor a_{ij} is given by

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \tag{13.60}$$

and θ_i and θ_j are given above in (13.53).

For the two-soliton solution we use $1 \leqslant i < j \leqslant 2$, and therefore we obtain

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12}\exp(\theta_1 + \theta_2), \tag{13.61}$$

where the phase factor a_{12} is given by

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}. \tag{13.62}$$

This in turn gives

$$f(x,t) = 1 + e^{k_1(x-k_1^2 t)} + e^{k_2(x-k_2^2 t)} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{(k_1+k_2)x-(k_1^3+k_2^3)t}. \tag{13.63}$$

To determine the two-soliton solutions explicitly, we use (13.39) for the function f in (13.63). Fig. 13.2 below shows a two-soliton solution for $k_1 = 1, k_2 = 2, -20 \leqslant x,t \leqslant 20$.

To determine f_3 we follow the discussion presented before. We therefore set

$$\begin{aligned} f_1(x,t) &= \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3), \\ f_2(x,t) &= a_{12}\exp(\theta_1 + \theta_2) + a_{23}\exp(\theta_2 + \theta_3) + a_{13}\exp(\theta_1 + \theta_3), \end{aligned} \tag{13.64}$$

and consequently we have

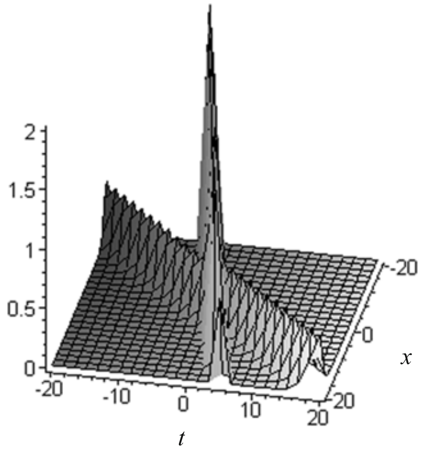

Fig. 13.2 A two-soliton solution graph for $k_1 = 1, k_2 = 2, -20 \leqslant x, t \leqslant 20$.

$$\begin{aligned} f(x,t) = {} & 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ & + a_{12}\exp(\theta_1 + \theta_2) + a_{23}\exp(\theta_2 + \theta_3) + a_{13}\exp(\theta_1 + \theta_3) \\ & + f_3(x,t). \end{aligned} \tag{13.65}$$

Substituting (13.65) into (13.49) and proceeding as before we find

$$f_3 = b_{123}\exp(\theta_1 + \theta_2 + \theta_3), \tag{13.66}$$

where

$$b_{123} = a_{12}a_{13}a_{23} = \frac{(k_1 - k_2)^2(k_1 - k_3)^2(k_2 - k_3)^2}{(k_1 + k_2)^2(k_1 + k_3)^2(k_2 + k_3)^2}, \tag{13.67}$$

and θ_1, θ_2 and θ_3 are given above in (13.53). For the three-soliton solutions we use $1 \leqslant i < j \leqslant 3$

$$\begin{aligned} f(x,t) = {} & 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ & + a_{12}\exp(\theta_1 + \theta_2) + a_{23}\exp(\theta_2 + \theta_3) + a_{13}\exp(\theta_1 + \theta_3) \\ & + b_{123}\exp(\theta_1 + \theta_2 + \theta_3). \end{aligned} \tag{13.68}$$

This in turn gives

$$\begin{aligned} f(x,t) = {} & 1 + e^{k_1(x - k_1^2 t)} + e^{k_2(x - k_2^2 t)} + e^{k_3(x - k_3^2 t)} \\ & + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{(k_1 + k_2)x - (k_1^3 + k_2^3)t} + \frac{(k_1 - k_3)^2}{(k_1 + k_3)^2} e^{(k_1 + k_3)x - (k_1^3 + k_3^3)t} \\ & + \frac{(k_2 - k_3)^2}{(k_2 + k_3)^2} e^{(k_2 + k_3)x - (k_2^3 + k_3^3)t} \\ & + \frac{(k_1 - k_2)^2(k_1 - k_3)^2(k_2 - k_3)^2}{(k_1 + k_2)^2(k_1 + k_3)^2(k_2 + k_3)^2} e^{(k_1 + k_2 + k_3)x - (k_1^3 + k_2^3 + k_3^3)t}. \end{aligned} \tag{13.69}$$

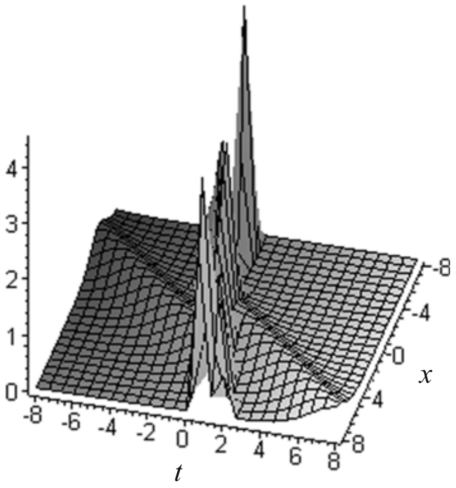

Fig. 13.3 Three-soliton solutions graph for $k_j = j, i \leqslant j \leqslant 3, -8 \leqslant x, t \leqslant 8$.

To determine the three-soliton solutions explicitly, we use (13.39) for the function $f(x,t)$ in (13.69). Fig. 13.3 above shows a three soliton solutions for $-8 \leqslant x, t \leqslant 8$.

In a parallel manner, we can determine $f_4(x,t)$ where we can easily show that

$$f_4(x,t) = c_{1234}\exp(\theta_1 + \theta_2 + \theta_3 + \theta_4), \tag{13.70}$$

where

$$\begin{aligned} c_{1234} &= a_{12}a_{13}a_{14}a_{23}a_{24}a_{34} \\ &= \frac{(k_1-k_2)^2(k_1-k_3)^2(k_1-k_4)^2(k_2-k_3)^2(k_2-k_4)^2(k_3-k_4)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_1+k_4)^2(k_2+k_3)^2(k_2+k_4)^2(k_3+k_4)^2}, \end{aligned} \tag{13.71}$$

and $\theta_i, 1 \leqslant i \leqslant 4$ are given above in (13.53). For the four-soliton solution we use

$$\begin{aligned} f(x,t) = {} & 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) + \exp(\theta_4) \\ & + a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{14}\exp(\theta_1+\theta_4) \\ & + a_{23}\exp(\theta_2+\theta_3) + a_{24}\exp(\theta_2+\theta_4) + a_{34}\exp(\theta_3+\theta_4) \\ & + b_{123}\exp(\theta_1+\theta_2+\theta_3) + b_{134}\exp(\theta_1+\theta_3+\theta_4) \\ & + b_{124}\exp(\theta_1+\theta_2+\theta_4) + b_{234}\exp(\theta_2+\theta_3+\theta_4) \\ & + c_{1234}\exp(\theta_1+\theta_2+\theta_3+\theta_4), \end{aligned} \tag{13.72}$$

and proceed as before to obtain

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, \quad 1 \leqslant i < j \leqslant 4, \quad b_{ijk} = a_{ij}a_{ik}a_{jk}. \tag{13.73}$$

This in turn gives

$$
\begin{aligned}
f(x,t) &= 1+e^{k_1(x-k_1^2t)}+e^{k_2(x-k_2^2t)}+e^{k_3(x-k_3^2t)}+e^{k_4(x-k_4^2t)}\\
&\quad +a_{12}e^{(k_1+k_2)x-(k_1^3+k_2^3)t}+a_{13}e^{(k_1+k_3)x-(k_1^3+k_3^3)t}\\
&\quad +a_{14}e^{(k_1+k_4)x-(k_1^3+k_4^3)t}+a_{23}e^{(k_2+k_3)x-(k_2^3+k_3^3)t}\\
&\quad +a_{24}e^{(k_2+k_4)x-(k_2^3+k_4^3)t}+a_{34}e^{(k_3+k_4)x-(k_3^3+k_4^3)t}\\
&\quad +b_{123}e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t}+b_{134}e^{(k_1+k_3+k_4)x-(k_1^3+k_3^3+k_4^3)t}\\
&\quad +b_{124}e^{(k_1+k_2+k_4)x-(k_1^3+k_2^3+k_4^3)t}+b_{234}e^{(k_2+k_3+k_4)x-(k_2^3+k_3^3+k_4^3)t}\\
&\quad +c_{1234}\,e^{(k_1+k_2+k_3+k_4)x-(k_1^3+k_2^3+k_3^3+k_4^3)t}.
\end{aligned}
\tag{13.74}
$$

To determine the four-soliton solutions explicitly, we use (13.39) for the function $f(x,t)$ in (13.74).

In conclusion we summarize the necessary steps needed to obtain the multiple soliton solutions for the completely integrable equations. The multiple-soliton solutions of the KdV equation can be formally constructed by using $u(x,t) = 2(\ln f(x,t))_{xx}$ where $f(x,t)$ is given as:
(i) for one-soliton solution:

$$f = 1+e^{\theta_1},$$

(ii) for two-soliton solutions:

$$f = 1+e^{\theta_1}+e^{\theta_2}+a_{12}e^{\theta_1+\theta_2},$$

(iii) for three-soliton solutions:

$$f = 1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+a_{12}e^{\theta_1+\theta_2}+a_{13}e^{\theta_1+\theta_3}+a_{23}e^{\theta_2+\theta_3}+a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3},$$

and so on.

Three facts should be emphasized as a result to the analysis presented above:
(i) The first is that soliton solutions are just polynomials of exponentials as emphasized by Hirota [10–12] and others such as in [6,7,15,22].
(ii) The three-soliton solutions and the higher level soliton solution as well, do not contain any new free parameters other than a_{ij} derived for the two-soliton solutions.
(iii) Every solitonic equation that has generic $N = 3$ soliton solutions, then it has also soliton solutions for any $N \geqslant 4$ [10–12]. In other words, completely integrable equations give multiple-soliton solutions. Some equations as the ninth-order KdV equation [25] has one and two-soliton solutions only, hence it is not completely integrable.

To summarize, if we set $k_j = j$ we obtain the following functions

$$
\begin{aligned}
f(x,t) &= 1+e^{x-t},\\
f(x,t) &= 1+e^{x-t}+e^{2(x-4t)}+\frac{1}{9}e^{3(x-3t)},
\end{aligned}
$$

$$
\begin{aligned}
f(x,t) &= 1+e^{x-t}+e^{2(x-4t)}+e^{3(x-9t)}\\
&\quad +\frac{1}{9}e^{3(x-3t)}+\frac{1}{4}e^{4(x-7t)}+\frac{1}{25}e^{5(x-7t)}+\frac{1}{900}e^{6(x-6t)},\\
f(x,t) &= 1+e^{x-t}+e^{2(x-4t)}+e^{3(x-9t)}+e^{4(x-16t)}+\frac{1}{9}e^{3(x-3t)}\\
&\quad +\frac{1}{4}e^{4(x-7t)}+\frac{9}{25}e^{5(x-13t)}+\frac{2}{25}e^{5(x-7t)}+\frac{1}{9}e^{6(x-12t)}\\
&\quad +\frac{1}{49}e^{7(x-13t)}+\frac{1}{450}e^{6(x-6t)}+\frac{1}{225}e^{(7x-73t)}\\
&\quad +\frac{9}{4900}e^{(8x-92t)}+\frac{1}{11025}e^{9(x-11t)}+\frac{1}{1102500}e^{10(x-10t)}.
\end{aligned}
\tag{13.75}
$$

The related soliton solutions can be easily obtained by substituting $f(x,t)$ from (13.75) into

$$u(x,t)=2(\ln(f))_{xx}, \tag{13.76}$$

to obtain the one, two, three, and four soliton solutions respectively. Other distinct values of k_j can be used. In practical, there is no need to derive the four soliton solutions if the three-soliton solutions are obtained. It is therefore sufficient to obtain soliton solutions for $N=1,2,3$ to show that multiple-soliton solutions exist. As stated before, the existence of the one and the two-soliton solutions only means that the evolution equation is not completely integrable.

Fig. 13.4 below shows graphs of one, two, three and four soliton solutions where $u(x,t)$ is plotted against the spatial variable x for fixed time t.

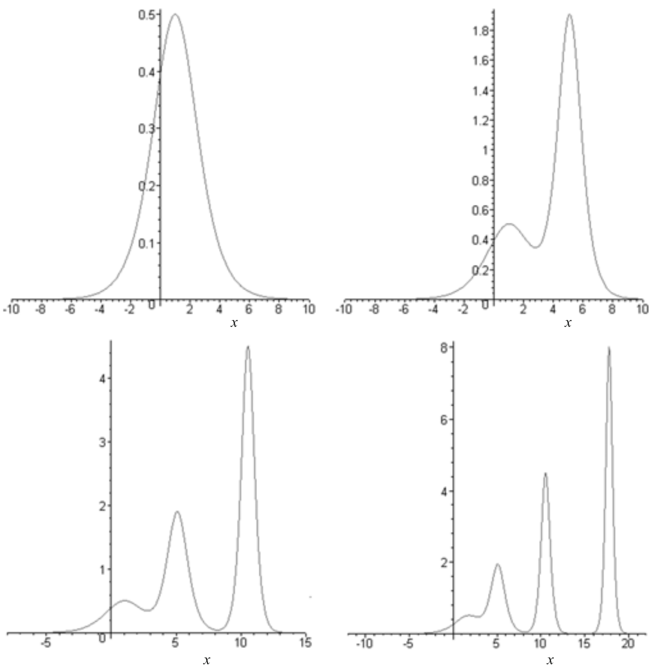

Fig. 13.4 One-soliton, two-soliton, three-soliton, and four-soliton solutions (from top to bottom).

13.4 The Modified KdV Equation

The modified KdV (mKdV) equation reads

$$u_t - 6u^2u_x + u_{xxx} = 0. \tag{13.77}$$

The mKdV equation is completely integrable [13,15,20], and is known to exhibit N-soliton solutions and an infinite number of conserved densities. Recall that a conservation law is given by the relation

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0,$$

where T and X are the density and flux respectively. This in turn gives the first three conservation laws:

$$T_1 = u, \quad X_1 = 2u^3 + u_{xx};$$
$$T_2 = \frac{1}{2}u^2, \quad X_2 = \frac{3}{2}u^4 + uu_{xx} - \frac{1}{2}u_x^2;$$
$$T_3 = \frac{1}{4}u^4 - \frac{1}{4}u_x^2, \quad X_3 = u^6 + u^3u_{xx} - 3u^2u_x^2 - \frac{1}{2}u_xu_{xxx} + \frac{1}{4}u_{xx}^2.$$

The modified KdV equation differs from the original KdV equation in the nonlinear term only, where it includes u^2u_x instead of uu_x but both include the dispersion term u_{xxx}. This change in the nonlinear term causes several substantial differences in the structures of the solutions. However, the KdV and the mKdV equations are linked at a deeper level by the so called Miura transformation [16], given by

$$u = v^2 + v_x, \tag{13.78}$$

that gives

$$\begin{aligned}
u_t &= 2vv_t + v_{xt},\\
u_x &= 2vv_x + v_{xx},\\
v_{xx} &= 2vv_{xx} + 2v_x^2 + v_{xxx},\\
v_{xxx} &= 2vv_{xxx} + 6v_xv_{xx} + v_{xxxx}.
\end{aligned} \tag{13.79}$$

Substituting the Miura transformation (13.78)–(13.79) into the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \tag{13.80}$$

leads to

$$\left(2v + \frac{\partial}{\partial x}\right)(v_t - 6v^2v_x + v_{xxx}) = 0. \tag{13.81}$$

This in turn gives the mKdV equation

$$v^2 - 6vv_x + v_{xxx} = 0. \tag{13.82}$$

This indicates that if v is a solution of the mKdV equation (13.82), then the solution of the KdV equation can be obtained by using the Miura transformation (13.78).

The modified KdV equation describes nonlinear wave propagation in systems with polarity symmetry. The mKdV equation is used in electrodynamics, wave propagation in size quantized films, and in elastic media. The mKdV equation is integrable and can be solved by the inverse scattering method.

Substituting the wave variable $\xi = x - ct$ into the mKdV equation

$$u_t + au^2u_x + u_{xxx} = 0. \tag{13.83}$$

and integrating once we find

$$-cu + \frac{a}{3}u^3 + u'' = 0. \tag{13.84}$$

In a manner parallel to the analysis presented to approach the KdV equation, we will use the tanh-coth method, sine-cosine method, and the Hirota's direct method to handle the mKdV equation.

13.4.1 Using the Tanh-coth Method

Balancing the nonlinear term u^3, that has the exponent $3M$, with the highest order derivative u'', that has the exponent $M+2$, in (13.84) yields

$$3M = M + 2, \tag{13.85}$$

that gives

$$M = 1. \tag{13.86}$$

The tanh-coth method allows us to use the substitution

$$u(x,t) = S(Y) = a_0 + a_1Y + b_1Y^{-1}. \tag{13.87}$$

Substituting (13.87) into (13.84), collecting the coefficients of each power of $Y^i, 0 \leqslant i \leqslant 6$, setting each coefficient to zero, and solving the resulting system we obtain the following sets of solutions

(i)

$$a_0 = a_1 = 0, \quad b_1 = \sqrt{\frac{3c}{a}}, \quad \mu = \sqrt{-\frac{c}{2}}, \quad c < 0. \tag{13.88}$$

(ii)

$$a_0 = b_1 = 0, \quad a_1 = \sqrt{\frac{3c}{a}}, \quad \mu = \sqrt{-\frac{c}{2}}, \quad c < 0. \tag{13.89}$$

(iii)

$$a_0 = 0, \quad a_1 = b_1 = \frac{1}{2}\sqrt{\frac{3c}{a}}, \quad \mu = \frac{1}{2}\sqrt{-\frac{c}{2}}, \quad c < 0. \tag{13.90}$$

(iv)

$$a_0 = 0, \quad a_1 = -b_1 = -\frac{1}{2}\sqrt{-\frac{3c}{2a}}, \quad \mu = \frac{1}{2}\sqrt{c}, \quad c > 0. \tag{13.91}$$

The following soliton and kink solutions

$$\begin{aligned} u_1(x,t) &= \sqrt{\frac{6c}{a}}\operatorname{sech}\left[\sqrt{c}(x-ct)\right], \quad c>0, a>0, \\ u_2(x,t) &= \sqrt{\frac{3c}{a}}\tanh\left[\sqrt{-\frac{c}{2}}(x-ct)\right], \quad c<0, a<0, \end{aligned} \tag{13.92}$$

respectively are readily obtained. Moreover, the following travelling wave solutions

$$\begin{aligned} u_3(x,t) &= \sqrt{\frac{3c}{a}}\coth\left[\sqrt{-\frac{c}{2}}(x-ct)\right], \quad c<0, a<0, \\ u_4(x,t) &= \sqrt{\frac{6c}{a}}\sec\left[\sqrt{-c}(x-ct)\right], \quad c<0, a<0, \\ u_5(x,t) &= \sqrt{-\frac{6c}{a}}\operatorname{csch}\left[\sqrt{c}(x-ct)\right], \quad c>0, a<0, \end{aligned} \tag{13.93}$$

can also be derived upon using the sign of the wave speed c and the parameter a.

13.4.2 Using the Sine-cosine Method

Substituting the cosine assumption into (13.84) yields

$$\begin{aligned} &-c\lambda\cos^{\beta}(\mu\xi) + \frac{a}{3}\lambda^3\cos^{3\beta}(\mu\xi) \\ &\quad -\lambda\mu^2\beta^2\cos^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0, \end{aligned} \tag{13.94}$$

We balance the exponents of the cosine functions. Equation (13.94) is satisfied only if the following system of algebraic equations holds:

$$\begin{aligned} \beta - 1 &\neq 0, \\ 3\beta &= \beta - 2, \\ \mu^2\beta^2\lambda &= -c\lambda, \\ \frac{a}{3}\lambda^3 &= -c\lambda\mu^2\beta(\beta-1), \end{aligned} \tag{13.95}$$

which leads to

$$\begin{aligned}\beta &= -1,\\ \mu &= \sqrt{-c},\\ \lambda &= \frac{6c}{a}.\end{aligned} \tag{13.96}$$

The results in (13.96) can be easily obtained if we also use the sine method. This in turn gives the periodic solutions for $c < 0, a < 0$:

$$\begin{aligned}u(x,t) &= \sqrt{\frac{6c}{a}}\sec\left[\sqrt{-c}(x-ct)\right], \quad c<0, a<0,\\ u(x,t) &= \sqrt{\frac{6c}{a}}\csc\left[\sqrt{-c}(x-ct)\right], \quad c<0, a<0.\end{aligned} \tag{13.97}$$

However, for $c > 0, a > 0$, we obtain the soliton solution

$$u(x,t) = \sqrt{\frac{6c}{a}}\operatorname{sech}\left[\sqrt{c}(x-ct)\right]. \tag{13.98}$$

13.4.3 Multiple-soliton Solutions of the mKdV Equation

In this section, we will examine multiple-soliton solutions of the modified KdV (mKdV) equation

$$u_t + 6\sigma u^2 u_x + u_{xxx} = 0, \quad \sigma = \pm 1. \tag{13.99}$$

Following the approach used by Hirota [10] and Hietarinta [9], we first consider the case where $\sigma = 1$, therefore Eq. (13.99) becomes

$$u_t + 6u^2 u_x + u_{xxx} = 0. \tag{13.100}$$

Proceeding as in the KdV equation, the dispersion relation is given by

$$c_i = k_i^3, \tag{13.101}$$

and as a result we obtain

$$\theta_i = k_i x - k_i^3 t. \tag{13.102}$$

In [9,10], it is shown that the multiple-soliton solutions of the mKdV equation is expressed by

$$u(x,t) = 2\partial_x\left(\arctan(f/g)\right) = 2\frac{f_x g - g_x f}{f^2+g^2}. \tag{13.103}$$

For the single soliton solution, it was found that

$$\begin{aligned}f(x,t) &= e^{\theta_1} = e^{k_1(x-k_1^2 t)},\\ g(x,t) &= 1.\end{aligned} \tag{13.104}$$

Substituting (13.104) into (13.103) gives the single soliton solution

$$u(x,t)=\frac{2k_1\,e^{k_1(x-k_1^2t)}}{1+e^{2k_1(x-k_1^2t)}}. \tag{13.105}$$

For the two-soliton solutions we find

$$\begin{aligned} f(x,t) &= e^{\theta_1}+e^{\theta_2}=e^{k_1(x-k_1^2t)}+e^{k_2(x-k_2^2t)},\\ g(x,t) &= 1-a_{12}e^{\theta_1+\theta_2}=1-a_{12}e^{(k_1+k_2)x-(k_1^3+k_2^3)t}. \end{aligned} \tag{13.106}$$

Using (13.106) in (13.103) and substituting the result in the mKdV equation (13.100), we find

$$a_{12}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{13.107}$$

and hence we set

$$a_{ij}=\frac{(k_i-k_j)^2}{(k_i+k_j)^2},\quad 1\leqslant i<j\leqslant 3. \tag{13.108}$$

Consequently, the two-soliton solutions are obtained by substitution (13.107) and (13.106) into (13.103).

For the three-soliton solutions, it was found that

$$\begin{aligned} f(x,t) &= e^{\theta_1}+e^{\theta_2}+e^{\theta_3}-a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3}\\ &= e^{k_1(x-k_1^2t)}+e^{k_2(x-k_2^2t)}+e^{k_3(x-k_3^2t)}\\ &\quad -a_{12}a_{13}a_{23}e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t},\\ g(x,t) &= 1-a_{12}e^{\theta_1+\theta_2}-a_{13}e^{\theta_1+\theta_3}-a_{23}e^{\theta_2+\theta_3}\\ &= 1-a_{12}e^{(k_1+k_2)x-(k_1^3+k_2^3)t}-a_{13}e^{(k_1+k_3)x-(k_1^3+k_3^3)t}\\ &\quad -a_{23}e^{(k_2+k_3)x-(k_2^3+k_3^3)t}, \end{aligned} \tag{13.109}$$

where a_{ij} is given in (13.108). Based on this result, the three–soliton solutions for the mKdV equation (13.100) is obtained by substituting (13.109) into (13.103). This shows that the mKdV equations is completely integrable and N-soliton solutions can be obtained for finite N, where $N\geqslant 1$.

It is interesting to point out that for $\sigma=-1$, the mKdV equation (13.100) becomes

$$u_t-6u^2u_x+u_{xxx}=0, \tag{13.110}$$

with a singular soliton

$$u(x,t)=\sqrt{c}\,\mathrm{csch}\left[\sqrt{c}(x-ct)\right],\quad c>0. \tag{13.111}$$

The dispersion relation remains the same. However, the solution of the mKdV equation (13.110) is expressed by

$$u(x,t)=2i\partial_x\left(\arctan(f/g)\right)=2i\frac{f_xg-g_xf}{f^2+g^2},\quad i=\sqrt{-1}. \tag{13.112}$$

In this case the solution is expressed in a complex form by

$$u(x,t) = i\frac{2k_1\, e^{k_1(x-k_1^2t)}}{1+e^{2k_1(x-k_1^2t)}}, \quad i=\sqrt{-1}. \tag{13.113}$$

Complex solutions will not be considered in this text.

13.5 Singular Soliton Solutions

It was proved in [17] that certain nonlinear evolution equations have not only soliton solutions, but also explode-decay mode solutions, or singular soliton solutions which can be expressed by closed form of analytic solutions. In this section, we will use the Hirota's sense, that we used before to obtain singular N-soliton solutions for the mKdV equation

$$u_t - 6u^2u_x + u_{xxx} = 0. \tag{13.114}$$

To achieve this goal, we combine the simplified version of Hereman *et. al.* [6,7], the Hirota's sense, and the Hietarinta approach that was introduced in [8,9], where the following assumption

$$F(x,t) = \frac{f(x,t)}{g(x,t)}, \quad g(x,t)\neq 0, \tag{13.115}$$

was first used. The bilinear form for the mKdV equation (13.99) is given by

$$(D_t + D_x^3)(f\cdot g) = 0, \quad D_x^2(f\cdot f + g\cdot g) = 0.$$

The solution of the mKdV equation is assumed to be of the form

$$u(x,t) = \frac{\partial \log F(x,t)}{\partial x} = \frac{gf_x - fg_x}{gf}. \tag{13.116}$$

We next assume that $f(x,t)$ and $g(x,t)$ have perturbation expansions of the form

$$\begin{aligned} f(x,t) &= 1+\textstyle\sum_{n=1}^{\infty} \varepsilon^n f_n(x,t),\\ g(x,t) &= 1+\textstyle\sum_{n=1}^{\infty} \varepsilon_1^n g_n(x,t), \end{aligned} \tag{13.117}$$

where ε and ε_1 are non small formal expansion parameters. Following the simplified form presented in [6,7], we define

$$\begin{aligned} f_1 &= \textstyle\sum_{i=1}^{N} \varepsilon \exp(\theta_i),\\ g_1 &= \textstyle\sum_{i=1}^{N} \varepsilon_1 \exp(\theta_i), \end{aligned} \tag{13.118}$$

where

$$\theta_i = k_i x - c_i t, \tag{13.119}$$

where k_i and c_i are arbitrary constants, k_i is called the wave number.

To obtain the first solution, we set $N=1$ into (13.118), and by using (13.117) we find

$$\begin{aligned} f(x,t) &= 1+\varepsilon f_1(x,t),\\ g(x,t) &= 1+\varepsilon_1 g_1(x,t), \end{aligned} \tag{13.120}$$

and hence

$$u(x,t)=\frac{\partial \log F(x,t)}{\partial x}=\frac{\partial}{\partial x}\log\left(\frac{1+\varepsilon f_1}{1+\varepsilon_1 g_1}\right). \tag{13.121}$$

This is a solution of the mKdV equation (13.99) if

$$\varepsilon_1=-\varepsilon. \tag{13.122}$$

The dispersion relation is given by

$$c_i=k_i^3, \tag{13.123}$$

and as a result we obtain

$$\theta_i=k_ix-k_i^3t. \tag{13.124}$$

The obtained results give a new definition to (13.117) in the form

$$\begin{aligned} f(x,t) &= 1+\sum_{n=1}^{\infty}\varepsilon^n f_n(x,t),\\ g(x,t) &= 1+\sum_{n=1}^{\infty}(-1)^n\varepsilon^n g_n(x,t), \end{aligned} \tag{13.125}$$

and consequently we obtain

$$\begin{aligned} f_1(x,t) &= \exp(\theta_1)=\exp\left(k_1(x-k_1^2t)\right),\\ g_1(x,t) &= -\exp(\theta_1)=-\exp\left(k_1(x-k_1^2t)\right). \end{aligned} \tag{13.126}$$

Accordingly, we find

$$F=\frac{1+f_1}{1+g_1}=\frac{1+\exp\left(k_1(x-k_1^2t)\right)}{1-\exp\left(k_1(x-k_1^2t)\right)}. \tag{13.127}$$

The singular soliton solution

$$u(x,t)=\frac{2k_1\exp(k_1(x-k_1^2t))}{1-\exp(k_1(x-k_1^2t))}, \tag{13.128}$$

follows immediately.

To determine the singular two-soliton solutions, we proceed as before to find that

$$\begin{aligned} f(x,t) &= 1+\exp(\theta_1)+\exp(\theta_2)+a_{12}\exp(\theta_1+\theta_2),\\ g(x,t) &= 1-\exp(\theta_1)-\exp(\theta_2)+b_{12}\exp(\theta_1+\theta_2). \end{aligned} \tag{13.129}$$

Substituting (13.129) into the mKdV equation (13.114), we find that (13.129) is a solution of this equation if a_{12} and b_{12}, and therefore a_{ij} and b_{ij}, are equal and given by

$$a_{ij} = b_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \tag{13.130}$$

where θ_i and θ_j are given above in (13.124). For the two-soliton solutions we use $1 \leqslant i < j \leqslant 2$ to obtain

$$\begin{aligned} f(x,t) &= 1 + e^{k_1(x-k_1^2 t)} + e^{k_2(x-k_2^2 t)} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x-(k_1^3+k_2^3)t}, \\ g(x,t) &= 1 - e^{k_1(x-k_1^2 t)} - e^{k_2(x-k_2^2 t)} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x-(k_1^3+k_2^3)t}. \end{aligned} \tag{13.131}$$

This in turn gives the singular two-soliton solutions explicitly

$$\begin{aligned} u(x,t) &= \\ &\frac{\partial}{\partial x}\left(\log\left[\frac{1 + e^{k_1(x-k_1^2 t)} + e^{k_2(x-k_2^2 t)} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x-(k_1^3+k_2^3)t}}{1 - e^{k_1(x-k_1^2 t)} - e^{k_2(x-k_2^2 t)} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x-(k_1^3+k_2^3)t}}\right]\right). \end{aligned} \tag{13.132}$$

We can proceed in a similar manner to obtain

$$\begin{aligned} f(x,t) &= 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ &\quad + a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_3) + a_{13}\exp(\theta_1+\theta_3) + f_3(x,t), \\ g(x,t) &= 1 - \exp(\theta_1) - \exp(\theta_2) - \exp(\theta_3) \\ &\quad + a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_3) + a_{13}\exp(\theta_1+\theta_3) + g_3(x,t). \end{aligned} \tag{13.133}$$

Substituting (13.133) into (13.114) to find that

$$\begin{aligned} f_3(x,t) &= b_{123}\exp(\theta_1+\theta_2+\theta_3), \\ g_3(x,t) &= -b_{123}\exp(\theta_1+\theta_2+\theta_3), \\ b_{123} &= a_{12}a_{13}a_{23}. \end{aligned} \tag{13.134}$$

For the singular three-soliton solutions we use $1 \leqslant i < j \leqslant 3$, we therefore obtain

$$\begin{aligned} f(x,t) &= 1 + e^{k_1(x-k_1^2 t)} + e^{k_2(x-k_2^2 t)} + e^{k_3(x-k_3^2 t)} \\ &\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x-(k_1^3+k_2^3)t} + \frac{(k_1-k_3)^2}{(k_1+k_3)^2} e^{(k_1+k_3)x-(k_1^3+k_3^3)t} \\ &\quad + \frac{(k_2-k_3)^2}{(k_2+k_3)^2} e^{(k_2+k_3)x-(k_2^3+k_3^3)t} \\ &\quad + \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2} e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t}, \\ g(x,t) &= 1 - e^{k_1(x-k_1^2 t)} - e^{k_2(x-k_2^2 t)} - e^{k_3(x-k_3^2 t)} \\ &\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x-(k_1^3+k_2^3)t} + \frac{(k_1-k_3)^2}{(k_1+k_3)^2} e^{(k_1+k_3)x-(k_1^3+k_3^3)t} \\ &\quad + \frac{(k_2-k_3)^2}{(k_2+k_3)^2} e^{(k_2+k_3)x-(k_2^3+k_3^3)t} \end{aligned} \tag{13.135}$$

$$-\frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t}.$$

The singular three-soliton solutions are therefore given by

$$u(x,t) = \frac{\partial}{\partial x}\left(\ln(\frac{f(x,t)}{g(x,t)})\right), \tag{13.136}$$

where $f(x,t)$ and $g(x,t)$ are given in (13.135). This confirms the conclusion in [17] that certain equations which have N-soliton solutions, have simultaneously, N-singular soliton solutions so far as the equation has self-similar symmetry.

13.6 The Generalized KdV Equation

The generalized KdV (gKdV) equation reads

$$u_t + au^n u_x + u_{xxx} = 0, \quad n > 2. \tag{13.137}$$

Using the wave variable $\xi = x - ct$ and integrating once, the gKdV equation can be transformed to the ODE

$$-cu + \frac{a}{n+1}u^{n+1} + u'' = 0, \tag{13.138}$$

Recall that for $n = 1$ and $n = 2$, the gKdV equation will be reduced to the completely integrable KdV and mKdV equations respectively. However, the gKdV equation is not completely integrable for $n > 2$. This means that the N-soliton solutions do not exist for this equation. The gKdV equation will be investigated by using the tanh-coth and the sine-cosine method only .

13.6.1 Using the Tanh-coth Method

Balancing the nonlinear term u^{n+1} with the highest order derivative u'' gives

$$M = \frac{2}{n}. \tag{13.139}$$

To obtain closed form solutions, M should be an integer. To achieve this goal we use the transformation

$$u(x,t) = v^{\frac{1}{n}}(x,t), \tag{13.140}$$

That will carry out Eq. (13.138) to the ODE

$$-cn^2(n+1)v^2 + an^2v^3 + n(n+1)vv'' + (1-n^2)(v')^2 = 0. \tag{13.141}$$

Balancing vv'', that has the exponent $2M+2$, with v^3, that has an exponent $3M$, gives $M=2$. Accordingly, the tanh-coth method admits the use of the substitution

$$u(x,t)=S(Y)=\sum_{i=0}^{2}a_iY^i+\sum_{j=1}^{2}b_jY^{-j}. \tag{13.142}$$

Substituting (13.142) into (13.138), collecting the coefficients of each power of $Y^i, 0\leqslant i\leqslant 12$, and solving the resulting system of algebraic equations we obtain $a_1=b_1=0$ and the following sets of solutions for $c>0$
(i)

$$a_0=\frac{c(n+1)(n+2)}{2a},\quad a_2=0,\quad b_2=-\frac{c(n+1)(n+2)}{2a},\quad \mu=\frac{n}{2}\sqrt{c}. \tag{13.143}$$

(ii)

$$a_0=\frac{c(n+1)(n+2)}{2a},\quad b_2=0,\quad a_2=-\frac{c(n+1)(n+2)}{2a},\quad \mu=\frac{n}{2}\sqrt{c}. \tag{13.144}$$

(iii)

$$a_0=\frac{c(n+1)(n+2)}{4a},\quad a_2=b_2=-\frac{c(n+1)(n+2)}{8a},\quad \mu=\frac{n}{4}\sqrt{c}. \tag{13.145}$$

Using these results, and noting that $u(x,t)=v^{\frac{1}{n}}(x,t)$, we obtain the following soliton solution

$$u_1(x,t)=\left\{\frac{c(n+1)(n+2)}{2a}\operatorname{sech}^2\left[\frac{n}{2}\sqrt{c}(x-ct)\right]\right\}^{\frac{1}{n}}, \tag{13.146}$$

and the solutions

$$\begin{aligned}
u_2(x,t)&=\left\{-\frac{c(n+1)(n+2)}{2a}\operatorname{csch}^2\left[\frac{n}{2}\sqrt{c}(x-ct)\right]\right\}^{\frac{1}{n}},\\
u_3(x,t)&=\left\{\Gamma\left(2-\tanh^2\left[\frac{n}{4}\sqrt{c}(x-ct)\right]-\coth^2\left[\frac{n}{4}\sqrt{c}(x-ct)\right]\right)\right\}^{\frac{1}{n}},
\end{aligned} \tag{13.147}$$

where

$$\Gamma=\frac{c(n+1)(n+2)}{8a}. \tag{13.148}$$

However, for $c<0$, we obtain the following plane periodic solutions

$$u_4(x,t)=\left\{\frac{c(n+1)(n+2)}{2a}\sec^2\left[\frac{n}{2}\sqrt{-c}(x-ct)\right]\right\}^{\frac{1}{n}},$$

$$
\begin{aligned}
u_5(x,t) &= \left\{ \frac{c(n+1)(n+2)}{2a} \csc^2 \left[\frac{n}{2}\sqrt{-c}(x-ct) \right] \right\}^{\frac{1}{n}}, \\
u_6(x,t) &= \left\{ \Gamma \left(2+\tan^2 \left[\frac{n}{4}\sqrt{-c}(x-ct) \right] + \cot^2 \left[\frac{n}{4}\sqrt{-c}(x-ct) \right] \right) \right\}^{\frac{1}{n}}.
\end{aligned}
\tag{13.149}
$$

13.6.2 Using the Sine-cosine Method

Substituting the cosine assumption into (13.138) yields

$$
\begin{aligned}
&-c\lambda \cos^{\beta}(\mu\xi) + \frac{a}{n+1}\lambda^{n+1}\cos^{(n+1)\beta}(\mu\xi) \\
&-\lambda\mu^2\beta^2\cos^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0,
\end{aligned}
\tag{13.150}
$$

Equation (13.150) is valid only if the following system of algebraic equations holds:

$$
\begin{aligned}
\beta - 1 &\neq 0, \\
(n+1)\beta &= \beta - 2, \\
\mu^2\beta^2\lambda &= -c\lambda, \\
\frac{a}{n+1}\lambda^{n+1} &= -\lambda\mu^2\beta(\beta-1).
\end{aligned}
\tag{13.151}
$$

Solving this system gives

$$
\begin{aligned}
\beta &= -\frac{2}{n}, \\
\mu &= \frac{n}{2}\sqrt{-c}, c<0 \\
\lambda &= \left(\frac{c(n+1)(n+2)}{2a} \right)^{\frac{1}{n}}.
\end{aligned}
\tag{13.152}
$$

These results give the soliton solutions $u_1(x,t)$ and the travelling wave solutions obtained above by using the tanh-coth method. We can easily observe that the sine-cosine method can be used directly and does not require the use of a transformation formula as required by the tanh-coth method when M is not an integer.

13.7 The Potential KdV Equation

In this section we will study the potential KdV equation

$$
u_t + au_x^2 + u_{3x} = 0. \tag{13.153}
$$

As stated before, the potential KdV equation can be obtained from the KdV equation by using the transformation $u = v_x$ and integrating once. The potential KdV equation (13.153) is completely integrable equation, and therefore gives rise to multiple-soliton solutions.

The potential KdV equation (13.153) can be converted to the ODE

$$-cu' + a(u')^2 + u''' = 0, \tag{13.154}$$

by using the wave variable $\xi = x - ct$. The potential KdV equation will be handled by the tanh-coth method for single soliton solution and by the Hirota's direct method for multiple soliton solutions.

13.7.1 Using the Tanh-coth Method

Balancing the nonlinear term $(u')^2$, that has the exponent $(M+1)^2$, with the highest order derivative u''', that has the exponent $M+3$, leads to

$$(M+1)^2 = M+3, \tag{13.155}$$

and hence

$$M = 1, -2. \tag{13.156}$$

Case (i):

For $M = 1$, the tanh-coth method admits the use of the substitution

$$u(x,t) = S(Y) = a_0 + a_1 Y + b_1 Y^{-1}, \tag{13.157}$$

into (13.154), and by proceeding as before we obtain the following sets of solutions
(i)

$$\begin{aligned} &a_0 = R, R \text{ is an arbitrary constant,} \\ &a_1 = \frac{3\sqrt{c}}{a}, \quad b_1 = 0, \quad \mu = \frac{\sqrt{c}}{2}, \quad c > 0. \end{aligned} \tag{13.158}$$

(ii)

$$\begin{aligned} &a_0 = R, R \text{ is an arbitrary constant,} \\ &a_1 = 0, \quad b_1 = \frac{3\sqrt{c}}{a}, \quad \mu = \frac{\sqrt{c}}{2}, \quad c > 0. \end{aligned} \tag{13.159}$$

This in turn gives the following kink solution for $c > 0$

$$u_1(x,t) = R + \frac{3\sqrt{c}}{a} \tanh\left[\frac{\sqrt{c}}{2}(x-ct)\right], \tag{13.160}$$

and the travelling wave solution

$$u_2(x,t) = R + \frac{3\sqrt{c}}{a} \coth\left[\frac{\sqrt{c}}{2}(x-ct)\right]. \tag{13.161}$$

However, for $c<0$, the obtained solutions are complex that are not presented in this text.

Case (ii):

For $M=-2$, the tanh-coth method applies the substitution

$$u(x,t)=S(Y)=1/(a_0+a_1Y+a_2Y^2+b_1Y^{-1}+b_2Y^{-2}), \tag{13.162}$$

into (13.154), and by proceeding as before we found that $a_2=b_2=0$. Therefore, the substitution (13.162) is reduced to

$$u(x,t)=S(Y)=1/(a_0+a_1Y+b_1Y^{-1}), \tag{13.163}$$

that will be substitutes into (13.154) to obtain the following sets of solutions

(i)

$$a_0=b_1=0,\quad a_1=\frac{a}{3\sqrt{c}},\quad \mu=\frac{\sqrt{c}}{2},\quad c>0. \tag{13.164}$$

(ii)

$$a_0=a_1=0,\quad b_1=\frac{a}{3\sqrt{c}},\quad \mu=\frac{\sqrt{c}}{2},\quad c>0. \tag{13.165}$$

(iii)

$$\begin{aligned} a_1&=R, R \text{ is an arbitrary constant},\\ a_0&=\sqrt{\frac{3cR^2-aR\sqrt{c}}{3c}},\quad b_1=0,\quad \mu=\frac{\sqrt{c}}{2},\quad c>0. \end{aligned} \tag{13.166}$$

(iv)

$$\begin{aligned} b_1&=R, R \text{ is an arbitrary constant},\\ a_0&=\sqrt{\frac{3cR^2-aR\sqrt{c}}{3c}},\quad a_1=0,\quad \mu=\frac{\sqrt{c}}{2},\quad c>0. \end{aligned} \tag{13.167}$$

This in turn gives the following solutions for $c>0$

$$\begin{aligned} u_3(x,t)&=\frac{3\sqrt{c}}{a}\tanh\left[\frac{\sqrt{c}}{2}(x-ct)\right],\\ u_4(x,t)&=\frac{3\sqrt{c}}{a}\coth\left[\frac{\sqrt{c}}{2}(x-ct)\right], \end{aligned}$$

$$\begin{aligned} u_5(x,t)&=\frac{1}{\sqrt{\frac{3cR^2-aR\sqrt{c}}{3c}}+R\tanh\left[\frac{\sqrt{c}}{2}(x-ct)\right]},\\ u_6(x,t)&=\frac{1}{\sqrt{\frac{3cR^2-aR\sqrt{c}}{3c}}+R\coth\left[\frac{\sqrt{c}}{2}(x-ct)\right]}. \end{aligned} \tag{13.168}$$

The solutions u_3 and u_4 are the same as u_1 and u_2 when we set $R=0$. Moreover, for $c<0$, we obtain complex solutions that will not be discussed in this text.

13.7.2 Multiple-soliton Solutions of the Potential KdV Equation

The multiple-soliton solutions of the potential KdV equation

$$u_t + 3(u_x)^2 + u_{xxx} = 0 \tag{13.169}$$

will be derived. We closely follow our approach presented before in the previous sections. To achieve our goal, we first introduce the change of dependent variable

$$u(x,t) = 2\frac{\partial \ln f(x,t)}{\partial x} = 2\frac{f_x}{f}, \tag{13.170}$$

that will convert (13.169) into

$$f_{xt} - f_x f_t + 3(f_{xx})^2 - 4 f_{xxx} f_t + f_{xxxx} = 0. \tag{13.171}$$

Equation (13.171) can be decomposed into linear operator L and nonlinear operator N defined by

$$\begin{aligned} L &= \frac{\partial^2}{\partial x \partial t} + \frac{\partial}{\partial x^4}, \\ N(f,f) &= -f_x f_t + 3 f_{2x} f_{2x} - 4 f_{xxx} f_t. \end{aligned} \tag{13.172}$$

The function $f(x,t)$ may be assumed to have a perturbation expansion of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \tag{13.173}$$

where ε is a non small formal expansion parameter.

The N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i), \tag{13.174}$$

where

$$\theta_i = k_i x - c_i t, \tag{13.175}$$

where k_i and c_i are arbitrary constants. Substituting (13.174) into (13.169) gives the dispersion relation

$$c_i = k_i^3. \tag{13.176}$$

Consequently, Eq. (13.175) becomes

$$\theta_i = k_i x - k_i^3 t. \tag{13.177}$$

This also gives

$$f_1 = \exp(\theta_1) = \exp\left(k_1(x - k_1^2 t)\right), \tag{13.178}$$

obtained by using $N = 1$ in (13.174). For the one-soliton solution we set

$$f = 1 + \exp(\theta_1) = 1 + \exp\left(k_1(x - k_1^2 t)\right), \tag{13.179}$$

where we used $\varepsilon = 1$. The single soliton solution is given by

$$u(x,t) = \frac{2k_1 \exp\left(k_1(x - k_1^2 t)\right)}{1 + \exp\left(k_1(x - k_1^2 t)\right)}, \tag{13.180}$$

obtained upon using (13.170).

To derive the two-soliton solutions, we set $N = 2$ in (13.174) to get

$$f_1 = \exp(\theta_1) + \exp(\theta_2). \tag{13.181}$$

To determine f_2, we follow our discussion presented before to find

$$f_2 = \sum_{1 \leqslant i < j \leqslant N} a_{ij} \exp\left(\theta_i + \theta_j\right), \tag{13.182}$$

where

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leqslant i < j \leqslant 2, \tag{13.183}$$

and θ_i and θ_j are given above in (13.44). For the two-soliton solutions we therefore use

$$f = 1 + \exp\left(\theta_1\right) + \exp\left(\theta_2\right) + a_{12} \exp\left(\theta_1 + \theta_2\right), \tag{13.184}$$

where

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}. \tag{13.185}$$

This in turn gives

$$f = 1 + e^{k_1(x - k_1^2 t)} + e^{k_2(x - k_2^2 t)} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{(k_1 + k_2)x - (k_1^3 + k_2^3)t}. \tag{13.186}$$

The two-soliton solutions are obtained by using (13.170) for the function f in (13.186).

We can determine f_3 similarly. Proceeding as before, we therefore use

$$\begin{aligned} f_1(x,t) &= \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3), \\ f_2(x,t) &= a_{12} \exp(\theta_1 + \theta_2) + a_{23} \exp(\theta_2 + \theta_3) + a_{13} \exp(\theta_1 + \theta_3), \end{aligned} \tag{13.187}$$

and this will give

$$\begin{aligned} f(x,t) = {} & 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ & + a_{12} \exp(\theta_1 + \theta_2) + a_{23} \exp(\theta_2 + \theta_3) + a_{13} \exp(\theta_1 + \theta_3) \\ & + f_3(x,t). \end{aligned} \tag{13.188}$$

Substituting (13.188) into (13.169) and proceeding as before we find

$$f_3 = b_{123} \exp\left(\theta_1 + \theta_2 + \theta_3\right), \tag{13.189}$$

where

$$b_{123} = a_{12}a_{13}a_{23} = \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}, \tag{13.190}$$

and θ_1, θ_2 and θ_3 are given above in (13.175). For the three-soliton solution we use $1 \leqslant i < j \leqslant 3$, we therefore obtain

$$\begin{aligned} f = {} & 1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3) \\ & +a_{12}\exp(\theta_1+\theta_2)+a_{13}\exp(\theta_1+\theta_3)+a_{23}\exp(\theta_2+\theta_3) \\ & +b_{123}\exp(\theta_1+\theta_2+\theta_3). \end{aligned} \tag{13.191}$$

To determine the three-soliton solutions explicitly, we use (13.170) for the function f in (13.191). This shows that the multiple-soliton solutions exist for $N \geqslant 1$. This result proves that the potential KdV equation is completely integrable.

13.8 The Gardner Equation

The standard Gardner equation, or the combined KdV-mKdV equation, reads

$$u_t + 2auu_x - 3bu^2u_x + u_{xxx} = 0, \quad a,b > 0, \tag{13.192}$$

where $u(x,t)$ is the amplitude of the relevant wave mode. The KdV equation was complemented with a higher-order cubic nonlinear term of the form u^2u_x to obtain the Gardner equation (13.192). The Gardner equation was first derived rigorously within the asymptotic theory for long internal waves in a two-layer fluid with a density jump at the interface [19]. The competition among dispersion, quadratic and cubic nonlinearities constitutes the main interest [22]. Equation (13.192), like the KdV equation, is completely integrable with a Lax pair and inverse scattering transform. It was found, as will be discussed later, that soliton solutions exist only for $b > 0$. Gardner equation is widely used in various branches of physics, such as plasma physics, fluid physics, quantum field theory. The equation plays a prominent role in ocean waves. The Gardner equation describes internal solitary waves in shallow seas, and admits quite interesting tanh and cn type solutions. The Gardner equation has been investigated in the literature because it is used to model a variety of nonlinear phenomena. The tanh method, the cosh ansatz, and the Hirota's method will be used to handle this problem.

13.8.1 The Kink Solution

Using the wave variable $\xi = x - ct$ and integrating the result will convert the Gardner equation (13.192) to the ODE

$$-cu + au^2 - bu^3 + u'' = 0, \tag{13.193}$$

Balancing u^3 with u'' gives

$$M = 1. \tag{13.194}$$

The tanh method uses the finite expansion

$$u(x,t) = S(Y) = a_0 + a_1 Y. \tag{13.195}$$

Inserting (13.195) into (13.193) and proceeding as before we obtain

$$a_0 = \frac{a}{3b}, \quad a_1 = \pm\frac{a}{3b}, \quad \mu = \frac{a}{3\sqrt{2b}}, \quad c = \frac{2a^2}{9b}. \tag{13.196}$$

The kink solution

$$u(x,t) = \frac{a}{3b}\left(1 \pm \tanh(\frac{a}{3\sqrt{2b}}(x - \frac{2a^2}{9b}t))\right), \tag{13.197}$$

and the travelling wave solution

$$u(x,t) = \frac{a}{3b}\left(1 \pm \coth(\frac{a}{3\sqrt{2b}}(x - \frac{2a^2}{9b}t))\right), \tag{13.198}$$

are readily obtained. This confirms the reality that Gardner equation has real solutions only for $b > 0$. For $b < 0$, we obtain complex solutions that are not required in this text.

13.8.2 The Soliton Solution

In this approach we assume the ansatz

$$u(x,t) = \frac{\alpha}{1 + \lambda\cosh(\mu(x - ct))}, \tag{13.199}$$

where α, λ and c are parameters that will be determined. Substituting (13.199) into (13.192), and following our analysis presented before we find

$$\alpha = \frac{3\mu^2}{a}, \quad \lambda = \frac{1}{a}\sqrt{\frac{2a^2 - 9b\mu^2}{2}}, \quad c = \mu^2. \tag{13.200}$$

Substituting (13.200) into (13.199) yields the soliton solution

$$u(x,t) = \frac{3\mu^2}{a + \sqrt{\frac{2a^2 - 9b\mu^2}{2}}\cosh\mu(x - \mu^2 t)}, \tag{13.201}$$

for $2a^2 > 9b\mu^2$. For the special case where $a = 3, b = 2$ the soliton solution (13.201) becomes

$$u(x,t) = \frac{\mu^2}{1+\sqrt{1-\mu^2}\cosh\mu(x-\mu^2 t)}, \tag{13.202}$$

where $\mu \in (0,1)$. This means that the soliton profile (13.202) will have positive polarity with an amplitude that ranges from zero to one. Moreover, we can easily conclude that the width of the soliton increases with the increase in its height. Moreover, the upper bound of the soliton in (13.202) is 1 as $\mu \to 1$. Fig. 13.5 above shows, from bottom to top, different solitary waves profiles given by (13.202) at $t = 0$ for $\mu = 0.4, 0.6, 0.8, 0.96$.

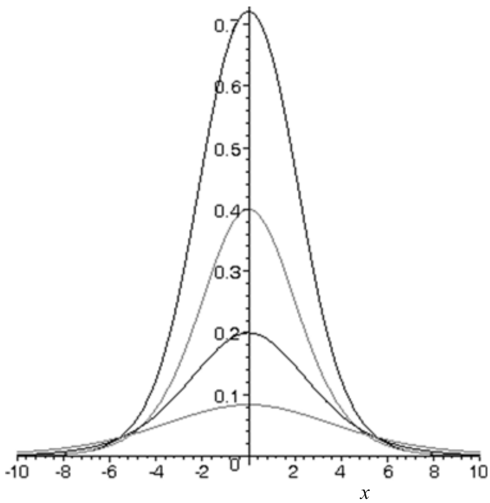

Fig. 13.5 Different solitary waves profiles given by (13.202) at $t = 0$ for $\mu = 0.4, 0.6, 0.8, 0.96$ (From bottom to top).

13.8.3 N-soliton Solutions of the Positive Gardner Equation

As stated before, the Gardner equation (13.192), like the KdV and the mKdV equation, is completely integrable with a Lax pair and inverse scattering transform. To show that the Gardner equation is completely integrable, without loss of generality, we consider the two models for the Gardner equation

$$u_t + 6uu_x \pm 6u^2 u_x + u_{xxx} = 0, \tag{13.203}$$

that describe internal solitary waves in shallow seas. The two models will be classified as *positive* Gardner equation and *negative* Gardner equation depending on the sign of the cubic nonlinear term. To show the complete integrability, we use the transformation

$$u = v \mp \frac{1}{2}, \tag{13.204}$$

to convert the Gardner models (13.203) into the modified KdV equations

$$v_t \mp \frac{3}{2}v_x + 6v^2 v_x + v_{xxx} = 0. \tag{13.205}$$

This shows that the Gardner equation, like the modified KdV equation is completely integrable.

To derive the N-soliton solutions, we consider first the positive Gardner equation

$$u_t + 6uu_x + 6u^2 u_x + u_{xxx} = 0, \tag{13.206}$$

that will be converted to the mKdV equation

$$v_t - \frac{3}{2}v_x + 6v^2 v_x + v_{xxx} = 0, \tag{13.207}$$

by using the transformation

$$u = v - \frac{1}{2}. \tag{13.208}$$

It is then normal to use the same approach used before to determine multiple soliton solutions for the modified KdV equation. Using only the linear terms gives the dispersion relation by

$$c_i = k_i^3 - \frac{3}{2}k_i. \tag{13.209}$$

and as a result we obtain

$$\theta_i = k_i x - (k_i^3 - \frac{3}{2}k_i)t. \tag{13.210}$$

The multiple-soliton solutions of the mKdV equation (13.207) is expressed by

$$v(x,t) = 2\partial_x \left(\arctan(f/g)\right) = 2\frac{f_x g - g_x f}{f^2 + g^2}. \tag{13.211}$$

For the single soliton solution, it was found that

$$\begin{aligned} f(x,t) &= e^{\theta_1} = e^{k_1(x-(k_1^2-\frac{3}{2})t)}, \\ g(x,t) &= 1. \end{aligned} \tag{13.212}$$

Substituting (13.212) into (13.211) gives the single soliton solution

$$v(x,t) = \frac{2k_1\, e^{k_1(x-(k_1^2-\frac{3}{2})t)}}{1 + e^{2k_1(x-(k_1^2-\frac{3}{2})t)}}. \tag{13.213}$$

Noting that $u(x,t)=v(x,t)-\dfrac{1}{2}$, therefore the single soliton solution for the Gardner equation (13.206)

$$u(x,t)=-\frac{1}{2}+\frac{2k_1\,e^{k_1(x-(k_1^2-\frac{3}{2})t)}}{1+e^{2k_1(x-(k_1^2-\frac{3}{2})t)}}. \tag{13.214}$$

For the two-soliton solutions we find

$$\begin{aligned} f(x,t) &= e^{\theta_1}+e^{\theta_2}=e^{k_1(x-(k_1^2-\frac{3}{2})t)}+e^{k_2(x-(k_2^2-\frac{3}{2})t)}, \\ g(x,t) &= 1-a_{12}e^{\theta_1+\theta_2}=1-a_{12}e^{(k_1+k_2)x-(k_1^3+k_2^3-\frac{3}{2}k_1-\frac{3}{2}k_2)t}. \end{aligned} \tag{13.215}$$

Using (13.215) in (13.211) and substituting the result in the mKdV equation (13.207), we find

$$a_{12}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{13.216}$$

and hence we set

$$a_{ij}=\frac{(k_i-k_j)^2}{(k_i+k_j)^2},\quad 1\leqslant i<j\leqslant 3. \tag{13.217}$$

Consequently, the two-soliton solutions are obtained by substitution (13.216) and (13.215) into (13.211) and noting that $u(x,t)=v(x,t)-\dfrac{1}{2}$.

For the three-soliton solutions, we set

$$\begin{aligned} f(x,t) &= e^{\theta_1}+e^{\theta_2}+e^{\theta_3}-a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3} \\ &= e^{k_1(x-(k_1^2-\frac{3}{2})t)}+e^{k_2(x-(k_2^2-\frac{3}{2})t)}+e^{k_3(x-(k_3^2-\frac{3}{2})t)} \\ &\quad -a_{12}a_{13}a_{23}e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3-\frac{3}{2}k_1-\frac{3}{2}k_2-\frac{3}{2}k_3)t}, \\ g(x,t) &= 1-a_{12}e^{\theta_1+\theta_2}-a_{13}e^{\theta_1+\theta_3}-a_{23}e^{\theta_2+\theta_3} \\ &= 1-a_{12}e^{(k_1+k_2)x-(k_1^3+k_2^3-\frac{3}{2}k_1-\frac{3}{2}k_2)t} \\ &\quad -a_{13}e^{(k_1+k_3)x-(k_1^3+k_3^3-\frac{3}{2}k_1-\frac{3}{2}k_3)t} \\ &\quad -a_{23}e^{(k_2+k_3)x-(k_2^3+k_3^3-\frac{3}{2}k_2-\frac{3}{2}k_3)t}. \end{aligned} \tag{13.218}$$

Based on this result, the three-soliton solutions for the mKdV equation (13.205) is obtained by substituting (13.218) into (13.211) and noting that $u(x,t)=v(x,t)-\frac{1}{2}$. This shows that the mKdV equations (13.207), and hence the Gardner equation (13.206), is completely integrable and N-soliton solutions can be obtained for finite N, where $N\geqslant 1$.

13.8.4 Singular Soliton Solutions

We now consider the negative Gardner equation

$$u_t+6uu_x-6u^2u_x+u_{xxx}=0. \tag{13.219}$$

Following [19], we use the Miura transformation

$$f = -u + u^2 + u_x, \tag{13.220}$$

that will convert (13.219) to the KdV equation

$$f_t - 6ff_x + f_{xxx} = 0. \tag{13.221}$$

This conversion of the Gardner equation to the KdV equation emphasizes the complete integrability of the Gardner equation. It is to be noted that Eq. (13.220) is the Riccati equation, but unfortunately its solution cannot be simplified.

As stated before, we use the transformation formula

$$u = v + \frac{1}{2}, \tag{13.222}$$

to convert the Gardner equation (13.219) into the modified KdV equation

$$v_t + \frac{3}{2}v_x - 6v^2 v_x + v_{xxx} = 0. \tag{13.223}$$

It is then normal to use the same approach used before to determine singular soliton solutions for the modified KdV equation.

We first introduce the following assumption

$$F(x,t) = \frac{f(x,t)}{g(x,t)}, \quad g(x,t) \neq 0. \tag{13.224}$$

The solution of the mKdV equation (13.223) is assumed to be of the form

$$v(x,t) = \frac{\partial \log F(x,t)}{\partial x} = \frac{gf_x - fg_x}{gf}. \tag{13.225}$$

We next assume that $f(x,t)$ and $g(x,t)$ have perturbation expansions of the form

$$\begin{aligned} f(x,t) &= 1 + \textstyle\sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \\ g(x,t) &= 1 + \textstyle\sum_{n=1}^{\infty} \varepsilon_1^n g_n(x,t), \end{aligned} \tag{13.226}$$

where ε and ε_1 are non small formal expansion parameters. We next define the N-soliton solution

$$\begin{aligned} f_1 &= \textstyle\sum_{i=1}^{N} \varepsilon \exp(\theta_i), \\ g_1 &= \textstyle\sum_{i=1}^{N} \varepsilon_1 \exp(\theta_i), \end{aligned} \tag{13.227}$$

where

$$\theta_i = k_i x - c_i t, \quad i = 1, 2, \cdots, N \tag{13.228}$$

where k_i and c_i are arbitrary constants, k_i is called the wave number.

To obtain the singular one-soliton solution, we set $N = 1$ into (13.227), and by using (13.226) we find

$$
\begin{aligned}
f(x,t) &= 1+\varepsilon f_1(x,t), \\
g(x,t) &= 1+\varepsilon_1 g_1(x,t),
\end{aligned}
\tag{13.229}
$$

and hence

$$
v(x,t) = \frac{\partial \log F(x,t)}{\partial x} = \frac{\partial}{\partial x} \log\left(\frac{1+\varepsilon f_1}{1+\varepsilon_1 g_1}\right). \tag{13.230}
$$

This is a solution of the mKdV equation (13.223) if

$$
\varepsilon_1 = -\varepsilon. \tag{13.231}
$$

This in turn gives the dispersion relation by

$$
c_i = k_i(\frac{3}{2}+k_i^2), \tag{13.232}
$$

and as a result we obtain

$$
\theta_i = k_i x - k_i(\frac{3}{2}+k_i^2)\,t. \tag{13.233}
$$

The obtained results give a new definition to (13.226) in the form

$$
\begin{aligned}
f(x,t) &= 1+\sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \\
g(x,t) &= 1+\sum_{n=1}^{\infty} (-1)^n \varepsilon^n g_n(x,t),
\end{aligned}
\tag{13.234}
$$

and consequently we obtain

$$
\begin{aligned}
f_1(x,t) &= \exp(\theta_1) = \exp\left(k_1(x-(\frac{3}{2}+k_1^2)t)\right), \\
g_1(x,t) &= -\exp(\theta_1) = -\exp\left(k_1(x-(\frac{3}{2}+k_1^2)t)\right).
\end{aligned}
\tag{13.235}
$$

Accordingly, we find

$$
F = \frac{1+f_1}{1+g_1} = \frac{1+\exp\left(k_1(x-(\frac{3}{2}+k_1^2)t)\right)}{1-\exp\left(k_1(x-(\frac{3}{2}+k_1^2)t)\right)}. \tag{13.236}
$$

The singular one-soliton solution

$$
v(x,t) = \frac{2k_1 \exp(k_1(x-(\frac{3}{2}+k_1^2)t))}{1-\exp(k_1(x-(\frac{3}{2}+k_1^2)t))}, \tag{13.237}
$$

follows immediately. Recall that the solution of the Gardner equation is given by

$$
u(x,t) = \frac{1}{2}+v(x,t), \tag{13.238}
$$

and consequently we obtain the singular soliton solution of the Gardner equation

$$u(x,t)=\frac{1}{2}+\frac{2k_1\exp(k_1(x-(\frac{3}{2}+k_1^2)t))}{1-\exp(k_1(x-(\frac{3}{2}+k_1^2)t))}. \tag{13.239}$$

To determine the singular two-soliton solution, we first set $N=2$ in (13.227) to get

$$\begin{aligned} f_1(x,t) &= \exp(\theta_1)+\exp(\theta_2),\\ g_1(x,t) &= -\exp(\theta_1)-\exp(\theta_2). \end{aligned} \tag{13.240}$$

To determine f_2 and g_2, we assume that

$$\begin{aligned} f_2(x,t) &= \sum_{1\leqslant i<j\leqslant N} a_{ij}\exp(\theta_i+\theta_j),\\ g_2(x,t) &= \sum_{1\leqslant i<j\leqslant N} b_{ij}\exp(\theta_i+\theta_j). \end{aligned} \tag{13.241}$$

This in turn gives

$$\begin{aligned} f(x,t) &= 1+\exp(\theta_1)+\exp(\theta_2)+a_{12}\exp(\theta_1+\theta_2),\\ g(x,t) &= 1-\exp(\theta_1)-\exp(\theta_2)+b_{12}\exp(\theta_1+\theta_2). \end{aligned} \tag{13.242}$$

Substituting (13.242) into the mKdV equation (13.223), we find that (13.242) is a solution of this equation if a_{12} and b_{12}, and therefore a_{ij} and b_{ij}, are equal and given by

$$a_{ij}=b_{ij}=\frac{(k_i-k_j)^2}{(k_i+k_j)^2}, \tag{13.243}$$

where θ_i and θ_j are given above in (13.233). For the two-soliton solutions we use $1\leqslant i<j\leqslant 2$ to obtain

$$\begin{aligned} f(x,t) &= 1+e^{k_1(x-(\frac{3}{2}+k_1^2)t)}+e^{k_2(x-(\frac{3}{2}+k_2^2)t)}\\ &\quad+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(\frac{3}{2}(k_1+k_2)+k_1^3+k_2^3)t},\\ g(x,t) &= 1-e^{k_1(x-(\frac{3}{2}+k_1^2)t)}-e^{k_2(x-(\frac{3}{2}+k_2^2)t)}\\ &\quad+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(\frac{3}{2}(k_1+k_2)+k_1^3+k_2^3)t}. \end{aligned} \tag{13.244}$$

Recall that the singular two-soliton solutions are obtained by using the formulas

$$\begin{aligned} v(x,t) &= \frac{\partial \log F(x,t)}{\partial x},\\ F(x,t) &= \frac{f(x,t)}{g(x,t)}. \end{aligned} \tag{13.245}$$

This in turn gives the singular two-soliton solutions explicitly upon using (13.244). Recall that

$$u(x,t)=\frac{1}{2}+v(x,t). \tag{13.246}$$

We can proceed in a similar manner to derive the three-soliton solutions. To determine f_3 and g_3 we set

$$\begin{aligned} f_1(x,t) &= \exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3),\\ f_2(x,t) &= a_{12}\exp(\theta_1+\theta_2)+a_{23}\exp(\theta_2+\theta_3)+a_{13}\exp(\theta_1+\theta_3),\\ g_1(x,t) &= -\exp(\theta_1)-\exp(\theta_2)-\exp(\theta_3),\\ g_2(x,t) &= a_{12}\exp(\theta_1+\theta_2)+a_{23}\exp(\theta_2+\theta_3)+a_{13}\exp(\theta_1+\theta_3), \end{aligned} \tag{13.247}$$

to obtain

$$\begin{aligned} f(x,t) &= 1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3)\\ &\quad +a_{12}\exp(\theta_1+\theta_2)+a_{23}\exp(\theta_2+\theta_3)+a_{13}\exp(\theta_1+\theta_3)\\ &\quad +f_3(x,t),\\ g(x,t) &= 1-\exp(\theta_1)-\exp(\theta_2)-\exp(\theta_3)\\ &\quad +a_{12}\exp(\theta_1+\theta_2)+a_{23}\exp(\theta_2+\theta_3)+a_{13}\exp(\theta_1+\theta_3)\\ &\quad +g_3(x,t). \end{aligned} \tag{13.248}$$

Substituting (13.248) into (13.223) to find that

$$\begin{aligned} f_3(x,t) &= b_{123}\exp(\theta_1+\theta_2+\theta_3),\\ g_3(x,t) &= -b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned} \tag{13.249}$$

where

$$b_{123} = a_{12}a_{13}a_{23} = \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}, \tag{13.250}$$

and θ_1, θ_2 and θ_3 are given before. For the singular three-soliton solutions we use $1 \leqslant i < j \leqslant 3$, we therefore obtain

$$\begin{aligned} f(x,t) &= 1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3)\\ &\quad +a_{12}\exp(\theta_1+\theta_2)+a_{13}\exp(\theta_1+\theta_3)+a_{23}\exp(\theta_2+\theta_3)\\ &\quad +b_{123}\exp(\theta_1+\theta_2+\theta_3),\\ g(x,t) &= 1-\exp(\theta_1)-\exp(\theta_2)-\exp(\theta_3)\\ &\quad +a_{12}\exp(\theta_1+\theta_2)+a_{13}\exp(\theta_1+\theta_3)+a_{23}\exp(\theta_2+\theta_3)\\ &\quad -b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned} \tag{13.251}$$

where

$$\begin{aligned} a_{ij} &= \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, \quad 1 \leqslant i < j \leqslant 3\\ b_{123} &= a_{12}a_{13}a_{23} = \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}. \end{aligned} \tag{13.252}$$

In view of this result we obtain

$$
\begin{aligned}
f(x,t) &= 1+e^{k_1(x-(\frac{3}{2}+k_1^2)t)}+e^{k_2(x-(\frac{3}{2}+k_2^2)t)}+e^{k_3(x-(\frac{3}{2}+k_3^2)t)} \\
&\quad +\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(\frac{3}{2}(k_1+k_2)+k_1^3+k_2^3)t} \\
&\quad +\frac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{(k_1+k_3)x-(\frac{3}{2}(k_1+k_3)+k_1^3+k_3^3)t} \\
&\quad +\frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{(k_2+k_3)x-(\frac{3}{2}(k_2+k_3)+k_2^3+k_3^3)t} \\
&\quad +\frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}e^{(k_1+k_2+k_3)x-(\frac{3}{2}(k_1+k_2+k_3)+k_1^3+k_2^3+k_3^3)t}, \\
g(x,t) &= 1-e^{k_1(x-(\frac{3}{2}+k_1^2)t)}-e^{k_2(x-(\frac{3}{2}+k_2^2)t)}-e^{k_3(x-(\frac{3}{2}+k_3^2)t)} \\
&\quad +\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(\frac{3}{2}(k_1+k_2)+k_1^3+k_2^3)t} \\
&\quad +\frac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{(k_1+k_3)x-(\frac{3}{2}(k_1+k_3)+k_1^3+k_3^3)t} \\
&\quad +\frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{(k_2+k_3)x-(\frac{3}{2}(k_2+k_3)+k_2^3+k_3^3)t} \\
&\quad -\frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}e^{(k_1+k_2+k_3)x-(\frac{3}{2}(k_1+k_2+k_3)+k_1^3+k_2^3+k_3^3)t}.
\end{aligned}
\tag{13.253}
$$

The singular three-soliton solutions are given by

$$
v(x,t)=\frac{\partial}{\partial x}\left(\log(\frac{f(x,t)}{g(x,t)})\right), \tag{13.254}
$$

where $f(x,t)$ and $g(x,t)$ are given in (13.253), and consequently $u(x,t)=\frac{1}{2}+v(x,t)$. This shows that the Gardner equation has singular multiple-soliton solutions for N finite, where $N \geqslant 1$.

13.9 Generalized KdV Equation with Two Power Nonlinearities

The generalized KdV equation with two power nonlinearities is of the form

$$
u_t+(au^n-bu^{2n})u_x+u_{xxx}=0. \tag{13.255}
$$

This equation describes the propagation of nonlinear long acoustic-type waves. The function f', where $f=(\frac{a}{n+1}u^{n+1}-\frac{b}{2n+1}u^{2n+1})$ is regarded as a nonlinear correction to the limiting long-wave phase speed c. If the amplitude is not supposed to be small, Eq. (13.255) serves as an approximate model for the description of weak dispersive effects on the propagation of nonlinear waves along a characteristic direction [19]. It is to be noted that for $n=1$, Eq. (13.255) is the well-known Gardner equation, the combined KdV-mKdV equation that was examined before.

13.9.1 Using the Tanh Method

We first apply the tanh method to the generalized KdV equation with two power nonlinearities given by

$$u_t + (au^n - bu^{2n})u_x + u_{xxx} = 0, \tag{13.256}$$

Using the wave variable $\xi = x - ct$ and integrating once, Eq. (13.256) will be converted to the ODE

$$-cu + \frac{a}{n+1}u^{n+1} - \frac{b}{2n+1}u^{2n+1} + u'' = 0, \tag{13.257}$$

Balancing u^{2n+1}, with exponent $(2n+1)M$, with u'', with exponent $M+2$ in (13.257) we find

$$M + 2 = (2n+1)M, \tag{13.258}$$

so that

$$M = \frac{1}{n}. \tag{13.259}$$

To get analytic closed solution, M should be an integer, hence we use the transformation

$$u = v^{\frac{1}{n}}. \tag{13.260}$$

Using (13.260) into (13.257) gives

$$\begin{aligned}
&-cn^2(2n+1)(n+1)v^2 + an^2(2n+1)v^3 - bn^2(n+1)v^4 \\
&\quad + n(2n+1)(n+1)vv'' + (1-n^2)(2n+1)(v')^2 = 0.
\end{aligned} \tag{13.261}$$

Balancing vv'' with v^4 gives $M = 1$. The tanh method admits the use of the finite expansion

$$v(\xi) = a_0 + a_1 Y. \tag{13.262}$$

Proceeding as before we obtain the following set of solutions

$$a_0 = \frac{a(2n+1)}{2b(n+2)}, \quad a_1 = \pm\frac{a(2n+1)}{2b(n+2)}, \quad \mu = \pm\frac{an}{2(n+2)}\sqrt{\frac{2n+1}{b(n+1)}}. \tag{13.263}$$

Consequently, we obtain the kink solution

$$v_1(x,t) = \frac{a(2n+1)}{2b(n+2)}\left(1 \pm \tanh\left[\frac{an}{2(n+2)}\sqrt{\frac{2n+1}{b(n+1)}}\left(x - \frac{a^2(2n+1)}{b(n+1)(n+2)^2}t\right)\right]\right), \tag{13.264}$$

and the travelling wave solution

$$v_2(x,t) = \frac{a(2n+1)}{2b(n+2)}\left(1 \pm \coth\left[\frac{an}{2(n+2)}\sqrt{\frac{2n+1}{b(n+1)}}(x - \frac{a^2(2n+1)}{b(n+1)(n+2)^2}t)\right]\right). \tag{13.265}$$

Recall that $u = v^{\frac{1}{n}}$. based on this we obtain the kink solution for the generalized KdV equation (13.256) by

$$u_1(x,t) = \left\{\frac{a(2n+1)}{2b(n+2)}\left(1 \pm \tanh\left[\frac{an}{2(n+2)}\sqrt{\frac{2n+1}{b(n+1)}}(x - \frac{a^2(2n+1)}{b(n+1)(n+2)^2}t)\right]\right)\right\}^{\frac{1}{n}}, \tag{13.266}$$

and the travelling wave solution

$$u_2(x,t) = \left\{\frac{a(2n+1)}{2b(n+2)}\left(1 \pm \coth\left[\frac{an}{2(n+2)}\sqrt{\frac{2n+1}{b(n+1)}}(x - \frac{a^2(2n+1)}{b(n+1)(n+2)^2}t)\right]\right)\right\}^{\frac{1}{n}}. \tag{13.267}$$

For $n = 1$, the solutions $u_1(x,t)$ and $u_2(x,t)$ are the solutions for the Gardner equation.

13.9.2 Using the Sine-cosine Method

Substituting the cosine assumption or the sine assumption as presented before, the method works only if $a = 0$ or $b = 0$. In either case, Eq. (13.256) will be reduced to the generalized KdV equation that was investigated in the previous section.

13.10 Compactons: Solitons with Compact Support

As stated before, the nonlinear term uu_x in the standard KdV equation

$$u_t + \alpha uu_x + u_{xxx} = 0, \tag{13.268}$$

causes the steepening of wave form. The dispersion effect term u_{xxx} in Eq. (13.268) makes the wave form spread. The balance between the weak nonlinearity of uu_x and the linear dispersion of u_{xxx} gives rise to soliton solutions. Soliton has been defined by Wadati [20] and many others as a nonlinear wave that has the following properties:

(1) A localized wave propagates without change of its properties (shape, velocity, etc.),

(2) Localized waves are stable against mutual collisions and retain their identities.

This means that the nonlinear KdV equation (13.268) with linear dispersion admits solitary waves that are infinite in extent or localized waves with exponential tails or wings.

Rosenau and Hyman [18] introduced a KdV like equation in the form

$$u_t + a(u^n)_x + (u^n)_{xxx} = 0, \quad n > 1, \tag{13.269}$$

that was called the K(n,n) equation. The convection term $(u^n)_x$ and the dispersion effect term $(u^n)_{xxx}$ in the nonlinear dispersive K(n,n) equation are both genuinely nonlinear. It is formally derived by Rosenau and Hyman [18] that the delicate interaction between nonlinear convection with genuine nonlinear dispersion generates solitary waves with exact compact support that are called *compactons*. The study conducted by Rosenau and Hyman revealed that Eq. (13.269) generates compactly supported solutions with non smooth fronts. In fact compactons are solitons with finite wavelength. This means that compactons are waves with compact support or solitons free of exponential wings [26]. Unlike soliton that narrows as the amplitude increases, the compacton's width is independent of the amplitude [18]. Compactons such as drops do not possess infinite wings, hence they interact among themselves only across short distances. A suffix-on is used in physics to indicate the particle property, such as in phonon, photon, and soliton. For this reason, the solitary wave with compact support is called *compacton* to indicate that it has the property of a particle.

A considerable size of studies has been conducted in the literature to show that purely nonlinear dispersion can cause a deep qualitative change in the genuinely nonlinear phenomenon [18,26]. It was shown that the compactons are nonanalytic solutions, whereas classical solitons are analytic solutions. The points of non analyticity at the edge of the compacton correspond to points of genuine nonlinearity for the differential equation and introduce singularities in the associated dynamical system for the traveling waves.

The K(n,n) equation (13.269) cannot be derived from a first order Lagrangian except for $n = 1$, and it does not possess the usual conservation laws of energy that KdV equation (13.268) possessed [18], hence the K(n,n) equation is not integrable. The stability analysis has shown that compacton solutions are stable, where the stability condition is satisfied for arbitrary values of the nonlinearity parameter. The stability of the compactons solutions was investigated by means of both linear stability and by Lyapunov stability criteria as well. Compactons were proved to collide elastically and vanish outside a finite core region. Two important features of compactons structures are observed:

(1) The compacton is a soliton characterized by the absence of exponential wings,
(2) The width of the compacton is independent of the amplitude.

The compactons discovery motivated a considerable work to make compactons be practically realized in scientific applications, such as the super deformed nuclei, preformation of cluster in hydrodynamic models, the fission of liquid drops (nuclear physics), inertial fusion and others.

It is interesting to note that solitary wave solutions may be expressed in terms of $\operatorname{sech}^{\alpha}$, or $\arctan(e^{\alpha(x-ct)})$. However, the compactons solutions may be expressed in terms of trigonometric functions $\cos\xi$ or $\sin\xi$ raised to an exponent. The cusps or infinite slopes solutions of the defocusing branches, where $a<0$, are expressed in terms of hyperbolic functions $\cosh\xi$ or $\sinh\xi$ raised to an exponent.

The pseudo spectral method and the tri-Hamiltonian operators, among other methods, were used to handle the K(n,n) equation. However, in this section we will use the tanh-coth method to handle the K(n,n) equation.

13.10.1 The K(n,n) Equation

The K(n,n) equation

$$u_t + a(u^n)_x + (u^n)_{xxx} = 0, \quad n>1, \tag{13.270}$$

will be investigated. To determine the traveling type wave solution $u(x,t)$ of Eq. (13.270) we use the wave variable $\xi=(x-ct)$ to convert (13.270) into the ODE

$$-cu + au^n + (u^n)'' = 0, \tag{13.271}$$

or equivalently

$$-cu + au^n + nu^{n-1}u'' + n(n-1)u^{n-2}(u')^2 = 0. \tag{13.272}$$

Balancing the terms $u^{n-1}u''$ with exponent $(n-1)M+4+M-2$ and u with exponent M gives

$$(n-1)M+4+M-2 = M, \tag{13.273}$$

so that

$$M = -\frac{2}{n-1}. \tag{13.274}$$

As stated before, M should be an integer to obtain a closed form analytic solution, the parameter M should be an integer. To achieve this goal we use a transformation formula

$$u(x,t) = v^{-\frac{1}{n-1}}(x,t). \tag{13.275}$$

This formula carries (13.272) into

$$-c(n-1)^2v^3 + a(n-1)^2v^2 - bn(n-1)vv'' + bn(2n-1)(v')^2 = 0. \tag{13.276}$$

Balancing the terms v^3 and vv'' we find

$$3M = M+M+2, \tag{13.277}$$

that gives $M=2$. The tanh-coth method allows us to use the substitution

$$v(x,t) = S(Y) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}. \tag{13.278}$$

Substituting (13.278) into (13.276), collecting the coefficients of each power of Y, and solving the resulting system of algebraic equations we obtain the following sets:
(i)

$$a_0 = \frac{a(n+1)}{2cn}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = -\frac{a(n+1)}{2cn}, \quad \mu = \frac{n-1}{2n}\sqrt{-a}, \tag{13.279}$$

and
(ii)

$$a_0 = \frac{a(n+1)}{2cn}, \quad a_1 = b_1 = a_2 = 0, \quad b_2 = -\frac{a(n+1)}{2cn}, \quad \mu = \frac{n-1}{2n}\sqrt{-a}, \tag{13.280}$$

Noting that $u = v^{-\frac{1}{n-1}}$, we first obtain the solitary patterns solutions

$$u_1(x,t) = \left\{-\frac{2cn}{a(n+1)}\sinh^2\left[\frac{n-1}{2n}\sqrt{-a}(x-ct)\right]\right\}^{\frac{1}{n-1}}, \tag{13.281}$$

$$u_2(x,t) = \left\{\frac{2cn}{a(n+1)}\cosh^2\left[\frac{n-1}{2n}\sqrt{-a}(x-ct)\right]\right\}^{\frac{1}{n-1}}, \tag{13.282}$$

for $a < 0$, where $\xi = x - ct$.

However, for $a > 0$ we obtain the compactons solutions

$$u_3(x,t) = \begin{cases} \left\{\dfrac{2cn}{a(n+1)}\sin^2[\dfrac{n-1}{2n}\sqrt{a}(x-ct)]\right\}^{\frac{1}{n-1}}, & |\,\xi\,| \leqslant \dfrac{\pi}{\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{13.283}$$

and

$$u_4(x,t) = \begin{cases} \left\{\dfrac{2cn}{a(n+1)}\cos^2[\dfrac{n-1}{2n}\sqrt{a}(x-ct)]\right\}^{\frac{1}{n-1}}, & |\,\xi\,| \leqslant \dfrac{\pi}{2\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{13.284}$$

The last results are in consistent with the results that other researchers obtained by using different approaches.

13.11 Variants of the K(n,n) Equation

In this section, three variants of the K(n,n) equations will be studied. The main goal of this work is to show that compactons arise from a variety of variants of the K(n,n) equations. The variants that will be examined involve both genuinely nonlinear convection and dispersion terms where the interaction between the effects of these concepts generates compactons. The tanh method will be used only in this study.

13.11.1 First Variant

A variant of the K(n,n) equation of the form

$$u_t + a(u^{n+1})_x + [u(u^n)_{xx}]_x = 0, \quad a > 0, n \geqslant 1, \tag{13.285}$$

was investigated by Rosenau [18]. This model emerges in nonlinear lattices and was used to describe the dispersion of dilute suspensions for $n = 1$. Equation (13.285) was considered as a variant of the KdV equation or of the K(n,n) equation.

To determine the traveling-type wave solution $u(x,t)$ of Eq. (13.285) we use the wave variable $\xi = x - ct$, and integrate the resulting ODE to transform (13.285) into an ODE

$$-cu + au^{n+1} + u(u^n)'' = 0, \tag{13.286}$$

or equivalently

$$-cu + au^{n+1} + nu^n u'' + n(n-1)u^{n-1}(u')^2 = 0. \tag{13.287}$$

Balancing the terms $u^n u''$ and u gives

$$nM + M + 2 = M, \tag{13.288}$$

so that

$$M = -\frac{2}{n}. \tag{13.289}$$

To obtain a closed form analytic solution, the parameter M should be an integer. For this reason we use a transformation formula

$$u(x,t) = v^{-\frac{1}{n}}(x,t), \tag{13.290}$$

that will carry (13.287) into

$$-cv^3 + av^2 - vv'' + 2(v')^2 = 0. \tag{13.291}$$

Balancing the terms v^3 and vv'' we find $M = 2$. The tanh-coth method allows us to use the substitution

$$v(x,t) = S(Y) = \sum_{i=0}^{2} a_i Y^i + \sum_{j=1}^{2} b_j Y^{-j} \tag{13.292}$$

into (13.291), and proceeding as before we obtain the following sets of solutions:
(i)

$$a_0 = \frac{a}{2c}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = -\frac{a}{2c}, \quad \mu = \frac{\sqrt{-a}}{2}, \tag{13.293}$$

and
(ii)

$$a_0 = \frac{a}{2c}, \quad a_1 = b_1 = a_2 = 0, \quad b_2 = -\frac{a}{2c}, \quad \mu = \frac{\sqrt{-a}}{2}. \tag{13.294}$$

Noting that $u = v^{-\frac{1}{n}}$, we first obtain the solitary patterns solutions

$$u_1(x,t) = \left\{\frac{2c}{a}\sinh^2\left[\frac{\sqrt{a}}{2}(x-ct)\right]\right\}^{\frac{1}{n}}, \tag{13.295}$$

$$u_2(x,t) = -\left\{\frac{2c}{a}\cosh^2\left[\frac{\sqrt{a}}{2}(x-ct)\right]\right\}^{\frac{1}{n}}, \tag{13.296}$$

for $a < 0$, where $\xi = x - ct$.

For $a > 0$, the following compactons solutions

$$u(x,t) = \begin{cases} \left\{\frac{2c}{a}\sin^2\left[\frac{\sqrt{a}}{2}(x-ct)\right]\right\}^{\frac{1}{n}}, & |\, x-ct \,| \leqslant \frac{\pi}{\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{13.297}$$

$$u(x,t) = \begin{cases} \left\{\frac{2c}{a}\cos^2\left[\frac{\sqrt{a}}{2}(x-ct)\right]\right\}^{\frac{1}{n}}, & |\, x-ct \,| \leqslant \frac{\pi}{2\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{13.298}$$

are readily obtained.

13.11.2 Second Variant

A second variant of the K(n,n) equation is of the form

$$u_t + a(u^{n+1})_x + [u^n(u)_{xx}]_x = 0, \quad a > 0, n \geqslant 2. \tag{13.299}$$

This variant was investigated by in [26].

To determine the traveling-type wave solution $u(x,t)$ of Eq. (13.299) we use the wave variable $\xi = x - ct$, and integrate the resulting ODE to transform (13.299) into an ODE

$$-cu + au^{n+1} + u^n u'' = 0. \tag{13.300}$$

Balancing the terms $u^n u''$ and u gives

$$nM + M + 2 = M, \tag{13.301}$$

so that

$$M = -\frac{2}{n}. \tag{13.302}$$

To obtain a closed form analytic solution, the parameter M should be an integer. For this reason we use a transformation formula

$$u(x,t) = v^{-\frac{1}{n}}(x,t), \tag{13.303}$$

that will carry (13.300) into

$$-cv^3 + av^2 + \alpha(\alpha - 1)(v')^2 + \alpha v v'' = 0, \quad \alpha = -\frac{1}{n}. \tag{13.304}$$

Balancing the terms v^3 and vv'' we find $M = 2$. The tanh-coth method uses the substitution

$$v(x,t) = S(Y) = \sum_{i=0}^{2} a_i Y^i + \sum_{j=1}^{2} b_j Y^{-j}, \tag{13.305}$$

into (13.304), and proceeding as before we obtain the following sets of solutions:
(i)

$$a_0 = -\frac{a(n-2)}{2c}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = -\frac{a(n-2)}{2c}, \quad \mu = \frac{n}{2}\sqrt{-a}, a < 0, \tag{13.306}$$

and
(ii)

$$a_0 = -\frac{a(n-2)}{2c}, \quad a_1 = b_1 = a_2 = 0, \quad b_2 = -\frac{a(n-2)}{2c}, \quad \mu = \frac{n}{2}\sqrt{-a}, a < 0. \tag{13.307}$$

Noting that $u = v^{-\frac{1}{n}}$, we first obtain the solitary patterns solutions

$$u(x,t) = \left\{ \frac{2c}{a(2-n)} \sinh^2 \left[n\frac{\sqrt{-a}}{2}(x - ct) \right] \right\}^{\frac{1}{n}}, \tag{13.308}$$

and

$$u(x,t) = \left\{ -\frac{2c}{a(2-n)} \cosh^2 \left[n\frac{\sqrt{-a}}{2}(x - ct) \right] \right\}^{\frac{1}{n}}, \tag{13.309}$$

for $a < 0$, where $\xi = x - ct$.

For $a > 0$, The following compactons solutions

$$u(x,t) = \begin{cases} \left\{ \dfrac{2c}{a(2-n)} \sin^2 \left[\dfrac{n}{2}\sqrt{a}(x - ct) \right] \right\}^{\frac{1}{n}}, & |\, x - ct \,| \leqslant \dfrac{\pi}{\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{13.310}$$

and

$$u(x,t)=\begin{cases}\left\{\dfrac{2c}{a(2-n)}\cos^2\left[\dfrac{n}{2}\sqrt{a}(x-ct)\right]\right\}^{\frac{1}{n}}, & |\,x-ct\,|\leqslant\dfrac{\pi}{2\mu},\\ 0, & \text{otherwise},\end{cases}\tag{13.311}$$

are readily obtained.

13.11.3 Third Variant

A third variant of the KdV equation

$$u_t+(au+bu^n)_x+k(u^n)_{xxx}=0, n>1, k\neq 0,\tag{13.312}$$

was studied in [26]. It is to be noted that Eq. (13.312) is the linear KdV equation for $b=0$ and $n=1$. However, for $a=0$, Eq. (13.312) will be reduced to the well known K(n,n) equation that was examined before. The convection and dispersion terms are both nonlinear.

As shown before, this equation can be transformed to the ODE

$$(a-c)u+bu^n+k(u^n)''=0,\tag{13.313}$$

upon using the wave variable $\xi=x-ct$ and integrating the resulting equation. Eq. (13.313) can be converted to

$$(a-c)u+bu^n+knu^{n-1}u''+kn(n-1)u^{n-2}(u')^2=0.\tag{13.314}$$

Balancing $u(x,t)$ with $u^{n-1}u''$ gives

$$M=-\frac{2}{n-1}.\tag{13.315}$$

As stated before, a transformation formula

$$u=v^{-\frac{1}{n-1}},\tag{13.316}$$

enables us to achieve a closed form solution. This transformation formula converts (13.314) to

$$(a-c)(n-1)^2v^3+b(n-1)v^2-kn(n-1)vv''+kn(2n-1)(v')^2=0.\tag{13.317}$$

Balancing vv'' with v^3 gives $M=2$. The tanh method gives the same results as the tanh-coth method. The tanh method allows us to use the finite expansion

$$v(x,t)=S(Y)=a_0+a_1Y+a_2Y^2.\tag{13.318}$$

Substituting (13.318) into (13.317), collecting the coefficients of each power of Y and proceeding as before we obtain

$$\begin{aligned} a_0 &= -\frac{(n+1)b}{2(a-c)n}, \\ a_1 &= 0, \\ a_2 &= \frac{(n+1)b}{2(a-c)n}, \\ M &= \frac{n-1}{2n}\sqrt{-\frac{b}{k}}, \quad \frac{b}{k} < 0. \end{aligned} \tag{13.319}$$

where c is selected as a free parameter. Noting that $u(x,t) = v^{-\frac{1}{n-1}}$, and using the previous results we find a family of solitary patterns solutions

$$u_1(x,t) = \left\{-\frac{2n(c-a)}{b(n+1)}\sinh^2\left[\frac{n-1}{2n}\sqrt{-\frac{b}{k}}(x-ct)\right]\right\}^{\frac{1}{n-1}}, \tag{13.320}$$

and

$$u_2(x,t) = \left\{\frac{2n(c-a)}{b(n+1)}\cosh^2\left[\frac{n-1}{2n}\sqrt{-\frac{b}{k}}(x-ct)\right]\right\}^{\frac{1}{n-1}}, \tag{13.321}$$

valid for $\frac{b}{k} < 0$.

However, for $\frac{b}{k} > 0$, we obtain a family of compactons solutions given by

$$u_3(x,t) = \begin{cases} \left\{\frac{2n(c-a)}{b(n+1)}\sin^2\left[\frac{n-1}{2n}\sqrt{\frac{b}{k}}(x-ct)\right]\right\}^{\frac{1}{n-1}}, & |\mu\xi| < \pi, \\ 0, & \text{otherwise}, \end{cases} \tag{13.322}$$

and

$$u_4(x,t) = \begin{cases} \left\{\frac{2n(c-a)}{b(n+1)}\cos^2\left[\frac{n-1}{2n}\sqrt{\frac{b}{k}}(x-ct)\right]\right\}^{\frac{1}{n-1}}, & |\mu\xi| < \frac{\pi}{2}, \\ 0, & \text{otherwise}, \end{cases} \tag{13.323}$$

The results obtained above by using the tanh method are consistent with the results obtained before by using the sine-cosine method.

13.12 Compacton-like Solutions

In previous sections, we examined the conditions needed for the existence of solitons. In this case, it was confirmed that solitons are due to the balance between weak nonlinearity of uu_x and the linear dispersion of u_{xxx}. Moreover, we have studied the $\mathrm{K}(n,n)$ and some of its variants where the interaction between nonlinear term of $(u^n)_x$ and the genuine nonlinear dispersive term $(u^n)_{xxx}$ gives rise to compactons: solitons with compact support. In this section we will study some of the nonlinear evolution equations that we discussed before that will give rise to compacton-like solutions.

The main idea is to assume that the compacton-like solution is of the form

$$u(x,t) = \frac{A\cos^2\mu(x-ct)}{1+B\cos^2\mu(x-ct)}, \tag{13.324}$$

or in the form

$$u(x,t) = \frac{A\sin^2\mu(x-ct)}{1+B\sin^2\mu(x-ct)}, \tag{13.325}$$

where A, B, μ and c are constants that will be determined.

13.12.1 The Modified KdV Equation

Recall that the mKdV equation is

$$u_t + au^2u_x + u_{xxx} = 0. \tag{13.326}$$

Substituting (13.324) or (13.324) into the mKdV equation (13.326), and proceeding as before we obtain

$$A = \frac{4\mu}{3}\sqrt{\frac{2}{a}},\ B = -\frac{2}{3},\ c = 4\mu^2, \tag{13.327}$$

where μ is left as a free parameter. In view of these results, we obtain the following compacton-like solutions

$$u(x,t) = \frac{4\mu}{3}\sqrt{\frac{2}{a}}\begin{cases} \dfrac{\cos^2\mu(x-4\mu^2t)}{1-\frac{2}{3}\cos^2\mu(x-4\mu^2t)}, & |(x-4\mu^2t)| \leqslant \dfrac{\pi}{2\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{13.328}$$

and

$$u(x,t) = \frac{4\mu}{3}\sqrt{\frac{2}{a}}\begin{cases} \dfrac{\sin^2\mu(x-4\mu^2t)}{1-\frac{2}{3}\sin^2\mu(x-4\mu^2t)}, & |(x-4\mu^2t)| \leqslant \dfrac{\pi}{\mu}, \\ 0, & \text{otherwise}. \end{cases} \tag{13.329}$$

13.12.2 The Gardner Equation

Recall that the Gardner equation, or the combined KdV-mKdV equation, is

$$u_t + 6uu_x + 6u^2u_x + u_{xxx} = 0. \tag{13.330}$$

We first substitute (13.324) or (13.324) into the Gardner equation (13.330), collect all coefficients of like cosine functions, equate these coefficients to zero, and solve the resulting system we obtain

$$A = -\frac{2B(1+B)}{2+3B}, \quad \mu = \frac{\sqrt{B+1}}{2+3B}, \quad c = -\frac{4(1+B)(1+3B)}{(2+3B)^2}, \quad B \neq -\frac{2}{3}, -1, \tag{13.331}$$

where B is left as a free parameter. In view of these results, we obtain the following compacton-like solutions

$$u(x,t) = \begin{cases} \dfrac{2B(1+B)\cos^2\mu(x-ct)}{(2+3B)\,(1+B\cos^2\mu(x-ct))}, & |\,(x-ct)\,| \leqslant \dfrac{\pi}{2\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{13.332}$$

and

$$u(x,t) = \begin{cases} \dfrac{2B(1+B)\sin^2\mu(x-ct)}{(2+3B)\left(1+B\sin^2\mu(x-ct)\right)}, & |\,(x-ct)\,| \leqslant \dfrac{\pi}{\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{13.333}$$

where μ and c are defined above.

13.12.3 The Modified Equal Width Equation

The modified equal width equation is

$$u_t + 3u^2u_x - u_{xxt} = 0. \tag{13.334}$$

Proceeding as before we find $A = \frac{2}{3}\sqrt{\frac{2c}{3}}$, $B = -\frac{2}{3}$, $\mu = \frac{1}{2}$, where c is left as a free parameter. This gives the following solutions

$$u(x,t) = \begin{cases} \dfrac{2\sqrt{6c}\cos^2(\frac{1}{2}(x-ct))}{3\left(3-2\cos^2(\frac{1}{2}(x-ct))\right)}, & |\,(x-ct)\,| \leqslant \dfrac{\pi}{2\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{13.335}$$

and

$$u(x,t) = \begin{cases} \dfrac{2\sqrt{6c}\sin^2(\frac{1}{2}(x-ct))}{3\left(3-2\sin^2(\frac{1}{2}(x-ct))\right)}, & |(x-ct)| \leqslant \dfrac{\pi}{\mu}, \\ 0, & \text{otherwise,} \end{cases} \tag{13.336}$$

References

1. M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge (1991).
2. M.J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia (1981).
3. R. Bullough and P. Caudrey, Solitons vol 17 in "Topics in Current Physics", Springer, Berlin (1980).
4. P.G. Drazin and R.S. Johnson, Solitons: an Introduction, Cambridge University Press, Cambridge, (1996).
5. C. Gardner, J. Greene, M. Kruskal and R. Miura, Method for solving the Korteweg-deVries equation, *Phys. Rev. Lett.*, 19, 1095–1097, (1967).
6. W. Hereman and W. Zhaung, Symbolic software for soliton theory, Acta Applicandae Mathematicae. *Phys. Lett. A*, 76, 95–96, (1980).
7. W. Hereman and A. Nuseir, Symbolic methods to construct exact solutions of nonlinear partial differential equations, *Math. Comput. Simulation*, 43 13–27, (1997).
8. J. Hietarinta, A search for bilinear equations passing Hirota's three-soliton condition. I. KdV-type bilinear equations, *J. Math. Phys.*, 28(8), 1732–1742, (1987).
9. J. Hietarinta, A search for bilinear equations passing Hirota's three-soliton condition. II. mKdV-type bilinear equations, J. Math. Phys., 28(9), 2094–2101, (1987).
10. R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge (2004).
11. R. Hirota, Exact solutions of the Korteweg-de Vries equation for multiple collisions of solitons, Phys. Rev. Lett., 27(18), 1192–1194, (1971).
12. R. Hirota, Exact N-soliton solutions of a nonlinear wave equation, *J. Math Phys.*, 14(7), 805–809, (1973).
13. M. Ito, An extension of nonlinear evolution equations of the KdV (mKdV) type to higher orders, *J. Phys. Soc. Japan*, 49(2), 771–778, (1980).
14. D.J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular channel, and a new type of long stationary waves, *Phil. Mag.* 39(5), 422-443, (1895).
15. Y. Matsuno, Bilinear Transformation Method, Academic Press, New York, (1984).
16. R.M. Miura, The Korteweg-de Vries Equation: a survey of results, *SIAM Rev.*, 18, 412–459, (1976).
17. A. Nakamura, Simple explode-decay mode solutions of a certain one-space dimensional nonlinear evolution equations, J. Phys. Soc. Japan, 33(5), 1456–1458, (1972).
18. P. Rosenau and J.M. Hyman, Compactons: Solitons with finite wavelengths, *Phys. Rev. Lett.*, 70(5), 564–567, (1993).
19. A.V. Slyunaev and E.N. Pelinovski, Dynamics of large-amplitude solitons, *J. Exper. Theor. Phys.*, 89(1), 173–181, (1999).
20. M. Wadati, The exact solution of the modified Korteweg-de Vries equation, *J. Phys. Soc. Japan*, 32, 1681–1687, (1972).
21. A.M. Wazwaz, New solitons and kink solutions for the Gardner equation, *Commun. Nonlinear Sci. Numer. Simul.*, 12(8), 1395–1404, (2007).
22. A.M. Wazwaz, New solitons and kinks solutions to the Sharma-Tasso-Olver equation, *Appl. Math. Comput.*, 188, 1205–1213, (2007).

23. A.M. Wazwaz, The tanh-coth method for solitons and kink solutions for nonlinear parabolic equations, *Appl. Math. Comput.*, 188, 1467–1475, (2007).
24. A.M. Wazwaz, The tanh-coth method for new compactons and solitons solutions for the $\mathrm{K}(n,n)$ and the $\mathrm{K}(n+1,n+1)$ equations. *Appl. Math. Comput.*, 188, 1930–1940, (2007).
25. A.M. Wazwaz, Multiple-soliton solutions for the KP equation by Hirota's bilinear method and by the tanh-coth method, *Appl. Math. Comput.*, 190, 633–640, (2007).
26. A.M. Wazwaz, Compact and noncompact structures for variants of the KdV equation, *Appl. Math. and Comput.*, 132 (1) 29–45, (2002).

Chapter 14
KdV and mKdV Equations of Higher-orders

14.1 Introduction

It is well known that the third order KdV equation is the generic model for studying weakly nonlinear waves. The equation models surface waves with small amplitude and long wavelength on shallow water. The KdV equation involves a balance between weak nonlinearity and linear dispersion. The KdV equation is completely integrable and the collision between solitary waves is elastic, which means that the solitons retain original identity after collision.

In Chapter 13, we have discussed modified and extensions of the KdV equations. The equations that we studied in Chapter 13 are of third-order and give soliton solutions. However, several other extensions of the standard KdV equation of higher orders appear in scientific applications. The bilinear form for the standard KdV equation may be extended to formulate the higher-order KdV equations. Other higher-order KdV equations were constructed from scientific applications. The nature of solitary wave interaction for these higher-order KdV equations has attracted a considerable size of research work.

In this chapter, we will follow the same analysis employed before to handle the higher-order KdV equations of fifth-order, seventh-order, and ninth-order for soliton solutions. Moreover, higher-order modified KdV equations of fifth-order and seventh-order that belong to Lax hierarchy will be also studied to determine N-soliton solutions. It is interesting to note that not all higher order KdV equations are completely integrable. The study will be focused on the determination of single solitons, and multiple-soliton solutions for integrable equations.

In addition, the Hirota-Satsuma system will also be studied to determine N-soliton solutions. The generalized models for shallow water wave equations are of significant interest that will be examined as well in this chapter.

14.2 Family of Higher-order KdV Equations

In this section we will study the family of higher-order KdV equations. The higher-order KdV equations involve more than one dispersive terms. The fifth-order, the seventh-order and the ninth-order KdV equations will be studied in the forthcoming sections. The tanh-coth method will be used for the determination of single soliton solutions and for periodic solutions. Moreover, the Hirota's bilinear method combined with Herman's simplified form will be used for the derivation of multiple-soliton solutions for completely integrable equations.

14.2.1 Fifth-order KdV Equations

The well-known fifth-order KdV (fKdV) equations in its standard form reads

$$u_t+\alpha u^2u_x+\beta u_xu_{xx}+\gamma uu_{3x}+u_{5x}=0, \tag{14.1}$$

where α,β and γ are arbitrary nonzero and real parameters, and $u=u(x,t)$ is a sufficiently smooth function. The fifth-order KdV equations involve two dispersive terms u_{3x} and u_{5x}. Because the parameters α,β and γ are arbitrary and take different values, this will drastically change the characteristics of the fKdV equation (14.1). A variety of the fKdV equations can be developed by changing the real values of the parameters α,β and γ. The most well-known fifth-order KdV equations that will be approached are the Sawada-Kotera (SK) equation, the Caudrey-Dodd-Gibbon equation, the Lax equation, the Kaup-Kuperschmidt (KP) equation, and the Ito equation. The derivation of these fifth-order forms are derived from specific bilinear forms of the so-called Hirota's D-operators.

As stated before, many forms of the fKdV equation can be constructed by changing the parameters α,β and γ. However, five well known forms of the fKdV that are of particular interest in the literature. These forms are:

(i) The Sawada-Kotera (SK) equation [12] is given by

$$u_t+5u^2u_x+5u_xu_{xx}+5uu_{3x}+u_{5x}=0, \tag{14.2}$$

and characterized by

$$\beta=\gamma,\quad \alpha=\frac{1}{5}\gamma^2. \tag{14.3}$$

Recall that the KdV equation

$$u_t+6uu_x+u_{xxx}=0 \tag{14.4}$$

can be expressed in terms of the bilinear operators by

$$D_x(D_t+D_x^3)(f\cdot f)=0. \tag{14.5}$$

The solution of Eq. (14.4) is of the form

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}. \tag{14.6}$$

Sawada and Kotera [12] generalized (14.5) and (14.6) into the form

$$D_x(D_t + D_x^5)(f \cdot f) = 0, \quad u(x,t) = 2(\ln f(x,t))_{xx}, \tag{14.7}$$

or equivalently

$$D_x D_t(f \cdot f) + D_x^6(f \cdot f) = 0, \quad u(x,t) = 2(\ln f(x,t))_{xx}. \tag{14.8}$$

The Hirota's bilinear operators have several properties, and two of these properties are given by

$$\begin{aligned} D_x D_t(f \cdot f) &= f^2(\ln f^2)_{xt}, \\ D_x^6(f \cdot f) &= f^2(u_{4x} + 15uu_{2x} + 15u^3). \end{aligned} \tag{14.9}$$

Substituting (14.9) into (14.8) gives

$$2(\ln f)_{xt} + u_{4x} + 15uu_{2x} + 15u^3 = 0. \tag{14.10}$$

Differentiate (14.10) with respect to x and using (14.6) gives the Sawada-Kotera equation

$$u_t + 45u^2u_x + 15u_xu_{2x} + 15uu_{3x} + u_{5x} = 0. \tag{14.11}$$

It is to be noted that the Sawada-Kotera equation (14.2) is obtained from (14.11) by using the scaling $u = \frac{1}{3}v$.

(ii) The Caudrey-Dodd-Gibbon equation (CDG) [2] is given by

$$u_t + 180u^2u_x + 30u_xu_{xx} + 30uu_{xxx} + u_{xxxxx} = 0, \tag{14.12}$$

with $u(x,t)$ is a sufficiently often differentiable function. The CDG equation is, like the SK equation, characterized by

$$\beta = \gamma, \quad \alpha = \frac{1}{5}\gamma^2. \tag{14.13}$$

The CDG equation is completely integrable and therefore it has multiple-soliton solutions, and is obtained from the SK equation (14.11) by using $u = 2v$. It was found that the CDG equation (14.12) possesses the Painlevé property.

(iii) The Lax equation [10] reads

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{3x} + u_{5x} = 0, \tag{14.14}$$

and characterized by

$$\beta = 2\gamma, \quad \alpha = \frac{3}{10}\gamma^2. \tag{14.15}$$

Lax [10] generalized (14.5) and (14.6) into the form

$$\begin{gathered}\left[D_x(D_t+D_x^5)-\frac{5}{3}D_s(D_s+D_x^3)\right](f\cdot f)=0,\quad u(x,t)=2(\ln f(x,t))_{xx},\\ D_x(D_s+D_x^3)(f\cdot f)=0,\end{gathered}\tag{14.16}$$

by involving an auxiliary variable s. Equations (14.16) can be written as

$$\begin{gathered}D_xD_t(f\cdot f)+D_x^6(f\cdot f)-\frac{5}{3}D_s^2-\frac{5}{3}D_sD_x^3=0,\quad u(x,t)=2(\ln f(x,t))_{xx},\\ D_xD_s(f\cdot f)+D_x^4(f\cdot f)=0.\end{gathered}\tag{14.17}$$

Using the properties of Hirota's D-operators, as presented before, gives

$$2(\ln f)_{xt}+10u^3+10uu_{2x}+5u_x^2+u_{4x}=0.\tag{14.18}$$

Differentiate (14.18) with respect to x and using (14.6) gives the Lax fifth-order equation

$$u_t+30u^2u_x+20u_xu_{2x}+10uu_{3x}+u_{5x}=0.\tag{14.19}$$

(iv) The Kaup-Kuperschmidt (KK) equation [8,9] reads

$$u_t+20u^2u_x+25u_xu_{xx}+10uu_{3x}+u_{5x}=0,\tag{14.20}$$

is characterized by

$$\beta=\frac{5}{2}\gamma,\quad \alpha=\frac{1}{5}\gamma^2.\tag{14.21}$$

Kaup and Kuperschmidt in [8] and [9] respectively generalized (14.5) and (14.6) into the bilinear form

$$\begin{gathered}\left[D_x(16D_t+D_x^5)\right](f\cdot f)-30D_x^2(f\cdot g)=0,\quad u(x,t)=-2(\ln f(x,t))_{xx},\\ D_x^4(f\cdot f)+2(f\cdot g)=0.\end{gathered}\tag{14.22}$$

Using the properties of Hirota's D-operators presented before gives the KK equation by

$$u_t+45u^2u_x-\frac{75}{2}u_xu_{xx}-15uu_{3x}+u_{5x}=0.\tag{14.23}$$

The KK form (14.20) is obtained by setting $u(x,t)=-\frac{2}{3}v(x,t)$.

(v) The Ito equation [7]

$$u_t+2u^2u_x+6u_xu_{xx}+3uu_{3x}+u_{5x}=0,\tag{14.24}$$

is characterized by

$$\beta=2\gamma,\quad \alpha=\frac{2}{9}\gamma^2.\tag{14.25}$$

It was found that the SK, CDG, Lax, and KK equations belong to the completely integrable hierarchy of higher-order KdV equations. These four equations have infinite sets of conservation laws, and therefore these equations have N-soliton solutions. However, the Ito equation is not completely integrable but has a limited number of special conservation laws.

It is important to note that there is another significant fifth-order KdV equation that appears in the literature in the form

$$u_t + auu_x + bu_{3x} - ru_{5x} = 0, \tag{14.26}$$

where a, b, r are constants. This equation is called the Kawahara equation. When $b = 1$, and $r = 0$, the Kawahara equation reduces to the standard third-order KdV equation. The Kawahara equation models the dynamics of long waves in a viscous fluid. It appears in the theory of shallow water waves with surface tension and the theory of magneto-acoustic waves in plasmas. Moreover, the Kawahara equation has also a modified form given by

$$u_t + au^2u_x + bu_{xxx} - ru_{xxxxx} = 0, \tag{14.27}$$

where the quadratic nonlinearity uu_x of the Kawahara equation (14.26) is replaced by the cubic nonlinearity u^2u_x. The Kawahara and the modified Kawahara equations will be studied in Chapter 15.

14.2.2 Seventh-order KdV Equations

The higher-order KdV equation may appear in a seventh-order KdV (sKdV) equation given by generalized form

$$u_t + au^3u_x + bu_x^3 + cuu_xu_{2x} + du^2u_{3x} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0, \tag{14.28}$$

where a, b, c, d, e, f, and g are nonzero parameters, and $u_{kx} = \dfrac{\partial^k}{\partial x^k}$. The parameters a, b, c, d, e, f, and g can take arbitrary values. However, there are three well-known special cases of Eq. (14.28) derived by using an extension to the bilinear form of the standard KdV equation. These forms are:

(i) The seventh-order Sawada-Kotera-Ito equation is given by

$$\begin{aligned} u_t + 252u^3u_x + 63u_x^3 + 378uu_xu_{2x} + 126u^2u_{3x} + 63u_{2x}u_{3x} \\ +42u_xu_{4x} + 21uu_{5x} + u_{7x} = 0. \end{aligned} \tag{14.29}$$

(ii)The seventh-order Lax equation reads

$$\begin{aligned} u_t + 140u^3u_x + 70u_x^3 + 280uu_xu_{2x} + 70u^2u_{3x} + 70u_{2x}u_{3x} \\ +42u_xu_{4x} + 14uu_{5x} + u_{7x} = 0. \end{aligned} \tag{14.30}$$

(iii) The seventh-order Kaup-Kuperschmidt equation is given by

$$\begin{aligned} u_t &+ 2016u^3u_x + 630u_x^3 + 2268uu_xu_{2x} + 504u^2u_{3x} + 252u_{2x}u_{3x} \\ &+147u_xu_{4x} + 42uu_{5x} + u_{7x} = 0. \end{aligned} \tag{14.31}$$

As stated before, these three cases of the seventh-order KdV equation are completely integrable. This means that each of these equations admits an infinite number of conservation laws, and as a result each gives rise to N-soliton solutions.

14.2.3 Ninth-order KdV Equations

The KdV equation may appear in a ninth-order KdV equation. The ninth-order Sawada-Kotera equation (nSK) reads

$$\begin{aligned} u_t &+ 45u_xu_{6x} + 45uu_{7x} + 210u_{3x}u_{4x} + 210u_{2x}u_{5x} + 1575u_x(u_{2x})^2 \\ &+ 3150uu_{2x}u_{3x} + 1260uu_xu_{4x} + 630u^2u_{5x} + 9450u^2u_xu_{2x} \\ &+ 3150u^3u_{3x} + 4725u^4u_x + u_{9x} = 0. \end{aligned} \tag{14.32}$$

Other forms of different coefficients may be derived. The ninth-order KdV equation (14.32) is not completely integrable. Any nonlinear evolution equation should give N-soliton solutions for $N \geqslant 1$ to be characterized as completely integrable. The ninth-order forms are derived from specific bilinear forms of the so-called Hirota's D-operators.

In what follows we will study the higher-order KdV equations. The tanh-coth method and the Hirota's direct method will be used to achieve our goal of this study.

14.3 Fifth-order KdV Equations

In this section we will study a class of fifth-order KdV equation (fKdV) given in a generalized form

$$u_t + \alpha u^2u_x + \beta u_xu_{xx} + \gamma uu_{3x} + u_{5x} = 0, \tag{14.33}$$

where α, β, and γ are arbitrary nonzero and real parameters, and $u = u(x,t)$ is a sufficiently-often differentiable function. As stated before, the tanh-coth method will be used to derive single soliton solutions. The multiple-soliton solutions will be determined for completely integrable equations by using the Hirota's bilinear formalism. The fKdV equation has wide applications in quantum mechanics and nonlinear optics. It is well known that nonlinear wave phenomena of plasma media and fluid dynamics are modelled by kink shaped tanh solution or by bell shaped sech solutions.

14.3.1 Using the Tanh-coth Method

We first rewrite (14.33) as

$$u_t + \frac{\alpha}{3}(u^3)_x + \gamma(uu_{xx})_x + \frac{\beta-\gamma}{2}((u_x)^2)_x + u_{5x} = 0. \tag{14.34}$$

Using the wave variable $\xi = \mu x - ct$ and integrating once, Eq. (14.34) will be converted to

$$-cu + \frac{\alpha}{3}\mu u^3 + \gamma\mu^3 uu'' + \frac{\beta-\gamma}{2}\mu^3(u')^2 + \mu^5 u^{(4)} = 0, \tag{14.35}$$

Balancing the terms $u^{(4)}$ with u^3 in (13.257) gives $M = 2$. The tanh-coth method admits the use of

$$u(\xi) = \sum_{i=0}^{2} a_i Y^i + \sum_{j=1}^{2} b_j Y^{-j}. \tag{14.36}$$

Substituting (14.36) into (14.35), collecting the coefficients of Y, and solving the resulting system we find the following sets of solutions

(i) The *first set* of parameters is given by

$$\begin{aligned} a_0 &= -\frac{2}{3}a_2,\ a_1 = b_1 = b_2 = 0,\\ c &= -\frac{2}{3}\beta\mu^3 a_2 - 24\mu^5,\\ \alpha &= -\frac{6\mu^2\left(60\mu^2 + \beta a_2 + 2\gamma a_2\right)}{a_2^2}. \end{aligned} \tag{14.37}$$

(ii) The *second set* of parameters is given by

$$\begin{aligned} a_0 &= A, A \text{ is a constant},\ a_1 = b_1 = b_2 = 0,\ a_2 = -\frac{60\mu^2}{\beta+\gamma},\\ c &= \frac{\mu\left[\gamma(\beta+\gamma)^2 a_0^2 - 80\gamma\mu^2(\beta+\gamma)a_0 + 80\mu^4(2\beta+17\gamma)\right]}{10(\beta+\gamma)},\\ \alpha &= \frac{\gamma(\beta+\gamma)}{10}. \end{aligned} \tag{14.38}$$

(iii) The *third set* of parameters is given by

$$\begin{aligned} a_0 &= -\frac{2}{3}a_2, \quad a_1 = b_1 = 0, \quad b_2 = a_2,\\ c &= -\frac{32}{3}\beta\mu^3 a_2 - 384\mu^5,\\ \alpha &= -\frac{6\mu^2\left(60\mu^2 + \beta a_2 + 2\gamma a_2\right)}{a_2^2}. \end{aligned} \tag{14.39}$$

(iv) The *fourth set* of parameters is given by

$$\begin{aligned}
a_0 &= A, A \text{ is a constant}, \quad a_1 = b_1 = 0, \quad a_2 = -\frac{60\mu^2}{\beta+\gamma}, \quad b_2 = -\frac{60\mu^2}{\beta+\gamma},\\
c &= \frac{\mu\left[\gamma(\beta+\gamma)^2 a_0^2 - 80\gamma\mu^2(\beta+\gamma)a_0 + 320\mu^4(8\beta-7\gamma)\right]}{10(\beta+\gamma)},\\
\alpha &= \frac{\gamma(\beta+\gamma)}{10}.
\end{aligned} \tag{14.40}$$

14.3.2 The First Condition

The first and the third sets of parameters are expressed in terms of μ and a_2. It is normal to examine the result obtained for α from these sets where we find

$$\alpha = -\frac{6\mu^2(60\mu^2 + (\beta+2\gamma)a_2)}{a_2^2}, \tag{14.41}$$

that gives

$$\alpha a_2^2 + 6\mu^2(\beta+2\gamma)a_2 + 360\mu^4 = 0. \tag{14.42}$$

This quadratic equation has real solutions only if

$$\left(6\mu^2(\beta+2\gamma)\right)^2 \geqslant 1440\alpha\mu^4, \tag{14.43}$$

that gives the first condition, that we are seeking, given by

$$\alpha \leqslant \frac{(\beta+2\gamma)^2}{40}. \tag{14.44}$$

The condition (14.44) enables us to use several real values for α, even for fixed values of the parameters β and γ. In what follows we will derive soliton solutions for all forms of fifth-order KdV that were presented above.

For fifth-order Lax equation, $\alpha = 30$, $\beta = 20$ and $\gamma = 10$. We first determine a_2 by using (14.41) and accordingly, the third set (14.39) gives $a_1 = b_1 = 0$ and

$$\begin{aligned}
a_2 &= b_2 = -2\mu^2, -6\mu^2,\\
a_0 &= \frac{4}{3}\mu^2, 4\mu^2,\\
c &= \frac{128}{3}\mu^5, 896\mu^5.
\end{aligned} \tag{14.45}$$

This in turn gives the solutions

$$u_1(x,t) = \frac{4}{3}\mu^2 - 2\mu^2\tanh^2\left(\mu x - \frac{128}{3}\mu^5 t\right) - 2\mu^2\coth^2\left(\mu x - \frac{128}{3}\mu^5 t\right), \tag{14.46}$$

$$u_2(x,t) = 4\mu^2 - 6\mu^2\tanh^2\left(\mu x - 896\,\mu^5 t\right) - 6\mu^2\coth^2\left(\mu x - 896\,\mu^5 t\right), \tag{14.47}$$

where μ is a nonzero real parameter.

For the fifth-order SK equation, $\alpha = 5$, $\beta = 5$ and $\gamma = 5$. We first determine a_2 by using (14.41) and proceeding as before, we find $a_1 = b_1 = 0$ and

$$\begin{aligned} a_2 &= b_2 = -6\mu^2, -12\mu^2, \\ a_0 &= 4\mu^2, 8\mu^2, \\ c &= -64\mu^5, 256\mu^5. \end{aligned} \tag{14.48}$$

This in turn gives the two solutions

$$u_1(x,t) = 4\mu^2 - 6\mu^2\tanh^2\left(\mu x + 64\,\mu^5 t\right) - 6\mu^2\coth^2\left(\mu x + 64\,\mu^5 t\right), \tag{14.49}$$

and

$$u_2(x,t) = 8\mu^2 - 12\mu^2\tanh^2\left(\mu x - 256\,\mu^5 t\right) - 12\mu^2\coth^2\left(\mu x - 256\,\mu^5 t\right), \tag{14.50}$$

where μ is a nonzero real free parameter.

For the Kaup-Kuperschmidt (KK) equation, $\alpha = 20$, $\beta = 25$ and $\gamma = 10$. Proceeding as before we find

$$\begin{aligned} a_2 &= b_2 = -\frac{3}{2}\mu^2, -12\mu^2, \\ a_1 &= b_1 = 0, \\ a_0 &= \mu^2, 8\mu^2, \\ c &= 16\mu^5, 2816\mu^5. \end{aligned} \tag{14.51}$$

This in turn gives the solutions

$$u_1(x,t) = \mu^2 - \frac{3}{2}\mu^2\tanh^2\left(\mu x - 16\mu^5 t\right) - \frac{3}{2}\mu^2\coth^2\left(\mu x - 16\mu^5 t\right), \tag{14.52}$$

$$u_2(x,t) = 8\mu^2 - 12\mu^2\tanh^2\left(\mu x - 2816\mu^5 t\right) - 12\mu^2\coth^2\left(\mu x - 2816\mu^5 t\right). \tag{14.53}$$

For the Ito equation, $\alpha = 2$, $\beta = 6$ and $\gamma = 3$. Proceeding as before, we find

$$\begin{aligned} a_2 &= b_2 = -6\mu^2, -30\mu^2, \\ a_1 &= b_1 = 0, \\ a_0 &= 4\mu^2, 20\mu^2, \\ c &= 0, 1536\mu^5. \end{aligned} \tag{14.54}$$

This in turn gives the solutions

$$u_1(x,t) = 20\mu^2 - 30\mu^2 \tanh^2\left(\mu x - 1536\mu^5 t\right) - 30\mu^2 \coth^2\left(\mu x - 1536\mu^5 t\right), \tag{14.55}$$

and the solutions

$$u_2(x) = 4\mu^2 - 6\mu^2 \tanh^2(\mu x) - 6\mu^2 \coth^2(\mu x). \tag{14.56}$$

For the fifth-order CDG equation, it is left to the reader.

14.3.3 The Second Condition

In (14.40), we derived the following set

$$\begin{aligned} a_0 &= A, A \text{ is a constant, } a_1 = b_1 = 0,\ a_2 = -\frac{60\mu^2}{\beta+\gamma},\ b_2 = -\frac{60\mu^2}{\beta+\gamma}, \\ c &= \frac{\mu\left[\gamma(\beta+\gamma)^2 a_0^2 - 80\gamma\mu^2(\beta+\gamma)a_0 + 320\mu^4(8\beta-7\gamma)\right]}{10(\beta+\gamma)}, \\ \alpha &= \frac{\gamma(\beta+\gamma)}{10}. \end{aligned} \tag{14.57}$$

as a fourth set of values for the parameters $a_0, a_1, a_2, b_1, b_2, c$ and α. It is obvious from this set that we have a unique value for α for fixed values of β and γ. This fixed value for α is only justified for Lax, SK, and CDG equations. A modification for values of α should be set for KK and Ito equations to obtain solutions for variants of these equations. It is obvious that only one soliton solution will be obtained for Lax and the SK equations.

For the Lax equation, $\alpha = 30$, $\beta = 20$ and $\gamma = 10$. Using (14.57) we find

$$\begin{aligned} a_0 &= a_0,\ a_0 \text{ is an arbitrary constant,} \\ a_1 &= b_1 = 0, \\ a_2 &= b_2 = -2\mu^2, \\ c &= 2\mu(48\mu^4 - 40a_0\mu^2 + 15a_0^2), \end{aligned} \tag{14.58}$$

where μ is left as a free parameter. This in turn gives the solution

$$\begin{aligned} u(x,t) = a_0 &- 2\mu^2 \tanh^2\left(\mu x - 2\mu(48\mu^4 - 40a_0\mu^2 + 15a_0^2)t\right) \\ &- 2\mu^2 \coth^2\left(\mu x - 2\mu(48\mu^4 - 40a_0\mu^2 + 15a_0^2)t\right), \end{aligned} \tag{14.59}$$

Similarly, for the SK equation we find

$$a_0 = a_0,\ a_0 \text{ is an arbitrary constant,}$$

$$\begin{aligned} a_2 &= b_2 = -6\mu^2, \\ a_1 &= b_1 = 0, \\ c &= \mu(16\mu^4 - 40a_0\mu^2 + 5a_0^2). \end{aligned} \tag{14.60}$$

This in turn gives the solution

$$\begin{aligned} u(x,t) = a_0 &- 6\mu^2 \tanh^2\left(\mu x - \mu(16\mu^4 - 40a_0\mu^2 + 5a_0^2)t\right) \\ &- 6\mu^2 \coth^2\left(\mu x - \mu(16\mu^4 - 40a_0\mu^2 + 5a_0^2)t\right). \end{aligned} \tag{14.61}$$

We leave it to the reader to obtain the soliton solution for the CDG equation.

14.3.4 N-soliton Solutions of the Fifth-order KdV Equations

In this section, we will examine multiple-soliton solutions of the fifth-order KdV equations. As stated before, Hirota [5,6] proposed a bilinear form where it was shown that soliton solutions are just polynomials of exponentials.

Hereman *et.al.* [3] introduced a simplified version of Hirota method, where exact solitons can be obtained by solving a perturbation scheme using a symbolic manipulation package. In what follows, we summarize the main steps of the simplified version of Hirota's method.

We first substitute

$$u(x,t) = R\frac{\partial^2 \ln f(x,t)}{\partial x^2} = R\frac{f f_{2x} - (f_x)^2}{f^2}, \tag{14.62}$$

into (14.33), where the auxiliary function $f = 1 + \exp(\theta), \theta = kx - wt$, and solving the equation we get

$$\begin{aligned} \alpha &= \frac{\gamma^2 + \gamma\beta}{10}, \\ R &= \frac{60}{\gamma + \beta}. \end{aligned} \tag{14.63}$$

The Lax Equation

For Lax equation, $R = 2$, therefor we use the transformation

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{f f_{2x} - (f_x)^2}{f^2}, \tag{14.64}$$

that will carry out the Lax equation (14.14) into a cubic equation in f given by

$$f^2(f_{xt} + f_{6x}) - f(f_x f_t + 6f_x f_{5x} - 5f_{2x}f_{4x}) + 10(f_x^2 f_{4x} - 2f_x f_{2x} f_{3x} + f_{2x}^3) = 0, \tag{14.65}$$

that can be decomposed into linear operator and two nonlinear operators.

Proceeding as before, we assume that $f(x,t)$ has a perturbation expansion of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \tag{14.66}$$

where ε is a non small formal expansion parameter. Following the simplified version of Hirota's method [5], we substitute (14.66) into (14.65) and equate to zero the powers of ε.

The N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp(k_i x - c_i t), \tag{14.67}$$

where

$$\theta_i = k_i x - c_i t, \tag{14.68}$$

where k_i and c_i are arbitrary constants. Substituting (14.67) into (14.65), and equate the coefficients of ε^1 to zero, we obtain the dispersion relation

$$c_i = k_i^5, \tag{14.69}$$

and in view of this result we obtain

$$\theta_i = k_i x - k_i^5 t. \tag{14.70}$$

This means that

$$f_1 = \exp(\theta_1) = \exp(k_1(x - k_1^4 t)), \tag{14.71}$$

obtained by using $N = 1$ in (14.67).

For the one-soliton solution, we set

$$f = 1 + \exp(\theta_1) = 1 + \exp(k_1(x - k_1^4 t)), \tag{14.72}$$

where we used $\varepsilon = 1$. The one soliton solution is therefore

$$u(x,t) = \frac{2k_1^2 \exp(k_1(x - k_1^4 t))}{(1 + \exp(k_1(x - k_1^4 t)))^2}. \tag{14.73}$$

To determine the two-soliton solutions, we first set $N = 2$ in (14.67) to get

$$f_1 = \exp(\theta_1) + \exp(\theta_2). \tag{14.74}$$

To determine f_2, we set

$$f_2 = \sum_{1 \leqslant i < j \leqslant N} a_{ij} \exp(\theta_i + \theta_j), \tag{14.75}$$

and therefore we substitute

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12}\exp(\theta_1 + \theta_2), \tag{14.76}$$

into (14.65) and proceed as before to obtain the phase factor a_{12} by

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{14.77}$$

and hence

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leqslant i < j \leqslant N. \tag{14.78}$$

This in turn gives

$$f = 1 + e^{k_1(x - k_1^4 t)} + e^{k_2(x - k_2^4 t)} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{(k_1 + k_2)x - (k_1^5 + k_2^5)t}. \tag{14.79}$$

The two-soliton solutions are obtained by using (14.64) for the function f in (14.79).

Similarly, we can determine f_3. Proceeding as before, we therefore set

$$\begin{aligned} f_1(x,t) &= \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3), \\ f_2(x,t) &= a_{12}\exp(\theta_1 + \theta_2) + a_{23}\exp(\theta_2 + \theta_3) + a_{13}\exp(\theta_1 + \theta_3), \end{aligned} \tag{14.80}$$

and accordingly we have

$$\begin{aligned} f(x,t) &= 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ &\quad + a_{12}\exp(\theta_1 + \theta_2) + a_{23}\exp(\theta_2 + \theta_3) + a_{13}\exp(\theta_1 + \theta_3) \\ &\quad + f_3(x,t). \end{aligned} \tag{14.81}$$

Substituting (14.81) into (14.65) and proceeding as before we find

$$f_3 = b_{123}\exp(\theta_1 + \theta_2 + \theta_3), \tag{14.82}$$

where

$$b_{123} = a_{12}a_{13}a_{23} = \frac{(k_1 - k_2)^2(k_1 - k_3)^2(k_2 - k_3)^2}{(k_1 + k_2)^2(k_1 + k_3)^2(k_2 + k_3)^2}, \tag{14.83}$$

and θ_1, θ_2 and θ_3 are given above in (14.70). For the three-soliton solution we use $1 \leqslant i < j \leqslant 3$, we therefore obtain

$$\begin{aligned} f &= 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ &\quad + a_{12}\exp(\theta_1 + \theta_2) + a_{13}\exp(\theta_1 + \theta_3) + a_{23}\exp(\theta_2 + \theta_3) \\ &\quad + b_{123}\exp(\theta_1 + \theta_2 + \theta_3), \end{aligned} \tag{14.84}$$

where

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \ \ 1 \leqslant i < j \leqslant 2, \quad b_{123} = a_{12}a_{13}a_{23}. \tag{14.85}$$

This in turn gives

$$
\begin{aligned}
f = {} & 1+e^{k_1(x-k_1^4t)}+e^{k_2(x-k_2^4t)}+e^{k_3(x-k_3^4t)}\\
&+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(k_1^5+k_2^5)t}+\frac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{(k_1+k_3)x-(k_1^5+k_3^5)t}\\
&+\frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{(k_2+k_3)x-(k_2^5+k_3^5)t}\\
&+\frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}e^{(k_1+k_2+k_3)x-(k_1^5+k_2^5+k_3^5)t}.
\end{aligned}
\tag{14.86}
$$

To determine the three-soliton solutions explicitly, we use (14.64) for the function f in (14.86).

As stated before, the Lax equation is characterized by

$$\beta = 2\gamma, \quad \alpha = \frac{3}{10}\gamma^2, \tag{14.87}$$

where γ is any arbitrary constant, then the transformation (14.64) can be generalized to

$$u = \frac{20}{\gamma}(\ln(f(x,t))_{xx}, \tag{14.88}$$

that works for every γ.

We again emphasize the three facts presented before:

(i) The first is that soliton solutions are just polynomials of exponentials as emphasized by Hirota.

(ii) The three-soliton solution and the higher level soliton solution as well, do not contain any new free parameters other than a_{ij} derived for the two-soliton solution.

(iii) Every solitonic equation that has generic $N = 3$ soliton solutions, then it has also soliton solutions [4,14,15] for any $N \geqslant 4$.

The Sawada-Kotera Equation

For the SK equation, $R = 6$, therefor the transformation

$$u(x,t) = 6\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 6\frac{ff_{2x}-(f_x)^2}{f^2}, \tag{14.89}$$

that will carry out the SK equation (14.2) into a quadratic equation in f given by

$$f(f_{xt}+f_{6x})+(15f_{2x}f_{4x}-10f_{3x}^2-6f_xf_{5x}-f_xf_t)=0. \tag{14.90}$$

We again assume that $f(x,t)$ has a perturbation expansion of the form

$$f(x,t) = 1+\sum_{n=1}^{\infty}\varepsilon^n f_n(x,t). \tag{14.91}$$

Substituting (14.91) into (14.90) and equate to zero the powers of ε.

The N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp(k_i x - c_i t), \tag{14.92}$$

where

$$\theta_i = k_i x - c_i t. \tag{14.93}$$

Substituting (14.92) into (14.90) and equate the coefficients of ε^1 to zero, we obtain the dispersion relation

$$c_i = k_i^5, \tag{14.94}$$

and in view of this result we obtain

$$\theta_i = k_i x - k_i^5 t. \tag{14.95}$$

This means that

$$f_1 = \exp(\theta_1) = \exp(k_1(x - k_1^4 t)), \tag{14.96}$$

obtained by using $N = 1$ in (14.92).

Consequently, for the one-soliton solution, we set

$$f = 1 + \exp(\theta_1) = 1 + \exp(k_1(x - k_1^4 t)). \tag{14.97}$$

The one soliton solution is therefore given by

$$u(x,t) = \frac{6k_1^2 \exp(k_1(x - k_1^4 t))}{(1 + \exp(k_1(x - k_1^4 t)))^2}. \tag{14.98}$$

To determine the two-soliton solution, we first set $N = 2$ in (14.92) to get

$$f_1 = \exp(\theta_1) + \exp(\theta_2). \tag{14.99}$$

To determine f_2, we set

$$f_2 = \sum_{1 \leqslant i < j \leqslant N} a_{ij} \exp(\theta_i + \theta_j), \tag{14.100}$$

and therefore we substitute

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12} \exp(\theta_1 + \theta_2), \tag{14.101}$$

into (14.90) and proceed as before to obtain the phase factor a_{12} by

$$a_{12} = \frac{(k_1 - k_2)^2 (k_1^2 - k_1 k_2 + k_2^2)}{(k_1 + k_2)^2 (k_1^2 + k_1 k_2 + k_2^2)}, \tag{14.102}$$

and hence

$$a_{ij} = \frac{(k_i-k_j)^2(k_i^2-k_ik_j+k_j^2)}{(k_i+k_j)^2(k_i^2-k_ik_j+k_j^2)}, 1 \leqslant i < j \leqslant N. \tag{14.103}$$

This in turn gives

$$f = 1 + e^{k_1(x-k_1^4t)} + e^{k_2(x-k_2^4t)} + \frac{(k_1-k_2)^2(k_1^2-k_1k_2+k_2^2)}{(k_1+k_2)^2(k_1^2+k_1k_2+k_2^2)} e^{(k_1+k_2)x-(k_1^5+k_2^5)t}. \tag{14.104}$$

The two-soliton solutions can be obtained by using (14.89) for the function f in (14.104).

Similarly, we can determine f_3. Proceeding as before, we therefore set

$$\begin{aligned} f_1(x,t) &= \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3), \\ f_2(x,t) &= a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_3) + a_{13}\exp(\theta_1+\theta_3), \end{aligned} \tag{14.105}$$

and accordingly we have

$$\begin{aligned} f(x,t) = &\ 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ &+ a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_3) + a_{13}\exp(\theta_1+\theta_3) \\ &+ f_3(x,t). \end{aligned} \tag{14.106}$$

Substituting (14.106) into (14.90) and proceeding as before we find

$$f_3 = b_{123}\exp(\theta_1+\theta_2+\theta_3), \tag{14.107}$$

where

$$b_{123} = a_{12}a_{13}a_{23}. \tag{14.108}$$

For the three-soliton solution we use $1 \leqslant i < j \leqslant 3$, we proceed as before to get

$$\begin{aligned} f = &\ 1 + e^{k_1(x-k_1^4t)} + e^{k_2(x-k_2^4t)} + e^{k_3(x-k_3^4t)} \\ &+ a_{12}e^{(k_1+k_2)x-(k_1^5+k_2^5)t} + a_{13}e^{(k_1+k_3)x-(k_1^5+k_3^5)t} \\ &+ a_{23}e^{(k_2+k_3)x-(k_2^5+k_3^5)t} \\ &+ b_{123}e^{(k_1+k_2+k_3)x-(k_1^5+k_2^5+k_3^5)t}. \end{aligned} \tag{14.109}$$

where a_{ij} and b_{123} are defined above in (14.103) and (14.108) respectively. To determine the three-solitons solution explicitly, we use (14.90) for the function f in (14.109).

As stated before, the Sawada-Kotera equation is characterized by

$$\beta = \gamma, \quad \alpha = \frac{1}{5}\gamma^2, \tag{14.110}$$

where γ is any arbitrary constant, then the transformation (14.89) can be generalized to

$$u = \frac{30}{\gamma}(\ln(f(x,t))_{xx}, \tag{14.111}$$

that works for every γ.

The Caudrey-Dodd-Gibbon Equation

For the CDG equation, $R=1$, therefore the transformation

$$u(x,t) = (\ln(f))_{xx}, \tag{14.112}$$

that transforms (14.12) to

$$\begin{aligned} &[f(f_x f_{6x} - f_t f_{2x} - 2 f_x f_{xt} - 5 f_{3x} f_{4x} + 9 f_{2x} f_{5x})] + \left[f^2(f_{7x} + f_{xxt})\right] \\ &\quad + \left[f_x(20 f_{3x}^2 - 30 f_{2x} f_{4x})\right] + \left[f_x^2(12 f_{5x} + 2 f_t)\right] = 0. \end{aligned} \tag{14.113}$$

Following the discussions introduced before, we assume that $f(x,t)$ has a perturbation expansion of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,t). \tag{14.114}$$

Substituting (14.114) into (14.113) and equate to zero the powers of ε.

The N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp(k_i x - c_i t), \tag{14.115}$$

where

$$\theta_i = k_i x - c_i t. \tag{14.116}$$

The dispersion relation is found to be

$$c_i = k_i^5. \tag{14.117}$$

In view of this result we obtain

$$\theta_i = k_i x - k_i^5 t. \tag{14.118}$$

This means that

$$f_1 = \exp(\theta_1) = \exp(k_1(x - k_1^4 t)), \tag{14.119}$$

obtained by using $N=1$ in (14.115).

Consequently, for the one-soliton solution, we set

$$f = 1 + \exp(\theta_1) = 1 + \exp(k_1(x - k_1^4 t)), \tag{14.120}$$

where we set $\varepsilon = 1$. The one soliton solution is therefore

$$u(x,t) = \frac{k_1^2 \exp\left(k_1\left(x - k_1^4 t\right)\right)}{\left(1 + \exp\left(k_1\left(x - k_1^4 t\right)\right)\right)^2}. \tag{14.121}$$

To determine the two-soliton solution, we first set $N = 2$ in (14.115) to get

$$f_1 = \exp(\theta_1) + \exp(\theta_2). \tag{14.122}$$

To determine f_2, we set

$$f_2 = \sum_{1 \leqslant i < j \leqslant N} a_{ij} \exp\left(\theta_i + \theta_j\right), \tag{14.123}$$

and therefore we substitute

$$f = 1 + \exp\left(\theta_1\right) + \exp\left(\theta_2\right) + a_{12} \exp\left(\theta_1 + \theta_2\right), \tag{14.124}$$

into (14.113) and proceed as before to obtain the phase factor a_{12} by

$$a_{12} = \frac{(k_1 - k_2)^2 (k_1^2 - k_1 k_2 + k_2^2)}{(k_1 + k_2)^2 (k_1^2 + k_1 k_2 + k_2^2)}, \tag{14.125}$$

and hence

$$a_{ij} = \frac{(k_i - k_j)^2 (k_i^2 - k_i k_j + k_j^2)}{(k_i + k_j)^2 (k_i^2 - k_i k_j + k_j^2)}, \quad 1 \leqslant i < j \leqslant N. \tag{14.126}$$

This in turn gives

$$f = 1 + e^{k_1(x - k_1^4 t)} + e^{k_2(x - k_2^4 t)} + \frac{(k_1 - k_2)^2 (k_1^2 - k_1 k_2 + k_2^2)}{(k_1 + k_2)^2 (k_1^2 + k_1 k_2 + k_2^2)} e^{(k_1 + k_2)x - (k_1^5 + k_2^5)t}. \tag{14.127}$$

To determine the two-soliton solutions explicitly, we use (14.112) for the function f in (14.127).

Proceeding as before we obtain

$$\begin{aligned} f = {} & 1 + e^{k_1(x - k_1^4 t)} + e^{k_2(x - k_2^4 t)} + e^{k_3(x - k_3^4 t)} \\ & + a_{12} e^{(k_1 + k_2)x - (k_1^5 + k_2^5)t} + a_{13} e^{(k_1 + k_3)x - (k_1^5 + k_3^5)t} \\ & + a_{23} e^{(k_2 + k_3)x - (k_2^5 + k_3^5)t} \\ & + b_{123} e^{(k_1 + k_2 + k_3)x - (k_1^5 + k_2^5 + k_3^5)t}. \end{aligned} \tag{14.128}$$

where a_{ij} and $b_{123} = a_{12} a_{13} a_{23}$ are defined above. To determine the three-solitons solution explicitly, we use (14.112) for the function f in (14.128). This means that the N-soliton solutions exist for the CDG equation, for $n \geqslant 1$.

The Kaup-Kuperschmidt Equation

For the KK equation we summarize the work in [3], where the transformation

$$u(x,t) = \frac{3}{2}\frac{\partial^2 \ln f(x,t)}{\partial x^2} = \frac{3}{2}\frac{f f_{2x} - (f_x)^2}{f^2} \tag{14.129}$$

is used.

The dispersion relation is given by

$$\theta_i = k_i x - k_i^5 t. \tag{14.130}$$

For the one-soliton solution it was found that

$$f = 1 + \exp(\theta_1) + \frac{1}{16}\exp(2\theta_1), \tag{14.131}$$

so that the one soliton solution is

$$u(x,t) = \frac{24\,{k_1}^2\, e^{(k_1\,(-t\,{k_1}^4+x))}\,(16+4\,e^{(k_1\,(-t\,{k_1}^4+x))} + e^{(2\,k_1\,(-t\,{k_1}^4+x))})}{(16+16\,e^{(k_1\,(-t\,{k_1}^4+x))} + e^{(2\,k_1\,(-t\,{k_1}^4+x))})^2}. \tag{14.132}$$

For the two-soliton solutions it was found that

$$\begin{aligned} f =\ & 1 + \exp(\theta_1) + \exp(\theta_2) + \frac{1}{16}\exp(2\theta_1) + \frac{1}{16}\exp(2\theta_2) \\ & + a_{12}\exp(\theta_1+\theta_2) + b_{12}\left[\exp(2\theta_1+\theta_2) + \exp(\theta_1+2\theta_2)\right] \\ & + b_{12}^2\exp(2\theta_1+2\theta_2), \end{aligned} \tag{14.133}$$

where

$$a_{12} = \frac{2k_1^4 - k_1^2k_2^2 + 2k_2^4}{2(k_1+k_2)^2(k_1^2+k_1k_2+k_2^2)}, \tag{14.134}$$

and

$$b_{12} = \frac{(k_1-k_2)^2(k_1^2-k_1k_2+k_2^2)}{16(k_1+k_2)^2(k_1^2+k_1k_2+k_2^2)}. \tag{14.135}$$

It is obvious that the two soliton solutions $u(x,t)$ can be obtained by substituting the last expression for $f(x,t)$ into (14.129). For the higher level solitons solution, it becomes more complicated.

It is well known that the Kaup-Kuperschmidt equation is characterized by

$$\beta = \frac{5}{2}\gamma,\ \alpha = \frac{1}{5}\gamma^2, \tag{14.136}$$

where γ is any arbitrary constant, then the transformation (14.129) can be generalized to

$$u = \frac{15}{\gamma}(\ln(f(x,t))_{xx}, \tag{14.137}$$

that works for every γ.

14.4 Seventh-order KdV Equations

The generalized seventh-order KdV equations are given by

$$u_t + au^2u_x + bu_x^3 + cuu_xu_{2x} + du^2u_{3x} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0, \tag{14.138}$$

where a,b,c,d,e,f, and g are nonzero parameters. In a manner parallel to our approach before, we will use the tanh-coth method and the Hirota's bilinear form combined with the Hereman's simplified form respectively.

14.4.1 Using the Tanh-coth Method

For single soliton solutions of the seventh-order KdV equations, the tanh-coth method will be used to achieve our goal.

The Sawada-Kotera-Ito Seventh-order Equation

The wave variable $\xi = x - ct$ converts (14.29) to an ODE where by using the balance method we obtain $M = 2$. The tanh-coth method gives

$$u(\xi) = \sum_{i=0}^{2} a_i Y^i + \sum_{i=1}^{2} b_i Y^{-i}. \tag{14.139}$$

Substituting this transformation formula into the reduced ODE, collecting the coefficients of Y, and solving the resulting system we find that $a_1 = b_1 = 0$ and the following sets of solutions

$$\begin{aligned}
a_0 &= \alpha, a_2 = -2\mu^2, b_2 = 0,\ \ c = -608\mu^6 + 1344\alpha\mu^4 - 1008\alpha^2\mu^2 + 252\alpha^3,\\
a_0 &= \frac{8}{3}\mu^2, \quad a_2 = -4\mu^2,\ \ b_2 = 0,\ \ c = -\frac{256}{3}\mu^6,\\
a_0 &= \alpha,\ \ a_2 = 0,\ \ b_2 = -2\mu^2,\ \ c = -608\mu^6 + 1344\alpha\mu^4 - 1008\alpha^2\mu^2 + 252\alpha^3,\\
a_0 &= \frac{8}{3}\mu^2,\ \ a_2 = 0,\ \ b_2 = -4\mu^2,\ \ c = -\frac{256}{3}\mu^6,
\end{aligned} \tag{14.140}$$

where α is an arbitrary constant and μ is left as a free parameter. This in turn gives the soliton solutions

$$\begin{aligned}
u_1(x,t) &= \alpha - 2\mu^2 \tanh^2\left[\mu(x - (-608\mu^6 + 1344\alpha\mu^4 - 1008\alpha^2\mu^2 + 252\alpha^3)t)\right],\\
u_2(x,t) &= \frac{8}{3}\mu^2 - 4\mu^2 \tanh^2\left[\mu(x + \frac{256}{3}\mu^6 t)\right],
\end{aligned} \tag{14.141}$$

and the solutions

$$
\begin{aligned}
u_3(x,t) &= \alpha - 2\mu^2 \coth^2\left[\mu(x-(-608\mu^6+1344\alpha\mu^4-1008\alpha^2\mu^2+252\alpha^3)t)\right],\\
u_4(x,t) &= \frac{8}{3}\mu^2 - 4\mu^2\coth^2\left[\mu(x+\frac{256}{3}\mu^6 t)\right].
\end{aligned}
\tag{14.142}
$$

The Lax Seventh-order Equation

Proceeding as before, we obtain $M=2$ and the following sets of solutions

$$
\begin{aligned}
&a_0=\alpha,\ a_2=-2\mu^2,\ b_2=0,\ c=-384\mu^6+784\alpha\mu^4-560\alpha^2\mu^2+140\alpha^3,\\
&a_0=\alpha,\ a_2=0,\ b_2=-2\mu^2,\ c=-384\mu^6+784\alpha\mu^4-560\alpha^2\mu^2+140\alpha^3,
\end{aligned}
\tag{14.143}
$$

where α is an arbitrary constant and μ is left as a free parameter. This in turn gives the soliton solutions

$$
u_1(x,t)=\alpha-2\mu^2\tanh^2\left[\mu(x-(-384\mu^6+784\alpha\mu^4-560\alpha^2\mu^2+140\alpha^3)t)\right],
\tag{14.144}
$$

and the solution

$$
u_2(x,t)=\alpha-2\mu^2\coth^2\left[\mu(x-(-384\mu^6+784\alpha\mu^4-560\alpha^2\mu^2+140\alpha^3)t)\right].
\tag{14.145}
$$

The Kaup-Kuperschmidt Seventh-order Equation

Using the wave variable $\xi=x-ct$ transforms (14.31) and using the balance method we find that $M=2$. The tanh-coth method

$$
u(\xi)=\sum_{i=0}^{2}a_iY^i+\sum_{i=1}^{2}b_iY^{-i},
\tag{14.146}
$$

is readily used as before to find that $a_1=b_1=0$ and the following sets of solutions

$$
\begin{aligned}
&a_0=\frac{1}{3}\mu^2,\quad a_2=-\frac{1}{2}\mu^2,\quad b_2=0,\quad c=-\frac{4}{3}\mu^6,\\
&a_0=\frac{1}{3}\mu^2,\quad a_2=0,\quad b_2=-\frac{1}{2}\mu^2,\quad c=-\frac{4}{3}\mu^6,
\end{aligned}
\tag{14.147}
$$

where μ is left as a free parameter. This in turn gives the soliton solution

$$
u_1(x,t)=\frac{1}{3}\mu^2-\frac{1}{2}\mu^2\tanh^2\left[\mu(x+\frac{4}{3}\mu^6 t)\right],
\tag{14.148}
$$

and the solution

$$u_2(x,t) = \frac{1}{3}\mu^2 - \frac{1}{2}\mu^2\coth^2\left[\mu(x+\frac{4}{3}\mu^6 t)\right]. \tag{14.149}$$

14.4.2 *N*-soliton Solutions of the Seventh-order KdV Equations

To obtain multiple-soliton solutions for the seventh-order KdV equations, we apply the Hirota's bilinear form combined with the Hereman's simplified approach where it was shown that soliton solutions are just polynomials of exponentials. We again summarize the main steps of the approach, where details can be found above in the previous section.

We first substitute

$$u(x,t) = e^{kx-ct}, \tag{14.150}$$

in the linear terms of the equation under discussion to determine the dispersion relation between k and c. We then substitute the single soliton solution

$$u(x,t) = R\frac{\partial^2 \ln f(x,t)}{\partial x^2} = R\frac{ff_{2x}-(f_x)^2}{f^2}, \tag{14.151}$$

into the equation under discussion, where the auxiliary function f is given by

$$f(x,t) = 1 + f_1(x,t) = 1 + e^{\theta_1}, \tag{14.152}$$

where

$$\theta_i = k_i x - c_i t, \quad i = 1,2,\cdots,N, \tag{14.153}$$

and solving the resulting equation to determine the numerical value for R. Notice that the N-soliton solutions can be obtained, for the Sawada-Kotera-Ito and Lax equations, by using the following forms for $f(x,t)$ into (14.151):

(i) For dispersion relation, we use

$$u(x,t) = e^{\theta_i}, \quad \theta_i = k_i x - c_i t. \tag{14.154}$$

(ii) For single soliton, we use

$$f = 1 + e^{\theta_1}. \tag{14.155}$$

(iii) For two-soliton solutions, we use

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}. \tag{14.156}$$

(iv) For three-soliton solutions, we use

$$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} + a_{13}e^{\theta_1+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3}. \tag{14.157}$$

Notice that we use (14.155) to determine the dispersion relation, (14.156) to determine the factor a_{12} to generalize for the other factors a_{ij} and finally we use (14.157) to determine b_{123}, which is mostly given by $b_{123} = a_{12}a_{23}a_{13}$. The determination of three-soliton solutions confirms the fact that N-soliton solutions exist for any order. In the following, we will apply the aforementioned steps to the three well-known seventh-order equations.

The Sawada-Kotera-Ito Seventh-order Equation

Substituting

$$u(x,t) = e^{\theta_i}, \quad \theta_i = k_i x - c_i t. \tag{14.158}$$

into the linear terms of the SK-Ito equation

$$u_t + 252u^2u_x + 63u_x^3 + 378uu_xu_{2x} + 126u^2u_{3x} + 63u_{2x}u_{3x} + 42u_xu_{4x} + 21uu_{5x} + u_{7x} = 0, \tag{14.159}$$

gives the dispersion relation

$$c_i = k_i^7. \tag{14.160}$$

To determine R, we substitute

$$u(x,t) = R\frac{\partial^2 \ln f(x,t)}{\partial x^2} = R\frac{ff_{2x} - (f_x)^2}{f^2}, \tag{14.161}$$

where $f(x,t) = 1 + e^{k_1x - k_1^7t}$ into the SK-Ito equation and solve to find that $R = 2$. This means that the single soliton solution is given by

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = \frac{2k_1^2 e^{k_1(x-k_1^6t)}}{(1+e^{k_1(x-k_1^6t)})^2}. \tag{14.162}$$

For two-soliton solutions, we substitute

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}, \tag{14.163}$$

where

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{14.164}$$

into the SK-Ito equation to obtain that

$$a_{12} = \frac{(k_1-k_2)^2(k_1^2 - k_1k_2 + k_2^2)}{(k_1+k_2)^2(k_1^2 + k_1k_2 + k_2^2)}, \tag{14.165}$$

and hence

$$a_{ij} = \frac{(k_i-k_j)^2(k_i^2 - k_ik_j + k_j^2)}{(k_i+k_j)^2(k_i^2 + k_ik_j + k_j^2)}, 1 \leqslant i < j \leqslant N. \tag{14.166}$$

This in turn gives

$$f = 1 + e^{k_1(x-k_1^6 t)} + e^{k_2(x-k_2^6 t)} + \frac{(k_1-k_2)^2(k_1^2-k_1k_2+k_2^2)}{(k_1+k_2)^2(k_1^2+k_1k_2+k_2^2)} e^{(k_1+k_2)x-(k_1^6+k_2^6)t}. \tag{14.167}$$

To determine the two-solitons solution explicitly, we substitute the last result for $f(x,t)$ into (14.163).

Similarly, to determine the three-soliton solutions, we set

$$\begin{aligned} f(x,t) = {} & 1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3) \\ & +a_{12}\exp(\theta_1+\theta_2)+a_{23}\exp(\theta_2+\theta_3)+a_{13}\exp(\theta_1+\theta_3) \\ & +b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned} \tag{14.168}$$

into (14.163) and substitute it in the SK-Ito equation to find that where

$$b_{123} = a_{12}a_{13}a_{23}. \tag{14.169}$$

The higher level soliton solution can be obtained in a parallel manner.

The Lax Seventh-order Equation

Using

$$u(x,t) = e^{\theta_i}, \quad \theta_i = k_i x - c_i t. \tag{14.170}$$

into the linear terms of the Lax seventh-order equation

$$u_t + 140u^2u_x + 70u_x^3 + 280uu_xu_{2x} + 70u^2u_{3x} + 70u_{2x}u_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x} = 0, \tag{14.171}$$

yields the dispersion relation

$$c_i = k_i^7. \tag{14.172}$$

Proceeding as before we find that $R = 2$, and hence the solution is given by

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{ff_{2x}-(f_x)^2}{f^2}, \tag{14.173}$$

where $f(x,t) = 1 + e^{k_1x-k_1^7t}$. This means that the single soliton solution is given by

$$u(x,t) = u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = \frac{2k_1^2 e^{k_1(x-k_1^6t)}}{(1+e^{k_1(x-k_1^6t)})^2}. \tag{14.174}$$

For two-soliton solutions, we substitute

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}, \tag{14.175}$$

where

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{14.176}$$

into the Lax equation to obtain that

$$a_{12}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{14.177}$$

and hence

$$a_{ij}=\frac{(k_i-k_j)^2}{(k_i+k_j)^2}, \quad 1\leqslant i<j\leqslant N. \tag{14.178}$$

This in turn gives

$$f=1+e^{k_1(x-k_1^6t)}+e^{k_2(x-k_2^6t)}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(k_1^6+k_2^6)t}. \tag{14.179}$$

To determine the two-solitons solution explicitly, we substitute the last result for $f(x,t)$ into (14.174).

Similarly, to determine the three-soliton solutions, we set

$$\begin{aligned} f(x,t) \;=\; & 1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3) \\ & +a_{12}\exp(\theta_1+\theta_2)+a_{23}\exp(\theta_2+\theta_3)+a_{13}\exp(\theta_1+\theta_3) \\ & +b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned} \tag{14.180}$$

into (14.175) and substitute it in the Lax equation to find that where

$$b_{123}=a_{12}a_{13}a_{23}=\frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}, \tag{14.181}$$

The higher level soliton solutions can be obtained in a parallel manner.

The Kaup-Kuperschmidt Seventh-order Equation

The KK seventh-order equation is given by

$$\begin{aligned} & u_t+2016u^2u_x+630u_x^3+2268uu_xu_{2x}+504u^2u_{3x}+252u_{2x}u_{3x} \\ & \quad +147u_xu_{4x}+42uu_{5x}+u_{7x}=0, \end{aligned} \tag{14.182}$$

It is interesting to point out here that the assumptions made above for $f(x,t)$ are not applicable here. We will follow the same approach presented above for the KK fifth-order equation. The dispersion relation is found to be

$$\theta_i=k_ix-k_i^7t. \tag{14.183}$$

We next use the transformation

$$u(x,t)=R\frac{\partial^2\ln f(x,t)}{\partial x^2}=R\frac{ff_{2x}-(f_x)^2}{f^2}, \tag{14.184}$$

where $f(x,t)$ in this case is given by

$$f = 1 + \exp(\theta_1) + \frac{1}{16}\exp(2\theta_1). \tag{14.185}$$

Substituting (14.184) into the KK equation gives $R = \frac{1}{2}$. This means that (14.184) becomes

$$u(x,t) = \frac{1}{2}\frac{\partial^2 \ln f(x,t)}{\partial x^2} = \frac{1}{2}\frac{ff_{2x} - (f_x)^2}{f^2}. \tag{14.186}$$

This in turn gives the one soliton solution by

$$u(x,t) = \frac{8k_1^2 e^{k_1(x-k_1^6 t)}\left(e^{2k_1(x-k_1^6 t)} + 4e^{k_1(x-k_1^6 t)} + 16\right)}{\left(e^{2k_1(x-k_1^6 t)} + 16e^{k_1(x-k_1^6 t)} + 16\right)^2}. \tag{14.187}$$

For the two-soliton solution it was found that

$$\begin{aligned} f = {} & 1 + \exp(\theta_1) + \exp(\theta_2) + \frac{1}{16}\exp(2\theta_1) + \frac{1}{16}\exp(2\theta_2) \\ & + a_{12}\exp(\theta_1 + \theta_2) + b_{12}\left[\exp(2\theta_1 + \theta_2) + \exp(\theta_1 + 2\theta_2)\right] \\ & + b_{12}^2\exp(2\theta_1 + 2\theta_2), \end{aligned} \tag{14.188}$$

where

$$a_{12} = \frac{2k_1^4 - k_1^2 k_2^2 + 2k_2^4}{2(k_1 + k_2)^2(k_1^2 + k_1 k_2 + k_2^2)}, \tag{14.189}$$

and

$$b_{12} = \frac{(k_1 - k_2)^2(k_1^2 - k_1 k_2 + k_2^2)}{16(k_1 + k_2)^2(k_1^2 + k_1 k_2 + k_2^2)}. \tag{14.190}$$

It is obvious that the two soliton solution $u(x,t)$ can be obtained by substituting (14.189) into (14.186). For the higher level solitons solution, it becomes more complicated and more details can be found in [3].

14.5 Ninth-order KdV Equations

It is well-known that the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{14.191}$$

can be expressed in bilinear form by

$$D_x(D_t + D_x^3)(f \cdot f) = 0. \tag{14.192}$$

The solution of Eq. (14.191) is of the form

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}. \tag{14.193}$$

Following Sawada and Kotera approach, we can generalize (14.192) and (14.193) into the form

$$D_x(D_t + D_x^9)(f \cdot f) = 0, \quad u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}, \tag{14.194}$$

or equivalently

$$D_x D_t(f \cdot f) + D_x^{10}(f \cdot f) = 0, \quad u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}. \tag{14.195}$$

The Hirota's bilinear operators have several properties such as

$$\begin{aligned} D_x D_t(f \cdot f)/f^2 &= (\ln f^2)_{xt}, \\ D_x^{10}(f \cdot f)/f^2 &= u_{8x} + 45uu_{6x} + 210u_{2x}u_{4x} + 1575u(u_{2x})^2 \\ &\quad +630u^2u_{4x} + 3150u^3u_{2x} + 945u^5. \end{aligned} \tag{14.196}$$

Substituting (14.196) into (14.195), and differentiating the resulting equation with respect to x we obtain the ninth-order KdV equation given by

$$\begin{aligned} &u_t + 45u_xu_{6x} + 45uu_{7x} + 210u_{3x}u_{4x} + 210u_{2x}u_{5x} + 1575u_x(u_{2x})^2 \\ &\quad +3150uu_{2x}u_{3x} + 1260uu_xu_{4x} + 630u^2u_{5x} + 9450u^2u_xu_{2x} + 3150u^3u_{3x} \\ &\quad +4725u^4u_x + u_{9x} = 0. \end{aligned} \tag{14.197}$$

14.5.1 Using the Tanh-coth Method

Balancing u_{9x} with any of the terms such as u_xu_{6x} gives $M = 2$. The tanh-coth method gives

$$u(\xi) = \sum_{i=0}^{2} a_i Y^i + \sum_{i=1}^{2} b_i Y^{-i}. \tag{14.198}$$

Substituting this transformation formula into (14.197), collecting the coefficients of Y, and solving the resulting system we find that $a_1 = b_1 = 0$ and the following sets of solutions

$$\begin{aligned} &a_0 = \alpha, \quad a_2 = -2\mu^2, \quad b_2 = 0, \\ &c = 9616\mu^8 - 37440\alpha\mu^6 + 47880\alpha^2\mu^4 - 25200\alpha^3\mu^2 + 4725\alpha^4, \end{aligned} \tag{14.199}$$

and

$$\begin{aligned} &a_0 = \alpha, \quad a_2 = 0, \quad b_2 = -2\mu^2, \\ &c = 9616\mu^8 - 37440\alpha\mu^6 + 47880\alpha^2\mu^4 - 25200\alpha^3\mu^2 + 4725\alpha^4, \end{aligned} \tag{14.200}$$

where α is an arbitrary constant and μ is left as a free parameter. This in turn gives the soliton solution

$$\begin{aligned} u_1(x,t) &= \alpha - 2\mu^2 \tanh^2 \\ &\times \left[\mu(x - (9616\mu^8 - 37440\alpha\mu^6 + 47880\alpha^2\mu^4 - 25200\alpha^3\mu^2 + 4725\alpha^4)t)\right], \end{aligned} \tag{14.201}$$

and the solution

$$\begin{aligned} u_2(x,t) &= \alpha 2\mu^2 \coth^2 \\ &\times \left[\mu(x - (9616\mu^8 - 37440\alpha\mu^6 + 47880\alpha^2\mu^4 - 25200\alpha^3\mu^2 + 4725\alpha^4)t)\right]. \end{aligned} \tag{14.202}$$

14.5.2 The Soliton Solutions

Substituting

$$u(x,t) = e^{\theta_i}, \quad \theta_i = k_i x - c_i t \tag{14.203}$$

into the linear terms of the ninth-order equation (14.197) gives the dispersion relation

$$c_i = k_i^9, \quad i = 1, 2, \cdots, N, \tag{14.204}$$

and hence θ_i becomes

$$\theta_i = k_i x - k_i^9 t. \tag{14.205}$$

To determine R, we substitute

$$u(x,t) = R\frac{\partial^2 \ln f(x,t)}{\partial x^2} = R\frac{f f_{2x} - (f_x)^2}{f^2}, \tag{14.206}$$

where $f(x,t) = 1 + e^{k_1 x - k_1^9 t}$ into the ninth-order KdV equation (14.197) and solve to find that $R = 2$. This means that the single soliton solution is given by

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = \frac{2k_1^2 e^{k_1(x - k_1^8 t)}}{(1 + e^{k_1(x - k_1^8 t)})^2}. \tag{14.207}$$

For two-soliton solutions, we substitute

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}, \tag{14.208}$$

where

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{14.209}$$

into the ninth-order KdV equation (14.197), and solve for a_{12} to find that

$$a_{12} = \frac{(k_1-k_2)^2(3k_1^6-9k_1^5k_2+19k_1^4k_2^2-23k_1^3k_2^3+19k_1^2k_2^4-9k_1k_2^5+3k_2^6)}{(k_1+k_2)^2(3k_1^6+9k_1^5k_2+19k_1^4k_2^2+23k_1^3k_2^3+19k_1^2k_2^4+9k_1k_2^5+3k_2^6)}, \tag{14.210}$$

and hence

$$a_{ij} = \frac{(k_i-k_j)^2(3k_i^6-9k_i^5k_j+19k_i^4k_j^2-23k_i^3k_j^3+19k_i^2k_j^4-9k_ik_j^5+3k_j^6)}{(k_i+k_j)^2(3k_i^6+9k_i^5k_j+19k_i^4k_j^2+23k_i^3k_j^3+19k_i^2k_j^4+9k_ik_j^5+3k_j^6)}. \tag{14.211}$$

This in turn gives

$$\begin{aligned} f(x,t) = {} & 1+e^{k_1(x-k_1^8)t}+e^{k_2(x-k_2^8)t} \\ & +\frac{(k_1-k_2)^2(3k_1^6-9k_1^5k_2+19k_1^4k_2^2-23k_1^3k_2^3+19k_1^2k_2^4-9k_1k_2^5+3k_2^6)}{(k_1+k_2)^2(3k_1^6+9k_1^5k_2+19k_1^4k_2^2+23k_1^3k_2^3+19k_1^2k_2^4+9k_1k_2^5+3k_2^6)}. \\ & e^{(k_1+k_2)x-(k_1^9+k_2^9)t}. \end{aligned} \tag{14.212}$$

To determine the two-soliton solutions explicitly, we substitute the last result for $f(x,t)$ into (14.208).

To determine the three-soliton solutions, we first substitute

$$\begin{aligned} f(x,t) = {} & 1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3) \\ & +a_{12}\exp(\theta_1+\theta_2)+a_{23}\exp(\theta_2+\theta_3)+a_{13}\exp(\theta_1+\theta_3) \\ & +b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned} \tag{14.213}$$

into (14.163) and then substitute it into the ninth-order KdV equation to find that

$$b_{123} \neq a_{12}a_{13}a_{23}. \tag{14.214}$$

Accordingly, the three-soliton solutions do not exist for the ninth-order equation. This result shows that the three-soliton solutions, and then the higher level soliton solutions do not exist. Based on this, we conclude that the ninth-order KdV equation is not completely integrable.

14.6 Family of Higher-order mKdV Equations

It was stated before that the modified KdV equation, where the quadratic nonlinearity uu_x of the KdV equation is replaced bt the cubic nonlinearity u^2u_x, appears in many physical applications. The mKdV equation appears in applications such as electrodynamics, electro-magnetic waves in size-quantized films, traffic flow and elastic media. It is like the KdV equation that the mKdV equation appears in higher-order versions as well. In [4] and [7], modified KdV equations of fifth-order, seventh-order, and ninth-order were formally derived. In this section, only the fifth-order mKdV and seventh-order mKdV equations will be investigated by using the

Hirota's bilinear method. The complete integrability of the third-order KdV equation applies equally well to the fifth-order and seventh-order mKdV equation.

14.6.1 *N*-soliton Solutions for Fifth-order mKdV Equation

The fifth-order modified KdV equation reads

$$u_t + \left\{ 6u^5 + 10\sigma(uu_x^2 + u^2 u_{2x}) + u_{4x} \right\}_x = 0, \quad \sigma = \pm 1. \tag{14.215}$$

We first consider the case where $\sigma = 1$. To determine the N-soliton solutions we follow the approach used for the third-order mKdV equation. We find that the dispersion relation is given by

$$c_i = k_i^5, \tag{14.216}$$

and as a result we obtain

$$\theta_i = k_i x - k_i^5 t. \tag{14.217}$$

In [4], it is shown that the multi-soliton solutions of the fifth-order mKdV equation (14.215) is expressed by

$$u(x,t) = 2\partial_x \left(\arctan(f/g)\right) = 2\frac{f_x g - g_x f}{f^2 + g^2}. \tag{14.218}$$

For the single soliton solution, it was found that

$$\begin{aligned} f(x,t) &= e^{\theta_1} = e^{k_1(x - k_1^4 t)}, \\ g(x,t) &= 1. \end{aligned} \tag{14.219}$$

Substituting (14.219) into (14.218) gives the single soliton solution

$$u(x,t) = \frac{2k_1\, e^{k_1(x-k_1^4 t)}}{1 + e^{2k_1(x-k_1^4 t)}}. \tag{14.220}$$

For the two-soliton solutions we find

$$\begin{aligned} f(x,t) &= e^{\theta_1} + e^{\theta_2} = e^{k_1(x-k_1^4 t)} + e^{k_2(x-k_2^4 t)}, \\ g(x,t) &= 1 - a_{12} e^{\theta_1+\theta_2} = 1 - a_{12} e^{(k_1+k_2)x - (k_1^5+k_2^5)t}. \end{aligned} \tag{14.221}$$

Using (14.221) in (14.218) and substituting the result into (14.215), we find

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{14.222}$$

and hence we set

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leqslant i < j \leqslant 3. \tag{14.223}$$

Consequently, the two-soliton solutions are obtained by substitution (14.222) and (14.221) into (14.218).

For the three-soliton solutions, it was found that

$$\begin{aligned} f(x,t) &= e^{\theta_1} + e^{\theta_2} + e^{\theta_3} - a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3} \\ &= e^{k_1(x-k_1^4 t)} + e^{k_2(x-k_2^4 t)} + e^{k_3(x-k_3^4 t)} \\ &\quad -a_{12}a_{13}a_{23}e^{(k_1+k_2+k_3)x-(k_1^5+k_2^5+k_3^5)t}, \\ g(x,t) &= 1 - a_{12}e^{\theta_1+\theta_2} - a_{13}e^{\theta_1+\theta_3} - a_{23}e^{\theta_2+\theta_3} \\ &= 1 - a_{12}e^{(k_1+k_2)x-(k_1^5+k_2^5)t} - a_{13}e^{(k_1+k_3)x-(k_1^5+k_3^5)t} \\ &\quad -a_{23}e^{(k_2+k_3)x-(k_2^5+k_3^5)t}, \end{aligned} \tag{14.224}$$

where a_{ij} is given in (14.223). Based on this result, the three-soliton solutions for the fifth-order mKdV equation (14.215) is obtained by substituting (14.224) into (14.218). This shows that the fifth-order mKdV equations is completely integrable and N-soliton solutions can be obtained for finite N, where $N \geqslant 1$.

14.6.2 Singular Soliton Solutions for Fifth-order mKdV Equation

It was proved in [11] that certain nonlinear evolution equations have not only soliton solutions, but also explode-decay mode solutions, or singular soliton solutions which can be expressed by closed form of analytic solutions. As investigated in the third-order mKdV equation, we now consider the negative fifth-order mKdV equation

$$u_t + \left\{6u^5 - 10(uu_x^2 + u^2 u_{2x}) + u_{4x}\right\}_x = 0. \tag{14.225}$$

The singular soliton solutions, can be obtained by using the sense of Hirota as used in the third-order mKdV equation. To achieve this goal we set

$$F(x,t) = \frac{f(x,t)}{g(x,t)}, \quad g(x,t) \neq 0. \tag{14.226}$$

The solution of the fifth-order mKdV equation (14.215) is assumed to be of the form

$$u(x,t) = \frac{\partial \log F(x,t)}{\partial x} = \frac{gf_x - fg_x}{gf}, \tag{14.227}$$

where the auxiliary functions $f(x,t)$ and $g(x,t)$ have perturbation expansions of the form

$$\begin{aligned} f(x,t) &= 1 + \textstyle\sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \\ g(x,t) &= 1 + \textstyle\sum_{n=1}^{\infty} \varepsilon_1^n g_n(x,t), \end{aligned} \tag{14.228}$$

where ε and ε_1 are non small formal expansion parameters. Following the simplified form presented in [3], we define the first solution by

$$f_1 = \sum_{i=1}^{N} \varepsilon \exp(\theta_i), \quad g_1 = \sum_{i=1}^{N} \varepsilon_1 \exp(\theta_i), \tag{14.229}$$

where

$$\theta_i = k_i x - c_i t. \tag{14.230}$$

To obtain the single singular soliton solution, we set $N = 1$ into (14.229), and by using (14.228) we find

$$f(x,t) = 1 + \varepsilon f_1(x,t), \quad g(x,t) = 1 + \varepsilon_1 g_1(x,t), \tag{14.231}$$

and hence

$$u(x,t) = \frac{\partial \log F(x,t)}{\partial x} = \frac{\partial}{\partial x} \log\left(\frac{1+\varepsilon f_1}{1+\varepsilon_1 g_1}\right). \tag{14.232}$$

This is a solution of the mKdV equation (14.215) if

$$\varepsilon_1 = -\varepsilon. \tag{14.233}$$

This in turn gives the dispersion relation by $c_i = k_i^5$, and as a result we set

$$\theta_i = k_i x - k_i^5 t. \tag{14.234}$$

The obtained results give a new definition to (14.228) in the form

$$\begin{aligned} f(x,t) &= 1 + \textstyle\sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \\ g(x,t) &= 1 + \textstyle\sum_{n=1}^{\infty} (-1)^n \varepsilon^n g_n(x,t), \end{aligned} \tag{14.235}$$

and consequently we obtain

$$\begin{aligned} f_1(x,t) &= \exp(\theta_1) = \exp(k_1(x - k_1^4 t)), \\ g_1(x,t) &= -\exp(\theta_1) = -\exp(k_1(x - k_1^4 t)). \end{aligned} \tag{14.236}$$

Accordingly, the singular soliton solution

$$u(x,t) = \frac{2k_1 \exp(k_1(x - k_1^4 t))}{1 - \exp(2k_1(x - k_1^4 t))} \tag{14.237}$$

follows immediately.

To determine the two singular soliton solutions, we proceed as before to find

$$\begin{aligned} f(x,t) &= 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12}\exp(\theta_1 + \theta_2), \\ g(x,t) &= 1 - \exp(\theta_1) - \exp(\theta_2) + b_{12}\exp(\theta_1 + \theta_2). \end{aligned} \tag{14.238}$$

Substituting (14.238) into the mKdV equation (14.215), we find that (14.238) is a solution of this equation if a_{12} and b_{12}, and therefore a_{ij} and b_{ij}, are equal and given by

$$a_{ij} = b_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \tag{14.239}$$

where θ_i and θ_j are given above in (14.234). For the second solution we use $1 \leqslant i < j \leqslant 2$ to obtain

$$\begin{aligned} f(x,t) &= 1 + e^{k_1(x-k_1^4 t)} + e^{k_2(x-k_2^4 t)} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x-(k_1^5+k_2^5)t}, \\ g(x,t) &= 1 - e^{k_1(x-k_1^4 t)} - e^{k_2(x-k_2^4 t)} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x-(k_1^5+k_2^5)t}. \end{aligned} \tag{14.240}$$

Recall that the two singular soliton solutions are obtained by using the formulas

$$\begin{aligned} u(x,t) &= \frac{\partial \log F(x,t)}{\partial x}, \\ F(x,t) &= \frac{f(x,t)}{g(x,t)}, \end{aligned} \tag{14.241}$$

We can proceed in a similar manner to derive a three singular soliton solutions, where we find

$$\begin{aligned} f_3(x,t) &= b_{123}\exp(\theta_1+\theta_2+\theta_3), \\ g_3(x,t) &= -b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned} \tag{14.242}$$

where

$$b_{123} = a_{12}a_{13}a_{23} = \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}, \tag{14.243}$$

and this gives

$$\begin{aligned} f(x,t) &= 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ &\quad + a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3) \\ &\quad + b_{123}\exp(\theta_1+\theta_2+\theta_3), \\ g(x,t) &= 1 - \exp(\theta_1) - \exp(\theta_2) - \exp(\theta_3) \\ &\quad + a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3) \\ &\quad - b_{123}\exp(\theta_1+\theta_2+\theta_3). \end{aligned} \tag{14.244}$$

The three singular soliton solutions are given by

$$u(x,t) = \frac{\partial}{\partial x}\left(\ln\left(\frac{f(x,t)}{g(x,t)}\right)\right), \tag{14.245}$$

where $f(x,t)$ and $g(x,t)$ are given in (14.244).

14.6.3 N-soliton Solutions for the Seventh-order mKdV Equation

The seventh-order modified KdV equation reads

$$\begin{aligned}&\left\{20u^7+70(u^4u_{2x}+2u^3u_x^2)+14(u^2u_{4x}+3uu_{2x}^2+4uu_xu_{3x}+5u_x^2u_{2x})+u_{6x}\right\}_x\\&\quad+u_t=0.\end{aligned} \tag{14.246}$$

The seventh-order mKdV equation is completely integrable. To determine the N-soliton solutions we proceed as before where the dispersion relation reads

$$c_i=k_i^7, \tag{14.247}$$

and as a result we obtain

$$\theta_i=k_ix-k_i^7t. \tag{14.248}$$

The multi-soliton solutions of the seventh-order mKdV equation (14.246) is expressed by

$$u(x,t)=2\partial_x\left(\arctan(f/g)\right)=2\frac{f_xg-g_xf}{f^2+g^2}. \tag{14.249}$$

For the single soliton solution, it was found that

$$\begin{aligned}f(x,t)&=e^{\theta_1}=e^{k_1(x-k_1^6t)},\\g(x,t)&=1.\end{aligned} \tag{14.250}$$

Substituting (14.250) into (14.249) gives the single soliton solution

$$u(x,t)=\frac{2k_1\,e^{k_1(x-k_1^6t)}}{1+e^{2k_1(x-k_1^6t)}}. \tag{14.251}$$

For the two-soliton solutions we find

$$\begin{aligned}f(x,t)&=e^{\theta_1}+e^{\theta_2}=e^{k_1(x-k_1^6t)}+e^{k_2(x-k_2^6t)},\\g(x,t)&=1-a_{12}e^{\theta_1+\theta_2}=1-a_{12}e^{(k_1+k_2)x-(k_1^7+k_2^7)t}.\end{aligned} \tag{14.252}$$

Using (14.252) in (14.249) and substituting the result into (14.246), we find

$$a_{12}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{14.253}$$

so that

$$a_{ij}=\frac{(k_i-k_j)^2}{(k_i+k_j)^2},\quad 1\leqslant i<j\leqslant 3. \tag{14.254}$$

Consequently, the two-soliton solutions are obtained by substitution (14.253) and (14.252) into (14.249).

For the three-soliton solutions, it was found that

$$\begin{aligned}f(x,t)&=e^{\theta_1}+e^{\theta_2}+e^{\theta_3}-a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3}\\&=e^{k_1(x-k_1^6t)}+e^{k_2(x-k_2^6t)}+e^{k_3(x-k_3^6t)}\\&\quad-a_{12}a_{13}a_{23}e^{(k_1+k_2+k_3)x-(k_1^7+k_2^7+k_3^7)t},\end{aligned}$$

$$
\begin{aligned}
g(x,t) &= 1 - a_{12}e^{\theta_1+\theta_2} - a_{13}e^{\theta_1+\theta_3} - a_{23}e^{\theta_2+\theta_3} \\
&= 1 - a_{12}e^{(k_1+k_2)x-(k_1^7+k_2^7)t} - a_{13}e^{(k_1+k_3)x-(k_1^7+k_3^7)t} \\
&\quad - a_{23}e^{(k_2+k_3)x-(k_2^7+k_3^7)t},
\end{aligned}
\tag{14.255}
$$

where a_{ij} is given in (14.254). Based on this result, the three-soliton solutions for the seventh-order mKdV equation (14.246) is obtained by substituting (14.255) into (14.249). This shows that the seventh-order mKdV equations is completely integrable and N-soliton solutions can be obtained for finite N, where $N \geqslant 1$.

14.7 Complex Solution for the Seventh-order mKdV Equations

Other solutions that blow up, or singular soliton solutions, can be obtained by following the discussion presented before. We first set

$$
F(x,t) = \frac{f(x,t)}{g(x,t)}, \quad g(x,t) \neq 0. \tag{14.256}
$$

The solution of the seventh-order mKdV equation (14.246) is assumed to be of the form

$$
u(x,t) = R\frac{\partial \log F(x,t)}{\partial x} = \frac{gf_x - fg_x}{gf}, \tag{14.257}
$$

where the auxiliary functions $f(x,t)$ and $g(x,t)$ have perturbation expansions of the form

$$
\begin{aligned}
f(x,t) &= 1 + \textstyle\sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \\
g(x,t) &= 1 + \textstyle\sum_{n=1}^{\infty} \varepsilon_1^n g_n(x,t),
\end{aligned}
\tag{14.258}
$$

where ε and ε_1 are non small formal expansion parameters. We next define

$$
\begin{aligned}
f_1 &= \textstyle\sum_{i=1}^{N} \varepsilon \exp(\theta_i), \\
g_1 &= \textstyle\sum_{i=1}^{N} \varepsilon_1 \exp(\theta_i),
\end{aligned}
\tag{14.259}
$$

where

$$
\theta_i = k_i x - c_i t, \tag{14.260}
$$

where k_i and c_i are arbitrary constants, k_i is called the wave number.

To obtain the single singular soliton solution we use

$$
\begin{aligned}
f(x,t) &= 1 + \varepsilon f_1(x,t), \\
g(x,t) &= 1 + \varepsilon_1 g_1(x,t),
\end{aligned}
\tag{14.261}
$$

and hence

$$
u(x,t) = \frac{\partial \log F(x,t)}{\partial x} = \frac{\partial}{\partial x} \log\left(\frac{1+\varepsilon f_1}{1+\varepsilon_1 g_1}\right). \tag{14.262}
$$

This is a solution of the mKdV equation (14.246) if

$$\varepsilon_1 = -\varepsilon. \tag{14.263}$$

This in turn gives the dispersion relation by

$$c_i = k_i^7, \tag{14.264}$$

and as a result we obtain

$$\theta_i = k_i x - k_i^7 t. \tag{14.265}$$

The obtained results give

$$\begin{aligned} f_1(x,t) &= \exp(\theta_1) = \exp\left(k_1(x - k_1^6 t)\right), \\ g_1(x,t) &= -\exp(\theta_1) = -\exp\left(k_1(x - k_1^6 t)\right). \end{aligned} \tag{14.266}$$

Accordingly, we find

$$F = \frac{1+f_1}{1+g_1} = \frac{1+\exp\left(k_1(x-k_1^6 t)\right)}{1-\exp\left(k_1(x-k_1^6 t)\right)}. \tag{14.267}$$

and $R = i, i = \sqrt{-1}$. The complex solution

$$u(x,t) = \frac{2ik_1 \exp(k_1(x-k_1^6 t))}{1-\exp(2k_1(x-k_1^6 t))}, \tag{14.268}$$

follows immediately. Other complex solutions were not obtained.

14.8 The Hirota-Satsuma Equations

Hirota and Satsuma [6] proposed a coupled KdV equation which describes interactions of two long waves with different dispersion relations. The Hirota-Satsuma equations are

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} + 3uu_x - 6vv_x, \\ v_t &= -v_{xxx} - 3uv_x. \end{aligned} \tag{14.269}$$

If $v = 0$, Eq. (14.269) reduces to the KdV equation. In this section we will use the tanh-coth method and the simplified version of the Hirota's bilinear formalism to handle the Hirota-Satsuma system. The following three conserved densities

$$\begin{aligned} I_1 &= u, \\ I_2 &= u^2 - 2v^2, \\ I_3 &= \frac{3}{2}(u^3 - \frac{1}{2}(u_x)^2) - 3(uv^2 - (v_x)^2), \end{aligned} \tag{14.270}$$

were confirmed.

14.8.1 Using the Tanh-coth Method

Using the wave variable $\xi = x - ct$, system (14.269) is converted to

$$\begin{aligned} -cu - \frac{1}{2}u'' - \frac{3}{2}u^2 + 3v^2 &= 0, \\ -cv' + v''' + 3uv' &= 0. \end{aligned} \tag{14.271}$$

Balancing the nonlinear term u^2 with the highest order derivative u'' in the first equation of the couple gives

$$2M = M + 2, \tag{14.272}$$

that gives

$$M = 2. \tag{14.273}$$

Substituting for u from the first equation into the second equation, and balancing the nonlinear term v^2v' with the highest order derivative v''' in the second equation of the couple gives

$$M_1 + 3 = 2M_1 + M_1 + 1, \tag{14.274}$$

that gives

$$M_1 = 1. \tag{14.275}$$

The tanh-coth method allows us to use the substitution

$$\begin{aligned} u(x,t) &= S(Y) = a_0 + a_1Y^2 + a_2Y^{-2}, \\ v(x,t) &= S_1(Y) = b_0 + b_1Y + b_2Y^{-1}, \end{aligned} \tag{14.276}$$

where we found that $u(x,t)$ does not include Y or Y^{-1} terms. Substituting (14.276) into (14.271), collecting the coefficients of each power of $Y^i, 0 \leqslant i \leqslant 8$, setting each coefficient to zero, and solving the resulting system of algebraic equations we obtain the following sets of solutions

(i)

$$\begin{aligned} &a_0 = \frac{c}{3} + \frac{2}{3}\lambda^2, \quad a_1 = -2\lambda^2, \quad a_2 = 0, \\ &b_0 = 0, \quad b_1 = \frac{1}{\sqrt{2}}c, \quad b_2 = 0, \\ &\mu = \lambda, \end{aligned} \tag{14.277}$$

where

$$\lambda = \frac{(2+\sqrt{10})c}{2}. \tag{14.278}$$

(ii)

$$\begin{aligned} &a_0 = \frac{c}{3} + \frac{2}{3}\lambda^2, \quad a_1 = 0, \quad a_2 = -2\lambda^2, \\ &b_0 = 0, \quad b_1 = 0, \quad b_2 = \frac{1}{\sqrt{2}}c, \\ &\mu = \lambda, \end{aligned} \tag{14.279}$$

(iii)

$$\begin{aligned}
&a_0 = \frac{c}{3} - \frac{1}{3}\lambda^2, \quad a_1 = -\frac{1}{2}\lambda^2, \quad a_2 = -\frac{1}{2}\lambda^2,\\
&b_0 = 0, \quad b_1 = \frac{1}{2\sqrt{2}}c, \quad b_2 = \frac{1}{2\sqrt{2}}c,\\
&\mu = \lambda.
\end{aligned} \tag{14.280}$$

In view of these results we obtain the following sets of solutions

$$\begin{aligned}
u_1(x,t) &= \frac{c}{3} + \frac{2}{3}\lambda^2 - 2\lambda^2 \tanh^2[\lambda(x-ct)],\\
v_1(x,t) &= \frac{1}{\sqrt{2}}c \tanh[\lambda(x-ct)].
\end{aligned} \tag{14.281}$$

$$\begin{aligned}
u_2(x,t) &= \frac{c}{3} + \frac{2}{3}\lambda^2 - 2\lambda^2 \coth^2[\lambda(x-ct)],\\
v_2(x,t) &= \frac{1}{\sqrt{2}}c \coth[\lambda(x-ct)].
\end{aligned} \tag{14.282}$$

and

$$\begin{aligned}
u_3(x,t) &= \frac{c}{3} - \frac{1}{3}\lambda^2 - \frac{1}{2}\lambda^2 \left(\tanh^2\left[\frac{1}{2}\lambda(x-ct)\right] + \coth^2\left[\frac{1}{2}\lambda(x-ct)\right]\right),\\
v_3(x,t) &= \frac{1}{2\sqrt{2}}c\left(\coth\left[\frac{1}{2}\lambda(x-ct)\right] + \coth\left[\frac{1}{2}\lambda(x-ct)\right]\right).
\end{aligned} \tag{14.283}$$

14.8.2 N-soliton Solutions of the Hirota-Satsuma System

In this section, we will examine multiple-soliton solutions of the Hirota-Satsuma system [6]

$$\begin{aligned}
u_t &= \frac{1}{2}u_{xxx} + 3uu_x - 6vv_x,\\
v_t &= -v_{xxx} - 3uv_x.
\end{aligned} \tag{14.284}$$

Hirota introduced the dependent variable transformation

$$\begin{aligned}
u(x,t) &= 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{ff_{2x} - (f_x)^2}{f^2},\\
v(x,t) &= \frac{g}{f},
\end{aligned} \tag{14.285}$$

that will convert (14.284) into the bilinear forms

$$\begin{aligned}
&D_x(D_t - \frac{1}{2}D_x^3)(f \cdot f) = -3g^2,\\
&(D_t + D_x^3)(f \cdot g) = 0.
\end{aligned} \tag{14.286}$$

We next assume that $f(x,t)$ and $g(x,t)$ have the perturbation expansions

$$\begin{aligned} f(x,t) &= 1+\sum_{n=0}^{\infty}\varepsilon^n f_n(x,t), \\ g(x,t) &= \sum_{n=0}^{\infty}\sigma^n g_n(x,t), \end{aligned} \tag{14.287}$$

where ε and σ are non small formal expansion parameter. Following Hirota's method and the simplified version in [3] we first set

$$\begin{aligned} f_1(x,t) &= \sum_{i=1}^{N} \exp(2\theta_i), \\ g_1(x,t) &= \sum_{i=1}^{N} \exp(\theta_i), \end{aligned} \tag{14.288}$$

where

$$\theta_i = k_i x - c_i t, \tag{14.289}$$

where k_i and c_i are arbitrary constants. Substituting (14.288) into (14.284) gives the dispersion relation

$$c_i = k_i^3, \tag{14.290}$$

and in view of this result we obtain

$$\theta_i = k_i x - k_i^3 t. \tag{14.291}$$

This means that

$$\begin{aligned} f_1(x,t) &= \exp(2\theta_1) = \exp(2k_1(x-k_1^2 t)), \\ g_1(x,t) &= \exp(\theta_1) = \exp(k_1(x-k_1^2 t)), \\ \varepsilon &= \frac{1}{8k_1^4}, \\ \sigma &= 1, \end{aligned} \tag{14.292}$$

obtained by using $N=1$ in (14.288). In what follows we list the solutions obtained by Hirota, and more details can be found there. For the one-soliton solution, it was found that

$$\begin{aligned} f &= 1+\frac{1}{8k_1^4}\exp(2\theta_1), \\ g &= \exp(\theta_1). \end{aligned} \tag{14.293}$$

The one soliton solution is therefore

$$u(x,t) = 2(\ln f)_{xx}, \quad v(x,t) = g/f. \tag{14.294}$$

For the two-soliton solution it was found that

$$\begin{aligned} f &= 1+\frac{1}{8k_1^4}e^{2\theta_1}+\frac{1}{8k_2^4}e^{2\theta_2} \\ &\quad +\frac{2}{(k_1+k_2)^2(k_1^2+k_2^2)}e^{\theta_1+\theta_2}+\frac{(k_1-k_2)^4}{64k_1^4k_2^4(k_1+k_2)^4}e^{2(\theta_1+\theta_2)}, \\ g &= e^{\theta_1}+e^{\theta_2}+\frac{1}{8k_1^4}\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{2\theta_1+\theta_2}+\frac{1}{8k_2^4}\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\theta_1+2\theta_2}. \end{aligned} \tag{14.295}$$

The two-soliton solution is obtained by substituting (14.295) into (14.294).

It is interesting to point out that Hirota and Satsuma derived the one and two-soliton solutions only and used this result to suggest the existence of the N-soliton solutions.

14.8.3 N-soliton Solutions by an Alternative Method

Tam *et.al.* [13] applied a slightly different approach and derived entirely new one, two and three-soliton solutions to the Hirota-Satsuma system.

The dependent variable transformation

$$\begin{aligned} u(x,t) &= 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{ff_{2x}-(f_x)^2}{f^2}, \\ v(x,t) &= \frac{g}{f} \end{aligned} \tag{14.296}$$

was applied to convert (14.284) into the bilinear forms

$$\begin{aligned} &D_x(D_t - \frac{1}{2}D_x^3)(f\cdot f) = -3g^2 + Cf^2, \\ &(D_t + D_x^3)(f\cdot g) = 0, \end{aligned} \tag{14.297}$$

where C in an integration constant. For $C=0$ we obtain the bilinear forms (14.286). In [13], $C=3$ was used to convert the last bilinear form to

$$\begin{aligned} &D_x(D_t - \frac{1}{2}D_x^3)(f\cdot f) = 3(f^2 - g^2), \\ &(D_t + D_x^3)(f\cdot g) = 0. \end{aligned} \tag{14.298}$$

It was obtained after some tests and guesses that for the one-soliton solution

$$\begin{aligned} f &= 1 + \exp(\theta_1) + \frac{1}{32}(4+k_1^4)\exp(2\theta_1), \\ g &= 1 + \frac{1}{2}(2+k_1^4)\exp(\theta_1) + \frac{1}{32}(4+k_1^4)\exp(2\theta_1). \end{aligned} \tag{14.299}$$

This result is distinct from that obtained in (14.293). Consequently, the one soliton solution is therefore

$$u(x,t) = 2(\ln f)_{xx}, \quad v(x,t) = g/f. \tag{14.300}$$

For the two-soliton solution it was found that, after correcting some of the coefficients in [13]

$$\begin{aligned} f = {} & 1 + \exp(\theta_1) + \exp(\theta_2) + A_1\exp(2\theta_1) + A_2\exp(2\theta_2) + A_3\exp(\theta_1+\theta_2) \\ & + A_4\exp(2\theta_1+\theta_2) + A_5\exp(\theta_1+2\theta_2) + A_6\exp(2(\theta_1+\theta_2)), \end{aligned}$$

$$
\begin{aligned}
g = {} & 1+\frac{1}{2}(2+k_1^4)\exp(\theta_1)+\frac{1}{2}(2+k_2^4)\exp(\theta_2)+B_1\exp(2\theta_1)+B_2\exp(2\theta_2)\\
& +B_3\exp(\theta_1+\theta_2)+B_4\exp(2\theta_1+\theta_2)+B_5\exp(\theta_1+2\theta_2)\\
& +B_6\exp(2(\theta_1+\theta_2)),
\end{aligned}
\tag{14.301}
$$

where

$$
\begin{aligned}
A_i &= \frac{1}{32}(4+k_i^4), \quad i=1,2,\\
A_3 &= \frac{2(k_1^4+k_2^4)+k_1^4k_2^4}{2(k_1+k_2)^2(k_1^2+k_2^2)},\\
A_{j+3} &= \frac{1}{32}(4+k_j^4)\frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \quad j=1,2,\\
A_6 &= A_1A_2\frac{(k_1-k_2)^4}{(k_1+k_2)^4},\\
B_i &= \frac{1}{32}(4+k_i^4), \quad i=1,2,\\
B_3 &= \frac{(k_1^8+k_2^8)-k_1^4k_2^4+2(k_1^4+k_2^4)}{2(k_1+k_2)^2(k_1^2+k_2^2)},\\
B_4 &= \frac{1}{2}(2+k_2^4)\times\frac{1}{32}(4+k_1^4)\frac{(k_1-k_2)^2}{(k_1+k_2)^2},\\
B_5 &= \frac{1}{2}(2+k_1^4)\times\frac{1}{32}(4+k_2^4)\frac{(k_1-k_2)^2}{(k_1+k_2)^2},\\
B_6 &= A_6.
\end{aligned}
\tag{14.302}
$$

The two-soliton solution is obtained by substituting (14.301) into (14.300). The explicit three-soliton solution is obtained in [13]. Because the three-soliton solutions were obtained, this clearly indicates that the N-soliton solutions, $N \geqslant 3$ exist for the coupled KdV equations.

14.9 Generalized Short Wave Equation

In this section, the generalized short water wave (GSWW) equation

$$
u_t-u_{xxt}-\alpha uu_t-\beta u_x\int^x u_t\,dx+u_x=0, \tag{14.303}
$$

where α and β are non-zero constants.

Ablowitz *et. al.* [1] studied the specific case $\alpha=4$ and $\beta=2$ where Eq. (14.303) is reduced to

$$
u_t-u_{xxt}-4uu_t-2u_x\int^x u_t\,dx+u_x=0. \tag{14.304}
$$

This equation was introduced as a model equation which reduces to the KdV equation in the long small amplitude limit. However, Hirota *et.al.* [5] examined the model equation for shallow water waves

$$u_t - u_{xxt} - 3uu_t - 3u_x \int^x u_t\,dx + u_x = 0, \tag{14.305}$$

obtained by substituting $\alpha = \beta = 3$ in (14.303).

The customary definition of the Hirota's bilinear operators [5,14,15] are given by

$$D_t^n D_x^m (a\cdot b) = (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^n (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m a(x,t)b(x',t')\Big| x' = x, t' = t. \tag{14.306}$$

Some of the properties of the D-operators are as follows

$$\begin{aligned}
\frac{D_t^2(f\cdot f)}{f^2} &= \iint u_{tt}\,dx\,dx,\\
\frac{D_t D_x^3(f\cdot f)}{f^2} &= u_{xt} + 3u\int x u_t\,dx',\\
\frac{D_x^2(f\cdot f)}{f^2} &= u,\\
\frac{D_x^4(f\cdot f)}{f^2} &= u_{2x} + 3u^2,\\
\frac{D_t D_x(f\cdot f)}{f^2} &= \ln(f^2)_{xt},\\
\frac{D_x^6(f\cdot f)}{f^2} &= u_{4x} + 15uu_{2x} + 15u^3,\\
\frac{D_t^2(f\cdot f)}{f^2} &= \iint u_{tt}\,dx\,dx,\\
\frac{D_t D_x^3(f\cdot f)}{f^2} &= u_{xt} + 3u\int u_t\,dx',
\end{aligned} \tag{14.307}$$

where

$$u(x,t) = 2(\ln f(x,t))_{xx}, \tag{14.308}$$

We first remove the integral term in (14.303) by introducing the potential

$$u(x,t) = v_x(x,t), \tag{14.309}$$

that will carry (14.303) into the equation

$$v_{xt} - v_{xxxt} - \alpha v_x v_{xt} - \beta v_{xx} v_t + v_{xx} = 0, \tag{14.310}$$

Substituting

$$v(x,t) = e^{\theta_i}, \quad \theta_i = k_i x - c_i t, \tag{14.311}$$

into the linear terms of equation (14.310), and solving the resulting equation we obtain the dispersion relation

$$c_i = \frac{k_i}{1-k_i^2}, \quad i = 1,2,\cdots,N, k_i \neq \pm 1, \tag{14.312}$$

and hence θ_i becomes

$$\theta_i = k_i x - \frac{k_i}{1-k_i^2} t. \tag{14.313}$$

To determine R, we substitute

$$v(x,t) = R\frac{\partial \ln f(x,t)}{\partial x} = R\frac{f_x}{f} \tag{14.314}$$

into Eq. (14.310) and solve to find that

$$R = \frac{12}{\alpha+\beta}, \alpha+\beta \neq 0, \tag{14.315}$$

where $f(x,t) = 1 + e^{k_1 x - \frac{k_1}{1-k_1^2} t}$.

It is interesting to point out that for single solitons solutions, the result (14.315) works for all values of α and β. However, the complete integrability of (14.303) requires that

$$(\alpha-\beta)(\alpha-2\beta) = 0. \tag{14.316}$$

This in turn suggests that $\alpha = \beta$ and $\alpha = 2\beta$ are the only integrable cases of the GSWW equation (14.303), and hence for (14.310).

Case i
Substituting $\alpha = \beta$ into (14.310) gives

$$v_{xt} - v_{xxxt} - \beta v_x v_{xt} - \beta v_{xx} v_t + v_{xx} = 0, \quad \beta \neq 0. \tag{14.317}$$

This means that

$$R = \frac{6}{\beta}, \quad \beta \neq 0. \tag{14.318}$$

Consequently, the solution is given by

$$v(x,t) = \frac{6}{\beta}\frac{\partial \ln f(x,t)}{\partial x} = \frac{6k_1 e^{k_1 x - \frac{k_1}{1-k_1^2} t}}{\beta\left(1 + e^{k_1 x - \frac{k_1}{1-k_1^2} t}\right)}. \tag{14.319}$$

Using $u(x,t) = v_x(x,t)$ yields the single soliton solution of the first shallow water wave equation by

$$u(x,t)=\frac{6}{\beta}(\ln f(x,t))_{xx}=\frac{6k_1^2e^{k_1x-\frac{k_1}{1-k_1^2}t}}{\beta\left(1+e^{k_1x-\frac{k_1}{1-k_1^2}t}\right)^2}. \tag{14.320}$$

For two-soliton solutions, we substitute

$$v(x,t)=\frac{6}{\beta}\frac{\partial \ln f(x,t)}{\partial x}, \tag{14.321}$$

where

$$f(x,t)=1+e^{\theta_1}+e^{\theta_2}+a_{12}e^{\theta_1+\theta_2}, \tag{14.322}$$

into (14.310), where θ_1 and θ_2 are given in (14.313), to obtain

$$a_{12}=\frac{(k_1^2-k_1k_2+k_2^2-3)(k_1-k_2)^2}{(k_1^2+k_1k_2+k_2^2-3)(k_1+k_2)^2}, \tag{14.323}$$

and hence

$$a_{ij}=\frac{(k_i^2-k_ik_j+k_j^2-3)(k_i-k_j)^2}{(k_i^2+k_ik_j+k_j^2-3)(k_i+k_j)^2},\quad 1\leqslant i<j\leqslant N. \tag{14.324}$$

Notice that the coefficients a_{ij} do not depend on β. This in turn gives

$$\begin{aligned} f(x,t) &= 1+e^{k_1x-\frac{k_1}{1-k_1^2}t}+e^{k_2x-\frac{k_2}{1-k_2^2}t}\\ &\quad+\frac{(k_1^2-k_1k_2+k_2^2-3)(k_1-k_2)^2}{(k_1^2+k_1k_2+k_2^2-3)(k_1+k_2)^2}\,e^{(k_1+k_2)x-(\frac{k_1}{1-k_1^2}+\frac{k_2}{1-k_2^2})t}. \end{aligned} \tag{14.325}$$

To determine the two-soliton solutions explicitly, we substitute (14.325) into the formula $u(x,t)=\dfrac{6}{\beta}(\ln f(x,t))_{xx}$.

For the three-soliton solutions, we set

$$\begin{aligned} f(x,t) &= 1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3)\\ &\quad+a_{12}\exp(\theta_1+\theta_2)+a_{23}\exp(\theta_2+\theta_3)+a_{13}\exp(\theta_1+\theta_3)\\ &\quad+b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned} \tag{14.326}$$

into (14.321) and substitute it into (14.310) to find that

$$b_{123}=a_{12}a_{13}a_{23}. \tag{14.327}$$

To determine the three-soliton solutions explicitly, we substitute the last result for $f(x,t)$ in the formula $u(x,t)=\dfrac{6}{\beta}(\ln f(x,t))_{xx}$. The higher level soliton solutions, for $n\geqslant 4$ can be obtained in a parallel manner. This confirms that the first shallow water wave equation (14.317) is completely integrable that admits multiple-soliton solutions of any order.

Case ii
Substituting $\alpha = 2\beta$ into (14.310) gives

$$v_{xt} - v_{xxxt} - 2\beta v_x v_{xt} - \beta v_{xx} v_t + v_{xx} = 0, \quad \beta \neq 0. \tag{14.328}$$

This means that

$$R = \frac{4}{\beta}, \quad \beta \neq 0. \tag{14.329}$$

Consequently, the solution is given by

$$v(x,t) = \frac{4}{\beta}\frac{\partial \ln f(x,t)}{\partial x} = \frac{4k_1 e^{k_1 x - \frac{k_1}{1-k_1^2}t}}{\beta\left(1 + e^{k_1 x - \frac{k_1}{1-k_1^2}t}\right)}. \tag{14.330}$$

Using $u(x,t) = v_x(x,t)$ yields the single soliton solution of the first shallow water wave equation by

$$u(x,t) = \frac{4}{\beta}(\ln f(x,t))_{xx} = \frac{4k_1^2 e^{k_1 x - \frac{k_1}{1-k_1^2}t}}{\beta\left(1 + e^{k_1 x - \frac{k_1}{1-k_1^2}t}\right)^2}. \tag{14.331}$$

For two-soliton solutions, we substitute

$$v(x,t) = \frac{4}{\beta}\frac{\partial \ln f(x,t)}{\partial x}, \tag{14.332}$$

where

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{14.333}$$

into (14.328), where θ_1 and θ_2 are given in (14.313), to obtain

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{14.334}$$

and hence

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leqslant i < j \leqslant N. \tag{14.335}$$

Notice that the coefficients a_{ij} do not depend on β. This in turn gives

$$f(x,t) = 1 + e^{k_1 x - \frac{k_1}{1-k_1^2}t} + e^{k_2 x - \frac{k_2}{1-k_2^2}t}$$

$$+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(\frac{k_1}{1-k_1^2}+\frac{k_2}{1-k_2^2})t}. \tag{14.336}$$

To determine the two-soliton solutions explicitly, we substitute (14.336) into the formula $u(x,t)=\frac{4}{\beta}(\ln f(x,t))_{xx}$.

Similarly, to determine the three-soliton solutions, we set

$$\begin{aligned} f(x,t) &= 1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3) \\ &\quad +a_{12}\exp(\theta_1+\theta_2)+a_{23}\exp(\theta_2+\theta_3)+a_{13}\exp(\theta_1+\theta_3) \\ &\quad +b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned} \tag{14.337}$$

into (14.332) and substitute it into (14.310) to find that

$$b_{123}=a_{12}a_{13}a_{23}. \tag{14.338}$$

To determine the three-soliton solutions explicitly, we substitute the last result for $f(x,t)$ in the formula $u(x,t)=\frac{4}{\beta}(\ln f(x,t))_{xx}$. The higher level soliton solutions, for $n\geqslant 4$ can be obtained in a parallel manner. This confirms that the generalized shallow water wave equation (14.328) is completely integrable that admits multiple-soliton solutions of any order. It is clear that β can be positive or negative numbers, but $\beta\neq 0$.

References

1. M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge (1991).
2. P.J. Caudrey, R.K. Dodd, and J.D. Gibbon, A new hierarchy of Korteweg-de Vries equations, *Proc. Roy. Soc. Lond.* A 351, 407–422, (1976).
3. W. Hereman and A. Nuseir, Symbolic methods to construct exact solutions of nonlinear partial differential equations, *Math. Comput. Simulation*, 43(1), 13–27, (1997).
4. J. Hietarinta, A search for bilinear equations passing Hirota's three-soliton condition. I. KdV-type bilinear equations, *J. Math. Phys.*, 28(8), 1732–1742, (1987).
5. R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge (2004).
6. R. Hirota and J. Satsuma, Soliton solutions of a coupled Korteweg-de Vries equation, *Phys. Lett.* A 85, 407–415, (1981).
7. M. Ito, An extension of nonlinear evolution equations of the KdV (mKdV) type to higher orders, *J. Phys. Soc. Japan*, 49(2), 771–778, (1980).
8. D.J. Kaup, On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx}+6Q\psi_x+6R\psi=\lambda\psi$, *Stud. Appl. Math.*, 62, 189–216, (1980).
9. B.A. Kuperschmidt, A super KdV equation: an integrable system, *Phys. Lett.* A 102, 213–215, (1984).
10. P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Commun. Pure Appl. Math.*, 21, 467–490, (1968).
11. A. Nakamura, Simple explode-decay mode solutions of a certain one-space dimensional nonlinear evolution equations, *J. Phys. Soc. Japan*, 33(5), 1456–1458, (1972).

12. K. Sawada and T. Kotera, A method for finding N-soliton solutions of the K.d.V. equation and the K.d.V.-like equation, *Progr. Theoret. Phys.*, 51, 1355–1367, (1974).
13. H.W. Tam, W.X. Ma, X.B. Hu, and D.L. Wang, The Hirota-Satsuma coupled KdV equation and a coupled Ito system revisited, *J.Phys.Soc.Japan*, 69(1), 45–52, (2000).
14. A.M. Wazwaz, The Hirota's direct method for multiple-soliton solutions for three model equations of shallow water waves, *Appl. Math. Comput.*, 201, 489–503, (2008).
15. A.M. Wazwaz, Multiple-soliton solutions of two extended model equations for shallow water waves, *Appl. Math. Comput.*, 201, 790–799, (2008).

Chapter 15
Family of KdV-type Equations

15.1 Introduction

In this chapter we will study a family of KdV-type equations. These equations appear in many scientific fields as will be examined for each model. This family of KdV-type equations contains the following forms:
(i) The complex modified KdV equation [16] (CMKdV) is of the form

$$w_t + w_{xxx} + \alpha(|w|^2 w)_x = 0, \tag{15.1}$$

where w is a complex valued function of the spatial coordinate x and the time t, and α is a real constant.
(ii) The Benjamin-Bona-Mahony (BBM) equation [2] is given by

$$u_t + au_x - 6uu_x + bu_{xxt} = 0, \tag{15.2}$$

where a and b are constants. This equation is also called the regularized long wave (RLW) equation.
(iii) The Modified Equal Width (MEW) equation [17] is given by

$$u_t + 3u^2 u_x - au_{xxt} = 0. \tag{15.3}$$

(iv) The Kawahara equation [9] is of fifth-order and given by

$$u_t + 6uu_x + u_{xxx} - u_{xxxxx} = 0, \tag{15.4}$$

and the modified Kawahara equation [9] is of the form

$$u_t + 6u^2 u_x + u_{xxx} - u_{xxxxx} = 0. \tag{15.5}$$

(v) The Kadomtsev-Petviashvili (KP) equation [8] is of the form

$$(u_t + auu_x + u_{xxx})_x + ku_{yy} = 0. \tag{15.6}$$

(vi) The Zakharov-Kuznetsov (ZK) equation [20] is given by

$$u_t + auu_x + b(\nabla^2 u)_x = 0, \tag{15.7}$$

where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the isotropic Laplacian. The (2+1)-dimensional Zakharov-Kuznetsov (ZK) equation is given by

$$u_t + auu_x + b(u_{xx} + u_{yy})_x = 0. \tag{15.8}$$

(vii) The Benjamin-Ono equation [1,13]

$$u_t + 4uu_x + H(u_{xx}) = 0, \tag{15.9}$$

where H is the Hilbert transform defined by

$$H[u(x,t)] = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(y,t)}{y-x} dy, \tag{15.10}$$

where P stands for the Cauchy principal value of the integral.
(viii) The KdV-Burgers equation reads

$$u_t + 6uu_x + au_{xx} + u_{xxx} = 0, \quad a \neq 0. \tag{15.11}$$

(ix) The seventh-order KdV equation (sKdV) is given by

$$u_t + 6uu_x + u_{3x} - u_{5x} + \alpha u_{7x} = 0, \tag{15.12}$$

where α is a nonzero constant.
(x) The ninth-order KdV equation (nKdV) is of the form

$$u_t + 6uu_x + u_{3x} - u_{5x} + \alpha u_{7x} + \beta u_{9x} = 0, \tag{15.13}$$

where α and β are arbitrary nonzero constants.

In what follows, we will study the KdV-related equations that were described above. In a manner parallel to the approach applied in the previous chapter, the sine-cosine method, the tanh-coth method, and the Hirota's bilinear formalism will be used here.

15.2 The Complex Modified KdV Equation

We first aim to cast light on the complex modified KdV equation [16]

$$w_t + w_{xxx} + \alpha(|w|^2 w)_x = 0, \tag{15.14}$$

where w is a complex valued function of the spatial coordinate x and the time t, α is a real parameter. This equation has been proposed as a model for the nonlinear evolution of plasma waves. The physical model (15.14) incorporates the propagation of transverse waves in a molecular chain model, and in a generalized elastic solid. The two-dimensional steady state distribution of lower hybrid waves is governed by the CMKdV equation (15.14).

The CMKdV equation (15.14) is completely integrable by the inverse scattering method and it admits sech-shaped soliton solutions whose amplitudes and velocities are free parameters. This equation will be solved by the sine-cosine and the tanh-coth methods.

15.2.1 Using the Sine-cosine Method

We begin our analysis by decomposing w into its real and imaginary parts, where we set

$$w = u + iv, i^2 = -1, \tag{15.15}$$

to obtain the coupled pair of the modified KdV (mKdV) equations

$$\begin{aligned} u_t + u_{xxx} + \alpha \left[(u^2+v^2)u\right]_x &= 0,\\ v_t + v_{xxx} + \alpha \left[(u^2+v^2)v\right]_x &= 0. \end{aligned} \tag{15.16}$$

These two coupled nonlinear equations describe the interaction of two orthogonally polarized transverse waves, where u and v represent y-polarized and z-polarized transverse waves respectively, propagating in the x-direction in an xyz coordinate system.

Using the wave variable $\xi = x - ct$ into the system (15.16) and integrating we obtain

$$\begin{aligned} -cu + \alpha u^3 + \alpha u v^2 + u'' &= 0,\\ -cv + \alpha v^3 + \alpha u^2 v + v'' &= 0. \end{aligned} \tag{15.17}$$

We then use the cosine assumption [18]

$$u(x,t) = \lambda \cos^{\beta}(\mu\xi), \quad v(x,t) = \tilde{\lambda}\cos^{\tilde{\beta}}(\mu\xi) \tag{15.18}$$

into (15.17) to get

$$\begin{aligned} &-c\lambda\cos^{\beta}(\mu\xi) + \alpha\lambda^3\cos^{3\beta}(\mu\xi) + \alpha\lambda\tilde{\lambda}^2\cos^{\beta}(\mu\xi)\cos^{2\tilde{\beta}}(\mu\xi)\\ &\qquad -\mu^2\lambda\beta^2\cos^{\beta}(\mu\xi) + \mu^2\lambda\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0,\\ &-c\tilde{\lambda}\cos^{\tilde{\beta}}(\mu\xi) + \alpha\tilde{\lambda}^3\cos^{3\tilde{\beta}}(\mu\xi) + \alpha\lambda^2\tilde{\lambda}\cos^{\tilde{\beta}}(\mu\xi)\cos^{2\beta}(\mu\xi)\\ &\qquad -\mu^2\tilde{\lambda}\tilde{\beta}^2\cos^{\tilde{\beta}}(\mu\xi) + \mu^2\tilde{\lambda}\tilde{\beta}(\tilde{\beta}-1)\cos^{\tilde{\beta}-2}(\mu\xi) = 0. \end{aligned} \tag{15.19}$$

Using the balance method, by equating the exponents and the coefficients of $\cos^j$, we get

$$\begin{array}{c}\beta-1\neq 0,\;\; \tilde{\beta}-1\neq 0,\\ 3\beta=\beta+2\tilde{\beta}=\beta-2,\;\; 3\tilde{\beta}=2\beta+\tilde{\beta}=\tilde{\beta}-2,\\ \mu^2\beta^2=-c,\;\; \mu^2\tilde{\beta}^2=-c,\\ \alpha\lambda^3+\alpha\lambda^2\tilde{\lambda}=-\lambda\mu^2\beta(\beta-1),\;\; \alpha\tilde{\lambda}^3+\alpha\lambda\tilde{\lambda}^2=-\tilde{\lambda}\mu^2\tilde{\beta}(\tilde{\beta}-1).\end{array} \tag{15.20}$$

Solving the system (15.20) leads to the results

$$\begin{aligned}\beta &= \tilde{\beta}=-1,\\ \mu &= \sqrt{-c}, \quad c<0,\\ \lambda &= \tilde{\lambda}=\sqrt{\frac{c}{\alpha}}.\end{aligned} \tag{15.21}$$

Consequently, for $c<0$, we obtain the following periodic solutions

$$u(x,t)=v(x,t)=\sqrt{\frac{c}{\alpha}}\csc\left(\sqrt{-c}(x-ct)\right), \quad 0<\mu(x-ct)<\pi, \tag{15.22}$$

and

$$u(x,t)=v(x,t)=\sqrt{\frac{c}{\alpha}}\sec\left(\sqrt{-c}(x-ct)\right), \quad |\mu(x-ct)|<\frac{\pi}{2}. \tag{15.23}$$

Noting that $w(x,t)=u(x,t)+iv(x,t)$, the solutions of the CMKdV equation read

$$w_1(x,t)=(1+i)\sqrt{\frac{c}{\alpha}}\csc\left(\sqrt{-c}(x-ct)\right), \quad 0<\mu(x-ct)<\pi, \tag{15.24}$$

and

$$w_2(x,t)=(1+i)\sqrt{\frac{c}{\alpha}}\sec\left(\sqrt{-c}(x-ct)\right), \quad |\mu(x-ct)|<\frac{\pi}{2}. \tag{15.25}$$

However, for $c>0$, we obtain the complex solutions

$$w(x,t)=(1+i)\sqrt{-\frac{c}{\alpha}}\operatorname{csch}\left(\sqrt{c}(x-ct)\right), \tag{15.26}$$

and

$$w(x,t)=(1+i)\sqrt{\frac{c}{\alpha}}\operatorname{sech}\left(\sqrt{c}(x-ct)\right). \tag{15.27}$$

15.2.2 Using the Tanh-coth Method

In this section, we will use the tanh-coth method [19] as presented before to handle the (CMKdV) equation. It was shown before that the CMKdV equation takes the form of two coupled nonlinear equations of the system

$$\begin{aligned} u_t + u_{xxx} + \alpha\left[(u^2+v^2)u\right]_x &= 0, \\ v_t + v_{xxx} + \alpha\left[(u^2+v^2)v\right]_x &= 0. \end{aligned} \tag{15.28}$$

Using the wave variable $\xi = x - ct$ into the system (15.28) and integrating once we find

$$\begin{aligned} -cu + \alpha u^3 + \alpha uv^2 + u'' &= 0, \\ -cv + \alpha v^3 + \alpha vu^2 + v'' &= 0. \end{aligned} \tag{15.29}$$

The tanh-coth method admits the use of finite series

$$\begin{aligned} u(x,t) = S(Y) &= \textstyle\sum_{m=0}^{M} a_m Y^m, \\ v(x,t) = \tilde{S}(Y) &= \textstyle\sum_{m=0}^{M_1} b_m Y^m, \end{aligned} \tag{15.30}$$

to express the solutions $u(x,t)$ and $v(x,t)$, where $Y = \tanh(\mu\xi)$. Substituting (15.30) into the ODE (15.28) gives

$$\begin{aligned} -cS + \alpha S^3 + \alpha S\tilde{S}^2 + \mu^2(1-Y^2)\left(-2YS' + (1-Y^2)S''\right) &= 0, \\ -c\tilde{S} + \alpha\tilde{S}^3 + \alpha S^2\tilde{S} + \mu^2(1-Y^2)\left(-2Y\tilde{S}' + (1-Y^2)\tilde{S}''\right) &= 0. \end{aligned} \tag{15.31}$$

Balancing the linear term of highest order with the nonlinear term in both equations we find

$$\begin{aligned} 3M &= M + 2M_1 = 4 + M - 2, \\ 3M_1 &= 2M + M_1 = 4 + M_1 - 2. \end{aligned} \tag{15.32}$$

which gives $M = M_1 = 1$. This means that

$$\begin{aligned} u(x,t) &= a_0 + a_1 Y + a_2 Y^{-1}, \\ v(x,t) &= b_0 + b_1 Y + b_2 Y^{-1}. \end{aligned} \tag{15.33}$$

Substituting (15.33) into the two components of (15.31), and collecting the coefficients of Y gives two systems of algebraic equations for $a_0, a_1, a_2, b_0, b_1, b_2$ and μ. Solving these systems leads to the following sets:

(i)

$$a_1 = \sqrt{\frac{c}{\alpha}}, \quad \mu = \sqrt{-\frac{c}{2}}, \quad c < 0, \alpha < 0, \quad \text{other parameters are zeros.} \tag{15.34}$$

(ii)

$$b_1 = \sqrt{\frac{c}{\alpha}}, \quad \mu = \sqrt{-\frac{c}{2}}, \quad c < 0, \alpha < 0, \quad \text{other parameters are zeros.} \tag{15.35}$$

(iii)

$$a_2 = \sqrt{\frac{c}{\alpha}}, \quad \mu = \sqrt{-\frac{c}{2}}, \quad c < 0, \alpha < 0, \quad \text{other parameters are zeros.} \tag{15.36}$$

(iv)

$$b_2=\sqrt{\frac{c}{\alpha}},\quad \mu=\sqrt{-\frac{c}{2}},\quad c<0,\alpha<0,\quad \text{other parameters are zeros.} \tag{15.37}$$

(v)

$$a_1=a_2=\frac{1}{2}\sqrt{\frac{c}{\alpha}},\quad \mu=\frac{1}{2}\sqrt{-\frac{c}{2}},\quad c<0,\alpha<0,\quad \text{other parameters are zeros.} \tag{15.38}$$

(vi)

$$b_1=b_2=\frac{1}{2}\sqrt{\frac{c}{\alpha}},\quad \mu=\frac{1}{2}\sqrt{-\frac{c}{2}},\quad c<0,\alpha<0,\quad \text{other parameters are zeros.} \tag{15.39}$$

(vii)

$$a_1=-a_2=\sqrt{-\frac{c}{2\alpha}},\quad \mu=\frac{1}{2}\sqrt{c},\quad c>0,\alpha<0,\quad \text{other parameters are zeros.} \tag{15.40}$$

(viii)

$$b_1=-b_2=\sqrt{-\frac{c}{2\alpha}},\quad \mu=\frac{1}{2}\sqrt{c},\quad c>0,\alpha<0,\quad \text{other parameters are zeros.} \tag{15.41}$$

(ix)

$$b_1=\gamma,\quad a_1=\sqrt{\frac{c-\alpha\gamma^2}{\alpha}},\quad \mu=\sqrt{-\frac{c}{2}},\quad c<\alpha\gamma^2<0,\quad \alpha<0, \\ \text{other parameters are zeros.} \tag{15.42}$$

(x)

$$b_2=\beta,\quad a_2=\sqrt{\frac{c-\alpha\beta^2}{\alpha}},\quad \mu=\sqrt{-\frac{c}{2}},\quad c<\alpha\beta^2<0,\quad \alpha<0, \\ \text{other parameters are zeros.} \tag{15.43}$$

(xi)

$$\begin{aligned} b_1 &= -b_2=\beta,\quad a_1=-a_2=\sqrt{-\frac{c+2\alpha\beta^2}{2\alpha}}, \\ \mu &= \frac{1}{2}\sqrt{c},\ \ c>0,\ \ c+2\alpha\beta^2>0,\ \ \alpha<0, \\ &\quad \text{other parameters are zeros.} \end{aligned} \tag{15.44}$$

(xii)

$$\begin{aligned} b_1 &= b_2=\beta,\quad a_1=a_2=\frac{1}{2}\sqrt{\frac{c-4\alpha\beta^2}{\alpha}}, \\ \mu &= \frac{1}{2}\sqrt{-\frac{c}{2}},\ \ c<0,\ \ c-4\alpha\beta^2<0,\ \ \alpha<0, \\ &\quad \text{other parameters are zeros,} \end{aligned} \tag{15.45}$$

where β and γ are constants. Based on these results, we obtain real and complex solutions. Recall that the solution is given by $w = u + iv$, hence we obtain the following solutions, where $c < 0, \alpha < 0$

$$
\begin{aligned}
w_1(x,t) &= \sqrt{\frac{c}{\alpha}} \tanh\left(\sqrt{-\frac{c}{2}}(x-ct)\right),\\
w_2(x,t) &= i\sqrt{\frac{c}{\alpha}} \tanh\left(\sqrt{-\frac{c}{2}}(x-ct)\right), \quad i^2=-1,\\
w_3(x,t) &= \sqrt{\frac{c}{\alpha}} \coth\left(\sqrt{-\frac{c}{2}}(x-ct)\right),\\
w_4(x,t) &= i\sqrt{\frac{c}{\alpha}} \coth\left(\sqrt{-\frac{c}{2}}(x-ct)\right),\\
w_5(x,t) &= \frac{1}{2}\sqrt{\frac{c}{\alpha}}\left(\tanh\left(\frac{1}{2}\sqrt{-\frac{c}{2}}(x-ct)\right)+\coth\left(\frac{1}{2}\sqrt{\frac{-c}{2}}(x-ct)\right)\right),\\
w_6(x,t) &= i\frac{1}{2}\sqrt{\frac{c}{\alpha}}\left(\tanh\left(\frac{1}{2}\sqrt{-\frac{c}{2}}(x-ct)\right)+\coth\left(\frac{1}{2}\sqrt{\frac{-c}{2}}(x-ct)\right)\right),\\
w_7(x,t) &= \sqrt{-\frac{c}{2\alpha}}\left(\tanh\left(\frac{1}{2}\sqrt{c}(x-ct)\right)-\coth\left(\frac{1}{2}\sqrt{c}(x-ct)\right)\right),\\
w_8(x,t) &= i\sqrt{-\frac{c}{2\alpha}}\left(\tanh\left(\frac{1}{2}\sqrt{c}(x-ct)\right)-\coth\left(\frac{1}{2}\sqrt{c}(x-ct)\right)\right),\\
w_9(x,t) &= \left(\sqrt{\frac{c-\alpha\gamma^2}{\alpha}}+i\gamma\right)\tanh\left(\sqrt{-\frac{c}{2}}(x-ct)\right),\\
w_{10}(x,t) &= \left(\sqrt{\frac{c-\alpha\beta^2}{\alpha}}+i\beta\right)]\coth\left(\sqrt{-\frac{c}{2}}(x-ct)\right),\\
w_{11}(x,t) &= \left(\sqrt{-\frac{c+2\alpha\beta^2}{2\alpha}}+i\beta\right)\left(\tanh\left(\frac{1}{2}\sqrt{c}(x-ct)\right)-\coth\left(\frac{1}{2}\sqrt{c}(x-ct)\right)\right),\\
w_{12}(x,t) &= \left(\frac{1}{2}\sqrt{\frac{c-4\alpha\beta^2}{\alpha}}+i\beta\right)\\
&\quad\times\left(\tanh\left(\frac{1}{2}\sqrt{-\frac{c}{2}}(x-ct)\right)+\coth\left(\frac{1}{2}\sqrt{-\frac{c}{2}}(x-ct)\right)\right),
\end{aligned}
$$

valid for the relations given above. Notice that some solutions are identical such as $w_5 = w_3$ and $w_6 = w_4$. It is obvious that the tanh-coth method gives different solutions compared to that obtained by the sine-cosine method. Moreover, other solutions can be obtained if $c > 0$ for some solutions, and if $c < 0$ for the remaining solutions. This is left as an exercise to the reader.

15.3 The Benjamin-Bona-Mahony Equation

The regularized long-wave (RLW) equation is given by

$$u_t + au_x + 2uu_x + bu_{xxt} = 0, \tag{15.46}$$

where a and b are constants. For $a = 1$ and $b = 1$, Eq. (15.46) is reduced to the Benjamin-Bona-Mahony (BBM) equation. The BBM equation [2] (or also the RLW equation) was introduced as an alternative to the KdV equation for describing unidirectional propagation of weakly long dispersive waves [2] on inviscid fluids. Eq. (15.46) is obtained from the KdV equation by replacing the third-order derivative u_{xxx} by a mixed derivative u_{xxt}. This change results in a bounded dispersion relation, whereas the KdV equation possesses an unbounded dispersion relation. The BBM equation is a superior model for long waves, and the boundedness of the dispersion relation of the BBM equation was used to utilize the regularity results for its solutions. Unlike the KdV equation which is completely solvable by the inverse scattering method, the BBM equation is found not to be completely integrable, and therefore it does not have N-soliton solutions. Moreover, the BBM equation has a stable solitary wave solution and dispersive property.

It is interesting to point out that like the KdV equation, the BBM equation was formally derived to describe an approximation for surface water waves in a uniform channel. Moreover, the BBM equation covers also, in addition to the surface waves of long wavelength in liquids, it covers hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, and acoustic-gravity waves in compressible fluids. The wide applicability of the BBM equation has attracted a considerable size of attention from researchers. In what follows we will apply the sine-cosine and the tanh-coth methods to handle the BBM equation.

15.3.1 Using the Sine-cosine Method

The wave variable $\xi = x - ct$ carries out Eq. (15.46) to the ODE

$$(a-c)u' + (u^2)' - bcu''' = 0. \tag{15.47}$$

Integrating (15.47), setting the constant of integration to zero, we obtain

$$(a-c)u + u^2 - bcu'' = 0. \tag{15.48}$$

Substituting the cosine assumption into (15.48) gives

$$\begin{aligned}(a-c)\lambda\cos^{\beta}(\mu\xi) + \lambda^2\cos^{2\beta}(\mu\xi) + bc\mu^2\beta^2\lambda\cos^{\beta}(\mu\xi) \\ - bc\lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0.\end{aligned} \tag{15.49}$$

Equating the exponents of the first and the last cosine functions, collecting the coefficients of each pair of cosine functions of like exponents, and setting it equal to zero, we obtain the following system of algebraic equations:

$$\begin{aligned} \beta-1 &\neq 0, \\ \beta-2 &= 2\beta, \\ bc\mu^2\beta^2 &= c-a, \\ bc\lambda\mu^2\beta(\beta-1) &= \lambda^2. \end{aligned} \tag{15.50}$$

This system gives

$$\begin{aligned} \beta &= -2, \\ \mu &= \frac{1}{2}\sqrt{\frac{c-a}{bc}}, \frac{c-a}{bc} > 0, \\ \lambda &= \frac{3}{2}(c-a), \end{aligned} \tag{15.51}$$

that can also be obtained by using the sine method. Based on this result, we obtain the following periodic solutions

$$u_1(x,t) = \frac{3}{2}(c-a)\sec^2\left[\frac{1}{2}\sqrt{\frac{c-a}{bc}}(x-ct)\right], \tag{15.52}$$

and

$$u_2(x,t) = \frac{3}{2}(c-a)\csc^2\left[\frac{1}{2}\sqrt{\frac{c-a}{bc}}(x-ct)\right], \tag{15.53}$$

valid for $\frac{c-a}{bc} > 0$. However, for $\frac{c-a}{bc} < 0$, we obtain the soliton solution

$$u_3(x,t) = \frac{3}{2}(c-a)\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{a-c}{bc}}(x-ct)\right], \tag{15.54}$$

and the travelling wave solution

$$u_4(x,t) = -\frac{3}{2}(c-a)\mathrm{csch}^2\left[\frac{1}{2}\sqrt{\frac{a-c}{bc}}(x-ct)\right]. \tag{15.55}$$

15.3.2 Using the Tanh-coth Method

Recall that the reduced BBM equation is

$$(a-c)u + u^2 - bcu'' = 0. \tag{15.56}$$

Balancing u^2 with u'' we find

$$2M = M + 2, \tag{15.57}$$

so that

$$M = 2. \tag{15.58}$$

The tanh-coth method introduces the finite expansion

$$u(x,t) = S(Y) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}. \tag{15.59}$$

Substituting (15.59) into (15.56), collecting the coefficients of each power of Y, and using *Mathematica* to solve the resulting system of algebraic equations we found that $a_1 = b_1 = 0$, where c is left as a free parameter. In addition we obtained the following sets:
(i)

$$a_0 = -\frac{3(a-c)}{2},\ \ a_2 = \frac{3(a-c)}{2},\ \ b_2 = 0,\ \ \mu = \frac{1}{2}\sqrt{\frac{a-c}{bc}},\ \ \frac{a-c}{bc} > 0. \tag{15.60}$$

(ii)

$$a_0 = \frac{(a-c)}{2},\ \ a_2 = -\frac{3(a-c)}{2},\ \ b_2 = 0,\ \ \mu = \frac{1}{2}\sqrt{\frac{c-a}{bc}},\ \ \frac{c-a}{bc} > 0. \tag{15.61}$$

(iii)

$$a_0 = -\frac{3(a-c)}{2},\ \ a_2 = 0,\ \ b_2 = \frac{3(a-c)}{2},\ \ \mu = \frac{1}{2}\sqrt{\frac{a-c}{bc}},\ \ \frac{a-c}{bc} > 0. \tag{15.62}$$

(iv)

$$a_0 = \frac{(a-c)}{2},\ \ a_2 = 0,\ \ b_2 = -\frac{3(a-c)}{2},\ \ \mu = \frac{1}{2}\sqrt{\frac{c-a}{bc}},\ \ \frac{c-a}{bc} > 0. \tag{15.63}$$

In view of these results, we obtain the following soliton solutions

$$\begin{aligned} u_1(x,t) &= \frac{3(c-a)}{2}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{\frac{a-c}{bc}}\,(x-ct)\right], \quad \frac{a-c}{bc} > 0,\\ u_2(x,t) &= \frac{(a-c)}{2}\left(1 - 3\tanh^2\left[\frac{1}{2}\sqrt{\frac{c-a}{bc}}\,(x-ct)\right]\right), \quad \frac{c-a}{bc} > 0, \end{aligned} \tag{15.64}$$

and the travelling wave solutions

$$\begin{aligned} u_3(x,t) &= -\frac{3(c-a)}{2}\operatorname{csch}^2\left[\frac{1}{2}\sqrt{\frac{a-c}{bc}}\,(x-ct)\right], \quad \frac{a-c}{bc} > 0,\\ u_4(x,t) &= \frac{(a-c)}{2}\left(1 - 3\coth^2\left[\frac{1}{2}\sqrt{\frac{c-a}{bc}}\,(x-ct)\right]\right), \quad \frac{c-a}{bc} > 0. \end{aligned} \tag{15.65}$$

Notice that $\frac{a-c}{bc}$ plays a major role in describing the structure of the obtained solution, therefore the last four solutions will give the following periodic solutions:

$$\begin{aligned}
u_5(x,t) &= \frac{3(c-a)}{2}\sec^2\left[\frac{1}{2}\sqrt{\frac{c-a}{bc}}(x-ct)\right], \quad \frac{c-a}{bc}>0,\\
u_6(x,t) &= \frac{(a-c)}{2}\left(1+3\tan^2\left[\frac{1}{2}\sqrt{\frac{a-c}{bc}}(x-ct)\right]\right), \quad \frac{a-c}{bc}>0,\\
u_7(x,t) &= \frac{3(c-a)}{2}\csc^2\left[\frac{1}{2}\sqrt{\frac{c-a}{bc}}(x-ct)\right], \quad \frac{c-a}{bc}>0,\\
u_8(x,t) &= \frac{(a-c)}{2}\left(1+3\cot^2\left[\frac{1}{2}\sqrt{\frac{a-c}{bc}}(x-ct)\right]\right), \quad \frac{a-c}{bc}>0.
\end{aligned} \tag{15.66}$$

Remark: It is interesting to point out that the solutions obtained by the tanh-coth method includes the solutions obtained by the sine-cosine method among others. This is true if M is even. However, for M is odd, the two methods give different solutions as discussed before and will be emphasized later.

15.4 The Medium Equal Width (MEW) Equation

The medium equal width equation [17] is given by

$$u_t+3u^2u_x-au_{xxt}=0. \tag{15.67}$$

The MEW equation (15.67), which is related to the RLW equation, has solitary waves with both positive and negative amplitudes, all of which have the same width. The MEW equation is a nonlinear wave equation with cubic nonlinearity with a pulselike solitary wave solution. This equation appears in many physical applications and is used as a model for nonlinear dispersive waves. The equation gives rise to equal width undular bore. As stated before, this equation will be approached by using the sine-cosine method and the tanh-coth method.

15.4.1 Using the Sine-cosine Method

Using the wave variable $\xi=x-ct$ converts (15.67) to an ODE

$$-cu+u^3+acu''=0, \tag{15.68}$$

obtained after integrating once and setting the constant of integration to zero. Substituting the cosine ansatz into (15.68) gives

$$
\begin{aligned}
-c\lambda \cos^{\beta}(\mu\xi) + \lambda^3 \cos^{3\beta}(\mu\xi) - ac\mu^2\beta^2\lambda \cos^{\beta}(\mu\xi) \\
+ac\lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0.
\end{aligned} \tag{15.69}
$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following system of algebraic equations:

$$
\beta - 1 \neq 0, \quad \beta - 2 = 3\beta, \quad ac\mu^2\beta^2 = -c, \quad ac\lambda\mu^2\beta(\beta-1) = -\lambda^3. \tag{15.70}
$$

Solving the last system yields

$$
\begin{aligned}
\beta &= -1, \\
\mu &= \frac{1}{\sqrt{-a}}, \quad a < 0, \\
\lambda &= \sqrt{2c}.
\end{aligned} \tag{15.71}
$$

The last results can be easily obtained if we also use the sine ansatz. Consequently, the following periodic solutions

$$
u_1(x,t) = \sqrt{2c}\csc\left[\frac{1}{\sqrt{-a}}(x-ct)\right], \quad 0 < \frac{1}{\sqrt{-a}}(x-ct) < \pi, \tag{15.72}
$$

and

$$
u_2(x,t) = \sqrt{2c}\sec\left[\frac{1}{\sqrt{-a}}(x-ct)\right], \quad \left|\frac{1}{\sqrt{-a}}(x-ct)\right| < \frac{\pi}{2}, \tag{15.73}
$$

are readily obtained. However, for $a > 0, c < 0$, we obtain the travelling wave solutions

$$
u_3(x,t) = \sqrt{-2c}\,\mathrm{csch}\left[\frac{1}{\sqrt{a}}(x-ct)\right], \tag{15.74}
$$

and for $a > 0$ and $c > 0$ we obtain the soliton solution

$$
u_4(x,t) = \sqrt{2c}\,\mathrm{sech}\left[\frac{1}{\sqrt{a}}(x-ct)\right]. \tag{15.75}
$$

15.4.2 Using the Tanh-coth Method

Recall that the wave variable $\xi = x - ct$ converts the MEW equation to an ODE

$$
-cu + u^3 + acu'' = 0, \tag{15.76}
$$

obtained after integrating once and setting the constant of integration to zero. Balancing u^3 with u'' gives $M = 1$. Consequently, the tanh-coth method introduces the finite expansion

$$
u(x,t) = S(Y) = a_0 + a_1 Y + \frac{b_1}{Y}. \tag{15.77}
$$

Substituting (15.77) into (15.76), collecting the coefficients of each power of Y, and proceeding as before, we obtain the following sets:

$$a_0 = 0, \quad a_1 = \sqrt{c}, \quad b_1 = 0, \quad \mu = \frac{1}{\sqrt{-2a}}, \quad a < 0, \tag{15.78}$$

and

$$a_0 = 0, \quad a_1 = 0, \quad b_1 = \sqrt{c}, \quad \mu = \frac{1}{\sqrt{-2a}}, \quad a < 0, \tag{15.79}$$

This in turn gives the following kink solution

$$u_5(x,t) = \sqrt{c}\tanh\left(\frac{1}{\sqrt{-2a}}(x-ct)\right), \quad a < 0, \tag{15.80}$$

and the travelling wave solution

$$u_6(x,t) = \sqrt{c}\coth\left(\frac{1}{\sqrt{-2a}}(x-ct)\right), \quad a < 0, \tag{15.81}$$

However, for $a > 0, c < 0$, we obtain the solutions

$$u_7(x,t) = \sqrt{-c}\tan\left(\frac{1}{\sqrt{2a}}(x-ct)\right), \tag{15.82}$$

and

$$u_8(x,t) = \sqrt{-c}\cot\left(\frac{1}{\sqrt{2a}}(x-ct)\right). \tag{15.83}$$

Remark It is interesting to emphasize the fact indicated before that the solutions obtained by the sine-cosine method are included in the solutions obtained by the tanh-coth method only if the parameter M is even. However, the two methods give different solutions if M is odd.

15.5 The Kawahara and the Modified Kawahara Equations

The standard Kawahara equation [9] is a fifth-order KdV equation of the form

$$u_t + 6uu_x + u_{xxx} - u_{xxxxx} = 0, \tag{15.84}$$

that describes a model for plasma waves, capillary-gravity water waves. The Kawahara equation appears in the theory of shallow water waves with surface tension and in the theory of magneto-acoustic waves in a cold collision free plasma. Kawahara studied this equation numerically and observed that this equation has both oscillatory and monotone solitary wave solutions. The inclusion of a fifth-order dispersive term u_{xxxxx} is necessary to model magneto acoustic waves. The Kawahara equation also contains the simplest kinematic term uu_x that appears also in KdV, Burgers, Kuramoto-Sivashinsky equations, and in the description of weak turbulence.

Moreover, we will also study the modified Kawahara equation

$$u_t + 6u^2u_x + u_{xxx} - u_{xxxxx} = 0. \tag{15.85}$$

Equation (15.85) was proposed first by Kawahara [9] as an important dispersive equation. It contains two dispersive terms u_{xxx} and u_{xxxxx} and also the nonlinear term $(u^3)_x$. It is also called the singularly perturbed KdV equation. It also appears in the theory of shallow water waves with surface tension and in the theory of magneto-acoustic waves in plasmas.

15.5.1 The Kawahara Equation

We first study the standard Kawahara equation (15.84). Using the transformation $u(x,t) = u(\xi), \xi = x - ct$ and integrating once, Eq. (15.84) is transformed to the ODE

$$-cu + 3u^2 + u'' - u^{(4)} = 0. \tag{15.86}$$

Balancing $u^{(4)}$ with u^2 gives $M = 4$. Because M is even, the tanh-coth method and the sine-cosine method gives the same solutions, therefore only the tanh-coth method will be used. The tanh-coth method admits the use of the finite expansion

$$u(x,t) = S(Y) = \sum_{i=0}^{4} a_i Y^i + \sum_{i=1}^{4} b_i Y^{-i}. \tag{15.87}$$

Substituting (15.87) into (15.86), collecting the coefficients of each power of Y, and proceeding as before, we found that $a_1 = a_3 = b_1 = b_3 = 0$, and the following sets:
(i)

$$a_0 = \frac{35}{338}, \quad a_2 = -\frac{35}{338}, \quad a_4 = \frac{35}{338}, \quad b_2 = b_4 = 0, \quad c = \frac{36}{169}, \quad \mu = \frac{1}{2\sqrt{13}}, \tag{15.88}$$

(ii)

$$a_0 = \frac{11}{338}, \quad a_2 = -\frac{35}{169}, \quad a_4 = \frac{35}{338}, \quad b_2 = b_4 = 0, \quad c = -\frac{36}{169}, \quad \mu = \frac{1}{2\sqrt{13}}, \tag{15.89}$$

(iii)

$$a_0 = \frac{35}{338}, \quad a_2 = a_4 = 0, \quad b_2 = -\frac{35}{338}, \quad b_4 = \frac{35}{338}, \quad c = \frac{36}{169}, \quad \mu = \frac{1}{2\sqrt{13}}, \tag{15.90}$$

(iv)

$$a_0 = \frac{11}{338}, \quad a_2 = a_4 = 0, \quad b_2 = -\frac{35}{169}, \quad b_4 = \frac{35}{338}, \quad c = -\frac{36}{169}, \quad \mu = \frac{1}{2\sqrt{13}}. \tag{15.91}$$

In view of these results we obtain the soliton solutions

$$\begin{aligned} u_1(x,t) &= \frac{35}{338}\operatorname{sech}^4\left[\frac{1}{2\sqrt{13}}\left(x - \frac{36}{169}t\right)\right], \\ u_2(x,t) &= \frac{1}{338}\left(11 - 70\tanh^2\left[\frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)\right]\right. \\ &\quad \left. +35\tanh^4\left[\frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)\right]\right), \end{aligned} \tag{15.92}$$

and the travelling wave solutions

$$\begin{aligned} u_3(x,t) &= \frac{35}{338}\operatorname{csch}^4\left[\frac{1}{2\sqrt{13}}\left(x - \frac{36}{169}t\right)\right], \\ u_4(x,t) &= \frac{1}{338}\left(11 - 70\coth^2\left[\frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)\right]\right. \\ &\quad \left. +35\coth^4\left[\frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)\right]\right). \end{aligned} \tag{15.93}$$

15.5.2 The Modified Kawahara Equation

We next consider the modified Kawahara equation (15.85). Using the transformation $u(x,t) = u(\xi), \xi = x - ct$ and integrating once, Eq. (15.85) is transformed to the ODE

$$-cu + 2u^3 + u'' - u^{(4)} = 0. \tag{15.94}$$

Balancing $u^{(4)}$ with u^3 gives $M = 2$. We then use the finite expansion

$$u(x,t) = S(Y) = \sum_{i=0}^{2} a_i Y^i + \sum_{i=1}^{2} b_i Y^{-i}. \tag{15.95}$$

Substituting (15.95) into (15.94), collecting the coefficients of each power of Y, and solving the system of coefficients, we found that $a_1 = b_1 = 0$, and the following sets:

(i)

$$a_0 = -\frac{1}{2}\sqrt{\frac{3}{5}}, \;\; a_2 = \frac{1}{2}\sqrt{\frac{3}{5}}, \;\; b_2 = 0, \;\; c = \frac{4}{25}, \;\; \mu = \frac{1}{2\sqrt{5}}, \tag{15.96}$$

(ii)

$$a_0 = -\frac{1}{2}\sqrt{\frac{3}{5}}, \;\; a_2 = 0, \;\; b_2 = \frac{1}{2}\sqrt{\frac{3}{5}}, \;\; c = \frac{4}{25}, \;\; \mu = \frac{1}{2\sqrt{5}}, \tag{15.97}$$

(iii)

$$a_0 = 0, \;\; a_2 = \frac{1}{4}\sqrt{\frac{3}{5}}, \;\; b_2 = -\frac{1}{4}\sqrt{\frac{3}{5}}, \;\; c = -\frac{11}{100}, \;\; \mu = \frac{i}{2\sqrt{10}}. \tag{15.98}$$

In view of these results we obtain the bell-shaped soliton solution

$$u_1(x,t) = -\frac{1}{2}\sqrt{\frac{3}{5}}\operatorname{sech}^2\left[\frac{1}{2\sqrt{5}}\left(x - \frac{4}{25}t\right)\right], \tag{15.99}$$

and the solutions

$$\begin{aligned} u_2(x,t) &= -\frac{1}{2}\sqrt{\frac{3}{5}}\operatorname{csch}^2\left[\frac{1}{2\sqrt{5}}\left(x - \frac{4}{25}t\right)\right], \\ u_3(x,t) &= -\frac{1}{4}\sqrt{\frac{3}{5}}\left(\tan^2\left[\frac{1}{2\sqrt{10}}\left(x + \frac{11}{100}t\right)\right]\right. \\ &\quad \left. - \cot^2\left[\frac{1}{2\sqrt{10}}\left(x + \frac{11}{100}t\right)\right]\right). \end{aligned} \tag{15.100}$$

15.6 The Kadomtsev-Petviashvili (KP) Equation

This section is concerned with the single soliton and the multiple-soliton solutions of the Kadomtsev-Petviashvili (KP) equation [8]

$$(u_t + 6uu_x + u_{xxx})_x + \lambda u_{yy} = 0, \tag{15.101}$$

where $u = u(x,y,t)$ is a real-valued function of two spatial variables x and y, and one time variable t, and λ is a constant scalar. When $\lambda = 0$, Eq. (15.101) reduces to the KdV equation. When $\lambda < 0$, the equation is known as the KP-I equation which is a good model when surface tension is strong and dominates in very shallow water. However, for $\lambda > 0$, the equation is called the KP-II equation which is a good model when surface tension is weak or absent. In other words, the coefficients $\lambda > 0$ and $\lambda < 0$ are used for weak surface tension and strong surface tension respectively. This means that the two KP equations have different physical structures and different properties.

The KP equation is used to model shallow-water waves with weakly non-linear restoring forces. It is also used to model waves in ferromagnetic media. The KP equation [8] is a model for shallow long waves in the x direction with some mild dispersion in the y direction. It is a natural generalization of the KdV equation and it is a completely integrable equation by the inverse scattering transform method. Kadomtsev and Petviashvili [8] generalized the KdV equation from (1+1) to (2+1) dimensions. They developed this equation when they relaxed the restriction that the waves be strictly one-dimensional of the KdV equation.

We first aim to use the tanh-coth method to determine single soliton and periodic solutions for the KP equation. Unlike previous sections where only the tanh-coth method is used, the Hirota's direct method, based on the bilinear formalism, combined with the simplified version [5,15] of this bilinear formalism, will be used to determine multi-soliton solutions for the KP equation.

15.6.1 Using the Tanh-coth Method

The KP equation

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0, \tag{15.102}$$

will be converted to the ODE

$$(r^2 - c)u + 3u^2 + u'' = 0, \tag{15.103}$$

obtained upon using $\xi = x + ry - ct$, integrating twice, and using $\lambda = 1$. Balancing the nonlinear term u^2 with the highest order derivative u'' gives $M = 2$. The tanh-coth method admits the use of the substitution

$$u(x,t) = S(Y) = \sum_{i=0}^{2} a_i Y^i + \sum_{i=1}^{2} b_i Y^{-i}. \tag{15.104}$$

Substituting (15.104) into (15.103), collecting the coefficients of each power of $Y^i, 0 \leqslant i \leqslant 8$, setting each coefficient to zero, and solving the resulting system of algebraic equations, we found that $a_1 = b_1 = 0$ and the following sets of solutions:
(i)

$$a_0 = \frac{c - r^2}{2},\ a_2 = -\frac{c - r^2}{2},\ b_2 = 0,\ \mu = \frac{1}{2}\sqrt{c - r^2},\ c > r^2. \tag{15.105}$$

(ii)

$$a_0 = -\frac{c - r^2}{6},\ a_2 = \frac{c - r^2}{2},\ b_2 = 0,\ \mu = \frac{1}{2}\sqrt{r^2 - c},\ c < r^2. \tag{15.106}$$

(iii)

$$a_0 = \frac{c - r^2}{2},\ a_2 = 0,\ b_2 = -\frac{c - r^2}{2},\ \mu = \frac{1}{2}\sqrt{c - r^2},\ c > r^2. \tag{15.107}$$

(iv)

$$a_0 = -\frac{c - r^2}{6},\ a_2 = 0,\ b_2 = \frac{c - r^2}{2},\ \mu = \frac{1}{2}\sqrt{r^2 - c},\ c < r^2. \tag{15.108}$$

This in turn gives the soliton solutions

$$
\begin{aligned}
u_1(x,y,t) &= \frac{c-r^2}{2}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{c-r^2}(x+ry-ct)\right], \quad c>r^2,\\
u_2(x,y,t) &= -\frac{c-r^2}{6}(1-3\tanh^2\left[\frac{1}{2}\sqrt{r^2-c}(x+ry-ct)\right], \quad c<r^2.
\end{aligned}
\tag{15.109}
$$

Moreover, we obtain the following travelling wave solutions

$$
\begin{aligned}
u_3(x,y,t) &= -\frac{c-r^2}{2}\operatorname{csch}^2\left[\frac{1}{2}\sqrt{c-r^2}(x+ry-ct)\right], \quad c>r^2,\\
u_4(x,y,t) &= -\frac{c-r^2}{6}(1-3\coth^2\left[\frac{1}{2}\sqrt{r^2-c}(x+ry-ct)\right], \quad c<r^2.
\end{aligned}
\tag{15.110}
$$

The tanh-coth method gives the solutions u_1 and u_2, each is a single soliton solution. The parameter c plays an important role in the physical structure of the solutions obtained in (15.110). Consequently, we obtain the following plane periodic solutions

$$
\begin{aligned}
u_5(x,y,t) &= \frac{c-r^2}{2}\sec^2\left[\frac{1}{2}\sqrt{r^2-c}(x+ry-ct)\right], \quad c<r^2,\\
u_6(x,y,t) &= -\frac{c-r^2}{6}\left(1+3\tan^2\left[\frac{1}{2}\sqrt{c-r^2}(x+ry-ct)\right]\right), \quad c>r^2,\\
u_7(x,y,t) &= \frac{c-r^2}{2}\csc^2\left[\frac{1}{2}\sqrt{r^2-c}(x+ry-ct)\right], \quad c<r^2,\\
u_8(x,y,t) &= -\frac{c-r^2}{6}\left(1+3\cot^2\left[\frac{1}{2}\sqrt{c-r^2}(x+ry-ct)\right]\right), \quad c>r^2.
\end{aligned}
\tag{15.111}
$$

15.6.2 Multiple-soliton Solutions of the KP Equation

We now examine multiple-soliton solutions of the KP equation

$$(u_t+6uu_x+u_{xxx})_x \pm u_{yy} = 0. \tag{15.112}$$

Hirota [6,7] introduced the direct method that usually leads to a bilinear form, if such a form exists. It was shown by this method that soliton solutions are just polynomials of exponentials. The direct method uses the dependent variable transformation

$$u(x,y,t) = 2(\ln(f))_{xx}, \tag{15.113}$$

that transforms (15.112) into the *bilinear form*

$$B(f,f) = (D_x^4 + D_xD_t \pm D_y^2)(f\cdot f) = 0, \tag{15.114}$$

or equivalently

$$[f(f_{xt}+f_{4x}\pm f_{2y})] - \left[f_xf_t+4f_xf_{3x}-3f_{2x}^2 \pm f_y^2\right] = 0. \tag{15.115}$$

Hereman *et. al.* [5] introduced a simplified version of Hirota's method, that was introduced in Chapter 13, where exact solitons can be obtained by solving a perturbation scheme using a symbolic manipulation package, and without any need to use bilinear forms. In what follows, we summarize the main steps of the simplified version of Hirota's method.

Equation (15.115) can be decomposed into linear operator L and nonlinear operator N defined by

$$\begin{aligned} L &= \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} \pm \frac{\partial^2}{\partial y^2}, \\ N(f,f) &= -f_x f_t - 4 f_x f_{3x} + 3 f_{2x} f_{2x} \pm f_y^2. \end{aligned} \tag{15.116}$$

We next assume that $f(x,y,t)$ has a perturbation expansion of the form

$$f(x,y,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,y,t), \tag{15.117}$$

where ε is a non small formal expansion parameter. Following Hirota's method [6,7] and the simplified version introduced in [5], we substitute (15.117) into (15.116) and equate to zero the powers of ε to get:

$$O(\varepsilon^1) : Lf_1 = 0, \tag{15.118}$$

$$O(\varepsilon^2) : Lf_2 = -N(f_1, f_1), \tag{15.119}$$

$$O(\varepsilon^3) : Lf_3 = -f_1 L f_2 - N(f_1, f_2) - N(f_2, f_1), \tag{15.120}$$

$$\begin{aligned} O(\varepsilon^4) : Lf_4 = &-f_1 L f_3 - f_2 L f_2 - f_3 L f_1 - N(f_1, f_3) \\ &- N(f_2, f_2) - N(f_2, f_1), \end{aligned} \tag{15.121}$$

$$\vdots \tag{15.122}$$

$$O(\varepsilon^n) : Lf_n = -\sum_{j=1}^{n-1} [f_j L f_{n-1} + N(f_j, f_{n-j})] = 0.$$

The N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i), \tag{15.123}$$

where

$$\theta_i = k_i x + m_i y - c_i t, \tag{15.124}$$

where k_i, m_i and c_i are arbitrary constants. Substituting (15.123) into (15.119) gives the dispersion relation

$$c_i = \frac{k_i^4 \pm m_i^2}{k_i}, \tag{15.125}$$

and in view of this result we obtain

$$\theta_i = k_i x + m_i y - \frac{k_i^4 \pm m_i^2}{k_i} t. \tag{15.126}$$

This means that

$$f_1 = \exp(\theta_1) = \exp\left(k_1 x + m_1 y - \frac{k_1^4 \pm m_1^2}{k_1}\right), \tag{15.127}$$

obtained by using $N = 1$ in (15.123).

Consequently, for the one-soliton solution, we set

$$f = 1 + \exp(\theta_1) = 1 + \exp\left(k_1 x + m_1 y - \frac{k_1^4 \pm m_1^2}{k_1} t\right), \tag{15.128}$$

where we set $\varepsilon = 1$. The one soliton solution for the KP equation is obtained by recalling that $u(x,t) = 2(\ln f)_{xx}$, therefore we obtain

$$u(x,y,t) = \frac{2k_1^2 \exp\left(k_1 x + m_1 y - \frac{k_1^4 \pm m_1^2}{k_1} t\right)}{\left(1 + \exp\left(k_1 x + m_1 y - \frac{k_1^4 \pm m_1^2}{k_1} t\right)\right)^2}, \tag{15.129}$$

or equivalently

$$u(x,y,t) = \frac{k_1^2}{2} \operatorname{sech}^2\left[\frac{1}{2}\left(k_1 x + m_1 y - \frac{k_1^4 \pm m_1^2}{k_1} t\right)\right]. \tag{15.130}$$

To determine the two-soliton solutions, we first set $N = 2$ in (15.123) to get

$$f_1 = \exp(\theta_1) + \exp(\theta_2), \tag{15.131}$$

and accordingly we have

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + f_2(x,y,t). \tag{15.132}$$

To determine f_2, we substitute the last equation into (15.120) to obtain

$$f_2 = \sum_{1 \leqslant i < j \leqslant 2} a_{ij} \exp(\theta_1 + \theta_2), \tag{15.133}$$

where

$$a_{12} = \frac{3k_1^2 k_2^2 (k_1 - k_2)^2 - (k_1 m_2 - k_2 m_1)^2}{3k_1^2 k_2^2 (k_1 + k_2)^2 - (k_1 m_2 - k_2 m_1)^2}, \tag{15.134}$$

and θ_1 and θ_2 are given above in (15.124). For the two-soliton solutions we use $1 \leqslant i < j \leqslant 2$, and therefore we obtain

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12} \exp(\theta_1 + \theta_2), \tag{15.135}$$

or equivalently

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + \frac{3k_1^2k_2^2(k_1-k_2)^2 - (k_1m_2 - k_2m_1)^2}{3k_1^2k_2^2(k_1+k_2)^2 - (k_1m_2 - k_2m_1)^2}\exp(\theta_1+\theta_2). \tag{15.136}$$

To determine the two-soliton solutions for the KP equation explicitly, we use (15.113) for the function f in (15.136).

Similarly, we can determine f_3. Proceeding as before, we therefore set

$$\begin{aligned} f_1(x,y,t) &= \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3), \\ f_2(x,y,t) &= a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_2) + a_{13}\exp(\theta_1+\theta_3), \end{aligned} \tag{15.137}$$

and accordingly we have

$$\begin{aligned} f(x,y,t) = {} & 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ & + a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_3) + a_{13}\exp(\theta_1+\theta_3) \\ & + f_3(x,y,t). \end{aligned} \tag{15.138}$$

Substituting (15.138) into (15.121) and proceeding as before we find

$$f_3 = b_{123}\exp(\theta_1+\theta_2+\theta_3), \tag{15.139}$$

where

$$a_{ij} = \frac{3k_i^2k_j^2(k_i-k_j)^2 - (k_im_j - k_jm_i)^2}{3k_i^2k_j^2(k_i+k_j)^2 - (k_im_j - k_jm_i)^2}, \quad 1 \leqslant i < j \leqslant 3, \tag{15.140}$$

and

$$b_{123} = a_{12}a_{13}a_{23}, \tag{15.141}$$

and θ_1, θ_2 and θ_3 are given above in (15.124). For the three-soliton solutions we use $1 \leqslant i < j \leqslant 3$, we therefore obtain

$$\begin{aligned} f = {} & 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ & + a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3) \\ & + b_{123}\exp(\theta_1+\theta_2+\theta_3). \end{aligned} \tag{15.142}$$

To determine the three-soliton solutions explicitly, we use (15.113) for the function f in (15.142).

Similarly, we can determine f_4. Proceeding as before we set

$$f_4 = c_{1234}\exp(\theta_1+\theta_2+\theta_3+\theta_4), \tag{15.143}$$

where

$$c_{1234} = a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}, \tag{15.144}$$

and $\theta_i, 1 \leqslant i \leqslant 4$ are given above in (15.124). For the four-soliton solution we use $1 \leqslant i < j \leqslant 4$, we therefore obtain

$$\begin{aligned} f = {} & 1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3)+\exp(\theta_4) \\ & +a_{12}\exp(\theta_1+\theta_2)+a_{13}\exp(\theta_1+\theta_3)+a_{14}\exp(\theta_1+\theta_4) \\ & +a_{23}\exp(\theta_2+\theta_3)+a_{24}\exp(\theta_2+\theta_4)+a_{34}\exp(\theta_3+\theta_4) \\ & +b_{123}\exp(\theta_1+\theta_2+\theta_3)+b_{124}\exp(\theta_1+\theta_2+\theta_4)+b_{134}\exp(\theta_1+\theta_3+\theta_4) \\ & +b_{234}\exp(\theta_2+\theta_3+\theta_4)+c_{1234}\exp(\theta_1+\theta_2+\theta_3+\theta_4), \end{aligned} \tag{15.145}$$

where

$$\begin{aligned} a_{ij} &= a_{ij} = \frac{3k_i^2k_j^2(k_i-k_j)^2-(k_im_j-k_jm_i)^2}{3k_i^2k_j^2(k_i+k_j)^2-(k_im_j-k_jm_i)^2}, \quad 1\leqslant i<j\leqslant 4, \\ b_{ijr} &= a_{ij}a_{ir}a_{jr}, \quad 1\leqslant i<j<r\leqslant 4, \\ c_{1234} &= a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}. \end{aligned} \tag{15.146}$$

To determine the four-solitons solution explicitly, we use (15.113) for the function f in (15.145).

We formally justified the N-soliton solutions for $N=1,2,3,4$. This means that the multiple-soliton solutions exist for $N\geqslant 1$, but the computational work becomes more and more exhausting. We again emphasize the following conclusions that were made before:

(i) the soliton solutions are just polynomials of exponentials as emphasized by Hirota [6,7], and

(ii) the three-soliton solutions and the higher level soliton solution as well, do not contain any new free parameters other than a_{ij} derived for the two-soliton solutions.

15.7 The Zakharov-Kuznetsov (ZK) Equation

The Zakharov-Kuznetsov (ZK) equation [20] is given by

$$u_t+auu_x+(\nabla^2 u)_x=0, \tag{15.147}$$

where $\nabla^2=\partial_x^2+\partial_y^2+\partial_z^2$ is the isotropic Laplacian. The ZK equation is a generalization of the KdV equation. The ZK equation governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field. The ZK equation, which is a more isotropic, was first derived for describing weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma in two dimensions. Unlike the KP equation, the ZK equation is not integrable by the inverse scattering transform method. This means that we cannot determine N-soliton solutions for this equation. It was found that the solitary wave solutions of the ZK equation are inelastic.

In this section we employ the tanh-coth method to the Zakharov-Kuznetsov equation in the $(2+1)$ dimensions, two spatial and one time variables:

$$u_t+auu_x+b\left(u_{xx}+u_{yy}\right)_x=0, \tag{15.148}$$

where a and b are constants. Using the wave variable $\xi = x+y-ct$ carries the PDE (15.148) into the ODE

$$-cu' + \frac{a}{2}(u^2)' + 2bu''' = 0, \tag{15.149}$$

where by integrating (15.149) and neglecting the constant of integration we obtain

$$-cu + \frac{a}{2}u^2 + 2bu'' = 0. \tag{15.150}$$

Balancing u'' with u^2 in (15.150) gives $M = 2$. The tanh-coth method admits the use of the finite expansion

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{2} a_k Y^k + \sum_{k=1}^{2} b_k Y^{-k}, \tag{15.151}$$

Substituting (15.151) into (15.150), collecting the coefficients of Y, setting it equal to zero, and solving the resulting system we find the following sets of solutions
(i)

$$a_0 = \frac{3c}{a},\ a_1 = 0,\ a_2 = -\frac{3c}{a},\ b_1 = 0,\ b_2 = 0,\ \mu = \frac{1}{2}\sqrt{\frac{c}{2b}}. \tag{15.152}$$

(ii)

$$a_0 = \frac{3c}{a},\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = -\frac{3c}{a},\ \mu = \frac{1}{2}\sqrt{\frac{c}{2b}}. \tag{15.153}$$

(iii)

$$a_0 = \frac{3c}{2a},\ a_1 = 0,\ a_2 = -\frac{3c}{4a},\ b_1 = 0,\ b_2 = -\frac{3c}{4a},\ \mu = \frac{1}{4}\sqrt{\frac{c}{2b}}. \tag{15.154}$$

(iv)

$$a_0 = -\frac{c}{a},\ a_1 = 0,\ a_2 = \frac{3c}{a},\ b_1 = 0,\ b_2 = 0,\ \mu = \frac{1}{2}\sqrt{-\frac{c}{2b}}. \tag{15.155}$$

(v)

$$a_0 = -\frac{c}{a},\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = \frac{3c}{a},\ \mu = \frac{1}{2}\sqrt{-\frac{c}{2b}}. \tag{15.156}$$

(vi)

$$a_0 = \frac{c}{2a},\ a_1 = 0,\ a_2 = \frac{3c}{4a},\ b_1 = 0,\ b_2 = \frac{3c}{4a},\ \mu = \frac{1}{4}\sqrt{-\frac{c}{2b}}. \tag{15.157}$$

For $\frac{c}{b} > 0$, the sets (15.152), (15.153), and (15.154) give the soliton solution

$$u_1(x,y,t) = \frac{3c}{a}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{\frac{c}{2b}}(x+y-ct)\right], \tag{15.158}$$

and the travelling wave solutions

$$u_2(x,y,t) = -\frac{3c}{a}\operatorname{csch}^2\left[\frac{1}{2}\sqrt{\frac{c}{2b}}(x+y-ct)\right], \tag{15.159}$$

and

$$u_3(x,y,t) = \frac{3c}{4a}\left(2-\tanh^2\left[\frac{1}{4}\sqrt{\frac{c}{2b}}(x+y-ct)\right]-\coth^2\left[\frac{1}{4}\sqrt{\frac{c}{2b}}(x+y-ct)\right]\right). \tag{15.160}$$

However, for $\frac{c}{b}<0$, the sets (15.152), (15.153), and (15.154) give the periodic solutions

$$u_4(x,y,t) = \frac{3c}{a}\sec^2\left[\frac{1}{2}\sqrt{-\frac{c}{2b}}(x+y-ct)\right], \tag{15.161}$$

$$u_5(x,y,t) = \frac{3c}{a}\csc^2\left[\frac{1}{2}\sqrt{-\frac{c}{2b}}(x+y-ct)\right], \tag{15.162}$$

and

$$u_6(x,y,t) = \frac{3c}{4a}\left(2+\tan^2\left[\frac{1}{4}\sqrt{-\frac{c}{2b}}(x+y-ct)\right]+\cot^2\left[\frac{1}{4}\sqrt{-\frac{c}{2b}}(x+y-ct)\right]\right). \tag{15.163}$$

On the other hand, for $\frac{c}{b}<0$, the sets (15.155), (15.156), and (15.157) give the soliton solution

$$u_7(x,y,t) = -\frac{c}{a}\left(1-3\tanh^2\left[\frac{1}{2}\sqrt{-\frac{c}{2b}}(x+y-ct)\right]\right), \tag{15.164}$$

and the travelling wave solutions

$$u_8(x,y,t) = -\frac{c}{a}\left(1-3\coth^2\left[\frac{1}{2}\sqrt{-\frac{c}{2b}}(x+y-ct)\right]\right), \tag{15.165}$$

and

$$\begin{aligned} u_9(x,y,t) = {} & \frac{c}{4a}\left(2+3\tanh^2\left[\frac{1}{4}\sqrt{-\frac{c}{2b}}(x+y-ct)\right]\right. \\ & \left.+3\coth^2\left[\frac{1}{4}\sqrt{-\frac{c}{2b}}(x+y-ct)\right]\right), \end{aligned} \tag{15.166}$$

However, for $\frac{c}{b}>0$, the sets (15.155), (15.156), and (15.157) also give the solitons solutions

$$u_{10}(x,y,t) = -\frac{c}{a}\left(1+3\tan^2\left[\frac{1}{2}\sqrt{\frac{c}{2b}}(x+y-ct)\right]\right), \tag{15.167}$$

$$u_{11}(x,y,t) = -\frac{c}{a}\left(1+3\cot^2\left[\frac{1}{2}\sqrt{\frac{c}{2b}}(x+y-ct)\right]\right), \tag{15.168}$$

and

$$u_{12}(x,y,t) = \frac{c}{4a}\left(2-3\tan^2\left[\frac{1}{4}\sqrt{\frac{c}{2b}}(x+y-ct)\right]-3\cot^2\left[\frac{1}{4}\sqrt{\frac{c}{2b}}(x+y-ct)\right]\right). \tag{15.169}$$

15.8 The Benjamin-Ono Equation

The Benjamin-Ono (BO) equation [1,13] is defined by

$$u_t + 4uu_x + H(u_{xx}) = 0, \tag{15.170}$$

where H is the Hilbert transform defined by

$$H[u(x,t)] = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{u(y,t)}{y-x}dy, \tag{15.171}$$

where P refers to the principal value of the integral. The BO equation describes internal waves. It is a completely integrable equation that gives N-soliton solutions. To obtain the multiple-soliton solutions, we follow the analysis developed by Drazin [3] and Matsuno [11,12]. A solution of the BO equation which is real and finite over all x and t is of the form

$$u(x,t) = \frac{i}{2}\frac{\partial}{\partial x}\left(\ln\frac{f^*(x,t)}{f(x,t)}\right), \tag{15.172}$$

where $f^*(x,t)$ is the complex conjugate of $f(x,t)$. Matsuno [11,12] formally proved that

$$H[u(x,t)] = iu(x,t) - \frac{1}{f(x,t)}\left(\frac{\partial f(x,t)}{\partial x}\right). \tag{15.173}$$

Substituting (15.172) and (15.173) into (15.170) gives the bilinear equation

$$\mathrm{Im}\,(f_t^* f) = f_x^* f_x - \mathrm{Re}\,(f_{xx}^* f). \tag{15.174}$$

The solution of (15.174) is expressed as [11]

$$f_N = \det M_{N\times N}, \tag{15.175}$$

where the square matrix $M_{N\times N}$ is defined by

$$M_{nm} = \begin{cases} i\theta_n + 1, & \text{for}\, n = m, \\ \dfrac{2\sqrt{k_n k_m}}{k_n - k_m}, & \text{for}\, n \neq m, \end{cases} \tag{15.176}$$

where

$$\theta_n = k_n(x - k_n t - b_n), \tag{15.177}$$

where k_n and b_n are arbitrary constants, and $k_n \neq k_m$ for $n \neq m$. Setting $N = 1$ in (15.175) we get

$$f_1 = 1 + i\theta_1, \tag{15.178}$$

where by using (15.172) the one-soliton solution

$$u(x,t) = \frac{k_1}{k_1^2(x - k_1 t - b_1)^2 + 1}. \tag{15.179}$$

Setting $N = 2$ in (15.175) we get

$$f_2 = -\theta_1\theta_2 + i(\theta_1 + \theta_2) + a_{12}, \tag{15.180}$$

where

$$a_{nm} = \left(\frac{k_n + k_m}{k_n - k_m}\right)^2. \tag{15.181}$$

where by using (15.172) the two-soliton solutions are readily obtained. Similarly, for $N = 3$, we find

$$f_3 = -(\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1) + i(k_{12}\theta_3 + k_{23}\theta_1 + k_{31}\theta_2 - \theta_1\theta_2\theta_3) + k_{12} + k_{23} + k_{31} - 2, \tag{15.182}$$

where the three-soliton solutions can be obtained by using (15.172).

15.9 The KdV-Burgers Equation

In this section we will study the KdV-Burgers equation

$$u_t + 6uu_x + au_{xx} + u_{xxx} = 0, \quad a \neq 0. \tag{15.183}$$

The Burgers-KdV equation (15.183) arises from many physical contexts such as the propagation of undular bores in shallow water, the flow of liquids containing gas bubbles, weakly nonlinear plasma waves with certain dissipative effect, theory of ferro electricity, nonlinear circuit, and the propagation of waves in an elastic tube filled with a viscous fluid. The KdV-Burgers equation incorporates the effects of dispersion u_{xxx} and dissipation u_{xx} as well as nonlinearity uu_x.

Using the wave variable $\xi = x - ct$, Eq. (15.183) will be carried to

$$-cu + 3u^2 + au' + u'' = 0. \tag{15.184}$$

Balancing u'' with u^2 gives $M=2$. This means that we can set

$$u(x,t)=S(Y)=\sum_{r=0}^{2}a_rY^r+\sum_{r=1}^{2}b_rY^{-r}. \tag{15.185}$$

Substituting (15.185) into (15.184) and solving the resulting system for $a_0,a_1,a_1,$ $b_1,b_2,c,$ and μ, we obtain the following sets of solutions:
(i)

$$a_0=-\frac{a^2}{50},\quad a_1=\frac{a^2}{25},\ a_2=-\frac{a^2}{50},\ b_1=b_2=0,\ c=-\frac{6a^2}{25},\ \mu=\frac{a}{10}. \tag{15.186}$$

(ii)

$$a_0=\frac{3a^2}{50},\quad a_1=\frac{a^2}{25},\ a_2=-\frac{a^2}{50},\ b_1=b_2=0,\ c=\frac{6a^2}{25},\ \mu=\frac{a}{10}. \tag{15.187}$$

(iii)

$$a_0=-\frac{a^2}{50},\quad a_1=a_2=0,\ b_1=\frac{a^2}{25},\ b_2=-\frac{a^2}{50},\ c=-\frac{6a^2}{25},\ \mu=\frac{a}{10}. \tag{15.188}$$

(iv)

$$a_0=\frac{3a^2}{50},\quad a_1=a_2=0,\ b_1=\frac{a^2}{25},\ b_2=-\frac{a^2}{50},\ c=\frac{6a^2}{25},\ \mu=\frac{a}{10}. \tag{15.189}$$

(v)

$$a_0=\frac{a^2}{20},\quad a_1=b_1=\frac{a^2}{25},\ a_2=b_2=-\frac{a^2}{200},\ c=\frac{6a^2}{25},\ \mu=\frac{a}{20}. \tag{15.190}$$

(vi)

$$a_0=-\frac{3a^2}{100},\quad a_1=b_1=\frac{a^2}{50},\ a_2=b_2=-\frac{a^2}{200},\ c=-\frac{6a^2}{25},\ \mu=\frac{a}{20}. \tag{15.191}$$

In view of these results we obtain the following soliton solutions

$$\begin{aligned}
u_1&=-\frac{a^2}{50}\left(1-\tanh\left[\frac{a}{10}(x+\frac{6a^2}{25}t)\right]\right)^2,\\
u_2&=\frac{a^2}{50}\left(3+2\tanh\left[\frac{a}{10}(x-\frac{6a^2}{25}t)\right]-2\tanh^2\left[\frac{a}{10}(x-\frac{6a^2}{25}t)\right]\right),
\end{aligned} \tag{15.192}$$

and the following travelling wave solutions

$$
\begin{aligned}
u_3 &= -\frac{a^2}{50}\left(1-\coth\left[\frac{a}{10}(x+\frac{6a^2}{25}t)\right]\right)^2,\\
u_4 &= \frac{a^2}{50}\left(3+2\coth\left[\frac{a}{10}(x-\frac{6a^2}{25}t)\right]-2\coth^2\left[\frac{a}{10}(x-\frac{6a^2}{25}t)\right]\right),\\
u_5 &= \frac{a^2}{200}\left(5+4\tanh\left[\frac{a}{20}(x-\frac{6a^2}{25}t)\right]-\tanh^2\left[\frac{a}{20}(x-\frac{6a^2}{25}t)\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{a^2}{200}\left(5+4\coth\left[\frac{a}{20}(x-\frac{6a^2}{25}t)\right]-\coth^2\left[\frac{a}{20}(x-\frac{6a^2}{25}t)\right]\right),\\
u_6 = &-\frac{a^2}{200}\left(3-4\tanh\left[\frac{a}{20}(x+\frac{6a^2}{25}t)\right]+\tanh^2\left[\frac{a}{20}(x+\frac{6a^2}{25}t)\right]\right)\\
&-\frac{a^2}{200}\left(3-4\coth\left[\frac{a}{20}(x+\frac{6a^2}{25}t)\right]+\coth^2\left[\frac{a}{20}(x+\frac{6a^2}{25}t)\right]\right).
\end{aligned}
\tag{15.193}
$$

15.10 Seventh-order KdV Equation

In this section we will study other forms of the seventh-order KdV equations that are built in the Kawahara sense. The seventh-order KdV equation (sKdV)

$$
u_t + 6uu_x + u_{3x} - u_{5x} + \alpha u_{7x} = 0, \tag{15.194}
$$

where α is a nonzero constant, and $u = u(x,t)$ is a sufficiently often differentiable function, will be studied. The sech method used in [4,10] will be used to study this equation. The sKdV equation (15.194) has been introduced by Pomeau et. al [14] for discussing the structural stability of the KdV equation under singular perturbation. The sKdV equation possesses the dispersion term u_{3x} and two higher order dispersion terms, namely, u_{5x} and u_{7x}. Moreover, Eq. (15.194) has three polynomial type conserved quantities given by:

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} u\,dx,\\
I_2 &= \int_{-\infty}^{\infty} u^2\,dx,\\
I_3 &= \int_{-\infty}^{\infty}\left(-u^3+\frac{1}{2}(u_x)^2-\frac{1}{2}(u_{xx})^2+\frac{1}{2}\alpha(u_{3x})^2\right)dx,
\end{aligned}
\tag{15.195}
$$

15.10.1 The Sech Method

We begin our analysis by rewriting (15.194) as

$$
-cu + 3u^2 + u'' - u^{(4)} + \alpha u^{(6)} = 0, \tag{15.196}
$$

by using the wave variable $\xi = \mu(x-ct)$ and integrating once. Balancing the terms $u^{(6)}$ with u^2 in (15.196) we find that $M=6$. Following [4,10], we assume that the solution is of the form

$$u(\xi) = a_0 + a_1 \operatorname{sech}^6(\mu\xi). \tag{15.197}$$

Substituting (15.197) into (15.196), collecting the coefficients of sech^j, and solving the resulting system we find the following two sets of solutions
(i)

$$a_0 = 0, \quad a_1 = \frac{86625}{591361}, \quad c = \frac{180000}{591361}, \quad \mu = \frac{5}{\sqrt{1538}}, \quad \alpha = \frac{769}{2500}. \tag{15.198}$$

(ii)

$$a_0 = -\frac{60000}{591361}, \quad a_1 = \frac{86625}{591361}, \quad c = -\frac{180000}{591361}, \quad \mu = \frac{5}{\sqrt{1538}}, \quad \alpha = \frac{769}{2500}. \tag{15.199}$$

This in turn gives the soliton solutions

$$u_1(x,t) = \frac{86625}{591361}\operatorname{sech}^6\left(\frac{5}{\sqrt{1538}}\left(x - \frac{180000}{591361}t\right)\right), \tag{15.200}$$

and

$$u_2(x,t) = -\frac{60000}{591361} + \frac{86625}{591361}\operatorname{sech}^6\left(\frac{5}{\sqrt{1538}}\left(x + \frac{180000}{591361}t\right)\right). \tag{15.201}$$

In addition, we obtain the following traveling wave solutions

$$u_3(x,t) = -\frac{86625}{591361}\operatorname{csch}^6\left(\frac{5}{\sqrt{1538}}\left(x - \frac{180000}{591361}t\right)\right), \tag{15.202}$$

and

$$u_4(x,t) = -\frac{60000}{591361} - \frac{86625}{591361}\operatorname{csch}^6\left(\frac{5}{\sqrt{1538}}\left(x + \frac{180000}{591361}t\right)\right). \tag{15.203}$$

It is interesting to point out that these travelling solitary wave solutions exist only if the signs of the coefficients of the are opposite. Moreover, the solutions exist only for fixed value of α given before in (15.198).

However, if the coefficients of terms u_{3x} and u_{5x} have identical positive signs, we obtain periodic solutions that include $\sec^6(\mu\xi)$. In this case, we assume that the solution is of the form

$$u(\xi) = a_0 + a_1 \sec^6(\mu\xi). \tag{15.204}$$

Substituting (15.204) into (15.196), collecting the coefficients of $\sec^j$, and solving the resulting system we find the following two sets of solutions
(i)

$$a_0=0, \quad a_1=-\frac{86625}{591361}, \quad c=-\frac{180000}{591361}, \quad \mu=\frac{5}{\sqrt{1538}}, \quad \alpha=\frac{769}{2500}. \tag{15.205}$$

(ii)

$$a_0=\frac{60000}{591361}, \quad a_1=-\frac{86625}{591361}, \quad c=\frac{180000}{591361}, \quad \mu=\frac{5}{\sqrt{1538}}, \quad \alpha=\frac{769}{2500}. \tag{15.206}$$

This in turn gives the solutions

$$u_5(x,t)=-\frac{86625}{591361}\sec^6\left(\frac{5}{\sqrt{1538}}\left(x+\frac{180000}{591361}t\right)\right), \tag{15.207}$$

$$u_6(x,t)=-\frac{86625}{591361}\csc^6\left(\frac{5}{\sqrt{1538}}\left(x+\frac{180000}{591361}t\right)\right), \tag{15.208}$$

$$u_7(x,t)=\frac{60000}{591361}-\frac{86625}{591361}\sec^6\left(\frac{5}{\sqrt{1538}}\left(x-\frac{180000}{591361}t\right)\right), \tag{15.209}$$

and

$$u_8(x,t)=\frac{60000}{591361}-\frac{86625}{591361}\csc^6\left(\frac{5}{\sqrt{1538}}\left(x-\frac{180000}{591361}t\right)\right). \tag{15.210}$$

15.11 Ninth-order KdV Equation

In this section we will study the ninth-order KdV equation (nKdV)

$$u_t+6uu_x+u_{3x}-u_{5x}+\alpha u_{7x}+\beta u_{9x}=0, \tag{15.211}$$

where α and β are arbitrary nonzero constants, and u is a sufficiently often differentiable function. The equation is constructed using the sense of Kawahara equation. The sech method will be used again to study this equation. The nKdV equation possesses the dispersion term u_{3x} and three higher order dispersion terms, namely, u_{5x}, u_{7x} and u_{9x} and possesses polynomial type conserved quantities.

15.11.1 The Sech Method

We begin our analysis by rewriting (15.211) as

$$-cu+3u^2+u''-u^{(iv)}+\alpha u^{(vi)}+\beta u^{viii}=0, \tag{15.212}$$

by using the wave variable $\xi=\mu(x-ct)$ and integrating once. Balancing the terms $u^{(viii)}$ with u^2 in (15.212) we find

$$M+8=2M, \tag{15.213}$$

so that $M=8$. Following our discussion above, we assume that the solution is of the form

$$u(\xi)=a_0+a_1\,\mathrm{sech}^8(\mu\xi). \tag{15.214}$$

Substituting (15.214) into (15.212) and proceeding as before we find the following two sets of solutions

(i)

$$\begin{aligned} a_0 &= 0, \quad a_1=\frac{3816888075}{22609585952}, \quad c=\frac{249120900}{706549561},\\ \mu &= \frac{1}{4}\sqrt{\frac{5649}{26581}}, \quad \alpha=\frac{212648}{506527}, \quad \beta=-\frac{11304792976}{180266374449}. \end{aligned} \tag{15.215}$$

(ii)

$$\begin{aligned} a_0 &= -\frac{83040300}{706549561}, \quad a_1=\frac{3816888075}{22609585952}, \quad c=-\frac{249120900}{706549561},\\ \mu &= \frac{1}{4}\sqrt{\frac{5649}{26581}}, \quad \alpha=\frac{212648}{506527}, \quad \beta=-\frac{11304792976}{180266374449}. \end{aligned} \tag{15.216}$$

This in turn gives the soliton solutions

$$u_1(x,t)=\frac{3816888075}{22609585952}\,\mathrm{sech}^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x-\frac{249120900}{706549561}t\right)\right), \tag{15.217}$$

and

$$\begin{aligned} u_2(x,t) = &-\frac{83040300}{706549561}\\ &+\frac{3816888075}{22609585952}\,\mathrm{sech}^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x+\frac{249120900}{706549561}t\right)\right), \end{aligned} \tag{15.218}$$

and the solutions

$$u_3(x,t)=\frac{3816888075}{22609585952}\,\mathrm{csch}^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x-\frac{249120900}{706549561}t\right)\right), \tag{15.219}$$

and

$$\begin{aligned} u_4(x,t) = &-\frac{83040300}{706549561}\\ &+\frac{3816888075}{22609585952}\,\mathrm{csch}^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x+\frac{249120900}{706549561}t\right)\right). \end{aligned} \tag{15.220}$$

The obtained travelling solitary wave solutions exist only if the signs of the coefficients of the terms u_{3x} and u_{5x} are opposite. Moreover, the solutions exist only for specific values of α and β obtained above in (15.215).

However, if the coefficients of the terms u_{3x} and u_{5x} have identical positive signs we obtain periodic solutions that include $\sec^8(\mu\xi)$. To achieve our goal, we assume that the solution is of the form

$$u(\xi) = a_0 + a_1 \sec^8(\mu\xi). \tag{15.221}$$

Substituting (15.221) into (15.212) and proceeding as before we find the following two sets of solutions

(i)

$$\begin{aligned} a_0 &= 0, \quad a_1 = -\frac{3816888075}{22609585952}, \quad c = -\frac{249120900}{706549561}, \\ \mu &= \frac{1}{4}\sqrt{\frac{5649}{26581}}, \quad \alpha = \frac{212648}{506527}, \quad \beta = \frac{11304792976}{180266374449}. \end{aligned} \tag{15.222}$$

(ii)

$$\begin{aligned} a_0 &= \frac{83040300}{706549561}, \quad a_1 = -\frac{3816888075}{22609585952}, \quad c = \frac{249120900}{706549561}, \\ \mu &= \frac{1}{4}\sqrt{\frac{5649}{26581}}, \quad \alpha = \frac{212648}{506527}, \quad \beta = \frac{11304792976}{180266374449}. \end{aligned} \tag{15.223}$$

This in turn gives the solutions

$$u_5(x,t) = -\frac{3816888075}{22609585952}\sec^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x + \frac{249120900}{706549561}t\right)\right), \tag{15.224}$$

$$u_6(x,t) = -\frac{3816888075}{22609585952}\csc^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x + \frac{249120900}{706549561}t\right)\right), \tag{15.225}$$

$$u_7(x,t) = \frac{83040300}{706549561} - \frac{3816888075}{22609585952}\sec^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x - \frac{249120900}{706549561}t\right)\right), \tag{15.226}$$

and

$$u_8(x,t) = \frac{83040300}{706549561} - \frac{3816888075}{22609585952}\csc^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x - \frac{249120900}{706549561}t\right)\right). \tag{15.227}$$

References

1. T.B. Benjamin, Internal waves of permanent form in fluids of great depth, *J. Fluid Mech.*, 29, 559–592, (1967).
2. R.T. Benjamin, J.L. Bona and J.J. Mahony, Model equations for long waves in nonlinear dispersive systems, *Philos. Trans. Roy. Soc.*, 272, 47–78, (1972).
3. P.G. Drazin and R.S. Johnson, Solitons: an Introduction, Cambridge University Press, Cambridge, (1996).
4. B.R. Duffy and E.J. Parkes, Travelling solitary wave solutions to a seventh-order generalized KdV equation, *Phys. Lett.* A, 214, 271–272,(1996).
5. W. Hereman and A. Nuseir, Symbolic methods to construct exact solutions of nonlinear partial differential equations, *Math. Comput. Simulation*, 43(1), 13–27, (1997).
6. R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge (2004).
7. R. Hirota, Exact N-soliton solutions of a nonlinear wave equation, *J. Math Phys.*, 14(7), 805–809, (1973).
8. B.B. Kadomtsev and V.I. Petviashvili, On the stability of solitary waves in weakly dispersive media, *Sov. Phys. Dokl.*, 15, 539–541, (1970).
9. T. Kawahara, Oscillatory solitary waves in dispersive media, *J. Phys. Soc. Japan*, 33, 260–264, (1972).
10. W-X Ma, Travelling wave solutions to a seventh order generalized KdV equation, *Phys. Lett. A*, 180, 221–224, (1993).
11. Y. Matsuno, Bilinear Transformation Method, Academic Press, New York, (1984).
12. Y. Matsuno, Exact multi-soliton solution of the Benjamin-Ono equation, *J. Phys. A: Math. Gen.*, 12(4), 619–621, (1979).
13. H. Ono, Algebraic solitary waves in stratified fluids, *J. Phys. Soc. Japan*, 39, 1082–1091, (1975).
14. Y. Pomeau, A. Ramani and B. Grammaticos, Structural stability of the Korteweg-de Vries solitons under a singular perturbation, *Physica D*, 31(1),127–134, (1988).
15. A.M. Wazwaz, New compactons, solitons and periodic solutions for nonlinear variants of the KdV and the KP equations, *Chaos, Solitons and Compactons*, 22(1) 249–260, (2004).
16. A.M. Wazwaz, The tanh and the sine-cosine methods for the complex modified KdV and the generalized KdV equations, *Comput. Math. Applic.*, 49, 1101-1112, (2005).
17. A.M. Wazwaz, The tanh and the sine–cosine methods for a reliable treatment of the modified equal width equation and its variants, *Commun. Nonlinear Sci. Numer. Simul.*, 11(2), 148–160, (2006).
18. A.M. Wazwaz, The sine-cosine and the tanh methods: reliable tools for analytic treatment of nonlinear dispersive equations, Appl. Math. Comput., 173(1), 150-164, (2006).
19. A.M. Wazwaz, The tanh–coth method for new compactons and solitons solutions for the K(n,n) and the K(n+1,n+1) equations. *Appl. Math. Comput.*, 188, 1930–1940, (2007).
20. V.E. Zakharov and V.E. Kuznetsov, On three-dimensional solitons, *Sov. Phys.*, 39, 285–288, (1974).

Chapter 16
Boussinesq, Klein-Gordon and Liouville Equations

16.1 Introduction

In the preceeding two chapters we examined the family of the KdV and the KdV-type of equations, where the first order partial derivative u_t was involved in all these equations [1]. In this chapter, we will study the nonlinear evolution equations where each contains the second order partial derivative u_{tt} in addition to other partial derivatives. This family of nonlinear equations gained its importance because it appears in many scientific applications and physical phenomena.

The new family is of the form

$$u_{tt} - u_{xx} + P(u) = 0, \tag{16.1}$$

where $u(x,t)$ is a function of space x and time variable t. The nonlinear term $P(u)$ appears in the following forms

$$P(u) = \begin{cases} -3(u^2)_{xx} + \lambda u_{xxxx}, \\ -3(u^2)_{xx} + \lambda u_{xxtt}, \\ u - u^2, \\ e^{\pm u}, \\ e^{\pm u} + e^{-2u}, \\ \sin u, \\ \sinh u. \end{cases} \tag{16.2}$$

(i) For $P(u) = -3(u^2)_{xx} + \lambda u_{xxxx}$ we obtain the nonlinear fourth order Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} + \lambda u_{xxxx} = 0. \tag{16.3}$$

The Boussinesq equation [2] is completely integrable. As stated before, a common feature of complete integrable equations is the existence of an infinite number of conservation laws, and therefore the Boussinesq equation has N-soliton solutions as will be derived later. The equation with $\lambda = -1$ is called the bad Boussinesq equation or the ill-posed classical equation, whereas with $\lambda = 1$ is known as the good Boussinesq equation or the well-posed equation.

(ii) For $P(u) = -3(u^2)_{xx} + \lambda u_{xxtt}$, the equation is called the improved Boussinesq equation [3] given by

$$u_{tt} - u_{xx} - 3(u^2)_{xx} + \lambda u_{xxtt} = 0. \qquad (16.4)$$

An analogous characterization to that used for the Boussinesq equation can be set here. In other words, there is a bad improved Boussinesq equation and a good improved Boussinesq equation for $\lambda = -1$ and $\lambda = 1$ respectively.

(iii) For $P(u) = u - u^2$, Eq. (16.1) is called the nonlinear Klein-Gordon equation with quadratic nonlinearity [4,5,10] and given by

$$u_{tt} - u_{xx} + u - u^2 = 0. \qquad (16.5)$$

The Klein-Gordon equation appears also with other order of nonlinearities, such as $P(u) = u - u^3$, and $P(u) = u - u^n$.

(iv) For $P(u) = e^{\pm u}$, Eq. (16.1) gives the well known Liouville equation [11] given by

$$u_{tt} - u_{xx} + e^{\pm u} = 0. \qquad (16.6)$$

(v) For $P(u) = \sin u$, Eq. (16.1) gives the well known sine-Gordon equation [12,13] given by

$$u_{tt} - u_{xx} + \sin u = 0. \qquad (16.7)$$

The sine-Gordon equation is complete integrable, and the N-soliton solutions of the sine-Gordon equation [12,13] will be examined in this chapter.

(vi) For $P(u) = \sinh u$, Eq. (16.1) gives the well known sinh-Gordon equation given by

$$u_{tt} - u_{xx} + \sinh u = 0. \qquad (16.8)$$

In addition, we will study a second family of equations of the form

$$u_{xt} + P_1(u) = 0, \qquad (16.9)$$

where $u(x,t)$ is a function of space x and time variable t. The nonlinear term $P_1(u)$ appears in the following forms

$$P(u) = \begin{cases} e^u + e^{-2u}, \\ e^{-u} + e^{-2u}, \\ pe^u + qu^{-u} + re^{-2u}. \end{cases} \qquad (16.10)$$

(i) For $P_1(u) = e^u + e^{-2u}$, Eq. (16.9) gives the well known Dodd-Bullough-Mikhailov equation (DBM) [11] given by

$$u_{xt} + e^u + e^{-2u} = 0. \qquad (16.11)$$

(ii) For $P_1(u) = e^{-u} + e^{-2u}$, Eq. (16.9) gives the well known Tzitzeica-Dodd-Bullough (TDB) [11] given by

$$u_{xt} + e^{-u} + e^{-2u} = 0. \tag{16.12}$$

(iii) For $P_1(u) = pe^{u} + qe^{-u} + re^{-2u}$, Eq. (16.9) gives the well known Zhiber-Shabat equation [14] given by

$$u_{xt} + pe^{u} + qe^{-u} + re^{-2u} = 0. \tag{16.13}$$

We first begin the study of the first family of equations. Our approach stems mainly from the tanh-coth method. The Hirota's bilinear formalism [8] will be used for completely integrable equations.

16.2 The Boussinesq Equation

In this section we will study only the bad Boussinesq equation [2]

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0, \tag{16.14}$$

with $u(x,t)$ is a sufficiently often differentiable function. Eq. (16.14) was introduced by Boussinesq to describe the propagation of long waves in shallow water under gravity propagating in both directions. It also arises in other physical applications such as nonlinear lattice waves, iron sound waves in a plasma, and in vibrations in a nonlinear string. It is used in many physical applications such as the percolation of water in porous subsurface of a horizontal layer of material. This particular form (16.14) is of special interest because it is completely integrable and admits inverse scattering formalism.

Many well known methods, such as the inverse scattering transform method, the bilinear formalism, and Bäcklund transformation method were used to handle the completely integrable Boussinesq equation. In this chapter we will follow the approaches used before, namely the tanh-coth method to establish single solitary wave solutions, and the Hirota's method that will be combined with the simplified version of Hereman et. al. in [6] to determine the N-soliton solutions for the bad Boussinesq equation (16.14). However, the good Boussinesq equation or the well-posed equation can be handled in a like manner. As stated before, many other methods are introduced in many texts.

16.2.1 Using the Tanh-coth Method

The Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0, \tag{16.15}$$

will be converted to the ODE

$$(c^2 - 1)u - 3u^2 - u'' = 0, \tag{16.16}$$

obtained upon using $\xi = x - ct$ and integrating twice. Balancing the nonlinear term u^2 with the highest order derivative u'' gives

$$2M = M + 2, \tag{16.17}$$

so that

$$M = 2. \tag{16.18}$$

The tanh-coth method admits the use of the substitution

$$u(x,t) = S(Y) = \sum_{i=0}^{2} a_i Y^i + \sum_{i=0}^{2} b_i Y^{-i}. \tag{16.19}$$

Substituting (16.19) into (16.16), collecting the coefficients of each power of $Y^i, 0 \leqslant i \leqslant 8$, setting each coefficient to zero, and solving the resulting system of algebraic equations, we found that $a_1 = b_1 = 0$ and the following sets of solutions:
(i)

$$a_0 = \frac{c^2-1}{2}, \quad a_2 = -\frac{c^2-1}{2}, \quad b_2 = 0, \quad \mu = \frac{1}{2}\sqrt{c^2-1}, \quad c^2 > 1. \tag{16.20}$$

(ii)

$$a_0 = -\frac{c^2-1}{6}, \quad a_2 = \frac{c^2-1}{2}, \quad b_2 = 0, \quad \mu = \frac{1}{2}\sqrt{1-c^2}, \quad c^2 < 1. \tag{16.21}$$

(iii)

$$a_0 = \frac{c^2-1}{2}, \quad b_2 = -\frac{c^2-1}{2}, \quad a_2 = 0, \quad \mu = \frac{1}{2}\sqrt{c^2-1}, \quad c^2 > 1. \tag{16.22}$$

(iv)

$$a_0 = -\frac{c^2-1}{6}, \quad b_2 = \frac{c^2-1}{2}, \quad a_2 = 0, \quad \mu = \frac{1}{2}\sqrt{1-c^2}, \quad c^2 < 1. \tag{16.23}$$

This in turn gives the solitons solutions

$$\begin{aligned} u_1(x,t) &= \frac{c^2-1}{2}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{c^2-1}(x-ct)\right], \quad c^2 > 1, \\ u_2(x,t) &= -\frac{c^2-1}{6}\left(1 - 3\tanh^2\left[\frac{1}{2}\sqrt{1-c^2}(x-ct)\right]\right), \quad c^2 < 1. \end{aligned} \tag{16.24}$$

Moreover, the travelling wave solutions

$$
\begin{aligned}
u_3(x,t) &= -\frac{c^2-1}{2}\operatorname{csch}^2\left[\frac{1}{2}\sqrt{c^2-1}(x-ct)\right], \quad c^2>1,\\
u_4(x,t) &= -\frac{c^2-1}{6}\left(1-3\coth^2\left[\frac{1}{2}\sqrt{1-c^2}(x-ct)\right]\right), \quad c^2<1.
\end{aligned} \tag{16.25}
$$

are readily obtained.

The wave speed c plays an important role in the physical structure of the solutions obtained above. Consequently, we obtain the following plane periodic solutions

$$
\begin{aligned}
u_5(x,t) &= \frac{c^2-1}{2}\sec^2\left[\frac{1}{2}\sqrt{1-c^2}(x-ct)\right], \quad c^2<1,\\
u_6(x,t) &= -\frac{c^2-1}{6}\left(1-3\tan^2\left[\frac{1}{2}\sqrt{c^2-1}(x-ct)\right]\right), \quad c^2>1,\\
u_7(x,t) &= \frac{c^2-1}{2}\csc^2\left[\frac{1}{2}\sqrt{1-c^2}(x-ct)\right], \quad c^2<1,\\
u_8(x,t) &= -\frac{c^2-1}{6}\left(1-3\cot^2\left[\frac{1}{2}\sqrt{c^2-1}(x-ct)\right]\right), \quad c^2>1.
\end{aligned} \tag{16.26}
$$

16.2.2 Multiple-soliton Solutions of the Boussinesq Equation

In this section, we will examine multiple-soliton solutions of the Boussinesq equation

$$u_{tt}-u_{xx}-3(u^2)_{xx}-u_{xxxx}=0. \tag{16.27}$$

The Hirota's method [7,8] uses the dependent variable transformation

$$u(x,t)=2(\ln(f))_{xx}, \tag{16.28}$$

that transforms (16.27) into the *bilinear form* [9]

$$B(f,f)=(D_t^2-D_x^2-D_x^4)(f\cdot f)=0, \tag{16.29}$$

or equivalently

$$[f(f_{2t}-f_{2x}-f_{4x})]-\left[f_t^2+f_x^2+4f_xf_{3x}-3f_{2x}^2\right]=0. \tag{16.30}$$

Hereman *et. al.* [6] introduced a simplified version of Hirota's method, where exact solitons can be obtained by solving a perturbation scheme using a symbolic manipulation package, and without any need to use bilinear forms. In what follows, we summarize the main steps of the simplified version of Hirota's method.

Equation (16.30) can be decomposed into linear operator L and nonlinear operator N defined by

$$\begin{aligned} L &= \partial_{2t} - \partial_{2x} - \partial_{4x}, \\ N(f,f) &= f_t^2 + f_x^2 + 4 f_x f_{3x} - 3 f_{2x}^2. \end{aligned} \tag{16.31}$$

We next assume that $f(x,t)$ has a perturbation expansion of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \tag{16.32}$$

where ε is a non small formal expansion parameter. Following Hirota's method and the simplified version [6], we substitute (16.32) into (16.31) and equate to zero the powers of ε to get:

$$O(\varepsilon^1) : Lf_1 = 0, \tag{16.33}$$
$$O(\varepsilon^2) : Lf_2 = -N(f_1,f_1), \tag{16.34}$$
$$O(\varepsilon^3) : Lf_3 = -f_1 L f_2 - f_2 L f_1 - N(f_1,f_2) - N(f_2,f_1), \tag{16.35}$$
$$\begin{aligned} O(\varepsilon^4) : Lf_4 = &-f_1 L f_3 - f_2 L f_2 - f_3 L f_1 - N(f_1,f_3) \\ &- N(f_2,f_2) - N(f_3,f_1), \end{aligned} \tag{16.36}$$
$$\begin{aligned} &\vdots \\ O(\varepsilon^n) : Lf_n &= -\sum_{j=1}^{n-1} [f_j L f_{n-j} + N(f_j, f_{n-j})] = 0. \end{aligned} \tag{16.37}$$

The N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i), \tag{16.38}$$

where

$$\theta_i = k_i x - c_i t, \tag{16.39}$$

where k_i and c_i are arbitrary constants. Substituting (16.38) into (16.31) gives the dispersion relation

$$c_i = -k_i \sqrt{1 + k_i^2}, \tag{16.40}$$

and in view of this result we obtain

$$\theta_i = k_i x + k_i \sqrt{1 + k_i^2}\, t. \tag{16.41}$$

This means that

$$f_1 = \exp(\theta_1) = \exp\left(k_1 x + k_1 \sqrt{1 + k_1^2}\, t\right), \tag{16.42}$$

obtained by using $N = 1$ in (16.38).

Consequently, for the one-soliton solution, we set

$$f = 1 + \exp(\theta_1) = 1 + \exp\left(k_1 x + k_1 \sqrt{1 + k_1^2}\, t\right), \tag{16.43}$$

where we set $\varepsilon = 1$. The one soliton solution is obtained by recalling that

$$u(x,t) = 2(\ln f)_{xx}, \tag{16.44}$$

therefore we obtain

$$u(x,t) = \frac{2k_1^2 e^{k_1(x+\sqrt{1+k_1^2}t)}}{\left(1+e^{k_1(x+\sqrt{1+k_1^2}t)}\right)^2}, \tag{16.45}$$

or equivalently

$$u(x,t) = \frac{k_1^2}{2}\operatorname{sech}^2\left[\frac{1}{2}(k_1(x+\sqrt{1+k_1^2}t))\right]. \tag{16.46}$$

To determine the two-soliton solutions, we first set $N = 2$ in (16.38) to get

$$f_1 = \exp(\theta_1) + \exp(\theta_2), \tag{16.47}$$

and accordingly we have

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + f_2(x,t). \tag{16.48}$$

To determine f_2, we substitute the last equation into (16.31) to obtain

$$f_2 = a_{12}\exp(\theta_1 + \theta_2), \tag{16.49}$$

where the coupling coefficient a_{12} is given by

$$a_{12} = \frac{\sqrt{1+k_1^2}\sqrt{1+k_2^2} - (2k_1^2 - 3k_1k_2 + 2k_2^2 + 1)}{\sqrt{1+k_1^2}\sqrt{1+k_2^2} - (2k_1^2 + 3k_1k_2 + 2k_2^2 + 1)}, \tag{16.50}$$

and θ_1 and θ_2 are given above in (16.39). For the two-soliton solutions we use $1 \leqslant i < j \leqslant 2$, and therefore we obtain

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12}\exp(\theta_1 + \theta_2), \tag{16.51}$$

or equivalently

$$\begin{aligned} f &= 1 + \exp(\theta_1) + \exp(\theta_2) \\ &\quad + \frac{\sqrt{1+k_1^2}\sqrt{1+k_2^2} - (2k_1^2 - 3k_1k_2 + 2k_2^2 + 1)}{\sqrt{1+k_1^2}\sqrt{1+k_2^2} - (2k_1^2 + 3k_1k_2 + 2k_2^2 + 1)}\exp(\theta_1 + \theta_2). \end{aligned} \tag{16.52}$$

To determine the two-soliton solutions for the Boussinesq equation explicitly, we use (16.28) for the function f in (16.52).

Similarly, we can determine f_3. Proceeding as before, we therefore set

$$\begin{aligned} f_1(x,t) &= \exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3), \\ f_2(x,t) &= a_{12}\exp(\theta_1+\theta_2)+a_{23}\exp(\theta_2+\theta_3)+a_{13}\exp(\theta_1+\theta_3), \end{aligned} \tag{16.53}$$

and accordingly we have

$$\begin{aligned} f(x,t) = & \ 1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3) \\ & +a_{12}\exp(\theta_1+\theta_2)+a_{23}\exp(\theta_2+\theta_3)+a_{13}\exp(\theta_1+\theta_3) \\ & +f_3(x,t). \end{aligned} \tag{16.54}$$

Proceeding as before we find

$$f_3 = b_{123}\exp(\theta_1+\theta_2+\theta_3), \tag{16.55}$$

where

$$a_{ij} = \frac{\sqrt{1+k_i^2}\sqrt{1+k_j^2}-(2k_i^2-3k_ik_j+2k_j^2+1)}{\sqrt{1+k_i^2}\sqrt{1+k_j^2}-(2k_i^2+3k_ik_j+2k_j^2+1)}, \quad 1 \leqslant i < j \leqslant 3, \tag{16.56}$$

and

$$b_{123} = a_{12}a_{13}a_{23}, \tag{16.57}$$

and θ_1, θ_2 and θ_3 are given above in (16.39). For the three-soliton solution we use $1 \leqslant i < j \leqslant 3$, we therefore obtain

$$\begin{aligned} f = & \ 1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3) \\ & +a_{12}\exp(\theta_1+\theta_2)+a_{13}\exp(\theta_1+\theta_3)+a_{23}\exp(\theta_2+\theta_3) \\ & +b_{123}\exp(\theta_1+\theta_2+\theta_3). \end{aligned} \tag{16.58}$$

To determine the three-solitons solution explicitly, we use (16.28) for the function f in (16.58). We can now easily conclude that because we derived the three-soliton solutions, then the N-soliton solutions exist for any integer $N \geqslant 1$.

16.3 The Improved Boussinesq Equation

In this section we will study the improved Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxtt} = 0. \tag{16.59}$$

The main difference between the Boussinesq equation and the improved Boussinesq equation is that the last one contains a mixed fourth-order space-time derivative u_{xxtt}. The improved Boussinesq equation appears in acoustic waves on elastic rods with circular cross-section when transverse motion and nonlinearity are examined. Moreover, the bad improved Boussinesq equation is used to describe the wave propagation at right angles to the magnetic field, to study the propagation of ion-sound waves in a plasma, to study nonlinear lattice waves, and also to approach the bad Boussinesq equation. In this section, we will examine the bad improved Boussinesq equation only, and the good one can be handled in a like manner.

The improved Boussinesq equation will be converted to the ODE

$$(c^2-1)u-3u^2-c^2u''=0, \tag{16.60}$$

obtained upon using $\xi = x-ct$ and integrating twice. Balancing the nonlinear term u^2 with the highest order derivative u'' gives $M=2$. The tanh-coth method admits the use of the substitution

$$u(x,t)=S(Y)=\sum_{i=0}^{2} a_iY^i+\sum_{i=0}^{2} b_iY^{-i}. \tag{16.61}$$

Substituting (16.61) into (16.60), and proceeding as before we obtain the following sets of solutions:

(i)

$$a_0=\frac{c^2-1}{2},\quad a_2=-\frac{c^2-1}{2},\quad b_2=0,\quad \mu=\frac{1}{2c}\sqrt{c^2-1},\quad c^2>1. \tag{16.62}$$

(ii)

$$a_0=-\frac{c^2-1}{6},\quad a_2=\frac{c^2-1}{2},\quad b_2=0,\quad \mu=\frac{1}{2c}\sqrt{1-c^2},\quad c^2<1. \tag{16.63}$$

(iii)

$$a_0=\frac{c^2-1}{2},\quad b_2=-\frac{c^2-1}{2},\quad a_2=0,\quad \mu=\frac{1}{2c}\sqrt{c^2-1},\quad c^2>1. \tag{16.64}$$

(iv)

$$a_0=-\frac{c^2-1}{6},\quad b_2=\frac{c^2-1}{2},\quad a_2=0,\quad \mu=\frac{1}{2c}\sqrt{1-c^2},\quad c^2<1. \tag{16.65}$$

This in turn gives the solitons solutions

$$\begin{aligned} u_1(x,t) &= \frac{c^2-1}{2}\operatorname{sech}^2\left[\frac{1}{2c}\sqrt{c^2-1}(x-ct)\right],\quad c^2>1,\\ u_2(x,t) &= -\frac{c^2-1}{6}\left(1-3\tanh^2\left[\frac{1}{2c}\sqrt{1-c^2}(x-ct)\right]\right),\quad c^2<1. \end{aligned} \tag{16.66}$$

Moreover, the travelling wave solutions

$$u_3(x,t)=-\frac{c^2-1}{2}\operatorname{csch}^2\left[\frac{1}{2c}\sqrt{c^2-1}(x-ct)\right],\quad c^2>1,$$

$$u_4(x,t) = -\frac{c^2-1}{6}\left(1-3\coth^2\left[\frac{1}{2c}\sqrt{1-c^2}(x-ct)\right]\right), \quad c^2<1 \tag{16.67}$$

are readily obtained.

The wave speed c plays an important role in the physical structure of the solutions obtained above. Consequently, we obtain the following plane periodic solutions

$$\begin{aligned}
u_5(x,t) &= \frac{c^2-1}{2}\sec^2\left[\frac{1}{2c}\sqrt{1-c^2}(x-ct)\right], \quad c^2<1,\\
u_6(x,t) &= -\frac{c^2-1}{6}\left(1-3\tan^2\left[\frac{1}{2c}\sqrt{c^2-1}(x-ct)\right]\right), \quad c^2>1,\\
u_7(x,t) &= \frac{c^2-1}{2}\csc^2\left[\frac{1}{2c}\sqrt{1-c^2}(x-ct)\right], \quad c^2<1,\\
u_8(x,t) &= -\frac{c^2-1}{6}\left(1-3\cot^2\left[\frac{1}{2c}\sqrt{c^2-1}(x-ct)\right]\right), \quad c^2>1.
\end{aligned} \tag{16.68}$$

16.4 The Klein-Gordon Equation

In this section we will study the Klein-Gordon equation, with quadratic nonlinearity, that reads

$$u_{tt}-u_{xx}+u-u^2=0. \tag{16.69}$$

The Klein-Gordon equations play a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory. Equation (16.69) can be transformed to

$$(c^2-1)u''+u-u^2=0, \tag{16.70}$$

by using the wave variable $\xi = x-ct$. Balancing u'' with u^2 in (16.70) gives $M=2$. The tanh-coth method allows to use the finite expansion

$$u(\xi) = \sum_{j=0}^{2} a_j Y^j + \sum_{j=1}^{2} b_j Y^{-j}. \tag{16.71}$$

Substituting (16.71) into (16.70), collecting the coefficients of Y we obtain a system of algebraic equations for a_0, a_1, a_2, b_1, b_2, and μ. Solving this system gives $a_1 = b_1 = 0$ and the following four sets of solutions

(i)

$$a_0=\frac{3}{2}, \quad a_2=-\frac{3}{2}, \quad b_2=0, \quad \mu=\frac{1}{2\sqrt{1-c^2}}, \quad c^2<1 \tag{16.72}$$

(ii)

$$a_0=-\frac{1}{2}, \quad a_2=\frac{3}{2}, \quad b_2=0, \quad \mu=\frac{1}{2\sqrt{c^2-1}}, \quad c^2>1 \tag{16.73}$$

(iii)

$$a_0 = \frac{3}{2}, \quad a_2 = 0, \quad b_2 = -\frac{3}{2}, \quad \mu = \frac{1}{2\sqrt{1-c^2}}, \quad c^2 < 1 \tag{16.74}$$

(iv)

$$a_0 = -\frac{1}{2}, \quad a_2 = 0, \quad b_2 = \frac{3}{2}, \quad \mu = \frac{1}{2\sqrt{c^2-1}}, \quad c^2 > 1 \tag{16.75}$$

The first two sets give solitons solutions

$$u_1(x,t) = \frac{3}{2}\operatorname{sech}^2\left[\frac{1}{2\sqrt{1-c^2}}(x-ct)\right], \quad c^2 < 1, \tag{16.76}$$

and

$$u_2(x,t) = -\frac{1}{2}\left(1 - 3\tanh^2\left[\frac{1}{2\sqrt{c^2-1}}(x-ct)\right]\right), \quad c^2 > 1. \tag{16.77}$$

The last two sets lead to the solutions

$$u_3(x,t) = -\frac{3}{2}\operatorname{csch}^2\left[\frac{1}{2\sqrt{1-c^2}}(x-ct)\right], \quad c^2 < 1, \tag{16.78}$$

and

$$u_4(x,t) = -\frac{1}{2}\left(1 - 3\coth^2\left[\frac{1}{2\sqrt{c^2-1}}(x-ct)\right]\right), \quad c^2 > 1. \tag{16.79}$$

It is obvious that the wave speed c plays a major role in the physical structure of the obtained solutions. Consequently, we obtain the periodic solutions

$$u_5(x,t) = \frac{3}{2}\sec^2\left[\frac{1}{2\sqrt{c^2-1}}(x-ct)\right], \quad c^2 > 1, \tag{16.80}$$

$$u_6(x,t) = -\frac{1}{2}\left(1 + 3\tan^2\left[\frac{1}{2\sqrt{1-c^2}}(x-ct)\right]\right), \quad c^2 < 1, \tag{16.81}$$

$$u_7(x,t) = \frac{3}{2}\csc^2\left[\frac{1}{2\sqrt{c^2-1}}(x-ct)\right], \quad c^2 > 1, \tag{16.82}$$

and

$$u_8(x,t) = -\frac{1}{2}\left(1 + 3\cot^2\left[\frac{1}{2\sqrt{1-c^2}}(x-ct)\right]\right), \quad c^2 < 1. \tag{16.83}$$

16.5 The Liouville Equation

In this section we will study the Liouville equation [14]

$$u_{tt} - u_{xx} + e^{\pm u} = 0, \tag{16.84}$$

that arises in hydrodynamics where $u(x,t)$ is the stream function. To apply the tanh-coth method, we first use the Painlevé transformation

$$v = e^{\pm u}, \tag{16.85}$$

so that

$$u = \pm \ln v, \tag{16.86}$$

that gives

$$\begin{aligned} u_{tt} &= \pm\frac{1}{v}v_{tt} \mp \frac{1}{v^2}(v_t)^2, \\ u_{xx} &= \pm\frac{1}{v}v_{xx} \mp \frac{1}{v^2}(v_x)^2. \end{aligned} \tag{16.87}$$

These transformations will convert Eq. (16.84) into the ODE form

$$\pm(v_{tt} - v_{xx}) \pm (v_t^2 - v_x^2) + v^3 = 0, \tag{16.88}$$

Using the wave variable $\xi = x - ct$ carries the last equation into

$$\pm(c^2-1)vv'' \pm (c^2-1)(v')^2 + v^3 = 0. \tag{16.89}$$

Balancing vv'' with v^3 gives $M = 2$. The tanh-coth method allows us to use

$$v(\xi) = \sum_{j=0}^{2} a_j Y^j + \sum_{j=1}^{2} b_j Y^{-j}. \tag{16.90}$$

Substituting (16.90) into (16.89), and proceeding as before we find that $a_1 = b_1 = 0$ and the following sets of solutions
(i)

$$a_0 = \pm 2\mu^2(c^2-1), \quad a_2 = 0, \quad b_2 = 0. \tag{16.91}$$

(ii)

$$a_0 = \pm 2\mu^2(c^2-1), \quad a_2 = 0, \quad b_2 = \mp 2\mu^2(c^2-1). \tag{16.92}$$

(iii)

$$a_0 = \pm 4\mu^2(c^2-1), \quad a_2 = \mp 2\mu^2(c^2-1), \quad b_2 = \mp 2\mu^2(c^2-1), \tag{16.93}$$

where μ and c are left as free parameters. This in turn gives the following solutions

$$\begin{aligned} v_1(x,t) &= \pm 2\mu^2(c^2-1)\operatorname{sech}^2(\mu(x-ct)), \quad \pm c^2 > 1, \\ v_2(x,t) &= \mp 2\mu^2(c^2-1)\operatorname{csch}^2(\mu(x-ct)), \quad \mp c^2 > 1, \\ v_3(x,t) &= \pm 8\mu^2(c^2-1)\operatorname{sech}^2(2\mu(x-ct)), \quad \pm c^2 > 1, \\ v_4(x,t) &= \mp 8\mu^2(c^2-1)\operatorname{csch}^2(2\mu(x-ct)), \quad \mp c^2 > 1. \end{aligned} \tag{16.94}$$

Consequently, the exact solutions can be obtained by noting that $u(x,t) = \pm \ln v(x,t)$.

16.6 The Sine-Gordon Equation

The sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0, \tag{16.95}$$

gained its importance when it gave *kink* and *antikink* solutions with the collisional behaviors of solitons. The first appearance of the sine-Gordon equation is not in wave problems, but in the study of differential geometry of surfaces with Gaussian curvature $K = -1$. This equation appeared in many scientific fields such as the propagation of fluxons in Josephson junctions between two superconductors, the motion of rigid pendulum attached to a stretched wire, solid state physics, nonlinear optics, and dislocations in metals. A kink is a solution with boundary values 0 and 2π at the left infinity and at the right infinity respectively. However, antikink is a solution with boundary values 0 and -2π at the left infinity and at the right infinity respectively. The two soliton solutions of the sine-Gordon equation can be interpreted as a collision of a kink and antikink, collision of two kinks, or collision of two antikinks. Most importantly, the sine-Gordon equation is a completely integrable equation, therefore it has multi-soliton solutions that will be examined later.

16.6.1 Using the Tanh-coth Method

We begin our analysis by introducing the transformations

$$v = e^{iu}, \tag{16.96}$$

so that

$$\sin u = \frac{v - v^{-1}}{2i}, \quad \cos u = \frac{v + v^{-1}}{2}, \tag{16.97}$$

that also gives

$$u = \arccos\left(\frac{v + v^{-1}}{2}\right). \tag{16.98}$$

This transformation will change the sine-Gordon equation (16.95) into the ODE form

$$2vv_{tt} - 2vv_{xx} - 2v_t^2 + 2v_x^2 + v^3 - v = 0, \tag{16.99}$$

or equivalently

$$v^3 - v + 2(c^2 - 1)vv'' + 2(1 - c^2)(v')^2 = 0, \tag{16.100}$$

by using the wave variable $\xi = x - ct$. Balancing vv'' with v^3 gives $M = 2$. The tanh-coth method allows us to use

$$v(\xi) = \sum_{j=0}^{2} a_j Y^j + \sum_{j=1}^{2} b_j Y^{-j}. \tag{16.101}$$

Proceeding as presented before we find that $a_1 = b_1 = 0$ and
(i)
$$a_0 = 0, \quad a_2 = -1, \quad b_2 = 0, \quad \mu = \frac{1}{2\sqrt{c^2-1}}, \quad c^2 > 1. \tag{16.102}$$
(ii)
$$a_0 = 0, \quad a_2 = 1, \quad b_2 = 0, \quad \mu = \frac{1}{2\sqrt{1-c^2}}, \quad c^2 < 1. \tag{16.103}$$
(iii)
$$a_0 = 0, \quad a_2 = 0, \quad b_2 = -1, \quad \mu = \frac{1}{2\sqrt{c^2-1}}, \quad c^2 > 1. \tag{16.104}$$
(iv)
$$a_0 = 0, \quad a_2 = 0, \quad b_2 = 1, \quad \mu = \frac{1}{2\sqrt{1-c^2}}, \quad c^2 < 1. \tag{16.105}$$
This in turn gives
$$v_1(x,t) = -\tanh^2\left[\frac{1}{2\sqrt{c^2-1}}(x-ct)\right], \quad c^2 > 1. \tag{16.106}$$
$$v_2(x,t) = \tanh^2\left[\frac{1}{2\sqrt{1-c^2}}(x-ct)\right], \quad c^2 < 1. \tag{16.107}$$
$$v_3(x,t) = -\coth^2\left[\frac{1}{2\sqrt{c^2-1}}(x-ct)\right], \quad c^2 > 1. \tag{16.108}$$
$$v_4(x,t) = \coth^2\left[\frac{1}{2\sqrt{1-c^2}}(x-ct)\right], \quad c^2 < 1. \tag{16.109}$$
We can also obtain the periodic solutions
$$v_5(x,t) = \tan^2\left[\frac{1}{2\sqrt{1-c^2}}(x-ct)\right], \quad c^2 < 1. \tag{16.110}$$
$$v_6(x,t) = -\tan^2\left[\frac{1}{2\sqrt{c^2-1}}(x-ct)\right], \quad c^2 > 1. \tag{16.111}$$
$$v_7(x,t) = \cot^2\left[\frac{1}{2\sqrt{1-c^2}}(x-ct)\right], \quad c^2 < 1. \tag{16.112}$$
$$v_8(x,t) = -\cot^2\left[\frac{1}{2\sqrt{c^2-1}}(x-ct)\right], \quad c^2 > 1. \tag{16.113}$$
Recall that
$$u = \arccos\left(\frac{v+v^{-1}}{2}\right), \tag{16.114}$$
therefore we obtain the solutions, for $c^2 > 1$,

$$u_1(x,t) = \arccos\left\{-\frac{1}{2}\frac{\tanh^4\left[\dfrac{1}{2\sqrt{c^2-1}}(x-ct)+1\right]}{\tanh^2\left[\dfrac{1}{2\sqrt{c^2-1}}(x-ct)\right]}\right\}, \tag{16.115}$$

$$u_2(x,t) = \arccos\left\{-\frac{1}{2}\frac{\coth^4\left[\dfrac{1}{2\sqrt{c^2-1}}(x-ct)+1\right]}{\coth^2\left[\dfrac{1}{2\sqrt{c^2-1}}(x-ct)\right]}\right\}, \tag{16.116}$$

$$u_3(x,t) = \arccos\left\{-\frac{1}{2}\frac{\tan^4\left[\dfrac{1}{2\sqrt{c^2-1}}(x-ct)+1\right]}{\tan^2\left[\dfrac{1}{2\sqrt{c^2-1}}(x-ct)\right]}\right\}, \tag{16.117}$$

and

$$u_4(x,t) = \arccos\left\{-\frac{1}{2}\frac{\cot^4\left[\dfrac{1}{2\sqrt{c^2-1}}(x-ct)+1\right]}{\cot^2\left[\dfrac{1}{2\sqrt{c^2-1}}(x-ct)\right]}\right\}. \tag{16.118}$$

On the other hand, for $c^2 < 1$ we find

$$u_5(x,t) = \arccos\left\{\frac{1}{2}\frac{\tanh^4\left[\dfrac{1}{2\sqrt{1-c^2}}(x-ct)+1\right]}{\tanh^2\left[\dfrac{1}{2\sqrt{1-c^2}}(x-ct)\right]}\right\}, \tag{16.119}$$

$$u_6(x,t) = \arccos\left\{\frac{1}{2}\frac{\coth^4\left[\dfrac{1}{2\sqrt{1-c^2}}(x-ct)+1\right]}{\coth^2\left[\dfrac{1}{2\sqrt{1-c^2}}(x-ct)\right]}\right\}, \tag{16.120}$$

$$u_7(x,t) = \arccos\left\{\frac{1}{2}\frac{\tan^4\left[\dfrac{1}{2\sqrt{1-c^2}}(x-ct)+1\right]}{\tan^2\left[\dfrac{1}{2\sqrt{1-c^2}}(x-ct)\right]}\right\}, \tag{16.121}$$

and

$$u_8(x,t) = \arccos\left\{\frac{1}{2}\frac{\cot^4\left[\dfrac{1}{2\sqrt{1-c^2}}(x-ct)+1\right]}{\cot^2\left[\dfrac{1}{2\sqrt{1-c^2}}(x-ct)\right]}\right\}. \tag{16.122}$$

16.6.2 Using the Bäcklund Transformation

The standard form of the sine-Gordon equation [5]

$$u_{TT} - u_{XX} + \sin u = 0, \tag{16.123}$$

can be converted to the characteristic form of the sine-Gordon equation

$$u_{xt} = \sin u, \tag{16.124}$$

by using the transformations

$$x = \frac{1}{2}(X - T),\ t = \frac{1}{2}(X + T). \tag{16.125}$$

Bäcklund introduced the well-known *auto-Bäcklund transformation* for the sine-Gordon equation

$$\begin{aligned} \frac{1}{2}(u-v)_x &= p\sin\frac{1}{2}(u+v), \\ \frac{1}{2}(u+v)_t &= p^{-1}\sin\frac{1}{2}(u-v), \quad p \neq 0. \end{aligned} \tag{16.126}$$

Differentiating this pair of transformations with respect to t and x respectively gives

$$\begin{aligned} \frac{1}{2}(u-v)_{xt} &= \frac{p}{2}(u+v)_t\cos\frac{1}{2}(u+v) = \sin\frac{1}{2}(u-v)\cos\frac{1}{2}(u+v), \\ \frac{1}{2}(u+v)_{xt} &= \frac{p^{-1}}{2}(u-v)_x\cos\frac{1}{2}(u-v) = \sin\frac{1}{2}(u+v)\cos\frac{1}{2}(u-v). \end{aligned} \tag{16.127}$$

Adding and subtracting the last equations gives

$$\begin{aligned} u_{xt} &= \sin u, \\ v_{xt} &= \sin v. \end{aligned} \tag{16.128}$$

This means that the Bäcklund transformation (16.126) gives two solutions u and v to the sine-Gordon equation. Setting $v = 0$ into Bäcklund transformation yields the two separable ODEs

$$\begin{aligned} u_x &= 2p\sin(\frac{1}{2}u), \\ u_t &= 2p^{-1}\sin(\frac{1}{2}u), \end{aligned} \tag{16.129}$$

where by integrating these equations with respect to x and t respectively we obtain

$$\begin{aligned} 2px &= 2\ln|\tan(\frac{1}{4}u)| + f(t), \\ \frac{2t}{p} &= 2\ln|\tan(\frac{1}{4}u)| + g(x), \end{aligned} \tag{16.130}$$

where $f(t)$ and $g(x)$ are arbitrary functions that result from integration. This in turn gives

$$\tan(\frac{1}{4}u) = Ae^{px+\frac{t}{p}}, \tag{16.131}$$

and hence the exact solution is given by

$$u(x,t) = 4\tan^{-1}\left(Ae^{px+\frac{t}{p}}\right), \tag{16.132}$$

where A is an arbitrary constant.

Drazin [5] used the last result to formally derive the two-soliton solutions of the characteristic equation (16.124). The two-soliton solutions were assumed to be of the form

$$u(x,t) = 4\tan^{-1}\left(\frac{g(x,t)}{f(x,t)}\right), \tag{16.133}$$

where

$$\begin{aligned} g(x,t) &= (p_1+p_2)e^{\theta_1} - e^{\theta_2}, \\ f(x,t) &= (p_1-p_2)(1+e^{\theta_1+\theta_2}), \end{aligned} \tag{16.134}$$

where

$$\theta_i = p_i x + \frac{t}{p_i}, \quad i = 1,2. \tag{16.135}$$

Notice that

$$\sin u = \frac{4gf(f^2-g^2)}{(f^2+g^2)^2}. \tag{16.136}$$

16.6.3 Multiple-soliton Solutions for Sine-Gordon Equation

In this section, we will examine multiple-soliton solutions of the standard sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0. \tag{16.137}$$

We first derive the dispersion relation by substituting

$$u(x,t) = e^{\theta},\ \theta = kx - ct, \tag{16.138}$$

into the linearized equation

$$u_{tt} - u_{xx} + u = 0, \tag{16.139}$$

where we approximated $\sin u$ by u. This gives the dispersion relation by

$$k^2 - c^2 = 1, \tag{16.140}$$

or equivalently

$$c = \pm\sqrt{k^2-1}, \tag{16.141}$$

and as a result we set

$$\theta_i = k_i x - c_i t = k_i x \pm \sqrt{k_i^2 - 1}\, t,\ i = 1,2,3,\cdots,N. \tag{16.142}$$

The bilinear form for the sine-Gordon equation is

$$(D_x^2 - D_t^2)(f \cdot f) - (f^2 - f^{*2})/2 = \lambda f^2 = 0, \tag{16.143}$$

together with its complex conjugate, where f^* is the complex conjugate of f.

Based on the obtained results by Bäcklund transformation, it was assumed in [7] that the N-soliton solutions are of the form

$$u(x,t) = 4\tan^{-1}\left(\frac{g(x,t)}{f(x,t)}\right), \quad f(x,t) \neq 0. \tag{16.144}$$

(i) For the one soliton solution: It was found that

$$\begin{aligned} f(x,t) &= 1, \\ g(x,t) &= e^{\theta}, \quad \theta = kx \pm \sqrt{k^2-1}\, t. \end{aligned} \tag{16.145}$$

This means that the one soliton solution is given by

$$u(x,t) = 4\tan^{-1}\left(e^{kx \pm \sqrt{k^2-1}\, t}\right). \tag{16.146}$$

(ii) For the two-soliton solutions: It was found that

$$\begin{aligned} f(x,t) &= 1 + a_{12} e^{\theta_1 + \theta_2}, \\ g(x,t) &= e^{\theta_1} + e^{\theta_2}, \quad \theta_i = k_i x \pm \sqrt{k^2-1}\, t, \end{aligned} \tag{16.147}$$

where

$$a_{ij} = -\frac{(k_i - k_j + c_i - c_j)^2}{(k_i - k_j + c_i - c_j)^2}. \tag{16.148}$$

(iii) For the three-soliton solutions: It was found that

$$\begin{aligned} f(x,t) &= 1 + a_{12} e^{\theta_1 + \theta_2} + a_{13} e^{\theta_1 + \theta_3} + a_{23} e^{\theta_2 + \theta_3}, \\ g(x,t) &= e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{123} e^{\theta_1 + \theta_2 + \theta_3}, \end{aligned} \tag{16.149}$$

where

$$a_{123} = a_{12} a_{13} a_{23}. \tag{16.150}$$

For $n \geqslant 4$, the last results can be generalized for N-soliton solutions. For justification, notice that

$$\sin u = \frac{4gf(f^2 - g^2)}{(f^2 + g^2)^2}. \tag{16.151}$$

16.7 The Sinh-Gordon Equation

We close our study of the first set of equations by investigating the sinh-Gordon equation

$$u_{tt} - u_{xx} + \sinh u = 0, \tag{16.152}$$

or equivalently

$$u_{tt} - u_{xx} + \frac{1}{2}(e^u - e^{-u}) = 0. \tag{16.153}$$

The Painlevé transformation

$$v = e^u,\, v = \ln u, \tag{16.154}$$

will convert the sinh-Gordon equation (16.153) into the ODE form

$$(c^2-1)vv'' - 2(c^2-1)(v')^2 + v^3 - v = 0. \tag{16.155}$$

Balancing vv'' with v^3 gives $M = 2$. The tanh-coth method allows us to use

$$v(\xi) = \sum_{j=0}^{2} a_j Y^j + \sum_{j=1}^{2} b_j Y^{-j}. \tag{16.156}$$

Proceeding as presented before we find that $a_1 = b_1 = 0$ and
(i)

$$a_0 = 0, \quad a_2 = -1, \quad b_2 = 0, \quad \mu = \frac{1}{2\sqrt{c^2-1}}, \quad c^2 > 1. \tag{16.157}$$

(ii)

$$a_0 = 0, \quad a_2 = 1, \quad b_2 = 0, \quad \mu = \frac{1}{2\sqrt{1-c^2}}, \quad c^2 < 1. \tag{16.158}$$

(iii)

$$a_0 = 0, \quad a_2 = 0, \quad b_2 = -1, \quad \mu = \frac{1}{2\sqrt{c^2-1}}, \quad c^2 > 1. \tag{16.159}$$

(iv)

$$a_0 = 0, \quad a_2 = 0, \quad b_2 = 1, \quad \mu = \frac{1}{2\sqrt{1-c^2}}, \quad c^2 < 1. \tag{16.160}$$

This in turn gives

$$v_1(x,t) = -\tanh^2\left[\frac{1}{2\sqrt{c^2-1}}(x-ct)\right], \quad c^2 > 1. \tag{16.161}$$

$$v_2(x,t) = \tanh^2\left[\frac{1}{2\sqrt{1-c^2}}(x-ct)\right], \quad c^2 < 1. \tag{16.162}$$

$$v_3(x,t) = -\coth^2\left[\frac{1}{2\sqrt{c^2-1}}(x-ct)\right], \quad c^2 > 1. \tag{16.163}$$

$$v_4(x,t) = \coth^2\left[\frac{1}{2\sqrt{1-c^2}}(x-ct)\right], \quad c^2 < 1. \tag{16.164}$$

We can also obtain the periodic solutions

$$v_5(x,t) = \tan^2\left[\frac{1}{2\sqrt{1-c^2}}(x-ct)\right], \quad c^2 < 1. \tag{16.165}$$

$$v_6(x,t) = -\tan^2\left[\frac{1}{2\sqrt{c^2-1}}(x-ct)\right], \quad c^2 > 1. \tag{16.166}$$

$$v_7(x,t) = \cot^2\left[\frac{1}{2\sqrt{1-c^2}}(x-ct)\right], \quad c^2 < 1. \tag{16.167}$$

$$v_8(x,t) = -\cot^2\left[\frac{1}{2\sqrt{c^2-1}}(x-ct)\right], \quad c^2 > 1. \tag{16.168}$$

The exact solution can be obtained by using

$$u(x,t) = \ln v(x,t). \tag{16.169}$$

We will next study the second family of equations. We will follow a parallel approach to that used before.

16.8 The Dodd-Bullough-Mikhailov Equation

The Dodd-Bullough-Mikhailov equation appears in problems varying from fluid flow to quantum field theory. It appears in scientific applications such as solid state physics and nonlinear optics. The Dodd-Bullough-Mikhailov equation is given by

$$u_{xt} + e^u + e^{-2u} = 0. \tag{16.170}$$

We first use the Painlevé transformation [11]

$$v = e^u, \tag{16.171}$$

so that

$$u = \ln v. \tag{16.172}$$

This transformation (16.171) transforms Eq. (16.170) into the ODE form

$$vv_{tx} - v_x v_t + v^3 + 1 = 0. \tag{16.173}$$

Using the wave variable $\xi = x - ct$ carries (16.173) into the ODE

$$v^3 - cvv'' + c(v')^2 + 1 = 0. \tag{16.174}$$

Balancing vv'' with v^3 gives $M = 2$. This gives the solution in the form

$$v(\xi) = \sum_{j=0}^{2} a_j Y^j + \sum_{j=1}^{2} b_j Y^{-j}. \tag{16.175}$$

Substituting this assumption into (16.174) and solving as before, we obtain $a_1 = b_1 = 0$ and the following sets
(i)

$$a_0 = \frac{1}{2}, \quad a_2 = -\frac{3}{2}, \quad b_2 = 0, \quad \mu = \frac{1}{2}\sqrt{-\frac{3}{c}}, \quad c < 0. \tag{16.176}$$

(ii)

$$a_0 = \frac{1}{2}, \quad a_2 = 0, \quad b_2 = -\frac{3}{2}, \quad \mu = \frac{1}{2}\sqrt{-\frac{3}{c}}, \quad c < 0, \tag{16.177}$$

where c is left as a free parameter. This gives

$$v_1(x,t) = \frac{1}{2}\left(1 - 3\tanh^2\left[\frac{1}{2}\sqrt{-\frac{3}{c}}(x-ct)\right]\right), \quad c < 0, \tag{16.178}$$

and

$$v_2(x,t) = \frac{1}{2}\left(1 - 3\coth^2\left[\frac{1}{2}\sqrt{-\frac{3}{c}}(x-ct)\right]\right), \quad c < 0. \tag{16.179}$$

On the other hand, for $c > 0$ we find the periodic solution

$$v(x,t) = \frac{1}{2}\left(1 + 3\tan^2\left[\frac{1}{2}\sqrt{\frac{3}{c}}(x-ct)\right]\right), \quad c > 0, \tag{16.180}$$

and

$$v(x,t) = \frac{1}{2}\left(1 + 3\cot^2\left[\frac{1}{2}\sqrt{\frac{3}{c}}(x-ct)\right]\right), \quad c > 0. \tag{16.181}$$

The exact solutions can be obtained by using

$$u(x,t) = \ln v(x,t), \tag{16.182}$$

and noting the domain of the natural logarithm.

16.9 The Tzitzeica-Dodd-Bullough Equation

We next consider the Tzitzeica-Dodd-Bullough (TDB) equation

$$u_{xt} = e^{-u} + e^{-2u}. \tag{16.183}$$

Using the transformation [11]

$$v(x,t) = e^{-u},\, u(x,t) = -\ln v(x,t), \tag{16.184}$$

carries (16.183) into the ODE

$$-vv_{xt} + v_x v_t - v^3 - v^4 = 0, \tag{16.185}$$

that can be transformed to the ODE

$$cvv'' - c(v')^2 - v^3 - v^4 = 0, \tag{16.186}$$

upon using the wave variable $\xi = x - ct$. Balancing vv'' with v^4 gives $M = 1$. This means that we can set the solution in the form

$$v(\xi) = S(Y) = a_0 + a_1 Y + \frac{b_1}{Y}. \tag{16.187}$$

Substituting the tanh-coth assumption into (16.186), and solving we find for $c > 0$

(i)

$$a_0 = -\frac{1}{2}, \quad a_1 = \frac{1}{2}, \quad b_1 = 0, \quad \mu = \frac{1}{2\sqrt{c}}. \tag{16.188}$$

(ii)

$$a_0 = -\frac{1}{2}, \quad a_1 = -\frac{1}{2}, \quad b_1 = 0, \quad \mu = \frac{1}{2\sqrt{c}}. \tag{16.189}$$

(iii)

$$a_0 = -\frac{1}{2}, \quad a_1 = 0, \quad b_1 = \frac{1}{2}, \quad \mu = \frac{1}{2\sqrt{c}}. \tag{16.190}$$

(iv)

$$a_0 = -\frac{1}{2}, \quad a_1 = 0, \quad b_1 = -\frac{1}{2}, \quad \mu = \frac{1}{2\sqrt{c}}. \tag{16.191}$$

(v)

$$a_0 = -\frac{1}{2}, \quad a_1 = -\frac{1}{4}, \quad b_1 = -\frac{1}{4}, \quad \mu = \frac{1}{4\sqrt{c}}. \tag{16.192}$$

(vi)

$$a_0 = -\frac{1}{2}, \quad a_1 = \frac{1}{4}, \quad b_1 = \frac{1}{4}, \quad \mu = \frac{1}{4\sqrt{c}}. \tag{16.193}$$

This gives, for $c > 0$:

$$v_1(x,t) = -\frac{1}{2}\left(1 - \tanh\left[\frac{1}{2\sqrt{c}}(x-ct)\right]\right), \tag{16.194}$$

$$v_2(x,t) = -\frac{1}{2}\left(1 + \tanh\left[\frac{1}{2\sqrt{c}}(x-ct)\right]\right), \tag{16.195}$$

$$v_3(x,t) = -\frac{1}{2}\left(1-\coth\left[\frac{1}{2\sqrt{c}}(x-ct)\right]\right), \tag{16.196}$$

$$v_4(x,t) = -\frac{1}{2}\left(1+\coth\left[\frac{1}{2\sqrt{c}}(x-ct)\right]\right), \tag{16.197}$$

$$v_5(x,t) = -\frac{1}{4}\left(2+\tanh\left[\frac{1}{4\sqrt{c}}(x-ct)\right]+\coth\left[\frac{1}{4\sqrt{c}}(x-ct)\right]\right), \tag{16.198}$$

$$v_6(x,t) = -\frac{1}{4}\left(2-\tanh\left[\frac{1}{4\sqrt{c}}(x-ct)\right]-\coth\left[\frac{1}{4\sqrt{c}}(x-ct)\right]\right). \tag{16.199}$$

The exact solutions for the TDB equation can be obtained by using $u(x,t) = -\ln v(x,t)$ and noting the domain of the natural logarithm. For $c<0$, complex solutions arise.

16.10 The Zhiber-Shabat Equation

We close this study of the second family of equations by investigating the Zhiber-Shabat equation [14]

$$u_{xt} + pe^{u} + qe^{-u} + re^{-2u} = 0, \tag{16.200}$$

where p, q, and r are arbitrary constants. For $q = 0$, Eq. (16.200) reduces to the Dodd-Bullough-Mikhailov equation that was studied before. For $p = 0, q = -1, r = 1$, Eq. (16.200) gives the Tzitzeica-Dodd-Bullough equation that was examined in the previous equation. For $r = 0$, we obtain the sinh-Gordon equation that also was studies before. The aforementioned equations arise in many applications such as solid state physics, plasma physics, nonlinear optics, chemical kinetics and quantum field theory.

We first use $u(x,t) = u(\xi)$ that will carry out the Zhiber-Shabat equation (16.200) into

$$-cu'' + pe^{u} + qe^{-u} + re^{-2u} = 0. \tag{16.201}$$

We use the Painlevé property

$$v = e^{u},\, u = \ln v, \tag{16.202}$$

that will transform (16.201) into the ODE

$$-c(vv'' - (v')^2) + pv^3 + qv + r = 0. \tag{16.203}$$

Balancing vv'' with v^3 gives $M = 2$. The tanh-coth method gives

$$v(\xi) = \sum_{j=0}^{2} a_j Y^j + \sum_{j=1}^{2} b_j Y^{-j}. \tag{16.204}$$

Without loss of generality, we set $p=q=r=1$. Substituting (16.204) into (16.203), and proceeding as before we obtain

$$\begin{aligned} a_0 &= \frac{\gamma_1}{24\alpha_1}, \\ a_1 &= b_1 = 0, \\ a_2 &= b_2 = -\frac{2\gamma_1^2-15\alpha_1\gamma_1+21\alpha_1^2}{432\alpha_1^2}, \\ \mu_1 &= \frac{1}{24\alpha_1}\sqrt{\frac{2(15\alpha_1\gamma_1-2\gamma_1^2-216\alpha_1^2)}{3c}}, \end{aligned} \tag{16.205}$$

where

$$\begin{aligned} \alpha_1 &= (188+36\sqrt{93})^{\frac{1}{3}}, \\ \gamma_1 &= \alpha_1^2+2\alpha_1-44. \end{aligned} \tag{16.206}$$

Notice that $15\alpha_1\gamma_1-2\gamma_1^2-216\alpha_1^2<0$ by using the numerical value of α_1. Recall that $u(x,t)=\ln v(x,t)$, hence we obtain the solution

$$u(x,t)=\ln\left\{\frac{\gamma_1}{24\alpha_1}-\frac{2\gamma_1^2-15\alpha_1\gamma_1+21\alpha_1^2}{432\alpha_1^2}\left(\tanh^2\left[\mu_1(x-ct)\right]+\coth^2\left[\mu_1(x-ct)\right]\right)\right\}, \tag{16.207}$$

for $c<0$, where μ_1 is given above by (16.205).

However, for $c>0$, we obtain the travelling wave solutions

$$u(x,t)=\ln\left\{\frac{\gamma_1}{24\alpha_1}+\frac{2\gamma_1^2-15\alpha_1\gamma_1+21\alpha_1^2}{432\alpha_1^2}\left(\tan^2\left[\overline{\mu_1}(x-ct)\right]+\cot^2\left[\overline{\mu_1}(x-ct)\right]\right)\right\}, \tag{16.208}$$

where

$$\overline{\mu_1}=\frac{1}{24\alpha_1}\sqrt{-\frac{2(15\alpha_1\gamma_1-2\gamma_1^2-216\alpha_1^2)}{3c}}. \tag{16.209}$$

References

1. M.J. Ablowitz and P.A. Clarkson, Solitons: Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge (1991).
2. J.Boussinesq, Theorie des ondes et des remous qui se propagent le long d Run canal rectangulaire horizontal, en communiquant au liquide continu dans ce canal des vitesses sensiblement pareilles de la surface au fond, *J.Math. Pures Appl.*, 17, 55–108, (1872).
3. J. Boyd, Weakly Nonlocal Solitary Waves and Beyond-All-Orders Asymptotics, Kluwer, Boston, (1998).
4. R. Bullough and P. Caudrey, Solitons vol 17 in “Topics in Current Physics”, Springer, Berlin (1980).
5. P.G. Drazin and R.S. Johnson, Solitons: an Introduction, Cambridge University Press, Cambridge, (1996).

6. W. Hereman and A. Nuseir, Symbolic methods to construct exact solutions of nonlinear partial differential equations, *Math. Comput. Simulation*, 43(1), 13–27, (1997).
7. R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge (2004).
8. R. Hirota, Exact N-soliton solutions of a nonlinear wave equation, *J. Math Phys.*, 14(7), 805–809, (1973).
9. Y. Matsuno, Bilinear Transformation Method, Academic Press, New York, (1984).
10. A.M. Wazwaz, The tanh and the sine-cosine methods for compact and noncompact solutions of the nonlinear Klein-Gordon equation, *Appl. Math. Comput.*, 167(2), 1179–1195, (2005).
11. A.M. Wazwaz, The tanh method: solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tzitzeica-Dodd-Bullough equations, *Chaos, Solitons and Fractals*, 25(1) 55–63, (2005).
12. A.M. Wazwaz, Exact solutions of the generalized sine-Gordon and the generalized sinh-Gordon equations, *Chaos, Solitons and Fractals*, 28, 127–135, (2006).
13. A.M. Wazwaz, The tanh method: exact solutions of the Sine-Gordon and the Sinh-Gordon equations, *Appl. Math. Comput.*, 167(2), 1196–1210, (2005).
14. A.M. Wazwaz, The tanh method for travelling wave solutions to the Zhiber-Shabat equation and other related equations, *Commun. Nonlinear Sci. Numer. Simul.*, 13, 584–592, (2008).

Chapter 17
Burgers, Fisher and Related Equations

17.1 Introduction

In the preceeding three chapters we examined the nonlinear evolution equations that include dispersion terms. In this chapter, we will study the nonlinear evolution equations where each equation contains the dissipative term u_{xx} in addition to other partial derivatives. This new family of nonlinear equations gained its importance because it appears in many scientific applications and physical phenomena.

The new family of nonlinear equations, that will be discussed in this chapter, is of the form

$$u_t - u_{xx} = P(u), \tag{17.1}$$

where $u(x,t)$ is a function of space x and time variable t. The nonlinear term $P(u)$ appears in the following forms

$$P(u) = \begin{cases} -uu_x, \\ u(1-u), \\ u(k-u)(u-1), \\ uu_x + u(1-u), \\ uu_x + u(k-u)(u-1), \\ -u(1-u)(a-u), \\ a + be^{n\lambda u}. \end{cases} \tag{17.2}$$

(i) For $P(u) = -uu_x$, we obtain the Burgers equation [1,3,11]

$$u_t + uu_x - u_{xx} = 0. \tag{17.3}$$

(ii) For $P(u) = u(1-u)$, we obtain the Fisher equation [4,5]

$$u_t - u_{xx} = u(1-u). \tag{17.4}$$

(iii) For $P(u) = uu_x + u(1-u)$, the Huxley equation [16] reads

$$u_t - u_{xx} = u(k-u)(u-1), k \neq 0. \tag{17.5}$$

(iv) For $P(u) = uu_x + u(1-u)$, we obtain the Burgers-Fisher equation [15]

$$u_t - u_{xx} = uu_x + u(1-u). \tag{17.6}$$

(v) For $P(u) = uu_x + u(k-u)(u-1)$, we obtain Burgers-Huxley equation [10,11]

$$u_t - u_{xx} = uu_x + u(k-u)(u-1). \tag{17.7}$$

(vi) For $P(u) = -u(1-u)(a-u)$, we obtain the FitzHugh-Nagumo equation [16]

$$u_t - u_{xx} = -u(1-u)(a-u). \tag{17.8}$$

(vii) For $P(u) = a + e^{n\lambda u}$, we obtain a parabolic equation with exponential nonlinearity of the form

$$u_t - u_{xx} = a + be^{n\lambda u}. \tag{17.9}$$

In addition to these equations, the coupled Burgers equation [13]

$$\begin{aligned} u_t - 2uu_x - u_{xx} - u_{yy} - 2vu_y &= 0, \\ v_t - 2uv_x - v_{xx} - v_{yy} - 2vv_y &= 0, \end{aligned} \tag{17.10}$$

will be examined for single and multiple-soliton solutions.

Moreover, the Kuramoto-Sivashinsky (KS) equation [2]

$$u_t + auu_x + bu_{2x} + ku_{4x} = 0, \tag{17.11}$$

will also be investigated. For $k = 0, a = 1, b = -1$, the KS equation (17.11) will be reduced to the Burgers equation.

It is interesting to point out that other equations of distinct forms of $P(u)$ exist in the literature such as Newell-Whitehead equation. These equations of distinct forms can be handled in a manner similar to the approach that we will use in this chapter; therefore we leave it as exercises. Our approach stems mainly from the tanh-coth method [9]. The Hirota's bilinear formalism [6,7,8] will be used for completely integrable equations.

17.2 The Burgers Equation

In this section we will study the Burgers equation [1,3,11]

$$u_t + uu_x - u_{xx} = 0. \tag{17.12}$$

Equation (17.12) is the lowest order approximation for the one-dimensional propagation of weak shock waves in a fluid. It is also used in the description of the variation in vehicle density in highway traffic. It is one of the fundamental model equations in fluid mechanics. The Burgers equation demonstrates the coupling between dissipation effect of u_{xx} and the convection process of uu_x. Unlike the KdV equation that combines the nonlinear uu_x and dispersion u_{xxx} effects, the Burgers equation combines the nonlinear uu_x and dissipation u_{xx} effects.

Burgers introduced this equation to capture some of the features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion. It is also used to describe the structure of shock waves, traffic flow, and acoustic transmission. Burgers equation is completely integrable. The wave solutions of Burgers equation are single and multiple-front solutions.

17.2.1 Using the Tanh-coth Method

The Burgers equation

$$u_t - 2uu_x - u_{xx} = 0, \tag{17.13}$$

will be converted to the ODE

$$cu + u^2 + u' = 0, \tag{17.14}$$

obtained upon using $\xi = x - ct$ and integrating once. Balancing the nonlinear term u^2 with the highest order derivative u' gives $M = 1$. The tanh-coth method admits the use of the substitution

$$u(x,t) = S(Y) = a_0 + a_1 Y + b_1 Y^{-1}. \tag{17.15}$$

Substituting (17.15) into (17.14), collecting the coefficients of each power of $Y^i, 0 \leqslant i \leqslant 4$, setting each coefficient to zero, and solving the resulting system of algebraic equations, we find the following sets of solutions:

(i)

$$a_0 = -\frac{c}{2}, \quad a_1 = \frac{c}{2}, \quad b_1 = 0, \quad \mu = \frac{c}{2}. \tag{17.16}$$

(ii)

$$a_0 = -\frac{c}{2}, \quad b_1 = \frac{c}{2}, \quad a_1 = 0, \quad \mu = \frac{c}{2}. \tag{17.17}$$

This in turn gives the front wave (kink) solution

$$u_1(x,t) = -\frac{c}{2}\left(1 - \tanh\left[\frac{c}{2}(x - ct)\right]\right), \quad c > 0, \tag{17.18}$$

and the travelling wave solutions

$$u_2(x,t) = -\frac{c}{2}\left(1 - \coth\left[\frac{c}{2}(x - ct)\right]\right), \quad c > 0. \tag{17.19}$$

For $c < 0$, we obtain the solutions

$$u_3(x,t) = -\frac{c}{2}\left(1 + \tan\left[\frac{c}{2}(x - ct)\right]\right), \tag{17.20}$$

and

$$u_4(x,t) = -\frac{c}{2}\left(1 - \cot\left[\frac{c}{2}(x - ct)\right]\right). \tag{17.21}$$

It is also to be noted that many other solutions, periodic and rational, exist for Burgers equation. These solutions can be found in Appendix C.

17.2.2 Using the Cole-Hopf Transformation

It is interesting to give the main steps of the Cole-Hopf transformation. The solution of the Burgers equation can be expressed by

$$u(x,t) = \frac{\partial}{\partial x}\ln f = \frac{f_x}{f}, \tag{17.22}$$

where $f(x,t)$ is given by the perturbation expansion

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \tag{17.23}$$

where ε is a formal expansion parameter. For the single front solution we set

$$f(x,t) = 1 + \varepsilon f_1, \tag{17.24}$$

and for the two-soliton solutions we set

$$f(x,t) = 1 + \varepsilon f_1 + \varepsilon^2 f_2, \tag{17.25}$$

and so on. The functions $f_1, f_2, f_3, \cdots$ can be determined by substituting the last equation into the appropriate equation as will be seen later.

As stated before, we use the Cole-Hopf transformation

$$u(x,t) = (\ln(f))_x, \tag{17.26}$$

that transforms the Burgers equation into

$$f(f_t - f_{xx})_x - f_x(f_t - f_{xx}) = 0. \tag{17.27}$$

We next follow the sense of the Hirota's direct method [7,8] and the Hereman method [6] where we assume that $f(x,t)$ has a perturbation expansion of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \tag{17.28}$$

where ε is a non small formal expansion parameter. The N-front solution is obtained by assuming that

$$f_i = \exp(\theta_i), \tag{17.29}$$

where

$$\theta_i = k_i x - c_i t, \tag{17.30}$$

where k_i and c_i are arbitrary constants. Substituting (17.28) into (17.27) and equate to zero the powers of ε, and by using (17.29) we obtain the relation

$$c_i = -k_i^2, \tag{17.31}$$

and in view of this result we obtain

$$\theta_i = k_i x + k_i^2 t. \tag{17.32}$$

This means that

$$f_1 = \exp(\theta_1) = \exp(k_1 x + k_1^2 t), \tag{17.33}$$

obtained by using (17.29).

Consequently, for the single front solution, we set

$$f = 1 + \exp(\theta_1) = 1 + \exp(k_1 x + k_1^2 t), \tag{17.34}$$

where we set $\varepsilon = 1$. The single front solution is obtained by recalling that

$$u(x,t) = (\ln f)_x, \tag{17.35}$$

therefore we obtain

$$u(x,t) = \frac{k_1 e^{k_1 x + k_1^2 t}}{1 + e^{k_1 x + k_1^2 t}}. \tag{17.36}$$

To determine the two-front solutions, we first set

$$f = 1 + \exp(\theta_1) + f_2. \tag{17.37}$$

To determine f_2, we substitute the last equation into (17.27) to find

$$f_2 = \exp(\theta_2), \tag{17.38}$$

and as a result we obtain

$$f = 1 + \exp(\theta_1) + \exp(\theta_2), \tag{17.39}$$

or equivalently

$$f = 1 + e^{k_1 x + k_1^2 t} + e^{k_2 x + k_2^2 t}. \tag{17.40}$$

This gives the two front solutions

$$u(x,t) = \frac{k_1 e^{k_1 x + k_1^2 t} + k_2 e^{k_2 x + k_2^2 t}}{1 + e^{k_1 x + k_1^2 t} + k_2 e^{k_2 x + k_2^2 t}}. \tag{17.41}$$

Proceeding in the same manner, we can generalize and set

$$f = 1 + \sum_{i=1}^{N} \exp(\theta_i), \tag{17.42}$$

to obtain the multiple-wave solutions

$$u(x,t) = \frac{\sum_{i=1}^{N} k_i e^{k_i x + k_i^2 t}}{1 + \sum_{i=1}^{N} e^{k_i x + k_i^2 t}}. \tag{17.43}$$

This result is consistent with the result obtained in [13].

17.3 The Fisher Equation

The Fisher equation describes the process of interaction between diffusion and reaction. This equation is encountered in chemical kinetics and population dynamics [11] which includes problems such as nonlinear evolution of a population in one-dimensional habitual, neutron population in a nuclear reaction. The Fisher equation

$$u_t - u_{xx} - u(1-u) = 0, \tag{17.44}$$

will be carried into the ODE

$$-cu' - u'' - u(1-u) = 0, \tag{17.45}$$

upon using $\xi = x - ct$. Balancing u'' with u^2 gives $M = 2$.

The tanh-coth method applies the finite expansion

$$u(\xi) = \sum_{j=0}^{2} a_j Y^j + \sum_{j=1}^{2} b_j Y^{-j}. \tag{17.46}$$

Substituting (17.46) into (17.45), and proceeding as before to solve for $a_0, a_1, a_2, b_1, b_2, c$, and μ we find

(i)

$$a_0 = \frac{1}{4}, \quad a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{4}, \quad b_1 = b_2 = 0, \quad c = \frac{5}{\sqrt{6}}, \quad \mu = \frac{1}{2\sqrt{6}}. \tag{17.47}$$

(ii)

$$a_0 = \frac{1}{4}, \quad a_1 = a_2 = 0, \quad b_1 = -\frac{1}{2}, \quad b_2 = \frac{1}{4}, \quad c = \frac{5}{\sqrt{6}}, \quad \mu = \frac{1}{2\sqrt{6}}. \tag{17.48}$$

(iii)

$$a_0 = \frac{3}{8}, \quad a_1 = -\frac{1}{4}, \quad a_2 = \frac{1}{16}, \quad b_1 = -\frac{1}{4}, \quad b_2 = \frac{1}{16}, \quad c = \frac{5}{\sqrt{6}}, \quad \mu = \frac{1}{4\sqrt{6}}. \tag{17.49}$$

The first set gives the kink solution

$$u_1(x,t) = \frac{1}{4}\left(1 - \tanh\left[\frac{1}{2\sqrt{6}}(x - \frac{5}{\sqrt{6}}t)\right]\right)^2. \tag{17.50}$$

As a result we obtain the following solutions

$$u_2(x,t) = \frac{1}{4}\left(1 - \coth\left[\frac{1}{2\sqrt{6}}(x - \frac{5}{\sqrt{6}}t)\right]\right)^2, \tag{17.51}$$

and

$$\begin{aligned} u_3(x,t) = {} & \frac{1}{16}\left(3 - 4\tanh\left[\frac{1}{4\sqrt{6}}(x - \frac{5}{\sqrt{6}}t)\right] + \tanh^2\left[\frac{1}{2\sqrt{6}}(x - \frac{5}{\sqrt{6}}t)\right]\right) \\ & + \frac{1}{16}\left(3 - 4\coth\left[\frac{1}{4\sqrt{6}}(x - \frac{5}{\sqrt{6}}t)\right] + \coth^2\left[\frac{1}{2\sqrt{6}}(x - \frac{5}{\sqrt{6}}t)\right]\right). \end{aligned} \tag{17.52}$$

17.4 The Huxley Equation

The Huxley equation [15,16] reads

$$u_t - u_{xx} - u(k-u)(u-1) = 0, \quad k \neq 0. \tag{17.53}$$

This equation is used for nerve propagation in neurophysics and wall propagation in liquid crystals. The wave variable $\xi = x - ct$ carries out Eq. (17.53) into the ODE

$$-cu' - u'' - u(k-u)(u-1) = 0. \tag{17.54}$$

Balancing u'' with u^3 gives $M = 1$. The tanh-coth method becomes

$$u(\xi) = a_0 + a_1 Y + \frac{b_1}{Y}. \tag{17.55}$$

Substituting (17.55) into (17.54), and proceeding as before we obtain the sets of solutions

(i)

$$a_0 = \frac{1}{2}, \quad a_1 = \pm\frac{1}{2}, \quad b_1 = 0, \quad c = \gamma, \quad \mu = \frac{1}{2\sqrt{2}}. \tag{17.56}$$

(ii)

$$a_0 = \frac{1}{2}, \quad a_1 = 0, \quad b_1 = \pm\frac{1}{2}, \quad c = \gamma, \quad \mu = \frac{1}{2\sqrt{2}}. \tag{17.57}$$

(iii)

$$a_0 = \frac{1}{2}, \quad a_1 = \pm\frac{1}{4}, \quad b_1 = \pm\frac{1}{4}, \quad c = \gamma, \quad \mu = \frac{1}{2\sqrt{2}}. \tag{17.58}$$

(iv)

$$a_0 = \frac{k}{2}, \quad a_1 = \pm\frac{k}{2}, \quad b_1 = 0, \quad c = \lambda, \quad \mu = \frac{k}{2\sqrt{2}}. \tag{17.59}$$

(v)

$$a_0 = \frac{k}{2}, \quad a_1 = 0, \quad b_1 = \pm\frac{k}{2}, \quad c = \lambda, \quad \mu = \frac{k}{2\sqrt{2}}. \tag{17.60}$$

(vi)

$$a_0 = \frac{k}{2}, \quad a_1 = \pm\frac{k}{4}, \quad b_1 = \pm\frac{k}{4}, \quad c = \lambda, \quad \mu = \mp\frac{k}{4\sqrt{2}}, \tag{17.61}$$

where

$$\begin{aligned} \gamma &= \frac{2k-1}{\sqrt{2}}, \\ \lambda &= \frac{2-k}{\sqrt{2}}. \end{aligned} \tag{17.62}$$

Based on this, the first three sets give the following solutions

$$u_1(x,t) = \frac{1}{2}\left(1 \pm \tanh\left[\frac{1}{2\sqrt{2}}(x-\gamma t)\right]\right). \tag{17.63}$$

$$u_2(x,t) = \left(1 \pm \coth\left[\frac{1}{2\sqrt{2}}(x-\gamma t)\right]\right). \tag{17.64}$$

$$u_3(x,t) = \frac{1}{4}\left(2 \pm \tanh\left[\frac{1}{4\sqrt{2}}(x-\gamma t)\right] \pm \coth\left[\frac{1}{4\sqrt{2}}(x-\gamma t)\right]\right). \tag{17.65}$$

$$u_4(x,t) = \frac{k}{2}\left(1 \pm \tanh\left[\frac{k}{2\sqrt{2}}(x-\lambda t)\right]\right). \tag{17.66}$$

$$u_5(x,t) = \frac{1}{2}\left(1 \pm \coth\left[\frac{k}{2\sqrt{2}}(x-\lambda t)\right]\right). \tag{17.67}$$

and

$$u_6(x,t) = \frac{k}{4}\left(2 \pm \tanh\left[\frac{k}{4\sqrt{2}}(x-\lambda t)\right] \pm \coth\left[\frac{k}{4\sqrt{2}}(x-\lambda t)\right]\right), \tag{17.68}$$

where γ and λ are defined above in (17.62). Notice that the solutions in (17.63) and (17.66) are kink solutions.

17.5 The Burgers-Fisher Equation

The Burgers-Fisher equation [15] reads

$$u_t - u_{xx} = uu_x + u(1-u), \tag{17.69}$$

that will be carried into the ODE

$$cu' + uu' + u'' + u(1-u) = 0, \tag{17.70}$$

obtained after using the wave variable $\xi = x - ct$. Balancing u^2 with u'' gives $M = 2$ that does not lead to any solution. However, balancing uu' with u'' gives $M = 1$. This means that we can set

$$u(\xi) = a_0 + a_1 Y + \frac{b_1}{Y}. \tag{17.71}$$

Substituting (17.71) into (17.70), and proceeding as before we obtain the sets of solutions

(i)

$$a_0 = \frac{1}{2},\ a_1 = \frac{1}{2},\ b_1 = 0,\ c = -\frac{5}{2},\ \mu = \frac{1}{4}. \tag{17.72}$$

(ii)

$$a_0 = \frac{1}{2},\ a_1 = 0,\ b_1 = \frac{1}{2},\ c = -\frac{5}{2},\ \mu = \frac{1}{4}. \tag{17.73}$$

(iii)

$$a_0 = \frac{1}{2},\ a_1 = \frac{1}{4},\ b_1 = \frac{1}{4},\ c = -\frac{5}{2},\ \mu = \frac{1}{8}. \tag{17.74}$$

This in turn gives the following kink solution

$$u_1(x,t) = \frac{1}{2}\left(1 + \tanh\left[\frac{1}{4}(x + \frac{5}{2}t)\right]\right), \tag{17.75}$$

and the following solutions

$$u_2(x,t) = \frac{1}{2}\left(1 + \coth\left[\frac{1}{4}(x + \frac{5}{2}t)\right]\right), \tag{17.76}$$

and

$$u_3(x,t) = \frac{1}{4}\left(2 + \tanh\left[\frac{1}{8}(x + \frac{5}{2}t)\right] + \coth\left[\frac{1}{8}(x + \frac{5}{2}t)\right]\right). \tag{17.77}$$

17.6 The Burgers-Huxley Equation

The Burgers-Huxley equation reads

$$u_t - u_{xx} = uu_x + u(k-u)(u-1). \tag{17.78}$$

The Burgers-Huxley equation is used for description of nonlinear wave processes in physics, economics and ecology [11]. The negative coefficients of u_{xx} and u^3 follows from the physical meanings of the problem. This equation can be converted to the ODE

$$cu' + uu' + u'' + u(k-u)(u-1) = 0, \tag{17.79}$$

upon using the wave variable $\xi = x - ct$. Balancing u^3 with u'' gives $M = 1$. This allows us to set

$$u(\xi) = a_0 + a_1 Y + b_1 Y^{-1}. \tag{17.80}$$

Substituting (17.80) into (17.79) collecting the coefficients of $Y^i, i \geqslant 0$, and setting these coefficients to zero we obtain a system of algebraic equations in a_0, a_1, b_1, μ and c. Solving the resulting system we find the following sets of solutions:
Case 1: We found that $b_1 = 0$, and

$$\begin{aligned}
&a_0 = \frac{1}{2}, \quad a_1 = -\frac{1}{2}, \quad \mu = \frac{1}{4}, \quad c = \frac{1-4k}{2},\\
&a_0 = \frac{k}{2}, \quad a_1 = -\frac{k}{2}, \quad \mu = \frac{k}{4}, \quad c = \frac{k-4}{2},\\
&a_0 = \frac{k+1}{2}, \quad a_1 = -\frac{k-1}{2}, \quad \mu = \frac{k-1}{4}, \quad c = \frac{k+1}{2},\\
&a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{2}, \quad \mu = \frac{1}{2}, \quad c = k-1,\\
&a_0 = \frac{k}{2}, \quad a_1 = \frac{k}{2}, \quad \mu = \frac{k}{2}, \quad c = 1-k,\\
&a_0 = \frac{k+1}{2}, \quad a_1 = \frac{k-1}{2}, \quad \mu = \frac{k-1}{2}, \quad c = -(1+k).
\end{aligned} \tag{17.81}$$

Case 2: We found that $a_1 = 0$, and

$$\begin{aligned}
&a_0 = \frac{1}{2}, \quad b_1 = -\frac{1}{2}, \quad \mu = \frac{1}{4}, \quad c = \frac{1-4k}{2},\\
&a_0 = \frac{k}{2}, \quad b_1 = -\frac{k}{2}, \quad \mu = \frac{k}{4}, \quad c = \frac{k-4}{2},\\
&a_0 = \frac{k+1}{2}, \quad b_1 = -\frac{k-1}{2}, \quad \mu = \frac{k-1}{4}, \quad c = \frac{k+1}{2},\\
&a_0 = \frac{1}{2}, \quad b_1 = \frac{1}{2}, \quad \mu = \frac{1}{2}, \quad c = k-1,\\
&a_0 = \frac{k}{2}, \quad b_1 = \frac{k}{2}, \quad \mu = \frac{k}{2}, \quad c = 1-k,\\
&a_0 = \frac{k+1}{2}, \quad b_1 = \frac{k-1}{2}, \quad \mu = \frac{k-1}{2}, \quad c = -(1+k).
\end{aligned} \tag{17.82}$$

The first case gives the kink solutions

$$u_1(x,t) = \frac{1}{2}\left(1 - \tanh\left[\frac{1}{4}\left(x - \frac{1-4k}{2}t\right)\right]\right),$$

$$
\begin{aligned}
u_2(x,t) &= \frac{k}{2}\left(1-\tanh\left[\frac{k}{4}\left(x-\frac{k-4}{2}t\right)\right]\right),\\
u_3(x,t) &= \frac{k+1}{2}-\frac{k-1}{2}\tanh\left[\frac{k-1}{4}\left(x-\frac{k+1}{2}t\right)\right],\\
u_4(x,t) &= \frac{1}{2}\left(1+\tanh\left[\frac{1}{2}\left(x-(k-1)t\right)\right]\right),\\
u_5(x,t) &= \frac{k}{2}\left(1+\tanh\left[\frac{k}{2}\left(x-(1-k)t\right)\right]\right),\\
u_6(x,t) &= \frac{k+1}{2}+\frac{k-1}{2}\tanh\left[\frac{k-1}{2}\left(x+(1+k)t\right)\right].
\end{aligned}
\tag{17.83}
$$

The second case gives the following travelling wave solutions

$$
\begin{aligned}
u_7(x,t) &= \frac{1}{2}\left(1-\coth\left[\frac{1}{4}\left(x-\frac{1-4k}{2}t\right)\right]\right),\\
u_8(x,t) &= \frac{k}{2}\left(1-\coth\left[\frac{k}{4}\left(x-\frac{k-4}{2}t\right)\right]\right),\\
u_9(x,t) &= \frac{k+1}{2}-\frac{k-1}{2}\coth\left[\frac{k-1}{4}\left(x-\frac{k+1}{2}t\right)\right],\\
u_{10}(x,t) &= \frac{1}{2}\left(1+\coth\left[\frac{1}{2}\left(x-(k-1)t\right)\right]\right),\\
u_{11}(x,t) &= \frac{k}{2}\left(1+\coth\left[\frac{k}{2}\left(x-(1-k)t\right)\right]\right),\\
u_{12}(x,t) &= \frac{k+1}{2}+\frac{k-1}{2}\coth\left[\frac{k-1}{2}\left(x+(1+k)t\right)\right].
\end{aligned}
\tag{17.84}
$$

17.7 The FitzHugh-Nagumo Equation

We next consider the FitzHugh-Nagumo equation

$$u_t = u_{xx} - u(1-u)(a-u), \tag{17.85}$$

that will be transformed to the ODE

$$cu' + u'' - u(1-u)(a-u) = 0, \tag{17.86}$$

by using the wave variable $\xi = x - ct$. Balancing the terms u'' with u^3 gives $M = 1$. The tanh-coth method admits the use of

$$u(\xi) = a_0 + a_1 Y + b_1 Y^{-1}. \tag{17.87}$$

Substituting (17.87) into (17.86), collecting the coefficients of Y, and solving the resulting system we find the following sets of solutions
(i) *Case 1:* We found that $b_1 = 0$, and

$$
\begin{aligned}
&a_0=\frac{1}{2},\quad a_1=\frac{1}{2},\quad c=\frac{2a-1}{\sqrt{2}},\quad \mu=\frac{1}{2\sqrt{2}},\\
&a_0=\frac{a}{2},\quad a_1=-\frac{a}{2},\quad c=\frac{a-2}{\sqrt{2}},\quad \mu=\frac{a}{2\sqrt{2}},\\
&a_0=\frac{a+1}{2},\quad a_1=-\frac{a-1}{2},\quad c=\frac{a+1}{\sqrt{2}},\quad \mu=\frac{a-1}{2\sqrt{2}}.
\end{aligned}
\tag{17.88}
$$

(ii) *Case 2:* We found that $a_1=0$, and

$$
\begin{aligned}
&a_0=\frac{1}{2},\quad b_1=\frac{1}{2},\quad c=\frac{2a-1}{\sqrt{2}},\quad \mu=\frac{1}{2\sqrt{2}},\\
&a_0=\frac{a}{2},\quad b_1=-\frac{a}{2},\quad c=\frac{a-2}{\sqrt{2}},\quad \mu=\frac{a}{2\sqrt{2}},\\
&a_0=\frac{a+1}{2},\quad b_1=-\frac{a-1}{2},\quad c=\frac{a+1}{\sqrt{2}},\quad \mu=\frac{a-1}{2\sqrt{2}}.
\end{aligned}
\tag{17.89}
$$

The first set gives the kink solution

$$u_1(x,t)=\frac{1}{2}\left(1+\tanh\left[\frac{1}{2\sqrt{2}}(x-\frac{2a-1}{\sqrt{2}}t)\right]\right). \tag{17.90}$$

$$u_2(x,t)=\frac{a}{2}\left(1-\tanh\left[\frac{a}{2\sqrt{2}}(x-\frac{a-2}{\sqrt{2}}t)\right]\right). \tag{17.91}$$

$$u_3(x,t)=\frac{a+1}{2}-\frac{a-1}{2}\tanh\left[\frac{a-1}{2\sqrt{2}}(x-\frac{a+1}{\sqrt{2}}t)\right]. \tag{17.92}$$

The second set gives the following solution

$$u_4(x,t)=\frac{1}{2}\left(1+\coth\left[\frac{1}{2\sqrt{2}}(x-\frac{2a-1}{\sqrt{2}}t)\right]\right). \tag{17.93}$$

$$u_5(x,t)=\frac{a}{2}\left(1-\coth\left[\frac{a}{2\sqrt{2}}(x-\frac{a-2}{\sqrt{2}}t)\right]\right). \tag{17.94}$$

$$u_6(x,t)=\frac{a+1}{2}-\frac{a-1}{2}\coth\left[\frac{a-1}{2\sqrt{2}}(x-\frac{a+1}{\sqrt{2}}t)\right]. \tag{17.95}$$

17.8 Parabolic Equation with Exponential Nonlinearity

As stated before, this equation is defined by

$$u_t=u_{xx}+a+be^{n\lambda u}, \tag{17.96}$$

or equivalently

$$cu'+u''+a+be^{n\lambda u}=0, \tag{17.97}$$

obtained by using $\xi = x - ct$.

We first use the transformation

$$v(x,t) = e^{n\lambda u}, \tag{17.98}$$

or equivalently

$$u(x,t) = \frac{1}{n\lambda}\ln v(x,t). \tag{17.99}$$

This transformation will carry out (17.97) into the ODE

$$cvv' + vv'' - (v')^2 + an\lambda v^2 + bn\lambda v^3 = 0. \tag{17.100}$$

Balancing the terms vv'' with v^3 we find $M = 2$, and hence the tanh-coth method uses

$$u(\xi) = \sum_{i=0}^{2} a_i Y^i + \sum_{i=1}^{2} b_i Y^{-i}. \tag{17.101}$$

Substituting (17.101) into (17.100), and proceeding as before we obtain
(i) *Case 1:* We find that $b_1 = b_2 = 0$ and

$$a_0 = -\frac{a}{4b},\ a_1 = \pm\frac{a}{2b},\ a_2 = \frac{a}{4b},\ c = \mp\sqrt{\frac{an\lambda}{2}},\ \mu = \frac{1}{2}\sqrt{\frac{an\lambda}{2}},\ a > 0. \tag{17.102}$$

(ii) *Case 2:* We find that $a_1 = a_2 = 0$ and

$$a_0 = -\frac{a}{4b},\ b_1 = \mp\frac{a}{2b},\ b_2 = -\frac{a}{4b},\ c = \pm\sqrt{\frac{an\lambda}{2}},\ \mu = \frac{1}{2}\sqrt{\frac{an\lambda}{2}},\ a > 0. \tag{17.103}$$

The first set gives the kink solutions $u(x,t) = \frac{1}{n\lambda}\ln v(x,t)$ for $a\lambda > 0$, where

$$v_1(x,t) = \frac{-a}{4b}\left(1 \pm \tanh\left[\frac{1}{2}\sqrt{\frac{an\lambda}{2}}\left(x \pm \sqrt{\frac{an\lambda}{2}}\,t\right)\right]\right)^2. \tag{17.104}$$

The second set gives the solution

$$v_2(x,t) = \frac{-a}{4b}\left(1 \pm \coth\left[\frac{1}{2}\sqrt{\frac{an\lambda}{2}}\left(x \pm \sqrt{\frac{an\lambda}{2}}\,t\right)\right]\right)^2. \tag{17.105}$$

The exact solutions can be obtained by substituting the last results into (17.99) and noting the domain of the natural logarithm.

On the other hand, for $a\lambda < 0$ we obtain complex solutions that are beyond the scope of this text.

17.9 The Coupled Burgers Equation

As stated before, the coupled Burgers equation [13] is given by

$$\begin{aligned} u_t - 2uu_x - u_{xx} - u_{yy} - 2vu_y &= 0,\\ v_t - 2uv_x - v_{xx} - v_{yy} - 2vv_y &= 0. \end{aligned} \tag{17.106}$$

To handle this coupled equation, we use the couple Cole-Hopf transformations

$$u(x,y,t) = (\ln(f))_x, \quad v(x,y,t) = (\ln(f))_y, \tag{17.107}$$

that transform the coupled Burgers equation into two equations given by

$$\begin{aligned} f(f_t - f_{xx} - f_{yy})_x - f_x(f_t - f_{xx} - f_{yy}) &= 0,\\ f(f_t - f_{xx} - f_{yy})_y - f_y(f_t - f_{xx} - f_{yy}) &= 0. \end{aligned} \tag{17.108}$$

We next assume that $f(x,y,t)$ has a perturbation expansion of the form

$$f(x,y,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,y,t), \tag{17.109}$$

where ε is a non small formal expansion parameter. The N-front solution is obtained by assuming that

$$f_i = \exp(\theta_i), \tag{17.110}$$

where

$$\theta_i = k_i x + m_i y - c_i t, \tag{17.111}$$

where k_i, m_i and c_i are arbitrary constants. Substituting (17.109) into (17.108) and equate to zero the powers of ε, and by using (17.110) we obtain the relation

$$c_i = -k_i^2 - m_i^2, \tag{17.112}$$

and in view of this result we obtain

$$\theta_i = k_i x + m_i y + (k_i^2 + m_i^2)t. \tag{17.113}$$

This means that

$$f_1 = \exp(\theta_1) = \exp(k_1 x + m_1 y + (k_1^2 + m_1^2)t), \tag{17.114}$$

obtained by using (17.110).

Consequently, for the single front solution, we set

$$f = 1 + \exp(\theta_1) = 1 + \exp(k_1 x + m_1 y + (k_1^2 + m_1^2)t), \tag{17.115}$$

where we set $\varepsilon = 1$. The single front solution is obtained by recalling that

$$u(x,y,t) = (\ln f)_x, \quad v(x,y,t) = (\ln f)_y, \tag{17.116}$$

therefore we obtain

$$\begin{aligned} u(x,y,t) &= \frac{k_1 e^{k_1 x + m_1 y + (k_1^2 + m_1^2)t}}{1 + e^{k_1 x + m_1 y + (k_1^2 + m_1^2)t}}, \\ v(x,y,t) &= \frac{m_1 e^{k_1 x + m_1 y + (k_1^2 + m_1^2)t}}{1 + e^{k_1 x + m_1 y + (k_1^2 + m_1^2)t}}. \end{aligned} \tag{17.117}$$

To determine the two-front solutions, we first set

$$f = 1 + \exp(\theta_1) + f_2. \tag{17.118}$$

To determine f_2, we substitute the last equation into (17.108) to find

$$f_2 = \exp(\theta_2), \tag{17.119}$$

and as a result we obtain

$$f = 1 + \exp(\theta_1) + \exp(\theta_2), \tag{17.120}$$

or equivalently

$$f = 1 + e^{k_1 x + m_1 y + (k_1^2 + m_1^2)t} + e^{k_2 x + m_2 y + (k_2^2 + m_2^2)t}. \tag{17.121}$$

This gives the two-front solutions

$$\begin{aligned} u(x,y,t) &= \frac{k_1 e^{k_1 x + m_1 y + (k_1^2 + m_1^2)t} + k_2 e^{k_2 x + m_2 y + (k_2^2 + m_2^2)t}}{1 + e^{k_1 x + m_1 y + (k_1^2 + m_1^2)t} + e^{k_2 x + m_2 y + (k_2^2 + m_2^2)t}}, \\ v(x,y,t) &= \frac{m_1 e^{k_1 x + m_1 y + (k_1^2 + m_1^2)t} + m_2 e^{k_2 x + m_2 y + (k_2^2 + m_2^2)t}}{1 + e^{k_1 x + m_1 y + (k_1^2 + m_1^2)t} + e^{k_2 x + m_2 y + (k_2^2 + m_2^2)t}}. \end{aligned} \tag{17.122}$$

Proceeding in the same manner, we can generalize and set

$$f = 1 + \sum_{i=1}^{N} \exp(\theta_i), \tag{17.123}$$

to obtain the multiple-wave solutions

$$\begin{aligned} u(x,y,t) &= \frac{\sum_{i=1}^{N} k_i e^{k_i x + m_i y + (k_i^2 + m_i^2)t}}{1 + \sum_{i=1}^{N} e^{k_i x + m_i y + (k_i^2 + m_i^2)t}}, \\ v(x,y,t) &= \frac{\sum_{i=1}^{N} m_i e^{k_i x + m_i y + (k_i^2 + m_i^2)t}}{1 + \sum_{i=1}^{N} e^{k_i x + m_i y + (k_i^2 + m_i^2)t}}. \end{aligned} \tag{17.124}$$

17.10 The Kuramoto-Sivashinsky (KS) Equation

The Kuramoto-Sivashinsky (KS) equation

$$u_t + auu_x + bu_{2x} + ku_{4x} = 0, \tag{17.125}$$

describes the fluctuations of the position of a flame front, the motion of a fluid going down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium [2]. This equation was examined as a prototypical example of spatiotemporal chaos in one space dimension. Moreover, this equation was originally derived in the context of plasma instabilities, flame front propagation, and phase turbulence in reaction-diffusion system.

Using the wave variable $\xi = x - ct$ carries the KS equation (17.125) into the ODE

$$-cu + \frac{a}{2}u^2 + bu'' + ku''' = 0, \tag{17.126}$$

obtained upon integrating the resulting ODE and setting the constant of integration to zero.

Balancing u''' with u^2 gives $M = 3$. The tanh-coth method admits the use of the finite expansion

$$u(\xi) = \sum_{i=0}^{3} a_i Y^i + \sum_{i=1}^{3} b_i Y^{-i}, Y = \tanh(\mu\xi). \tag{17.127}$$

Substituting (17.127) into (17.126), collecting the coefficients of Y, proceeding as before, and solving the resulting system of algebraic equations for $a_0, a_1, a_2, a_3, b_1, b_2, b_3$ and μ, we obtain the following two sets of solutions

$$\begin{aligned}
a_0 = \frac{30b}{19a}\sqrt{\frac{-b}{19k}}, \quad a_1 &= \frac{135b}{152a}\sqrt{\frac{-b}{19k}}, \quad a_2 = 0, \quad a_3 = -\frac{15b}{152a}\sqrt{\frac{-b}{19k}},\\
b_1 &= \frac{135b}{152a}\sqrt{\frac{-b}{19k}}, \quad b_2 = 0, \quad b_3 = -\frac{15b}{152a}\sqrt{\frac{-b}{19k}},\\
\mu &= \frac{1}{4}\sqrt{\frac{-b}{19k}}, \quad c = \frac{30b}{19}\sqrt{\frac{-b}{19k}}, \quad \frac{b}{k} < 0,
\end{aligned} \tag{17.128}$$

and

$$\begin{aligned}
a_0 = \frac{30b}{19a}\sqrt{\frac{11b}{19k}}, \quad a_1 &= -\frac{45b}{152a}\sqrt{\frac{11b}{19k}}, \quad a_2 = 0, \quad a_3 = \frac{165b}{152a}\sqrt{\frac{11b}{19k}},\\
b_1 &= -\frac{45b}{152a}\sqrt{\frac{11b}{19k}}, \quad b_2 = 0, \quad b_3 = \frac{165b}{152a}\sqrt{\frac{11b}{19k}},\\
\mu &= \frac{1}{4}\sqrt{\frac{11b}{19k}}, \quad c = \frac{30b}{19}\sqrt{\frac{11b}{19k}}, \quad \frac{b}{k} > 0.
\end{aligned} \tag{17.129}$$

The first set gives the soliton solution for $\frac{b}{k} < 0$:

$$u(x,t) = \frac{15b}{152a}\sqrt{\frac{-b}{19k}}\left(16+9Y-Y^3+9Y^{-1}-Y^{-3}\right), \tag{17.130}$$

where μ and c are given above. For $\frac{b}{k} > 0$, complex solution can be derived.

The second set gives the soliton solution for $\frac{b}{k} > 0$:

$$u(x,t) = \frac{15b}{152a}\sqrt{\frac{11b}{19k}}\left(16-3Y+11Y^3-3Y^{-1}+11Y^{-3}\right), \tag{17.131}$$

where $Y = \tanh\left[\mu(x-ct)\right]$.

References

1. J.M. Burgers, A mathematical model illustrating the theory of turbulence, *Adv.Appl.Mech.* 1, 171–199, (1948).
2. R. Conte, Exact solutions of nonlinear partial differential equations by singularity analysis, *Lecture Notes in Physics*, Springer 1–83 (2003).
3. P.G. Drazin and R.S. Johnson, Solitons: an Introduction, Cambridge University Press, Cambridge, (1996).
4. R.A. Fisher, The wave of advance of advantageous genes, *Ann.Eugenics*, 7, 335–369, (1936).
5. C.K. Hayes, A new traveling-wave solution of Fisher's equation with density-dependent diffusivity, *J.Math.Biol.*, 29,531–537, (1991).
6. W. Hereman and A. Nuseir, Symbolic methods to construct exact solutions of nonlinear partial differential equations, *Math.Comput. Simulation*, 43(1), 13–27, (1997).
7. R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge (2004).
8. R. Hirota, Exact N-soliton solutions of a nonlinear wave equation, *J. Math. Phys.*, 14(7), 805–809, (1973).
9. Malfliet W., Solitary wave solutions of nonlinear wave equations, *Am. J. Phys.*, 60(7), 650–654, (1992).
10. M. McAsey and L.A. Rubel, Some closed-form solutions of Burgers' equation, *Stu. Appl. Math.*, 88, 173–190, (1993).
11. J.D.Murray, Mathematical Biology, Springer, New York, (1993).
12. S. Tang and R.O. Weber, Numerical study of Fisher's equation by a Petrov-Galerkin finite element method, *J. Austral. Math. Soc. Ser.*, B 33,27 – 38, (1991).
13. A.Veksler and Y. Zarmi, Wave interactions and the analysis of the perturbed Burgers equation, *Physica D*, 211, 57–73, (2005).
14. A.M. Wazwaz, An analytic study of compactons solutions for variants of Kuramoto–Sivanshinsky equation, *Appl. Math. Comput.*, 148, 571– 585, (2004).
15. A.M. Wazwaz, The tanh method for generalized forms of nonlinear heat conduction and Burgers–Fisher equation, *Appl. Math. Comput.*, 169, 321–338, (2005).
16. A.M. Wazwaz, Travelling wave solutions of generalized forms of Burgers, Burgers-KdV and Burgers–Huxley equations, *Appl. Math. Comput.*, 169, 639–656, (2005).

Chapter 18
Families of Camassa-Holm and Schrodinger Equations

18.1 Introduction

In this chapter, we will study two families of nonlinear evolution equations that gained its importance because of its appearance in many scientific applications and physical phenomena. These are the family of Camassa-Holm equations and the family of Schrodinger equations.

The Camassa-Holm family of equations is of the form

$$u_t - u_{xxt} + au_x + buu_x = ku_x u_{xx} + uu_{xxx}, \tag{18.1}$$

where a, b, and k are constants, and $u(x,t)$ is the unknown function depending on temporal variable t and spatial variable x. For $b = 3$ and $k = 2$, Eq. (18.1) reduces to the Camassa-Holm equation

$$u_t - u_{xxt} + au_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{18.2}$$

that models the unidirectional propagation of nonlinear shallow-water waves over a flat bottom [1]. In (18.2), $u(x,t)$ is the fluid velocity, or, equivalently, the water's free surface. One aspect of the Camassa-Holm equation is its bi-Hamiltonian structure and the existence of many conservation laws. This feature indicates that the CH equation is completely integrable. Another aspect of interest is that for $a = 0$, it admits a new kind of of solitary waves which have a discontinuous slope at their crest. The peaked solitary wave is termed *peakon* which has nonanalytic nature unlike the smooth soliton. Peakon has discontinuities in the spatial derivative. Both one-sided spatial derivatives exist and differ only by a sign. For $a = 0$, the CH equation gives multi-soliton solutions in the form of peaked solitary waves.

We will study in this chapter four well-known equations of this family for the following specific values of the constants b and k. The set of constants

$$\begin{aligned} b &= 3, k = 2, \\ b &= 4, k = 3, \end{aligned}$$

$$\begin{aligned} &b=1, k=3, a=1, \\ &b=3, k=2, a \text{ is replaced by } 2a, \end{aligned} \tag{18.3}$$

are used to obtain the following equations

$$\begin{aligned} u_t - u_{xxt} + au_x + 3uu_x &= 2u_x u_{xx} + uu_{xxx}, \\ u_t - u_{xxt} + au_x + 4uu_x &= 3u_x u_{xx} + uu_{xxx}, \\ u_t - u_{xxt} + u_x + uu_x &= 3u_x u_{xx} + uu_{xxx}, \\ u_t - u_{xxt} + 2au_x + 3uu_x &= 2u_x u_{xx} + uu_{xxx}, \end{aligned} \tag{18.4}$$

known as the Camassa-Holm equation (CH) [1], the Degasperis-Procesi (DP) equation [2], the Fornberg-Whitham (FW) equation [12], and Fuchssteiner-Fokas-Camassa-Holm (FFCH) equation [4,5] respectively. It is obvious that these four equations contain both linear dispersion term u_{xxt} and the nonlinear dispersion term uu_{xxx}.

As stated before, the completely integrable wave Camassa-Holm (CH) equation describes the unidirectional propagation of shallow water waves over a flat bottom and possesses peakons solutions if $a=0$. The CH equation is bi-Hamiltonian and has an infinite number of conservation laws. It has been shown that the CH equation has peaked solitary wave solutions of the form

$$u(x,t) = ce^{(-|x-ct|)}, \tag{18.5}$$

if $a=0$, where c is the wave speed. Solitary wave solutions of the CH equation have discontinuous first derivative at the wave peak in contrast to the smoothness of most previously known species of solitary waves and thus are called peakons. The name "peakons", that is, solitary waves with slope discontinuities, was used to single them from general solitary wave solutions since they have a corner at the peak of height c.

The completely integrable Degasperis-Procesi (DP) equation [2,8] also possesses peakons solutions if $a=0$. The CH and the DP equations are bi-Hamiltonian and have an associated isospectral problem. Both equations are formally integrable [2] by means of the scattering/inverse scattering approach. Moreover, both equations admit peaked solitary wave solutions. The CH and the DP equations present similarities although they are truly different. The isospectral problem for the DP equation is of third order, whereas the CH equation admits a second order isospectral problem [2].

Degasperis and Procesi [2] showed that the family of CH equations (18.1) cannot be integrable unless $b=3$ or $b=4$ by using the method of asymptotic integrability. The CH equation is a shallow water equation and was originally derived as an approximation to the incompressible Euler equation and found to be completely integrable with a Lax pair.

The Fornberg-Whitham (FW) equation appeared in the study of the qualitative behaviors of wave-breaking. This equation admits a peaked solution of the form

$$u(x,t) = Ae^{-\frac{1}{2}|x-\frac{4}{3}t|}. \tag{18.6}$$

The Fuchssteiner-Fokas-Camassa-Holm (FFCH) equation first arose in the work of Fuchssteiner and Fokas using a bi-Hamiltonian approach [5].

A second family of nonlinear equations will be examined in this chapter. This family of equations consists of the following fourth order nonlinear Schrodinger equations [10,11]

$$i\frac{\partial w}{\partial t}+a\frac{\partial^2 w}{\partial x^2}-b\frac{\partial^4 w}{\partial x^4}+c|w|^2 w=0, \tag{18.7}$$

and

$$i\frac{\partial w}{\partial t}+a\frac{\partial^2 w}{\partial x^2}-b\frac{\partial^4 w}{\partial x^4}+c|w|^{2n} w=0, \tag{18.8}$$

with a cubic and power law nonlinearities respectively. The function w is a complex valued function of the spatial coordinate x and the time t. The function w is a sufficiently differentiable function. The physical models (18.7) and (18.8) occur in various areas of physics, including nonlinear optics, plasma physics, superconductivity and quantum mechanics. With $b=0$, equations (18.7) and (18.8) collapse to the standard second order NLS equation. The fields of application of the standard NLS equation varies from optics, propagation of the electric field in optical fibers, self-focusing and collapse of Langmuir waves in plasma physics, to modelling deep water waves and freak waves in the ocean.

Moreover, the cubic complex Ginzburg-Landau equation [3,7,9] is given by

$$w_t=(1+ia)w_{xx}+Rw-(1+ib)|w|^2 w, \tag{18.9}$$

where $w(x,t):R^2\mapsto C$ is a complex function, x is real, $t\geqslant 0$, and the coefficients a,b, and R are real. The CGL (18.9) is the generic amplitude model describing the slow phase and amplitude modulations of a spatially distributed assembly of coupled oscillators near its Hopf bifurcation. The complex field w describes the modulations of the oscillator field, and b and R are two real control parameters [7].

The cubic CGL equation (18.9) has been used to study many practical problems such as chemical turbulence, Poiseuille flow, Taylor-Coutte flow, Rayleigh-Bénard convection, reaction-diffusion systems, nonlinear optics, and hydrodynamical stability problems. It exhibits rich dynamics and has become a paradigm for the transition to spatio-temporal chaos. The CGL can be thought of as a normal form for a Hopf bifurcation in a variety of spatially extended systems. In fact, the amplitude w describes slow modulations in space and time of the underlying bifurcating spatially periodic pattern.

In addition, the generalized complex Ginzburg-Landau equation with a nonlinearity of order $(2n+1)$ is given by

$$w_t=(1+ia)w_{xx}+Rw-(1+ib)|w|^{2n} w, \tag{18.10}$$

will be examined as well.

On the other hand, the generalized form of the quintic complex Ginzburg-Landau equation

$$w_t = (1+ia)w_{xx} + Rw - (1+ib)|w|^{2n}w - (1+id)|w|^{4n}w, \tag{18.11}$$

with nonlinearity of order $(4n+1)$ will be investigated. For $n=1$, Eq. (18.11) is a one-dimensional model of the large scale behavior of many non equilibrium pattern forming systems [9].

18.2 The Family of Camassa-Holm Equations

The family of the Camassa-Holm equations

$$u_t - u_{xxt} + au_x + buu_x = ku_xu_{xx} + uu_{xxx}, \tag{18.12}$$

will be handled by using the tanh-coth method [6]. Moreover, an exponential algorithm will be used as well to obtain peakon solutions.

18.2.1 Using the Tanh-coth Method

Equation (18.12) will be converted to the ODE

$$(a-c)u + cu'' + \frac{b}{2}u^2 - \frac{k-1}{2}(u')^2 - uu'' = 0, \tag{18.13}$$

obtained upon using the transformation $\xi = x - ct$. The only balance that works in this case is between uu'' and $(u')^2$, where we obtain $M = -1$. The tanh-coth method admits the use of

$$u(x,t) = (a_0 + a_1Y + b_1Y^{-1})^{-1}. \tag{18.14}$$

Substituting (18.14) into (18.13), proceeding as before to get
(i)

$$\begin{aligned} a_0 &= \frac{bk}{2\left[(k+1)(c-a)-bc\right]}, \quad a_1 = \pm\frac{bk}{2\left[(k+1)(c-a)-bc\right]},\\ b_1 &= 0, \quad \mu = \frac{1}{2}\sqrt{\frac{b}{k+1}}, \ \ k \neq -1. \end{aligned} \tag{18.15}$$

(ii)

$$\begin{aligned} a_0 &= \frac{bk}{2\left[(k+1)(c-a)-bc\right]}, \quad a_1 = 0,\\ b_1 &= \pm\frac{bk}{2\left[(k+1)(c-a)-bc\right]}, \quad \mu = \frac{1}{2}\sqrt{\frac{b}{k+1}}, \ \ k \neq -1, \end{aligned} \tag{18.16}$$

where c is left as a free parameter for these two cases. Consequently, we obtain the solutions

$$u_1(x,t) = \frac{2\left[(k+1)(c-a)-bc\right]}{bk\left(1\pm\tanh\left[\frac{1}{2}\sqrt{\frac{b}{k+1}}(x-ct)\right]\right)}, \quad k\neq -1, \tag{18.17}$$

that blows up for $x \to -\infty$ for $+\tanh$, and blows up for $x \to \infty$ for $-\tanh$. This in turn gives the solutions

$$\begin{aligned}
u(x,t) &= -\frac{a}{\left(1\pm\tanh\left[\frac{1}{2}(x-ct)\right]\right)},\\
u(x,t) &= -\frac{2a}{3\left(1\pm\tanh\left[\frac{1}{2}(x-ct)\right]\right)},\\
u(x,t) &= \frac{2(3c-4a)}{3\left(1\pm\tanh\left[\frac{1}{4}(x-ct)\right]\right)},\\
u(x,t) &= -\frac{2a}{\left(1\pm\tanh\left[\frac{1}{2}(x-ct)\right]\right)},
\end{aligned} \tag{18.18}$$

for the CH, DP, FW, and FFCH equations respectively. The previous solutions are valid only for $b>0, k>0$. However, for $b<0, k>0$, we obtain complex solutions.

The second set gives the general solution for the CH family given by

$$u_2(x,t) = \frac{2\left[(k+1)(c-a)-bc\right]}{bk\left(1\pm\coth\left[\frac{1}{2}\sqrt{\frac{b}{k+1}}(x-ct)\right]\right)}, \quad k\neq -1. \tag{18.19}$$

However, if c is not left as a free parameter, we obtain the following sets of solutions:
(iii)

$$\begin{aligned}
&a_0 = -\frac{b+k-1}{4a}, \quad a\neq 0, \quad a_1 = \pm\frac{b+k-1}{4a}, \quad b_1 = 0,\\
&c = \frac{a(k-1)}{b+k-1}, \quad \mu = \frac{1}{2}\sqrt{\frac{b}{k+1}}, \ \ k\neq -1.
\end{aligned} \tag{18.20}$$

(iv)

$$\begin{aligned}
&a_0 = -\frac{b+k-1}{4a}, \quad a\neq 0, \quad a_1 = 0, \quad b_1 = \pm\frac{b+k-1}{4a},\\
&c = \frac{a(k-1)}{b+k-1}, \quad \mu = \frac{1}{2}\sqrt{\frac{b}{k+1}}, \ \ k\neq -1.
\end{aligned} \tag{18.21}$$

for $b+k\neq 1$. The last two sets give the solutions

$$u_3(x,t) = -\frac{4a}{(b+k-1)\left(1\pm\tanh\left[\frac{1}{2}\sqrt{\frac{b}{k+1}}\left(x-\frac{a(k-1)}{b+k-1}t\right)\right]\right)}, \tag{18.22}$$

and

$$u_4(x,t) = -\frac{4a}{(b+k-1)\left(1 \pm \coth\left[\frac{1}{2}\sqrt{\frac{b}{k+1}}\left(x - \frac{a(k-1)}{b+k-1}t\right)\right]\right)}, \tag{18.23}$$

for $b+k \neq 1, \dfrac{b}{k+1} > 0$. It is important to note that the specific solutions for the CH, DP, FW and FFCH equations can be obtained by substituting the prescribed values of b and k, i.e.

$$b = 3, k = 2,\ b = 4, k = 3,,\ b = 1, k = 3, a = 1,\ b = 3, k = 2, a \text{ is replaced by } 2a, \tag{18.24}$$

for the CH, DP, FW and FFCH equations respectively.

It is interesting to point out that the following set of solutions

$$u_5(x,t) = \lambda_0 + \left(-2\lambda_0 + \frac{2\left[(k+1)(c-a)-bc\right]}{bk}\right)\cosh^2\left[\frac{1}{2}\sqrt{\frac{b}{k+1}}(x-ct)\right], \tag{18.25}$$

$$u_6(x,t) = \lambda_0 + \left(-2\lambda_0 + \frac{2\left[(k+1)(c-a)-bc\right]}{bk}\right)\cos^2\left[\frac{1}{2}\sqrt{-\frac{b}{k+1}}(x-ct)\right], \tag{18.26}$$

$$u_7(x,t) = \lambda_0 + \left(2\lambda_0 - \frac{2\left[(k+1)(c-a)-bc\right]}{bk}\right)\sinh^2\left[\frac{1}{2}\sqrt{\frac{b}{k+1}}(x-ct)\right], \tag{18.27}$$

$$u_8(x,t) = \lambda_0 - \left(2\lambda_0 - \frac{2\left[(k+1)(c-a)-bc\right]}{bk}\right)\sin^2\left[\frac{1}{2}\sqrt{-\frac{b}{k+1}}(x-ct)\right], \tag{18.28}$$

can be shown to satisfy the family of the Camassa-Holm equations.

18.2.2 Using an Exponential Algorithm

The exponential algorithm assumes that the solution $u(x,t)$ can be set in the form

$$u(x,t) = \lambda + \alpha e^{-\mu|x-ct|}, \tag{18.29}$$

where λ, α, and μ are parameters that will be determined.

Substituting the exponential algorithm (18.29) into

$$u_t - u_{xxt} + au_x + buu_x = ku_xu_{xx} + uu_{xxx}, \tag{18.30}$$

and proceeding as before we find

$$
\begin{aligned}
\alpha &= \alpha_0, \alpha_0 \text{ is arbitrary constant,} \\
\lambda &= \frac{(k+1)(c-a)-bc}{bk}, \\
\mu &= \sqrt{\frac{b}{k+1}}, \quad k \neq -1.
\end{aligned} \tag{18.31}
$$

This in turn gives the peakon solution

$$
u(x,t) = \frac{(k+1)(c-a)-bc}{bk} + \alpha_0 e^{-\sqrt{\frac{b}{k+1}}|x-ct|}, \quad k \neq -1. \tag{18.32}
$$

This means that we obtain the following peakons solutions

$$
\begin{aligned}
u(x,t) &= -\frac{a}{2} + \alpha_0 e^{-|x-ct|}, \\
u(x,t) &= -\frac{a}{3} + \alpha_0 e^{-|x-ct|}, \\
u(x,t) &= -a + \alpha_0 e^{-|x-ct|}, \\
u(x,t) &= \frac{3c-4}{3} + \alpha_0 e^{-\frac{1}{2}|x-ct|},
\end{aligned} \tag{18.33}
$$

for the CH, DP, FFCH and FW equations respectively. For $a=0$, we obtain the well known solutions obtained by other methods.

18.3 Schrodinger Equation of Cubic Nonlinearity

The nonlinear Schrodinger equation with cubic nonlinearity [10,11] is given by

$$
i\frac{\partial w}{\partial t} + a\frac{\partial^2 w}{\partial x^2} - b\frac{\partial^4 w}{\partial x^4} + c|w|^2 w = 0, \tag{18.34}
$$

will be studied in this section by using the tanh-coth method only.

We first set the transformation

$$
w(x,t) = u(x)e^{i\alpha t}, i^2 = -1, \tag{18.35}
$$

where α is a real parameter. The assumption (18.35) transforms (18.34) into the ODE

$$
bu^{(4)} - au'' + \alpha u - cu^3 = 0. \tag{18.36}
$$

Balancing $u^{(4)}$ with u^3 in (18.36) gives $M=2$. This means that

$$
u(x) = \sum_{i=0}^{2} a_i Y^i + \sum_{i=1}^{2} b_i Y^{-i}. \tag{18.37}
$$

Substituting (18.37) into (18.36), collecting the coefficients of Y, and solving the resulting system we find

$$\begin{aligned} &a_0 = -\sqrt{\frac{3a^2}{10bc}}, \quad a_1 = 0, \quad a_2 = \sqrt{\frac{3a^2}{10bc}}, \\ &b_1 = b_2 = 0, \quad \mu = \sqrt{\frac{a}{20b}}, \ \frac{a}{b} > 0, \ \alpha = \frac{4a^2}{25b}, \end{aligned} \tag{18.38}$$

and

$$\begin{aligned} &a_0 = -\sqrt{\frac{3a^2}{10bc}}, \quad a_1 = a_2 = 0, \\ &b_1 = 0, \quad b_2 = \sqrt{\frac{3a^2}{10bc}}, \quad \mu = \sqrt{\frac{a}{20b}}, \ \frac{a}{b} > 0, \ \alpha = \frac{4a^2}{25b}, \end{aligned} \tag{18.39}$$

For $\frac{a}{b} < 0$, we obtain the following solutions

$$w(x,t) = \sqrt{\frac{3a^2}{10bc}} \sec^2(\sqrt{-\frac{a}{20b}}x)e^{i\alpha t}, \tag{18.40}$$

and

$$w(x,t) = \sqrt{\frac{3a^2}{10bc}} \csc^2(\sqrt{-\frac{a}{20b}}x)e^{i\alpha t}. \tag{18.41}$$

However, for $\frac{a}{b} > 0$, we obtain solitons solutions for $u(x)$, and therefore we find

$$w(x,t) = \sqrt{\frac{3a^2}{10bc}} \operatorname{sech}^2(\sqrt{\frac{a}{20b}}x)e^{i\alpha t}, \tag{18.42}$$

and

$$w(x,t) = \sqrt{\frac{3a^2}{10bc}} \operatorname{csch}^2(\sqrt{\frac{a}{20b}}x)e^{i\alpha t}, \tag{18.43}$$

where $\alpha = \frac{4a^2}{25b}$.

18.4 Schrodinger Equation with Power Law Nonlinearity

We next consider the fourth-order nonlinear Schrodinger equation with power law nonlinearity given by

$$i\frac{\partial w}{\partial t} + a\frac{\partial^2 w}{\partial x^2} - b\frac{\partial^4 w}{\partial x^4} + c|w|^{2n}w = 0, \tag{18.44}$$

by following the same approach used before.

We first set the assumption

$$w(x,t) = u(x)e^{i\alpha t}, i^2 = -1, \tag{18.45}$$

where α is a real parameter. The assumption (18.45) transforms (18.44) into the ODE

$$bu^{(4)} - au'' + \alpha u - cu^{2n+1} = 0. \tag{18.46}$$

Balancing $u^{(4)}$ with u^{2n+1} in (18.46) we find

$$8 + M - 4 = (2n+1)M, \tag{18.47}$$

so that $M = \frac{2}{n}$. To obtain an analytic solution, we should use the transformation

$$u(x) = v^{\gamma}(x), \quad \gamma = \frac{1}{n}, \tag{18.48}$$

so that (18.46) becomes

$$\begin{array}{l} b\gamma v^3 v^{(4)} + 4b\gamma v^2 v' v''' + 3b\gamma(\gamma-1)v^2(v'')^2 \\ 6b\gamma(\gamma-1)(\gamma-2)v(v')^2 v'' + b\gamma(\gamma-1)(\gamma-2)(\gamma-3)(v')^4 \\ -a\gamma v^3 v'' - a\gamma(\gamma-1)v^2(v')^2 + \alpha v^4 - cv^6 = 0. \end{array} \tag{18.49}$$

Balancing v^6 with $v^3 v^{(4)}$ gives $M = 2$. This admits the use of the finite expansion

$$v(x) = \sum_{i=0}^{2} a_i Y^i + \sum_{i=1}^{2} b_i Y^{-i}. \tag{18.50}$$

Substituting (18.50) into (18.46), collecting the coefficients of Y, and solving the resulting system we find

$$\begin{aligned} a_0 &= -\sqrt{\frac{a^2(n+1)(n+2)(3n+2)}{4bc(n^2+2n+2)^2}}, \\ a_1 &= 0, \\ a_2 &= \sqrt{\frac{a^2(n+1)(n+2)(3n+2)}{4bc(n^2+2n+2)^2}}, \\ b_1 &= b_2 = 0, \\ \mu &= \frac{n}{2}\sqrt{\frac{a}{b(n^2+2n+2)}}, \frac{a}{b} > 0, \\ \alpha &= \frac{(n+1)^2 a^2}{b(n^2+2n+2)^2}, \end{aligned} \tag{18.51}$$

and

$$
\begin{aligned}
a_0 &= -\sqrt{\frac{a^2(n+1)(n+2)(3n+2)}{4bc(n^2+2n+2)^2}},\\
a_1 &= a_2 = 0,\\
b_1 &= 0,\\
b_2 &= \sqrt{\frac{a^2(n+1)(n+2)(3n+2)}{4bc(n^2+2n+2)^2}},\\
\mu &= \frac{n}{2}\sqrt{\frac{a}{b(n^2+2n+2)}}, \frac{a}{b} > 0,\\
\alpha &= \frac{(n+1)^2a^2}{b(n^2+2n+2)^2}.
\end{aligned}
\tag{18.52}
$$

For $\frac{a}{b} < 0$, we obtain the following periodic solutions

$$
w(x,t) = \left\{\sqrt{\frac{a^2(n+1)(n+2)(3n+2)}{4bc(n^2+2n+2)^2}}\sec^2\left(\frac{n}{2}\sqrt{-\frac{a}{b(n^2+2n+2)}}x\right)\right\}^{\frac{1}{n}} e^{i\alpha t},
\tag{18.53}
$$

and

$$
w(x,t) = \left\{\sqrt{\frac{a^2(n+1)(n+2)(3n+2)}{4bc(n^2+2n+2)^2}}\csc^2\left(\frac{n}{2}\sqrt{-\frac{a}{b(n^2+2n+2)}}x\right)\right\}^{\frac{1}{n}} e^{i\alpha t}.
\tag{18.54}
$$

However, for $\frac{a}{b} > 0$, we obtain solitons solutions for $u(x)$, and therefore we find

$$
w(x,t) = \left\{\sqrt{\frac{a^2(n+1)(n+2)(3n+2)}{4bc(n^2+2n+2)^2}}\operatorname{sech}^2\left(\frac{n}{2}\sqrt{\frac{a}{b(n^2+2n+2)}}x\right)\right\}^{\frac{1}{n}} e^{i\alpha t},
\tag{18.55}
$$

and

$$
w(x,t) = \left\{\sqrt{\frac{a^2(n+1)(n+2)(3n+2)}{4bc(n^2+2n+2)^2}}\operatorname{csch}^2\left(\frac{n}{2}\sqrt{-\frac{a}{b(n^2+2n+2)}}x\right)\right\}^{\frac{1}{n}} e^{i\alpha t},
\tag{18.56}
$$

where $\alpha = \frac{(n+1)^2a^2}{b(n^2+2n+2)^2}$.

18.5 The Ginzburg-Landau Equation

The cubic complex Ginzburg-Landau (cGL) equation is given by

$$
w_t = (1+ia)w_{xx} + Rw - (1+ib)|w|^2 w,
\tag{18.57}
$$

where $w(x,t): R^2 \mapsto C$ is a complex function, x is real, $t \geqslant 0$, and the coefficients a, b, and R are real. The cGL (18.57) is the generic amplitude model describing the slow phase and amplitude modulations of a spatially distributed assembly of coupled oscillators near its Hopf bifurcation [3]. The complex field w describes the modulations of the oscillator field, and b and R are two real control parameters.

The cubic cGL equation (18.57) has been used to study many practical problems such as chemical turbulence, Poiseuille flow, Taylor-Coutte flow, Rayleigh-Bénard convection, reaction-diffusion systems, nonlinear optics, and hydrodynamical stability problems. It exhibits rich dynamics and has become a paradigm for the transition to spatio-temporal chaos . The cGL can be thought of as a normal form for a Hopf bifurcation in a variety of spatially extended systems. In fact, the amplitude w describes slow modulations in space and time of the underlying bifurcating spatially periodic pattern [7].

The generalized complex Ginzburg-Landau equation with a nonlinearity of order $(2n+1)$ is given by

$$w_t = (1+ia)w_{xx} + Rw - (1+ib)|w|^{2n}w, \tag{18.58}$$

where n is a positive integer, $n \geqslant 1$. The function $w(x,t)$ and the parameters are the same as in (18.57).

On the other hand, the generalized form of the quintic equation

$$w_t = (1+ia)w_{xx} + Rw - (1+ib)|w|^{2n}w - (1+id)|w|^{4n}w, \tag{18.59}$$

with nonlinearity of order $(4n+1)$, will be examined as well.

We aim to derive explicit and implicit complex solutions for each equation by using the method of separation of variables to reduce each equation to an equivalent separable ODE that can be easily solved.

18.5.1 The Cubic Ginzburg-Landau Equation

We begin our analysis by investigating the cubic cGL

$$w_t = (1+ia)w_{xx} + Rw - (1+ib)|w|^2 w. \tag{18.60}$$

We assume that the complex field $w(x,t)$ can be expressed as

$$w(x,t) = u(t)e^{i\alpha x}, \quad i^2 = -1, \tag{18.61}$$

so that

$$\begin{aligned} w_t &= u'(t)e^{i\alpha x}, \\ w_{xx} &= -\alpha^2 u(t)e^{i\alpha x}. \end{aligned} \tag{18.62}$$

Substituting (18.62) into (18.60) gives

$$u'(t) = \beta u(t) + \gamma u^3(t), \tag{18.63}$$

where

$$\begin{aligned} \beta &= (1-\alpha^2) - ia\alpha^2, \\ \gamma &= -1 - ib. \end{aligned} \tag{18.64}$$

Solving the separable ODE (18.63) yields

$$\frac{1}{\beta}\ln u - \frac{1}{2\beta}\ln(\beta + \gamma u^2) = t + \delta, \tag{18.65}$$

form which we find

$$u(t) = \frac{\sqrt{\beta}e^{\beta(t+\delta)}}{\sqrt{1-\gamma e^{2\beta(t+\delta)}}}, \tag{18.66}$$

where δ is a constant of integration. Combining (18.66) with (18.61) we obtain the exact complex solution

$$w(x,t) = \frac{\sqrt{\beta}e^{\beta(t+\delta)}}{\sqrt{1-\gamma e^{2\beta(t+\delta)}}}\, e^{i\alpha x}, \tag{18.67}$$

where β and γ are complex defined before in (18.64).

However, plane wave solutions of the form

$$w(x,t) = Ce^{-iAt+iBx}, \tag{18.68}$$

can be assumed, where A, B and C are constants. Substituting (18.68) into (18.60) and solving for A we find

$$A = (B^2 a + b) + i(R - 1 - B^2). \tag{18.69}$$

This in turn gives the plane wave solutions

$$w(x,t) = Ce^{\left((R-1-B^2) - i(B^2 a + b)\right)t + iBx}. \tag{18.70}$$

18.5.2 The Generalized Cubic Ginzburg-Landau Equation

We now consider the generalized cGL

$$w_t = (1+ia)w_{xx} + Rw - (1+ib)|w|^{2n}w, \tag{18.71}$$

with a nonlinearity of order $(2n+1)$. Proceeding as before we assume that the complex field $w(x,t)$ can be expressed as

$$w(x,t) = u(t)e^{i\alpha x}, \quad i^2 = -1, \tag{18.72}$$

so that

$$u'(t) = \beta u(t) + \gamma u^{2n+1}(t), \tag{18.73}$$

where

$$\begin{aligned} \beta &= (1-\alpha^2) - ia\alpha^2, \\ \gamma &= -1 - ib. \end{aligned} \tag{18.74}$$

Solving the separable ODE (18.73) yields

$$\frac{2n+1}{2n\beta}\ln u - \frac{1}{2n\beta}\ln(\beta u + \gamma u^{2n+1}) = t + \delta_1, \tag{18.75}$$

form which we find

$$u(t) = \left\{ \frac{\sqrt{\beta}e^{2n\beta(t+\delta_1)}}{\sqrt{1-\gamma e^{2n\beta(t+\delta_1)}}} \right\}^{\frac{1}{2n}}, \tag{18.76}$$

where δ_1 is a constant of integration. Combining (18.76) with (18.72) we obtain the exact solution

$$w(x,t) = \left\{ \frac{\sqrt{\beta}e^{2n\beta(t+\delta_1)}}{\sqrt{1-\gamma e^{2n\beta(t+\delta_1)}}} \right\}^{\frac{1}{2n}} e^{i\alpha x}, \tag{18.77}$$

where β and γ are complex defined before in (18.74).

However, plane wave solutions of the form

$$w(x,t) = Ce^{-iAnt+iBnx}, \tag{18.78}$$

can be assumed, where A, B and C are constants. Substituting (18.78) into (18.71) and solving for A we find

$$A = \frac{(n^2B^2a+b) + i(R-1-n^2B^2)}{n}. \tag{18.79}$$

This in turn gives the plane wave solutions

$$w(x,t) = Ce^{\left((R-1-n^2B^2)-i(n^2B^2a+b)\right)t+iBx}. \tag{18.80}$$

18.5.3 The Generalized Quintic Ginzburg-Landau Equation

Finally, the generalized quintic cGL

$$w_t = (1+ia)w_{xx} + Rw - (1+ib)|w|^{2n}w - (1+id)|w|^{4n}w, \tag{18.81}$$

with nonlinearity of order $(4n+1)$ will be investigated. We first set

$$w(x,t) = u(t)e^{i\alpha x}, \quad i^2 = -1, \tag{18.82}$$

and proceeding as before we obtain

$$u'(t) = \beta u(t) + \gamma u^{2n+1}(t) + \lambda u^{4n+1}, \tag{18.83}$$

where

$$\beta = (R - \alpha^2) - ia\alpha^2, \quad \gamma = -1 - ib, \quad \lambda = -1 - id. \tag{18.84}$$

Solving the separable ODE (18.83) yields

$$\frac{1}{\beta}\ln u - \frac{1}{4n\beta}\ln(\beta + \gamma u^{2n} + \lambda u^{4n}) - \frac{\gamma}{2n\beta\sqrt{4\beta\lambda - \gamma^2}}\arctan\left[\frac{\gamma + 2\lambda u^{2n}}{\sqrt{4\beta\lambda - \gamma^2}}\right] = t + \delta_3, \tag{18.85}$$

hence we find the implicit solution

$$\ln\left(\frac{u^{4n}}{\beta + \gamma u^{2n} + \lambda u^{4n}}\right) - \frac{2\gamma}{\sqrt{4\beta\lambda - \gamma^2}}\arctan\left[\frac{\gamma + 2\lambda u^2}{\sqrt{4\beta\lambda - \gamma^2}}\right] = 4n\beta(t + \delta_3) \tag{18.86}$$

where δ_3 is a constant of integration. In view of (18.86), the amplitude function $w(x,t)$ can be obtained implicitly.

Plane wave solutions of the form

$$w(x,t) = Ce^{-iAnt + iBnx}, \tag{18.87}$$

can be assumed, where A, B and C are constants. Substituting (18.68) into (18.81) and solving for A we find

$$A = \frac{(n^2B^2a + b + d) + i(R - 2 - n^2B^2)}{n}. \tag{18.88}$$

This in turn gives the plane wave solutions

$$w(x,t) = Ce^{\left((R-2-n^2B^2) - i(n^2B^2a+b+d)\right)t + iBx}. \tag{18.89}$$

References

1. R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, 71 (11), 1661–1664, (1993).
2. A. Degasperis and M. Procesi, Asymptotic Integrability, Symmetry and Perturbation Theory, World Scientific, 23–37, (1999).
3. V. Eguiluz, E. Hernandez-Garcia and O. Piro, Boundary effects in the complex Ginzburg-Landau equation, *Internat. J. Bifur. Chaos*, 9(11), 2209–2214, (1999).
4. B. Fuchssteiner, Some tricks from the symmetry-toolbox for nonlinear equations: Generalization of the Camassa-Holm equation, *Physica D*, 95(3/4), 229–243, (1996).
5. B. Fuchssteiner and A. Fokas, Sympletic structures, their Backlund transformations and hereditary symmetries, *Physica D*, 4(1), 47–66 (1981).
6. Malfliet W., Solitary wave solutions of nonlinear wave equations, *Am. J. Phys.*, 60(7), 650–654, (1992).

7. P. Marcq, H. Chate and R. Conte, Exact solutions of the one-dimensional quintic complex Ginzburg-Landau equation, *Physica D*, 73 305–317 (1994).
8. Y. Matsuno, Cusp and loop soliton solutions of short-wave models for the Camassa-Holm and Degasperis-Procesi equations, *Phys. lett. A*, 359, 451–457 (2006).
9. W. Van Saarloos and P.C. Hohenberg, Fronts, pulses, sources and sinks in generalized complex Ginzburg-Landau equation, *Physica D*, 56 303–311 (1992).
10. A.M. Wazwaz, Reliable analysis for the nonlinear Schrodinger equations with a cubic and a power law nonlinearities, *Math. Comput. Modelling*, 43, 178–184, (2006).
11. A.M. Wazwaz, Exact solutions for the fourth order nonlinear Schrodinger equations with a cubic and a power law nonlinearities, *Math. Comput. Modelling*, 43, 802–808, (2006).
12. G.B. Whitham, Linear and Nonlinear Waves, John Wiley, New York, (1976).

Appendix A
Indefinite Integrals

A.1 Fundamental Forms

1. $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1.$
2. $\int \frac{1}{x} \, dx = \ln|x| + C.$
3. $\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C.$
4. $\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$
5. $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C.$
6. $\int \frac{1}{x\sqrt{x^2 - 1}} \, dx = \sec^{-1} x + C.$
7. $\int \cos ax \, dx = \frac{1}{a} \sin ax + C.$
8. $\int \sin ax \, dx = -\frac{1}{a} \cos ax + C.$
9. $\int \tan ax \, dx = -\frac{1}{a} \ln|\cos ax| + C.$
10. $\int \cot ax \, dx = \frac{1}{a} \ln \sin ax + C.$
11. $\int \tan ax \sec ax \, dx = \frac{1}{a} \sec ax + C.$
12. $\int \sec x \, dx = -\ln(\sec x - \tan x) + C.$
13. $\int \csc x \, dx = -\ln(\csc x + \cot x) + C.$
14. $\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C.$
15. $\int \csc^2 ax \, dx = -\frac{1}{a} \cot ax + C.$

A.2 Trigonometric Forms

1. $\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$
2. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$
3. $\int \sin^3 x\,dx = -\frac{1}{3}\cos x\left(2+\sin^2 x\right) + C.$
4. $\int \cos^3 x\,dx = \frac{1}{3}\sin x\left(2+\cos^2 x\right) + C.$
5. $\int \tan^2 x\,dx = \tan x - x + C.$
6. $\int \cot^2 x\,dx = -\cot x - x + C.$
7. $\int x\sin x\,dx = \sin x - x\cos x + C.$
8. $\int x\cos x\,dx = \cos x + x\sin x + C.$
9. $\int x^2\sin x\,dx = 2x\sin x - \left(x^2-2\right)\cos x + C.$
10. $\int x^2\cos x\,dx = 2x\cos x + \left(x^2-2\right)\sin x + C.$
11. $\int \sin x\cos x\,dx = \frac{1}{2}\sin^2 x + C.$
12. $\int \frac{1}{1+\sin x}\,dx = -\tan(\frac{1}{4}\pi - \frac{1}{2}x) + C.$
13. $\int \frac{1}{1-\sin x}\,dx = \tan(\frac{1}{4}\pi + \frac{1}{2}x) + C.$
14. $\int \frac{1}{1+\cos x}\,dx = \tan(\frac{1}{2}x) + C.$
15. $\int \frac{1}{1-\cos x}\,dx = -\cot(\frac{1}{2}x) + C.$

A.3 Inverse Trigonometric Forms

1. $\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C.$
2. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2} + C.$
3. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C.$
4. $\int x\sin^{-1} x\,dx = \frac{1}{4}[(2x^2-1)\sin^{-1} x + x\sqrt{1-x^2}] + C.$
5. $\int x\cos^{-1} x\,dx = \frac{1}{4}[(2x^2-1)\cos^{-1} x - x\sqrt{1-x^2}] + C.$

6. $\int x\tan^{-1}x\,dx = \frac{1}{2}[(x^2+1)\tan^{-1}x - x] + C.$
7. $\int x\cot^{-1}x\,dx = \frac{1}{2}[(x^2+1)\cot^{-1}x + x] + C.$
8. $\int \sec^{-1}x\,dx = x\sec^{-1}x - \ln(x+\sqrt{x^2-1}) + C.$
9. $\int x\sec^{-1}x\,dx = \frac{1}{2}[x^2\sec^{-1}x - \sqrt{x^2-1}] + C.$

A.4 Exponential and Logarithmic Forms

1. $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C.$
2. $\int xe^{ax}\,dx = \frac{1}{a^2}(ax-1)e^{ax} + C.$
3. $\int x^2e^{ax}\,dx = \frac{1}{a^3}(a^2x^2 - 2ax + 2)e^{ax} + C.$
4. $\int x^3e^{ax}\,dx = \frac{1}{a^4}(a^3x^3 - 3a^2x^2 + 6ax - 6)e^{ax} + C.$
5. $\int e^x \sin x\,dx = \frac{1}{2}(\sin x - \cos x)e^x + C.$
6. $\int e^x \cos x\,dx = \frac{1}{2}(\sin x + \cos x)e^x + C.$
7. $\int \ln x\,dx = x\ln x - x + C.$
8. $\int x\ln x\,dx = \frac{1}{2}x^2(\ln x - \frac{1}{2}) + C.$

A.5 Hyperbolic Forms

1. $\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C.$
2. $\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C.$
3. $\int x\sinh x\,dx = x\cosh x - \sinh x + C.$
4. $\int x\cosh x\,dx = x\sinh x - \cosh x + C.$
5. $\int \sinh^2 x\,dx = \frac{1}{2}(\sinh x\cosh x - x) + C.$
6. $\int \cosh^2 x\,dx = \frac{1}{2}(\sinh x\cosh x + x) + C.$
7. $\int \tanh ax\,dx = \frac{1}{a}\ln\cosh ax + C.$

8. $\displaystyle\int \coth ax\,dx = \frac{1}{a}\ln \sinh ax + C.$

9. $\displaystyle\int \operatorname{sech}^2 x\,dx = \tanh x + C.$

10. $\displaystyle\int \operatorname{csch}^2 x\,dx = -\coth x + C.$

A.6 Other Forms

1. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\frac{x}{a} + C.$

2. $\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C.$

3. $\displaystyle\int \frac{1}{\sqrt{2ax - x^2}}\,dx = \arccos\frac{a - x}{a} + C.$

4. $\displaystyle\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\frac{x + a}{x - a} + C.$

Appendix B
Series

B.1 Exponential Functions

1. $e^{ax} = 1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \cdots.$
2. $e^{-ax} = 1 - ax + \frac{(ax)^2}{2!} - \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \cdots.$
3. $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots.$
4. $a^x = 1 + x\ln a + \frac{1}{2!}(x\ln a)^2 + \frac{1}{3!}(x\ln a)^3 + \cdots, a > 0.$
5. $e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \cdots.$
6. $e^{\cos x} = e(1 - \frac{x^2}{2!} - \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots).$
7. $e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{57x^5}{5!} + \cdots.$
8. $e^{\sin^{-1} x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \cdots.$

B.2 Trigonometric Functions

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$
3. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots.$
4. $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \frac{1385x^8}{8!} + \cdots.$

B.3 Inverse Trigonometric Functions

1. $\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots, x^2 < 1.$
2. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$

B.4 Hyperbolic Functions

1. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots.$
2. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots.$
3. $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots.$

B.5 Inverse Hyperbolic Functions

1. $\sinh^{-1}x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots.$
2. $\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots.$

Appendix C
Exact Solutions of Burgers' Equation

1. $u(x,t) = 2\tan x$.

2. $u(x,t) = -2\cot x$.

3. $u(x,t) = -2\tanh x$.

4. $u(x,t) = -\dfrac{2}{x}$.

5. $u(x,t) = \dfrac{x}{t}$.

6. $u(x,t) = -2\coth x$.

7. $u(x,t) = \dfrac{x}{t}$.

8. $u(x,t) = \dfrac{x}{t} - \dfrac{2}{x+t}$.

9. $u(x,t) = \dfrac{x}{t} - \dfrac{2}{x+nt}$, n is an integer.

10. $u(x,t) = \dfrac{2}{x+n}$, n is an integer.

11. $u(x,t) = \dfrac{x}{t} + \dfrac{2}{x+t} + \dfrac{x+t}{2t^2-t}$.

12. $u(x,t) = -\dfrac{2}{1 \pm e^{-t-x}}$.

13. $u(x,t) = \dfrac{2}{1 \pm e^{-t+x}}$.

14. $u(x,t) = 1 + 2k\tan(k(x-t))$.

15. $u(x,t) = 1 - 2k\tanh(k(x-t))$.

16. $u(x,t) = 1 - \dfrac{1}{x-t}$.

17. $u(x,t) = \dfrac{2\sin x}{\cos x \pm e^t}$.

18. $u(x,t) = -\dfrac{2\cos x}{\sin x \pm e^t}$.

19. $u(x,t) = \dfrac{x}{t} + \dfrac{2}{t}\tan\dfrac{x}{t}$.

20. $u(x,t) = \dfrac{x}{t} - \dfrac{2}{t}\cot\dfrac{x}{t}$.

21. $u(x,t) = \dfrac{x}{t} - \dfrac{t}{x}$.

Appendix D
Padé Approximants for Well-Known Functions

D.1 Exponential Functions

D.1.1. $f(x) = e^x$,

$$[2/2] = \frac{12 + 6x + x^2}{12 - 6x + x^2},$$

$$[3/3] = \frac{120 + 60x + 12x^2 + x^3}{120 - 60x + 12x^2 - x^3},$$

$$[4/4] = \frac{1680 + 840x + 180x^2 + 20x^3 + x^4}{1680 - 840x + 180x^2 - 20x^3 + x^4}.$$

D.1.2. $f(x) = e^{-x}$,

$$[2/2] = \frac{12 - 6x + x^2}{12 + 6x + x^2},$$

$$[3/3] = \frac{120 - 60x + 12x^2 - x^3}{120 + 60x + 12x^2 + x^3},$$

$$[4/4] = \frac{1680 - 840x + 180x^2 - 20x^3 + x^4}{1680 + 840x + 180x^2 + 20x^3 + x^4}.$$

D.2 Trigonometric Functions

D.2.1. $f(x) = \sin x$,

$$[2/2] = \frac{6x}{6 + x^2},$$

$$[3/3] = \frac{60x - 7x^3}{60 + 3x^2},$$

$$[4/4] = \frac{5880x - 620x^3}{5880 + 360x^2 + 11x^4}.$$

D.2.2. $f(x) = \cos x$,

$$[2/2] = \frac{12 - 5x^2}{12 + 5x^2},$$

$$[3/3] = \frac{12 - 5x^2}{12 + 5x^2},$$

$$[4/4] = \frac{15120 - 6900x^2 + 313x^4}{15120 + 660x^2 + 13x^4}.$$

D.2.3. $f(x) = \tan x$,

$$[2/2] = \frac{3x}{3 - x^2},$$

$$[3/3] = \frac{15x - x^3}{15 - 6x^2},$$

$$[4/4] = \frac{105x - 10x^3}{105 - 45x^2 + x^4}.$$

D.2.4. $f(x) = \sec x$,

$$[2/2] = \frac{12 + x^2}{12 - 5x^2},$$

$$[3/3] = \frac{12 + x^2}{12 - 5x^2},$$

$$[4/4] = \frac{15120 + 660x^2 + 13x^4}{15120 - 6900x^2 + 313x^4}.$$

D.2.5. $f(x) = \tan^{-1} x$,

$$[2/2] = \frac{3x}{3 + x^2},$$

$$[3/3] = \frac{15x + 4x^3}{15 + 9x^2},$$

$$[4/4] = \frac{105x + 55x^3}{105 + 90x^2 + 9x^4}.$$

D.3 Hyperbolic Functions

D.3.1. $f(x) = \sinh x$,

$$[2/2] = \frac{6x}{6 - x^2},$$

$$[3/3] = \frac{60x + 7x^3}{60 - 3x^2},$$

$$[4/4] = \frac{5880x + 620x^3}{5880 - 360x^2 + 11x^4}.$$

D.3.2. $f(x) = \cosh x$,

$$[2/2] = \frac{12 + 5x^2}{12 - x^2},$$

$$[3/3] = \frac{12 + 5x^2}{12 - x^2},$$

$$[4/4] = \frac{15120 + 6900x^2 + 313x^4}{15120 - 660x^2 + 13x^4}.$$

D.4 Logarithmic Functions

D.4.1. $f(x) = \ln(1 + x)$,

$$[2/2] = \frac{6x + 3x^2}{6 + 6x + x^2},$$

$$[3/3] = \frac{60x + 60x^2 + 11x^3}{60 + 90x + 36x^2 + 3x^3},$$

$$[4/4] = \frac{420x + 630x^2 + 260x^3 + 25x^4}{420 + 840x + 540x^2 + 120x^3 + 6x^4}.$$

D.4.2. $f(x) = \ln(1 - x)$,

$$[2/2] = \frac{-6x + 3x^2}{6 - 6x + x^2},$$

$$[3/3] = \frac{60x - 60x^2 + 11x^3}{-60 + 90x - 36x^2 + 3x^3},$$

$$[4/4] = \frac{-420x + 630x^2 - 260x^3 + 25x^4}{420 - 840x + 540x^2 - 120x^3 + 6x^4}.$$

D.4.3. $f(x) = \ln(1+x)/x$,

$$[2/2] = \frac{30 + 21x + x^2}{30 + 36x + 9x^2},$$

$$[3/3] = \frac{420 + 510x + 140x^2 + 3x^3}{420 + 720x + 360x^2 + 48x^3},$$

$$[4/4] = \frac{3780 + 6510x + 3360x^2 + 505x^3 + 6x^4}{3780 + 8400x + 6300x^2 + 1800x^3 + 150x^4}.$$

D.4.4. $f(x) = \ln(1-x)/x$,

$$[2/2] = \frac{30 - 21x + x^2}{-30 + 36x - 9x^2},$$

$$[3/3] = \frac{420 - 510x + 140x^2 - 3x^3}{-420 + 720x - 360x^2 + 48x^3},$$

$$[4/4] = \frac{3780 - 6510x + 3360x^2 - 505x^3 + 6x^4}{-3780 + 8400x - 6300x^2 + 1800x^3 - 150x^4}.$$

Appendix E
The Error and Gamma Functions

E.1 The Error function

The error function erf (x) is defined by:

1.erf $(x) = \dfrac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$

2.erf $(x) = \dfrac{2}{\sqrt{\pi}} \left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5 \cdot 2!} - \dfrac{x^7}{7 \cdot 3!} + \cdots \right).$

The complementary error function erf $c(x)$ is defined by :

3.erf $c(x) = \dfrac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du.$

4.erf $(x) +$ erf $c(x) = 1.$

5.erf $c(x) = 1 - \dfrac{2}{\sqrt{\pi}} \left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5 \cdot 2!} - \dfrac{x^7}{7 \cdot 3!} + \cdots \right).$

E.2 The Gamma function $\Gamma(x)$

1. $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$

2. $\Gamma(x+1) = x\Gamma(x), \Gamma(1) = 1, \Gamma(n+1) = n!,$ n is an integer .

3. $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin \pi x}.$

4. $\Gamma(1/2) = \sqrt{\pi}.$

Appendix F
Infinite Series

F.1 Numerical Series

1. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln 2.$

2. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$

5. $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$

6. $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1.$

7. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} = 1 - 2\ln 2.$

9. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}.$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+2)} = -\frac{1}{4}.$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}.$

12. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{n^{2k}} = \frac{n^2}{n^2+1}.$

13. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} = 2 = 2.718281828....$

14. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e} = 0.3678794412....$

F.2 Trigonometric Series

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\sin(kx) = \frac{1}{2}(\pi - x), \quad 0 < x < 2\pi.$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\sin(kx) = \frac{1}{2}x, \quad -\pi < x < \pi.$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin[(2k-1)x]}{2k-1} = \frac{\pi}{4}, \quad 0 < x < \pi.$

4. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1}\frac{\sin[(2k-1)x]}{2k-1} = \frac{1}{2}\ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right), \quad -\frac{1}{2}\pi < x < \frac{1}{2}\pi.$

5. $\displaystyle\sum_{k=1}^{\infty} (-1)^k\frac{\sin[(2k-1)x]}{(2k+1)^2} = \begin{cases} \frac{1}{4}\pi x, & \text{if} \quad -\frac{1}{2}\pi < x < \frac{1}{2}\pi, \\ \frac{1}{4}\pi(\pi - x), & \text{if} \quad \frac{1}{2}\pi < x < \frac{3}{2}\pi. \end{cases}$

6. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\cos(kx) = -\ln\left(2\sin\frac{1}{2}x\right), \quad 0 < x < 2\pi.$

7. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\cos(kx) = \ln\left(2\cos\frac{1}{2}x\right), \quad -\pi < x < \pi.$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}\cos(kx) = \frac{1}{12}\left(3x^2 - 6\pi x + 2\pi^2\right), \quad 0 < x < 2\pi.$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\cos(kx) = \frac{1}{12}\left(3x^2 - \pi^2\right), \quad -\pi < x < \pi.$

10. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos[(2k-1)x]}{2k-1}=\frac{\pi}{4},\ \ 0<x<\pi.$

11. $\displaystyle\sum_{k=1}^{\infty}\frac{\cos[(2k-1)x]}{(2k-1)^2}=\frac{\pi}{4}\left(\frac{\pi}{2}-|x|\right),\ \ -\pi<x<\pi.$

Answers

Exercises 1.2

1. (a) 3 (b) 2 (c) 1 (d) 4
2. (a) linear (b) nonlinear
   (c) linear (d) nonlinear
3. (a) inhomogeneous (b) homogeneous
   (c) inhomogeneous (d) homogeneous

Exercises 1.3

1. Hyperbolic 2. Elliptic 3. Parabolic
4. Hyperbolic 5. Parabolic 6. Hyperbolic
7. Elliptic 8. Parabolic 9. Hyperbolic
10. Elliptic if $y > 0$, Parabolic if $y = 0$, Hyperbolic if $y < 0$

Exercises 2.2

1. $u(x,y) = x^2y^2$ 2. $u(x,y) = x^2 + y^2$
3. $u(x,y) = e^{-xy}$ 4. $u(x,y) = e^x + e^y$
5. $u(x,y) = e^{x+y}$ 6. $u(x,y) = x - y$
7. $u(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$ 8. $u(x,y) = x + e^y$
9. $u(x,y) = y + e^x$ 10. $u(x,y) = xe^y$
11. $u(x,y) = ye^x$ 12. $u(x,y) = e^{x-y}$
13. $u(x,y,z) = x + y + z$ 14. $u(x,y,z) = e^{x+y+z}$
15. $u(x,y,z) = yze^x$ 16. $u(x,y,z) = e^x + y + z$
17. $u(x,y) = 3x^2 + 2x + 4xy + y + y^2$ 18. $u(x,y) = x^2 + x + 2xy + y + y^2$

19. $u(x,y) = \sin(x+y)$ 20. $u(x,y) = \cosh(x+y)$

Exercises 2.3

1. $u(x,y) = x^3 + y^3$ 2. $u(x,y) = \cosh x + \cosh y$
3. $u(x,y) = xy$ 4. $u(x,y) = \sin x + \cos y$
5. $u(x,y) = x\sin y + y\sin x$ 6. $u(x,y) = x\cos y - y\cos x$
7. $u(x,y) = xe^y + ye^x$ 8. $u(x,y) = xe^{-y} - ye^{-x}$
9. $u(x,y) = x^2y^2$ 10. $u(x,y) = xy^2 + yx^2$
11. $u(x,y) = \sin x + \cosh y$ 12. $u(x,y) = xe^y$
13. $u(x,y) = x^2y^3 + x^3y^2$ 14. $u(x,y) = x^3y^4 + x^4y^3$
15. $u(x,y) = (x+y)^2$ 16. $u(x,y) = (x+y)^2$
17. $u(x,y) = x^2 - xy + y^2$ 18. $u(x,y) = e^x + e^y + x$

Exercises 2.4

1. $u(x,y) = x^3 + y^3$ 2. $u(x,y) = x^2 - y^2$
3. $u(x,y) = (x+y)^2$ 4. $u(x,y) = \cosh x + \cosh y$
5. $u(x,y) = \sin x + \cos y$ 6. $u(x,y) = x^2y^2$
7. $u(x,y) = \sin x + \sin y$ 8. $u(x,y) = e^x - e^y$
9. $u(x,y) = x^2 + y^3$ 10. $u(x,y) = 1 + x^2 + \sin y$
11. $u(x,y) = 1 + x + \sinh y$ 12. $u(x,y) = \sin x + \sinh y$

Exercises 2.5

Answers are the same as in Exercises 2.2

Exercises 2.6

1. $u(x,y) = x^2 + y^2$ 2. $u(x,y) = e^x + e^y$
3. $u(x,y) = e^{-xy}$ 4. $u(x,y) = 2x + 3y - \cos x$
5. $u(x,y) = e^{x+y}$ 6. $u(x,y) = xe^y$
7. $u(x,y) = ye^x$ 8. $u(x,y) = \cosh x + \cosh y$
9. $u(x,y) = 2xy$ 10. $u(x,y) = x^2y^2$
11. $u(x,y) = y - kx$ 12. $u(x,y) = x - y$
13. $u(x,y) = x + \sin y$ 14. $u(x,y) = 4xe^y$

15. $u(x,y) = xy$

16. $u(x,y) = e^x + e^y$

17. $u(x,y) = e^{xy}$

18. $u(x,y) = e^{x+y}$

19. $u(x,y) = x\sinh y$

20. $u(x,y) = y\cosh x$

Exercises 2.7

1. $u(x,t) = \sinh(x+t), \quad \upsilon(x,t) = \cosh(x+t)$
2. $u(x,t) = \sin(x+t), \quad \upsilon(x,t) = \cos(x+t)$
3. $u(x,t) = \cos(x+t), \quad \upsilon(x,t) = \cos(x+t)$
4. $u(x,t) = (x+t), \quad \upsilon(x,t) = (x-t)$
5. $u(x,t) = \sin x + \sin t, \quad \upsilon(x,t) = \sin x - \sin t$
6. $u(x,t) = \sinh(x-t), \quad \upsilon(x,t) = \cosh(x-t)$
7. $u(x,y,t) = -w = \sin(x+y-t), \quad \upsilon(x,y,t) = \cos(x+y-t)$
8. $u(x,y,t) = \sin(x+y+t), \quad \upsilon(x,y,t) = -w = \cos(x+y+t)$
9. $u(x,y,t) = (x+y+t), \quad \upsilon(x,y,t) = (x+y-t), \quad w(x,y,t) = x-y+t$
10. $u(x,y,t) = e^x, \quad \upsilon(x,y,t) = e^y, \quad w(x,y,t) = e^t$

Exercises 3.2

1. $u(x,t) = x + e^{-t}\sin x$

2. $u(x,t) = 4 + e^{-t}\cos x$

3. $u(x,t) = e^{-t}\sin x$

4. $u(x,t) = e^{-5t}\sin x$

5. $u(x,t) = e^{-t}\sinh x$

6. $u(x,t) = e^{-t}\cosh x$

7. $u(x,t) = e^{-t}\sin x + \frac{1}{4}\sin 2x$

8. $u(x,t) = x^2 + e^{-t}\sin x$

9. $u(x,t) = x^3 + e^{-t}\sin x$

10. $u(x,t) = 3x^2 + e^{-t}\cos x$

11. $u(x,t) = x^2 + e^{-t}\cos x$

12. $u(x,t) = x^3 + e^{-t}\cos x$

13. $u(x,t) = 1 + e^{-\pi^2 t}\sin(\pi x)$

14. $1 + e^{-4\pi^2 t}\sin(\pi x)$

15. $u = e^{-4t}\cos x$

16. $u = x + e^{-2t}\sin x$

17. $u = e^{-t}\cos x$

18. $u = 2 + e^{-t}\cos x$

Exercises 3.3

The answers are the same as in Exercises 3.2.

Exercises 3.4.1

1. $u(x,t) = e^{-t}\sin x + 2e^{-9t}\sin(3x)$
2. $u(x,t) = e^{-\pi^2 t}\sin(\pi x) + e^{-4\pi^2 t}\sin(2\pi x)$

3. $u(x,t) = e^{-16t}\sin(2x)$

4. $u(x,t) = e^{-2\pi^2 t}\sin(\pi x)$

5. $u(x,t) = 1 + e^{-t}\cos x$

6. $u(x,t) = 3 + 4e^{-2t}\cos x$

7. $u(x,t) = 1 + e^{-3t}\cos x + e^{-12t}\cos(2x)$

8. $u(x,t) = 2 + 2e^{-16\pi^2 t}\cos(2\pi x)$

9. $u(x,t) = \dfrac{8}{\pi}\sum_{m=0}^{\infty}\dfrac{1}{(2m+1)}e^{-4(2m+1)^2 t}\sin(2m+1)x$

10. $u(x,t) = \dfrac{6}{\pi}\sum_{m=0}^{\infty}\dfrac{1}{(2m+1)}e^{-2(2m+1)^2 t}\sin(2m+1)x$

11. $u(x,t) = \pi - \dfrac{8}{\pi}\sum_{m=0}^{\infty}\dfrac{1}{(2m+1)^2}e^{-(2m+1)^2 t}\cos(2m+1)x$

12. $u(x,t) = (1 + \dfrac{\pi}{2}) - \dfrac{4}{\pi^2}\sum_{m=0}^{\infty}\dfrac{1}{(2m+1)}e^{-(2m+1)^2 t}\cos(2m+1)x$

Exercises 3.4.2

1. $u(x,t) = 1 + 2x + 3e^{-\pi^2 t}\sin(\pi x)$
2. $u(x,t) = 1 + 4e^{-t}\sin x$
3. $u(x,t) = x + e^{-4t}\sin 2x$
4. $u(x,t) = 4 - 4x + e^{-9t}\sin(3x)$
5. $u(x,t) = 2 + 3x + e^{-t}\sin x$
6. $u(x,t) = 1 + 2x + 3e^{-4\pi^2 t}\sin(2\pi x)$

Exercises 3.4.3

1. $u(x,t) = e^{-2t}\sin x$
2. $u(x,t) = e^{-5\pi^2 t}\sin(2\pi x)$
3. $u(x,t) = e^{-4t}\sin x$
4. $u(x,t) = e^{-3\pi^2 t}\sin(\pi x)$
5. $u(x,t) = 1 + e^{-2t}\sin x$
6. $u(x,t) = 3 + 3e^{-3\pi^2 t}\sin(\pi x)$

Exercises 4.2.1

1. $u(x,y,t) = e^{-4t}\sin x\sin y$
2. $u(x,y,t) = 2e^{-2t}\sin x\sin y$
3. $u(x,y,t) = e^{-4t}\cos(x+y)$
4. $u(x,y,t) = e^{-6t}\sin(x-y)$
5. $u(x,y,t) = e^{-5t}\sin x\sin y$
6. $u(x,y,t) = e^{-8t}\sin(x+y)$
7. $u(x,y,t) = e^{-4t}\sin x\sin y + \sin x$

8. $u(x,y,t) = e^{-6t}\sin x \sin y + \cos x$

9. $u(x,y,t) = e^{-2t}\sin(x+y) + \cos(x+y)$

10. $u(x,y,t) = e^{-2t}\sin x \sin y + \sin x + \sin y$

11. $u(x,y,t) = e^{-2t}\sin x \sin y + x^2$

12. $u(x,y,t) = e^{-2t}\sin x \sin y + y^2$

Exercises 4.2.2

1. $u = e^{-6t}\sin x \sin y \sin z$

2. $u = 2e^{-3t}\sin x \sin y \sin z$

3. $u = e^{-3t}\sin(x+y+z)$

4. $u = e^{-4t}\sin x \sin y \sin z$

5. $u = 2x^2 + e^{-3t}\sin x \sin y \sin z$

6. $u = y^2 + e^{-3t}\sin x \sin \sin z$

7. $u = \sin x + e^{-2t}\sin(y+z)$

8. $u = x^2 + e^{-3t}(\sin x + \sin y + \sin z)$

Exercises 4.3.1

1. $u(x,y,t) = e^{-13t}\sin 2x \sin 3y$

2. $u(x,y,t) = e^{-6t}\sin x \sin y + e^{-24t}\sin 2x \sin 2y$

3. $u(x,y,t) = e^{-8t}\sin x \sin y + e^{-20t}\sin x \sin 2y + e^{-20t}\sin 2x \sin y$

4. $u(x,y,t) = e^{-2t}\cos x \sin y$

5. $u(x,y,t) = e^{-2t}\sin x \cos y$

6. $u(x,y,t) = e^{-2t}\cos x \sin y + e^{-8t}\cos 2x \sin 2y$

7. $u(x,y,t) = e^{-4t}\sin x \cos y + e^{-16t}\sin 2x \cos 2y$

8. $u(x,y,t) = e^{-13t}\cos 2x \cos 3y$

9. $u(x,y,t) = 1 + e^{-5t}\cos x \cos 2y$

10. $u(x,y,t) = 4 + e^{-32t}\cos 2x \cos 2y$

Exercises 4.3.2

1. $u(x,y,z,t) = e^{-29t}\sin 2x \sin 3y \sin 4z$

2. $u(x,y,z,t) = e^{-3t}\sin x \sin y \sin z + e^{-12t}\sin 2x \sin 2y \sin 2z$

3. $u(x,y,z,t) = e^{-6t}\sin x \sin y \sin 2z + e^{-14t}\sin x \sin 2y \sin 3z$

4. $u(x,y,z,t) = e^{-3t}\cos x \sin y \sin z$

5. $u(x,y,z,t) = e^{-3t}\sin x \cos y \cos z$

6. $u(x,y,z,t) = e^{-3t}\cos x \sin y \cos z$

7. $u(x,y,z,t) = e^{-3t}\sin x\cos y\sin z$

8. $u(x,y,z,t) = 2 + 3e^{-6t}\cos x\cos 2y\cos z$

9. $u(x,y,z,t) = 1 + e^{-3t}\cos x\cos y\cos z + e^{-12t}\cos 2x\cos 2y\cos 2z$

10. $u(x,y,z,t) = 1 + 2e^{-3t}\cos x\cos y\cos z + 3e^{-29t}\cos 2x\cos 3y\cos 4z$

Exercises 5.2.2

1. $u(x,t) = \sin(2x)\cos(4t)$
2. $u(x,t) = \sin x\sin t + \sin x\cos t$
3. $u(x,t) = 2 + \cos x\cos t$
4. $u(x,t) = 1 + x + \sin x\sin t$
5. $u(x,t) = \sin x\cos(3t)$
6. $u(x,t) = \sin x\sin(2t)$
7. $u(x,t) = \cos x\cos t$
8. $u(x,t) = x + \cos x\cos t$
9. $u(x,t) = \cos x + \sin x\sin t$
10. $u(x,t) = \sin x + \sin x\cos t$
11. $u(x,t) = 1 + \sin x\sin(2t)$
12. $u(x,t) = x^4 + \sin x\cos t$
13. $u(x,t) = x^3 + \sin x\sin t$
14. $u(x,t) = 1 + \cos x + \sin x\sin t$
15. $u(x,t) = 2x^2 + \sin x\cos t$
16. $u(x,t) = x^2 + \cos x\sin t$
17. $u(x,t) = \sin x + \cos x\sin t$
18. $u(x,t) = x^2 + \cos x\cos t$
19. $u(x,t) = t^4 + t^3x + \cos x\sin t$
20. $u(x,t) = t^2 + x^3 + \sin x\sin t$
21. $u(x,t) = x^2\cosh t$
22. $u(x,t) = x^2e^t$
23. $u(x,t) = x^4\sinh t$
24. $u(x,t) = x^3\cosh t$

Exercises 5.2.3

1. $u(x,t) = 4t + \sin x\sin t$
2. $u(x,t) = \sin(x+t)$
3. $u(x,t) = \cos(x+t)$
4. $u(x,t) = \sin(x-t)$
5. $u(x,t) = 6t + \sin x\cos t$
6. $u(x,t) = 6t + 2xt^2$
7. $u(x,t) = x^2t + t^3 + e^x\sinh t$
8. $u(x,t) = \cos x\sin t + xe^t$
9. $u(x,t) = x^2 + t^2 + \sin x\sin t$
10. $u(x,t) = t + 2xt + \cos x(\cos t - 1)$
11. $u(x,t) = x^2 + t^2 - \sin x + \sin x\sin t$
12. $u(x,t) = \cos x + \cos x\cos t$

Exercises 5.3.2

The answers are given in 5.2.2.

Exercises 5.3.3

The answers are given in 5.2.3.

Exercises 5.4.1

1. $u(x,t) = \sin(3x)\sin(3t)$
2. $u(x,t) = \sin x \cos t$
3. $u(x,t) = 2\sin(2x)\sin(2t)$
4. $u(x,t) = \sin x \cos(2t)$
5. $u(x,t) = \sin(2x)\cos(4t)$
6. $u(x,t) = \sin x \sin(3t)$
7. $u(x,t) = 1 + \cos x \sin(3t)$
8. $u(x,t) = 2 + \cos x \cos(2t)$
9. $u(x,t) = \cos x \sin(3t)$
10. $u(x,t) = \cos x \sin t + \cos x \cos t$
11. $u(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5n^2} \sin(nx)\sin(nt)$
12. $u(x,t) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2} \sin(nx)\sin 2(2n+1)t$

Exercises 5.4.2

1. $u(x,t) = 1 + \sin(\pi x)\sin(\pi t)$
2. $u(x,t) = 2 + x + 2\sin(\pi x)\cos(\pi t)$
3. $u(x,t) = 3x + 4\sin(\pi x)\sin(\pi t)$
4. $u(x,t) = 4 - 3x + \sin(\pi x)\sin(\pi t)$
5. $u(x,t) = 3 + 4x + \sin(\pi x)\sin(2\pi t)$
6. $u(x,t) = 1 + x + \sin(2\pi x)\sin(4\pi t)$
7. $u(x,t) = 3x + x^2 + t^2 + \cos(\pi x)\cos(\pi t)$
8. $u(x,t) = 4x + \cos(\pi x)\sin(\pi t)$
9. $u(x,t) = 2x + 2x^2 + 2t^2 + \cos(\pi x)\cos(\pi t)$
10. $u(x,t) = x + \cos(\pi x)\sin(2\pi t)$
11. $u(x,t) = 2x + \cos(\pi x)\cos(2\pi t)$
12. $u(x,t) = x + x^2 + t^2 + \cos(\pi x)\sin(3\pi t)$

Exercises 5.5.1

1. $u(x,t) = 2t + \sin x \sin t$
2. $u(x,t) = \sin(x+t)$
3. $u(x,t) = \cos(x+t)$
4. $u(x,t) = \sin(x+4t)$
5. $u(x,t) = 2t + \sin x \cos t$
6. $u(x,t) = \cos(x+2t)$
7. $u(x,t) = \sinh(x+t)$
8. $u(x,t) = x + e^{-x} \sinh t$
9. $u(x,t) = \cosh x \cosh t$
10. $u(x,t) = \sinh x \sinh(2t)$
11. $u(x,t) = t + 2xt + \cos x \cos t$
12. $u(x,t) = 4t + 4xt + \sin x \cos t$

Exercises 6.2.1

1. $u(x,y,t) = \sin x \sin y \sin(2t)$
2. $u(x,y,t) = \sin(2x)\sin(2y)\sin(4t)$
3. $u(x,y,t) = \sin(2x)\sin(2y)\cos(4t)$
4. $u(x,y,t) = 2 + \sin x \sin y \sin t$
5. $u(x,y,t) = 1 + y + \sin x \sin y \sin(2t)$
6. $u(x,y,t) = 1 + x + \sin y \sin t$
7. $u(x,y,t) = \sin x \sin y \sin(2t)$
8. $u(x,y,t) = \sin x \sin y \cos(3t)$
9. $u(x,y,t) = \sin x + \sin x \sin y \sin t$
10. $u(x,y,t) = \cos x + \sin y \sin t$
11. $u(x,y,t) = x^2 + y^2 + \sin x \sin t$
12. $u(x,y,t) = 2x^2 + 2y^2 + 2\sin x \sin y \cos(2t)$
13. $u(x,y,t) = t^2 + tx + ty + \sin x \sin y \sin t$
14. $u(x,y,t) = t^3 + t^2 x + ty + \sin x \sin y \cos(2t)$
15. $u(x,y,t) = y^2 + \sin x \cos t$
16. $u(x,y,t) = x^2 + \sin y \sin t$
17. $u(x,y,t) = \sin x + \sin y \sin t$
18. $u(x,y,t) = \cos x \cos y \sin(2t)$
19. $u(x,y,t) = t^4 + t^2 y + \sin x \sin y \sin t$
20. $u(x,y,t) = t^2 + x^2 + y^2 + \sin x \sin y \sin t$
21. $u(x,y,t) = x^2 y^2 \sinh t$
22. $u(x,y,t) = x^2 y^2 e^t$
23. $u(x,y,t) = x^2 \sinh t + y^2 \cosh t$

24. $u(x,y,t) = x^2e^{-t} + y^2e^t$

25. $u(x,y,t) = y^2e^{-t} - x^2e^t$

Exercises 6.2.2

1. $u(x,y,z,t) = \sin 2x \sin 2y \sin 2z \sin 6t$
2. $u(x,y,z,t) = 1 + \sin x \sin y \sin z \sin t$
3. $u(x,y,z,t) = 3 + \sin x \sin y \sin z \cos 3t$
4. $u(x,y,z,t) = \sin x \sin y \sin z \sin 3t$
5. $u(x,y,z,t) = \sin(x+2y)\sin(z+2t)$
6. $u(x,y,z,t) = \sin x \sin 2y \sin 3z \cos 4t$
7. $u(x,y,z,t) = \cos(x+y)\sin(z+t)$
8. $u(x,y,z,t) = 1 + z + \sin x \sin y \sin z \sin t$
9. $u(x,y,z,t) = \sin x + \sin y + \sin z \sin t$
10. $u(x,y,z,t) = \cos x + \cos y + \sin z \sin t$
11. $u(x,y,z,t) = x^2 + y^2 + z^2 + \sin t$
12. $u(x,y,z,t) = t^2 + tx + ty + tz + \sin t$
13. $u(x,y,z,t) = t^2(x+y+z) + \sin y \sin t$
14. $u(x,y,z,t) = x^2 + y^2 + z^2 + \cos y \cos t$
15. $u(x,y,z,t) = x^2 + \sin y \sin z \cos t$
16. $u(x,y,z,t) = \cos x \cos y \cos z \sin 2t$
17. $u(x,y,z,t) = 1 + \sin x \sin y \sin z \cos 2t$
18. $u(x,y,z,t) = 1 + \sin x \sin y + \sin z \sin t$
19. $u(x,y,z,t) = t^2 + x^2 + y^2 + z^2 + \cos x \cos y \cos z \cos t$
20. $u(x,y,z,t) = x^4 + y^4 + \cos z \cos t$
21. $u(x,y,z,t) = x^2y^2z^2 \cosh t$
22. $u(x,y,z,t) = x^3 \sinh t + (y^3 + z^3)\cosh t$
23. $u(x,y,z,t) = x^2e^t + y^2e^{-t} + z^2e^t$
24. $u(x,y,z,t) = (x^3 + y^3 + z^3)\sinh t$

Exercises 6.3.1

1. $u(x,y,t) = \sin 2x \sin 2y \cos 4t$
2. $u(x,y,t) = \sin x \sin 2y \cos 5t$
3. $u(x,y,t) = \sin x \sin y \sin 2t$

4. $u(x,y,t) = \sin x \sin 2y \sin 5t$

5. $u(x,y,t) = 2 + \cos x \cos y \sin 2t$

6. $u(x,y,t) = 1 + \cos x \cos y \cos 4t$

7. $u(x,y,t) = \sin x \cos y \sin 2t$

8. $u(x,y,t) = \cos x \sin y \cos 2t$

9. $u(x,y,t) = \cos x \sin y \sin 2t$

10. $u(x,y,t) = 3 + \cos x \cos 2y \sin 5t$

11. $$u(x,y,t) = \frac{32}{\pi^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(2n+1)(2m+1)} \times \sin(2n+1)x \sin(2m+1)y \cos(\sqrt{2}\lambda_{nm} t),$$
$\lambda_{nm} = \sqrt{(2n+1)^2 + (2m+1)^2}$

12. $$u(x,y,t) = \frac{48}{\pi^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(2n+1)(2m+1)} \times \sin(2n+1)x \sin(2m+1)y \sin(\sqrt{2}\lambda_{nm} t),$$
$\lambda_{nm} = \sqrt{(2n+1)^2 + (2m+1)^2}$

Exercises 6.3.2

1. $u(x,y,z,t) = \sin 2x \sin 2y \sin 2z \sin 12t$

2. $u(x,y,z,t) = \sin x \sin 2y \sin 3z \cos 14t$

3. $u(x,y,z,t) = \sin x \sin y \sin 2z \sin 6t$

4. $u(x,y,z,t) = \cos x \cos 2y \cos 2z \sin 6t$

5. $u(x,y,z,t) = 3 + \cos x \cos y \cos z \sin 6t$

6. $u(x,y,z,t) = 4 + \cos x \cos y \cos z \cos 6t$

7. $u(x,y,z,t) = \sin x \cos y \cos z \sin 3t$

8. $u(x,y,z,t) = \sin x \sin y \cos z \sin 3t$

9. $u(x,y,z,t) = \cos x \sin y \sin z \cos 6t$

10. $u(x,y,z,t) = \sin x \sin y \cos 2z \cos 6t$

11. $u(x,y,z,t) = \sin x \sin y \cos z \sin \sqrt{3}t$

12. $u(x,y,z,t) = \cos x \cos y \cos 2z \sin 6t$

Exercises 7.2.1

1. $u(x,y) = \sinh x \cos y$ 2. $u(x,y) = \cosh x \sin y$

3. $u(x,y)=\cosh x\cos y$
4. $u(x,y)=\sin 2x\sinh 2y$
5. $u(x,y)=\sin 2x\cosh 2y$
6. $u(x,y)=\cos 3x\cosh 3y$
7. $u(x,y)=\sinh 2x\cos 2y$
8. $u(x,y)=\cosh 2x\cos 2y$
9. $u(x,y)=\cos x\cosh y$
10. $u(x,y)=\sin x\cosh y$
11. $u(x,y)=\sinh x\cos y$
12. $u(x,y)=\sin x\cosh y$
13. $u(x,y)=x+\sin x\sinh y$
14. $u(x,y)=y+\sin x\cosh y$
15. $u(x,y)=1+\sin x\sinh y$
16. $u(x,y)=1+\cos x\sinh y$

Exercises 7.3.1

Answers are the same as in Exercises 7.2.1.

Exercises 7.4.1

1. $u(x,y)=\sin 2x\sinh 2y$
2. $u(x,y)=\sinh 3x\sin 3y$
3. $u(x,y)=4\sinh 2x\sin 2y$
4. $u(x,y)=\cos x\cosh y$
5. $u(x,y)=\sin x\cosh y$
6. $u(x,y)=\cos 2x\cosh 2y$
7. $u(x,y)=\sinh x\sin(\pi-y)$
8. $u(x,y)=\sin 2x\sinh(2\pi-2y)$
9. $u(x,y)=C_0+\cos 2x\cosh 2y$
10. $u(x,y)=C_0+\cosh 2x\cos 2y$

Exercises 7.4.2

1. $u(x,y,z)=\sin x\sin 2y\sinh\sqrt{5}z$
2. $u(x,y,z)=\sin 6x\sin 8y\sinh 10z$
3. $u(x,y,z)=\sin 2x\sin 2y\sinh\sqrt{8}z$
4. $u(x,y,z)=\sin x\sin 2y\sinh\sqrt{5}(\pi-z)$
5. $u(x,y,z)=\sin 3x\sin 4y\sinh 5(\pi-z)$
6. $u(x,y,z)=\sin 5x\sin 12y\sinh 13(\pi-z)$
7. $u(x,y,z)=\cos x\cos 2y\cosh\sqrt{5}z+C_0$
8. $u(x,y,z)=\cos 5x\cos 12y\cosh 13z+C_0$
9. $u(x,y,z)=\cos 3x\cos 4y\cosh 5(\pi-z)+C_0$
10. $u(x,y,z)=\cos 2x\cos 2y\cosh\sqrt{8}+C_0(\pi-z)$
11. $u(x,y,z)=\sin 8x\sin 15y\cosh 17z$
12. $u(x,y,z)=\sin 3x\cos 4y\sinh 5z$

Exercises 7.5.1

1. $u(r,\theta)=2+3r\sin\theta+4r\cos\theta$
2. $u(r,\theta)=1+r^4\cos 4\theta$
3. $u(r,\theta)=1-r^6\cos 6\theta$
4. $u(r,\theta)=r^2\sin 2\theta+r^2\cos 2\theta$

5. $u(r,\theta) = C_0 + 2r^4 \sin 4\theta$

6. $u(r,\theta) = C_0 + r(\sin\theta - \cos\theta)$

7. $u(r,\theta) = C_0 + r^2 \sin 2\theta$

8. $u(r,\theta) = C_0 + r^3 \cos 3\theta$

9. $u(r,\theta) = C_0 + r^2 \sin 2\theta + r^3 \cos 3\theta$

10. $u(r,\theta) = C_0 + r^2 \cos 2\theta$

Exercises 7.5.2

1. $u(r,\theta) = 1 + \ln r + \left(r - \frac{1}{r}\right)\cos\theta + \left(r - \frac{1}{r}\right)\sin\theta$

2. $u(r,\theta) = 1 + \ln r + r\cos\theta + r\sin\theta$

3. $u(r,\theta) = 1 + \left(r - \frac{1}{r}\right)\sin\theta$

4. $u(r,\theta) = 1 + \left(r - \frac{2}{r}\right)\cos\theta + \left(r - \frac{2}{r}\right)\sin\theta$

5. $u(r,\theta) = 1 + \ln r + \left(r - \frac{1}{r}\right)\cos\theta$

6. $u(r,\theta) = 1 + \ln r + \left(r - \frac{1}{r}\right)\sin\theta$

7. $u(r,\theta) = C_0 + \ln r + \left(r + \frac{1}{r}\right)\sin\theta$

8. $u(r,\theta) = C_0 + \left(r - \frac{1}{r}\right)\cos\theta$

9. $u(r,\theta) = C_0 + \left(3r + \frac{2}{r}\right)\sin\theta$

10. $u(r,\theta) = C_0 + \ln r + \left(r + \frac{1}{r}\right)\cos\theta + \left(r + \frac{1}{r}\right)\sin\theta$

11. $u(r,\theta) = C_0 + 2\left(r - \frac{1}{r}\right)\cos\theta$

12. $u(r,\theta) = C_0 + \left(3r + \frac{2}{r}\right)\sin\theta$

Exercises 8.2

1. $$\begin{aligned} A_0 &= u_0^4 \\ A_1 &= 4u_0^3u_1 \\ A_2 &= 4u_0^3u_2 + 6u_0^2u_1^2 \\ A_3 &= 4u_0^3u_3 + 12u_0^2u_1u_2 + 4u_0u_1^3 \end{aligned}$$

$$
\begin{aligned}
2.\ A_0 &= u_0^2 + u_0^3 \\
A_1 &= u_1(2u_0 + 3u_0^2) \\
A_2 &= u_2(2u_0 + 3u_0^2) + u_1^2(1 + 3u_0) \\
A_3 &= u_3(2u_0 + 3u_0^2) + u_1u_2(2 + 6u_0) + u_1^3
\end{aligned}
$$

$$
\begin{aligned}
3.\ A_0 &= \cos 2u_0 \\
A_1 &= -2u_1 \sin 2u_0 \\
A_2 &= -2u_2 \sin 2u_0 - 2u_1^2 \cos 2u_0 \\
A_3 &= -2u_3 \sin 2u_0 - 4u_1u_2 \cos 2u_0 + \frac{4}{3}u_1^3 \sin 2u_0
\end{aligned}
$$

$$
\begin{aligned}
4.\ A_0 &= \sinh 2u_0 \\
A_1 &= 2u_1 \cosh 2u_0 \\
A_2 &= 2u_2 \cosh 2u_0 + 2u_1^2 \sinh 2u_0 \\
A_3 &= 2u_3 \cosh 2u_0 + 4u_1u_2 \sinh 2u_0 + \frac{4}{3}u_1^3 \cosh 2u_0
\end{aligned}
$$

$$
\begin{aligned}
5.\ A_0 &= e^{2u_0} \\
A_1 &= 2u_1 e^{2u_0} \\
A_2 &= 2(u_2 + u_1^2)e^{2u_0} \\
A_3 &= 2\left(u_3 + 2u_1u_2 + \frac{2}{3}u_1^3\right) e^{2u_0}
\end{aligned}
$$

$$
\begin{aligned}
6.\ A_0 &= u_0^2 u_{0_x} \\
A_1 &= 2u_0u_1u_{0_x} + u_0^2 u_{1_x} \\
A_2 &= 2u_0u_2u_{0_x} + u_1^2 u_{0_x} + 2u_0u_1u_{1_x} + u_0^2 u_{2_x} \\
A_3 &= 2u_0u_3u_{0_x} + 2u_1u_2u_{0_x} + 2u_0u_2u_{1_x} \\
&\quad + u_1^2 u_{1_x} + 2u_0u_1u_{2_x} + u_0^2 u_{3_x}
\end{aligned}
$$

$$
\begin{aligned}
7.\ A_0 &= u_0 u_{0_x}^2 \\
A_1 &= 2u_0u_{0_x}u_{1_x} + u_1 u_{0_x}^2 \\
A_2 &= 2u_0u_{0_x}u_{2_x} + u_0 u_{1_x}^2 + 2u_1u_{0_x}u_{1_x} + u_2 u_{0_x}^2 \\
A_3 &= 2u_0u_{0_x}u_{3_x} + 2u_0u_{1_x}u_{2_x} + 2u_1u_{0_x}u_{2_x} \\
&\quad + u_1 u_{1_x}^2 + 2u_2u_{0_x}u_{1_x} + u_{0_x}^2 u_3
\end{aligned}
$$

$$
\begin{aligned}
8.\ A_0 &= u_0 e^{u_0} \\
A_1 &= (u_0u_1 + u_1)e^{u_0} \\
A_2 &= \left(u_0u_2 + \frac{1}{2}u_0u_1^2 + u_1^2 + u_2\right) e^{u_0} \\
A_3 &= \left(u_0u_3 + u_0u_1u_2 + \frac{1}{6}u_0u_1^3 + 2u_1u_2 + \frac{1}{2}u_1^3 + u_3\right) e^{u_0}
\end{aligned}
$$

9. $$\begin{aligned} A_0 &= u_0 \sin u_0 \\ A_1 &= u_0 u_1 \cos u_0 + u_1 \sin u_0 \\ A_2 &= u_0 u_2 \cos u_0 - \frac{1}{2} u_0 u_1^2 \sin u_0 + u_1^2 \cos u_0 + u_2 \sin u_0 \\ A_3 &= u_0 u_3 \cos u_0 - u_0 u_1 u_2 \sin u_0 - \frac{1}{6} u_0 u_1^3 \cos u_0 \\ &\quad + 2 u_1 u_2 \cos u_0 - \frac{1}{2} u_1^3 \sin u_0 + u_3 \sin u_0 \end{aligned}$$

10. $$\begin{aligned} A_0 &= u_0 \cosh u_0 \\ A_1 &= u_0 u_1 \sinh u_0 + u_1 \cosh u_0 \\ A_2 &= u_0 u_2 \sinh u_0 + \frac{1}{2} u_0 u_1^2 \cosh u_0 + u_1^2 \sinh u_0 + u_2 \cosh u_0 \\ A_3 &= u_0 u_3 \sinh u_0 + u_0 u_1 u_2 \cosh u_0 + \frac{1}{6} u_0 u_1^3 \sinh u_0 \\ &\quad + 2 u_1 u_2 \sinh u_0 + \frac{1}{2} u_1^3 \cosh u_0 + u_3 \cosh u_0 \end{aligned}$$

11. $$\begin{aligned} A_0 &= u_0^2 + \sin u_0 \\ A_1 &= 2 u_0 u_1 + u_1 \cos u_0 \\ A_2 &= 2 u_0 u_2 + u_1^2 + u_2 \cos u_0 - \frac{1}{2} u_1^2 \sin u_0 \\ A_3 &= 2 u_0 u_3 + 2 u_1 u_2 + u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{6} u_1^3 \cos u_0 \end{aligned}$$

12. $$\begin{aligned} A_0 &= u_0 + \cos u_0 \\ A_1 &= u_1 - u_1 \sin u_0 \\ A_2 &= u_2 - u_2 \sin u_0 - \frac{1}{2} u_1^2 \cos u_0 \\ A_3 &= u_3 - u_3 \sin u_0 - u_1 u_2 \cos u_0 + \frac{1}{6} u_1^3 \sin u_0 \end{aligned}$$

13. $$\begin{aligned} A_0 &= u_0 + \ln u_0 \\ A_1 &= u_1 + \frac{u_1}{u_0} \\ A_2 &= u_2 + \frac{u_2}{u_0} - \frac{1}{2} \frac{u_1^2}{u_0^2} \\ A_3 &= u_3 + \frac{u_3}{u_0} - \frac{u_1 u_2}{u_0^2} + \frac{1}{3} \frac{u_1^3}{u_0^3} \end{aligned}$$

14. $$\begin{aligned} A_0 &= u_0 \ln u_0 \\ A_1 &= u_1 (1 + \ln u_0) \\ A_2 &= u_2 (1 + \ln u_0) + \frac{1}{2} \frac{u_1^2}{u_0} \end{aligned}$$

$$A_3 = u_3(1+\ln u_0) + \frac{u_1u_2}{u_0} - \frac{1}{6}\frac{u_1^3}{u_0^2}$$

15. $A_0 = u_0^{\frac{1}{2}}$

$$A_1 = \frac{1}{2}u_1u_0^{-\frac{1}{2}}$$
$$A_2 = \frac{1}{2}u_2u_0^{-\frac{1}{2}} - \frac{1}{8}u_1^2u_0^{-\frac{3}{2}}$$
$$A_3 = \frac{1}{2}u_3u_0^{-\frac{1}{2}} - \frac{1}{4}u_1u_2u_0^{-\frac{3}{2}} + \frac{1}{16}u_1^3u_0^{-\frac{5}{2}}$$

16. $A_0 = u_0^{-1}$

$$A_1 = -u_1u_0^{-2}$$
$$A_2 = -u_2u_0^{-2} + u_1^2u_0^{-3}$$
$$A_3 = -u_3u_0^{-2} + 2u_1u_2u_0^{-3} - u_1^3u_0^{-4}$$

Exercises 8.3

1. $y = \tan 3x$
2. $y = \tanh 4x$
3. $y = 1 + x$
4. $y = 1 - \ln(1+ex), -1 < ex \leqslant 1$
5. $y = x - 2$
6. $y = 1 + \dfrac{1}{1-x}$
7. $y = x$
8. $y = -x$
9. $y = \dfrac{\pi}{2} + \left(1 - \dfrac{\pi}{2}\right)x - \dfrac{1}{2!}\left(1 - \dfrac{\pi}{2}\right)x^2 + \dfrac{1}{3!}\dfrac{\pi}{2}\left(1 - \dfrac{\pi}{2}\right)x^3 + \cdots$
10. $y = 1 + x + x^2 + \dfrac{4}{3}x^3 + \cdots$
11. $y = 2 + 2x + 3x^2 + \dfrac{12}{3}x^3 + \cdots$
12. $y = 1 - x + \dfrac{3}{2}x^2 - \dfrac{8}{3}x^3 + \cdots, y = e^{-xy}$
13. $y = \cos x$
14. $y = \tan x$
15. $y = \tanh x$

16. $y = 1 - e^{-x}$

17. $y = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{72}x^6 + \cdots$

18. $y + 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{3}{80}x^6 + \cdots$

19. $y = \frac{\pi}{2} + \frac{1}{2}x^2 - \frac{1}{240}x^6 + \frac{1}{19200}x^{10} + \cdots$

20. $y = 1 + \frac{e}{2}x^2 + \frac{e^2}{12}x^4 + \frac{13e^3}{720}x^6 + \cdots$

21. $y = x^3$

22. $y = x + \sin x$

23. $y = 1 - \cos x$

24. $y = \sinh x$

Exercises 8.4

The answers are given in Exercises 8.3.

Exercises 8.5

1. $u = x^2 + xy$

2. $u = y^2 + xy$

3. $u = x + t$

4. $u = 1 + xt$

5. $u = \dfrac{x}{1-t}$

6. $u = t + \sin x$

7. $u = 2x^2 \tanh t$

8. $u = x^3 \tanh t$

9. $u = 3x - 27x^2t + 486x^3t^2 - \cdots$

$u = 3x$, for $t = 0$, $\frac{1}{6t}(\sqrt{1+36xt} - 1)$, for $t > 0$

10. $u = 2y + \arctan x$

11. $u = t + \tanh x$ 12. $u = t + \tan x$

13. $u = x^2y^2$ 14. $u = y + e^x$

15. $u = x + \ln y$ 16. $u = y + \ln x$

17. $u = ye^{-x}$ 18. $u = y\cos x$

19. $u = x + \ln(1+y)$ 20. $u = x + \dfrac{1}{y}$

21. $u = y + \arctan x$ 22. $u = t + \sinh x$

23. $u = t + \sin x$ 24. $u = t + \cos x$

25. $u = \sinh x - t\sinh x\cosh x + \frac{1}{2}(2\sinh x\cosh^2 x + \sinh^3 x)t^2 + \cdots$

26. $u = \cos x + t\sin x\cos x + (\sin^2 x\cos x)t^2 + \cdots$

27. $u = x^2 t - \frac{2}{3}x^2 t^3 + \frac{8}{15}x^2 t^5 + \cdots$

28. $u = x - t$

29. $u = x - x^2 t + 2x^3 t^2 + \cdots$

30. $u = x - xt + \frac{3}{2}xt^2 + \cdots$

Exercises 8.6

The answers are given in Exercises 8.5.

Exercises 8.7

1. $u(x,t) = x + t$ $v(x,t) = x - t$

2. $u(x,t) = e^{-x+t}$ $v(x,t) = e^{x-t}$

3. $u(x,t) = e^{x+t}$ $v(x,t) = e^{-x-t}$

4. $u(x,t) = e^{2x+3t}$ $v(x,t) = e^{-2x-3t}$

5. $u(x,y,t) = x + y + t$ $v(x,y,t) = x - y + t,\ w = -x + y + t$

6. $u(x,y,t) = e^{x+y-t}$ $v(x,y,t) = e^{x-y+t},\ w = e^{-x+y+t}$

7. $u(x,y,t) = e^{x+y-t}$ $v(x,y,t) = e^{x-y+t},\ w = e^{-x+y+t}$

8. $u(x,y,t) = x + y + e^t$ $v(x,y,t) = x - y + e^{-t},\ w = -x + y + e^{-t}$

Exercises 8.8

The answers are given in Exercises 8.7.

Exercises 9.2

1. $u(x,t) = t + e^{-x}$ 2. $u(x,t) = t^2 + xt$

3. $u(x,t) = 1 + x^2 t^2$ 4. $u(x,t) = t + \sin x$

5. $u(x,t) = \dfrac{x}{t-1}$ 6. $u(x,t) = x\tanh t - \operatorname{sech} t$

7. $u(x,t) = (1 + x)\tanh t$ 8. $u(x,t) = x + e^t$

9. $u(x,t) = \begin{cases} 4x, & \text{for } t = 0 \\ \frac{1}{8t}(\sqrt{1+64xt} - 1), & \text{for } t > 0 \end{cases}$

10. $u = x^2 - 2x^3t + 5x^4t^2 + \cdots$

Exercises 9.3

1. $u(x,y) = y + e^{x+y}$
2. $u(x,y) = xy + e^{x+y}$
3. $u(x,y) = -(x+y) + e^{x+y}$
4. $u(x,y) = x^2 + e^{x+y}$
5. $u(x,y) = (x+y)e^{x+y}$
6. $u(x,y) = e^{x+y}$
7. $u(x,y) = \dfrac{x+y}{2} - \ln(e^x + e^y)$
8. $u(x,y) = \dfrac{x-y}{2} - \ln(e^x + e^{-y})$
9. $u(x,y) = \dfrac{x}{2} - \ln(e^x + e^y)$
10. $u(x,y) = \dfrac{y}{2} - \ln(e^x + e^y)$
11. $u(x,y) = \dfrac{x}{5} - \dfrac{2}{5}\ln(e^x + e^y)$
12. $u(x,y) = -\ln(e^x + e^y)$

Exercises 9.4

1. $u(x,t) = \cos x \cos t$
2. $u(x,t) = \cos x \sin t$
3. $u(x,t) = \sin x \sin t$
4. $u(x,t) = t \sin x$
5. $u(x,t) = t \cosh x$
6. $u(x,t) = 1 + xt$
7. $u(x,t) = 1 + xt$
8. $u(x,t) = x^3t^3$
9. $u(x,t) = t^2 + x^2$
10. $u(x,t) = x \cos t$
11. $\phi_3 = \dfrac{\pi}{6} + \dfrac{1}{4}t^2 + \dfrac{\sqrt{3}}{96}t^4$
12. $\phi_3 = \dfrac{\pi}{4} + \dfrac{1}{2\sqrt{2}}t^2 + \dfrac{1}{48}t^4$
13. $\phi_3 = t + \dfrac{1}{3}t^3$
14. $\phi_3 = \pi + t - \dfrac{1}{3}t^3$
15. $\phi_3 = \dfrac{3\pi}{2} + t - \dfrac{1}{4}t^2$

Exercises 9.5

1. $u(x,t) = \dfrac{x}{1+t}$
2. $u(x,t) = \dfrac{x}{t-1}$
3. $u(x,t) = \dfrac{2x}{1+2t}$
4. $u(x,t) = \dfrac{2x}{2t-1}$
5. $u(x,t) = \dfrac{1}{1+x}\left(1 + \dfrac{1}{(1+x)^2}t + \dfrac{2}{(1+x)^4}t^2 - \cdots\right)$
6. $u(x,t) = \dfrac{x}{t-1}$
7. $u(x,t) = \dfrac{2x}{1+2t}$
8. $u(x,t) = 4\tan 2x$

9. $u(x,t) = \frac{x}{t} + \frac{x+t}{2t^2 - t}$ 10. $u(x,t) = \frac{x}{t} - \frac{2}{x+3t}$

Exercises 9.6

1. $u(x,t) = e^{x-t}$ 2. $u(x,t) = e^{x+t}$
3. $u(x,t) = e^x + e^{-t}$ 4. $u(x,t) = e^{2x-t}$
5. $u(x,t) = e^x - e^{-t}$ 6. $u(x,t) = \sinh x + e^{-t}$
7. $u(x,t) = \cosh x - e^{-t}$ 8. $u(x,t) = e^x + e^{-3t}$
9. $u(x,t) = e^{2x} + e^{-2t}$ 10. $u(x,t) = \sinh 2x + e^{-2t}$

Exercises 9.7

1. $u(x,t) = e^{i(2x-4t)}$ 2. $u(x,t) = \sin x e^{-it}$
3. $u(x,t) = \cosh x e^{it}$ 4. $u(x,t) = 1 + \cos 3x e^{-9it}$
5. $u(x,t) = \sin 2x e^{-4it}$ 6. $u(x,t) = e^{i(2x-3t)}$
7. $u(x,t) = e^{i(t-x)}$ 8. $u(x,t) = e^{i(3x-3t)}$
9. $u(x,t) = e^{i(2x-6t)}$ 10. $u(x,t) = e^{i(3x+8t)}$

Exercises 9.8

6. $u(x,t) = \frac{x}{1+6t}$
7. $u(x,t) = \frac{2}{x^2}$
8. $u(x,t) = \frac{1}{6}\left(\frac{x-2}{2-t}\right)$
9. $u(x,t) = \frac{2}{(x-3)^2}$
10. $u(x,t) = \frac{1}{6}\left(\frac{x-4}{3-t}\right)$

Exercises 9.9

1. $u(x,t) = \sin(x+t)$ 2. $u(x,t) = \sin x \cos t$
3. $u(x,t) = \cos x \cos t$ 4. $u(x,t) = 1 + \cos(x+t)$
5. $u(x,t) = 2 + \sin x \sin t$ 6. $u(x,t) = \sin 2x \cos t$
7. $u(x,t) = e^{x+t}$ 8. $u(x,t) = \sin 2x \sin 2t$
9. $u(x,t) = e^{x-t}$ 10. $u(x,t) = \sin(x+2t)$
11. $u(x,t) = (x - \sin x)e^{-t}$ 12. $u(x,t) = \frac{x^6}{6!} \sin t$

13. $u(x,t) = (x - \cos x)e^{-t}$ 14. $u(x,t) = \frac{5}{6!}x^6 \sin t$

15. $u(x,t) = \frac{6}{7!}x^7(\sin t + \cos t)$

Exercises 10.2

1. $\phi_3 = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6\right) + \varepsilon\left(\frac{1}{3!}x^3 - \frac{1}{30}x^5\right) + \varepsilon^2\left(\frac{1}{180}x^6\right)$

2. $\phi_3 = \left(1 + x + \frac{1}{2!}x^2\right) + \varepsilon\left(\frac{1}{2!}x^2 + \frac{1}{2}x^3\right) + \varepsilon^2\left(\frac{1}{8}x^4\right)$

3. $\phi_4 = \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6\right) + \varepsilon\left(\frac{2}{3!}t^3 - \frac{4}{5!}t^5\right) + \varepsilon^2\left(-\frac{4}{4!}t^6\right)$

4. $\phi_4 = \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6\right) + \varepsilon\left(-\frac{6}{5!}t^5\right)$

5. $\phi_3 = 1 + \frac{1}{\varepsilon}\sin(\varepsilon x) - \frac{1}{2\varepsilon^2}(1 - \cos(2\varepsilon x))$

6. $\phi_4 = x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!}, y = \frac{\sqrt{\pi}}{2}\mathrm{erf}(x)$

7. $\phi_4 = \frac{4}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!}\right), y = 2\mathrm{erf}(x)$

8. $y(x) = x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42}$

9. $\phi_4 = x^4 - \frac{1}{90}x^{10} + \frac{1}{10800}x^{16}$

10. $\phi_4 = x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^6 + \frac{13}{630}x^7 + \frac{1}{1350}x^{10}$

Exercises 10.3

1. $\phi_4 = \cos x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}\right)$

2. $\phi_4 = \cos x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}\right)$

3. $\phi_4 = x^3\left(t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7\right)$

4. $\phi_4 = 4x\left(1 - 16xt + 512x^2t^2 - 20480x^3t^3\right)$

5. $\phi_4 = 2\left(1 - \frac{1}{2}t^2 + \frac{5}{24}t^4 - \frac{61}{120}t^6\right) + x\left(t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7\right)$

6. $\phi_3 = x - xt + \frac{3}{2}xt^2$

7. $\phi_3 = x^2\left(t - \frac{2}{3}xt^3 + \frac{2}{3}x^2t^5\right)$

8. $\phi_3 = x - (1+x^2)t + (2x+x^3)t^2$

9. $\phi_3 = \left(1 + (it) + \frac{1}{2!}(it)^2\right)\cosh x$

10. $\phi_3 = 1 + \left(\frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!}\right)\sinh x$

Exercises 10.4

1. (a) $[2/2] = \dfrac{1+7x+11x^2}{1+5x+5x^2}$

$[3/3] = \dfrac{1+10x+31x^2+29x^3}{1+8x+19x^2+13x^3}$

(b) $\sqrt{5} \approx 2.2$, $\sqrt{5} \approx 2.230769$

2. (a) $[2/2] = \dfrac{1+17x+61x^2}{1+11x+19x^2}$

$[3/3] = \dfrac{1+24x+171x^2+337x^3}{1+18x+87x^2+97x^3}$

(b) $\sqrt{13} \approx 3.210526$, $\sqrt{13} \approx 3.474227$

3. (a) $[2/2] = \dfrac{6x}{6+x^2}$

$[3/3] = \dfrac{60x-7x^3}{60+3x^2}$

4. (a) $[2/2] = \dfrac{12+6x+x^2}{12-6x+x^2}$

$[3/3] = \dfrac{120+60x+12x^2+x^3}{120-60x+12x^2-x^3}$

5. (a) $[2/2] = \dfrac{-30+21x-x^2}{30-36x+9x^2}$

$[3/3] = \dfrac{420-510x+140x^2-3x^3}{-420+720x-360x^2+48x^3}$

6. (a) $[2/2] = \dfrac{-15+x^2}{-15+6x^2}$

$[3/3] = \dfrac{-15+x^2}{-15+6x^2}$

7. (a) $[3/3] = \dfrac{-15x+4x^3}{-15+9x^2}$

$$[4/4] = \frac{105x-55x^3}{105-90x^2+9x^4}$$

8. (a) $[3/3] = \dfrac{60x+7x^3}{60-3x^2}$

$$[4/4] = \frac{5880x+620x^3}{5880-360x^2+11x^4}$$

9. (a) $[3/3] = \dfrac{1}{1+x}$

$$[4/4] = \frac{1}{1+x}$$

10. (a) $[3/3] = \dfrac{15+4x^2}{15+9x^2}$

$$[4/4] = \frac{945+735x^2+64x^4}{945+1050x^2+225x^4}$$

11. (a) $[3/3] = \dfrac{120+60x+28x^2-x^3}{120-60x+28x^2+x^3}$

$$[4/4] = \frac{240+120x-140x^2-100x^3-49x^4}{240-120x-140x^2+100x^3-49x^4}$$

12. (a) $[3/3] = \dfrac{120+60x-28x^2+x^3}{120-60x-28x^2-x^3}$

$$[4/4] = \frac{240+120x-500x^2-220x^3+111x^4}{240-120x-500x^2+220x^3+111x^4}$$

Exercises 10.5

1. $u(x) = .2+1.6x+4.6x^2-.933333x^3-44.466667x^4$
$-101.506667x^5+157.857778x^6+1291.071746x^7+O(x^8)$

$$u_{[4/4]} = \frac{.2+.922403x+1.802939x^2+1.820630x^3+.841658x^4}{1-3.387987x+13.118589x^2-13.255201x^3+15.045083x^4}$$

2. $u(x) = .1+.45x+.875x^2+.716667x^3-.540104x^4$
$-2.260417x^5-2.507231x^6+.377294x^7+O(x^8)$

$$u_{[4/4]} = \frac{.1+.221916x+.201328x^2+.087574x^3+.014724x^4}{1-2.280845x+3.527084x^2-2.205410x^3+.956694x^4}$$

3. $u(x) = \alpha+\dfrac{1}{4}(\alpha+\alpha^2)r^2+\dfrac{1}{4}(\alpha+3\alpha^2+2\alpha^2)r^4$
$+\dfrac{1}{2034}(\alpha+9\alpha^2+16\alpha^3+8\alpha^4)r^6+O(r^8)$

$\alpha = -2.392$

Exercises 11.2

1. $u(x,t) = -2\,\text{sech}^2(x-4t)$
2. $u(x,t) = 8\,\text{sech}^2 2(x-16t)$
3. $u(x,t) = \sqrt{2}\,\text{sech}\,2(x-4t)$
4. $u(x,t) = 4\arctan\left[\exp\left(-2\left(x-\frac{\sqrt{3}}{2}t\right)\right)\right]$
5. $u(x,y,t) = \frac{1}{2}\,\text{sech}^2\left[\frac{1}{2}(x+y-4t)\right]$

Exercises 11.3

1. $u(x,t) = 3\cos\left[\frac{1}{3}(x-6t)\right]$
2. $u(x,t) = \left\{2\sin\left[\frac{3}{8}\left(x-\frac{5}{2}t\right)\right]\right\}^{\frac{2}{3}}$
3. $u(x,y,t) = 3\cos\left[\frac{1}{3\sqrt{2}}(x+y-6t)\right]$
4. $u(x,t) = \cos^{\frac{1}{2}}(x-2t)$
5. $u(x,y,t) = \cos^{\frac{1}{2}}\left[\frac{1}{\sqrt{2}}(x+y-2t)\right]$

Exercises 11.4

1. $u(x,t) = 3\sinh\left[\frac{1}{3}(x-6t)\right]$
2. $u(x,t) = -\left[2\cosh\left(\frac{3}{8}\left(x-\frac{5}{2}t\right)\right)\right]^{\frac{2}{3}}$
3. $u(x,y,t) = 3\sinh\left[\frac{1}{3\sqrt{2}}(x+y-6t)\right]$
4. $u(x,t) = \sinh^{\frac{1}{2}}(x-2t)$
5. $u(x,y,t) = \sinh^{\frac{1}{2}}(x+y-4t)$

Index

Nonlinear Physical Science

(*Series Editors*: *Albert C.J. Luo*, *Nail H. Ibragimov*)

Nail. H. Ibragimov/ Vladimir. F. Kovalev: Approximate and Renormgroup Symmetries

Abdul-Majid Wazwaz: Partial Differential Equations and Solitary Waves Theory

Albert C.J. Luo: Discontinuous Dynamical Systems on Time-varying Domains

Anjan Biswas/ Daniela Milovic/ Matthew Edwards: Mathematical Theory of Dispersion-Managed Optical Solitons

Hanke,Wolfgang/ Kohn, Florian P.M./ Wiedemann, Meike: Self-organization and Pattern-formation in Neuronal Systems under Conditions of Variable Gravity