# 1 Fibonacci Number

## Problem Introduction

Recall the definition of Fibonacci sequence: $F_0 = 0, F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$. Your goal in this problem is to implement an efficient algorithm for computing Fibonacci numbers. The starter files for this problem contain an implementation of the following naïve recursive algorithm for computing Fibonacci numbers in `C++`, `Java`, and `Python3`:

```
FIBONACCI(n):
if n ≤ 1:
  return n
return FIBONACCI(n − 1) + FIBONACCI(n − 2)
```

Try compiling and running a starter solution on your machine. You will see that computing, say, $F_{40}$ already takes noticeable time.

Another way to appreciate the dramatic difference between an exponential time algorithm and a polynomial time algorithm is to use the following visualization by David Galles: http://www.cs.usfca.edu/~galles/visualization/DPFib.html. Try computing $F_{20}$ by a recursive algorithm by entering "20" and pressing the "Fibonacci Recursive" button. You will see an endless number of recursive calls. Now, press "Skip Forward" to stop the current algorithm and call the iterative algorithm by pressing "Fibonacci Table". This will compute $F_{20}$ very quickly. (Note that the visualization uses a slightly different definition of Fibonacci Numbers: $F_0 = 1$ instead of $F_0 = 0$. This of course has almost no influence on the running time.)

## Problem Description

**Task.** Given an integer $n$, find the $n$th Fibonacci number $F_n$.

**Input Format.** The input consists of a single integer $n$.

**Constraints.** $0 \leq n \leq 45$.

**Output Format.** Output $F_n$.

**Sample 1.**

Input:

```
10
```

Output:

```
55
```

$F_{10} = 55$.

# 2 Last Digit of a Large Fibonacci Number

## Problem Introduction

Your goal in this problem is to find the last digit of $n$-th Fibonacci number. Recall that Fibonacci numbers grow exponentially fast. For example,

$$F_{200} = 280\,571\,172\,992\,510\,140\,037\,611\,932\,413\,038\,677\,189\,525.$$

Therefore, a solution like

```
F[0] ← 0
F[1] ← 1
for i from 2 to n:
  F[i] ← F[i − 1] + F[i − 2]
print(F[n] mod 10)
```

Will turn out to be too slow, because as $i$ grows the $i$th iteration of the loop computes the sum of longer and longer numbers. Also, for example, $F_{1000}$ does not fit into the standard `C++ int` type. To overcome this difficulty, you may want to store in $F[i]$ not the $i$th Fibonacci number itself, but just its last digit (that is, $F_i \ mod\ 10$). Computing the last digit of $F_i$ is easy: it is just the last digit of the sum of the last digits of $F_{i-1}$ and $F_{i-2}$:

```
F[i] ← (F[i − 1] + F[i − 2]) mod 10
```

This way, all $F[i]$'s are just digits, so they fit properly into any standard integer type, and computing a sum of $F[i-1]$ and $F[i-2]$ is performed very quickly.

## Problem Description

**Task.** Given an integer $n$, find the last digit of the $n$th Fibonacci number $F_n$ (that is, $F_n \ mod\ 10$).

**Input Format.** The input consists of a single integer $n$.

**Constraints.** $0 \le n \le 10^7$.

**Output Format.** Output the last digit of $F_n$.

**Sample 1.**

Input:

```
3
```

Output:

```
2
```

$F_3 = 2$.

**Sample 2.**

Input:

```
331
```

Output:

```
9
```

$F_{331} = 668\,996\,615\,388\,005\,031\,531\,000\,081\,241\,745\,415\,306\,766\,517\,246\,774\,551\,964\,595\,292\,186\,469$.

**Sample 3.**

Input:

```
327305
```

Output:

```
5
```

$F_{327305}$ does not fit into one line of this pdf, but its last digit is equal to 5.

# 3 Greatest Common Divisor

## Problem Introduction

The greatest common divisor $GCD(a, b)$ of two non-negative integers $a$ and $b$ (which are not both equal to 0) is the greatest integer $d$ that divides both $a$ and $b$. Your goal in this problem is to implement the Euclidean algorithm for computing the greatest common divisor.

Efficient algorithm for computing the greatest common divisor is an important basic primitive of commonly used cryptographic algorithms like RSA.

$$\begin{aligned}&\mathrm{GCD}(1344, 217)\\ =&\mathrm{GCD}(217, 42)\\ =&\mathrm{GCD}(42, 7)\\ =&\mathrm{GCD}(7, 0)\\ =&7\end{aligned}$$

## Problem Description

**Task.** Given two integers $a$ and $b$, find their greatest common divisor.

**Input Format.** The two integers $a, b$ are given in the same line separated by space.

**Constraints.** $1 \leq a, b \leq 2 \cdot 10^9$.

**Output Format.** Output $GCD(a, b)$.

**Sample 1.**

Input:

```
18 35
```

Output:

```
1
```

18 and 35 do not have common non-trivial divisors.

**Sample 2.**

Input:

```
28851538 1183019
```

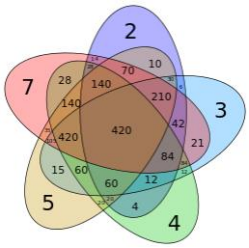
Output:

```
17657
```

$28851538 = 17657 \cdot 1634, 1183019 = 17657 \cdot 67.$

# 4 Least Common Multiple

## Problem Introduction




The least common multiple of two positive integers $a$ and $b$ is the least positive integer $m$ that is divisible by both $a$ and $b$.

## Problem Description

**Task.** Given two integers $a$ and $b$, find their least common multiple.

**Input Format.** The two integers $a, b$ are given in the same line separated by space.

**Constraints.** $1 \leq a, b \leq 10^7$.

**Output Format.** Output the least common multiple of $a$ and $b$.

**Sample 1.**

Input:

```
6 8
```

Output:

```
24
```

Among all the positive integers that are divisible by both 6 and 8 (e.g., 48, 480, 24), 24 is the smallest one.

**Sample 2.**

Input:

```
761457 614573
```

Output:

```
467970912861
```

# 5 Fibonacci Number Again

## Problem Introduction

In this problem, your goal is to compute $F_n$ modulo $m$, where $n$ may be really huge: up to $10^{14}$. For such values of $n$, an algorithm looping for $n$ iterations will not fit into one second for sure. Therefore we need to avoid such a loop.

To get an idea how to solve this problem without going through all $F_i$ for $i$ from 0 to $n$, let's see what happens when $m$ is small – say, $m = 2$ or $m = 3$.

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_i$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 | 377 | 610 |
| $F_i \bmod 2$ | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| $F_i \bmod 3$ | 0 | 1 | 1 | 2 | 0 | 2 | 2 | 1 | 0 | 1 | 1 | 2 | 0 | 2 | 2 | 1 |

Take a detailed look at this table. Do you see? Both these sequences are periodic! For $m = 2$, the period is $011$ and has length 3, while for $m = 3$, the period is $01120221$ and has length 8. Therefore, to compute, say, $F_{2015}$ mod 3 we just need to find the remainder of $2015$ when divided by 8. Since $2015 = 251 \cdot 8 + 7$, we conclude that $F_{2015} \bmod 3 = F_7 \bmod 3 = 1$.

This is true in general: for any integer $m \geq 2$, the sequence $F_n$ mod $m$ is periodic. The period always starts with $01$ and is known as Pisano period.

## Problem Description

**Task.** Given two integers $n$ and $m$, output $F_n$ mod $m$ (that is, the remainder of $F_n$ when divided by $m$).

**Input Format.** The two integers $n, m$ are given in the same line separated by space.

**Constraints.** $1 \leq n \leq 10^{14}$, $2 \leq m \leq 10^3$.

**Output Format.** output $F_n$ mod $m$.

**Sample 1.**

Input:

```
239 1000
```

Output:

```
161
```

$F_{239}$ mod $1000 = 39\,679\,027\,332\,006\,820\,581\,608\,740\,953\,902\,289\,877\,834\,488\,152\,161\ (mod\ 1\,000) = 161$.

**Sample 2.**

Input:

```
2816213588 239
```

Output:

```
151
```

$F_{2816213588}$ does not fit into one page of this file, but $F_{2816213588} \bmod 239 = 151$.

# 6 Last Digit of the Sum of Fibonacci Numbers

## Problem Introduction

The goal in this problem is to find the last digit of a sum of the first $n$ Fibonacci numbers.

## Problem Description

**Task.** Given an integer $n$, find the last digit of the sum $F_0 + F_1 + \cdots + F_n$.

**Input Format.** The input consists of a single integer $n$.

**Constraints.** $0 \leq n \leq 10^{14}$.

**Output Format.** Output the last digit of $F_0 + F_1 + \cdots + F_n$.

**Sample 1.**

Input:

```
3
```

Output:

```
4
```

$F_0 + F_1 + F_2 + F_3 = 0 + 1 + 1 + 2 = 4$.

**Sample 2.**

Input:

```
100
```

Output:

```
5
```

The sum is equal to $927\,372\,692\,193\,078\,999\,175$, the last digit is $5$.

## What To Do

Instead of computing this sum in a loop, try to come up with a formula for $F_0 + F_1 + \cdots + F_n$. For this, play with small values of $n$. Then, use a solution for the previous problem.

# 7 Last Digit of the Sum of Fibonacci Numbers Again

## Problem Introduction

Now, we would like to find the last digit of a *partial* sum of Fibonacci numbers: $F_m + F_{m+1} + \cdots + F_n$.

## Problem Description

**Task.** Given two non-negative integers $m$ and $n$, where $m \leq n$ find the last digit of the sum $F_m + F_{m+1} + \cdots + F_n$.

**Input Format.** The input consists of two non-negative integers $m$ and $n$ separated by a space.

**Constraints.** $0 \leq m \leq n \leq 10^{14}$.

**Output Format.** Output the last digit of $F_m + F_{m+1} + \cdots + F_n$.

**Sample 1.**

Input:

```
3 7
```

Output:

```
1
```

$F_3 + F_4 + F_5 + F_6 + F_7 = 2 + 3 + 5 + 8 + 13 = 31$.

**Sample 2.**

Input:

```
10 10
```

Output:

```
5
```

$F_{10} = 55$.

**Sample 3.**

Input:

```
10 200
```

Output:

```
2
```

$F_{10} + F_{11} + \cdots + F_{200} = 734\,544\,867\,157\,818\,093\,234\,908\,902\,110\,449\,296\,423\,262$.

# 8 Last Digit of the Sum of Squares of Fibonacci Numbers

## Problem Description

**Task.** Compute the last digit of the sum $F_0^2 + F_1^2 + \cdots + F_n^2$.

**Input Format.** The input consists of a single integer $n$.

**Constraints.** $0 \leq n \leq 10^{14}$.

**Output Format.** Output the last digit of $F_0^2 + F_1^2 + \cdots + F_n^2$.

**Sample 1.**

Input:

```
7
```

Output:

```
3
```

$F_0^2 + F_1^2 + \cdots + F_7^2 = 0 + 1 + 1 + 4 + 9 + 25 + 64 + 169 = 273.$

**Sample 2.**

Input:

```
73
```

Output:

```
1
```

$F_0^2 + F_1^2 + \cdots + F_{73}^2 = 1\,052\,478\,208\,141\,359\,608\,061\,842\,155\,201.$

**Sample 3.**

Input:

```
1234567890
```

Output:

```
0
```

## What To Do

Since the brute force search algorithm for this problem is too slow ($n$ may be as large as $10^{14}$), we need to come up with a simple formula for $F_0^2 + F_1^2 + \cdots + F_n^2$. The figure below represents the sum $F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2$ as the area of a rectangle with vertical side $F_5 = 5$ and horizontal side $F_5 + F_4 = 5 + 3 = F_6$.