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OLG Life Cycle Model Transition Paths: Alternate Model Forecast Method

Richard W. Evans · Kerk L. Phillips

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Abstract The overlapping generations (OLG) model is an important framework for analyzing any type of question in which age cohorts are affected differently by exogenous shocks. However, as the dimensions and degree of heterogeneity in these models increase, the computational burden imposed by rational expectations solution methods for nonstationary equilibrium transition paths increases exponentially. As a result, these models have been limited in the scope of their use to a restricted set of applications and a relatively small group of researchers. In addition to providing a detailed description of the benchmark rational expectations computational method, this paper presents an alternative method for solving for equilibrium transition paths in OLG life cycle models that is new to this class of model. The key insight is that even naïve limited information forecasts within the model produce aggregate time series similar to full information rational expectations time series as long as the naïve forecasts are updated each period. We find that our alternate model forecast method reduces computation time by 85 percent, and the approximation error is small.

Keywords Computable general equilibrium models · Heterogeneous agents · Overlapping generations model · Distribution of savings

JEL Classification C63 · C68 · D31 · D91

R. W. Evans (✉)
Department of Economics, Brigham Young University, 167 FOB, Provo, UT 84602, USA
e-mail: revans@byu.edu

K. L. Phillips
Department of Economics, Brigham Young University, 166 FOB, Provo, UT 84602, USA
e-mail: kerk_phillips@byu.edu

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1 Introduction

In 2008, the overlapping generations (OLG) model proposed by Samuelson (1958) turned 50.[1] OLG models provide a dynamic general equilibrium setting with heterogeneous agents that looks more simple and intuitive on the surface than the more widely used heterogeneous agent models with infinitely lived agents. The OLG framework is invaluable for analyzing any type of question in which age cohorts are affected differently by exogenous shocks. However, the intuitive structure of new generations of finitely lived agents being born in each period comes with at least two costs that are closely related. The fundamental welfare theorems of economics do not hold in OLG models, in general,[2] and the computation of equilibrium transition paths can require a tremendous computational burden.

It is the latter complexity of OLG models—namely, the increased computational burden of computing equilibrium transition paths—that we wish to address in this paper. In particular, we detail a new method for computing an approximation of the rational expectations equilibrium transition path in OLG models with only idiosyncratic uncertainty, which we call the alternate model forecast (AMF) method. Compared to the benchmark rational expectations solution method for this class of models, we find that our AMF method reduces computation times by about 85 percent and the approximation error in terms of mean absolute percent deviation is less than 1 percent. This result is robust to both the degree of heterogeneity in the model and the distance of the initial state from the steady-state equilibrium.

A requirement of a rational expectations equilibrium is that agents' beliefs about the future be correct. The only difference between our AMF solution method and the benchmark rational expectations method is that the AMF method relaxes this requirement of rational expectations in a similar way to Krusell and Smith (1998). We assume that agents' beliefs about the evolution of the aggregate capital stock—and therefore future interest rates and wages—are a simple function of the current capital stock, which is a summary statistic of the entire distribution of capital. These beliefs need not be correct. They just must be updated every period in a way that is consistent with updated information on the current state and a correct belief that the economy will eventually reach the steady-state. It is this AMF assumption, that agents' beliefs are a function of only of the aggregate capital stock and not the entire distribution of capital, that provides the tremendous reduction in agents' informational burden and computational times.

The AMF solution method has two equally plausible interpretations. First, the AMF method can be interpreted solely as a computational approximation to a rational expectations equilibrium in which agents' beliefs are assumed to be correct. The validity of this interpretation relies on the computed AMF transition paths being close to the rational expectations transition paths. We make this comparison in Sect. 3.3 and find that the two computed equilibrium time paths of the economy are very close.

[1] See Weil (2008) and Solow (2006) for good surveys of the origins, properties, and advantages of OLG models.

[2] See Weil (2008) for a careful description of the mechanism in OLG models that causes competitive equilibria to not be Pareto-optimal, in general.

A second interpretation of the AMF solution method is to take the alternate model forecasts of agents' beliefs about the future as their true beliefs. That is, we assume that agents have adaptive expectations rather than rational expectations. A large literature on adaptive, incorrect, or naïve expectations exists. This idea of agents not using all available information is similar to the rational inattention concept of Mankiw and Reis (2002) and Sims (2003). It might be reasonable to assume that it is either too costly for agents to acquire all the information about the entire distribution of capital or that the agents are capacity constrained on information processing.

Benhabib and Day (1982) and Michel and de la Croix (2000) showed that when agents have myopic beliefs, complex chaotic dynamics can result in an OLG environment. Chen et al. (2008) build an OLG model with capital accumulation and two-period-lived agents. Their consumers have additively separable preferences with a constant elasticity of intertemporal substitution an no aggregate shocks. They show that in this context when the intertemporal elasticity is small, myopic dynamics can be used to approximate economic behavior under perfect foresight. However, when the elasticity becomes large, myopia produces chaotic cycles.

Williams (2003) considers a variety of different learning rules in an otherwise standard real business cycle model and finds that the dynamics are not substantially different from a rational expectations assumption. However, Huang et al. (2009) build an infinitely lived agent model with both neutral and investment-specific technology shocks. They find, in contrast to Williams (2003), that while the steady states for rational expectations and adaptive expectations are identical, the dynamics about this steady state are quite different. They trace this difference to weakening of the wealth effects of shocks and a strengthening of the intertemporal substitution effects.

Lastly, Fuster et al. (2010) provide a good survey of the literature on agents' beliefs. They develop a model of quasi-rational expectations, which they term "natural expectations" and show that this hybrid of myopic and rational expectations can generate the hump shape observed in many macroeconomic time-series. They trace their model's ability to replicate this to fact that agent's beliefs do not adequately account for the reversion of the models exogenous driving processes to their mean.

With the adaptive expectations interpretation of our AMF solution method, we find that the much less costly informational burden on agents implies equilibrium transition paths that are very close to the rational expectations transition paths. This interpretation and our findings provide support for the rational inattention literature of Mankiw and Reis (2002) and Sims (2003) and for the results of Chen et al. (2008) and Williams (2003) that a weakening of the rationality assumption closely approximates rational behavior.

The benchmark conventional solution method for the non-stationary rational expectations equilibrium transition path in OLG models is outlined in Auerbach and Kotlikoff (1987, chapt. 4) for the perfect foresight case and in Nishiyama and Smetters (2007, Appendix II) for the case with idiosyncratic uncertainty. We call this benchmark solution method time path iteration (TPI). We will show the conventional TPI method is a rational expectations equilibrium concept in which each agent correctly forecasts the decisions of the other agents, thereby correctly forecasting the future distribution of savings.

As the AMF solution method represents an approximation of the benchmark rational expectations time path iteration (TPI) solution method, we compare the AMF method to the TPI method in terms of both speed and accuracy in a model with enough heterogeneity to match the computational issues in many current life cycle models. We limit our comparisons of the AMF solution method to OLG models because this method does not work in an infinitely lived agent model with idiosyncratic uncertainty.[3]

As a recent example of the constraints exacted by the computational requirements for equilibrium transition paths in OLG models with a significant degree of heterogeneity and idiosyncratic uncertainty, Nishiyama and Smetters (2007) lament that "[t]he more extensive model contained in this paper requires the addition of another state variable, which significantly increases the...required computation time from several hours to typically several days per simulation." Even a computation time of several hours makes procedures like forecasting prohibitive if the model must be simulated numerous times in order to approximate the distribution of forecasts.

This paper is organized as follows. Section 2 describes a model with S-period lived agents with heterogeneous stochastic ability and defines both the steady-state equilibrium and the non-steady-state transition path equilibrium. Section 3 describes the benchmark equilibrium TPI solution method, the AMF solution method, and compares the two methods in terms of speed and accuracy. Section 4 concludes.

2 Model

2.1 Household Problem

A measure $1/S$ of individuals with heterogeneous working ability $e \in \mathcal{E} \subset \mathbf{R}$ is born in each period t and live for $S \geq 3$ periods. Their working ability evolves over their lifetime according to an age-dependent i.i.d. process $e_s \sim f_s(e)$, where $f_s(e)$ is the age-dependent probability distribution function over working ability e. Individuals are endowed with a measure of time in each period t that they supply inelastically to the labor market. Let s represent the periods that an individual has been alive. The fixed labor supply in each period t by each age-s individual is denoted by $n(s)$.

At time t, all generation s agents with ability e know the real wage rate w_t and know the one-period real net interest rate $r_t - \delta$ on bond holdings $b_{s,t}$ that mature at the beginning of period t. For ease of notation, we subtract the depreciation rate δ from the net interest rate r_t to represent the fact that any depreciation is passed through directly from firms to households. In each period t, age-s agents with working ability e choose how much to consume $c_{s,t}$ and how much to save for the next period by loaning capital to firms in the form of a one-period bond $b_{s+1,t+1}$ in order to maximize expected lifetime utility of the following form,

$$U_{s,t} = E\left[\sum_{u=0}^{S-s} \beta^u u\left(c_{s+u,t+u}\right)\right] \quad \text{where} \quad u\left(c_{s,t}\right) = \frac{\left(c_{s,t}\right)^{1-\sigma} - 1}{1-\sigma} \quad \forall s,t \tag{2.1}$$

[3] We detail in Appendix 1 the fundamental difference between the OLG and infinite horizon models with idiosyncratic uncertainty but no aggregate uncertainty.

where $u(c)$ is a constant relative risk aversion utility function, $\sigma > 0$ is the coefficient of relative risk aversion, $\beta \in (0,1)$ is the agent's discount factor, and E is the expectations operator. As in Krusell and Smith (1998), we assume the idiosyncratic ability shocks are uninsurable because of the existence of only one asset with which to save.

Because agents are born without any bonds maturing and because they purchase no bonds in the last period of life S, the per-period budget constraints for each agent normalized by the price of consumption are the following,

$$w_t e_{s,t} n(s) \geq c_{s,t} + b_{s+1,t+1} \quad \text{for} \quad s = 1 \quad \forall t \tag{2.2}$$

$$(1 + r_t - \delta)\, b_{s,t} + w_t e_{s,t} n(s) \geq c_{s,t} + b_{s+1,t+1} \quad \text{for} \quad 2 \leq s \leq S-1 \quad \forall t \tag{2.3}$$

$$(1 + r_t - \delta)\, b_{s,t} + w_t e_{s,t} n(s) \geq c_{s,t} \quad \text{for} \quad s = S \quad \forall t \tag{2.4}$$

where $e_{s,t} \in \mathcal{E} \subset \mathbf{R}$ is an age-specific working ability shock that is i.i.d. and has a discrete support with age-dependent probability mass function $f_s(e)$.

$$e_{s,t} \in \mathcal{E}_s = \{e_{s,1}, e_{s,2}, \dots e_{s,J}\} \subset \mathbf{R}^J \sim f_s(e) = \theta_s(e) \tag{2.5}$$

The i.i.d. age-dependent probability mass function $f_s(e)$ assigns probabilities $\theta_s(e)$ to each possible ability level $e_{s,t}$ such that $\sum_{e=e_1}^{e_J} \theta_s(e) = 1$ for all ages s. The law of large numbers ensures that $\theta_s(e_j)$ percent of all age-s households have working ability e_j. The expected value of ability at age s is $\bar{e}_s \equiv E(e_{s,t}) = \sum_{e=e_1}^{e_J} \theta_s(e) e_{s,t}$. In addition to the budget constraints in each period, we impose a borrowing constraint.[4]

$$b_{s,t} \geq 0 \quad \forall s, t \tag{2.6}$$

We next describe the Euler equations that govern the choices of consumption $c_{s,t}$ and savings $b_{s+1,t+1}$ by household of age s and ability e in each period t. We work backward from the last period of life $s = S$. Because households do not save in the last period of life $b_{s+1,t+1} = 0$ due to our assumption of no bequest motive, the household's final-period maximization problem is given by the following.

$$\max_{c_{S,t}} \frac{\left(c_{S,t}\right)^{1-\sigma} - 1}{1-\sigma} \quad \text{s.t.} \quad (1 + r_t - \delta)\, b_{S,t} + w_t e_{S,t} n(S) \geq c_{S,t} \quad \forall t \tag{2.7}$$

Because the $s = S$ problem (2.7) involves only time-t variables that are known with certainty, the solution to the problem is trivially that the household consumes all of its income in the last period of life.

$$c_{S,t} = (1 + r_t - \delta)\, b_{S,t} + w_t e_{S,t} n(S) \quad \forall t \tag{2.8}$$

[4] This borrowing constraint is not too restrictive given the OLG environment. Some type of borrowing constraint must be imposed either exogenously or endogenously in order to constrain borrowing at the end of life. We set our exogenous constraint arbitrarily at zero, but it could be negative and age-dependent without changing the computation speed of the equilibrium. The nonnegativity constraint on bonds is consistent with holding physical capital instead of financial capital.

In general, maximizing (2.1) with respect to (2.2), (2.3), (2.4), and (2.6) gives the following set of $S-1$ intertemporal Euler equations.

$$\left(c_{s,t}\right)^{-\sigma} = \beta E\left[(1+r_{t+1}-\delta)\left(c_{s+1,t+1}\right)^{-\sigma}\right] \quad \text{for} \quad 1 \leq s \leq S-1, \quad \forall t \tag{2.9}$$

Note from (2.3) that $c_{s,t}$ in (2.9) depends on the household's age s, which ability shock $e_{s,t}$ the particular age-s household received, and the initial wealth with which the household entered the period $b_{s,t}$. The expectations operator E on the right-hand side of (2.9) integrates out any expected heterogeneity in ability $e_{s+1,t+1}$ in the next period. However, the presence of next period prices r_{t+1} and w_{t+1} on the right-hand-side of (2.9) requires an assumption about the housdhold's beliefs about the distribution of capital in the next period and, therefore, about other households' choices.

As will be shown in Sect. 2.3, equilibrium prices depend on the entire distribution of capital. Let the object $\Gamma_t = \{\gamma_t(s,e,b)\} \subset \mathbf{R}^S \times \mathbf{R}^J \times \mathbf{R}^B$ represent the entire distribution of capital in period t among all types of households s, e, and b, where each $\gamma_t(s,e,b)$ represents the fraction of the total population that is age s, ability e, and wealth b. Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\Gamma^e_{t+u} = \Omega^u\left(\Gamma_t\right) \quad \forall t, \quad u \geq 1 \tag{2.10}$$

where the e superscript signifies that Γ^e_{t+u} is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.

Now we can express the policy function for savings in the next period from (2.9) as a function of the state and beliefs $b' = \phi(s,e,b|\Omega)$, where $s \in \{1,2,\ldots S-1\}$, $e \in \{e_1,e_2,\ldots e_J\}$, and $b = \{b_1,b_2,\ldots b_B\}$. Discretizing the support of the current period wealth in bond holdings b allows us to not have to account for the history of ability shocks received up to age s. That is, Eqs. (2.8) and (2.9) are perfectly identified but represent $\sum_{v=1}^{S-1} J^v$ equations and $\sum_{v=1}^{S-1} J^v$ unknowns. If agents only live $S=10$ periods and there are only $J=5$ different abilities, then (2.8) and (2.9) represent 2,441,405 equations and 2,441,405 unknowns. Discretizing the possible values of current wealth to B points such that $b_{s,t} \in \{b_1,b_2,\ldots b_B\}$ allows us to deal with only $(S-1)\times J\times B$ equations and unknowns. If the number of points in the support of b is $B=100$, then (2.8) and (2.9) only represent 4,500 equations and 4,500 unknowns.

2.2 Firm Problem

A unit measure of identical, perfectly competitive firms exist in this economy. The representative firm is characterized by the following Cobb–Douglas production technology,

$$Y_t = AK_t^\alpha N_t^{1-\alpha} \quad \forall t \tag{2.11}$$

where A is the fixed technology process and $\alpha \in (0, 1)$ and N_t is measured in efficiency units of labor. The cost function includes both the net real interest rate r_t paid for aggregate capital K_t as well as the depreciation rate δ of the capital.

$$\text{Real profits} = AK_t^{\alpha}N_t^{1-\alpha} - r_t K_t - w_t N_t \tag{2.12}$$

Profit maximization results in the real wage w_t and the real rental rate of capital r_t being determined by the marginal products of labor and capital, respectively.

$$w_t = (1-\alpha)\frac{Y_t}{N_t} \quad \forall t \tag{2.13}$$

$$r_t = \alpha\frac{Y_t}{K_t} \quad \forall t \tag{2.14}$$

There is no expectations operator in (2.13) and (2.14) because the ability shock to households is i.i.d., and we assume that firms know the distribution of the shock. So if the wage clears the labor market, then all the firm needs to know is the average ability of each age cohort $\bar{e}_s$.

2.3 Market Clearing and Equilibrium

Labor market clearing requires that aggregate labor demand N_t measured in efficiency units equal the sum of individual efficiency labor supplied $e_{s,t}n(s)$. The supply side of market clearing in the labor market is trivial because household labor is supplied inelastically. Capital market clearing requires that aggregate capital demand K_t equal the sum of capital investment by households $b_{s,t}$. Aggregate consumption C_t is defined in (2.17), and investment is defined by the standard $Y=C+I$ constraint as shown in (2.18).

$$N_t = \frac{1}{S}\sum_{s=1}^{S}\sum_{e=e_1}^{e_J}\theta_s(e)e_{s,t}n(s) \quad \forall t \tag{2.15}$$

$$K_t = \frac{S-1}{S}\sum_{s=2}^{S}\sum_{e=e_1}^{e_J}\sum_{b=b_1}^{b_B}\gamma_t(s,e,b)b \quad \forall t \tag{2.16}$$

$$C_t \equiv \frac{S-1}{S}\sum_{s=2}^{S}\sum_{e=e_1}^{e_J}\sum_{b=b_1}^{b_B}\gamma_t(s,e,b)c(s,e,b) + \frac{1}{S}\sum_{e=e_1}^{e_J}\theta_1(e)c(1,e,0) \quad \forall t \tag{2.17}$$

$$Y_t = C_t + K_{t+1} - (1-\delta)K_t \tag{2.18}$$

where $c(s,e,b)$ in (2.17) is the optimal consumption rule for each household resulting from the optimal savings rule $b' = \phi(s,e,b|\Omega)$ through the period budget constraints (2.2), (2.3), and (2.4). Then the steady-state rational expectations equilibrium for this economy is defined as follows.

Definition 1 (*Steady-state rational expectations equilibrium*) A non-autarkic steady-state rational expectations equilibrium in the overlapping generations model with

S-period lived agents and heterogeneous ability e is defined as a constant distribution of capital

$$\Gamma_t = \bar{\Gamma} = \bar{\gamma}(s, e, b) \quad \forall t,$$

a savings decision rule given beliefs $b' = \phi(s, e, b|\Omega)$, consumption allocations $c_{s,t}$ for all s and t, aggregate firm production Y_t, aggregate labor demand N_t, aggregate capital demand K_t, real wage w_t, and real interest rate r_t for all t such that the following conditions hold:

i. households optimize according to (2.6), (2.8) and (2.9),
ii. firms optimize according to (2.13) and (2.14),
iii. markets clear according to (2.15), (2.16), and (2.18),
iv. and the steady-state distribution $\Gamma_t = \bar{\Gamma}$ is induced by the policy rule $b' = \phi(s, e, b|\Omega)$.

Note that the steady-state rational expectations equilibrium definition has no constraint that beliefs be correct $\Gamma_{t+1} = \Gamma^e_{t+1} = \Omega(\Gamma_t)$. The steady-state assumption that $\Gamma_t = \Gamma_{t+1} = \bar{\Gamma}$ removes the need for beliefs about other households' actions because Γ_{t+1} is known.

The steady-state rational expectations equilibrium is computed by guessing a steady-state distribution $\bar{\Gamma}_i$, where i is the index of the guess, and finding the implied steady-state real wage and real interest rate $\bar{w}_i$ and $\bar{r}_i$.[5] The policy function for each individual can then be found by backward induction by first solving the age-$S-1$ savings problem for b_S for all values of b_{S-1} and e_{S-1},

$$\begin{aligned}&\Big([1+\bar{r}_i-\delta]\,b_{S-1}+\bar{w}_i e_{S-1} n(S-1)-b_S\Big)^{-\sigma}\\&\quad=\ldots\beta(1+\bar{r}_i-\delta)E\Big[\Big([1+\bar{r}_i-\delta]\,b_S+\bar{w}_i e_S n(S)\Big)^{-\sigma}\Big]\end{aligned} \tag{2.19}$$

and then using the solution for b_S to solve the previous period problem for b_{S-1}.The process is repeated from the age $S-1$ Euler equation (2.19) backward to the age-1 Euler equation.

$$\begin{aligned}&\Big([1+\bar{r}_i-\delta]\,b_s+\bar{w}_i e_s n(s)-b_{s+1}\Big)^{-\sigma}\\&\quad=\ldots\beta(1+\bar{r}_i-\delta)E\Big[\Big([1+\bar{r}_i-\delta]\,b_{s+1}+\bar{w}_i e_{s+1} n(s+1)-b_{s+2}\Big)^{-\sigma}\Big]\\&\quad\text{for}\quad s\in\{S-2,S-3,\ldots 2,1\}\end{aligned} \tag{2.20}$$

Once a policy function is found $b' = \phi_i(s, e, b|\Omega)$ given the guess for the steady-state distribution of wealth $\bar{\Gamma}_i$ and the corresponding steady-state real wage $\bar{w}_i$ and real interest rate $\bar{r}_i$, the policy function can be used to check if the next period distribution of wealth $\bar{\Gamma}'_i$ is equal to the initial guess for the steady-state distribution of wealth

[5] Wendner (2004) provides an analytical proof for the existence and uniqueness of the steady-state rational expectations equilibrium.

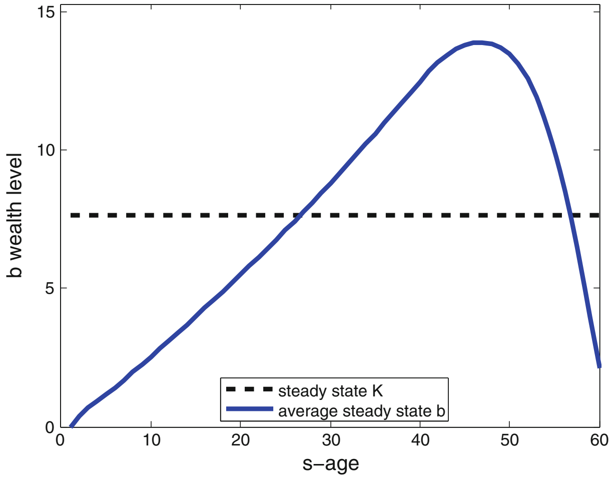

Fig. 1 Steady-state distribution of savings $\bar{b}_s : S = 60$

$\bar{\Gamma}_i$. If they are equal, then $\bar{\Gamma} = \bar{\Gamma}_i = \bar{\Gamma}'_i$. If they are not equal, then choose another steady-state distribution that is a convex combination of the initial guess and the new implied distribution $\bar{\Gamma}_{i+1} = \rho\bar{\Gamma}'_i + (1-\rho)\bar{\Gamma}_i$ where $\rho \in (0, 1)$.[6]

Figure 1 shows the computed steady-state equilibrium distribution of savings $\bar{\Gamma}$ over the life cycle for a particular calibration of the model parameters $[S, \beta, \sigma, \alpha, A, \delta] = [60, 0.96, 3, 0.35, 1, 0]$ and computation parameter $\rho = 0.2$. We assume that labor is supplied inelastically, and we calibrate the labor supply at each age to match the average labor supply reported by age in the CPS monthly survey.[7] The steady-state aggregate capital stock shown in Fig. 1 is $\bar{K}\left(\bar{\Gamma}\right) = 7.62$, and the steady-state equilibrium real wage and real interest rate are $\bar{w}\left(\bar{\Gamma}\right) = 1.43$ and $\bar{r}\left(\bar{\Gamma}\right) = 0.08$, respectively.

Outside of the steady state, an age-s household's intertemporal consumption decision in each period from (2.9) also depends on both the current period's distribution of capital Γ_t and the expected value of next period's distribution of capital Γ_{t+1}. But $\Gamma_t \neq \Gamma_{t+1}$ in general outside of the steady state.

$$\begin{aligned}
&\left(\Big[1 + r\left(\Gamma_t\right) - \delta\Big]b_{s,t} + w\left(\Gamma_t\right)e_{s,t}n(s) - b_{s+1,t+1}\right)^{-\sigma} \\
&= \ldots \beta E\Bigg[\Big(1 + r\left(\Gamma_{t+1}\right) - \delta\Big)\Bigg(\Big[1 + r\left(\Gamma_{t+1}\right) - \delta\Big]b_{s+1,t+1} \\
&\quad + w\left(\Gamma_{t+1}\right)e_{s+1,t+1}n(s+1) - b_{s+2,t+2}\Bigg)^{-\sigma}\Bigg] \\
&\text{for} \quad 1 \leq s \leq S-1, \quad \text{and} \quad b_{1,t} = b_{S+1,t} = 0, \quad \forall t
\end{aligned} \tag{2.21}$$

[6] A detailed description of the algorithm for computing the steady-state distribution is given in Appendix 2.

[7] A person's lifespan here is defined as the duration from the period they start working until the period they die. We ignore childhood. The exact calibration of $n(s)$ is reported in Appendix 2.

The non-steady-state equilibrium in this economy is much more complicated because the savings policy rule depends not only on age s, ability e, individual wealth b and beliefs Ω, but also on the current distribution of capital Γ.

$$b' = \phi(s, e, b, \Gamma|\Omega) \tag{2.22}$$

In contrast to the steady-state equilibrium, this means that each household must be able to forecast future prices, and therefore future capital distributions, in order to make its own savings decisions with the added complication that the capital distribution is changing over time. Let general beliefs about the future distribution of capital in period $t + u$ be characterized by the operator $\Omega(\cdot)$ as in (2.10).

The expression of individual beliefs in (2.10) is a weak assumption in the sense that it does not constrain the beliefs to be correct. However, it is a strong assumption in that it implies the following two properties. First (2.10) implies that each household knows the entire distribution of savings Γ_t at time t. It also implies that each household has symmetric beliefs about the savings policy function of all the other households. That is, $\Omega(\cdot)$ has no s subscript. We can now define a general non-steady-state rational expectations equilibrium.

Definition 2 (*Non-steady-state rational expectations equilibrium*) A non-steady-state rational expectations equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability e is defined as a distribution of capital Γ_t, household beliefs about how the distribution of capital will evolve $\Omega(\Gamma_t)$, a policy function $b' = \phi(s, e, b, \Gamma|\Omega)$, consumption allocations $c_{s,t}$, aggregate firm production Y_t, aggregate labor demand N_t, aggregate capital stock K_t, real wage w_t, and real rental rate r_t for all t such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the future savings decisions of the other agents described in (2.10), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\Gamma_{t+u} = \Gamma^e_{t+u} = \Omega^u(\Gamma_t) \quad \forall t, \quad u \geq 1$$

ii. households policy function $b' = \phi(s, e, b, \Gamma|\Omega)$ maximizes utility according to (2.6) and (2.9),
iii. firms choose aggregate labor demand N_t and aggregate capital demand K_t optimally according to (2.13) and (2.14), respectively,
iv. and markets clear according to (2.15), (2.16), and (2.18).

One implication of households having symmetric beliefs is that they will have symmetric policy functions. In other words, (2.10) implies the following.

$$\Gamma^e_{t+u} = \Omega^u(\Gamma_t) \quad \forall t, \quad u \geq 1 \Rightarrow b' = \phi(s, e, b, \Gamma|\Omega) \quad \perp \quad t \tag{2.23}$$

That is, if the equilibrium savings choice is $b_{s+1,t+1}$ according to Definition 2 for an age-s household given the state $(s, e_{s,t}, b_{s,t}, \Gamma_t)$, then an age-$s$ household in a different period $t + u$ will choose the same equilibrium savings rate if the same state $(s, e_{s,t+u}, b_{s,t+u}, \Gamma_{t+u})$ occurs. The intuition is that if a household knows what savings level b_{s+1} it would choose at any age s, ability e_s, wealth b_s, and distribution of wealth Γ, then the symmetry of the problem implies that the household knows what all the other households would choose at any age s, ability e_s, wealth b_s, and distribution Γ.

With Definition 2, the non-steady-state equilibrium can be computed by rewriting the set of $S-1$ intertemporal Euler equations from (2.21) in the following way.

$$\begin{aligned}
&\left(\Big[1+r\left(\Gamma_t\right)-\delta\Big]b_{s,t}+w\left(\Gamma_t\right)e_{s,t}n(s)-b_{s+1,t+1}\right)^{-\sigma}\\
&=\ldots\beta E\left[\Big(1+r\big(\Omega(\Gamma_t)\big)-\delta\Big)\left(\Big[1+r\big(\Omega(\Gamma_t)\big)-\delta\Big]b_{s+1,t+1}\right.\right.\\
&\quad\left.\left.+\,w\big(\Omega(\Gamma_t)\big)e_{s+1,t+1}n(s+1)-b_{s+2,t+2}\right)^{-\sigma}\right]\\
&\quad\text{for}\quad 1\le s\le S-1,\quad\text{and}\quad b_{1,t}=b_{S+1,t}=0,\quad\forall t
\end{aligned}\tag{2.24}$$

The rational expectations equilibrium assumption (i) in Definition 2 that beliefs be correct $\Gamma^e_{t+u}=\Gamma_{t+u}$ implies that a new agent $s=1$ at time t can correctly forecast all future wages and interest rates given the current distribution of capital.

$$w_{t+u}=w\left(\Omega^u(\Gamma_t)\right)\quad\text{and}\quad r_{t+u}=r\left(\Omega^u(\Gamma_t)\right)\quad 1\le u\le S-1\tag{2.25}$$

Knowing the path of wages and interest rates will allow each household to backward induct their non-steady-state equilibrium savings policy function $b'=\phi(s,e,b,\Gamma|\Omega)$ in the same way as the steady-state distribution of capital. The solution to this non-steady-state equilibrium problem is a fixed point in which the savings policy function $b'=\phi(s,e,b,\Gamma|\Omega)$ induces the transition path for the distribution of capital Γ_{t+u} consistent with the path implied by beliefs $\Omega^u(\Gamma_t)$.

3 Transition Path Solution Methods

This section outlines the benchmark TPI method for solving the non-steady-state rational expectations equilibrium transition path of the distribution of savings and then details our new AMF method for computing the equilibrium transition path. Because the AMF method represents an approximation of the rational expectations assumption, we compare the AMF method to the benchmark TPI method in terms of both speed and accuracy in the context of the model from Sect 2. This model has enough heterogeneity to match the varied richness and computational complexity of equilibrium transition paths in current life cycle models.

3.1 Benchmark: TPI

The most common method of solving for non-steady-state equilibrium transition path for the capital distribution in OLG models is finding a fixed point for the transition path of the distribution of capital for a given initial state of the distribution of capital. This solution method is detailed for the perfect foresight case in Auerbach and Kotlikoff (1987, chapt. 4) and for the stochastic case in Nishiyama and Smetters (2007, Appendix II). The idea is that the economy is infinitely lived, even though the agents that make up the economy are not. Rather than recursively solving for equilibrium policy functions by iterating on individual value functions, one must recursively solve for the policy functions by iterating on the entire transition path of the endogenous objects in the economy (see Stokey and Lucas (1989, chapt. 17)).

The key assumption is that the economy will reach the steady-state equilibrium $\bar{\Gamma}$ described in Definition 1 in a finite number of periods $T < \infty$ regardless of the initial state Γ_0. The first step is to assume a transition path for aggregate capital $\mathbf{K}^i = \left\{K_1^i, K_2^i, \dots K_T^i\right\}$ such that T is sufficiently large to ensure that $\Gamma_T = \bar{\Gamma}$ and $K_T^i\,(\Gamma_T) = \bar{K}\left(\bar{\Gamma}\right)$. The superscript i is an index for the iteration number. The transition path for aggregate capital determines the transition path for both the real wage $\mathbf{w}^i = \left\{w_1^i, w_2^i, \dots w_T^i\right\}$ and the real return on investment $\mathbf{r}^i = \left\{r_1^i, r_2^i, \dots r_T^i\right\}$. The exact initial distribution of capital in the first period Γ_1 can be arbitrarily chosen as long as it satisfies $K_1^i = \frac{S-1}{S}\sum_{s=2}^{S}\sum_{e=e_1}^{e_J}\sum_{b=b_1}^{b_B}\gamma_1(s,e,b)b$ according to market clearing condition (2.16). One could also first choose the initial distribution of capital Γ_1 and then choose an initial aggregate capital stock K_1^i that corresponds to that distribution. As mentioned earlier, the only other restriction on the initial transition path for aggregate capital is that it equal the steady-state level $K_T^i = \bar{K}\left(\bar{\Gamma}\right)$ by period T. But the aggregate capital stocks K_t^i for periods $1 < t < T$ can be any level.

Given the initial capital distribution Γ_1 and the transition paths of aggregate capital $\mathbf{K}^i = \left\{K_1^i, K_2^i, \dots K_T^i\right\}$, the real wage $\mathbf{w}^i = \left\{w_1^i, w_2^i, \dots w_T^i\right\}$, and the real return to investment $\mathbf{r}^i = \left\{r_1^i, r_2^i, \dots r_T^i\right\}$, one can solve for the optimal savings policy rule for each type of $S-1$-aged agent for the last period of his life $b_{S,2} = \phi_1(S-1,e,b)$ using his intertemporal Euler equation, where the "1" subscript on ϕ represents the time $t=1$ savings decision with the real wage w_1^i and real interest rate r_1^i.[8]

$$\left([1+r_1^i-\delta]b_{S-1,1}+w_1^i e_{S-1,1}n(S-1)-b_{S,2}\right)^{-\sigma}$$
$$=\dots\beta\left(1+r_2^i-\delta\right)E\left[\left([1+r_2^i-\delta]b_{S,2}+w_2^i e_{S,2}n(S)\right)^{-\sigma}\right] \qquad (3.1)$$

The final two savings decisions of each type of $S-2$-aged household in period 1, $b_{S-1,2}$ and $b_{S,3}$, are characterized by the following two intertemporal Euler equations and are solved by backward induction.

$$\left([1+r_1^i-\delta]b_{S-2,1}+w_1^i e_{S-2,1}n(S-2)-b_{S-1,2}\right)^{-\sigma}$$
$$=\dots\beta\left(1+r_2^i-\delta\right)E\left[\left([1+r_2^i-\delta]b_{S-1,2}+w_2^i e_{S-1,2}n(S-1)-b_{S,3}\right)^{-\sigma}\right]$$
$$\left([1+r_2^i-\delta]b_{S-1,2}+w_2^i e_{S-1,2}n(S-1)-b_{S,3}\right)^{-\sigma}$$
$$=\dots\beta\left(1+r_3^i-\delta\right)E\left[\left([1+r_3^i-\delta]b_{S,3}+w_3^i e_{S,3}n(S)\right)^{-\sigma}\right] \qquad (3.2)$$

8 Note that the Γ and Ω that usually appear in the policy functions ϕ have been dropped because they are assumed in the guess of the transition path $\mathbf{K}^i$.

The solution to the second equation delivers the savings policy function for $b_{S,3} = \phi_2(S-1, e, b)$. This policy function is then used in the first equation of (3.2) in order to solve for the policy function $b_{S-1,2} = \phi_1(S-2, e, b)$.

This process is repeated for every age of household in $t = 1$ down to the age-1 household at time $t = 1$. This household solves the full set of $S-1$ savings decisions characterized by the following equations.

$$
\begin{aligned}
&\left(w_1^i e_{1,1} n(1) - b_{2,2}\right)^{-\sigma} \\
&\quad = \ldots \beta \left(1 + r_2^i - \delta\right) E\left[\left(\left[1 + r_2^i - \delta\right] b_{2,2} + w_2^i e_{2,2} n(2) - b_{3,3}\right)^{-\sigma}\right] \\
&\left(\left[1 + r_2^i - \delta\right] b_{2,2} + w_2^i e_{2,2} n(2) - b_{3,3}\right)^{-\sigma} \\
&\quad = \ldots \beta \left(1 + r_3^i - \delta\right) E\left[\left(\left[1 + r_3^i - \delta\right] b_{3,3} + w_3^i e_{3,3} n(3) - b_{4,4}\right)^{-\sigma}\right] \\
&\qquad\qquad \vdots \\
&\left(\left[1 + r_{S-1}^i - \delta\right] b_{S-1,S-1} + w_{S-1}^i e_{S-1,S-1} n(S-1) - b_{S,S}\right)^{-\sigma} \\
&\quad = \ldots \beta \left(1 + r_S^i - \delta\right) E\left[\left(\left[1 + r_S^i - \delta\right] b_{S,S} + w_S^i e_{S,S} n(S)\right)^{-\sigma}\right]
\end{aligned}
\tag{3.3}
$$

Once the remaining lifetime decision rules have been solved for all households alive in period $t = 1$, the set of first period policy functions $\phi_1(s, e, b)$ is complete. The first period policy function $\phi_1(s, e, b)$ is then combined with the first period distribution of capital Γ_1 to compute the second period distribution of capital Γ_2.The second period distribution of capital Γ_2 implies an aggregate capital stock $K_2^{i'}$ through (2.16) which is not equal to the originally assumed second period aggregate capital stock $K_2^{i'} \neq K_2^i$, in general.

For every $1 < t < T$, the set of period-t policy functions $\phi_t(s, e, b)$ is computed by solving the full set of $S-1$ savings decisions for the age-1 household at period t. The policy rule $\phi_t(s, e, b)$ is then combined with the distribution of savings Γ_t computed in period $t-1$ in order to compute the distribution of savings in the next period Γ_{t+1}. The new Γ_{t+1} implies an aggregate capital stock $K_{t+1}^{i'}$ that, in general, is not equal to the originally assumed aggregate capital stock $K_{t+1}^{i'} \neq K_{t+1}^i$.[9]

Once this process has been completed for all $1 < t < T$, a new transition path for the aggregate capital stock has been computed $\mathbf{K}^{i'} = \{K_1^{i'}, K_2^{i'}, \ldots K_T^{i'}\}$. Let $|\cdot|$ be the sup norm. Then the fixed point necessary for the equilibrium transition path from Definition 2 has been found when the distance between $\mathbf{K}^{i'}$ and $\mathbf{K}^i$ is arbitrarily close to zero.

$$
|\mathbf{K}^{i'} - \mathbf{K}^i| < \varepsilon \quad \text{for} \quad \varepsilon > 0 \tag{3.4}
$$

[9] A check here for whether T is large enough is if $K_T^{i'} = \bar{K}\left(\bar{\Gamma}\right)$ as well as $K_{T+1}^{i'}$ and $K_{T+2}^{i'}$. If not, then T needs to be larger.

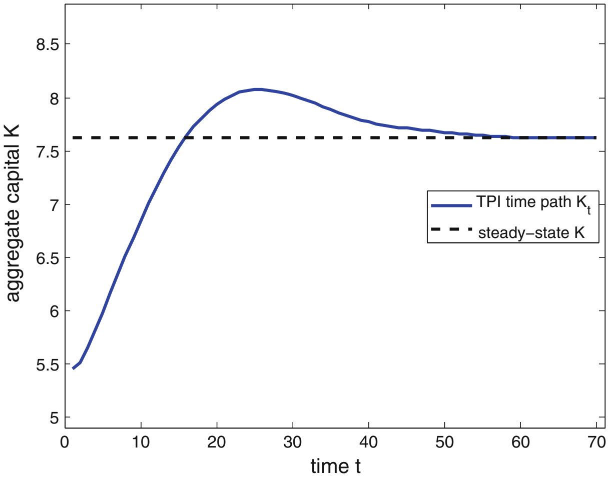

Fig. 2 TPI computed equilibrium transition path for aggregate capital stock K_t

If the fixed point has not been found $|\mathbf{K}^{i'} - \mathbf{K}^{i}| > \varepsilon$, then a new transition path for the aggregate capital stock is generated as a convex combination of $\mathbf{K}^{i'}$ and $\mathbf{K}^{i}$.

$$\mathbf{K}^{i+1} = \rho\mathbf{K}^{i'} + (1-\rho)\mathbf{K}^{i} \quad \text{for} \quad \rho \in (0, 1) \tag{3.5}$$

This process is repeated until the initial transition path for the aggregate capital stock is consistent with the transition path implied by those beliefs and household and firm optimization.

In essence, the TPI method iterates on beliefs represented by a transition path for the aggregate capital stock $\mathbf{K}^{i}$ until a fixed point in beliefs is found that are consistent with the transition path implied by optimization based on those beliefs.

Figure 2 shows the TPI computed transition path of the aggregate capital stock using the same calibrated example from the steady-state computation in Sect. 2.3. A detailed outline of the computational algorithm is given in Appendix 3. The benchmark TPI computational method shows the aggregate capital stock K_t converging to its steady state $\bar{K}$ in roughly 60 periods. It took 31 h, 59 min, and 39 s to compute the solution by time path iteration.

The equilibrium transition path overshoots the steady state and takes 60 periods to arrive at the new steady state because the initial state is so different from the new steady state. The initial state $K_1 = 5.45$ in the time path in Fig. 2 is 28 percent less than the steady-state aggregate capital stock level $\bar{K} = 7.62$ to which it must converge. Also, the initial distribution of savings across household types Γ_1 as defined in Sect. 2.1 is very different from the steady-state distribution of savings across household types $\bar{\Gamma}$.[10]

[10] More specifically, we assumed a uniform initial distribution of savings across all types, which resulted in an initial aggregate capital stock of $K_1 = 5.45$. The steady-state distribution $\bar{\Gamma}$ that results in $\bar{K} = 7.62$ is not uniform.

3.2 Relax Rational Expectations: AMF

We propose an alternative method for computing non-steady-state rational expectations transition paths in OLG life cycle models that we call the AMF method. AMF relaxes the rational expectations requirement from part (i) of Definition 2 that each agent knows the policy function of all other agents. Instead, AMF uses a weaker assumption that agents use some general alternative model to forecast in each period the transition path of the aggregate capital stock $\left\{K_u^f, w_u^f, r_u^f\right\}_{u=t}^{t+S}$ for the remaining periods of their lives, where the "f" superscript represents forecast values. This forecasted series is then updated each period when the value of the capital stock next period is realized.

As discussed in the introduction, this relaxation of rational expectations of the AMF method can have two interpretations. The first interpretation is that the AMF solution method is solely a computational approximation of a the rational expectations equilibrium described in Definition 2. In this interpretation, the agents are assumed to have correct beliefs, but the computation is simplified by approximating those beliefs. The validity of this interpretation depends on how close the AMF approximation is to the TPI rational expectations time path. Section 3.3 details the comparison of the TPI and AMF solution methods.

The other interpretation of the AMF method is to assume that the adaptive expectations implied by the approximation of beliefs are not an approximation but are rather a characterization of how individuals behave. Either interpretation can be valid, but the latter interpretation implies an adaptive expectations equilibrium different from the rational expectations equilibrium from Definition 2.

With respect to the rational expectations equilibrium time path, the approximation error in the AMF method comes from agents' beliefs about the future trajectory of the distribution of capital not being exactly correct, $\Gamma_{t+u}^e \neq \Gamma_{t+u}$ for all $u \geq 1$. But the size of the error is limited because beliefs are updated after each period as new information becomes available. In contrast to the benchmark TPI method from Sect. 3.1, AMF is faster because the household decision rules only have to be computed for one transition path rather than iterating until beliefs equal the truth.

This approach of using a forecasting method from outside the model is analogous to the approach taken by Krusell and Smith (1998). They conjectured a law of motion for the moments of the distribution of wealth in an infinitely lived heterogeneous agent environment, and the moments determined the levels of the aggregate variables. They found that a simple log-linear law of motion was enough to closely approximate the benchmark rational expectations equilibrium.

The AMF method also has some of the flavor of the rational inattention concept of Mankiw and Reis (2002) and Sims (2003) who justify relaxing the information burden of rational expectations on the grounds that agents update their information infrequently and agents have limited information-processing capacity. We use this type of assumption in the AMF method in order to streamline computation time. However, we find that the resulting equilbrium transition path has a very small approximation error relative to the benchmark TPI rational expectations transition path.

Let $\Omega_a(\cdot)$ represent the general form of the alternative model each agent uses to forecast the transition paths of the aggregate capital stock, real wage, and real rental rate $\{K_u^f, w_u^f, r_u^f\}_{u=t}^{t+S}$ which are functions of the distribution of capital Γ_u at time u. Then let the forecast for the aggregate capital stock be generated by the following general alternative model:

$$\Gamma_{t+u}^f = \Omega_a^u(\Gamma_t) \quad \text{s.t.} \quad \lim_{u\to\infty} \Omega^u(\Gamma_t) = \bar{\Gamma} \tag{3.6}$$

where w_{t+u}^f and r_{t+u}^f are just functions of $K_{t+u}^f\left(\Omega_a^u(\Gamma_t)\right)$. The only condition that must be imposed on the alternative model is that the forecasts must go to the steady state in the limit $\lim_{u\to\infty} \Omega^u(\Gamma_t) = \bar{\Gamma}$. With $\left\{K_u^f, w_u^f, r_u^f\right\}_{u=t}^S$, each agent can choose their savings for the next period $b_{s+1,t+1}$ as well as planned savings levels $b_{s+u,t+u}^p$ for $u \in \{2, 3, ..S - s\}$ for the remaining periods of life given the forecasted transition path of the aggregate variables in the same way as described in Eqs. (3.1) through (3.3) in Sect. 3.1. The "p" superscript refers to a planned policy decision because that policy will likely change by the time the household needs to make that choice due to the updating of the forecast.

At the end of period t, the distribution of capital for the next period Γ_{t+1} has been decided and implies an aggregate capital stock that is not equal to the forecasted capital stock $K_{t+1}(\Gamma_{t+1}) \neq K_{t+1}^f$, in general. With the new aggregate capital stock K_{t+1}, each agent repeats the process of forecasting the future values of the aggregate variables using the alternative model $\Omega_a(\cdot)$ until the transition path reaches the steady state in period T. Each distribution of capital Γ_{t+u} calculated using the alternative model from (3.7) and the corresponding time-t allocations and prices in the computed equilibrium transition path $\{\mathbf{b}_t\}_{t=1}^T$ represents an alternate model forecast non-steady-state equilibrium transition path.

Definition 3 (*AMF equilibrium transition path of the distribution of capital*) Given some initial distribution of capital Γ_1 and a steady-state distribution of capital arrived at after T periods $\Gamma_t = \bar{\Gamma}$ for $t \geq T$, the AMF equilibrium transition path of the distribution of capital $\{\Gamma_t\}_{t=1}^T$ is defined as the individual distributions of capital Γ_t calculated by forecasting future aggregate variables using the alternative forecasting model $\Omega_a(\cdot)$ specified in equation (3.7) that is specified as follows:

$$\Gamma_{t+1}^f = \Omega_a(\Gamma_t) \Rightarrow \Gamma_{t+u}^f = \Omega_a^u(\Gamma_t) \quad \text{s.t.} \quad \lim_{u\to\infty} \Omega^u(\Gamma_t) = \bar{\Gamma}$$

Each individual distribution of capital is calculated using the remaining life forecasts of aggregate variables K_{t+u}^f according to Eqs. (3.1) through (3.3).

Figure 3 shows the AMF computed transition path of the aggregate capital stock using the same calibrated example from the steady-state computation in Sect. 2.3. To this point, the alternative model $\Omega_a(\cdot)$ has been generally specified. In practice, it could be a complex econometric model based on observed data, or it could be an extremely simple interpolation. We use a naïve linear forecast between the current

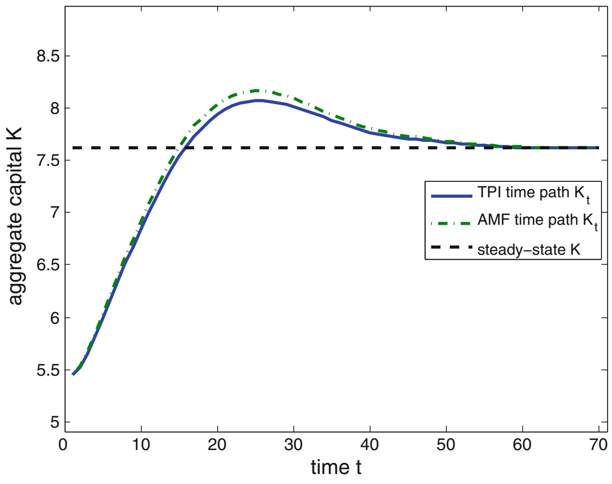

Fig. 3 AMF computed equilibrium transition path for aggregate capital stock K_t

aggregate capital stock and the steady state to forecast the future aggregate capital stock,

$$K_{t+1} = K_t + \frac{\bar{K} - K_t}{T - t} \tag{3.7}$$

where $\bar{K}$ is the steady-state capital stock, t is the current period, and T is the period in which the economy has reached the steady state.[11] The AMF rule (3.7) is simply a linear forecast between the current state K_t and the steady state $\bar{K}$. The naïve alternative model is conservative in that our computed approximation errors should represent an upper bound. A detailed outline of the computational algorithm is given in Appendix 4.

The AMF transition path is very close to the benchmark TPI transition path even though we use a naïve alternative model to forecast future wage rates and interest rates. In terms of mean percent deviation from the TPI path, the approximation error of the AMF path was only 0.7 percent. It took 4 h, 50 min, and 9 s to compute the solution by the AMF method, which is roughly 15 percent of the TPI computation time.

3.3 Comparison of Solution Methods and Robustness

Because the AMF method is an approximation of the benchmark TPI method, the goal of this paper is to compare the AMF method to the TPI method in terms of both computing time and accuracy. The benchmark TPI method for computing the non-steady-state equilibrium transition path of the distribution of capital

[11] We tried this with a log-linear forecast between the current state and the steady state, similar to Krusell and Smith (1998), and the transition path was nearly identical to the one from Fig. 3 using the more naïve linear forecast.

is the exact rational expectations equilibrium concept. The AMF method approximates the benchmark TPI method by using the alternative model $\Omega_a(\cdot)$ to forecast future aggregate variables rather than the TPI method's rational expectations requirement. In addition to comparing computation speed and accuracy for the calibration given previously, we show how these comparisons change with different calibrations.

Table 1 shows the computation times and accuracy comparisons of the AMF method to the TPI method for Calibration 1 from Sects. 3.1 and 3.2 that generated the transition paths in Fig. 3 as well as for two additional calibrations that differ in terms of initial states and degree of heterogeneity.

The accuracy and speed comparison for the example presented in Sects. 3.1 and 3.2 are presented in the first line of Table 1 as Calibration 1. The AMF method reduces the computation time from almost 32 h to just under 5 h, an 85 % reduction. To measure the approximation error of the AMF transition path from the benchmark TPI transition path shown in Fig. 3, we use the mean absolute percent deviation (MAPD) of the AMF path from the TPI path over the first τ periods where $\tau = 60$ in this case.

$$MAPD = \frac{1}{\tau}\sum_{t=1}^{\tau}\frac{|K_t^{AMF} - K_t^{TPI}|}{K_t^{TPI}} \tag{3.8}$$

The approximation error in Calibration 1 is less than 1 percent (0.7 %).

In Calibration 2, we test the speed and accuracy of the AMF method in a version of the model with less heterogeneity. We use a basic simplification of reducing the grid points in the discretized continuum of possible wealth levels to $B = 200$ with the same bounds. The computation times for both the TPI and AMF methods are less, and the speed reduction of the AMF method over the TPI method is about the same as the baseline calibration. The approximation error of the AMF method in Calibration 2 is a little bit smaller than that of the baseline calibration. Figure 4 shows the TPI and AMF transition paths for Calibration 2.

In Calibration 3, we test the speed and accuracy of the AMF method in a version of the model with an initial state ($K_0 = 6.5$) that is closer to the steady state than in the baseline calibration. We keep the same number if grid points $B = 350$ as in

Table 1 Computation times and accuracy of TPI and AMF methods

Calibration	Speed (h)			MAPD from TPI path (%)
	TPI	AMF	% reduction	
1 ($B = 350$, $K_0 = 5.5$)	32.0	4.8	84.9	0.7
2 ($B = 200$, $K_0 = 5.5$)	11.7	1.6	86.0	0.6
3 ($B = 350$, $K_0 = 6.5$)	32.0	5.0	84.5	0.3

All computations were performed using MatLab on a Dell PowerEdge 2950 with 8 Intel Xeon E5345 2.33 GHz processor cores, 16 GB of RAM, and 500 GB RAID hard drive

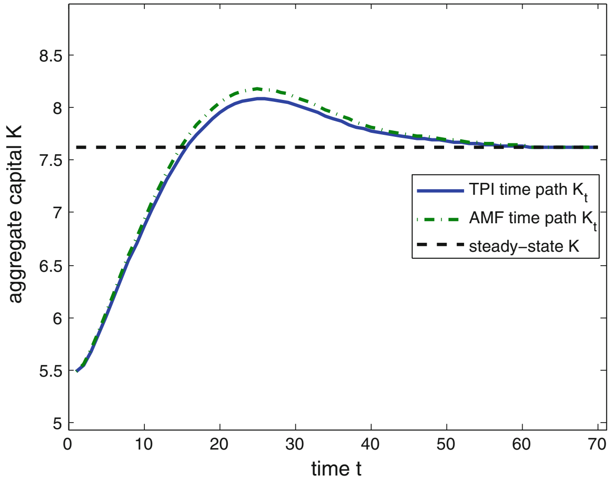

Fig. 4 TPI and AMF equilibrium transition paths for Calibration 2

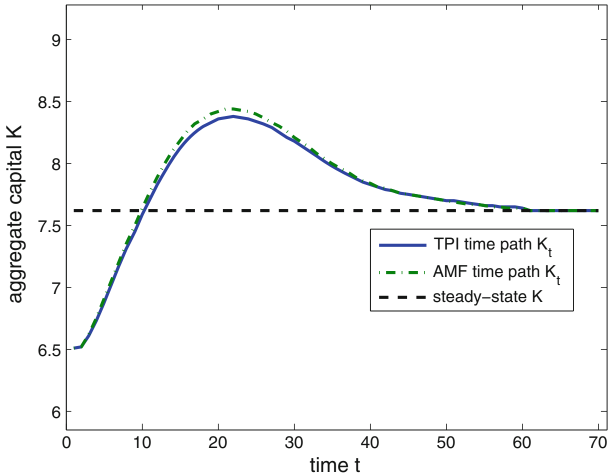

Fig. 5 TPI and AMF equilibrium transition paths for Calibration 3

the baseline calibration. The computation times for both the TPI and AMF methods are comparable to the baseline calibration with a reduction in computation time of 84.5 %. The mean percent deviation of the AMF transition path is smaller than that of the TPI method in Calibration 3 as would be expected with an initial state that is closer to the steady state. Figure 5 shows the TPI and AMF transition paths for Calibration 3.

In each calibration, the AMF method reduces computation times by about 85 percent, and the mean absolute percent deviations are less than 1 percent. It is important to note that the calibrations with the highest approximation error used an initial state that was relatively far away from the new steady state. In practice, most policy experiments study changes that imply a much smaller difference between the initial state and the new steady state. The results of this paper suggest that the approximation error of the AMF method will be significantly less than 1 percent in terms of mean percent deviation in more realistic policy experiments.

4 Conclusion

We propose a method for computing rational expectations equilibrium transition paths for OLG life cycle models that reduces computation time relative to the benchmark TPI method by 85 percent and has an approximation error of less than 1 percent. For our main calibration, the AMF method reduced the computation time from 32 hours to less than 5 h.

The AMF method presented in this paper used extremely naïve alternative models for forecasting future prices. When these models are actually taken to the data to perform policy experiments, more sophisticated alternative models could further reduce the approximation error without increasing computation time. For example, a VAR could be used to forecast future prices based on past observables in the data.

An obvious extension of our AMF method is to use it to calculate transition paths for infinite horizon models. Krusell and Smith (1998) use a similar idea to estimate the parameters of a stationary equilibrium in an environment with infinitely lived heterogeneous agents. The AMF method extends this idea and could simplify the equilibrium transition path computation in this class of models.

Another characteristic common to the infinitely lived heterogenous agent models that is often missing in large OLG life cycle models is an uninsurable aggregate shock. A transition path in an environment with aggregate uncertainty would be a stochastic object and would require some notion of confidence intervals computed by simulation. Because each computation of a transition path by the benchmark TPI method can take more than a day, simulation of confidence intervals can be computationally impractical. The increased computational speed of the AMF method makes simulation more practical.[12]

Lastly, linearization methods are most commonly use for computing equilibrium solutions to dynamic general equilibrium models, but they have not been applied to OLG life cycle models with occasionally binding constraints. Uhlig (1999) and Christiano (2002) present the standard method of undetermined coefficients linearization method for these types of models. Computers are particularly well suited for dealing with linear systems, and few methods can match linearization in speed. However, both Christiano and Uhlig note that the method of undetermined coefficients only works in models in which there are no occasionally binding constraints.[13] Borrowing constraints are a leading example of occasionally binding constraints and are an important characteristic of OLG life cycle models. A linearization method for

[12] As an example, 1,000 simulations of the TPI transition path in main calibration presented in Sect. 3.1 would take 3.65 computer years. The same simulations would take only 0.55 computer years using the AMF method.

[13] Christiano and Fisher (2000) detail a parameterized expectations algorithm for solving infinite horizon DSGE models with occasionally binding constraints. But it is not a linearization method.

solving OLG life cycle models with occasionally binding constraints has the potential to increase computation speeds enough to easily simulate the models.

Much research has been dedicated to solution methods for DSGE models with infinitely lived agents. Taylor and Uhlig (1990) survey a number of papers dedicated to various solution methods to the nonlinear rational expectations stochastic growth model. More recently, Aruoba et al. (2006) compare perturbation methods, finite elements methods, Chebyshev polynomial approximation, and value function iteration solution methods on the stochastic neoclassical growth model. Fernández-Villaverde and Rubio-Ramírez (2006) focus on the value of perturbation methods in solving the growth model. It is our hope that more efforts are dedicated to transition path solution methods for the valuable OLG life cycle model.

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Appendix 1: Description of Differences Between OLG and Infinite Horizon Models

In this section, we describe the difference between the OLG model with idiosyncratic uncertainty and no aggregate uncertainty and its infinite horizon counterpart. Chatterjee (1994) and Krusell and Ríos-Rull (1999) show that infinite horizon models with standard period utility functions and only idiosyncratic uncertainty with no aggregate uncertainty exhibit an aggregation theorem. That is, the law of motion for the aggregate capital stock, which is a function of the entire distribution of capital among idiosyncratically heterogeneous households, can be derived as a function solely of aggregate variables in the current information set.

This means that the infinite horizon analogue of our OLG model with idiosyncratic uncertainty has no need of adaptive expectations. Household beliefs about future interest rates and wages in the rational expectations infinite horizon model are simply a function of the aggregate capital stock in the current period. No alternate model forecast rule can be used here because the exact rational expectations law of motion can already be calculated with limited information.

For this reason, we only use our AMF approximation solution method for the OLG model with idiosyncratic uncertainty. The aggregation theorem does not hold in the OLG model with only idiosyncratic uncertainty, because each agent type within each generation cannot perfectly smooth consumption over its lifetime. This is the reason that the fundamental welfare theorems of economics do not hold in OLG models, in general. It is also the reason that the aggregation theorem does not hold, and forecasts of the aggregate capital stock remain a function of the entire distribution of capital rather than just a summary statistic (the aggregate capital stock) in the current information set.

For heterogeneous agent models with both idiosyncratic and aggregate uncertainty, other solution methods have been developed, such as Krusell and Smith (1998), Krueger and Kubler (2004), and Judd et al. (2011). But the OLG model with only idiosyncratic uncertainty remains important for answering many economic questions. Therefore, solution methods for this class of models are relevant.

Appendix 2: Computational Algorithm for Steady-State Equilibrium

The computation of the steady-state equilibrium described in Definition 1 requires the following steps. The MatLab code for this steady-state computation is available upon request.

1. Calibrate the exogenous parameters of the model S, β, σ, α, A, δ, the computation parameter ρ, the distribution of the ability shock $f_s(e)$, and the inelastic labor supply function as a function of age $n(s)$.
   - We chose the parameter values $[S, \beta, \sigma, \alpha, \rho, A, \delta] = [60, 0.96, 3, 0.35, 0.2, 1, 0]$.
   - The inelastic labor supply function of age $n(s)$ was calibrated to match the average labor supply by age reported in the CPS monthly survey, where the maximum average hours worked is normalized to unity.

$$n(s) = \begin{cases} [0.87, 0.89, 0.91, 0.93, 0.96, 0.98] & \text{for} \quad 1 \leq s \leq 6 \\ 1 & \text{for} \quad 7 \leq s \leq 40 \\ \big[0.95, 0.89, 0.84, 0.79, 0.73, 0.68, 0.63, 0.57, 0.52, \ldots \\ \quad 0.47, 0.40, 0.33, 0.26, 0.19, 0.12, 0.11, 0.11, 0.10, 0.10, 0.09\big] \\ & \text{for} \quad 41 \leq s \leq 60 \end{cases}$$

   - The discretized approximation of the ability shock is the following seven ability types $e_{s,t} \in \{0.1, 0.5, 0.8, 1.0, 1.2, 1.5, 1.9\}$ with a mass function $f_s(e) = \{0.04, 0.09, 0.20, 0.34, 0.20, 0.09, 0.04\}$ for all s. We could have just as easily made the probability distribution be conditional on s but that does not increase the computation time.
2. Discretize the space of possible wealth levels into B possible values such that $b \in \{b_1, b_2, \ldots b_B\}$, where $b_1 = 0$ and $b_B < \infty$.
   - We chose a discretized support of $B = 350$ equally spaced points between $b_1 = 0$ and $b_B = 15$.
   - Note that we have to impose a savings maximum constraint due to there being some states in which the household wants to save more than their current wealth level. Setting $b_{max} = 15$ is high enough to minimize the number of states in which the upper bound binds. The smoothness of Fig. 1 at its peak shows that the upper bound creates a minimal distortion.
3. Choose an arbitrary initial guess for the steady-state distribution of wealth $\bar{\Gamma}_0 = \bar{\gamma}_0(s, e, b)$ such that $\bar{\gamma}_0(s, e, b) \in [0, 1]$ and $\sum_s \sum_e \sum_b \bar{\gamma}_0(s, e, b) = 1$.
   - Our initial guess was simply the distribution across abilities by age spread across each possible wealth level $\gamma_0(s, e, b) = f_s(e)/[(S-1)B]$ for all s, e, and b.

4. Use $\bar{\Gamma}_0$ to calculate steady-state values for $\bar{K}_0$, $\bar{Y}_0$, $\bar{r}_0$ and $\bar{w}_0$ using Eqs. (2.11), (2.13), (2.14), (2.15), and (2.16).
5. Taking $\bar{r}_0$ and $\bar{w}_0$ as given each period, solve for the optimal policy rule of each agent $b' = \phi(s, e, b|\Omega)$ by backward induction.
6. Use $b' = \phi(s, e, b|\Omega)$ and $\bar{\Gamma}_0$ to calculate the distribution of wealth in the next period $\bar{\Gamma}'_0$.
7. Generate a new guess for the steady-state distribution of wealth $\bar{\Gamma}_1$ as a convex combination of the two distributions from the previous step $\bar{\Gamma}_1 = \rho\bar{\Gamma}'_0 + (1-\rho)\bar{\Gamma}_0$, where $\rho \in (0, 1)$.
8. Repeat steps (4) through (7) until the distance between $\bar{\Gamma}_i$ and $\bar{\Gamma}'_i$ is arbitrarily close to zero, where i is the index of the iteration number. Let $|\cdot|$ be the sup norm and let $\varepsilon > 0$ be some scalar arbitrarily close to zero. Then the steady state $\bar{\Gamma}$ is found when $\left|\bar{\Gamma}_i - \bar{\Gamma}'_i\right| < \varepsilon$.

In our example, the computation of the steady-state equilibrium took 3 h, 5 min, and 25 s. Figure 1 shows the steady-state aggregate capital stock $\bar{K}$ and the average wealth $\bar{b}_s$ as a function of age s.

Appendix 3: Computational Algorithm for TPI Transition Path

The computation of the TPI transition path described in Sect. 3.1 requires the following steps. The MatLab code for this TPI transition path computation is available upon request.

1. Using the parameterization from the steady-state computation, and choose the value for T at which the non-steady-state transition path should have converged to the steady state. We used $T = 60$.
2. Choose an initial state of the aggregate capital stock K_1. Choose an initial distribution of capital Γ_1 consistent with K_1 according to (2.16).
   - We chose an initial capital stock of $K_1 = 5.45$, which is consistent with a simple initial distribution of wealth—the distribution of ability by age spread across all possible wealth levels $\gamma_1(s, e, b) = f_s(e)/[(S-1)B]$ for all s, e, and b.
3. Conjecture a transition path for the aggregate capital stock $\mathbf{K}^i = \{K_t^i\}_{t=1}^{\infty}$ where the only requirements are that $K_1^i = K_1$ is your initial state and that $K_t^i = \bar{K}$ for all $t \geq T$. The conjectured transition path of the aggregate capital stock $\mathbf{K}^i$, along with the exogenous aggregate labor supply from (2.15), implies specific transition paths for the real wage $\mathbf{w}^i = \{w_t^i\}_{t=1}^{\infty}$ and the real interest rate $\mathbf{r}^i = \{r_t^i\}_{t=1}^{\infty}$ through expressions (2.11), (2.13), and (2.14).
4. With the conjectured transition paths $\mathbf{w}^i$ and $\mathbf{r}^i$, one can solve for the lifetime policy functions of each household alive at time $t = 1$ by backward induction using the Euler equations of the form (3.3). Rows 1 through 5 of Table 2 illustrate this process.
   - The first line is solving for the solution of the individual who is age $S-1$ at time $t = 1$ obtaining $b_{2,2} = \phi_1(S-1, e, b)$ from Eq. (3.1).
   - Each subsequent row from Table 2 represents the solution of the lifetime savings policy functions of an individual with more years remaining in their life

at time $t = 1$, down the the person who is age $s = 1$ at time $t = 1$ and has the entire set of $S - 1$ policy functions characterized by (3.3).

5. In similar fashion to step (4), solve for the lifetime policy functions by backward induction for the age $s = 1$ household at times $2 \le t \le T$. In Table 2, this means solving for the policy functions in the last two rows down to the age $s = 1$ household at time $t = T$.
6. Each column in Table 2 represents a complete set of policy functions for the corresponding period. Using the initial distribution of wealth Γ_1 and all the period $t = 1$ policy functions $\phi_1(s, e, b)$ for the households alive at time $t = 1$, the next period distribution of wealth Γ_2 and the corresponding aggregate capital stock $K_2^{i'}$ can be calculated. Consecutively repeat this procedure for each time period (column of Table 2) until a new transition path for the aggregate capital stock has been computed $\mathbf{K}^{i'} = \{K_t^{i'}\}_{t=1}^{T}$.
7. Generate a new guess for the transition path of the aggregate capital stock $\mathbf{K}^{i+1}$ as a convex combination of the initially conjectured transition path $\mathbf{K}^i$ and the newly generated transition path $\mathbf{K}^{i'}$.

$$\mathbf{K}^{i+1} = \rho\mathbf{K}^{i'} + (1-\rho)\mathbf{K}^i \quad \text{where} \quad \rho \in (0, 1)$$

8. Repeat steps (4) through (7) until the distance between $\mathbf{K}^{i'}$ and $\mathbf{K}^i$ is arbitrarily close to zero, where i is the index of the iteration number. Let $|\cdot|$ be the sup norm and let $\varepsilon > 0$ be some scalar arbitrarily close to zero. Then the rational expectations equilibrium transition path of the economy is found when when $\left|\mathbf{K}^{i'} - \mathbf{K}^i\right| < \varepsilon$.

In our example, the computation of the TPI transition path took 31 h, 59 min, and 39 s. Figure 2 shows the transition path of the aggregate capital stock from its initial state at K_1 to the steady state $\bar{K}$. The aggregate capital stock arrived at its steady state in about 60 periods.

Appendix 4: Computational Algorithm for AMF Transition Path

The computation of the AMF transition path described in Definition 3 requires the following steps. The MatLab code for this AMF transition path computation is available upon request.

1. Conjecture an alternative model forecast method Ω_a.
   - We use a linear trend from the current state K_t to the steady state $\bar{K}$.

$$K_{t+1} = \Omega_a(K_t) \Rightarrow K_{t+1} = K_t + \frac{\bar{K} - K_t}{T - t} \tag{3.7}$$

   - Our specific alternative model is written as a law of motion for the aggregate capital stock, but it implies a law of motion for the average wealth. From (2.16)

Table 2 TPI backward induction policy function solution method

$t=1$	$t=2$	$t=3$	$\cdot$	$t=S-2$	$t=S-1$	$t=S$
$\phi_1(S-1,e,b)$						
$\phi_1(S-2,e,b)$	$\phi_2(S-1,e,b)$					
$\phi_1(S-3,e,b)$	$\phi_2(S-2,e,b)$	$\phi_3(S-1,e,b)$				
$\vdots$	$\vdots$	$\vdots$				
$\phi_1(2,e,b)$	$\phi_2(3,e,b)$	$\phi_3(4,e,b)$	$\cdots$			
$\phi_1(1,e,b)$	$\phi_2(2,e,b)$	$\phi_3(3,e,b)$	$\cdots$	$\phi_{S-2}(S-1,e,b)$		
	$\phi_2(1,e,b)$	$\phi_3(2,e,b)$	$\cdots$	$\phi_{S-2}(S-2,e,b)$	$\phi_{S-1}(S-1,e,b)$	
		$\phi_3(1,e,b)$	$\cdots$	$\phi_{S-2}(S-3,e,b)$	$\phi_{S-1}(S-2,e,b)$	$\phi_S(S-1,e,b)$
				$\vdots$	$\vdots$	$\vdots$
Γ_2, K_2	Γ_3, K_3	Γ_4, K_4	$\cdots$	Γ_{S-1}, K_{S-1}	Γ_S, K_S	Γ_{S+1}, K_{S+1}

we know that aggregate capital K_t is just a function of the average wealth.

$$K_t = \frac{S-1}{S}\bar{b}_t \tag{4.1}$$

So the alternative model Ω_a implies a similar linear law of motion for the moments by combining (3.7) with (4.1).

$$\bar{b}_{t+1} = \Omega_a\left(\bar{b}_t\right) \Rightarrow \bar{b}_{t+1} = \bar{b}_t + \frac{\bar{b}_{ss} - \bar{b}_t}{T-t} \tag{4.2}$$

2. Solve the lifetime savings policy functions $\phi(s, e, b)$ for each agent alive at time t by backward induction using the alternate model forecast method (3.7) to obtain the forecasted series of prices over those lifetimes. (This step is the same as step 4 in Appendix 4.)
3. Use the complete set of policy functions for the current period in order to calculate the next period's distribution of wealth Γ_{t+1} and the corresponding aggregate capital stock K_{t+1}.
4. Repeat this process until the distribution of wealth Γ_T and the aggregate capital stock K_T have been computed for time T. Make sure that $\Gamma_T = \bar{\Gamma}$ and $K_T = \bar{K}$.

In our example, the computation of the AMF transition path took 4 h, 50 min, and 9 s. Figure 3 shows the transition path of the aggregate capital stock from its initial state at K_1 to the steady state $\bar{K}$ as compared to the benchmark TPI transition path. The aggregate capital stock arrived at its steady state in about 60 periods.

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