A Macroeconomic Model for Dynamic Scoring of Tax Policy *

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Abstract

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1 Introduction

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2 Model with Endogenous Labor

This is the basic OLG model in which households live S periods and are one of J ability types. The ability process is calibrated to match the wage distribution by age in the United States, and labor is endogenously supplied by individuals. The production side of the economy is characterized by a unit measure of identical, perfectly competitive firms.

2.1 Individual problem

A measure $\omega_{1,t}$ of individuals with heterogeneous working ability $e \in \mathcal{E} \subset \mathbb{R}_{++}$ is born in each period t and live for $S \geq 3$ periods. The population of age-s individuals in any period t is $\omega_{s,t}$. The population of agents of each age in each period $\omega_{s,t}$ evolves according to the following function,

$$\begin{aligned} \omega_{1,t+1} &= \sum_{s=1}^{S} f_s \omega_{s,t} \quad \forall t \\ \omega_{s+1,t+1} &= (1 + i_s - \rho_s)\omega_{s,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq S-1 \end{aligned} \tag{1}$$

where $f_s \geq 0$ is an age-specific fertility rate, i_s is an age-specific immigration rate, ρ_s is an age specific mortality rate, and $1 + i_s - \rho_s$ is constrained to be nonnegative. The total population in the economy N_t at any period is simply the sum of individuals in the economy and the population growth rate in any period t from the previous period $t-1$ is $g_{n,t}$.[1]

$$N_t \equiv \sum_{s=1}^{S} \omega_{s,t} \quad \forall t \tag{2}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{3}$$

[1]Appendix A-3 describes in detail the exogenous population dynamics.

Their working ability evolves over their lifetime according to an age-dependent deterministic process. At birth, an equal fraction $1/J$ of the $\omega_{s,t}$ measure of new agents are randomly assigned to each of the J ability types indexed by $j = 1, 2, ...J$. Once ability type is determined, that measure $\omega_{s,t}/J$ of individuals' ability evolves deterministically according to $e_{j,s}$. The process for the evolution of the population weights $\omega_{s,t}$ is an exogenous input to the model. We calibrate the matrix of lifetime ability paths $e_{j,s}$ for all types j using CPS hourly wage by age distribution data.[2]

Individuals are endowed with a measure of time $\tilde{l}$ in each period t, and they choose each period how much of that time to allocate between labor $n_{j,s,t}$ and leisure $l_{j,s,t}$.

$$n_{j,s,t} + l_{j,s,t} = \tilde{l} \tag{4}$$

At time t, all generation-s agents with ability $e_{j,s}$ know the real wage rate w_t and know the one-period real net interest rate r_t on bond holdings $b_{j,s,t}$ that mature at the beginning of period t. In each period t, age-s agents with working ability $e_{j,s}$ choose how much to consume $c_{j,s,t}$, how much to save for the next period by loaning capital to firms in the form of a one-period bond $b_{j,s+1,t+1}$, and how much to work $n_{j,s,t}$ in order to maximize expected lifetime utility of the following form,

$$\begin{aligned} U_{j,s,t} &= \sum_{v=0}^{S-s} \beta^u u\left(c_{j,s+u,t+u}, n_{j,s+u,t+u}\right) \\ \text{where} \quad u\left(c_{j,s,t}, n_{j,s,t}\right) &= \frac{\left(c_{j,s,t}\right)^{1-\sigma} - 1}{1-\sigma} + \chi e^{g_y t(1-\sigma)} \frac{\left(\tilde{l} - n_{j,s,t}\right)^{1-\eta}}{1-\eta} \quad \forall j, s, t \end{aligned} \tag{5}$$

where $\sigma \geq 1$ is the coefficient of relative risk aversion on consumption, $\eta \geq 1$ is proportional to the Frisch elasticity of labor supply, $\beta \in (0,1)$ is the agent's discount factor, χ is a constant term influencing the disutility of labor, and g_y is a constant growth rate of labor augmenting technological progress, which we explain in Section 2.2. [3]

[2] Appendix A-1 gives a detailed description of the calibration of the deterministic ability process by age s and type j, as well as alternative specifications and calibrations.

[3] The term with the growth rate $e^{g_y t(1-\sigma)}$ must be included in the period utility function because consumption will be growing at rate g_y and this term stationarizes the household Euler equation by making the marginal disutility of labor grow at the same rate as the marginal benefit of consumption.

Because agents are born without any bonds maturing and because they purchase no bonds in the last period of life $s = S$, the per-period budget constraints for each agent normalized by the price of consumption are the following.

$$w_t e_{j,s} n_{j,s,t} \geq c_{j,s,t} + b_{j,s+1,t+1} \quad \text{for} \quad s = 1 \quad \forall j,t \tag{6}$$

$$(1+r_t)\, b_{j,s,t} + w_t e_{j,s} n_{j,s,t} \geq c_{j,s,t} + b_{j,s+1,t+1} \quad \text{for} \quad 2 \leq s \leq S-1 \quad \forall j,t \tag{7}$$

$$(1+r_t)\, b_{j,s,t} + w_t e_{j,s} n_{j,s,t} \geq c_{j,s,t} \quad \text{for} \quad s = S \quad \forall j,t \tag{8}$$

Note that the price of consumption is normalized to one, so w_t is the real wage and r_t is the real net interest rate.

In addition to the budget constraints in each period, the utility function imposes nonnegative consumption through inifinite marginal utility and individual labor and leisure must be nonnegative $n_{j,s,t}, l_{j,s,t} \geq 0$. We allow the possibility for individual agents to borrow $b_{j,s,t} < 0$ for some j and s in period t. However, the borrowing must satisfy a series of individual feasibility constraints as well as a strict constraint that the aggregate capital stock $K_t > 0$ be positive in every period.[4]

We next describe the Euler equations that govern the choices of consumption $c_{j,s,t}$ and savings $b_{j,s+1,t+1}$ by household of age s and ability $e_{j,s}$ in each period t. We work backward from the last period of life $s = S$. Because households do not save in the last period of life $b_{j,s+1,t+1} = 0$ due to our assumption of no bequest motive, the household's final-period maximization problem is given by the following.

$$\begin{aligned} &\max_{c_{j,S,t}, n_{j,S,t}, b_{j,s+1,t+1}} \frac{(c_{j,S,t})^{1-\sigma} - 1}{1-\sigma} + \chi e^{g_y t(1-\sigma)} \frac{(\tilde{l} - n_{j,s,t})^{1-\eta}}{1-\eta} \\ &\qquad \text{s.t.} \quad (1+r_t)\, b_{j,S,t} + w_t e_{j,S} n_{j,S,t} \geq c_{j,S,t} \quad \forall t \end{aligned} \tag{9}$$

Because $u(c)$ is monotonically increasing in c, the $s = S$ consumption part of the maximization problem (9) is simply to choose the maximum amount of consumption possible. The household trivially consumes all of its income in the last period of life. However, the household must choose labor to balance its benefits in extra consumption

[4] We describe these constraints in detail in Appendix A-2.

with its costs in disutility.

$$c_{j,S,t} = (1+r_t)\, b_{j,S,t} + w_t e_{j,S} n_{j,S,t} \quad \forall t \tag{10}$$

$$w_t e_{j,S}\Big[(1+r_t)\, b_{j,S,t} + w_t e_{j,S} n_{j,S,t}\Big]^{-\sigma} = \chi e^{g_y t(1-\sigma)}(\tilde{l} - n_{j,S,t})^{-\eta} \quad \forall t \tag{11}$$

An individual in his second-to-last period of life $s = S-1$ must choose how much to consume and how much to save for the last period of life $b_{j,S,t+1}$ as well as how much to work in the current period $n_{j,S-1,t}$ and how much to work in the final period $n_{j,S,t+1}$. The $S-1$ individual optimization problem is governed by two static first order conditions for labor $n_{j,S-1,t}$ and $n_{j,S,t+1}$ and an intertemporal Euler equation for the savings decision.

$$\begin{aligned} w_t e_{j,S-1}\Big[(1+r_t)\, b_{j,S_1,t} + w_t e_{j,S-1} n_{j,S-1,t} - b_{j,S,t+1}\Big]^{-\sigma} &= ... \\ &\chi e^{g_y t(1-\sigma)}(\tilde{l} - n_{j,S-1,t})^{-\eta} \quad \forall t \end{aligned} \tag{12}$$

$$\begin{aligned} w_{t+1} e_{j,S}\Big[(1+r_{t+1})\, b_{j,S,t+1} + w_{t+1} e_{j,S} n_{j,S,t+1}\Big]^{-\sigma} &= ... \\ &\chi e^{g_y (t+1)(1-\sigma)}(\tilde{l} - n_{j,S,t+1})^{-\eta} \quad \forall t \end{aligned} \tag{13}$$

$$\begin{aligned} \Big[(1+r_t)\, b_{j,S_1,t} + w_t e_{j,S-1} n_{j,S-1,t} - b_{j,S,t+1}\Big]^{-\sigma} &= ... \\ &\beta(1+r_{t+1})\Big[(1+r_{t+1})\, b_{j,S,t+1} + w_{t+1} e_{j,S} n_{j,S,t+1}\Big]^{-\sigma} \quad \forall t \end{aligned} \tag{14}$$

In general, maximizing (5) with respect to (6), (7), (8), and the implied individual and aggregate borrowing constraints gives the following set of $S-1$ intertemporal Euler equations and S static first order conditions characterizing lifetime savings $b_{j,s,t}$ for all j and $2 \leq s \leq S$ and labor supply $n_{j,s,t}$ for all j and $1 \leq s \leq S$.

$$\begin{aligned} w_t e_{j,s}\Big[(1+r_t)\, b_{j,s,t} + w_t e_{j,s} n_{j,s,t} - b_{j,s+1,t+1}\Big]^{-\sigma} &= \chi e^{g_y t(1-\sigma)}(\tilde{l} - n_{j,s,t})^{-\eta} \\ &\forall j,t \quad \text{and} \quad 1 \leq s \leq S \quad \text{with} \quad b_{j,1,t}, b_{j,S+1,t} = 0 \end{aligned} \tag{15}$$

$$\begin{aligned} \Big[(1+r_t)\, b_{j,s,t} + w_t e_{j,s} n_{j,s,t} - b_{j,s+1,t+1}\Big]^{-\sigma} &= ... \\ \beta(1+r_{t+1})\Big[(1+r_{t+1})\, b_{j,s+1,t+1} &+ w_{t+1} e_{j,s+1} n_{j,s+1,t+1} - b_{j,s+2,t+2}\Big]^{-\sigma} \\ &\forall j,t \quad \text{and} \quad 1 \leq s \leq S-1 \quad \text{with} \quad b_{j,1,t}, b_{j,S+1,t} = 0 \end{aligned} \tag{16}$$

2.2 Firm problem

A unit measure of identical, perfectly competitive firms exist in this economy. The representative firm is characterized by the following Cobb-Douglas production technology,

$$Y_t = AK_t^\alpha \left(e^{g_y t}L_t\right)^{1-\alpha} \quad \forall t \tag{17}$$

where A is a constant level effect on the technology process, $\alpha \in (0,1)$ is the capital share of income, g_y is the constant growth rate of labor augmenting technological change, and L_t is measured in efficiency units of labor. The interest rate r_t in the cost function is a net real interest rate because depreciation δ is paid by the firms. The real wage is w_t. The real profit function of the firm is the following.

$$\text{Real Profits} = AK_t^\alpha \left(e^{g_y t}L_t\right)^{1-\alpha} - (r_t + \delta)K_t - w_t L_t \tag{18}$$

As in the budget constraints (6), (7), and (8), note that the price of the good has been normalized to one.

Profit maximization results in the real wage w_t and the real rental rate of capital r_t being determined by the marginal products of labor and capital, respectively.

$$w_t = (1-\alpha)\frac{Y_t}{L_t} \quad \forall t \tag{19}$$

$$r_t = \alpha\frac{Y_t}{K_t} - \delta \qquad \forall t \tag{20}$$

2.3 Market clearing and stationary equilibrium

Labor market clearing requires that aggregate labor demand L_t measured in efficiency units equal the sum of individual efficiency labor supplied $e_{j,s}n_{j,s,t}$. Capital market clearing requires that aggregate capital demand K_t equal the sum of capital investment by households $b_{j,s,t}$. Aggregate consumption C_t is defined as the sum of all individual consumptions, and aggregate investment is defined by the standard

$Y = C + I$ constraint as shown in (23).

$$L_t = \frac{1}{J}\sum_{s=1}^{S}\sum_{j=1}^{J}\omega_{s,t}e_{j,s}n_{j,s,t} \qquad \forall t \tag{21}$$

$$K_t = \frac{1}{J}\sum_{s=1}^{S}\sum_{j=1}^{J}\omega_{s,t}b_{j,s,t} \qquad \forall t \tag{22}$$

$$\begin{split} &Y_t = C_t + K_{t+1} - (1-\delta)K_t \quad \forall t \\ &\quad \text{where} \quad C_t \equiv \frac{1}{J}\sum_{s=1}^{S}\sum_{j=1}^{J}\omega_{s,t}c_{j,s,t} \end{split} \tag{23}$$

The usual definition of equilibrium would be allocations and prices such that households optimize (15) and (16), firms optimize (19) and (20), and markets clear (21) and (22). However, the variables in these characterizing equations are potentially not stationary due to the possible growth rate in the total population $g_{n,t}$ each period coming from the cohort growth rates in (1) and from the deterministic growth rate of labor augmenting technological change g_y in (17).

Define the following stationary versions of the variables of the model in Table 1 in which the variables are represented in per-capita terms and in which the growth rate from labor augmenting technical change has been removed.

Table 1: Stationary variable definitions

Individual variables	Aggregate variables	
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$	$\hat{Y}_t \equiv \frac{Y_t}{N_t e^{g_y t}}$	$\hat{K}_t \equiv \frac{K_t}{N_t e^{g_y t}}$
$\hat{b}_{j,s,t} \equiv \frac{b_{j,s,t}}{e^{g_y t}}$	$\hat{L}_t \equiv \frac{L_t}{N_t}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{N_t}$

Note: The interest rate r_t in (20) is already stationary because Y_t and K_t grow at the same rate. Individual labor supply $n_{j,s,t}$ is stationary.

With the definitions in Table 1, it can be shown that the equilibrium characterizing equations can be written in stationary form in the following way. The static and intertemporal Euler equations from the individual's optimization problem corresponding to (15) and (16) are the following.

$$\hat{w}_t e_{j,s}\Big[(1+r_t)\,\hat{b}_{j,s,t}+\hat{w}_t e_{j,s} n_{j,s,t} - e^{g_y}\hat{b}_{j,s+1,t+1}\Big]^{-\sigma} = \chi(\tilde{l} - n_{j,s,t})^{-\eta} \quad \forall j,t \quad \text{and} \quad 1\leq s\leq S \quad \text{with} \quad \hat{b}_{j,1,t}, \hat{b}_{j,S+1,t}=0 \tag{24}$$

$$\Big[(1+r_t)\,\hat{b}_{j,s,t}+\hat{w}_t e_{j,s} n_{j,s,t} - e^{g_y} b_{j,s+1,t+1}\Big]^{-\sigma} = ... \quad \beta(1+r_{t+1})e^{-\sigma g_y}\Big[(1+r_{t+1})\,\hat{b}_{j,s+1,t+1}+\hat{w}_{t+1} e_{j,s+1} n_{j,s+1,t+1} - e^{g_y}\hat{b}_{j,s+2,t+2}\Big]^{-\sigma} \quad \forall j,t \quad \text{and} \quad 1\leq s\leq S-1 \quad \text{with} \quad \hat{b}_{j,1,t}, \hat{b}_{j,S+1,t}=0 \tag{25}$$

The stationary firm first order conditions for optimal labor and capital demand corresponding to (19) and (20) are the following.

$$\hat{w}_t = (1-\alpha)\frac{\hat{Y}_t}{\hat{L}_t} \quad \forall t \tag{26}$$

$$r_t = \alpha\frac{\hat{Y}_t}{\hat{K}_t} - \delta = \alpha\frac{Y_t}{K_t} - \delta \quad \forall t \tag{20}$$

And the two stationary market clearing conditions corresponding to (21) and (22)—with the goods market clearing by Walras' Law—are the following.

$$\hat{L}_t = \frac{1}{J}\sum_{s=1}^{S}\sum_{j=1}^{J}\hat{\omega}_{s,t} e_{j,s} n_{j,s,t} \quad \forall t \tag{27}$$

$$\hat{K}_t = \frac{1}{J}\sum_{s=1}^{S}\sum_{j=1}^{J}\hat{\omega}_{s,t} \hat{b}_{j,s,t} \quad \forall t \tag{28}$$

$$\tag{29}$$

We can now define the stationary steady-state equilibrium for this economy in the following way.

Definition 1 (**Stationary steady-state equilibrium**). A non-autarkic stationary steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability $e_{j,s}$ is defined as constant allocations $\hat{b}_{j,s+1,t+1} = \bar{b}_{j,s+1}$ and $\hat{n}_{j,s,t} = \bar{n}_{j,s}$ and constant prices $\hat{w}_t = \bar{w}$ and $\hat{r}_t = \bar{r}$ for all j, s, and t such that the following conditions hold:

i. households optimize according to (24), and (25),

ii. firms optimize according to (26) and (20),

iii. markets clear according to (27) and (28), and

iv. the population has reached its stationary steady state distribution $\bar{\omega}_s$ for all ages s, characterized in Appendix A-3.

The steady-state equilibrium is characterized by the system of $J(2S-1)$ equations and $J(2S-1)$ unknowns $\bar{n}_{j,s}$ and $\bar{b}_{j,s+1}$ along with the individual borrowing constraints and aggregate borrowing constraint described in Appendix A-2.

$$\bar{w}e_{j,s}\Big[(1+\bar{r})\,\bar{b}_{j,s}+\bar{w}e_{j,s}\bar{n}_{j,s} - e^{g_y}\bar{b}_{j,s+1}\Big]^{-\sigma} = \chi(\tilde{l}-\bar{n}_{j,s})^{-\eta} \quad \forall j \quad \text{and} \quad 1\leq s\leq S \quad \text{with} \quad \bar{b}_{j,1},\bar{b}_{j,S+1}=0 \tag{30}$$

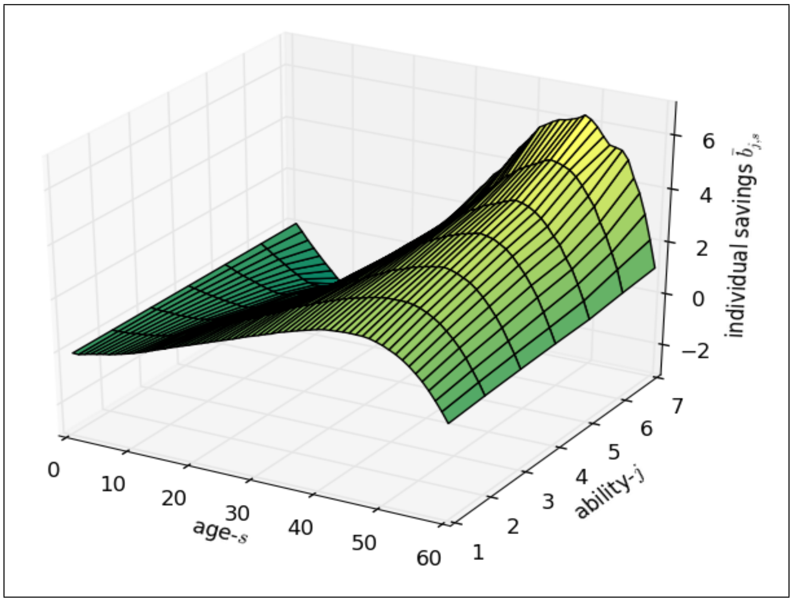
$$\Big[(1+\bar{r})\,\bar{b}_{j,s}+\bar{w}e_{j,s}\bar{n}_{j,s} - e^{g_y}\bar{b}_{j,s+1}\Big]^{-\sigma} = ... \\ \beta(1+\bar{r})e^{-\sigma g_y}\Big[(1+\bar{r})\,\bar{b}_{j,s+1}+\bar{w}e_{j,s+1}\bar{n}_{j,s+1} - e^{g_y}\bar{b}_{j,s+2}\Big]^{-\sigma} \\ \forall j \quad \text{and} \quad 1\leq s\leq S-1 \quad \text{with} \quad \bar{b}_{j,1},\bar{b}_{j,S+1}=0 \tag{31}$$

Define $\hat{\mathbf{\Gamma}}_t$ as the distribution of stationary savings across individuals at time t.

$$\hat{\mathbf{\Gamma}}_t \equiv \{\hat{b}_{j,s,t}\}_{j=1,s=2}^{J,S} \quad \forall t \tag{32}$$

In equilibrium, the steady-state individual labor supplies $\bar{n}_{j,s}$ for all j and s, the steady-state real wage $\bar{w}$, and the steady-state real rental rate $\bar{r}$ are simply functions of the steady-state distribution of savings $\bar{\Gamma}$. This is clear from the steady-state version of the capital market clearing condition (28) and the fact that aggregate labor supply is a function of the sum of exogenous efficiency units of labor in the labor market clearing condition (27). And the two firm first order conditions for the real wage $\hat{w}_t$ (26) and real rental rate r_t (20) are only functions of the aggregate capital stock

Figure 1: Stationary steady-state distribution of savings $\bar{\mathbf{\Gamma}}$ for $S = 60$ and $J = 7$



$\hat{K}_t$ and aggregate labor $\hat{L}_t$. Appendix A-4 details how to solve for the steady-state equilibrium.

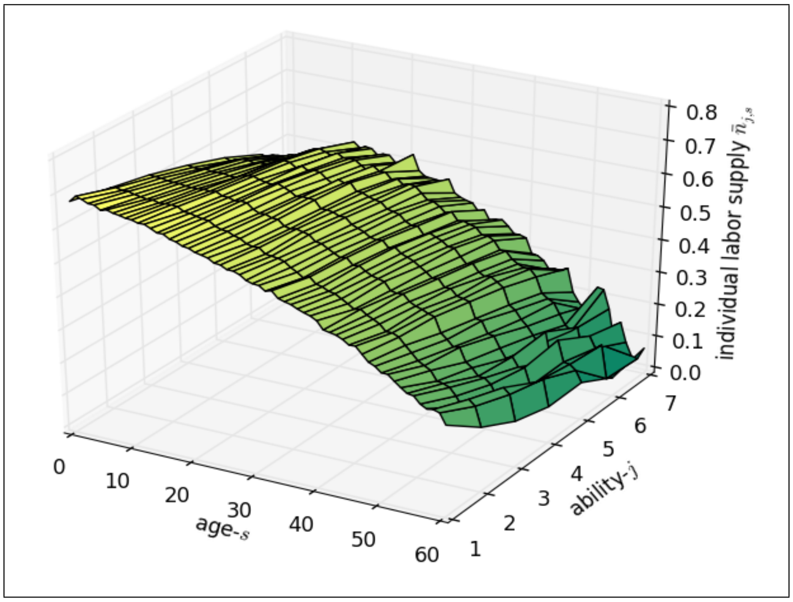
Figure 1 shows the stationary steady-state distribution of individual savings $\bar{\mathbf{\Gamma}}$ and Figure 2 shows the stationary steady-state distribution of individual labor supply $\bar{n}_{j,s}$ for a particular calibration of the model. The deterministic individual ability process $e_{j,s}$ is calibrated from CPS wage distribution data as described in Appendix A-1, with $S = 60$ and $J = 7$. We calibrate the other parameters to $[\beta, \sigma, \delta, \chi, \eta, \alpha, A] = [0.96, 3, 0.05, 1, 2.5, 0.35, 1]$. Notice ....

The definition of the stationary non-steady-state equilibrium is similar to Definition 1, with the stationary steady-state equilibrium definition being a special case of the stationary non-steady-state equilibrium.

Definition 2 (**Stationary non-steady-state equilibrium**). A non-autarkic stationary non-steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability $e_{j,s}$ is defined as allocations $n_{j,s,t}$ and $\hat{b}_{j,s+1,t+1}$ and prices $\hat{w}_t$ and r_t for all j, s, and t such that the following conditions hold:

i. households optimize according to (24), and (25),

Figure 2: Stationary steady-state distribution of individual labor supply $\bar{n}_{j,s}$ for $S = 60$ and $J = 7$



ii. firms optimize according to (26) and (20), and

iii. markets clear according to (27) and (28).

The household labor-leisure decision in the last period of life shows that the optimal labor supply for age $s = S$ is a function of individual holdings of savings and the prices in that period $n_{j,S,t} = \phi\big(\hat{b}_{j,S,t}, \hat{w}_t, r_t\big)$. This decision is characterized by final-age version of thet static Euler equation (24). Households in their second-to-last period of life in period t have three decisions to make. They must choose how much to work this period $n_{j,S-1,t}$ and next $n_{j,S,t+1}$ and how much to save this period for next period $\hat{b}_{j,S,t+1}$. The optimal responses for this individual are characterized by the $s = S-1$ and $s = S$ versions of the static Euler equations (24) and the $s = S-1$ version of the intertemporal Euler equation (25), respectively.

Optimal savings in the second-to-last period of life $s = S-1$ is a function of the current savings and the prices in the current period and in the next period $\hat{b}_{j,S,t+1} = \psi\big(\hat{b}_{j,S-1,t}, \hat{w}_t, r_t, \hat{w}_{t+1}, r_{t+1}\big)$. As before, the optimal labor supply at age $s = S$ is a function of the next period's savings and prices $n_{j,S,t+1} = \phi\big(\hat{b}_{j,S,t+1}, \hat{w}_{t+1}, r_{t+1}\big)$. But the optimal labor supply at age $s = S-1$ is a function of the current savings and the current prices as well as the future prices because of the dependence on the savings decision in that same period $n_{j,S-1,t} = \phi\big(\hat{b}_{j,S-1,t}, \hat{w}_t, r_t, \hat{w}_{t+1}, r_{t+1}\big)$. By induction, we can show that the optimal labor supply and savings functions for any individual with ability j, age s, and in period t is a function of current holdings of savings and the lifetime path of prices.

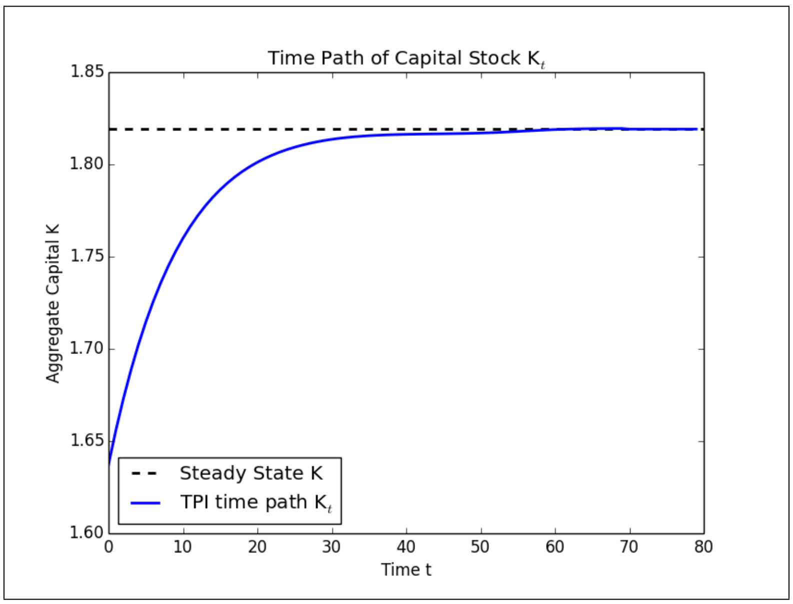
$$n_{j,s,t} = \phi\Big(\hat{b}_{j,s,t}, (\hat{w}_v, r_v)_{v=t}^{t+S-s}\Big) \quad \forall j, s, t \tag{33}$$

$$\hat{b}_{j,s+1,t+1} = \psi\Big(\hat{b}_{j,s,t}, (\hat{w}_v, r_v)_{v=t}^{t+S-s}\Big) \quad \forall j, t \quad \text{and} \quad 1 \leq s \leq S-1 \tag{34}$$

Each optimal saving decision for each household requires knowledge of at least todays prices and tomorrows prices and at most S periods of prices. In equilibrium, one can see that the prices $(\hat{w}_t, r_t)$ in each period t are functions of the entire distribution of savings $\mathbf{\Gamma}_t$. The requirement that individuals be able to forecast prices with perfect foresight over their lifetimes implies that each individual has correct information and beliefs about all the other individuals optimization problems and information. It also implies that the equilibrium allocations and prices are really just functions of the entire distribution of savings at a particular period, as well as a law of motion for that distribution of savings.

To solve for any non-steady-state equilibrium time path of the economy from an arbitrary current state to the steady state, we follow the time path iteration (TPI) method of Auerbach and Kotlikoff (1987). Appendix A-5 details how to solve for the non-steady-state equilibrium time path using the TPI method. Figure 3 shows the equilibrium time path of the aggregate capital stock for the calibration used in Figure 3 for $T = 80$ periods starting from an initial distribution of savings in which $b_{j,s,1} = (0.9\bar{K})/[(S-1)J]$ for all j and s. We used $\nu = 0.2$ as our time-path updating dampening parameter (see Equation (A.5.6) in Appendix A-5.)

Figure 3: Equilibrium time path of K_t for $S = 60$ and $J = 7$



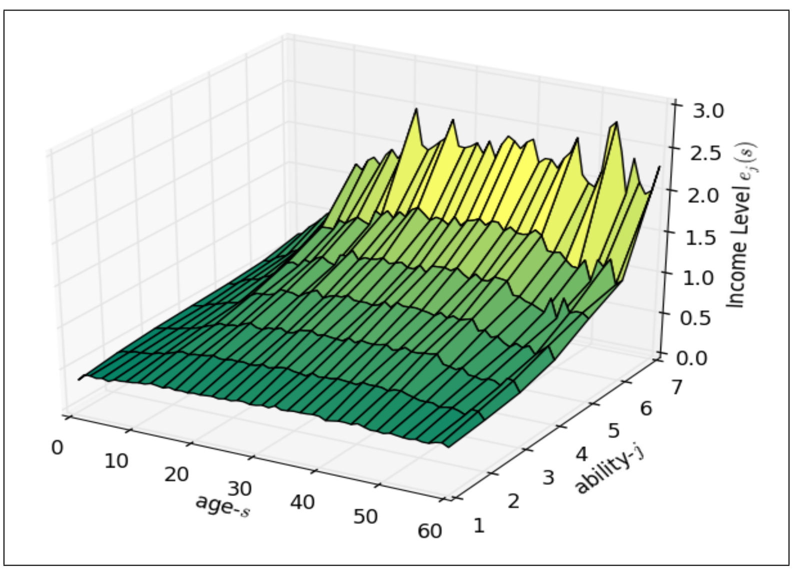
APPENDIX

A-1 Calibration of ability process

The calibration of the ability process $e_{j,s}$ is as follows. First, the ability types themselves must be calibrated. For each age group $s \in S$, the hourly wage rates are sorted into J percentile groups. The ability type for each percentile group is the median wage for the percentile group, divided by the average wage of all individuals in the data set.

The data used to calibrate the ability types were obtained from the Current Population Survey.[5] Individuals younger than 20 and older than 79 are dropped from the data. This is due to the extremely small amount of observations for ages outside of those bounds. Due to a limited number of observations in the survey who included their hourly wage, data was taken from the months of January, February, March, April, and May 2014. Population weights were also used to obtain the correct percentile groups of individuals. The income levels for the J ability types were then calculated for each month, and then an average was taken of the five calibrations of the ability types in order to produce a final calibration to be used in the model. Figure 4 shows this income distribution across age and ability type.

Figure 4: Distribution of Income where $S = 60$ **and** $J = 7$



In this paper, individuals are assigned ability types at the beginning of their life, and cannot change types later on.

[5] U.S. Census Bureau, Dataferret, Current Population Survey, 2014. The variables $PRTAGE$, $PTERNHLY$, and $PWCMPWGT$ were used for the age, hourly wage rate, and population weight of individuals, respectively.

A-2 Constraints on individual borrowing

As described in Section 2.1, individuals are allowed to borrow $b_{j,s,t}$ for some j and s in period t. However, two constraints must hold. First, the individual must be able to pay back the balance with interest r_{t+1} in the next period without driving consumption in the next period $c_{j,s+1,t+1}$ to be nonpositive. Let $\tilde{b}_{j,s,t}$ be the minimum value of savings in a period.

$$b_{j,s,t} \geq \tilde{b}_{j,s,t} \quad \forall j, s, t \tag{A.2.1}$$

Rearranging the bugdet constraints in (6), (7), and (8) and using backward induction gives the following expressions for $\bar{b}_{j,s,t}$,

$$\begin{aligned} \tilde{b}_{j,S,t} &= \frac{\tilde{c} - w_t e_{j,S}\tilde{l}}{1+r_t} \\ \tilde{b}_{j,S-1,t-1} &= \frac{\tilde{c} + \tilde{b}_{j,S,t} - w_{t-1}e_{j,S-1}\tilde{l}}{1+r_{t-1}} \\ &\vdots \\ \tilde{b}_{j,2,t-S+2} &= \frac{\tilde{c} + \tilde{b}_{j,3,t-S+3} - w_{t-S+2}e_{j,2}\tilde{l}}{1+r_{t-S+2}} \end{aligned} \tag{A.2.2}$$

where $\tilde{c} > 0$ is some minimum amount of consumption and $\tilde{l}$ is the maximum amount an individual can work from the time constraint (4). With endogenous labor supply $n_{j,s,t}$, it is less likely that the individual borrowing constraints every bind. This is because the disutility of labor increases exponentially according to $\eta > 1$ in the period utility function (5).

In addition to the individual borrowing constraint (A.2.1), a strict aggregate borrowing constraint must be met. That is, the aggregate capital stock must be strictly positive.

$$K_t > 0 \quad \forall t \tag{A.2.3}$$

A-3 Characteristics of exogenous population growth assumptions

In this appendix, we describe in detail the exogenous population growth assumptions in the model and their implications. In Section 2.1, we define the laws of motion for the population of each cohort $\omega_{s,t}$ to be the following.

$$\begin{aligned} \omega_{1,t+1} &= \sum_{s=1}^{S} f_s\omega_{s,t} \quad \forall t \\ \omega_{s+1,t+1} &= (1+i_s-\rho_s)\omega_{s,t} \quad \forall t \quad \text{and} \quad 1\leq s\leq S-1 \end{aligned} \tag{1}$$

We can transform the nonstationary equations in (1) into stationary laws of motion by dividing both sides by the total populations N_t and N_{t+1} in both periods,

$$\begin{aligned} \hat{\omega}_{1,t+1} &= \frac{\sum_{s=1}^{S} f_s\hat{\omega}_{s,t}}{1+g_{n,t+1}} \quad \forall t \\ \hat{\omega}_{s+1,t+1} &= \frac{(1+\phi_s-\rho_s)\hat{\omega}_{s,t}}{1+g_{n,t+1}} \quad \forall t \quad \text{and} \quad 1\leq s\leq S-1 \end{aligned} \tag{A.3.1}$$

where $\hat{\omega}_{s,t}$ is the percent of the total population in age cohort s and the population growth rate $g_{n,t+1}$ between periods t and $t+1$ is defined in (3),

$$\begin{bmatrix} \hat{\omega}_{1,t+1} \\ \hat{\omega}_{2,t+1} \\ \hat{\omega}_{2,t+1} \\ \vdots \\ \hat{\omega}_{S-1,t+1} \\ \hat{\omega}_{S,t+1} \end{bmatrix} = \frac{1}{1+g_{n,t+1}} \begin{bmatrix} f_1 & f_2 & f_3 & \dots & f_{S-1} & f_S \\ 1+i_1-\rho_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1+i_2-\rho_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1+i_3-\rho_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1+i_{S-1}-\rho_{S-1} & 0 \end{bmatrix} \begin{bmatrix} \hat{\omega}_{1,t} \\ \hat{\omega}_{2,t} \\ \hat{\omega}_{2,t} \\ \vdots \\ \hat{\omega}_{S-1,t} \\ \hat{\omega}_{S,t} \end{bmatrix} \tag{A.3.2}$$

where we restrict $1+i_s-\rho_s\geq 0$ for all s.

We write (A.3.2) in matrix notation as the following.

$$\hat{\boldsymbol{\omega}}_{t+1} = \frac{1}{1+g_{n,t+1}}\boldsymbol{\Omega}\hat{\boldsymbol{\omega}}_t \quad \forall t \tag{A.3.3}$$

The stationary steady state population distribution $\bar{\boldsymbol{\omega}}$ is the eigenvector $\boldsymbol{\omega}$ with eigenvalue $(1+\bar{g}_n)$ of the matrix $\boldsymbol{\Omega}$ that satisfies the following version of (A.3.3).

$$(1+\bar{g}_n)\bar{\boldsymbol{\omega}} = \boldsymbol{\Omega}\bar{\boldsymbol{\omega}} \tag{A.3.4}$$

TODO:

- We need to show the conditions under which the matrix $\boldsymbol{\Omega}$ has only one eigenvector associated with one positive eigen value with no complex part.
- Another approach is to simply simulate the problem from the initial population distribution $\boldsymbol{\omega}_0$ and what the steady state $\bar{\boldsymbol{\omega}}$ is and how many periods it takes to get there.
  - We can use the number of periods to arrive at the steady state as a lower bound for T in the time path iteration algorithm.

A-4 Solving for stationary steady-state equilibrium

This section describes the solution method for the stationary steady-state equilibrium described in Definition 1.

1. Choose an initial guess for the stationary steady-state distribution of capital $\bar{b}_{j,s+1}$ for all j and $s = 1, 2, ...S-1$ and labor supply $\bar{n}_{j,s}$ for all j and s.
   - A good first guess is a large positive number for all the $\bar{n}_{j,s}$ that is slightly less than $\tilde{l}$ and to choose some small positive number that is small enough to be less than the minimum income that an individual might have $\bar{w}e_{j,s}\bar{n}_{j,s}$.
2. Perform a constrained root finder that chooses $\bar{b}_{j,s+1}$ and $\bar{n}_{j,s}$ that solves the $J(2S-1)$ stationary steady-state Euler equations (30) and (31).
3. Make sure none of the implied steady-state consumptions $\bar{c}_{j,s}$ is less-than-or-equal-to zero.
   - If one consumption is less-than-or-equal-to zero $\bar{c}_{j,s} \leq 0$, then try different starting values.
4. Make sure that none of the Euler errors is too large in absolute value for interior stationary steady-state values. A steady-state Euler error is the following, which is supposed to be close to zero for all j and $s = 1, 2, ...S-1$:

$$\frac{\beta\left(1+\bar{r}\right)\left(\bar{c}_{j,s+1}\right)^{-\sigma}}{\left(\bar{c}_{j,s}\right)^{-\sigma}} - 1 \tag{A.4.1}$$

5. Make sure that the unconstrained solution satisfies the individual borrowing constraints in (A.2.1) and (A.2.2).
   - If any individual's borrowing constraint is not satisfied using the unconstrained root finding operation, rerun the root finding operation in step (ii) as a constrained minimization problem with the borrowing constraints imposed on those individuals.
   - Repeat steps (ii) through (v) until all the individual borrowing constraints are met.
6. Make sure that the solution satisfies the aggregate borrowing constraint (A.2.3).
   - If it does not, what is the least distortionary upward adjustment to individual steady-state savings $\bar{b}_{j,s+1}$?

A-5 Solving for stationary non-steady-state equilibrium by time path iteration

This section outlines the benchmark time path iteration (TPI) method of Auerbach and Kotlikoff (1987) for solving the stationary non-steady-state equilibrium transition path of the distribution of savings. TPI finds a fixed point for the transition path of the distribution of capital for a given initial state of the distribution of capital. The idea is that the economy is infinitely lived, even though the agents that make up the economy are not. Rather than recursively solving for equilibrium policy functions by iterating on individual value functions, one must recursively solve for the policy functions by iterating on the entire transition path of the endogenous objects in the economy (see Stokey and Lucas (1989, ch. 17)).

The key assumption is that the economy will reach the steady-state equilibrium described in Definition 1 in a finite number of periods $T < \infty$ regardless of the initial state. Let $\hat{\mathbf{\Gamma}}_t$ represent the distribution of savings at time t.

$$\hat{\mathbf{\Gamma}}_t \equiv \{\hat{b}_{j,s,t}\}_{j=1,s=1}^{J,S} \quad \forall t \tag{32}$$

In Section 2.3, we describe how the stationary non-steady-state equilibrium time path of allocations and price is described by functions of the state $\hat{\mathbf{\Gamma}}_t$ and its law of motion. TPI starts the economy at any initial distribution of savings $\hat{\mathbf{\Gamma}}_0$ and solves for its equilibrium time path over T periods to the steady-state distribution $\bar{\mathbf{\Gamma}}_T$.

The first step is to assume an initial transition path for aggregate capital $\hat{\mathbf{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$ such that T is sufficiently large to ensure that $\hat{\mathbf{\Gamma}}_T = \bar{\mathbf{\Gamma}}$ and $\hat{K}_T^i\left(\mathbf{\Gamma}_T\right) = \bar{K}\left(\bar{\mathbf{\Gamma}}\right)$. The superscript i is an index for the iteration number. The transition path for aggregate capital determines the transition path for both the real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$ and the real return on investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. The exact initial distribution of capital in the first period $\hat{\mathbf{\Gamma}}_1$ can be arbitrarily chosen as long as it satisfies $\hat{K}_1^i = \frac{1}{J}\sum_{s=1}^S\sum_{j=1}^J \hat{\omega}_{s,1}\hat{b}_{j,s,1}$ according to market clearing condition (28). One could also first choose the initial distribution of savings $\hat{\mathbf{\Gamma}}_1$ and then choose an initial aggregate capital stock $\hat{K}_1^i$ that corresponds to that distribution. As mentioned earlier, the only other restriction on the initial transition path for aggregate capital is that it equal the steady-state level $\hat{K}_T^i = \bar{K}\left(\bar{\mathbf{\Gamma}}\right)$ by period T. Evans and Phillips (2014) have shown that the initial guess for the aggregate capital stocks $\hat{K}_t^i$ for periods $1 < t < T$ can take on almost any positive values.

Given the initial capital distribution $\hat{\mathbf{\Gamma}}_1$ and the transition paths of aggregate capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$, the real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$, and the real return to investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, one can solve for the optimal labor supply for each type j of $s = S$-aged agents in the last period of their lives $n_{j,S,1} = \phi(\hat{b}_{j,S,1}, \hat{w}_1, r_1)$ using his one static Euler equation, which is the $s = S$ version of (24).

$$\hat{w}_1^i e_{j,S}\left[\left(1 + r_1^i\right)\hat{b}_{j,S,1} + \hat{w}_1^i e_{j,S} n_{j,S,1}\right]^{-\sigma} = \chi(\tilde{l} - n_{j,S,1})^{-\eta} \tag{A.5.1}$$

[HAVEN'T UPDATED FROM HERE DOWN]

We then solve the problem for all j types of $s = S-1$-aged individuals in period $t=1$, each of which entails labor supply decisions in the current period $n_{j,S-1,1}$ and in the next period $n_{j,S,2}$ and a savings decision in the current period for the next period $b_{j,S,t+1}$. The labor supply decision in the initial period for each type j of $S-1$-aged individual savings policy rule for each type j of $S-1$-aged agent for the last period of his life $\hat{b}_{j,S,2} = \psi_{j,S-1}(\hat{b}_{j,S-1,1}, \{\hat{w}_t, r_t\}_{t=1}^2)$ using his one intertemporal Euler equation (A.5.2), where the "j,S-1" subscript on ψ represents the function for type j in age $s = S-1$ savings decision.

$$\begin{aligned} \Big([1+r_1^i]b_{j,S-1,1} + w_1^i e_{j,S-1} l(S-1) - b_{j,S,2}\Big)^{-\sigma} &= \\ \beta(1+r_2^i)\Big([1+r_2^i]b_{j,S,2} + w_2^i e_{j,S} l(S)\Big)^{-\sigma} & \quad \forall j \end{aligned} \tag{A.5.2}$$

The final two savings decisions of each type j of $S-2$-aged household in period 1, $b_{j,S-1,2}$ and $b_{j,S,3}$, are characterized by the following two intertemporal Euler equations.

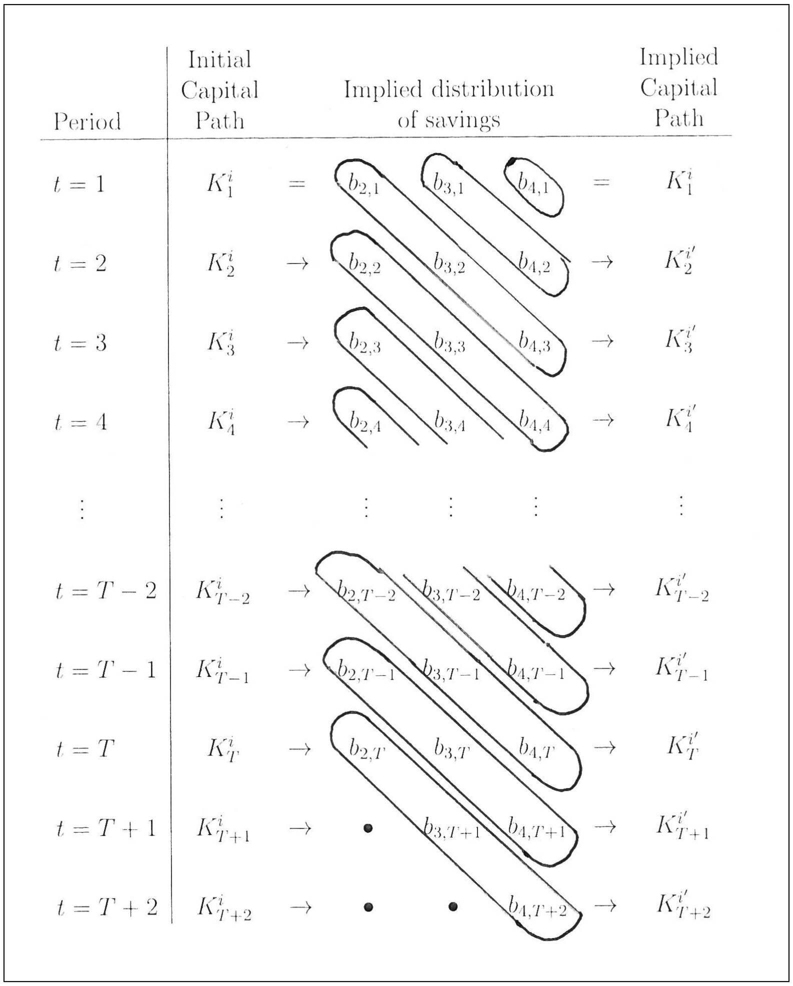
$$\begin{aligned} \Big([1+r_1^i]b_{j,S-2,1} + w_1^i e_{j,S-2} l(S-2) - b_{j,S-1,2}\Big)^{-\sigma} &= \\ \beta(1+r_2^i)\Big([1+r_2^i]b_{j,S-1,2} + w_2^i e_{j,S-1} l(S-1) - b_{j,S,3}\Big)^{-\sigma}, & \\ \Big([1+r_2^i]b_{j,S-1,2} + w_2^i e_{j,S-1} l(S-1) - b_{j,S,3}\Big)^{-\sigma} &= \\ \beta(1+r_3^i)\Big([1+r_3^i]b_{j,S,3} + w_3^i e_{j,S} l(S)\Big)^{-\sigma} & \end{aligned} \quad \forall j \tag{A.5.3}$$

This process is repeated for every age of household alive in $t=1$ down to the age $s=1$ household at time $t=1$. Each of these households j solve the full set of $S-1$ savings decisions characterized by the following equations.

$$\begin{aligned} \Big(w_1^i e_{j,1} l(1) - b_{j,2,2}\Big)^{-\sigma} &= ... \\ \beta\left(1+r_2^i\right)\Big(\left[1+r_2^i\right]b_{j,2,2} + w_2^i e_{j,2} l(2) - b_{j,3,3}\Big)^{-\sigma} & \\ \Big(\left[1+r_2^i\right]b_{j,2,2} + w_2^i e_{j,2} l(2) - b_{j,3,3}\Big)^{-\sigma} &= ... \\ \beta\left(1+r_3^i\right)\Big(\left[1+r_3^i\right]b_{j,3,3} + w_3^i e_{j,3} l(3) - b_{j,4,4}\Big)^{-\sigma} & \quad \forall j \\ \vdots & \\ \Big(\left[1+r_{S-1}^i\right]b_{j,S-1,S-1} + w_{S-1}^i e_{j,S-1} l(S-1) - b_{S,S}\Big)^{-\sigma} &= ... \\ \beta\left(1+r_S^i\right)\Big(\left[1+r_S^i\right]b_{j,S,S} + w_S^i e_{j,S} l(S)\Big)^{-\sigma} & \end{aligned} \tag{A.5.4}$$

We can then solve for the entire lifetime of savings decisions for each age $s=1$ individual in periods $t = 2, 3, ...T$. The central part of the schematic diagram in Figure 5 shows how this process is done in order to solve for the equilibrium time path of the economy from period $t = 1$ to T. Note that for each full lifetime savings path solved for an individual born in period $t \geq 2$, we can solve for the aggregate capital stock implied by those savings decisions $K_t^{i'} = \frac{1}{SJ}\sum_{s=1}^{S}\sum_{j=1}^{J} b_{j,s,t}$.

Figure 5: Diagram of TPI solution method within each iteration for $S = 4$ and $J = 1$



Once the set of lifetime saving decisions has been computed for all individuals alive in $1 \leq t \leq T$, we use the household decisions to compute a new implied time path of the aggregate capital stock. The implied path of the aggregate capital stock $\boldsymbol{K}^{i'} = \{K_1^i, K_2^{i'}, ...K_T^{i'}\}$ in general does not equal the initial path of the aggregate capital stock $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_T^i\}$ that was used to compute the household savings decisions $\boldsymbol{K}^{i'} \neq \boldsymbol{K}^i$.

Let $\|\cdot\|$ be a norm on the space of time paths of the aggregate capital stock $\boldsymbol{K} \in \mathcal{K} \subset \mathbb{R}_{++}^T$. Then the fixed point necessary for the equilibrium transition path from Definition 2 has been found when the distance between $\boldsymbol{K}^{i'}$ and $\boldsymbol{K}^i$ is arbitrarily close to zero.

$$\|\boldsymbol{K}^{i'} - \boldsymbol{K}^i\| \leq \varepsilon \quad \text{for} \quad \varepsilon > 0 \tag{A.5.5}$$

If the fixed point has not been found $\|\boldsymbol{K}^{i'} - \boldsymbol{K}^i\| > \varepsilon$, then a new transition path for the aggregate capital stock is generated as a convex combination of $\boldsymbol{K}^{i'}$ and $\boldsymbol{K}^i$.

$$\boldsymbol{K}^{i+1} = \nu \boldsymbol{K}^{i'} + (1-\nu)\boldsymbol{K}^i \quad \text{for} \quad \nu \in (0,1) \tag{A.5.6}$$

This process is repeated until the initial transition path for the aggregate capital stock is consistent with the transition path implied by those beliefs and household and firm optimization.

In essence, the TPI method iterates on individual beliefs about the time path of prices represented by a time path for the aggregate capital stock $\boldsymbol{K}^i$ until a fixed point in beliefs is found that are consistent with the transition path implied by optimization based on those beliefs.

The following are the steps for computing a non-steady-state equilibrium time path for the economy.

1. Using the parameterization from the steady-state computation, and choose the value for T at which the non-steady-state transition path should have converged to the steady state.

2. Choose an initial state of the aggregate capital stock K_1. Choose an initial distribution of savings $\boldsymbol{\Gamma}_1$ consistent with K_1 according to (22).

3. Conjecture a transition path for the aggregate capital stock $\boldsymbol{K}^i = \{K_t^i\}_{t=1}^{\infty}$ where the only requirements are that $K_1^i = K_1$ is your initial state and that $K_t^i = \bar{K}$ for all $t \geq T$. The conjectured transition path of the aggregate capital stock $\boldsymbol{K}^i$, along with the exogenous aggregate labor supply from (21), implies specific transition paths for the real wage $\boldsymbol{w}^i = \{w_t^i\}_{t=1}^{\infty}$ and the real interest rate $\boldsymbol{r}^i = \{r_t^i\}_{t=1}^{\infty}$ through expressions (17), (19), and (20).

4. With the conjectured transition paths $\boldsymbol{w}^i$ and $\boldsymbol{r}^i$, one can solve for the lifetime policy functions of each household alive at time $1 \leq t \leq T$ using the systems of Euler equations of the form (A.5.4).

   (a) Make sure that the individual borrowing constraints (A.2.1) are satisfied for each individual in every period.

   (b) Increase any individual savings to the minimum $\tilde{b}_{j,s,t}$ if the borrowing constraint is not satisfied.

5. Use the implied distribution of savings in each period (each row of $b_{j,s,t}$ in Figure 5) to compute the new implied time path for the aggregate capital stock $\boldsymbol{K}^{i'} = \{K_1^i, K_2^{i'}, ...K_T^{i'}\}$.

   (a) Make sure that the aggregate borrowing constraint (A.2.3) is satisfied in each period t.

   (b) If the aggregate borrowing constraint is not satisfied, increase every individual's savings by the fraction that makes the aggregate capital stock slightly greater than zero.

6. Check the distance between the two time paths $\|\boldsymbol{K}^{i'} - \boldsymbol{K}^{i}\|$.

   (a) If the distance between the initial time path and the implied time path is less-than-or-equal-to some convergence criterion $\varepsilon > 0$, then the fixed point has been achieved and the equilibrium time path has been found (A.5.5).

   (b) If the distance between the initial time path and the implied time path is greater than some convergence criterion $\|\| > \varepsilon$, then update the guess for the time path of the aggregate capital stock according to (A.5.6) and repeat steps (4) through (6) until a fixed point is reached.

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TECHNICAL APPENDIX

T-1 Comments and Notes

Structures to add to the model and order

1. Stationarize the model
2. Add household tax structures
3. Add firm structures
4. Add small open economy feature