OSPC's Dynamic General Equilibrium Tax Scoring Model [1]

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Abstract

This document details the large scale, overlapping-generations model developed by the Open Source Policy Center (OSPC). The model allows for dynamic scoring of federal tax policy. In particular, the model specifies the fundamental parameters defining the preferences and technologies of heterogenous individuals and firms and links them together in a dynamic, general equilibrium framework. This framework allows for detailed evaluation of tax policy, including revenue, distributional, and macroeconomic impacts. The model is open source, meaning that all documentation and files needed to reproduce and execute the model are available freely. This documents and other supporting files are available at https://github.com/OpenSourcePolicyCenter/dynamic. We encourage other interested parties to use and contribute to the model.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

This document details the large scale, overlapping-generations model developed by the Open Source Policy Center (OSPC). The model allows for dynamic scoring of federal tax policy. In particular, the model specifies the fundamental parameters defining the preferences and technologies of heterogenous individuals and firms and links them together in a dynamic, general equilibrium framework. This framework allows for detailed evaluation of tax policy, including revenue, distributional, and macroeconomic impacts.

The household sector consists of individuals of seven lifetime income groups, each of which has a different life-cycle earnings profiles. This allow us to consider the lifetime incidence of taxation on households. These individuals are intertemporal optimizers who allocate income between investment in financial assets and the consumption of 17 private consumption goods. The consumption goods are produced by 48 different production sectors, which include 24 production industries with corporate and non-corporate firms in each. In this way we can see the distributional impacts of consumption taxes and capital taxes levied on business entities as the taxes pass through to the individuals of different ages and income levels through changes in relative prices. Finally, we specify a government sector that derives revenue from taxes and government enterprise and uses those revenues to subsidize government produced private and public goods and fund transfers. The government is not bound by a balanced budget any particular period, but we do impose sustainable fiscal policy in the long run through a government reaction function that adjusts government purchases to maintain a specified debt-to-GDP ratio in the steady-state.

Our model is a general equilibrium model, meaning the taxes in one area of the economy result in effects on other sectors through changes in relative prices. For example, the simulation of a policy that slows the rate of depreciation allowed under tax law would increase the cost of capital in capital intensive industries to a greater extent than it would in other industries. This would have the effect of pushing up prices for goods produced from capital intensive industries and in turn move the economy back along the demand curve for those goods. This happens as individuals substitute towards other goods that are relatively cheaper. Thus demand for those goods produced from less capital intensive production increase. Capturing general equilibrium feedback effects such as these can be very important for the evaluation of the distributional, revenue, and macroeconomic impacts of policies and is why dynamic scoring is important.

Our model is intended to provide year-by-year revenue estimates for the budget window. To do this, we solve for not only the model's steady-state equilibrium, but also the entire transition path from the current state to the steady-state. It's in this way that we are able to see the revenue and macroeconomic impacts over the budget window.

The remainder of this document provides a detailed description of the model. We start by specifying households and then outline the firm's problem. We next turn to the specification of the government. Finally we define the equilibrium concept used to close the model and the numerical solution methods used to solve for this equilibrium.

A future extension to this document will detail how the model is calibrated.

Chapter 2

Households

2.1 Demographics

A measure $\omega_{1,t}$ of individuals with heterogeneous working ability $e \in \mathcal{E} \subset \mathbb{R}_{++}$ is born in each period t and live for $E+S$ periods, with $S \geq 4$.[1] The population of age-s individuals in any period t is $\omega_{s,t}$. Households are termed "youth", and do not participate in market activity, during ages $1 \leq s \leq E$. The households enter the workforce and economy in period $E+1$ and remain in the workforce until they unexpectedly die or live until age $s = E+S$.[2] The population of agents of each age in each period, $\omega_{s,t}$, evolves according to the following function,

$$\begin{aligned} \omega_{1,t+1} &= \sum_{s=1}^{E+S} f_s \omega_{s,t} \quad \forall t \\ \omega_{s+1,t+1} &= (1 + i_s - \rho_s)\omega_{s,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{2.1}$$

where $f_s \geq 0$ is an age-specific fertility rate, i_s is an age-specific immigration rate, ρ_s is an age specific mortality hazard rate,[3] and $1 + i_s - \rho_s$ is constrained to be nonnegative. The total population in the economy N_t at any period is simply the sum of individuals in the economy, the population growth rate in any period t from the previous period $t-1$ is $g_{n,t}$, $\tilde{N}_t$ is the working age population, and $\tilde{g}_{n,t}$ is the working age population growth rate in any period t from the previous period $t-1$.

$$N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{2.2}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{2.3}$$

[1]Theoretically, the model exposition of the model works without loss of generality for $S \geq 3$. However, because we are calibrating the ages outside of the economy to be one-fourth of S (e.g., ages 21 to 100 in the economy, and ages 1 to 20 outside of the economy), we need S to be at least 4.

[2]We model the population with households age $s \leq E$ outside of the workforce and economy in order most closely match the empirical population dynamics.

[3]The parameter ρ_s is the probability that a household of age s dies before age $s+1$.

$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{2.4}$$

$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{2.5}$$

2.2 Households

Consumer's are forward-looking, intertemporal optimizers. Their objective is the maximize the expected, discounted value of lifetime utility. Expectations are taken over mortality risk, the only source of uncertainty in the model. Individuals are heterogenous with repeat to age and lifetime income group. We define the expected, discounted lifetime utility at time t for an individual in lifetime income group j and age s to be $U_{j,s,t}$. We assume that utility is additively separable across periods and thus write expected, discounted lifetime utility as:

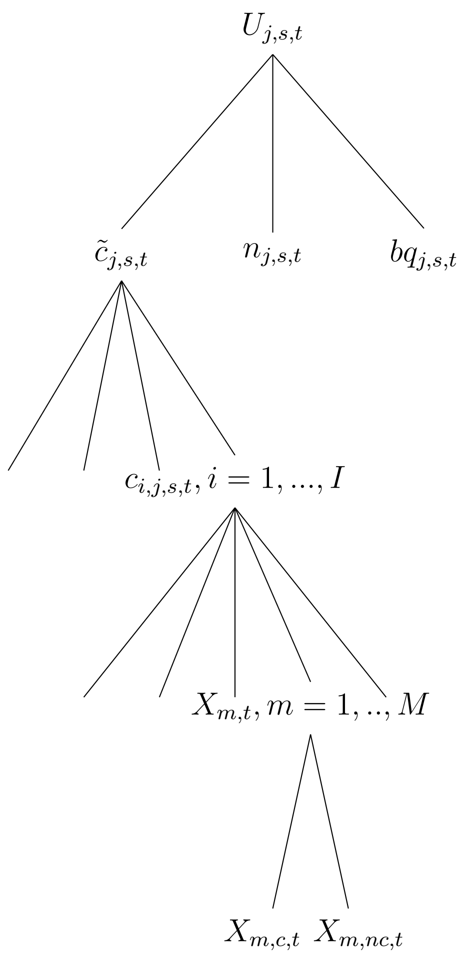
$$\begin{aligned}
U_{j,s,t} &= \sum_{u=0}^{E+S-s} \beta^u \left[\prod_{v=s-1}^{s+u-1} (1-\rho_v)\right] u\left(c_{j,s+u,t+u}, n_{j,s+u,t+u}, b_{j,s+u+1,t+u+1}\right) \\
\text{where} &\quad \rho_{s-1} = 0 \\
\text{and} &\quad u\left(c_{j,s,t}, n_{j,s,t}, b_{j,s+1,t+1}\right) = \frac{\left(c_{j,s,t}\right)^{1-\sigma} - 1}{1-\sigma} \ldots \\
&\quad + e^{g_y t(1-\sigma)} \chi^n_s \left(b\left[1 - \left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} + k\right) + \rho_s \chi^b \frac{\left(b_{j,s+1,t+1}\right)^{1-\sigma} - 1}{1-\sigma} \\
&\quad \forall j, t \quad \text{and } E+1 \leq s \leq E+S
\end{aligned} \tag{2.6}$$

The parameter $\beta \in (0,1)$ represents the individual's rate of time preference. The quantities $c_{j,s,t}$, $n_{j,s,t}$, and $b_{j,s,t}$ are total consumption of a composite consumption good, labor supply, and asset holdings, respectively. The parameter $\sigma \geq 1$ is the coefficient of relative risk aversion, υ is a measure of the elasticity of labor supply, and $\tilde{l}$ is the total time endowment of the individual. The utility weight for the disutility of labor is given by the age-dependent parameters χ^n_s. The parameter g_y is a constant growth rate of labor augmenting technological progress, which we explain in more detail in the firm's problem .[4] The disutility of labor term in the utility function looks nonstandard, but is simply the upper quadrant of an ellipse that closely approximates the standard constant relative risk aversion utility of leisure functional form.[5] The utility weight on bequests (both intentional and accidental) is given by χ^b.

[4]The term with the growth rate $e^{g_y t(1-\sigma)}$ must be included in the period utility function because consumption and bequests will be growing at rate g_y and this term stationarizes the individual Euler equation by making the marginal disutility of labor grow at the same rate as the marginal benefits of consumption and bequests. This is the same balanced growth technique as that used in **?**).

[5]Appendix **??** describes how the elliptical function closely matches the more standard utility

Figure 2.1: Summary of the Individual Problem



Utility, $U_{j,s,t}$, is a CRRA function of consumption, leisure, and bequests

Composite consumption, $\tilde{c}_{j,s,t}$, is a Stone-Geary function of distinct consumption goods

Consumption goods, $c_{i,j,s,t}$, are determined by a fixed coefficient mix of production goods

Production goods for each industry, $X_{m,t}$, are determined by a CES function of production goods from each sector

Households choose consumption of a composite consumption good, $c_{j,s+u,t+u}$, labor supply, $n_{j,s+u,t+u}$, and asset holdings, $b_{j,s+u+1,t+u+1}$, to maximize the expected, discounted, lifetime utility subject to their per-period budget constraint. Total consumption of the composite good is made up of discretionary consumption, $\tilde{c}_{j,s,t}$, and minimum required purchases of each consumption good, $\bar{c}_{i,s}$. Thus the consumer's choice is over $\tilde{c}_{j,s,t}$, which together with the minimum required purchases equal determine total composite consumption: $c_{j,s,t} = \tilde{c}_{j,s,t} + \sum_{i=1}^{I} c_{i,s}$. It is therefore the case that there minimum required purchases affect the household's ability to smooth consumption over time. We discuss the composite consumption good in more detail in Section **??**. This composite good is age dependent, thus the price of the composite consumption good varies with age s. We denote the gross-of-tax price of the composite consumption good for households of age s in period t as $\tilde{p}_{s,t}$ and the gross-of-tax

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of leisure of the form $\frac{(\tilde{l}-n_{j,s,t})^{1-\eta}-1}{1-\eta}$. The parameters b and k are the scale and shift parameters of describing the elliptical form. This elliptical utility function forces an interior solution that automatically respects both the upper and lower bound of labor supply, which greatly simplifies the computation of equilibrium. For a more in-depth discussion see **?**)

price for good i at time t as $p_{i,t}$. The households' per period budget constraint is:

$$\sum_{i=1}^{I} p_{i,t}\bar{c}_{i,s} + \tilde{p}_{s,t}\tilde{c}_{s,t} + b_{j,s+1,t+1} \leq (1+r_t)\, b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \frac{BQ_{j,t}}{\lambda_j \tilde{N}_t} - T_{j,s,t}$$
$$\text{where} \quad b_{j,s,1} = 0$$
$$\text{for} \quad E+1 \leq s \leq E+S \quad \forall j,t \tag{2.7}$$

Here, r_t and w_t are the real interest rate and the wages rate on a unit of effective labor. The variable $e_{j,s}$ denotes the effective labor units of an individual from lifetime income group s and age j. An individual's labor income is thus determined by her choice of $n_{j,s,t}$ units of labor times her measure of effective labor units, $e_{j,s}$, times the wage per unit of effect labor. An individuals effective labor units vary over the life-cycle, as the age subscript implies. $BQ_{j,t}$ denote aggregate bequests left from those in lifetime income group j at time t. We divide this number by the number of individuals in lifetime income group j at time t, given by $\lambda_j \tilde{N}_t$, to determine the amount of bequests received by each household in lifetime income group j.[6] The last term in the budget constraint, $T_{j,s,t}$ are total taxes paid by the individual. These include all non-consumption taxes and are based on tax functions for separate tax sources that we estimate based on a microsimulation model. We discuss the parameterization and calibration of these functions below.

The Lagrangian for the individual's problem can be written as:

$$\begin{aligned}\mathcal{L} = \max_{\left\{\substack{\tilde{c}_{j,s+u,t+u},\\ n_{j,s+u,t+u},\\ b_{j,s+u,t+u}}\right\}_{u=0}^{E+S-s}} & \sum_{u=0}^{E+S-s} \beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right] \frac{(c_{j,s+u,t+u})^{1-\sigma}-1}{1-\sigma} + \ldots \\ & e^{g_y t(1-\sigma)}\chi^n_s + u\left(b\left[1-\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)^v\right]^{\frac{1}{v}} + k\right) + \rho_s\chi^b \frac{(b_{j,s+u+1,t+u+1})^{1-\sigma}-1}{1-\sigma} + \ldots \\ & \lambda_{j,s+u,t+u}\Bigg\{(1+r_{t+u})b_{j,s+u,t+u} + w_{t+u}e_{j,s+u}n_{j,s+u,t+u} + \frac{BQ_{j,t+u}}{\lambda_j\tilde{N}_{t+u}} - T_{j,s+u,t+u} - \ldots. \\ & \sum_{i=1}^{I} p_{i,t+u}\bar{c}_{i,s+u} - \tilde{p}_{s+u,t+u}\tilde{c}_{j,s+u,t+u} - b_{j,s+u+1,t+u+1}\Bigg\}\end{aligned} \tag{2.8}$$

taking derivatives with respect to $\{\tilde{c}_{j,s,t}, n_{j,s,t+u}, b_{j,s,t+1}\}$ gives us the necessary conditions for each j, s and t. The necessary condition with respect to the discretionary consumption of the composite consumption good, $\tilde{c}_{j,s+u,t+u}$, labor supply, $n_{j,s+u,t+u}$, and asset holdings, $b_{j,s+u+1,t+u+1}$, are:

[6] This distribution of bequests is just place holder. The goal is to find suitable data to calibrate the process describing the transmission of bequests between individuals of different ages and lifetime income groups.

$$\frac{\partial U}{\partial \tilde{c}_{j,s+u,t+u}} = \beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right] c_{j,s+u,t+u}^{-\sigma} - \beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right] \lambda_{j,s+u,t+u}\tilde{p}_{s+u,t+u} = 0, \forall u \tag{2.9}$$

$$\begin{aligned}\frac{\partial U}{\partial n_{j,s+u,t+u}} = &\beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right] e^{g_y(t+u)(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}} \\ &- \beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right] \lambda_{j,s+u,t+u}\left(w_{t+u}e_{j,s+u} - \frac{\partial T_{j,s+u,t+u}}{\partial n_{j,s+u,t+u}}\right) = 0, \forall u\end{aligned} \tag{2.10}$$

$$\begin{aligned}\frac{\partial U}{\partial b_{j,s+u+1,t+u+1}} = &\beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right] \rho_s\chi^b\big(b_{j,s+u+1,t+u+1}\big)^{-\sigma} - \beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right] \lambda_{j,s+ut+u} \\ &+ \beta^{u+1}\left[\prod_{v=s-1}^{s+u}(1-\rho_v)\right]\lambda_{j,s+u+1,t+u+1}\left(1+r_{t+u+1} - \frac{\partial T_{j,s+u+1,t+u+1}}{\partial b_{j,s+u+1,t+u+1}}\right) = 0, \forall u\end{aligned} \tag{2.11}$$

Note that the term $\frac{\partial T_{j,s+u+1,t+u+1}}{\partial n_{j,s+U+1,t+u+1}}$ give the change in total taxes for additional labor supply $\frac{\partial T_{j,s+u+1,t+u+1}}{\partial b_{j,s+U+1,t+u+1}}$ gives the change in total taxes for additional savings. The tax functions that define the total taxes paid will take into account the interactions, for example how increasing capital income by saving more impacts the marginal tax rate on labor income in a system that progressively taxes labor income. Rearranging the equations above to solve each for λ_{t+u}, we get the following:

$$\lambda_{j,s+u,t+u} = \frac{c_{j,s+u,t+u}^{-\sigma}}{\tilde{p}_{s,t+u}}$$

$$\lambda_{j,s+u,t+u} = \frac{e^{g_y(t+u)(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}}{w_{+u}e_{j,s+u} - \frac{\partial T_{j,s+u,t+u}}{\partial n_{j,s+u,t+u}}}$$

$$\lambda_{j,s+u,t+u} = \rho_s\chi^b\big(b_{j,s+u+1,t+u+1}\big)^{-\sigma} - \beta(1-\rho_{s+u})\lambda_{t+u+1}\left(1+r_{t+u+1} - \frac{\partial T_{j,s+u+1,t+u+1}}{\partial b_{j,s+u+1,t+u+1}}\right)$$

These three equations can then be reduced to just two equations that must hold for all j, s, and t. The first relates the marginal utility of consumption of the composite good to the marginal utility of labor:

$$\frac{c_{j,s+u,t+u}^{-\sigma}}{\tilde{p}_{s+u,t+u}} = \frac{e^{g_y(t+u)(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}}{w_{t+u}e_{j,s+u}-\frac{\partial T_{j,s+u,t+u}}{\partial n_{j,s+u,t+u}}} \tag{2.12}$$

The second equation is the intertemporal Euler equation for savings, including the utility effects of bequests:

$$\frac{c_{j,s+u,t+u}^{-\sigma}}{\tilde{p}_{s+u,t+u}} = \rho_s\chi^b\left(b_{j,s+U+1,t+U+1}\right)^{-\sigma} + \frac{\beta(1-\rho_{s+u})c_{j,s+u+1,t+u+1}^{-\sigma}}{\tilde{p}_{s+u+1,t+u+1}} \times \left(1+r_{t+u+1}-\frac{\partial T_{j,s+u+1,t+U+1}}{\partial b_{j,s+u+1,t+u+1}}\right) \tag{2.13}$$

2.2.1 Household's Portfolio Problem

Household's are assumed to have constant elasticity of substitution (CES) preferences over debt and equity in their portfolio. These preferences allows for investor portfolios that are a mix of debt and equity despsite the two assets having differential returns. Thus, while our model does not include idiosyncratic or aggregate uncertainty, these CES preferences account for the premium paid to the more risky assets through the preference parameters. To match lifecycle portfolio changes, we may consider allowing to the CES preference parameters to vary by age. Given the CES preferences, the household's total assets, $a_{j,s,t}$ are given by (NOTE that if we include these notations we should change our notation for assets from b to a. We need to also think about the notion for equity - I use e below, but we are already using that for effective labor units.):

$$a_{j,s,t} = \left[\gamma_{a,s}^{\frac{-1}{\varepsilon_{a,s}}} b_{j,s,t}^{\frac{1+\varepsilon_{a.s}}{\varepsilon_{a,s}}} + (1-\gamma_{a,s})^{\frac{-1}{\varepsilon_{a,s}}} e_{j,s,t}^{\frac{1+\varepsilon_{a,s}}{\varepsilon_{a,s}}}\right]^{\frac{\varepsilon_{a,s}}{1+\varepsilon_{a,s}}} \tag{2.14}$$

The parameter $\varepsilon_{a,s}$ is the elasticity of substitution between bonds and stocks in the asset portfolio of an age s household. $\gamma_{a,s}$ is the taste parameter in these preferences (I've written this in the most flexible way, where both parameters depend upon age, but perhaps we just want the taste parameter to vary by age the rate of substitution is constant.).

Since neither firms of governments default in our model, the rate of return on bonds issued by firms and government will all have the same pretax rate of return, $r_{b,t}$.

It's not clear whether firm equity will all have the same return. It seems that multinationals, with the ability to shift profits, may have higher returns than domestic corporations. In general, firms will earn economic profits and thus have a rate of return in excess of the risk free return, $r_{b,t}$. If it is the case that some firms earn greater returns than others, I think we may just assume that households own an diversified equity portfolio of all firms and thus earn the average return on equity. Thus we'll have either a market return for equity that is the average of the heterogeneous returns across firms or the return that is the same for all firms. We will denote the pretax market turn on equity as $r_{e,t}$.

We introduce further notation that is the after-tax gross return on the each asset composite. The after-tax gross return on bonds for a household of age s, lifetime income group j, and in year t is given by:

$$\rho_{b,j,s,t} = (1 + r_{b,t}(1 - \tau^{int}(y_{j,s,t}))) \tag{2.15}$$

Note that I've again shifted some notation. We had used a to denote total income for tax purposes, while I use y above (since I've used a for total assets). Also, I'm writing the marginal tax rate on interest income as a function of total income, $\tau^{int}(y_{j,s,t})$, rather than using the total tax function. I think this is helpful for expository purposes. Is it ok to use the marginal rather than the average tax function? Seems like we could use either interchangeably. But if we want to use the total tax function, the notation for the above would be:

$$\rho_{b,j,s,t} = (1 + r_{b,t}) - \frac{\partial T_{j,s,t}}{\partial b_{j,s,t}} \tag{2.16}$$

Either way, we'll want to be sure that when we calibrate this functions we account for the mix of tax exempt and taxable interests realized by household and how that changes across age and income group. The micro simulation model should be able to help us here.

The after-tax gross return on equity for a household of age s, lifetime income group j, and in year t is given by:

$$\rho_{e,j,s,t} = (1 + r_{e,t}(1 - \tau^{cap}(y_{j,s,t}))) \tag{2.17}$$

Here, τ^{cap} will be the tax rate on capital income - some mix of the tax on dividends and capital gains from corporate and non-corporate entities. Recall that we do not explicitly model capital gains realizations nor do we track dividend issues from each representative firm. Instead, the return to the firms (both from dividends and capital gains) is put into the per period return on equity, $r_{e,t}$. Thus the tax rate on these returns must be a weighted average of the taxes on dividends and capital gains, where the weighting is given by data on the share of income from capital gains and dividends by age and income group in the data.

We can now write the gross after-tax return on the households total portfolio:

$$\rho_{a,j,s,t} a_{j,s,t} = \rho_{b,j,s,t} b_{j,s,t} + \rho_{e,j,s,t} e_{j,s,t} \tag{2.18}$$

The optimal portfolio is given by the household choosing bonds and stocks to maximize Equation **??** subject to Equation **??**. The Lagrangian for this problem is:

$$\mathcal{L} = \max_{b_{j,s,t}, e_{j,s,t}} \rho_{b,j,s,t} b_{j,s,t} + \rho_{e,j,s,t} e_{j,s,t} + \lambda_{j,s,t} \left(a_{j,s,t} - \left[\gamma_{a,s}^{\frac{-1}{\varepsilon_{a,s}}} b_{j,s,t}^{\frac{1+\varepsilon_{a.s}}{\varepsilon_{a,s}}} + (1-\gamma_{a,s})^{\frac{-1}{\varepsilon_{a,s}}} e_{j,s,t}^{\frac{1+\varepsilon_{a,s}}{\varepsilon_{a,s}}} \right]^{\frac{\varepsilon_{a,s}}{1+\varepsilon_{a,s}}} \right) \tag{2.19}$$

Since every variable is subscripted by j, s, t we drop these and write the necessary conditions as:

$$\frac{\partial \mathcal{L}}{\partial b} : \rho_b = \lambda \gamma_{a,s}^{\frac{-1}{\varepsilon_{a,s}}} b^{\frac{1}{\varepsilon_{a,s}}} \left[\gamma_{a,s}^{\frac{-1}{\varepsilon_{a,s}}} b^{\frac{1+\varepsilon_{a.s}}{\varepsilon_{a,s}}} + (1-\gamma_{a,s})^{\frac{-1}{\varepsilon_{a,s}}} e^{\frac{1+\varepsilon_{a,s}}{\varepsilon_{a,s}}} \right]^{\frac{-1}{1+\varepsilon_{a,s}}} \tag{2.20}$$

$$\frac{\partial \mathcal{L}}{\partial e} : \rho_e = \lambda (1-\gamma_{a,s})^{\frac{-1}{\varepsilon_{a,s}}} e^{\frac{1}{\varepsilon_{a,s}}} \left[\gamma_{a,s}^{\frac{-1}{\varepsilon_{a,s}}} b^{\frac{1+\varepsilon_{a.s}}{\varepsilon_{a,s}}} + (1-\gamma_{a,s})^{\frac{-1}{\varepsilon_{a,s}}} e^{\frac{1+\varepsilon_{a,s}}{\varepsilon_{a,s}}} \right]^{\frac{-1}{1+\varepsilon_{a,s}}} \tag{2.21}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} : a = \left[\gamma_{a,s}^{\frac{-1}{\varepsilon_{a,s}}} b^{\frac{1+\varepsilon_{a.s}}{\varepsilon_{a,s}}} + (1-\gamma_{a,s})^{\frac{-1}{\varepsilon_{a,s}}} e^{\frac{1+\varepsilon_{a,s}}{\varepsilon_{a,s}}} \right]^{\frac{\varepsilon_{a,s}}{1+\varepsilon_{a,s}}} \tag{2.22}$$

We can use Equations **??** and **??** to find the ratio of bonds to stocks as (again, suppressing the subscripts):

$$\frac{b}{e} = \left(\frac{\rho_b}{\rho_e} \right)^{\varepsilon_{a,s}} \frac{\gamma_{a,s}}{(1-\gamma_{a,s})} \tag{2.23}$$

Using Equation **??** can then be used with **??** to find demand for bonds and stocks separately. The solution should yield (We should work this out here, I had some trouble, but we know what the solution should be...):

$$b_{j,s,t} = \left(\frac{\rho_{b,j,s,t}}{\rho_{e,j,s,t}} \right)^{\varepsilon_{a,s}} \gamma_{a,s} a_{j,s,t} \tag{2.24}$$

$$e_{j,s,t} = \left(\frac{\rho_{e,j,s,t}}{\rho_{b,j,s,t}} \right)^{\varepsilon_{a,s}} (1-\gamma_{a,s}) a_{j,s,t} \tag{2.25}$$

We can thus find the return to the portfolio as:

$$\rho_{a,j,s,t} = \left(\gamma_{a,s} \rho_{b,j,s,t}^{\varepsilon_{a,s}} + (1-\gamma_{a,s}) \rho_{e,j,s,t}^{\varepsilon_{a,s}} \right)^{\frac{1}{1+\varepsilon_{a,s}}} \tag{2.26}$$

After-tax return differentials

I don't think we have any problem in that household have different after tax returns. Each will hold a different portfolio because of this, but none will have corner solutions because of the CES preferences. So it's not problem if, for example, the after-tax return to stocks exceed the return to bonds. This just means households hold relatively more stocks.

When considering the firms' problems we do need to think about different household having different after-tax returns to equity. The solution there will be that the firms just chooses a representative household when thinking about maximizing firm value. The household will be termed the "marginal investor". However, we will need to think a bit about which household in the microsimulation model represents this investor.

2.2.2 Household's Subutility Function

Household preferences over the composite consumption good are modeled as a Stone-Geary function. The aggregate discretionary consumption of the composite good is defined as follows.

$$\tilde{c}_{j,s,t} = \prod_{i=1}^{I} (c_{i,j,s,t} - \bar{c}_{i,s})^{\alpha_{i,s}} \tag{2.27}$$

Where, $c_{i,j,s,t}$ is consumption of good i by household of type j, age s, at time t. There are I total goods and $\bar{c}_{i,s}$ represents the minimum consumption amount for each good at each age. The parameters $\alpha_{i,s}$ are the share parameters (and $\sum_{i=1}^{I} \alpha_{i,s} = 1$). They correspond to the share of income, after minimum expenditure amounts, that are spent on each good at each age. Allowing the minimum consumption amounts and the share parameters to vary by age helps to incorporate life-cycle profiles of consumption into the model. For example, we do not explicitly model household formation decisions, but they will be some of the effects of changes in household composition over the life-cycle are obtained through the parameters of the Stone-Geary function. For example, the minimum required expenditure on shelter may be higher in the middle of the life-cycle when household size is larger. The minimum consumption amounts also mean that the composition of consumption will vary with income, even though all households have the same utility function.

The consumer chooses $c_{i,j,s,t}$ to maximize Equation **??** subject to the budget constraint:

$$\sum_{i=1}^{I} p_{i,t}(c_{i,j,s,t} - \bar{c}_{i,s}) = \tilde{p}_{s,t}\tilde{c}_{j,s,t} \tag{2.28}$$

where $p_{i,t}$ is the gross of tax price of good i at time t and $\tilde{p}_{s,t}$ is the gross of tax price of the the discretionary component of the composite consumption good consumed by those of age s at time t. Maximization of **??** subject to **??** yields:

$$\mathcal{L} = \max_{\{c_{i,j,s,t}\}_{i=1}^{I}} \prod_{i=1}^{I} (c_{i,j,s,t} - \bar{c}_{i,s})^{\alpha_{i,s}} + \lambda \left(\tilde{p}_{s,t}\tilde{c}_{j,s,t} - \sum_{i=1}^{I} p_{i,t}(c_{i,j,s,t} - \bar{c}_{i,j,s,t}) \right) \tag{2.29}$$

Which as I FOCs (for each j, s, t):

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{i,j,s,t}} &= \frac{\alpha_{i,s}\prod_{i=1}^{I}\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_{i,s}}}{\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)} - \lambda p_{i,t} = 0, \forall\ i \\
\implies & \frac{\alpha_{i,s}\prod_{i=1}^{I}\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_{i,s}}}{\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)} = \lambda p_{i,t}, \forall\ i \\
\implies & \frac{\alpha_{i,s}\prod_{i=1}^{I}\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_{i,s}}}{p_{i,t}\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)} = \lambda, \forall\ i \\
\implies & \frac{\alpha_{i,s}}{p_{i,t}\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)} = \frac{\alpha_{j,s}}{p_{k,t}\left(c_{k,j,s,t}-\bar{c}_{k,s}\right)}, \forall\ i,k \\
\implies & c_{i,j,s,t} = \frac{\alpha_{i,s}p_{k,t}\left(c_{k,j,s,t}-\bar{c}_{k,s}\right)}{\alpha_{k,s}p_{i,t}} + \bar{c}_{i,s}\forall i,k
\end{aligned}
\tag{2.30}
$$

Now substitute the last line of **??** into the budget constraint (Equation **??**):

$$
\begin{aligned}
\tilde{p}_{s,t}\tilde{c}_{j,s,t} &= \sum_{i=1}^{I} p_{i,t}\left(c_{i,j,s,t}-\bar{c}_{i,s}\right) \\
\implies \tilde{p}_{s,t}\tilde{c}_{j,s,t} &= \sum_{i=1}^{I} p_{i,t}\left[\frac{\alpha_{i,s}p_{k,t}\left(c_{k,j,s,t}-\bar{c}_{k,s}\right)}{\alpha_{k,s}p_{i,t}} + \bar{c}_{i,s} - \bar{c}_{i,s}\right] \\
\implies \tilde{p}_{s,t}\tilde{c}_{j,s,t} &= \sum_{i=1}^{I} \left[\frac{\alpha_{i,s}p_{k,t}\left(c_{k,s}-\bar{c}_{k,s}\right)}{\alpha_{k,s}}\right] \\
\implies \tilde{p}_{s,t}\tilde{c}_{j,s,t} &= \frac{p_{k,t}\left(c_{k,j,s,t}-\bar{c}_{k,s}\right)}{\alpha_{k,s}} \underbrace{\sum_{i=1}^{I}\alpha_{i,s}}_{=1} \\
\implies \tilde{p}_{s,t}\tilde{c}_{j,s,t} &= \frac{p_{k,t}\left(c_{k,j,s,t}-\bar{c}_{k,s}\right)}{\alpha_{k,s}} \\
\implies \frac{p_{k,t}\left(c_{k,j,s,t}-\bar{c}_{k,s}\right)}{\alpha_{k,s}} &= \tilde{p}_{s,t}\tilde{c}_{j,s,t} \\
\implies c_{k,j,s,t} &= \frac{\alpha_{k,s}\tilde{p}_{s,t}\tilde{c}_{j,s,t}}{p_{k,t}} + \bar{c}_{k,s}, \forall\ k
\end{aligned}
\tag{2.31}
$$

Thus, total consumption of each good i, $c_{i,j,s,t}$, is given by the the amount of minimum consumption plus the share of total expenditures remaining after making the minimum expenditures on all goods (this is called the "supernumerary" expenditure). We derive the prices of the age s composite consumption good in period t, $\tilde{p}_{s,t}$ by using the demand for good i provided in Equation **??** in the function defining aggregate discretionary consumption, Equation **??**:

$$
\begin{aligned}
\tilde{c}_{j,s,t} &= \prod_{i=1}^{I} (c_{i,j,s,t} - \bar{c}_{i,s})^{\alpha_{i,s}} \\
\implies \tilde{c}_{j,s,t} &= \prod_{i=1}^{I} \left(\frac{\alpha_{i,s}\tilde{p}_{s,t}\tilde{c}_{j,s,t}}{p_{i,t}} + \bar{c}_{i,s} - \bar{c}_{i,s} \right)^{\alpha_{i,s}} \\
\implies \tilde{c}_{j,s,t} &= \prod_{i=1}^{I} \left(\frac{\alpha_{i,s}\tilde{p}_{s,t}\tilde{c}_{j,s,t}}{p_{i,t}} \right)^{\alpha_{i,s}} \\
\implies \tilde{c}_{j,s,t} &= \tilde{p}_{s,t}\tilde{c}_{j,s,t} \prod_{i=1}^{I} \left(\frac{\alpha_{i,s}}{p_{i,t}} \right)^{\alpha_{i,s}} \\
\implies \frac{\tilde{p}_{s,t}\tilde{c}_{j,s,t}}{\tilde{c}_{j,s,t}} &= \prod_{i=1}^{I} \left(\frac{p_{i,t}}{\alpha_{i,s}} \right)^{\alpha_{i,s}} \\
\implies \tilde{p}_{s,t} &= \prod_{i=1}^{I} \left(\frac{p_{i,t}}{\alpha_{i,s}} \right)^{\alpha_{i,s}}
\end{aligned}
\tag{2.32}
$$

This composite good price is then used in the household's intertemporal optimization problem described in Equation **??**. With the parameters and endogenous variables, we then use **??** to find the $c_{i,j,s,t}$.

2.2.3 Relating Consumption and Production Goods

Our model contains I consumption goods and M production goods. We denote the quantity of production good m in period t as $X_{m,t}$. We relate the output of the production sectors and the consumption goods using a fixed coefficient model. That is, we assume each consumption good is made up of a mix of the outputs of different production sectors. This means that the composition of these consumption goods do not respond to prices. The weights that determine the mix for each consumption goods are given in the matrix Π. Element $\pi_{i,m}$ of the matrix Π corresponds to the percentage contribute of the output of sector m in the production of good i. The total supply of good i in the economy at time t is thus given by:

$$
c_{i,t} = \sum_{m=1}^{M} \pi_{i,m} X_{m,t} \tag{2.33}
$$

And the price of a unit of consumption good i at time t is:

$$
p_{i,t} = \sum_{m=1}^{M} \pi_{i,m} p_{m,t}, \tag{2.34}
$$

Where p_m is the price of output of production sector m at time t.

2.2.4 Preferences for Corporate vs. Noncorporate Goods

Production sectors may contain corporate and non-corporate producers, each facing different tax treatment. If the output from corporate and non-corporate entities are perfect substitutes, then if the producers have the same production technology, consumers will end up consuming only the output from the sector with lowest after tax cost of producing. **?**) propose a model where different production sectors use different technologies, which can give rise to an equilibrium where both the corporate and non-corporate sector produce the same good. We take a different track, following **?**) we allow production technologies to vary across industry, but not across sectors within industry. Both sectors produce output in equilibrium, because output across sectors are not perfect substitutes. For example, food outside the home from a corporate, chain restaurant chain is not the same as food outside the home from a small, family-owned restaurant. Specifically, we define consumer preferences such that demand for the composite production good (combing output from the corporate and non-corporate sector) for production sector m at time t, $X_{m,t}$, is a constant elasticity of substitution (CES) function of the output from the corporate and non-corporate sectors, $X_{m,t,C}$ and $X_{m,t,NC}$, respectively:

$$X_{m,t} = \left[\gamma_m^{\frac{1}{\varepsilon_3}} X_{m,t,C}^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + (1-\gamma_m)^{\frac{1}{\varepsilon_3}} X_{m,t,NC}^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + \right]^{\frac{\varepsilon_3}{(\varepsilon_3-1)}}, \tag{2.35}$$

where ε_3 is the elasticity of substitution between corporate and non-corporate output and is assumed to be constant across industries. The share parameter in the CES function, γ_m is allowed to vary across industry and will be identified by the fraction of corporate produced output across industries. The CES function thus explains the existence of corporate and non-corporate production within each industry as well as the different shares of corporate output across industries. Because of these preferences, changes in corporate and non-corproate tax treatment will have differential impacts across consumers of different ages and income levels. Consumers choose $X_{m,t,C}$ and $X_{m,t,NC}$ to maximize **??** subject to:

$$p_{m,t}X_{m,t} = p_{m,t,C}X_{m,t,C} + p_{m,t,NC}X_{m,t,NC}, \tag{2.36}$$

where $p_{m,t,C}$ and $p_{m,t,NC}$ are the prices of output from the corporate and non-corporate firms in production industry m, respectively. Note that these prices are determined through the firm's profit maximization problem and the zero economic profit condition for firms. The constrained optimization problem consumers face is:

$$\mathcal{L} = \left[\gamma_m^{\frac{1}{\varepsilon_3}} X_{m,t,C}^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + (1-\gamma_m)^{\frac{1}{\varepsilon_3}} X_{m,t,NC}^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + \right]^{\frac{\varepsilon_3}{(\varepsilon_3-1)}} + \lambda_{m,t,C}\left(p_{m,t}X_{m,t} - p_{m,t,C}X_{m,t,C} + p_{m,t,NC}X_{m,t,NC}\right) \tag{2.37}$$

FOCs are:

$$\frac{\partial \mathcal{L}}{\partial X_{m,t,C}} = \gamma_m^{\frac{1}{\varepsilon_3}} X_{m,t,C}^{\frac{-1}{\varepsilon_3}} \left[\gamma_m^{\frac{1}{\varepsilon_3}} X_{m,t,C}^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + (1-\gamma_m)^{\frac{1}{\varepsilon_3}} X_{m,t,NC}^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + \right]^{\frac{1}{(\varepsilon_3-1)}} - \lambda_{m,t,C} p_{m,t,C} = 0 \tag{2.38}$$

and

$$\frac{\partial \mathcal{L}}{\partial X_{m,t,NC}} = (1-\gamma_m)^{\frac{1}{\varepsilon_3}} X_{m,t,NC}^{\frac{-1}{\varepsilon_3}} \left[\gamma_m^{\frac{1}{\varepsilon_3}} X_{m,t,C}^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + (1-\gamma_m)^{\frac{1}{\varepsilon_3}} X_{m,t,NC}^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + \right]^{\frac{1}{(\varepsilon_3-1)}} - \lambda_{m,t,C} p_{m,t,NC} = 0 \tag{2.39}$$

Solving the two necessary conditions, we can find the equations for the demand for the corporate and non-corporate output in industry m as a function of the prices out output from each sector of industry m, price of the composite production good, the demand for the composite production good, and the parameters:

$$X_{m,t,C} = \frac{\gamma_m p_{m,t} X_{m,t}}{p_{m,t,C}^{\varepsilon_3}\left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m) p_{m,t,NC}^{1-\varepsilon_3}\right]} \tag{2.40}$$

and

$$X_{m,t,NC} = \frac{(1-\gamma_m) p_{m,t} X_{m,t}}{p_{m,t,NC}^{\varepsilon_3}\left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m) p_{m,t,NC}^{1-\varepsilon_3}\right]} \tag{2.41}$$

To determine $p_{m,t}$, note that the CES subutility function defining preferences over corporate and non-corporate output within a production industry is linearly homogenous. Because the subutility function is linearly homogenous, we know that the associated indirect utility function is homogenous of degree one in $X_{m,t}$. Letting $V(\cdot)$ represent the indirect utility function, this means that $V(p_{m,t,C}, p_{m,t,NC}, \lambda X_{m,t}) = \lambda V(p_{m,t,C}, p_{m,t,NC}, X_{m,t})$. The linear homogeneity of the utility function also means that the indirect utility function is homogenous of degree -1 in prices. That is, $V(\lambda p_{m,t,C}, \lambda p_{m,t,NC}, X_{m,t}) = \frac{V(p_{m,t,C}, p_{m,t,NC}, X_{m,t})}{\lambda}$. Linear homogeneity of the utility function means that:

$$V(p_{m,t,C}, p_{m,t,NC}, X_{m,t}) = \frac{p_{m,t} X_{m,t}}{e(p_{m,t,C}, p_{m,t,NC})}, \tag{2.42}$$

where $e(p_{m,t,C}, p_{m,t,NC})$ is the minimum expenditure for a unit of the composite good given prices. Rearranging, we have:

$$
\begin{aligned}
&e(p_{m,t,C}, p_{m,t,NC}) = \frac{p_{m,t}X_{m,t}}{V(p_{m,t,C}, p_{m,t,NC}, X_{m,t})} \\
&\implies e(p_{m,t,C}, p_{m,t,NC}) = p_{m,t}X_{m,t}/ \\
&\left[\gamma_m^{\frac{1}{\varepsilon_3}} \left(\frac{\gamma_m p_{m,t}X_{m,t}}{p_{m,t,C}^{\varepsilon_3}\left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m)p_{m,t,NC}^{1-\varepsilon_3}\right]}\right)^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + (1-\gamma_m)^{\frac{1}{\varepsilon_3}} \left(\frac{(1-\gamma_m)p_{m,t}X_{m,t}}{p_{m,t,NC}^{\varepsilon_3}\left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m)p_{m,t,NC}^{1-\varepsilon_3}\right]}\right)\right. \\
&\implies e(p_{m,t,C}, p_{m,t,NC}) = p_{m,t}X_{m,t}/ \\
&p_{m,t}X_{m,t}\left[\gamma_m^{\frac{1}{\varepsilon_3}} \left(\frac{\gamma_m}{p_{m,t,C}^{\varepsilon_3}\left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m)p_{m,t,NC}^{1-\varepsilon_3}\right]}\right)^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + (1-\gamma_m)^{\frac{1}{\varepsilon_3}} \left(\frac{(1-\gamma_m)}{p_{m,t,NC}^{\varepsilon_3}\left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m)p\right.}\right.\right. \\
&\implies e(p_{m,t,C}, p_{m,t,NC}) = 1/ \\
&\left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m)p_{m,t,NC}^{1-\varepsilon_3}\right]\left[\gamma_m^{\frac{1}{\varepsilon_3}} \left(\frac{\gamma_m}{p_{m,t,C}^{\varepsilon_3}}\right)^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + (1-\gamma_m)^{\frac{1}{\varepsilon_3}} \left(\frac{(1-\gamma_m)}{p_{m,t,NC}^{\varepsilon_3}}\right)^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} +\right]^{\frac{\varepsilon_3}{(\varepsilon_3-1)}} \\
&\implies e(p_{m,t,C}, p_{m,t,NC}) = \frac{\left[\gamma_m^{\frac{1}{\varepsilon_3}} \left(\frac{\gamma_m}{p_{m,t,C}^{\varepsilon_3}}\right)^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + (1-\gamma_m)^{\frac{1}{\varepsilon_3}} \left(\frac{(1-\gamma_m)}{p_{m,t,NC}^{\varepsilon_3}}\right)^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} +\right]^{\frac{\varepsilon_3}{(1-\varepsilon_3)}}}{\left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m)p_{m,t,NC}^{1-\varepsilon_3}\right]} \\
&\implies e(p_{m,t,C}, p_{m,t,NC}) = \frac{\left[\gamma_m \left(\frac{1}{p_{m,t,C}^{\varepsilon_3}}\right)^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} + (1-\gamma_m) \left(\frac{1}{p_{m,t,NC}^{\varepsilon_3}}\right)^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} +\right]^{\frac{\varepsilon_3}{(1-\varepsilon_3)}}}{\left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m)p_{m,t,NC}^{1-\varepsilon_3}\right]} \\
&\implies e(p_{m,t,C}, p_{m,t,NC}) = \frac{\left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m)p_{m,t,NC}^{1-\varepsilon_3}\right]^{\frac{\varepsilon_3}{(1-\varepsilon_3)}}}{\left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m)p_{m,t,NC}^{1-\varepsilon_3}\right]} \\
&\implies e(p_{m,t,C}, p_{m,t,NC}) = \left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m)p_{m,t,NC}^{1-\varepsilon_3}\right]^{\frac{1}{(1-\varepsilon_3)}}
\end{aligned}
\tag{2.43}
$$

Thus we have the price of the corporate-non-corporate composite good from production industry m at time t as:

$$
e(p_{m,t,C}, p_{m,t,NC}) = p_{m,t} = \left[\gamma_m p_{m,t,C}^{1-\varepsilon_3} + (1-\gamma_m)p_{m,t,NC}^{1-\varepsilon_3}\right]^{\frac{1}{(1-\varepsilon_3)}} \tag{2.44}
$$

Chapter 3

Firms

3.1 The Firm's Problem

The objective of the firm is to maximize firm value. Firms do this by choosing investment and labor demand, as well as financial policies such as new equity issues, dividend distributions, and borrowing. The problem of the firm is the same in each industry, m, and in each sector, $C \in \{\text{corporate}, \text{non-corporate}\}$, though the parameters defining the problem vary across industry and sector. Finally, we assume that each industry and sector is competitive, meaning that firms earn zero economic profits. Since the problem of the firm is the same in each sector and industry, we omit the subscripts m, and C that accompany each variable and parameter for the description of the firm's problem.

3.1.1 The Value of the Firm

Without aggregate uncertainty, asset market equilibrium requires that the after-tax returns on all assets be equalized if households are to simultaneously hold equity in firms and risk-free bonds from firms and government. The after-tax, nominal return on holding a risk-free government bond is:

$$i_t = (1 - \tau_t^i) r_t, \tag{3.1}$$

Where r_t is the real interest rate on bonds. Thus the return on holding corporate equity must equal i_t in equilibrium:

$$i_t = (1 - \tau_t^i) r_t = \frac{(1 - \tau_t^d) DIV_t + (1 - \tau_t^g)(V_{t+1} - V_t - VN_t)}{V_t} \tag{3.2}$$

The first part of the numerator in Equation **??** are the dividends from holding equity shares in the firm. The second part are the capital gains from holding equity, which are diluted by the issuance of new shares, VN_t. We can rearrange this equation **??** to solve for V_{t+1}:

$$
\begin{aligned}
V_{t+1} &= \frac{V_t(1-\tau_t^i)r_t - (1-\tau_t^d)DIV_t}{(1-\tau_t^g)} + V_t + VN_t \\
&= V_t \underbrace{\left(1 + \frac{(1-\tau_t^i)r_t}{(1-\tau_t^g)}\right)}_{\text{Let this be } 1+\theta_t} + VN_t - \frac{(1-\tau_t^d)}{(1-\tau_t^g)}DIV_t \\
&= V_t(1+\theta_t + VN_t - \left(\frac{1-\tau_t^d}{1-\tau_t^g}\right)DIV_t
\end{aligned}
\tag{3.3}
$$

Now we then solve for V_t by repeatedly substituting for V_{t+1} and applying the transversality condition ($\lim_{T\to\infty}\prod_{t=1}^{T}(1+\theta_t)V_T = 0$):

$$
\begin{aligned}
& V_t = \frac{V_{t+1}}{(1+\theta_t)} - \frac{VN_t}{(1+\theta_t)} + \frac{\left(\frac{1-\tau_t^d}{1-\tau_t^g}\right)DIV_t}{(1+\theta_t)} \\
\Longrightarrow\ & V_t = \frac{V_{t+2}}{(1+\theta_t)(1+\theta_{t+1})} - \frac{VN_{t+1}}{(1+\theta_t)(1+\theta_{t+1})} + \frac{\left(\frac{1-\tau_{t+1}^d}{1-\tau_{t+1}^g}\right)DIV_{t+1}}{(1+\theta_t)(1+\theta_{t+1})} - \frac{VN_t}{(1+\theta_t)} + \frac{\left(\frac{1-\tau_t^d}{1-\tau_t^g}\right)DIV_t}{(1+\theta_t)} \\
\Longrightarrow\ & V_t = \frac{V_{t+3}}{(1+\theta_t)(1+\theta_{t+1})(1+\theta_{t+2})} - \frac{VN_{t+2}}{(1+\theta_t)(1+\theta_{t+1})(1+\theta_{t+2})} + \frac{\left(\frac{1-\tau_{t+2}^d}{1-\tau_{t+2}^g}\right)DIV_{t+2}}{(1+\theta_t)(1+\theta_{t+1})(1+\theta_{t+2})} \\
& \quad - \frac{VN_{t+1}}{(1+\theta_t)(1+\theta_{t+1})} + \frac{\left(\frac{1-\tau_{t+1}^d}{1-\tau_{t+1}^g}\right)DIV_{t+1}}{(1+\theta_t)(1+\theta_{t+1})} - \frac{VN_t}{(1+\theta_t)} + \frac{\left(\frac{1-\tau_t^d}{1-\tau_t^g}\right)DIV_t}{(1+\theta_t)} \\
& \text{and so on...} \\
\Longrightarrow\ & V_t = \underbrace{\prod_{\nu=t}^{\infty}\left(\frac{1}{1+\theta_\nu}\right)V_\infty}_{=0 \text{ by transversality condition}} - \sum_{u=t}^{\infty}\prod_{\nu=t}^{u}\left(\frac{1}{1+\theta_\nu}\right)\left[VN_u - \left(\frac{1-\tau_u^d}{1-\tau_u^g}\right)DIV_u\right] \\
\Longrightarrow\ & V_t = \sum_{u=t}^{\infty}\prod_{\nu=t}^{u}\left(\frac{1}{1+\theta_\nu}\right)\left[\left(\frac{1-\tau_u^d}{1-\tau_u^g}\right)DIV_u - VN_u\right]
\end{aligned}
\tag{3.4}
$$

Thus, firm value is equal to the discounted, after-tax value of dividends, less the discounted value of new share issuance, which dilutes the value of the shares held at time t.

3.1.2 The Sequence Problem of the Firm

To solve for the equation defining the dynamic optimization problem of the firm as a function of demand for labor and capital, we first solve for VN_t, the value of shares issued in period t. To do this, we use the cash flow identity of the firm:

$$EARN_t + BN_t + VN_t = DIV_t + I_t(p_t^K + \Phi_t) + TE_t, \tag{3.5}$$

where $EARN_t$ are earnings before depreciation, corporate income taxes, and adjustment costs, but after property taxes; BN_t are new bond issues, I_t is investment, p_t^K is the price of capital, Φ_t are adjustment costs, and TE_t are total corporate income taxes (all in period t). Earnings are the difference between the revenues from selling firm output, X_t, and the costs of labor, debt, and property taxes. Specifically:

$$EARN_t = p_t X_t - w_t EL_t - r_t B_t - \tau_t^P K_t, \tag{3.6}$$

where p_t is the price of output, w_t the wage rate per unit of effective labor, and r_t the real interest rate. The stock of bonds outstanding at the start of period t is given by B_t and τ_t^P is the property tax rate on capital. Output, X_t, is determined by a constant elasticity of substitution (CES) production function that takes capital, K_t, and effective labor, EL_t as inputs. Note that we denote the capital stock that is determined when period t begins at K_t. Labor is augmented by a labor-augmenting technology with growth rate g_y. The CES production function for the firm is:

$$F(K_t, EL_t) = X_t = \left[(\gamma)^{1/\epsilon}(K_t)^{(\epsilon-1)/\epsilon} + (1-\gamma)^{1/\epsilon}(e^{g_y t} EL_t)^{(\epsilon-1)/\epsilon}\right]^{(\epsilon/(\epsilon-1))}, \tag{3.7}$$

where γ and ϵ give the share of capital and the elasticity of capital for labor in the production function, respectively. New debt issues are solved for by the assumption of a constant debt-to-capital ratio (and the law of motion for the capital stock):

$$BN_t = B_{t+1} - B_t \text{ and } B_t = bK_t \text{ by assumption} \tag{3.8}$$

The parameter b gives the exogenous debt-to-capital ratio that determines firm debt issuance. The law of motion of the capital stock is given by:

$$K_{t+1} = (1-\delta)K_t + I_t, \tag{3.9}$$

where δ is the economic rate of depreciation on physical capital and I_t is investment in capital. Adjustment costs are assumed to be a quadratic function of deviations from the steady-state investment rate:

$$\Phi_t = \frac{p_t^K \left(\frac{\beta}{2}\right)\left(\frac{I_t}{K_t} - \mu\right)^2}{\left(\frac{I_t}{K_t}\right)} \tag{3.10}$$

The parameter β is the scaling parameter for the adjustment cost function and μ is the steady-state investment rate, which is determined as $\mu = \delta + g_y + g_n$. Taxes on firm profits are given by:

$$\begin{aligned} TE_t = &\tau_t^b \left[p_t X_t - w_t EL_t - f_e p_t^K I_t - \Phi_t I_t - f_i i_t B_t - f_p \delta b K_t + f_b b p_t^K I_t - f_d \delta^\tau K_t^\tau - \tau_t^p K_t\right] \\ &+ \tau_t^{ic} p_t^K I_t, \end{aligned} \tag{3.11}$$

where τ_t^b is the tax rate on business income will be used to represent either an entity level tax or the tax rate on the distributions of income to owners for those firms not subject to an entity level tax. Note that we are assuming that investment may or may not be deductible (depending upon the dummy variable f_e), but that investment adjustment costs are always deductible (i.e., they are not preceded by f_e). Under a pre-pay consumption tax system, investments are not deductible from the tax base. Whether or not adjustment costs are deductible under a pre-pay consumption tax depends upon what you think these costs derive from. For example, if adjustment costs are from retraining employees to use new equipment, then these costs may be deductible under a consumption tax system (pre or post-pay) because they would likely be in the form of wage/labor costs.[1] The other indicator variables, f_i, f_p, f_b, and f_d, allow for various consumption tax policies to be incorporated into the model. The parameter $f_i = 1$ if interest on debt is deductible and 0 if not. The parameter f_p is equal to one the principle on corporate borrowing is deductible from the corporate income tax based. Principle on loans would be deductible in a post-pay consumption tax system. The parameter f_b is equal to one if the proceeds from firm borrowing is included in the corporate tax base. Such proceeds would be included in a pre-pay consumption tax system. The parameter f_d is equal to one if capital can be depreciated and zero if not. For example, in a post-pay consumption tax framework, $f_e = 1$ and $f_d = 0$.

The tax basis of the capital stock is given by K_t^τ. The law of motion for the tax basis of the capital stock is given by:

$$K_{t+1}^\tau = (1-\delta^\tau)(K_t^\tau + (1-f_e)p_t^K I_t), \tag{3.12}$$

where δ^τ is the rate if depreciation for tax purposes. Note how we form the law of motion for the tax basis. The above formulation accounts for the fact that investment in year t receives a depreciation deduction in year t.[2] We can think about modifying this so that you get no deduction in the year the investment is made, which may or may not be more consistent with the "time to build" built into the law of motion for the physical capital stock.

Dividends are determined by the assumption that dividends are a constant fraction of after-tax earnings, net of economic depreciation. In particular,

$$DIV_t = \zeta(EARN_t - TE_t - p_t^K \delta K_t) \tag{3.13}$$

The parameter ζ defines the exogenous dividend payment rules, specifying the fraction of earnings distributed as dividends.

Substituting Equations **??** - **??** into Equation **??** (and letting $\Omega_t = 1 - \zeta + \zeta\left(\frac{1-\tau_t^d}{1-\tau_t^g}\right) = \left[\zeta(1-\tau_t^d) + (1-\zeta)(1-\tau_t^g)\right]/(1-\tau_t^g)$), one can write the value of the firm at time t as:

[1] It's not clear how best to handle this and **?**) are vague on this point.

[2] The IRS specifies a partial year rule, where one deducts the value of investment proportional to the amount of the year in which the asset was in place. We ignore this detail and assume all assets are in place for the entire year.

$$
\begin{aligned}
V_t =& \sum_{u=t}^{\infty}\prod_{\nu=t}^{u}\left(\frac{1}{1+\theta_\nu}\right)(1-\tau_u^b)\Omega_u(p_u X_u - w_s E L_s) \\
&- K_t\left\{(1-\tau_u^b)\Omega_u\tau_u^p + (1-f_i\tau_u^i)i_u\Omega_u b - \delta(p_u - b - \Omega_u(p_u - f_p\tau_u^b b))\right\} \\
&- I_u\left\{p_t^K - b + \Omega_u f_b\tau_u^b b - \Omega_u f_e\tau_u^b + (1-\Omega_u\tau_u^b)\Phi_u\right\} \\
&- \Omega_u f_d\tau_u^b\delta^\tau K_u^\tau
\end{aligned}
\tag{3.14}
$$

Note that K_u^τ tracks depreciation deductions in in all periods $u = t, ..., \infty$. Future depreciation deductions on the tax basis of the capital stock in existence at time u do not affect investment decisions at time u (or forward) since the tax basis is predetermined.[3] However, future depreciation deductions for investments made at time u do affect investment decisions (since they lower the after-tax cost of investment). Therefore it's useful to distinguish between old and new capital.

The time u value of future depreciation deductions on the capital stock existing at the beginning of period u is given by K_{u-1}^τ. We can determine this value as:

$$
\begin{aligned}
f_d Z_u K_{u-1}^\tau &= \sum_{j=u}^{\infty}\prod_{\nu=u}^{j}\left(\frac{1}{1+\theta_\nu}\right) f_d\Omega_j\tau_j^b\delta^\tau(1-\delta^\tau)^{j-u}K_u^\tau \\
&= f_d K_{u-1}^\tau \underbrace{\sum_{j=u}^{\infty}\prod_{\nu=u}^{j}\left(\frac{1}{1+\theta_\nu}\right) f_d\Omega_j\tau_j^b\delta^\tau(1-\delta^\tau)^{j-u}}_{Z_u} \\
&= f_d K_{u-1}^\tau Z_u,
\end{aligned}
\tag{3.15}
$$

where Z_u is the net present value of future depreciation deductions per dollar of investment. With this, we derive the time u value of future depreciation deductions on investments made at time u, I_u^τ. These are given by $f_d(1-f_e)Z_u I_u$. Now we can rewrite Equation **??** describing the value of the firm at time t as:

$$
\begin{aligned}
V_t =& \sum_{u=t}^{\infty}\prod_{\nu=t}^{u}\left(\frac{1}{1+\theta_\nu}\right)(1-\tau_u^b)\Omega_u(p_u X_u - w_u E L_u) \\
&- K_t\left\{(1-\tau_u^b)\Omega_u\tau_u^p + (1-f_i\tau_u^i)i_u\Omega_u b - \delta(p_u - b - \Omega_u(p_u - f_p\tau_u^b b))\right\} \\
&- I_u\left\{1 - b + \Omega_u f_b\tau_u^b b - \Omega_u f_e\tau_u^b - f_d(1-f_e)Z_u + (1-\Omega_u\tau_u^b)\Phi_u\right\} \\
&+ f_d Z_t K_{t-1}^\tau
\end{aligned}
\tag{3.16}
$$

Using the above equations, we see all endogenous variables determining the value of the firm result from the firm's choice of investment and effective labor demand. The sequence problem of the firm is thus:

[3] Note that if there were financial frictions (e.g. a borrowing constraint or costly external finance), then investment would be dependent on cash flow and would then be affected by changes in the value of deductions for the existing capital basis.

$$
\begin{aligned}
V_t = \max_{\{I_u, EL_u\}_{u=t}^{\infty}} \sum_{u=t}^{\infty} \prod_{\nu=t}^{u} \left(\frac{1}{1+\theta_\nu}\right) & (1-\tau_u^b)\Omega_u(p_u X_u - w_u EL_u) \\
& - K_t \left\{(1-\tau_u^b)\Omega_u \tau_u^p + (1-f_i\tau_u^i)i_u\Omega_u b - \delta(p_u - b - \Omega_u(p_u - f_p\tau_u^b b))\right\} \\
& - I_u \left\{1 - b + \Omega_u f_b \tau_u^b b - \Omega_u f_e \tau_u^b - f_d(1-f_e)Z_u + (1-\Omega_u\tau_u^b)\Phi_u\right\} \\
& + f_d Z_t K_{t-1}^{\tau}
\end{aligned}
\tag{3.17}
$$

The Lagrangian to the firm's problem at time t can be written as:

$$
\begin{aligned}
\mathcal{L}_t = \max_{\{I_u, EL_u\}_{u=t}^{\infty}} \sum_{u=t}^{\infty} \prod_{\nu=t}^{u} \left(\frac{1}{1+\theta_\nu}\right) & (1-\tau_u^b)\Omega_u(p_u X_u - w_s EL_s) \\
& - K_s \left\{(1-\tau_u^b)\Omega_u \tau_u^{pC} + (1-f_i\tau_u^i)i_u\Omega_u b - \delta(p_u - b - \Omega_u(p_u - f_p\tau_u^b b))\right\} \\
& - I_u \left\{1 - b + \Omega_u f_b \tau_u^b b - \Omega_u f_e \tau_u^b - f_d(1-f_e)Z_u + (1-\Omega_u\tau_u^b)\Phi_u\right\} \\
& + f_d Z_s K_{s-1}^{\tau} + q_u((1-\delta)K_u + I_u - K_{u+1})
\end{aligned}
\tag{3.18}
$$

The first order conditions of the firm with respect to investment (which hold $\forall\ u$) are given by:

$$
\begin{aligned}
& \frac{\partial \mathcal{L}_t}{\partial I_u} = -\left\{1 - b + \Omega_u f_b \tau_u^b b - \Omega_u f_e \tau_u^b - f_d(1-f_e)Z_u + (1-\Omega_u\tau_u^b)\Phi_u\right\} - I_u(1-\Omega_u\tau_u^b)\frac{\partial \Phi_u}{\partial I_u} + q_u = 0 \\
& \Longrightarrow q_u = 1 - b + \Omega_u f_b \tau_u^b b - \Omega_u f_e \tau_u^b - f_d(1-f_e)Z_u + (1-\Omega_u\tau_u^b)\Phi_u + I_u(1-\Omega_u\tau_u^b)\frac{\partial \Phi_u}{\partial I_u} \\
& \Longrightarrow q_u = 1 - b - \Omega_u \tau_u^b(f_e - f_b b) - f_d(1-f_e)Z_u + (1-\Omega_u\tau_u^b)\Phi_u + I_u(1-\Omega_u\tau_u^b)\frac{\partial \Phi_u}{\partial I_u}
\end{aligned}
\tag{3.19}
$$

The Euler equation described in Equation **??** relates Tobin's q, given by q_u, to the marginal costs of investment. Tobin's q defines the marginal change in firm value for a dollar of investment. It is the shadow price of additional capital. The FOC for investment says that the firm invests until the marginal benefit (the LHS of Equation **??**) is equal to the marginal cost of investment (the RHS of Equation **??**). The cost of investment in the absence of taxes and frictions is equal to 1 (the first term on the RHS of Equation **??**) since investment goods are the numeraire. The second term reflects the reduction in the cost of capital due to debt financing. The third term on the RHS of Equation **??** is the change in the cost of capital due to debt being included or excluded from business entity-level income taxes. The fourth term reflects the reduction in the cost of capital due to depreciation deductions. The last term reflects the component of the cost of capital that is due to adjustment costs (net of the expensing of adjustment costs for tax purposes).

At times it is helpful to write this choice in terms of capital one period ahead rather than investment. In this case, the first order conditions are given by:

$$\frac{\partial \mathcal{L}_s}{\partial K_{u+1}} = \prod_{\nu=s}^{u}\left(\frac{1}{1+\theta\nu}\right)[-q_u] + \prod_{\nu=s}^{u+1}\left(\frac{1}{1+\theta\nu}\right)\Big[(1-\delta)q_{u+1} + p_{u+1}\frac{\partial X_{u+1}}{\partial K_{u+1}} - \big\{(1-\tau^b_{u+1})\Omega_{u+1}\tau^p_{u+1}$$
$$+(1-f_i\tau^i_{u+1})i_{u+1}\Omega_{u+1}b - \delta(p_{u+1} - b - \Omega_{u+1}(p_{u+1} - f_p\tau^b_{u+1}b))\big\}\Big] = 0$$
$$\implies q_u = \left(\frac{1}{1+\theta_{u+1}}\right)\Big[(1-\delta)q_{u+1} + p_{u+1}\frac{\partial X_{u+1}}{\partial K_{u+1}} - \big\{(1-\tau^b_{u+1})\Omega_{u+1}\tau^p_{u+1}$$
$$+(1-f_i\tau^i_{u+1})i_u\Omega_{u+1}b - \delta(p_{u+1} - b - \Omega_{u+1}(p_{u+1} - f_p\tau^b_{u+1}b))\big\}\Big] \tag{3.20}$$

The marginal product of labor is given by:

$$\frac{\partial X_{u+1}}{\partial K_{u+1}} = \gamma^{1/\epsilon}K_{u+1}^{-1/\epsilon}\left[(\gamma)^{1/\epsilon}(K_{u+1})^{(\epsilon-1)/\epsilon} + (1-\gamma)^{1/\epsilon}(e^{g_y(u+1)}EL_{u+1})^{(\epsilon-1)/\epsilon}\right]^{1/(\epsilon-1)} \tag{3.21}$$

Finally, the firm also chooses its demand for effective labor units. The necessary condition for this choice is give by:

$$p^C_u\frac{\partial F(K^C_u, EL^C_u)}{\partial EL^C_u} = w_u, \forall\ u \tag{3.22}$$

Labor demand is determined through this intratemporal trade off between the costs and benefits of employing additional labor in the production process. The left hand side gives the marginal revenue, or benefits from employing more labor, and the right hand save gives the costs, which are the wages paid to the additional labor.
The marginal product of labor is given by:

$$\frac{\partial X_u}{\partial EL_u} = (1-\gamma)^{1/\epsilon}\frac{(e^{g_y u}EL_u)^{(\epsilon-1)/\epsilon}}{EL_u}\left[(\gamma)^{1/\epsilon}(K_u)^{(\epsilon-1)/\epsilon} + (1-\gamma)^{1/\epsilon}(e^{g_y u}EL_u)^{(\epsilon-1)/\epsilon}\right]^{1/(\epsilon-1)} \tag{3.23}$$

The choice of capital and labor must satisfy Equations **??** and **??**. Together, capital and labor imply the output of front the production process through Equation **??**. The other endogenous quantity variables in the firm's problem are then determined through the relationships given in Equations **??** to **??**.

The price of firm output will be determined by the firm's zero profit condition. With competitive firms, and free entry and exit, output prices, p_u, are such that:

$$p_u X_u = w_u EL_u + (r_u + \delta)K_u \tag{3.24}$$

The final endogenous variable to solve for is the value of the firm at any point in time, V_u. As **?**) shows, with a constant returns to scale production function and quadratic adjustment costs, there is an equivalence between marginal q and average q. Note that in our case, we must make an adjustment for the value of depreciation deductions on the tax basis of the capital stock already in place at time u. The relation between marginal q, given by q_u, and average q, given by Q_u is:

$$q_u = \frac{[V_u - f_d Z_u K^\tau_{u-1}]}{K_u} \text{ and } Q_u = \frac{V_u}{K_u} \tag{3.25}$$

This relationship thus allows use to determine the value of the firm as:

$$V_u = q_u K_u + f_d Z_u K_{u-1}^{\tau} \tag{3.26}$$

3.1.3 Relating Firm Investment and Production Goods

Our model contains M production industries, each of which chooses investment that is a composite good from these production processes. We denote the quantity of production good m in period t as $X_{m,t}$. We relate the output of the production sectors to their inputs using a fixed coefficient model. That is, each investment good is made up of a mix of the outputs of different production sectors. This means that the composition of these investment goods do not respond to prices. The weights that determine the mix for each consumption goods are given in the matrix Ξ. Element $\xi_{j,m}$ of the matrix Ξ corresponds to the percentage contribute of the output of industry m in the production of the investment good for industry j. The total supply of investment good j in the economy at time t is thus given by:

$$I_{j,t} = \sum_{m=1}^{M} \xi_{j,m} X_{m,t} \tag{3.27}$$

And thus the price of a unit of investment good for industry m at time t is:

$$p_{j,t}^{K} = \sum_{m=1}^{M} \xi_{j,m} p_{m,t}, \tag{3.28}$$

Where p_m is the price of output of production sector m at time t.

Chapter 4

Government

4.1 Overview of government in the model

The government has four functions in the model. First, it runs the tax and social security systems as described in the households' and firms' problems. That is, it collects revenues and makes payments in accordance with the parameterized tax and social security policy functions. The tax functions are user determined, but will default to a current law baseline if the user does not specify a policy proposal in affecting the relevant tax function. Second, the government makes direct transfers to households outside of the social security system. Third, the government produces a non-excludable public good. Finally, the government contributes to the production of a private consumption good. We discuss each of these function in more detail below after first defining the government's budget constraint. Government will have four functions in our model:

4.2 Government budgeting

The government's per-period budget constraint is given by:

$$D_{t+1} + T_t^{\tau} = (1 + r_t)D_t + T_t^H + G_t^{subs} + w_t EL_t^G + p_g^K I_t^G, \tag{4.1}$$

where D_t denotes the government's outstanding debt, T_t is total tax revenue across all tax sources and net of social security transfers, T_t^H is total direct transfers to households, G_t^{subs} are government subsides to the production private goods, EL_t is are effective labor units employed by government in the production of the public good, and I_t is government investment in capital used to produce the government provided public good. The price of capital for the government sector is given by p_g^K and the price of an effective labor unit is w_t. Note that we do not impose a balanced budget. In any particular period, the government may run a surplus or deficit. The government finances any gaps using debt, D_t.

4.2.1 Rule for long-term fiscal stability

While the government can use debt to finance temporary budget shortfalls, the government cannot finance infinite amounts of debt. For example, it cannot be the case that the debt level grows to such an extent that interest payments on the debt exceed GDP. To ensure that the government budget is sustainable in the long run, we impose a rule that returns the debt-to-GDP ratio to some predetermined value after a set amount of time. In particular, the user will select (or the model will default to) a particular steady-state debt-to-GDP ratio, $\bar{d} = \frac{\bar{D}}{\bar{Y}}$. The model will then specify some period, T, such that after this time, the government budget adjusts to return to $\bar{d}$. In particular, if the debt-to-GDP ratio exceeds $\bar{d}$, then government provision of the public good, G_t is reduced. This government debt rule takes the form:

$$G_t - \bar{G} = \rho_t(\hat{d}_t - \bar{d}); \quad \rho_t < 0, \tag{4.2}$$

where $\hat{G}$ is the steady state level of public good (given steady-state debt-to-GDP ratio $\bar{d}$) and ρ is a parameter that determines how quickly government debt returns to its steady-state value. The parameter ρ will be estimated from historical data on the response of government spending to debt. This calibration is discussed in the calibration chapters to be added to this document.

4.3 Direct transfers

Direct transfers to individuals, T_t^H, will be modeled as a polynomial function of age and income. In that sense, this function will be similar to those used to determine individual income taxes. This function will be estimated from data on government transfers by age and income....

4.4 Government production of public goods

The government engages in the production of a non-excludable public good. Utility from the public good enters the individuals' utility functions in an additively separable way, and thus is excluded from the description above since it does not impact consumer decisions. To produce the public good, the government uses a capital and labor in a constant returns to scale Cobb-Douglas production function. In particular, the total quantity of the public good is given by:

$$G_t = (K_t^G)^{\alpha}(EL_t^G)^{1-\alpha} \tag{4.3}$$

The parameter α thus represents capital's share of output of the public good. The government's capital stock follows the standard law of motion: $K_{t+1}^G = (1-\delta^G)K_t^G + I_t^G$.

We determine G_0 as the the total amount of spending on government goods and services less the purchase of inputs for government production of private goods in the model's base year. We then set $G_t = G_0$ for all $t < T$. At period T, the government budget may need to adjust G_t to make debt sustainable. Thus the government supply of the public good is exogenous in periods 0 to T and determined by the steady state value of debt there after. We use this to solve for the expenditures on labor and capital to produce this public good. The government solves:

$$\min_{\{EL^G_u, I^G_u\}^\infty_{u=t}} \sum_{u=t}^{\infty} \prod_{\nu=t}^{u} \left(\frac{1}{1+r_\nu}\right) w_u EL^G_u + p^K_{g,u} I^G_u \tag{4.4}$$

$$\text{subject to: } G_t = (K^G_t)^\alpha (EL^G_t)^{1-\alpha} \text{ and} \tag{4.5}$$

$$K^G_{t+1} = (1-\delta^G)K^G_t + I^G_t, \forall\ t \tag{4.6}$$

The equation above can be solved for government's demand for capital and labor as a function of the model parameters, factor prices, and government supply of public goods. In particular, we can find the demand for capital as:

$$K^G_{t+1} = \frac{\alpha}{1-\alpha} \frac{w_t}{p^K_{g,t}} (1+r_t) L_t \frac{G_{t+1}}{G_t} \tag{4.7}$$

4.5 Government production of private goods

The government is one of the M production sectors. Its problem is thus described in Chapter **??**. However, the government firm differs from the private firms in that the gross-of-tax output price, $p_{g,t}$, includes a subsidy. This means that the government sells output at a price that is below the cost of production. Instead of the zero-profit condition, the condition for government firms is:

$$G^{subs}_t = p_{g,t} X_{g,t} - w_t EL_{g,t} - r_t K_{g,t} \tag{4.8}$$

Thus the government subsidy towards the production of private goods is the difference between government revenues at the subsidized price and the costs of inputs to production.

Chapter 5

Equilibrium

This chapter defines the model equilibria. We define both the stationary-steady-state equilibrium as well as the stationary non-steady state equilibrium. Our solution method, described in the next chapter, uses both of these concepts.

5.1 Market clearing and stationary equilibrium

Labor market clearing requires that aggregate demand for effective labor units EL_t equal the sum of individual efficiency units of labor supplied, $e_{j,s}n_{j,s,t}$. Asset market clearing requires that aggregate asset holdings equal the total about of debt and equity asset outstanding. Aggregate consumption C_t is defined as the sum of all individual consumption, and aggregate investment is defined by the resource constant, $Y = C+I$. In particular, we have market clearing in the labor, asset, and goods markets:

$$\sum_{m=1}^{M} EL_{m,c,t} + \sum_{m=1}^{M} EL_{m,nc,t} + EL_t^G = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j e_{j,s}n_{j,s,t} \quad \forall t \tag{5.1}$$

$$D_t + \sum_{m=1}^{M} B_{m,c,t} + \sum_{m=1}^{M} B_{m,nc,t} + \sum_{m=1}^{M} V_{m,c,t} + \sum_{m=1}^{M} V_{m,nc,t} = \sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\omega_{s-1,t-1}\lambda_j b_{j,s,t} \quad \forall t \tag{5.2}$$

$$\sum_{m=1}^{M} X_{m,c,t} + \sum_{m=1}^{M} X_{m,nc,t} = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\sum_{i=1}^{I}\omega_{s,t}\lambda_j c_{i,j,s,t} + \sum_{m=1}^{M} I_{m,c,t} + \sum_{m=1}^{M} I_{m,nc,t} + I_t^G \quad \forall t \tag{5.3}$$

$$\tag{5.4}$$

The usual definition of equilibrium would be allocations and prices such that households optimize (**??**), (**??**), and (**??**), firms optimize (**??**) and (**??**), and markets clear (**??**), (**??**), and (**??**). However, the variables in these characterizing equations are potentially not stationary due to the possible growth rate in the total population $g_{n,t}$ each period coming from the cohort growth rates in (**??**) and from the deterministic growth rate of labor augmenting technological change g_y in (**??**).

Table 5.1: Stationary variable definitions

Sources of growth			Not
$e^{g_y t}$	$\tilde{N}_t$	$e^{g_y t}\tilde{N}_t$	growing[a]
$\hat{c}_{j,s,t} \equiv \frac{\tilde{c}_{j,s,t}}{e^{g_y t}}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{\tilde{N}_t}$	$\hat{X}_t \equiv \frac{X_t}{e^{g_y t}\tilde{N}_t}$	$n_{j,s,t}$
$\hat{b}_{j,s,t} \equiv \frac{b_{j,s,t}}{e^{g_y t}}$	$\hat{EL}_t \equiv \frac{EL_t}{\tilde{N}_t}$	$\hat{K}_t \equiv \frac{K_t}{e^{g_y t}\tilde{N}_t}$	r_t
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$		$\hat{BQ}_{j,t} \equiv \frac{BQ_{j,t}}{e^{g_y t}\tilde{N}_t}$	
$\hat{y}_{j,s,t} \equiv \frac{y_{j,s,t}}{e^{g_y t}}$		$\hat{I}_t \equiv \frac{I_t}{e^{g_y t}\tilde{N}_t}$	
$\hat{T}_{j,s,t} \equiv \frac{T_{j,s,t}}{e^{g_y t}}$			
$\hat{p}_{s,t} \equiv \frac{\tilde{p}_{s,t}}{e^{g_y t}}$			
$\hat{p}_{i,t} \equiv \frac{p_{i,t}}{e^{g_y t}}$			

[a] The interest rate r_t is already stationary because X_t and K_t grow at the same rate. Individual labor supply, $n_{j,s,t}$, is stationary.

Table **??** characterizes the stationary versions of the variables of the model in terms of the variables that grow due to labor augmenting technological change, population growth, both, or none. With the definitions in Table **??**, it can be shown that the equilibrium characterizing equations can be written in stationary form in the following way. The static and intertemporal Euler equations from the individual's optimization problem corresponding to
eqrefEqcfoc, (**??**), and (**??**) are the following:

NEED TO UPDATE EQUATIONS...need individual and firm conditions, stationarized...

We can now define the stationary steady-state equilibrium for this economy in the following way.

Definition 1 (**Stationary steady-state equilibrium**)**.** A non-autarkic stationary steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability $e_{j,s}$ is defined as constant allocations $n_{j,s,t} = \bar{n}_{j,s}$ and $\hat{b}_{j,s+1,t+1} = \bar{b}_{j,s+1}$ and constant prices $\hat{w}_t = \bar{w}$ and $r_t = \bar{r}$ for all j, s, and t such that the following conditions hold:

i. households optimize according to (**??**), (**??**), and (**??**),

ii. firms optimize according to (**??**) and (**??**),

iii. markets clear according to(**??**), (**??**), and (**??**), and

iv. the population has reached its stationary steady state distribution $\bar{\omega}_s$ for all ages s, characterized in Chapter **??**.

The steady-state equilibrium is characterized by the system of $2JS + 4M + ...$ equations and $2JS + 4M + ...$ unknowns $\bar{n}_{j,s}$, $\bar{b}_{j,s+1}$, $EL_{m,c,t}$, $I_{m,c,t}$,... . Chapter **??** details how to solve for the steady-state equilibrium.

The definition of the stationary non-steady-state equilibrium is similar to Definition **??**, with the stationary steady-state equilibrium definition being a special case of the stationary non-steady-state equilibrium.

WHAT FOLLOWS NEEDS UPDATING TO INCLUDE RICHER FIRM AND GOV'T, BUT IS HELPFUL IN SEEING THAT NON-SS EQ'M WILL LOOK LIKE...

Definition 2 (**Stationary non-steady-state equilibrium**). A non-autarkic stationary non-steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability $e_{j,s}$ is defined as allocations $n_{j,s,t}$ and $\hat{b}_{j,s+1,t+1}$ and prices $\hat{w}_t$ and r_t for all j, s, and t such that the following conditions hold:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings, and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\hat{\mathbf{\Gamma}}_{t+u} = \hat{\mathbf{\Gamma}}^e_{t+u} = \Omega^u\left(\hat{\mathbf{\Gamma}}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (**??**), (**??**), and (**??**)

iii. firms optimize according to (**??**) and (**??**), and

iv. markets clear according to (**??**) and (**??**).

Taken together, the household labor-leisure and intended bequest decisions in the last period of life show that the optimal labor supply and optimal intended bequests for age $s = E+S$ are each functions of individual holdings of savings, total bequests received, and the prices in that period $n_{j,E+S,t} = \phi\big(\hat{b}_{j,E+S,t}, \hat{BQ}_{j,t}, \hat{w}_t, r_t\big)$ and $\hat{b}_{j,E+S+1,t+1} = \psi\big(\hat{b}_{j,E+S,t}, \hat{BQ}_{j,t}, \hat{w}_t, r_t\big)$. These two decisions are characterized by final-age version of the static labor supply Euler equation (**??**) and the static intended bequests Euler equation (**??**). Households in their second-to-last period of life in period t have four decisions to make. They must choose how much to work this period $n_{j,E+S-1,t}$ and next period $n_{j,E+S,t+1}$, how much to save this period for next period $\hat{b}_{j,E+S,t+1}$, and how much to bequeath next period $\hat{b}_{j,E+S+1,t+2}$. The optimal responses for this individual are characterized by the $s = E+S-1$ and $s = E+S$ versions of the static Euler equations (**??**), the $s = E+S-1$ version of the intertemporal Euler equation (**??**), and the $s = E+S$ static bequest Euler equation (**??**), respectively.

Optimal savings in the second-to-last period of life $s = E+S-1$ is a function of the current savings as well as the total bequests received and prices in the current period and in the next period $\hat{b}_{j,E+S,t+1} = \psi\big(\hat{b}_{j,E+S-1,t}, \hat{BQ}_{j,t}, \hat{w}_t, r_t, \hat{BQ}_{j,t+1}, \hat{w}_{t+1}, r_{t+1}|\Omega\big)$ given beliefs Ω. As before, the optimal labor supply at age $s = E+S$ is a function of the next period's savings, bequests received, and prices $n_{j,E+S,t+1} = \phi\big(\hat{b}_{j,E+S,t+1}, \hat{BQ}_{j,t+1}, \hat{w}_{t+1}, r_{t+1}\big)$. But the optimal labor supply at age $s = E+S-1$ is a function of the current savings, current bequests received, and the current prices as well as the future bequests received and future prices because of the dependence on the savings decision in that same period $n_{j,E+S-1,t} = \phi\big(\hat{b}_{j,E+S-1,t}, \hat{BQ}_{j,t}, \hat{w}_t, r_t, \hat{BQ}_{j,t+1}, \hat{w}_{t+1}, r_{t+1}|\Omega\big)$ given beliefs Ω. By induction, we can show that the optimal labor supply, savings, and intended bequests functions for any individual with ability j, age s, and in period t is a function of current holdings of savings and the lifetime path of total bequests received and prices given beliefs Ω.

$$n_{j,s,t} = \phi\Big(\hat{b}_{j,s,t}, \big(\hat{BQ}_{j,v}, \hat{w}_v, r_v\big)_{v=t}^{t+S-s}|\Omega\Big) \quad \forall j,s,t \tag{5.5}$$

$$\hat{b}_{j,s+1,t+1} = \psi\Big(\hat{b}_{j,s,t}, \big(\hat{BQ}_{j,v}, \hat{w}_v, r_v\big)_{v=t}^{t+S-s}|\Omega\Big) \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{5.6}$$

If one knows the current distribution of households savings and intended bequests $\hat{\mathbf{\Gamma}}_t$ and has a beliefs function that predicts the law of motion over time for $\hat{\mathbf{\Gamma}}_t$, then one can predict time series for total bequests received $\hat{BQ}_{j,t}$, real wages $\hat{w}_t$ and real interest rates r_t necessary for solving each household's optimal decisions. Characteristic (i) in equilibrium definition **??** that individuals be able to forecast prices with perfect foresight over their lifetimes implies that each individual has correct information and beliefs about all the other individuals optimization problems and information. It also implies that the equilibrium allocations and prices are really just functions of the entire distribution of savings at a particular period, as well as a law of motion for that distribution of savings.

In equilibrium, the steady-state household labor supplies $\bar{n}_{j,s}$ for all j and s, the steady-state savings $\bar{b}_{j,E+S+1}$, the steady-state real wage $\bar{w}$, and the steady-state real rental rate $\bar{r}$ are simply functions of the steady-state distribution of savings $\bar{\Gamma}$. This is clear from the steady-state version of the capital market clearing condition (**??**) and the fact that aggregate labor supply is a function of the sum of exogenous efficiency units of labor in the labor market clearing condition (**??**). And the two firm first order conditions for the real wage $\hat{w}_t$ (**??**) and real rental rate r_t (**??**) are only functions of the stationary aggregate capital stock $\hat{K}_t$ and aggregate labor $\hat{L}_t$.

Figure 5.1: Equilibrium time path of K_t for $S = 80$ and $J = 7$

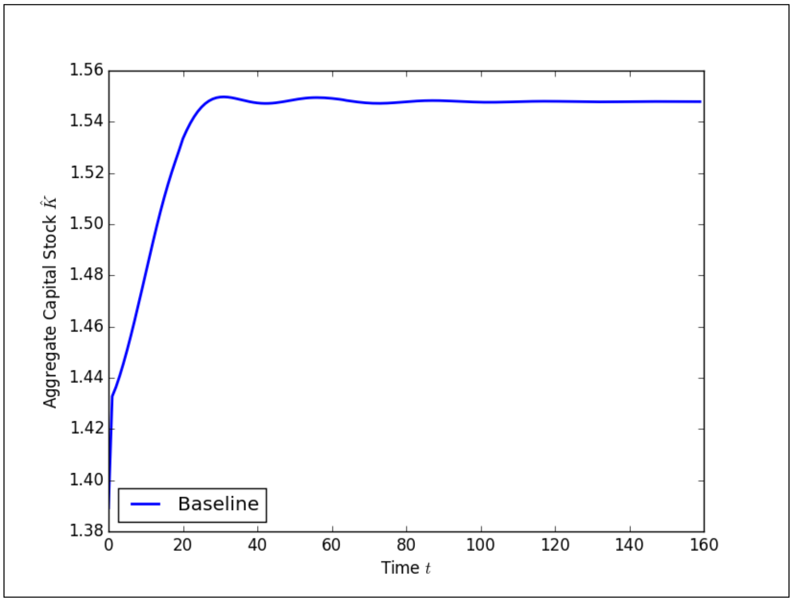
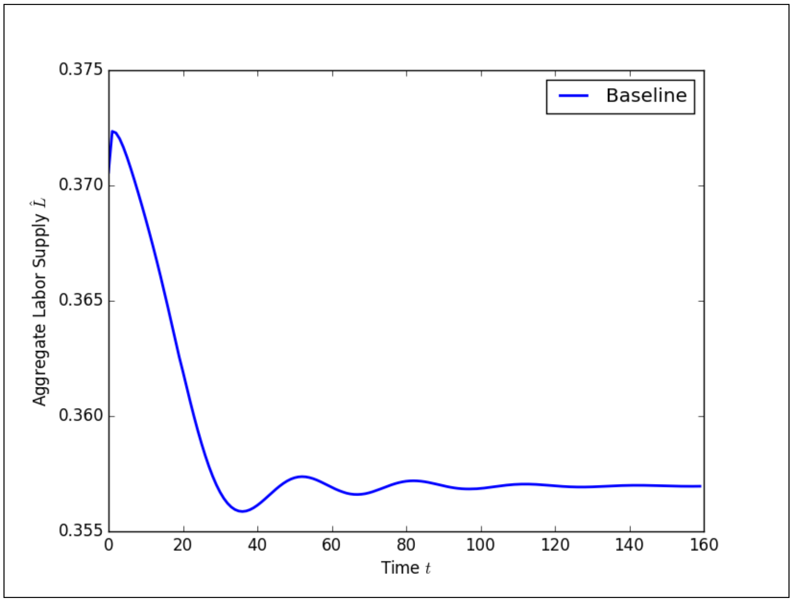


Figure 5.2: Equilibrium time path of L_t for $S = 80$ and $J = 7$



Chapter 6

Numerical Solution

When solving for the full transition path, we solve the model in two steps. First, we solve for the steady state prices and allocations. Next, we iterate backwards solving for prices and allocations along the transition path to the steady state. Here we define the solution to the model, starting with the steady-state solution.

6.1 Solving for stationary steady-state equilibrium

This section describes the solution method for the stationary steady-state equilibrium described in Definition **??**. To obtain the steady-state equilibrium, we do the following:

i. Use the techniques in Chapter **??** to solve for the steady-state population distribution vector $\bar{\boldsymbol{\omega}}$ of the exogenous population process.

ii. Choose an initial guess for the stationary steady-state wage rate, $\bar{w}$ and real interest rate, $\bar{r}$.

iii. With $\bar{w}$ and $\bar{r}$, use an unconstrained root finder to solve the $4 \times 2 \times M$ equations defining the steady-state version of the firms' problem. This will yield $\bar{I}_{m,c}$, $\bar{EL}_{m,c}$, $\bar{V}_{m,c}$, $\bar{p}_{m,c}$.

iv. Using $\bar{p}_{m,c}$ and the fixed coefficient matrix Π, we can determine $\bar{p}_i$, the price of consumption goods.

v. $\bar{p}_i$ and maximization of the consumer's subutility function imply $\bar{p}_s$, the price of the composite consumption good.

vi. Perform an unconstrained root finder that chooses $\bar{c}_{j,s}$ and $\bar{b}_{j,s+1}$ that solves the $2JS$ stationary steady-state Euler equations.

vii. Make sure none of the implied steady-state consumptions $\bar{c}_{j,s} + \sum_{i=1}^{I} c_{i,s}$ exceeds income.

- If consumption exceeds income, the individual can not afford the minimum required consumption amounts. We then...

viii. Given consumption demand and the fixed coefficient matrix Ξ, find the implied demand for output, X_m.

ix. Use the consumer's subutility function over corporate and non-corporate goods to determine demand for corporate and non-corporate production goods to get $X_{m,n}$ and $X_{m,nc}$.

x. Make sure that demand for these production goods matches the supply given firm's decisions in (iii).

xi. Make sure that none of the Euler errors is too large in absolute value for interior stationary steady-state values. A steady-state Euler error is the following, which is supposed to be close to zero for all j and s:

$$\frac{\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}}{(\bar{c}_{j,s})^{-\sigma}\left(\bar{w}e_{j,s}-\frac{\partial\bar{T}_{j,s}}{\partial\bar{n}_{j,s}}\right)}-1 \qquad \forall j \quad \text{and} \quad E+1\leq s\leq E+S \tag{6.1}$$

$$\frac{e^{-g_y\sigma}\left(\rho_s\chi^b\left(\bar{b}_{j,s+1}\right)^{-\sigma}+\beta(1-\rho_s)(\bar{c}_{j,s+1})^{-\sigma}\left[(1+\bar{r})-\frac{\partial\bar{T}_{j,s+1}}{\partial\bar{b}_{j,s+1}}\right]\right)}{(\bar{c}_{j,s})^{-\sigma}}-1 \qquad \forall j \quad \text{and} \quad E+1\leq s\leq E+S-1 \tag{6.2}$$

$$\frac{\chi^b e^{-g_y\sigma}(\bar{b}_{j,E+S+1})^{-\sigma}}{\left(\bar{c}_{j,E+S}\right)^{-\sigma}}-1 \quad \forall j \tag{6.3}$$

6.2 Solving for stationary non-steady-state equilibrium by time path iteration

This section outlines the benchmark time path iteration (TPI) method of **?**) for solving the stationary non-steady-state equilibrium transition path of the distribution of savings. TPI finds a fixed point for the transition path of the distribution of capital for a given initial state of the distribution of capital. The idea is that the economy is infinitely lived, even though the agents that make up the economy are not. Rather than recursively solving for equilibrium policy functions by iterating on individual value functions, one must recursively solve for the policy functions by iterating on the entire transition path of the endogenous objects in the economy (see **?**, ch. 17)).

The key assumption is that the economy will reach the steady-state equilibrium described in Definition **??** in a finite number of periods $T<\infty$ regardless of the initial state. Let $\hat{\mathbf{\Gamma}}_t$ represent the distribution of stationary savings at time t.

$$\hat{\mathbf{\Gamma}}_t\equiv\left\{\left\{\hat{b}_{j,s,t}\right\}_{j=1}^J\right\}_{s=E+2}^{E+S+1},\quad\forall t \tag{??}$$

In Section **??**, we describe how the stationary non-steady-state equilibrium time path of allocations and price is described by functions of the state $\hat{\mathbf{\Gamma}}_t$ and its law of motion. TPI starts the economy at any initial distribution of savings $\hat{\mathbf{\Gamma}}_1$ and solves for its equilibrium time path over T periods to the steady-state distribution $\bar{\mathbf{\Gamma}}_T$.

The first step is to assume an initial transition path for aggregate stationary capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$, aggregate stationary labor $\hat{\boldsymbol{L}}^i = \left\{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\right\}$, and total bequests received $\hat{\boldsymbol{BQ}}_j^i = \{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^i, ...\hat{BQ}_{j,T}^i\}$ for each ability type j such that T is sufficiently large to ensure that $\hat{\mathbf{\Gamma}}_T = \bar{\mathbf{\Gamma}}$, $\hat{K}_T^i(\mathbf{\Gamma}_T)$, $\hat{L}_T^i(\mathbf{\Gamma}_T) = \bar{L}\left(\bar{\mathbf{\Gamma}}\right)$, and $\hat{BQ}_{j,T}^i(\mathbf{\Gamma}_T) = \bar{BQ}_j\left(\bar{\mathbf{\Gamma}}\right)$ for all $t \geq T$. The superscript i is an index for the iteration number. The transition paths for aggregate capital and aggregate labor determine the transition paths for both the real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$ and the real return on investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. The time paths for the total bequests received also figure in each period's budget constraint and are determined by the distribution of savings and intended bequests.

The exact initial distribution of capital in the first period $\hat{\mathbf{\Gamma}}_1$ can be arbitrarily chosen as long as it satisfies the stationary capital market clearing condition (**??**).

$$\hat{K}_1 = \frac{1}{1+\tilde{g}_{n,1}}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\hat{\omega}_{s-1,0}\lambda_j\hat{b}_{j,s,1} \tag{6.1}$$

Simiilarly, each initial value of total bequests received $\hat{BQ}_{j,1}^i$ must be consistent with the initial distribution of capital through the stationary version of (**??**).

$$\hat{BQ}_{j,1} = \frac{(1+r_1)\lambda_j}{1+\tilde{g}_{n,1}}\sum_{s=E+1}^{E+S}\rho_s\hat{\omega}_{s,0}\hat{b}_{j,s+1,1} \quad \forall j \tag{6.2}$$

However, this is not the case with $\hat{L}_1^i$. Its value will be endogenously determined in the same way the K_2^i is. For this reason, a logical initial guess for the time path of aggregate labor is the steady state in every period $L_t^1 = \bar{L}$ for all $1 \leq t \leq T$.

It is easiest to first choose the initial distribution of savings $\hat{\mathbf{\Gamma}}_1$ and then choose an initial aggregate capital stock $\hat{K}_1^i$ and initial total bequests received $\hat{BQ}_{j,1}^i$ that correspond to that distribution. As mentioned earlier, the only other restrictions on the initial transition paths for aggregate capital, aggregate labor, and total bequests received is that they equal their steady-state levels $\hat{K}_T^i = \bar{K}\left(\bar{\mathbf{\Gamma}}\right)$, $\hat{L}_T^i = \bar{L}\left(\bar{\mathbf{\Gamma}}\right)$, and $\hat{BQ}_{j,T}^i = \bar{BQ}_j\left(\bar{\mathbf{\Gamma}}\right)$ by period T. **?**) have shown that the initial guess for the aggregate capital stocks $\hat{K}_t^i$ for periods $1 < t < T$ can take on almost any positive values satisfying the constraints above and still have the time path iteration converge.

Given the initial savings distribution $\hat{\mathbf{\Gamma}}_1$ and the transition paths of aggregate capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$, aggregate labor $\hat{\boldsymbol{L}}^i = \left\{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\right\}$, and total bequests received $\hat{\boldsymbol{BQ}}_j^i = \left\{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^i, ...\hat{BQ}_{j,T}^i\right\}$, as well as the resulting real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$, and real return to savings $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, one can solve for the period-1 optimal labor supply and intended bequests for each type j of $s = E+S$-aged agents in the last period of their lives $n_{j,E+S,1} = \phi_{j,E+S}(\hat{b}_{j,E+S,1}, \hat{BQ}_{j,E+S,1}, \hat{w}_1, r_1)$ and $\hat{b}_{j,E+S+1,2} = \psi_{j,E+S}(\hat{b}_{j,E+S,1}, \hat{BQ}_{j,E+S,1}, \hat{w}_1, r_1)$ using his two $s = E+S$ static Euler equations (**??**) and (**??**).

$$
\begin{split}
&(\hat{c}_{j,E+S,1})^{-\sigma}\left(\hat{w}_1^i e_{j,E+S} - \frac{\partial \hat{T}_{j,E+S,1}}{\partial n_{j,E+S,1}}\right) = ... \\
&\qquad \chi^n_{E+S}\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,E+S,1}}{\tilde{l}}\right)^{v-1}\left[1 - \left(\frac{n_{j,E+S,1}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}} \quad \forall j \\
&\text{where} \quad \hat{c}_{j,E+S,1} = ... \\
&\qquad \left(1 + r_1^i\right)\hat{b}_{j,E+S,1} + \hat{w}_1^i e_{j,E+S} n_{j,E+S,1} + \frac{\hat{BQ}_{j,1}}{\lambda_j} - e^{g_y}\hat{b}_{j,E+S+1,2} - \hat{T}_{j,E+S,1} \\
&\text{and} \quad \frac{\partial \hat{T}_{j,E+S,1}}{\partial n_{j,E+S,1}} = ... \\
&\qquad \hat{w}_1^i e_{j,E+S}\left[\tau^I\left(F\hat{a}_{j,E+S,1}\right) + \frac{\hat{a}_{j,E+S,1} CDF\left[2A(F\hat{a}_{j,E+S,1}) + B\right]}{\left[A(F\hat{a}_{j,E+S,1})^2 + B(F\hat{a}_{j,E+S,1}) + C\right]^2} + \tau^P\right]
\end{split}
\tag{6.3}
$$

$$
(\hat{c}_{j,E+S,1})^{-\sigma} = \chi^b e^{-g_y\sigma}(\hat{b}_{j,E+S+1,2})^{-\sigma} \quad \forall j \tag{6.4}
$$

Note that this is simply two equations (**??**) and (**??**) and two unknowns $n_{j,E+S,1}$ and $\hat{b}_{j,E+S+1,2}$.

We then solve the problem for all j types of $E+S-1$-aged individuals in period $t = 1$, each of which entails labor supply decisions in the current period $n_{j,E+S-1,1}$ and in the next period $n_{j,E+S,2}$, a savings decision in the current period for the next period $\hat{b}_{j,E+S,2}$ and an intended bequest decision in the last period $\hat{b}_{j,E+S+1,3}$. The labor supply decision in the initial period and the savings period in the initial period for the next period for each type j of $E+S-1$-aged individuals are policy functions of the current savings and the total bequests received and prices in this period and the next $\hat{b}_{j,E+S,2} = \psi_{j,E+S-1}(\hat{b}_{j,E+S-1,1}, \{\hat{BQ}_{j,t}, \hat{w}_t, r_t\}_{t=1}^2)$ and $\hat{n}_{j,E+S-1,1} = \phi_{j,E+S-1}(\hat{b}_{j,E+S-1,1}, \{\hat{BQ}_{j,t}, \hat{w}_t, r_t\}_{t=1}^2)$. The labor supply and intended bequests decisions in the next period are simply functions of the savings, total bequests received, and prices in that period $\hat{n}_{j,E+S,2} = \phi_{j,E+S}(\hat{b}_{j,E+S,2}, \hat{BQ}_{j,2}, \hat{w}_2, r_2)$ and $\hat{b}_{j,E+S+1,3} = \psi_{j,E+S}(\hat{b}_{j,E+S,2}, \hat{BQ}_{j,2}, \hat{w}_2, r_2)$. These four functions are characterized by the following versions of equations (**??**), (**??**), and (**??**).

$$
\begin{split}
&(\hat{c}_{j,E+S-1,1})^{-\sigma}\left(\hat{w}_1^i e_{j,E+S-1} - \frac{\partial \hat{T}_{j,E+S-1,1}}{\partial n_{j,E+S-1,1}}\right) = ... \\
&\qquad \chi^n_{E+S-1}\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,E+S-1,1}}{\tilde{l}}\right)^{v-1}\left[1 - \left(\frac{n_{j,E+S-1,1}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}} \quad \forall j
\end{split}
\tag{6.5}
$$

$$(\hat{c}_{j,E+S-1,1})^{-\sigma} = ...$$
$$e^{-g_y\sigma}\left(\rho_{E+S-1}\chi^b\big(\hat{b}_{j,E+S,2}\big)^{-\sigma} + \beta(1-\rho_{E+S-1})(\hat{c}_{j,E+S,2})^{-\sigma}\left[(1+r^i_2) - \frac{\partial T_{j,E+S,2}}{\partial b_{j,E+S,2}}\right]\right)$$
$$\forall j$$

$$\text{where}\quad \frac{\partial T_{j,E+S,2}}{\partial b_{j,E+S,2}} = ...$$
$$r^i_2\left(\tau^I(F\hat{a}_{j,E+S,2}) + \frac{F\hat{a}_{j,E+S,2}CD\left[2A(F\hat{a}_{j,E+S,2}) + B\right]}{\left[A(F\hat{a}_{j,E+S,2})^2 + B(F\hat{a}_{j,E+S,2}) + C\right]^2}\right)...$$
$$\tau^W(\hat{b}_{j,E+S,2}) + \frac{\hat{b}_{j,E+S,2}PHM}{\left(H\hat{b}_{j,E+S,2} + M\right)^2} \tag{6.6}$$

$$(\hat{c}_{j,E+S,2})^{-\sigma}\left(\hat{w}^i_2 e_{j,E+S} - \frac{\partial \hat{T}_{j,E+S,2}}{\partial n_{j,E+S,2}}\right) = ...$$
$$\chi^n_{E+S}\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,E+S,2}}{\tilde{l}}\right)^{v-1}\left[1 - \left(\frac{n_{j,E+S,2}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}} \quad \forall j \tag{6.7}$$

$$(\hat{c}_{j,E+S,2})^{-\sigma} = \chi^b e^{-g_y\sigma}(\hat{b}_{j,E+S+1,3})^{-\sigma} \quad \forall j \tag{6.8}$$

Note that this is four equations (**??**), (**??**), (**??**), and (**??**) and four unknowns $n_{j,E+S-1,1}$, $\hat{b}_{j,E+S,2}$, $n_{j,E+S,2}$, and $\hat{b}_{j,E+S+1,3}$.

This process is repeated for every age of household alive in $t = 1$ down to the age $s = E + 1$ household at time $t = 1$. Each of these households j solves the full set of remaining $S - s + 1$ labor supply decisions, $S - s$ savings decisions, and one intended bequest decision at the end of life. After the full set of lifetime decisions has been solved for all the households alive at time $t = 1$, each ability j household born in period $t \geq 2$ can be solved for, the solution to which is characterized by the following full set of Euler equations analogous to (**??**), (**??**), and (**??**).

$$(\hat{c}_{j,s,t})^{-\sigma}\left(\hat{w}^i_t e_{j,s} - \frac{\partial \hat{T}_{j,s,t}}{\partial n_{j,s,t}}\right) = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{v-1}\left[1 - \left(\frac{n_{j,s,t}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}} \tag{6.9}$$
$$\forall j \quad \text{and} \quad E+1 \leq s \leq E+S \quad \text{and} \quad t \geq 2$$

$$(\hat{c}_{j,s,t})^{-\sigma} = ...$$
$$e^{-g_y\sigma}\left(\rho_s\chi^b\big(\hat{b}_{j,s+1,t+1}\big)^{-\sigma} + \beta(1-\rho_s)(\hat{c}_{j,s+1,t+1})^{-\sigma}\left[(1+r^i_{t+1}) - \frac{\partial T_{j,s+1,t+1}}{\partial b_{j,s+1,t+1}}\right]\right) \tag{6.10}$$
$$\forall j \quad \text{and} \quad E+1 \leq s \leq E+S-1 \quad \text{and} \quad t \geq 2$$

$$(\hat{c}_{j,E+S,t})^{-\sigma} = \chi^b e^{-g_y\sigma}(\hat{b}_{j,E+S+1,t+1})^{-\sigma} \quad \forall j \quad \text{and} \quad t \geq 2 \tag{6.11}$$

For each household of ability type j entering the economy in period $t \geq 1$, the entire set of $2S$ lifetime decisions is characterized by the $2S$ equations represented in (**??**), (**??**), and (**??**).

We can then solve for the entire lifetime of savings and labor supply decisions for each age $s = 1$ individual in periods $t = 2, 3, ...T$. The central part of the schematic diagram in Figure **??** shows how this process is done in order to solve for the equilibrium time path of the economy from period $t = 1$ to T. Note that for each full lifetime savings and labor supply path solved for an individual born in period $t \geq 2$, we can solve for the aggregate capital stock and total bequests received implied by those savings decisions $\hat{\boldsymbol{K}}^{i'}$ and $\hat{\boldsymbol{BQ}}_j^{i'}$ and aggregate labor implied by those labor supply decisions $\hat{\boldsymbol{L}}^{i'}$.

Once the set of lifetime saving and labor supply decisions has been computed for all individuals alive in $1 \leq t \leq T$, we use the household decisions to compute a new implied time path of the aggregate capital stock and aggregate labor. The implied paths of the aggregate capital stock $\hat{\boldsymbol{K}}^{i'} = \{\hat{K}_1^i, \hat{K}_2^{i'}, ...\hat{K}_T^{i'}\}$, aggregate labor $\hat{\boldsymbol{L}}^{i'} = \{\hat{L}_1^i, \hat{L}_2^{i'}, ...\hat{L}_T^{i'}\}$, and total bequests received $\hat{\boldsymbol{BQ}}_j^{i'} = \{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^{i'}, ...\hat{BQ}_{j,T}^{i'}\}$ in general do not equal the initial guessed paths $\hat{\boldsymbol{K}}^i = \{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\}$, $\hat{\boldsymbol{L}}^i = \{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\}$, and $\hat{\boldsymbol{BQ}}_j^i = \{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^i, ...\hat{BQ}_{j,T}^i\}$ used to compute the household savings and labor supply decisions $\hat{\boldsymbol{K}}^{i'} \neq \hat{\boldsymbol{K}}^i$, $\hat{\boldsymbol{L}}^{i'} \neq \hat{\boldsymbol{L}}^i$, and $\hat{\boldsymbol{BQ}}_j^{i'} \neq \hat{\boldsymbol{BQ}}_j^i$.

Let $\|\cdot\|$ be a norm on the space of time paths of the aggregate capital stock $\hat{\boldsymbol{K}} \in \mathcal{K} \subset \mathbb{R}_{++}^T$, aggregate labor supply $\hat{\boldsymbol{L}} \in \mathcal{L} \subset \mathbb{R}_{++}^T$, and J paths of total bequests received $\hat{\boldsymbol{BQ}}_j \in \mathcal{B} \subset \mathbb{R}_{++}^T$. Then the fixed point necessary for the equilibrium transition path from Definition **??** has been found when the distance between these $J+2$ paths is arbitrarily close to zero.

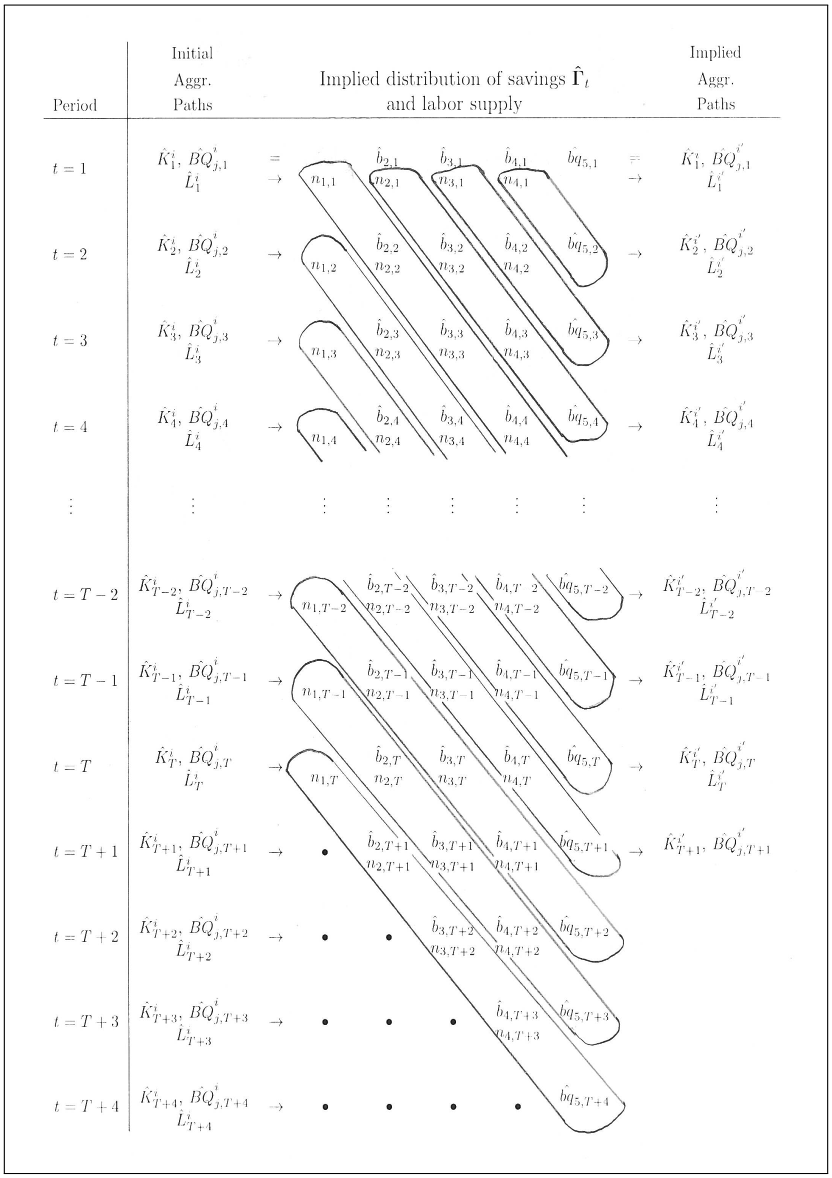
$$\left\|\left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}, \left\{\hat{\boldsymbol{BQ}}_j^{i'}\right\}_{j=1}^J\right] - \left[\hat{\boldsymbol{K}}^i, \hat{\boldsymbol{L}}^i, \left\{\hat{\boldsymbol{BQ}}_j^i\right\}_{j=1}^J\right]\right\| \leq \varepsilon \quad \text{for} \quad \varepsilon > 0 \tag{6.12}$$

If the fixed point has not been found $\left\|\left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}, \left\{\hat{\boldsymbol{BQ}}_j^{i'}\right\}_{j=1}^J\right] - \left[\hat{\boldsymbol{K}}^i, \hat{\boldsymbol{L}}^i, \left\{\hat{\boldsymbol{BQ}}_j^i\right\}_{j=1}^J\right]\right\| > \varepsilon$, then new transition paths for the aggregate capital stock and aggregate labor are generated as a convex combination of $\left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}, \left\{\hat{\boldsymbol{BQ}}_j^{i'}\right\}_{j=1}^J\right]$ and $\left[\hat{\boldsymbol{K}}^i, \hat{\boldsymbol{L}}^i, \left\{\hat{\boldsymbol{BQ}}_j^i\right\}_{j=1}^J\right]$.

$$\begin{aligned}
\hat{\boldsymbol{K}}^{i+1} &= \nu\hat{\boldsymbol{K}}^{i'} + (1-\nu)\hat{\boldsymbol{K}}^i \\
\hat{\boldsymbol{L}}^{i+1} &= \nu\hat{\boldsymbol{L}}^{i'} + (1-\nu)\hat{\boldsymbol{L}}^i \\
\hat{\boldsymbol{BQ}}_1^{i+1} &= \nu\hat{\boldsymbol{BQ}}_1^{i'} + (1-\nu)\hat{\boldsymbol{BQ}}_1^i \qquad \text{for} \quad \nu \in (0,1] \\
&\vdots \\
\hat{\boldsymbol{BQ}}_J^{i+1} &= \nu\hat{\boldsymbol{BQ}}_J^{i'} + (1-\nu)\hat{\boldsymbol{BQ}}_J^i
\end{aligned} \tag{6.13}$$

This process is repeated until the initial transition paths for the aggregate capital stock, aggregate labor, and total bequests received are consistent with the transition paths implied by those beliefs and household and firm optimization.

Figure 6.1: Diagram of TPI solution method within each iteration for $S = 4$ and $J = 1$



In essence, the TPI method iterates on individual beliefs about the time path of prices represented by a time paths for the aggregate capital stock $\hat{\boldsymbol{K}}^i$, aggregate labor $\hat{\boldsymbol{L}}^i$, and total bequests received $\hat{\boldsymbol{BQ}}_j^i$ until a fixed point in beliefs is found that are consistent with the transition paths implied by optimization based on those beliefs.

The following are the steps for computing a stationary non-steady-state equilibrium time path for the economy.

i. Input all initial parameters. See Table **??**.

   (a) The value for T at which the non-steady-state transition path should have converged to the steady state should be at least as large as the number of periods it takes the population to reach its steady state $\bar{\boldsymbol{\omega}}$ as described in Appendix **??**.

ii. Choose an initial distribution of savings and intended bequests $\hat{\boldsymbol{\Gamma}}_1$ and then calculat the initial state of the stationarized aggregate capital stock $\hat{K}_1$ and total bequests received $\hat{BQ}_{j,1}$ consistent with $\hat{\boldsymbol{\Gamma}}_1$ according to (**??**) and (**??**).

   (a) Note that you must have the population weights from the previous period $\hat{\omega}_{s,0}$ and the growth rate between period 0 and period 1 $\tilde{g}_{n,1}$to calculate $\hat{BQ}_{j,1}$.

iii. Conjecture transition paths for the stationarized aggregate capital stock $\hat{\boldsymbol{K}}^1 = \{\hat{K}_t^1\}_{t=1}^\infty$, stationarized aggregate labor $\hat{\boldsymbol{L}}^1 = \{\hat{L}_t^1\}_{t=1}^\infty$, and total bequests received $\hat{\boldsymbol{BQ}}_j^1 = \{\hat{BQ}_{j,t}^1\}_{t=1}^\infty$ where the only requirements are that $\hat{K}_1^i$ and $\hat{BQ}_{j,1}^i$ are functions of the initial distribution of savings $\hat{\boldsymbol{\Gamma}}_1$ for all i is your initial state and that $\hat{K}_t^i = \bar{K}$, $\hat{L}_t^i = \bar{L}$, and $\hat{BQ}_{j,t}^i = \bar{BQ}_j$ for all $t \geq T$. The conjectured transition paths of the aggregate capital stock $\hat{\boldsymbol{K}}^i$ and aggregate labor $\hat{\boldsymbol{L}}^i$ imply specific transition paths for the real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_t^i\}_{t=1}^\infty$ and the real interest rate $\boldsymbol{r}^i = \{r_t^i\}_{t=1}^\infty$ through expressions (**??**) and (**??**).

   (a) An intuitive choice for the time path of aggregate labor is the steady-state in every period $\hat{L}_t^1 = \bar{L}$ for all t.

iv. With the conjectured transition paths $\hat{\boldsymbol{w}}^i$, $\boldsymbol{r}^i$, and $\hat{\boldsymbol{BQ}}_j^i$ one can solve for the lifetime policy functions of each household alive at time $1 \leq t \leq T$ using the systems of Euler equations of the form (**??**), (**??**), and (**??**) and following the diagram in Figure **??**.

v. Use the implied distribution of savings and labor supply in each period (each row of $\hat{b}_{j,s,t}$ and $n_{j,s,t}$ in Figure **??**) to compute the new implied time paths for the aggregate capital stock $\hat{\boldsymbol{K}}^{i'} = \{\hat{K}_1^i, \hat{K}_2^{i'}, ...\hat{K}_T^{i'}\}$, aggregate labor supply $\hat{\boldsymbol{L}}^{i'} = \{\hat{L}_1^i, \hat{L}_2^{i'}, ...\hat{L}_T^{i'}\}$, and total bequests received $\hat{\boldsymbol{BQ}}_j^{i'} = \{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^{i'}, ...\hat{BQ}_{j,T}^{i'}\}$.

vi. Check the distance between the two sets time paths.

$$\left\| \left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}, \left\{\hat{\boldsymbol{BQ}}_j^{i'}\right\}_{j=1}^J\right] - \left[\hat{\boldsymbol{K}}^{i}, \hat{\boldsymbol{L}}^{i}, \left\{\hat{\boldsymbol{BQ}}_j^{i}\right\}_{j=1}^J\right] \right\|$$

(a) If the distance between the initial time paths and the implied time paths is less-than-or-equal-to some convergence criterion $\varepsilon > 0$, then the fixed point has been achieved and the equilibrium time path has been found (**??**).

(b) If the distance between the initial time paths and the implied time paths is greater than some convergence criterion $\|\cdot\| > \varepsilon$, then update the guess for the time paths according to (**??**) and repeat steps (4) through (6) until a fixed point is reached.

Chapter 7

Miscellaneous

7.1 Characteristics of exogenous population growth assumptions

In this appendix, we describe in detail the exogenous population growth assumptions in the model and their implications. In Section **??**, we define the laws of motion for the population of each cohort $\omega_{s,t}$ to be the following.

$$\begin{aligned} \omega_{1,t+1} &= \sum_{s=1}^{E+S} f_s \omega_{s,t} \quad \forall t \\ \omega_{s+1,t+1} &= (1 + i_s - \rho_s)\omega_{s,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{??}$$

We can transform the nonstationary equations in (**??**) into stationary laws of motion by dividing both sides by the total populations N_t and N_{t+1} in both periods,

$$\begin{aligned} \hat{\omega}_{1,t+1} &= \frac{\sum_{s=1}^{E+S} f_s \hat{\omega}_{s,t}}{1 + g_{n,t+1}} \quad \forall t \\ \hat{\omega}_{s+1,t+1} &= \frac{(1 + \phi_s - \rho_s)\hat{\omega}_{s,t}}{1 + g_{n,t+1}} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{7.1}$$

where $\hat{\omega}_{s,t}$ is the percent of the total population in age cohort s and the population growth rate $g_{n,t+1}$ between periods t and $t+1$ is defined in (**??**),

$$\begin{bmatrix} \hat{\omega}_{1,t+1} \\ \hat{\omega}_{2,t+1} \\ \hat{\omega}_{2,t+1} \\ \vdots \\ \hat{\omega}_{E+S-1,t+1} \\ \hat{\omega}_{E+S,t+1} \end{bmatrix} = \frac{1}{1+g_{n,t+1}} \times ...$$

$$\begin{bmatrix} f_1 & f_2 & f_3 & \dots & f_{E+S-1} & f_{E+S} \\ 1+i_1-\rho_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1+i_2-\rho_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1+i_3-\rho_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1+i_{E+S-1}-\rho_{E+S-1} & 0 \end{bmatrix} \begin{bmatrix} \hat{\omega}_{1,t} \\ \hat{\omega}_{2,t} \\ \hat{\omega}_{2,t} \\ \vdots \\ \hat{\omega}_{E+S-1,t} \\ \hat{\omega}_{E+S,t} \end{bmatrix} \tag{7.2}$$

where we restrict $1+i_s-\rho_s \geq 0$ for all s.

We write (**??**) in matrix notation as the following.

$$\hat{\boldsymbol{\omega}}_{t+1} = \frac{1}{1+g_{n,t+1}} \boldsymbol{\Omega} \hat{\boldsymbol{\omega}}_t \quad \forall t \tag{7.3}$$

The stationary steady state population distribution $\bar{\boldsymbol{\omega}}$ is the eigenvector $\boldsymbol{\omega}$ with eigenvalue $(1+\bar{g}_n)$ of the matrix $\boldsymbol{\Omega}$ that satisfies the following version of (**??**).

$$(1+\bar{g}_n)\bar{\boldsymbol{\omega}} = \boldsymbol{\Omega}\bar{\boldsymbol{\omega}} \tag{7.4}$$

Proposition 1. There exists a unique positive real eigenvector $\bar{\omega}$ of the matrix $\boldsymbol{\Omega}$, and it is a stable equilibrium.

Proof. First, note that the matrix $\boldsymbol{\Omega}$ is square and non-negative. This is enough for a general version of the Perron-Frobenius Theorem to state that a positive real eigenvector exists with a positive real eigenvalue. This is not yet enough for uniqueness. For it to be unique by a version of the Perron-Fobenius Theorem, we need to know that the matrix is irreducible. This can be easily shown. The matrix is of the form

$$\boldsymbol{\Omega} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & * & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & * & 0 \end{bmatrix}$$

Where each * is strictly positive. It is clear to see that taking powers of the matrix causes the sub-diagonal positive elements to be moved down a row and another row of positive entries is added at the top. None of these go to zero since the elements were all non-negative to begin with.

$$\mathbf{\Omega^2} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & * & 0 & 0 \end{bmatrix}; \quad \mathbf{\Omega^{S+E-1}} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ * & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{\Omega^{S+E}} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \end{bmatrix}$$

Existence of an $m \in \mathbb{N}$ such that $(\mathbf{\Omega^m})_{ij} \neq 0 \quad (> 0)$ is one of the definitions of an irreducible (primitive) matrix. It is equivalent to saying that the directed graph associated with the matrix is strongly connected. Now the Perron-Frobenius Theorem for irreducible matrices gives us that the equilibrium vector is unique.

We also know from that theorem that the eigenvalue associated with the positive real eigenvector will be real and positive. This eigenvalue, p, is the Perron eigenvalue and it is the steady state population growth rate of the model. By the PF Theorem for irreducible matrices, $|\lambda_i| \leq p$ for all eigenvalues λ_i and there will be exactly h eigenvalues that are equal, where h is the period of the matrix. Since our matrix $\mathbf{\Omega}$ is aperiodic, the steady state growth rate is the unique largest eigenvalue in magnitude. This implies that almost all initial vectors will converge to this eigenvector under iteration. □

For a full treatment and proof of the Perron-Frobenius Theorem, see **?**). Because the population growth process is exogenous to the model, we calibrate it to annual age data for age years $s = 1$ to $s = 100$. As is shown in Figure **??**, period $s = 1$ corresponds to the first year of life between birth and when an individual turns one year old.

Our initial population distribution $\{\omega_{s,1}\}_{s=1}^{100}$ in Figure **??** comes from **?**) population estimates for both sexes for 2013. The fertility rates $\{f_s\}_{s=1}^{100}$ in Figure **??** come from **?**, Table 1). The mortality rates $\{\rho_s\}_{s=1}^{99}$ in Figure **??** come from the 2010 death probabilities in **?**). We enforce a strict maximum age mortality rate of $\rho_{100} = 1$ in our model.

The immigration rates $\{i_s\}_{s=1}^{99}$ in Figure **??** are essentially residuals. We take total population for two consecutive years N_t and N_{t+1} and the population distribution

Figure 7.1: Correspondence of model timing to data timing for model periods of one year

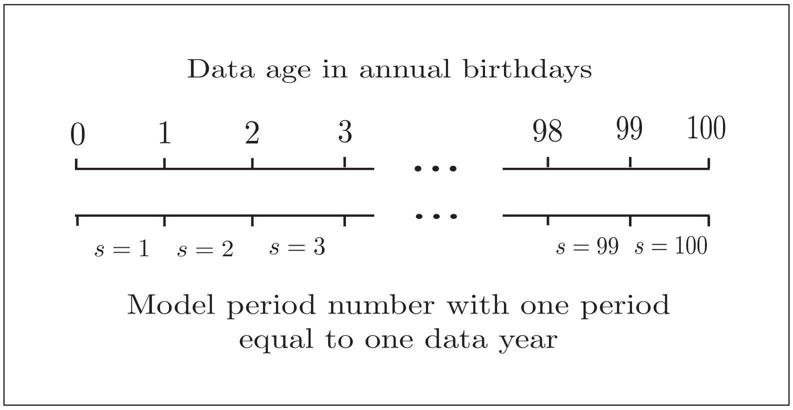


Figure 7.2: Initial population distribution $\omega_{s,1}$ by year, $1 \leq s \leq 100$

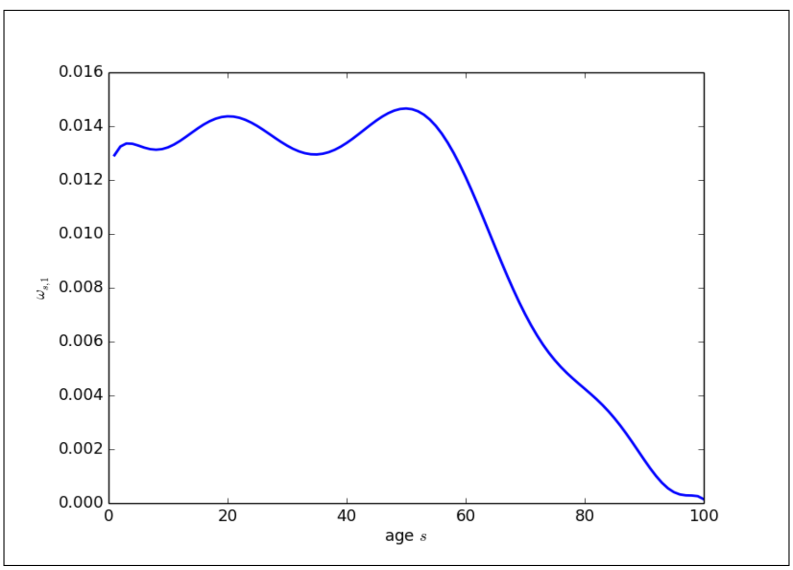


Figure 7.3: Fertility rates f_s by year, $1 \leq s \leq 100$

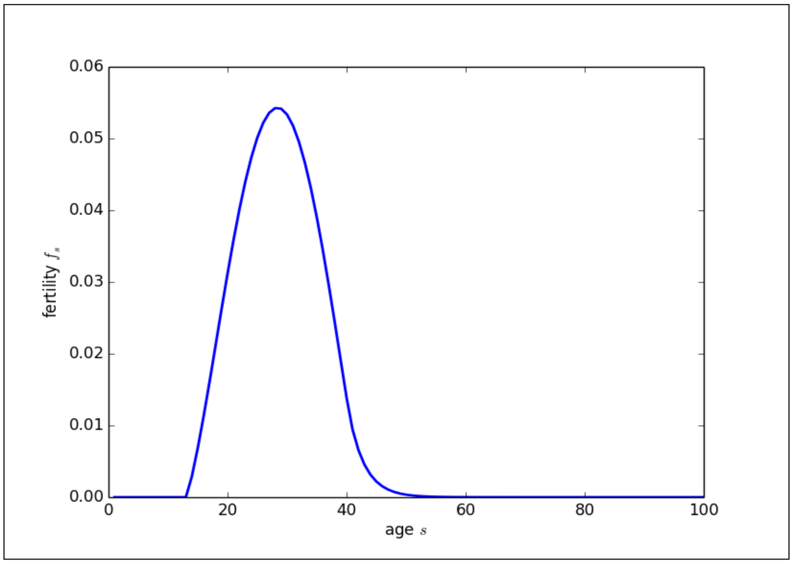
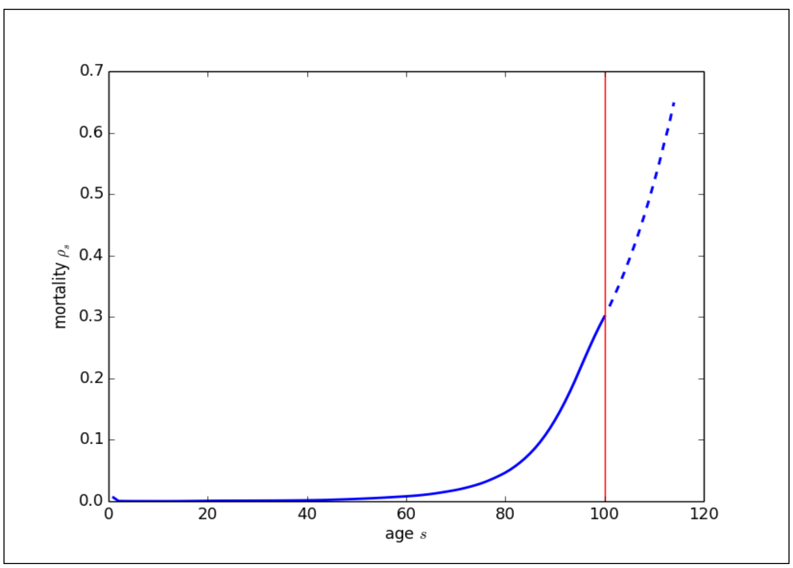


Figure 7.4: Mortality rates ρ_s by year, $1 \leq s \leq 100$



by age in both of those years $\boldsymbol{\omega}_t$ and $\boldsymbol{\omega}_{t+1}$from the **?**) data. We then deduce the immigration rates $\{i_s\}_{s=1}^{99}$ using equation (**??**). We do this for three consecutive sets of years, so that our calibrated immigration rates by age are the average of our three years of deduced rates from the data for each age.

Figure 7.5: Immigration rates i_s by year, $1 \leq s \leq 100$

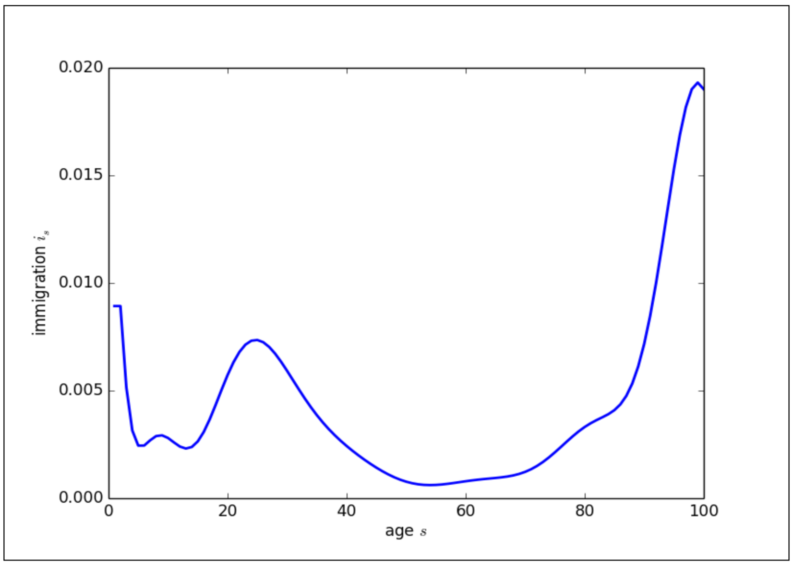


Figure **??** shows the predicted time path of the total population N_t given $\omega_{s,1}$ f_s, i_s, and ρ_s. Notice that the population approaches a constant growth rate. This is a result of the stationary population percent distribution $\bar{\boldsymbol{\omega}}$ eventually being reached. Figure **??** shows the steady-state population percent distribution by age $\bar{\boldsymbol{\omega}}$.

Figure 7.6: Forecast time path of population growth rate $g_{n,t}$

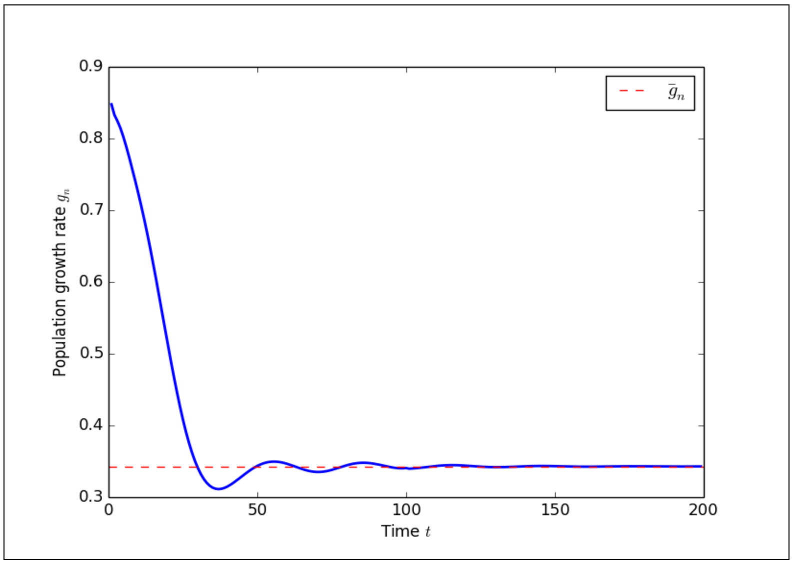
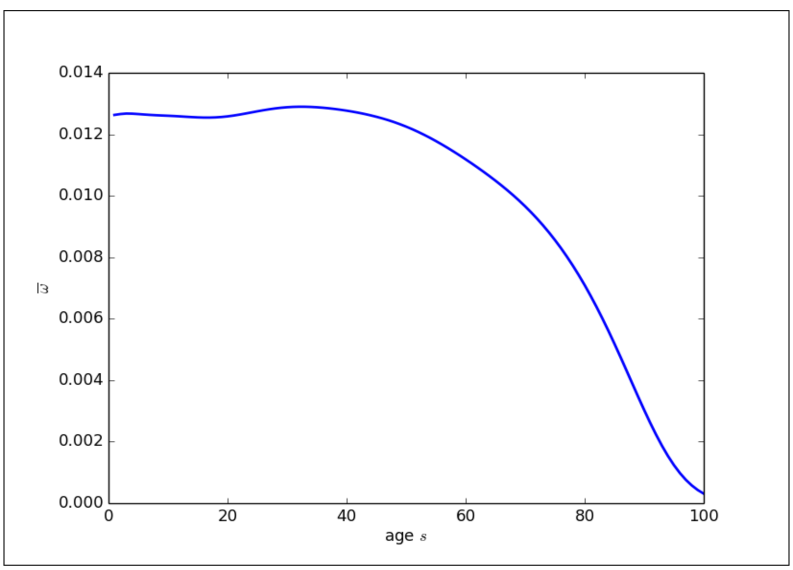


Figure 7.7: Steady-state population percent distribution by age $\bar{\omega}$



7.2 Incorporating Feedbacks with Micro Tax Simulations

Follow this algorithm:

- Period 1
  - Use current IRS public use sample.
  - Run the following within-period routine
    * Do the static tax analysis of this sample, save the results
    * Summarize the public use sample by aggregating into bins over age and earnings ability
    * Use this as a starting point for the dynamic macro model
    * Get values for fundamental interest rates and effective wages for next period
- Period 2
  - Age the public use data demographically by one year.
  - Let wages and interest rates rise by the amounts predicted in the macro model.
  - Rerun the within-period routine
- Iterate over periods until end of forecast period is reached.

7.3 Calibration

7.3.1 Tax Bend Points

We use IRS data which summarizes individual tax returns for 2011 by 19 income categories and 4 filing statuses. For each filing status we fit the mapping from reported income into adjusted gross income (AGI) using a sufficiently high-order polynomial. We then use this function to solve for the income level which corresponds to each of the five bend points in the tax code for each filing type.

We then fit a bivariate probability density function over income and filing type from the data. For each bendpoint we calculate the probability density at that bendpoint and use these as weights in a weighted average over filing types to generate an aggregate bendpoint.

Table 7.1: AGI and Income Bend Points

AGI Bend Points

Tax rate	Married Joint	Married Separate	Head of Household	Single
10%	17,400	8700	12,400	8700
15%	70,700	35,350	47,350	35,350
25%	142,700	71,350	122,300	85,650
28%	217,450	108,725	198,050	178,650
33%	388,350	194,175	388,350	388,350

Corresponding Reported Income Bendpoints

Tax rate	Married Joint	Married Separate	Head of Household	Single
0%	5850	91	756	1435
10%	22,932	8591	12,911	9956
15%	75,181	34,592	47,023	36,021
25%	145,866	69,768	120,200	85,244
28%	219,162	106,245	194,176	176,270
33%	386,798	189,674	380,043	381,524

Table 7.2: Aggregated Bend Points

Tax rate	Bend Point
0%	2889
10%	15,116
15%	52,580
25%	114,552
28%	196,201
33%	380,657