# A Macroeconomic Model for Dynamic Scoring *

Richard W. Evans[†] Evan Magnusson[‡] Kerk L. Phillips[§]
Isaac Swift[¶]

July 2014
(version 14.07.b)

**Abstract**

Put abstract here.

*keywords:* Put keywords here.

*JEL classification:* Put JEL codes here.

*The collection of data used in this study was partly supported by the National Institutes of Health under grant number R01 HD069609 and the National Science Foundation under award number 1157698. Put thanks here.

[†]Brigham Young University, Department of Economics, 167 FOB, Provo, Utah 84602, (801) 422-8303, revans@byu.edu.

[‡]Brigham Young University, Department of Economics, 163 FOB, Provo, Utah 84602, evanmag42@gmail.com.

[§]Brigham Young University, Department of Economics, 166 FOB, Provo, Utah 84602, (801) 422-5928, kerk_phillips@byu.edu.

[¶]Brigham Young University, Department of Economics, 163 FOB, Provo, Utah 84602, isaacdswift@gmail.com.

# 1 Introduction

Put introduction here.

# 2 Model

Model intro here.

## 2.1 Individual problem

A measure $1/S$ of individuals with heterogeneous working ability $e \in \mathcal{E} \subset \mathbb{R}_{++}$ is born in each period $t$ and live for $S \geq 3$ periods. Their working ability evolves over their lifetime according to an age-dependent deterministic process. At birth, a fraction $1/J$ of the $1/S$ measure of new agents are randomly assigned to one of $J$ ability types indexed by $j = 1, 2, ...J$. Once ability type is determined, that measure $1/(SJ)$ of individuals' ability evolves deterministically according to $e_j(s)$. We calibrate the matrix of lifetime ability paths $e_j(s)$ for all types $j$ using CPS hourly wage by age distribution data.[1]

Individuals are endowed with a measure of time in each period $t$ that they supply inelastically to the labor market. Let $s$ represent the periods that an individual has been alive. The fixed labor supply in each period $t$ by each age-$s$ individual is denoted by $l(s)$.

At time $t$, all generation $s$ agents with ability $e_j(s)$ know the real wage rate $w_t$ and know the one-period real net interest rate $r_t$ on bond holdings $b_{j,s,t}$ that mature at the beginning of period $t$. In each period $t$, age-$s$ agents with working ability $e$ choose how much to consume $c_{j,s,t}$ and how much to save for the next period by loaning capital to firms in the form of a one-period bond $b_{j,s+1,t+1}$ in order to maximize expected

[1] Appendix A-1 gives a detailed description of the calibration of the deterministic ability process by age $s$ and type $j$, as well as alternative specifications and calibrations.

lifetime utility of the following form,

$$U_{j,s,t} = \sum_{v=0}^{S-s} \beta^u u\left(c_{j,s+u,t+u}\right) \quad \text{where} \quad u\left(c_{j,s,t}\right) = \frac{\left(c_{j,s,t}\right)^{1-\sigma} - 1}{1-\sigma} \quad \forall j,s,t \tag{1}$$

where $u(c)$ is a constant relative risk aversion utility function, $\sigma \geq 1$ is the coefficient of relative risk aversion, and $\beta \in (0,1)$ is the agent's discount factor.

Because agents are born without any bonds maturing and because they purchase no bonds in the last period of life $s = S$, the per-period budget constraints for each agent normalized by the price of consumption are the following.

$$w_t e_j(s) l(s) \geq c_{j,s,t} + b_{j,s+1,t+1} \quad \text{for} \quad s = 1 \quad \forall j,t \tag{2}$$

$$\left(1 + r_t\right) b_{j,s,t} + w_t e_j(s) l(s) \geq c_{j,s,t} + b_{j,s+1,t+1} \quad \text{for} \quad 2 \leq s \leq S-1 \quad \forall j,t \tag{3}$$

$$\left(1 + r_t\right) b_{j,s,t} + w_t e_j(s) l(s) \geq c_{j,s,t} \quad \text{for} \quad s = S \quad \forall j,t \tag{4}$$

Note that the price of consumption is normalized to one, so $w_t$ is the real wage and $r_t$ is the real net interest rate.

In addition to the budget constraints in each period, the utility function imposes nonnegative consumption through inifinite marginal utility. We allow the possibility for individual agents to borrow $b_{j,s,t} < 0$ for some $j$ and $s$ in period $t$. However, the borrowing must satisfy a series of individual feasibility constraints as well as a strict constraint that the aggregate capital stock $K_t > 0$ be positive in every period.[2]

We next describe the Euler equations that govern the choices of consumption $c_{j,s,t}$ and savings $b_{j,s+1,t+1}$ by household of age $s$ and ability $e_j(s)$ in each period $t$. We work backward from the last period of life $s = S$. Because households do not save in the last period of life $b_{j,s+1,t+1} = 0$ due to our assumption of no bequest motive, the household's final-period maximization problem is given by the following.

$$\max_{c_{j,S,t}} \frac{\left(c_{j,S,t}\right)^{1-\sigma} - 1}{1-\sigma} \quad \text{s.t.} \quad \left(1 + r_t\right) b_{j,S,t} + w_t e_j(S) l(S) \geq c_{j,S,t} \quad \forall t \tag{5}$$

[2]We describe these constraints in detail in Appendix A-2.

Because $u(c)$ is monotonically increasing in $c$, the $s = S$ problem (5) is simply to choose the maximum amount of consumption possible. The household trivially consumes all of its income in the last period of life.

$$c_{j,S,t} = (1 + r_t)\, b_{j,S,t} + w_t e_j(S) l(S) \quad \forall t \tag{6}$$

In general, maximizing (1) with respect to (2), (3), (4), and the implied individual and aggregate borrowing constraints gives the following set of $S-1$ intertemporal Euler equations.

$$\begin{aligned} (c_{j,s,t})^{-\sigma} &= \beta\,(1 + r_{t+1})\,(c_{j,s+1,t+1})^{-\sigma} \\ &\text{for} \quad 1 \leq s \leq S-1, \quad \forall t \end{aligned} \tag{7}$$

Note from (3) that $c_{j,s,t}$ in (7) depends on the household's age $s$, his ability $e_j(s)$, and the initial wealth with which the household entered the period $b_{j,s,t}$.

## 2.2 Firm problem

A unit measure of identical, perfectly competitive firms exist in this economy. The representative firm is characterized by the following Cobb-Douglas production technology,

$$Y_t = A K_t^{\alpha} L_t^{1-\alpha} \quad \forall t \tag{8}$$

where $A$ is the fixed technology process and $\alpha \in (0,1)$ and $L_t$ is measured in efficiency units of labor. The interest rate $r_t$ in the cost function is a net real interest rate because depreciation $\delta$ is paid by the firms. The real wage is $w_t$. The real profit function of the firm is the following.

$$\text{Real Profits} = A K_t^{\alpha} L_t^{1-\alpha} - (r_t + \delta) K_t - w_t L_t \tag{9}$$

As in the budget constraints (2), (3), and (4), note that the price of the good has been normalized to one.

Profit maximization results in the real wage $w_t$ and the real rental rate of capital $r_t$ being determined by the marginal products of labor and capital, respectively.

$$w_t = (1-\alpha)\frac{Y_t}{L_t} \quad \forall t \tag{10}$$

$$r_t = \alpha\frac{Y_t}{K_t} - \delta \quad \forall t \tag{11}$$

### 2.3 Market clearing and equilibrium

Labor market clearing requires that aggregate labor demand $L_t$ measured in efficiency units equal the sum of individual efficiency labor supplied $e_j sl(s)$. The supply side of market clearing in the labor market is trivial because household labor is supplied inelastically. Capital market clearing requires that aggregate capital demand $K_t$ equal the sum of capital investment by households $b_{j,s,t}$. Aggregate consumption $C_t$ is defined as the sum of all individual consumptions, and investment is defined by the standard $Y = C + I$ constraint as shown in (14).

$$L_t = \frac{1}{SJ}\sum_{s=1}^{S}\sum_{j=1}^{J} e_j(s)l(s) \qquad \forall t \tag{12}$$

$$K_t = \frac{1}{SJ}\sum_{s=1}^{S}\sum_{j=1}^{J} b_{j,s,t} \qquad \forall t \tag{13}$$

$$\begin{aligned} Y_t &= C_t + K_{t+1} - (1-\delta)K_t \quad \forall t \\ \text{where} \quad C_t &\equiv \frac{1}{SJ}\sum_{s=1}^{S}\sum_{j=1}^{J} c_{j,s,t} \end{aligned} \tag{14}$$

The steady-state equilibrium for this economy is defined as follows.

---

**Definition 1** (**Steady-state equilibrium**)**.** A non-autarkic steady-state equilibrium in the overlapping generations model with $S$-period lived agents and heterogeneous ability $e_j(s)$ is defined as constant allocations $c_{j,s,t} = \bar{c}_{j,s}$ and $b_{j,s+1,t+1} = \bar{b}_{j,s+1}$ and constant prices $w_t = \bar{w}$ and $r_t = \bar{r}$ for all $j$, $s$, and $t$ such that the following conditions hold:

i. households optimize according to (5), (6) and (7),

ii. firms optimize according to (10) and (11), and

iii. markets clear according to (12), (13), and (14).

---

The steady-state equilibrium is characterized by the system of $J \times (S-1)$ equations and $J \times (S-1)$ unknowns $\bar{b}_{j,s+1}$ along with the individual borrowing constraints and aggregate borrowing constraint described in Appendix A-2.

$$
\begin{split}
&\left(\bar{c}_{j,s}\right)^{-\sigma} = \beta\left(1+\bar{r}\right)\left(\bar{c}_{j,s+1}\right)^{-\sigma}, \quad 1 \leq s \leq S-1 \\
&\qquad\qquad\qquad\qquad \text{or} \\
&\Big([1+\bar{r}]\bar{b}_{j,s} + \bar{w}e_j(s)l(s) - \bar{b}_{j,s+1}\Big)^{-\sigma} = \\
&\qquad\qquad \beta(1+\bar{r})\Big([1+\bar{r}]\bar{b}_{j,s+1} + \bar{w}e_j(s+1)l(s+1) - \bar{b}_{j,s+2}\Big)^{-\sigma}, \\
&\qquad\qquad \text{for} \quad 1 \leq s \leq S-1, \quad \text{where} \quad \bar{b}_{j,1}, \bar{b}_{j,S+1} = 0 \quad \forall j
\end{split}
\tag{15}
$$

In equilibrium, the steady-state real wage $\bar{w}$ and the steady-state real rental rate $\bar{r}$ are simply functions of the steady-state distribution of capital $\bar{b}_{j,s+1}$. This is clear from the steady-state version of the capital market clearing condition (13) and the fact that aggregate labor supply is a function of the sum of exogenous efficiency units of labor in the labor market clearing condition (12). And the two firm first order conditions for the real wage $w_t$ (10) and real rental rate $r_t$ (11) are only functions of the aggregate capital stock $K_t$ and aggregate labor $L_t$. Appendix A-3 details how to solve for the steady-state equilibrium.

# APPENDIX

## A-1 Calibration of ability process

The calibration of the ability process $e_j(s)$ is as follows. First, the ability types themselves must be calibrated. For each age group $s \in S$, the hourly wage rates are sorted into $J$ percentile groups. The ability type for each percentile group is the median wage for the percentile group, divided by the average wage of all individuals in the data set.

The data used to calibrate the ability types were obtained from the Current Population Survey.[3] Individuals younger than 20 and older than 79 are dropped from the data. This is due to the extremely small amount of observations for ages outside of those bounds. The data was truncated to allow for either method. Due to a limited number of observations in the survey who included their hourly wage, data was taken from the months of January, February, March, April, and May 2014. The ability types were then calculated for each month, and then an average was taken of the five calibrations of the ability types in order to produce a final calibration to be used in the model.

The ability types evolve according to a Markov process, which is altered depending on the specifications of the model. An individual in period $s$ and of ability type $j$ faces a distribution which will determine in which ability type they fall in the period $s+1$. In this paper, individuals are assigned ability types at the beginning of their life, and cannot change types later on. This Markov process is simply an identity matrix for each age group.

However, the model will readily accept a more complex Markov process. The data used to create the Markov process comes from the Panel Study of Income Dynamics (PSID) for 1999 and 2001. [4] Again, since $S = 60$, only individuals with ages 20 through 79 are included. Wages in 1999 are multiplied by 1.06303 to convert them to 2001 wages. Individuals without wage information in both years are dropped. Due to the low number of observations per age group, one Markov process is generated which is applied to all age groups in the model. Nishiyama (2003) noted that this Markov transition matrix was reasonably consistent across age cohorts, and so using the same process for all cohorts should not present a problem in the model.

To generate the transition matrix, the 1999 and 2001 wages are sorted into $J$ percentile groups. For each percentile group in 2001, the number of individuals that came from each of the percentile groups in 1999 are counted. This generates $J**2$ summations of individuals, $J$ for each percentile group. Then, the $J$ summations for

[3] U.S. Census Bureau, Dataferret, Current Population Survey, 2014. The variables $PRTAGE$ and $PTERNHLY$ were used for the age and hourly wage rate of individuals, respectively.

[4] Panel Study of Income Dynamics, public use dataset. Produced and distributed by the Survey Research Center, Institute for Social Research, University of Michigan, Ann Arbor, MI (2014). The variables for age and wage in 1999 and age and wage in 2001 are $ER33504$, $ER33537O$, $ER33604$, and $ER33628O$, respectively.

the first percentile group in 2001 are divided by the count of individuals in the first percentile group in 1999. Next, the $J$ summations for the second percentile group in 2001 are divided by the count of individuals in the second percentile group in 1999. This continues through the $J^{th}$ percentile group. We then have a $J \times J$ matrix of probabilities, where the first row is the probability of being in the first percentile group, given that one is in the $j^{th}$ percentile group before (where $j$ indicates the column of the matrix), and so on.

Because this Markov transitional matrix represents the probabilities of changing abilities types after two years, we must take the "square root" of the matrix. This process is described in the Python code for the model, and involves diagonalizing the matrix and taking the square root of the diagonalized matrix. Finally, the Markov matrix is then raised to the $60/S$ power, denoting the number of years that separate each age cohort.

## A-2 Constraints on individual borrowing

As described in Section 2.1, individuals are allowed to borrow $b_{j,s,t}$ for some $j$ and $s$ in period $t$. However, two constraints must hold. First, the individual must be able to pay back the balance with interest $r_{t+1}$ in the next period without driving consumption in the next period $c_{j,s+1,t+1}$ to be nonpositive. Let $\bar{b}_{j,s,t}$ be the minimum value of savings in a period.

$$b_{j,s,t} \geq \bar{b}_{j,s,t} \quad \forall j,s,t \tag{A.2.1}$$

Rearranging the bugdet constraints in (2), (3), and (4) and using backward induction gives the following expressions for $\bar{b}_{j,s,t}$,

$$\begin{aligned}
\bar{b}_{j,S,t} &= \frac{\varepsilon - w_t e_j(S) l(S)}{1+r_t} \\
\bar{b}_{j,S-1,t-1} &= \frac{\varepsilon + \bar{b}_{j,S,t} - w_{t-1} e_j(S-1) l(S-1)}{1+r_{t-1}} \\
&\vdots \\
\bar{b}_{j,2,t-S+2} &= \frac{\varepsilon + \bar{b}_{j,3,t-S+3} - w_{t-S+2} e_j(2) l(2)}{1+r_{t-S+2}}
\end{aligned} \tag{A.2.2}$$

In addition to the individual borrowing constraint (A.2.1), a strict aggregate borrowing constraint must be met. That is, the aggregate capital stock must be strictly positive.

$$K_t > 0 \quad \forall t \tag{A.2.3}$$

## A-3 Solving for steady-state equilibrium

This section describes the solution method for the steady-state equilibrium described in Definition 1.

i. Choose an initial guess for the steady-state distribution of capital $\bar{b}_{j,s+1}$ for all $j$ and $s = 1, 2, ...S-1$.

   - choose some small positive number that is strictly less that is small enough to be less than the minimum income that an individual might have $\bar{w}e_j(s)l(s)$.

ii. Perform an unconstrained root finder that chooses $\bar{b}_{j,s+1}$ that solves the $J \times (S-1)$ steady-state Euler equations (15).

iii. Make sure none of the implied steady-state consumptions $\bar{c}_{j,s,t}$ is less-than-or-equal-to zero.

   - If one consumption is less-than-or-equal-to zero $\bar{c}_{j,s} \leq 0$, then try different starting values.
   - If that does not work, then we must perform the root finding operation as a constrained minimization problem that puts a maximum value on $\bar{b}_{j,s+1}$.

iv. Make sure that none of the Euler errors is too large in absolute value. A steady-state Euler error is the following, which is supposed to be close to zero for all $j$ and $s = 1, 2, ...S-1$:

$$\frac{\beta\left(1+\bar{r}\right)\left(\bar{c}_{j,s+1}\right)^{-\sigma}}{\left(\bar{c}_{j,s}\right)^{-\sigma}} - 1 \tag{A.3.1}$$

v. Make sure that the unconstrained solution satisfies the individual borrowing constraints in (A.2.1) and (A.2.2).

   - If any individual's borrowing constraint is not satisfied using the unconstrained root finding operation, rerun the root finding operation in step (ii) as a constrained minimization problem with the borrowing constraints imposed on those individuals.
   - Repeat steps (ii) through (v) until all the individual borrowing constraints are met.

vi. Make sure that the solution satisfies the aggregate borrowing constraint (A.2.3).

   - If it does not, what is the least distortionary upward adjustment to individual steady-state savings $\bar{b}_{j,s+1}$?

# References

**Nishiyama, S.**, “Analyzing Tax Policy Changes Using a Stochastic OLG Model with Heterogeneous Households,” Technical Paper Series 2003-12, Congressional Budget Office December 2003.

# TECHNICAL APPENDIX

## T-1 Structures to add to the model and order

i. Endogenize labor

ii. Make sure bond holdings are correct

iii. Add demographics

iv. Add household tax structures

v. Add firm structures