Lagrange Polynomial Interpolation

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Overview

Introduction - Tanya 19/78006

Introduction - Tanya 19/78006

Interpolation

The process of deriving a function $y = f(x)$ from a set of tabulated data points so that the derived function passes through all given data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ and can then be used to approximate the values at data points within the interval (say where some $x \in [x_0, x_n]$).

Polynomial Interpolation

Polynomial Interpolation involves coming up with a polynomial of the lowest possible degree that passes through the given points. This polynomial can then be used to find the value of the function, and thereby, the ordinate at a given abscissa.

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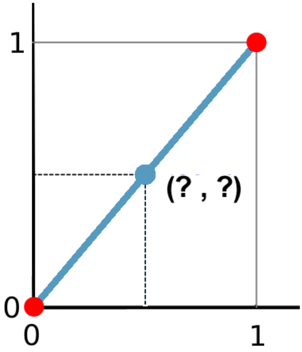


Introduction - Tanya 19/78006

Advantages

There is no need to solve a system of equations as in the case of direct method or construct difference tables as in the case of Newton interpolation. Lagrange interpolation can be used for both equally and unequally spaced data points. It is efficient when we have to interpolate on fixed data points.

Drawback

Whenever a new data point is added, the interpolation coefficients need to be recalculated. Newton's interpolation overcomes this drawback when we have to interpolate data incrementally.

Theory - Anmol 19/78009; Devraj 19/78004

Theory - Anmol 19/78009

Consider values $f(x_0)$, $f(x_1)$, ..., $f(x_n)$ at $n+1$ distinct, unevenly-spaced points or non-uniform points x_0, x_1, ..., x_n, such that $a = x_0 < x_1 < x_2 < \ldots < x_n = b$,

x	x_0	x_1	x_2	...	x_n
f(x)	$f(x_0)$	$f(x_1)$	$f(x_2)$	...	$f(x_n)$

We can find a polynomial of at most degree n defined as

$$f(x) = l_0(x) \cdot f(x_0) + l_1(x) \cdot f(x_1) + \ldots + l_n(x) \cdot f(x_n) \tag{1}$$

For a general point x_i, we have,

$$f(x_i) = l_0(x_i) \cdot f(x_0) + \ldots + l_i(x_i) \cdot f(x_i) + \ldots + l_n(x_i) \cdot f(x_n) \tag{2}$$

The above equation is true only when $l_i(x_i) = 1$ and $l_j(x_i) = 0, i \neq j$. Thus, the functions $l_i(x)$ can be represented as

$$l_i(x) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{3}$$

By the above result we can say that

$$l_i(x) = 0 \text{ at } x = x_0, x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \tag{4}$$

Thus, $l_i(x)$ can be represented as a product of its factors as

$$l_i(x) = C(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n) \tag{5}$$

where C is a constant.

Since $l_i(x_i) = 1$, we get,

$$l_i(x_i) = 1 = C(x_i - x_0)(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n) \tag{6}$$

Hence,

$$C = \frac{1}{(x_i - x_0)(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)} \tag{7}$$

Therefore,

$$l_i(x) = \frac{(x - x_0)(x - x_1)\dots(x - x_{i-1})(x - x_{i+1})\dots(x - x_n)}{(x_i - x_0)(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)} \tag{8}$$

Theory - Devraj 19/78004

Given (N + 1) data points in the form of $(x_i,\ f(x_i))$ for some $i \in [0, N]$, the Lagrangian interpolating polynomial is therefore given by

$$y = f(x) = \sum_{i=0}^{N} l_i(x) \cdot f(x_i) \tag{9}$$

where $f(x)$ is a polynomial of degree at most N and

$$l_i(x) = \prod_{j=0;\, j \neq i}^{N} \frac{x - x_j}{x_i - x_j} \tag{10}$$

is the Lagrangian fundamental polynomial.

Theory - Devraj 19/78004

Linear Lagrange Interpolation

The problem is concerned with determining a polynomial of degree at most one that passes through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

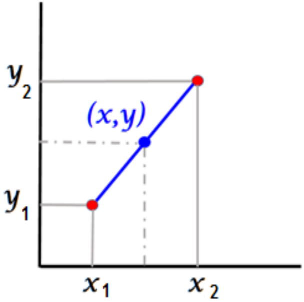
We define the Lagrange fundamental polynomials $l_0(x)$ and $l_1(x)$ as

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \text{ and } l_1(x) = \frac{x - x_0}{x_1 - x_0} \tag{11}$$

Then the Linear Lagrange Interpolating Polynomial is given by

$$y = f(x) = l_0(x) \cdot f(x_0) + l_1(x) \cdot f(x_1) \tag{12}$$

$$= \frac{x - x_1}{x_0 - x_1} \cdot f(x_0) + \frac{x - x_0}{x_1 - x_0} \cdot f(x_1) \tag{13}$$



y_2
(x,y)
y_1
x_1
x_2

Theory - Devraj 19/78004

Quadratic Lagrange Interpolation

The problem is concerned with determining a polynomial of degree at most two that passes through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$. We define the Lagrange fundamental polynomials $l_0(x)$, $l_1(x)$ and $l_2(x)$ as

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \tag{14}$$

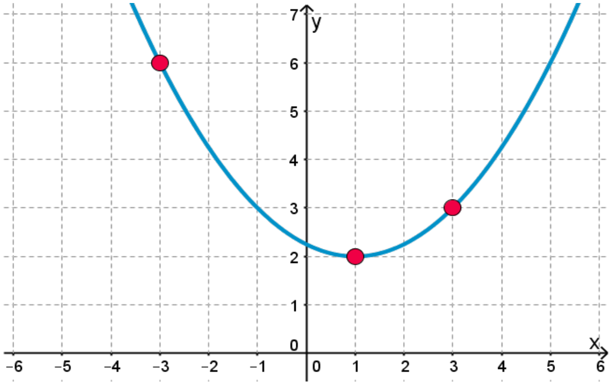
$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \tag{15}$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \tag{16}$$

Then the Quadratic Lagrange Interpolating Polynomial is given by

$$y = f(x) = l_0(x) \cdot f(x_0) + l_1(x) \cdot f(x_1) + l_2(x) \cdot f(x_2) \tag{17}$$

Theory - Devraj 19/78004



Algorithm - Abhimanyu 19/78014

Algorithm - Abhimanyu 19/78014

Interpolate (x $= [x_0, x_1, \ldots, x_n]$, y $= [y_0, y_1, \ldots, y_n]$)

N $\leftarrow$ length(x) // or length(y)
Y $\leftarrow$ empty expression
for $i = 0$ to N **do**
    $l_i \leftarrow$ empty expression
    for $j = 0$ to N **do**
        if i $\neq$ j **then**
            $l_i \leftarrow l_i \cdot \frac{x - x_j}{x_i - x_j}$
        end if
    end for
    $Y \leftarrow Y + l_i \cdot y_i$
end for
return Y

Solved Examples - Lakshay 19/78013

Q Construct the Lagrange interpolation polynomial for the data.

x	-1	1	4	7
$f(x)$	-2	0	63	342

Hence, intepolate at $x=5$.

Ans We need to construct the Lagrange Interpolation polynomials,

firstly, we find the Lagrange's fundamental polynomials, given by

$$l_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$$

now, put $i=0$,

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$\Rightarrow \frac{(x-1)(x-4)(x-7)}{(-1-1)(-1-4)(-1-7)}$$ on simplifying,

we get, $l_0(x) = -\frac{1}{80}(x^3-12x^2+39x-28)$ — (1)

Now, put $i=1$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$\Rightarrow \frac{(x+1)(x-4)(x-7)}{(1+1)(1-4)(1-7)}$$, on simplifying

we get, $\Rightarrow \frac{1}{36}(x^3-10x^2+17x+28)$ — 2

Now, put $i=2$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$\Rightarrow \frac{(x+1)(x-1)(x-7)}{(4+1)(4-1)(4-7)}$$, on simplifying

$l_2(x) \Rightarrow \frac{-1}{45}(x^3-7x^2-x+7)$ — 3

put $i=3$,

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$\Rightarrow \frac{(x+1)(x-1)(x-4)}{(7+1)(7-1)(7-4)} \Rightarrow \text{ on simplifying}$$

we get, $l_3(x) = \frac{1}{144}(x^3 - 4x^2 - x + 4)$ — (4)

Now, the lagrange interpolation polynomial is given by,

$$P_3(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2) + l_3(x)f(x_3) \quad \text{— (5)}$$

Now, from equation (1), (2), (3) & (4) we can subsitute the values in (5) and get the values of $f(x_i)$ from the given table.

$$\boxed{\begin{aligned} P(x) \Rightarrow & -\frac{1}{80}(x^3 - 12x^2 + 39x - 28)(-2) \\ & + 0 + \left(-\frac{1}{45}\right)(x^3 - 7x^2 - x + 7)(63) \\ & + \frac{1}{144}(x^3 - 4x^3 - x + 4)(342) \end{aligned}}$$

Note that the second term is zero because the value of $f(x_1) = 0$ given in the table.

Now, simplifying the interpolating polynomial we got,

$$\rightarrow \left(\frac{1}{40}-\frac{7}{5}+\frac{171}{72}\right)x^3+\left(\frac{-3}{10}+\frac{49}{5}-\frac{171}{18}\right)x^2$$

$$+\left(\frac{39}{40}+\frac{7}{5}\cdot+\left(-\frac{171}{72}\right)\right)]\,x+\left(\frac{-7}{10}-\frac{49}{5}+\frac{171}{8}\right)$$

finally we get,

$$\boxed{P_3(x) = x^3-1 = f(x)}$$

It is the required interpolating polynomial

Now, we need to find $f(5)$ so we put $x=5$ in above equation we get,

$$\boxed{f(5) = 5^3-1 \Rightarrow 124}$$

Q The following values of function $f(x) = \sin x + \cos x$ are given

x	10°	20°	30°
$f(x)$	1.1585	1.2817	1.3660

Constructs the quadratic Lagrange interpolating polynomial that fits the data. Hence find $f(\pi/12)$.

Ans Since the value of f at $\pi/12$ radians is required, we convert the data into radian measure. we have

$$x_0 = 10^\circ = \frac{\pi}{18} = 0.1745 \qquad x_1 = 20^\circ = \frac{\pi}{9} = 0.3491$$

$$x_2 = 30^\circ = \frac{\pi}{6} = 0.5236$$

We find the Lagrange's fundamental polynomial given by,

$$l_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

now, put $i=0$,

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \Rightarrow \frac{(x-0.3491)(x-0.5236)}{(-0.1746)(-0.3491)}$$

$$l_0(x) \Rightarrow 16.4061(x^2 - 0.8727x + 0.1828) \quad (1)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0.1745)(x-0.5236)}{(0.1746)(-0.1745)}$$

$$l_1(x) \Rightarrow -32.8216\,(x^2 - 0.6981x + 0.0914) \quad \text{—} \ (2)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0.1745)(x-0.3491)}{(0.3491)(-0.1745)}$$

$$l_2(x) \Rightarrow 16.4155\,(x^2 - 0.5236x + 0.0609) \quad \text{—} \ (3)$$

The lagrange's quadratic polynomial is given by

$$P_2(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)$$

$$P_2(x) \Rightarrow 16.4061\,(x^2 - 0.8727x + 0.1828)(1.1585)$$
$$- 32.8616\,(x^2 - 0.6981x + 0.0914)(1.2817)$$
$$+ 16.4155\,(x^2 - 0.5236x + 0.0609)(1.3660)$$

$$\Rightarrow \boxed{P_2(x) = f(x) = -0.6374x^2 + 1.0394x + 0.9950}$$

$$f(\pi/12) = f(0.2618) = 1.2234.$$

is the required value

Practical - Method 1 - Sudipto 19/78003

Using Iterations - Sudipto 19/78003

Refer Maxima File.

Practical - Method 2 - Amartya 19/78002

Using *interpol* Package - Amartya 19/78002

Refer Maxima File.

The End