Consumers Problem with Many Goods

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The consumer's maximization problem is:

$$
\begin{aligned}
&U_{j,s,t} = \sum_{u=0}^{E+S-s} \beta^u \left[\prod_{v=s-1}^{s+u-1} (1-\rho_v)\right] u\left(c_{j,s+u,t+u}, n_{j,s+u,t+u}, b_{j,s+u+1,t+u+1}\right) \\
&\text{where} \quad \rho_{s-1} = 0 \\
&\text{and} \quad u\left(c_{j,s,t}, n_{j,s,t}, b_{j,s+1,t+1}\right) = \frac{\left(c_{j,s,t}\right)^{1-\sigma}-1}{1-\sigma}... \\
&\qquad + e^{g_y t(1-\sigma)}\chi^n_s\left(b\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^v\right]^{\frac{1}{v}} + k\right) + \rho_s \chi^b \frac{\left(b_{j,s+1,t+1}\right)^{1-\sigma}-1}{1-\sigma} \\
&\text{and} \quad c_{j,s,t} = \prod_{i=1}^{I}\left(c_{i,j,s,t} - \bar{c}_{i,s}\right)^{\alpha_i}; \quad \sum_{i=1}^{I}\alpha_i = 1 \\
&\qquad \forall j,t \quad \text{and } E+1 \leq s \leq E+S
\end{aligned}
\tag{1}
$$

They maximize subject to the following budget constraint.

$$
\begin{aligned}
&\sum_{i=1}^{I} p_{i,t} c_{i,j,s,t} + b_{j,s+1,t+1} \leq \left(1+r_t\right) b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \frac{BQ_{j,t}}{\lambda_j \tilde{N}_t} - T_{j,s,t} \\
&\qquad \text{where} \quad b_{j,s,1} = 0 \\
&\qquad \text{for} \quad E+1 \leq s \leq E+S \quad \forall j,t
\end{aligned}
\tag{2}
$$

We set up a Lagrangian and solve by taking derivatives with respect to $\{c_{i,j,s,t}, n_{j,s,t+u}, b_{j,s,t+1}\}$ for all i, j, s and t.

With respect to each consumption good i:

$$
\begin{aligned}
\frac{\partial U}{\partial c_{i,j,s+u,t+u}} = &\beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right]\left[\prod_{i=1}^{I}\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_i}\right]^{-\sigma} \alpha_i\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_i - 1} \\
&- \lambda_{t+u}\left(p_{i,t+u} + \frac{\partial T_{j,s+u,t+u}}{\partial c_{i,j,s+u,t+u}}\right) = 0
\end{aligned}
\tag{3}
$$

With respect to labor:

$$
\begin{aligned}
\frac{\partial U}{\partial n_{j,s+u,t+u}} = &\beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right] e^{g_y(t+u)(1-\sigma)}\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}} \\
&- \lambda_{t+u}\left(w_{+u} e_{j,s+u} - \frac{\partial T_{j,s+u,t+u}}{\partial n_{j,s+u,t+u}}\right) = 0
\end{aligned}
\tag{4}
$$

With respect to savings:

$$\frac{\partial U}{\partial b_{j,s+u+1,t+u+1}} = \beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right]\rho_s\chi^b\big(b_{j,s+U+1,t+U+1}\big)^{-\sigma} - \lambda_{t+u} - \lambda_{t+u+1}\left(1+r_{t+u+1}-\frac{\partial T_{j,s+U+1,t+U+1}}{\partial b_{j,s+U+1,t+u+1}}\right) = 0 \tag{5}$$

We can solve each of these for λ_{t+u} to get the following.

$$\lambda_{t+u} = \frac{\beta^u\left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right]\left[\prod_{i=1}^{I}\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_i}\right]^{-\sigma}\alpha_i\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_i-1}}{p_{i,t+u}+\frac{\partial T_{j,s+u,t+u}}{\partial c_{i,j,s+u,t+u}}}$$

$$\lambda_{t+u} = \frac{\beta^u\left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right]e^{g_y(t+u)(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}}{w_{+u}e_{j,s+u}-\frac{\partial T_{j,s+u,t+u}}{\partial n_{j,s+u,t+u}}}$$

$$\lambda_{t+u} = \beta^u\left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right]\rho_s\chi^b\big(b_{j,s+U+1,t+U+1}\big)^{-\sigma} - \lambda_{t+u+1}\left(1+r_{t+u+1}-\frac{\partial T_{j,s+U+1,t+U+1}}{\partial b_{j,s+U+1,t+u+1}}\right)$$

These then reduce to the following $I+1$ Euler equations for each j, s and t:

Marginal utility of consumption for each good i compared to the marginal utility of labor:

$$\frac{\left[\prod_{i=1}^{I}\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_i}\right]^{-\sigma}\alpha_i\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_i-1}}{p_{i,t+u}+\frac{\partial T_{j,s+u,t+u}}{\partial c_{i,j,s+u,t+u}}} = \frac{e^{g_y(t+u)(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}}{w_{t+u}e_{j,s+u}-\frac{\partial T_{j,s+u,t+u}}{\partial n_{j,s+u,t+u}}} \tag{6}$$

Intertemporal Euler equation for savings, including the utility effects of bequests:

$$\frac{e^{g_y(t+u)(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}}{w_{t+u}e_{j,s+u}-\frac{\partial T_{j,s+u,t+u}}{\partial n_{j,s+u,t+u}}} = \rho_s\chi^b\big(b_{j,s+U+1,t+U+1}\big)^{-\sigma} - \frac{\beta(1-\rho_{s+u})e^{g_y(t+u+1)(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u+1,t+u+1}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u+1,t+u+1}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}}{w_{t+u+1}e_{j,s+u+1}-\frac{\partial T_{j,s+u+1,t+u+1}}{\partial n_{j,s+u+1,t+u+1}}}\times\left(1+r_{t+u+1}-\frac{\partial T_{j,s+U+1,t+U+1}}{\partial b_{j,s+U+1,t+u+1}}\right) \tag{7}$$

The aggrgeate consumption good is defined as follows.

$$c_{j,s,t} = \prod_{i=1}^{I} (c_{i,j,s,t} - \bar{c}_{i,s})^{\alpha_i} \tag{8}$$

The price of this aggregate good is given by the following.

$$p_{j,s,t} = \sum_{i=1}^{I} p_{i,t} c_{i,j,s,t} \tag{9}$$

An Euler equation that compares marginal utilities of two goods ($n\&m$) is given below.

$$\frac{\alpha_n \left(c_{n,j,s,t} - \bar{c}_{n,s}\right)^{\alpha_n - 1}}{p_{n,t+u} + \frac{\partial T_{j,s+u,t+u}}{\partial c_{n,j,s+u,t+u}}} = \frac{\alpha_m \left(c_{m,j,s,t} - \bar{c}_{m,s}\right)^{\alpha_m - 1}}{p_{m,t+u} + \frac{\partial T_{j,s+u,t+u}}{\partial c_{m,j,s+u,t+u}}} \tag{10}$$

We can use this equation for $m \in \{1, 2, ..., I\}$ solving for $c_{m,j,s,t} - \bar{c}_{m,s}$.

$$c_{m,j,s,t} - \bar{c}_{m,s} = \left(c_{n,j,s,t} - \bar{c}_{n,s}\right)^{\frac{1-\alpha_n}{1-\alpha_m}} \left(\frac{\alpha_m}{\alpha_n} \frac{p_{n,t+u} + \frac{\partial T_{j,s+u,t+u}}{\partial c_{n,j,s+u,t+u}}}{p_{m,t+u} + \frac{\partial T_{j,s+u,t+u}}{\partial c_{m,j,s+u,t+u}}} \right)^{\frac{1}{1-\alpha_m}}$$

We then substitute this into equations (8) and (9) to get:
The aggrgeate consumption good is defined as follows.

$$c_{j,s,t} = \prod_{i=1}^{I} \left[\left(c_{n,j,s,t} - \bar{c}_{n,s}\right)^{\frac{1-\alpha_n}{1-\alpha_i}} \left(\frac{\alpha_i}{\alpha_n} \frac{p_{n,t+u} + \frac{\partial T_{j,s+u,t+u}}{\partial c_{n,j,s+u,t+u}}}{p_{i,t+u} + \frac{\partial T_{j,s+u,t+u}}{\partial c_{i,j,s+u,t+u}}} \right)^{\frac{1}{1-\alpha_i}} \right]^{\alpha_i} \tag{11}$$

The price of this aggregate good is given by the following.

$$p_{j,s,t} = \sum_{i=1}^{I} p_{i,t} \left[\left(c_{n,j,s,t} - \bar{c}_{n,s}\right)^{\frac{1-\alpha_n}{1-\alpha_i}} \left(\frac{\alpha_i}{\alpha_n} \frac{p_{n,t+u} + \frac{\partial T_{j,s+u,t+u}}{\partial c_{n,j,s+u,t+u}}}{p_{i,t+u} + \frac{\partial T_{j,s+u,t+u}}{\partial c_{i,j,s+u,t+u}}} \right)^{\frac{1}{1-\alpha_i}} + \bar{c}_{i,s} \right] \tag{12}$$