Documentation for `OG-USA` *

(version 16.03.a)

Abstract

This document describes the `OG-USA` dynamic general equilibrium (DGE) overlapping generations macroeconomic model, its derivation, characterizing equations, calibration, and solution method. This document also details the integration of the tax information from the `Tax-Calculator` package into the tax functions in the `OG-USA` model.

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This document describes the `OG-USA` overlapping generations macroeconomic model. Section 1 characterizes the model. Section 2 details the integration of the tax information from the `Tax-Calculator` microsimulation model into the `OG-USA` dynamic general equilibrium (DGE) macroeconomic model.

1 Dynamic General Equilibrium Model

The DGE model is comprised of overlapping generations of homogenous households, perfectly competitive firms, and a government with a balanced budget requirement. A unit measure of identical firms make a static profit maximization decision in which they rent capital and hire labor to maximize profits given a Cobb-Douglas production function. The government levies taxes on households and makes lump sum transfers to households according to a balanced budget constraint.

Households are assumed to live for a maximum of $E+S$ periods. We define an age-s household as being in youth and out of the workforce during ages $1 \leq s \leq E$. We implement this dichotomy of being economically relevant by age in order to more easily match true population dynamics. Households enter the workforce at age $E+1$ and remain in the workforce until they die or until the maximum age $E+S$. Because of mortality risk, households can leave both intentional bequests at the end of life ($s = E+S$) as well as accidental bequests if they die before the maximum age of $E+S$.

Households face deterministic hourly earnings process. Hourly earnings may vary over the lifecycle. Given this exogenous hourly earnings process, households choose their labor supply, savings, and consumption in each year of their lifetime. The only uncertainty in the model is from the mortality risk the households face. The economic environment is one of incomplete markets because the overlapping generations structure prevents households from perfectly smoothing consumption.

1.1 Population dynamics and lifetime earnings profiles

We define $\omega_{s,t}$ as the number of households of age s alive at time t. We normalize the population of newborns each period t to $\omega_{1,t} = 1$ and live for up to $E+S$ periods, with $S \geq 4$.[1] Households are termed "youth", and do not participate in market activity during ages $1 \leq s \leq E$. The households enter the workforce and economy in period $E+1$ and remain in the workforce until they unexpectedly die or live until age $s = E+S$.[2] The population of agents of each age in each period $\omega_{s,t}$ evolves

[1] Theoretically, the model works without loss of generality for $S \geq 3$. However, because we are calibrating the ages outside of the economy to be one-fourth of S (e.g., ages 21 to 100 in the economy, and ages 1 to 20 outside of the economy), we need S to be at least 4.

[2] We model the population with households age $s \leq E$ outside of the workforce and economy in order to most closely match the empirical population dynamics. Appendix A-1 gives more detail on the population process and its calibration.

according to the following function,

$$
\begin{aligned}
\omega_{1,t} &= 1 && \forall t \\
\omega_{s+1,t+1} &= (1-\rho_s)\omega_{s,t} && \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1
\end{aligned}
\tag{1}
$$

where ρ_s is an age specific mortality hazard rate.[3] The total population in the economy N_t at any period is simply the sum of households in the economy. Given the law of motion for the population and the constant measure of households born in each period, the population growth rate is zero and the population distribution is stationary. This stationary distribution of households over age is shown in Figure 1. We define parameters are defined as:

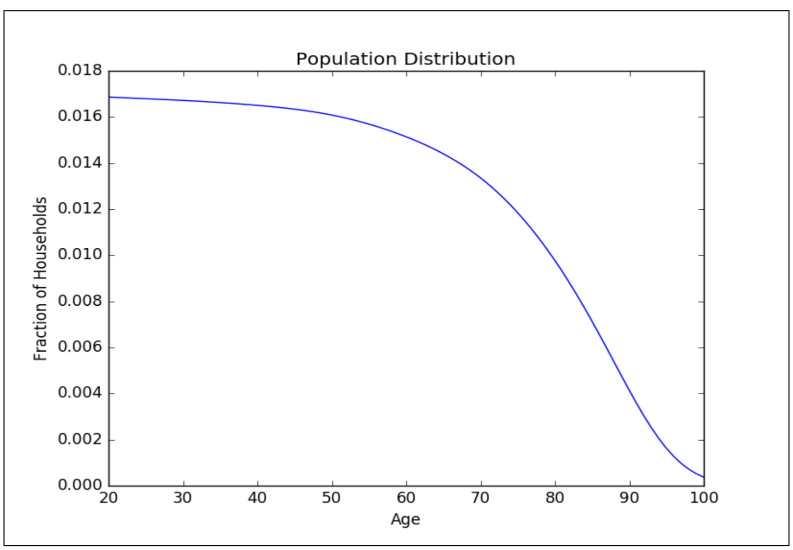
$$
N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{2}
$$

$$
g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{3}
$$

$$
\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{4}
$$

$$
\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{5}
$$

Figure 1: Exogenous population distribution

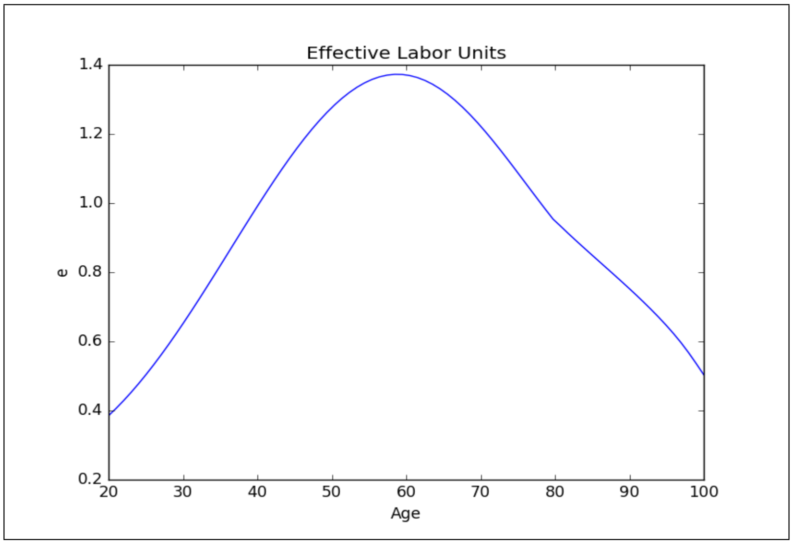


A household's working ability evolves over its working-age lifetime $E+1 \leq s \leq E+S$ according to an age-dependent deterministic process. The population weights

[3]The parameter ρ_s is the probability that a household of age s dies before age $s+1$.

$\omega_{s,t}$ as well as lifetime earnings are exogenous inputs to the model. Figure 2 shows the calibrated trajectory of effective labor units (ability), e_s, by age s.[4] The exogenous earnings process is taken from DeBacker et al. (2015).[5]

Figure 2: Exogenous life cycle profile of effective labor units



1.2 Household problem

Households are endowed with a measure of time $\tilde{l}$ in each period t, and they choose how much of that time to allocate between labor $n_{s,t}$ and leisure $l_{s,t}$ in each period. That is, a household's labor and leisure choice is constrained by its total time endowment.

$$n_{s,t} + l_{s,t} = \tilde{l} \tag{6}$$

At time t, all age-s households with ability e_s know the real wage rate, w_t, and know the one-period real net interest rate, r_t, on bond holdings, $b_{s,t}$, that mature at the beginning of period t. They also receive accidental and intentional bequests. They choose how much to consume $c_{s,t}$, how much to save for the next period by loaning capital to firms in the form of a one-period bond $b_{s+1,t+1}$, and how much to work $n_{s,t}$

[4]The units for this figure or normalized effective labor units, where the normalization is that which makes the weighted average effective labor units equal to one. In this way, the model wage will represent the compensation for a single effective labor hour.

[5]We collapse their heterogenous profiles into a single profile for our representative agent.

in order to maximize expected lifetime utility of the following form,

$$\begin{aligned}
U_{s,t} = &\sum_{u=0}^{E+S-s}\beta^u\left[\prod_{v=s}^{s+u-1}(1-\rho_v)\right]u\left(c_{s+u,t+u},n_{s+u,t+u},b_{s+u+1,t+u+1}\right)\\
\text{and}\quad &u\left(c_{s,t},n_{s,t},b_{s+1,t+1}\right) = \frac{(c_{s,t})^{1-\sigma}-1}{1-\sigma}...\\
&\quad + e^{g_y t(1-\sigma)}\chi^n_s\left(b\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}}+k\right)+\rho_s\chi^b\frac{(b_{s+1,t+1})^{1-\sigma}-1}{1-\sigma}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\forall t\quad\text{and } E+1\leq s\leq E+S
\end{aligned}\tag{7}$$

where $\sigma \geq 1$ is the coefficient of relative risk aversion on consumption and on intended (precautionary) bequests, $\beta \in (0,1)$ is the agent's discount factor, and the product term in brackets depreciates the household's discount factor by the cumulative mortality rate. The disutility of labor term in the period utility function looks nonstandard, but is simply the upper right quadrant of an ellipse that closely approximates the standard CRRA utility of leisure functional form.[6] The term χ^n_s is a constant term that varies by age s influencing the disutility of labor relative to the other arguments in the period utility function,[7] and g_y is a constant growth rate of labor augmenting technological progress, which we explain in Section 1.3.[8]

The last term in (7) incorporates a warm-glow bequest motive in which households value having savings to bequeath to the next generation in the chance they die before the next period. As was mentioned in Section 1.1, households in the model have no income uncertainty because each lifetime earnings path e_s deterministic, model agents thus hold no precautionary savings. Savings is thus motivated by the households preference to smooth consumption over their lifecycle and to provide for intentional and unintentional bequests. The parameter χ^b adjusts the level of the warm-glow utility of bequests.

The parameter $\sigma \geq 1$ is the coefficient of relative risk aversion on bequests, and the mortality rate ρ_s appropriately discounts the value of this term.[9] Note that, because of this bequest motive, households in the last period of their lives ($s = S$)

[6] Appendix A-2 describes how the elliptical function closely matches the more standard constant Frisch elasticity disutility of labor of the form $-\frac{(n_{j,s,t})^{1+\theta}}{1+\theta}$. This elliptical utility function forces an interior solution that automatically respects both the upper and lower bound of labor supply, which greatly simplifies the computation of equilibrium. In addition, the elliptical disutility of labor has a Frisch elasticity that asymptotes to a constant rather than increasing to infinity as it does in the CRRA case. For a more in-depth discussion see Evans and Phillips (2015)

[7] DeBacker et al. (2015) calibrate χ^n_s and χ^b to match average labor hours by age and some moments of the distribution of wealth.

[8] The term with the growth rate $e^{g_y t(1-\sigma)}$ must be included in the period utility function because consumption and bequests will be growing at rate g_y and this term stationarizes the individual Euler equation by making the marginal disutility of labor grow at the same rate as the marginal benefits of consumption and bequests. This is the same balanced growth technique as that used in Mertens and Ravn (2011).

[9] It is necessary for the coefficient of relative risk aversion σ to be the same on both the utility of consumption and the utility of bequests. If not, the resulting Euler equations are not stationarizable.

will die with positive savings $b > 0$. Also note that the CRRA utility of bequests term prohibits negative wealth holdings in the model, but is not a strong restriction since none of the wealth data for the lifetime income group and age s cohorts is negative except for the lowest quartile of incomes.

The per-period budget constraints for each agent normalized by the price of consumption are the following,

$$\begin{aligned} c_{s,t} + b_{s+1,t+1} \leq (1+r_t)\,b_{s,t} + w_t e_s n_{s,t} + \frac{BQ_t}{\tilde{N}_t} - T_{s,t} \\ \text{where} \quad b_{E+1,t} = 0 \quad \text{for} \quad E+1 \leq s \leq E+S \quad \forall t \end{aligned} \tag{8}$$

where $\tilde{N}_t$ is the total working age population at time t defined in (4). Note that the price of consumption is normalized to one, so w_t is the real wage and r_t is the net real interest rate. The term BQ_t represents total bequests from households who died at the end of period $t-1$. $T_{s,t}$ is a function representing net taxes paid, which we specify more fully below in equation (10).

Because the form of the period utility function in (7) ensures that $b_{s,t} > 0$ for all j, s, and t, total bequests will always be positive $BQ_t > 0$ for all j and t.

$$BQ_{t+1} = (1+r_{t+1})\left(\sum_{s=E+1}^{E+S} \rho_s \omega_{s,t} b_{s+1,t+1}\right) \quad \forall t \tag{9}$$

In addition to each the budget constraint in each period, the utility function (7) imposes nonnegative consumption through infinite marginal utility, and the elliptical utility of leisure ensures household labor and leisure must be strictly nonnegative $n_{s,t}, l_{s,t} > 0$. Because household savings or wealth is always strictly positive, the aggregate capital stock is always positive.[10] An interior solution to the household's problem (7) is assured.

In reality, each household is subject to many different taxes, all of which cannot be modeled in a DGE framework. It is the net tax liability function $T_{s,t}$ that we estimate from the microsimulation model output and characterize in a simple way. This tax liability output includes information on all federal individual income and payroll taxes in the U.S. tax code. We also assume that every individual also receives an equal lump sum transfer T^H_t which is generated from a balanced budget constraint on the government. We represent the net tax liability function as a constant effective tax rate τ_t for all ages s in a particular period t times total labor and capital income and minus the lump sum transfer.

$$T_{s,t} = \tau_t\big(w_t e_s n_{s,t} + r_t b_{s,t}\big) - T^H_t \quad \forall s,t \tag{10}$$

We detail the estimation of the effective tax rate in each year τ_t in Section 2.

The solution to the lifetime maximization problem (7) of the household subject to the per-period budget constraint (8) and the specification of taxes in (10) is a system

[10] An alternative would be to allow for household borrowing as long as an aggregate capital constraint $K_t > 0$ for all t is satisfied.

of $2S$ Euler equations. The S static first order conditions for labor supply $n_{s,t}$ are the following,

$$
\begin{aligned}
(c_{s,t})^{-\sigma}\left(w_t e_s - \frac{\partial T_{s,t}}{\partial n_{s,t}}\right) = e^{g_y t(1-\sigma)}\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \\
\forall t, \quad \text{and} \quad E+1 \leq s \leq E+S
\end{aligned}
\tag{11}
$$

$$
\text{where} \quad c_{s,t} = (1+r_t)\,b_{s,t} + w_t e_s n_{s,t} + \frac{BQ_t}{\tilde{N}_t} - b_{s+1,t+1} - T_{s,t}
$$

$$
\text{and} \quad b_{E+1,t} = 0 \quad \forall t
$$

where the marginal tax rate with respect to labor supply $\frac{\partial T_{s,t}}{\partial n_{s,t}}$ is described in equation (34) Section 2.

A household also has $S-1$ dynamic Euler equations that govern his saving decisions, $b_{s+1,t+1}$, with the included precautionary bequest saving in case of unexpected death. These are given by:

$$
\begin{aligned}
(c_{s,t})^{-\sigma} = \rho_s\chi^b\big(b_{s+1,t+1}\big)^{-\sigma} + \beta(1-\rho_s)(c_{s+1,t+1})^{-\sigma}\left[1 + r_{t+1} - \frac{\partial T_{s+1,t+1}}{\partial b_{s+1,t+1}}\right] \\
\forall t, \quad \text{and} \quad E+1 \leq s \leq E+S-1
\end{aligned}
\tag{12}
$$

where the marginal tax rate with respect to savings $\frac{\partial T_{s,t}}{\partial b_{s,t}}$ is described in (35) in Section 2. Lastly, Each household also has one static first order condition for the last period of life $s = E+S$, which governs how much to bequeath given that the household will die with certainty. This condition is simply equation (12) with $\rho_s = 1$.

$$
(c_{E+S,t})^{-\sigma} = \chi^b(b_{E+S+1,t+1})^{-\sigma} \quad \forall t \tag{13}
$$

Define $\hat{\mathbf{\Gamma}}_t$ as the distribution of stationary household savings across households at time t, including the intentional bequests of the oldest cohort.

$$
\hat{\mathbf{\Gamma}}_t \equiv \left\{\hat{b}_{s,t}\right\}_{s=E+2}^{E+S+1} \quad \forall t \tag{14}
$$

As will be shown in Section 1.5, the state in every period t for the entire equilibrium system described in the stationary, non-steady-state equilibrium characterized in Definition 2 is the stationary distribution of household savings $\hat{\mathbf{\Gamma}}_t$ from (14). Because households must forecast wages, interest rates, and aggregate bequests received in every period in order to solve their optimal decisions and because each of those future variables depends on the entire distribution of savings in the future, we must assume some household beliefs about how the entire distribution will evolve over time. Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\hat{\boldsymbol{\Gamma}}^{\boldsymbol{e}}_{\boldsymbol{t+u}} = \Omega^u\left(\hat{\boldsymbol{\Gamma}}_{\boldsymbol{t}}\right) \quad \forall t, \quad u \geq 1 \tag{15}$$

where the e superscript signifies that $\hat{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[11]

1.3 Firm problem

A unit measure of identical, perfectly competitive firms exist in the economy. The representative firm is characterized by the following Cobb-Douglas production technology,

$$Y_t = ZK_t^\alpha\left(e^{g_y t}L_t\right)^{1-\alpha} \quad \forall t \tag{16}$$

where Z is the measure of total factor productivity, $\alpha \in (0,1)$ is the capital share of income, g_y is the constant growth rate of labor augmenting technological change, and L_t is aggregate labor measured in efficiency units. The firm uses this technology to produce a homogeneous output which is consumed by households and used in firm investment. The interest rate r_t paid to the owners of capital is the real interest rate net of depreciation. The real wage is w_t. The real profit function of the firm is the following.

$$\text{Real Profits} = ZK_t^\alpha\left(e^{g_y t}L_t\right)^{1-\alpha} - (r_t+\delta)K_t - w_t L_t \tag{17}$$

As in the household budget constraint (8), note that the price output has been normalized to one.

Profit maximization results in the real wage, w_t, and the real rental rate of capital r_t being determined by the marginal products of labor and capital, respectively:

$$w_t = (1-\alpha)\frac{Y_t}{L_t} \quad \forall t \tag{18}$$

$$r_t = \alpha\frac{Y_t}{K_t} - \delta \quad \forall t \tag{19}$$

1.4 Government fiscal policy

The government is represented by a balanced budget constraint. The government collects taxes $\tau_t\left(w_t e_s n_{s,t} + r_t b_{s,t}\right)$ from all households and divides total revenues equally among households in the economy to determine the lump-sum transfer.

$$T_t^H = \frac{1}{\tilde{N}_t}\sum_s \omega_{s,t}\tau_t\big(w_t e_s n_{s,t} + r_t b_{s,t}\big) \tag{20}$$

[11] In Section 1.5 we will assume that beliefs are correct (rational expectations) for the stationary non-steady-state equilibrium in Definition 2.

1.5 Market clearing and stationary equilibrium

Labor market clearing requires that aggregate labor demand L_t measured in efficiency units equal the sum of household efficiency labor supplied $e_s n_{s,t}$. Capital market clearing requires that aggregate capital demand K_t equal the sum of capital investment by households $b_{s,t}$. Aggregate consumption C_t is defined as the sum of all household consumptions, and aggregate investment is defined by the resource constraint $Y_t = C_t + I_t$ as shown in (23). That is, the following conditions must hold:

$$L_t = \sum_{s=E+1}^{E+S} \omega_{s,t} e_s n_{s,t} \quad \forall t \tag{21}$$

$$K_t = \sum_{s=E+2}^{E+S+1} \omega_{s-1,t-1} b_{s,t} \quad \forall t \tag{22}$$

$$\begin{aligned} Y_t &= C_t + K_{t+1} - (1-\delta)K_t \quad \forall t \\ \text{where} \quad C_t &\equiv \sum_{s=E+1}^{E+S} \omega_{s,t} c_{s,t} \end{aligned} \tag{23}$$

An equilibrium would be defined by allocations and prices such that households optimize (11), (12), and (13), firms optimize (18) and (19), and markets clear (21) and (22). However, the variables in the equations characterizing the equilibrium are potentially non-stationary due to the growth rate in the total population $g_{n,t}$ each period coming from the cohort growth rates in (1) and from the deterministic growth rate of labor augmenting technological change g_y in (16).

Table 1: Stationary variable definitions

Sources of growth			Not
$e^{g_y t}$	$\tilde{N}_t$	$e^{g_y t}\tilde{N}_t$	growing[a]
$\hat{c}_{s,t} \equiv \frac{c_{s,t}}{e^{g_y t}}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{\tilde{N}_t}$	$\hat{Y}_t \equiv \frac{Y_t}{e^{g_y t}\tilde{N}_t}$	$n_{s,t}$
$\hat{b}_{s,t} \equiv \frac{b_{s,t}}{e^{g_y t}}$	$\hat{L}_t \equiv \frac{L_t}{\tilde{N}_t}$	$\hat{K}_t \equiv \frac{K_t}{e^{g_y t}\tilde{N}_t}$	r_t
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$		$\hat{BQ}_t \equiv \frac{BQ_t}{e^{g_y t}\tilde{N}_t}$	
$\hat{y}_{s,t} \equiv \frac{y_{s,t}}{e^{g_y t}}$		$\hat{C}_t \equiv \frac{C_t}{e^{g_y t}\tilde{N}_t}$	
$\hat{T}_{s,t} \equiv \frac{T_{s,t}}{e^{g_y t}}$			

[a] The interest rate r_t in (19) is already stationary because Y_t and K_t grow at the same rate. Household labor supply $n_{s,t}$ is stationary.

Table 1 characterizes the stationary versions of the variables of the model in terms of the variables that grow because of labor augmenting technological change, population growth, both, or none. With the definitions in Table 1, it can be shown that the equations characterizing the equilibrium can be written in stationary form in the following way. The static and intertemporal first-order conditions from the household's optimization problem corresponding to (11), (12), and (13) are the following:

$$
\begin{aligned}
&(\hat{c}_{s,t})^{-\sigma}\left(\hat{w}_t e_s - \frac{\partial \hat{T}_{s,t}}{\partial n_{s,t}}\right) = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall t, \quad \text{and} \quad E+1 \leq s \leq E+S
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\text{where}\quad & \hat{c}_{s,t} = (1+r_t)\,\hat{b}_{s,t} + \hat{w}_t e_s n_{s,t} + \hat{BQ}_t - e^{g_y}\hat{b}_{s+1,t+1} - \hat{T}_{s,t} \\
\text{and}\quad & \hat{b}_{E+1,t} = 0 \quad \forall t
\end{aligned}
$$

$$
\begin{aligned}
&(\hat{c}_{s,t})^{-\sigma} = \ldots \\
&e^{-g_y\sigma}\left(\rho_s\chi^b\left(\hat{b}_{s+1,t+1}\right)^{-\sigma} + \beta(1-\rho_s)(\hat{c}_{s+1,t+1})^{-\sigma}\left[1 + r_{t+1} - \frac{\partial \hat{T}_{s+1,t+1}}{\partial \hat{b}_{s+1,t+1}}\right]\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall t, \quad \text{and} \quad E+1 \leq s \leq E+S-1
\end{aligned}
\tag{25}
$$

$$
(\hat{c}_{E+S,t})^{-\sigma} = \chi^b e^{-g_y\sigma}(\hat{b}_{E+S+1,t+1})^{-\sigma} \quad \forall t \tag{26}
$$

The stationary firm first order conditions for optimal labor and capital demand corresponding to (18) and (19) are the following.

$$
\hat{w}_t = (1-\alpha)\frac{\hat{Y}_t}{\hat{L}_t} \quad \forall t \tag{27}
$$

$$
r_t = \alpha\frac{\hat{Y}_t}{\hat{K}_t} - \delta = \alpha\frac{Y_t}{K_t} - \delta \quad \forall t \tag{19}
$$

And the two stationary market clearing conditions corresponding to (21) and (22)—with the goods market clearing by Walras' Law—are the following.

$$
\hat{L}_t = \sum_{s=E+1}^{E+S}\hat{\omega}_{s,t}e_s n_{s,t} \quad \forall t \tag{28}
$$

$$
\hat{K}_t = \frac{1}{1+\tilde{g}_{n,t}}\left(\sum_{s=E+2}^{E+S+1}\hat{\omega}_{s-1,t-1}\hat{b}_{s,t}\right) \quad \forall t \tag{29}
$$

where $\tilde{g}_{n,t}$ is the growth rate in the working age population between periods $t-1$ and t described in (5). The stationary version of the goods market clearing condition (aggregate resource constraint) is the following.

$$
\hat{Y}_t = \hat{C}_t + e^{g_y}(1+\tilde{g}_{n,t+1})\hat{K}_{t+1} - (1-\delta)\hat{K}_t \quad \forall t \tag{30}
$$

It is also important to note the stationary version of the characterization of total bequests BQ_{t+1} from (9) and for the government budget constraint in (20).

$$\hat{BQ}_{t+1} = \frac{(1+r_{t+1})}{1+\tilde{g}_{n,t+1}}\left(\sum_{s=E+1}^{E+S}\rho_s\hat{\omega}_{s,t}\hat{b}_{s+1,t+1}\right) \quad \forall t \tag{31}$$

$$\hat{T}^H_t = \sum_s \hat{\omega}_{s,t}\hat{T}_{s,t} \tag{32}$$

We can now define the stationary steady-state equilibrium for this economy in the following way.

Definition 1 (**Stationary steady-state equilibrium**). A non-autarkic stationary steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability e_s is defined as constant allocations $n_{s,t} = \bar{n}_s$ and $\hat{b}_{s+1,t+1} = \bar{b}_{s+1}$ and constant prices $\hat{w}_t = \bar{w}$ and $r_t = \bar{r}$ for all s, and t such that the following conditions hold:

i. households optimize according to (24), (25), and (26),

ii. Firms optimize according to (27) and (19),

iii. Markets clear according to (28) and (29), and

iv. The population has reached its stationary steady state distribution $\bar{\omega}_s$ for all ages s, characterized in Appendix A-1.

The steady-state equilibrium is characterized by the system of $2S$ equations and $2S$ unknowns $\bar{n}_s$ and $\bar{b}_{s+1}$. Appendix A-3 details how to solve for the steady-state equilibrium.

The non-steady state equilibrium is characterized by $2ST$ equations and $2ST$ unknowns, where T is the number of periods along the transition path from the current state to the steady state. The definition of the stationary non-steady-state equilibrium is similar to Definition 1, with the stationary steady-state equilibrium definition being a special case of the stationary non-steady-state equilibrium.

Definition 2 (**Stationary non-steady-state equilibrium**). A non-autarkic stationary non-steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability e_s is defined as allocations $n_{s,t}$ and $\hat{b}_{s+1,t+1}$ and prices $\hat{w}_t$ and r_t for all s, and t such that the following conditions hold:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings, and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\hat{\boldsymbol{\Gamma}}_{t+u} = \hat{\boldsymbol{\Gamma}}^e_{t+u} = \Omega^u\left(\hat{\boldsymbol{\Gamma}}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (24), (25), and (26)

iii. Firms optimize according to (27) and (19), and

iv. Markets clear according to (28) and (29).

We describe the methodology to compute the solution to the non-steady-state equilibrium to Appendix A-4. We use the equilibrium transition path solution to find effects of tax policies on macroeconomic variables over the budget window.

1.6 Calibration

Table 2 shows the calibrated values for the exogenous variables and parameters. Note that the disutility of labor weight parameter, χ^n_s, takes on 80 values (one for each model age) that increase with age, representing an increasing disutility of labor that is not modeled anywhere else in the utility function. An hour of labor for an older person becomes more costly due to biological reasons related to aging. Such a parametrization helps to fit fact that hours worked decline much more sharply later in life than do hourly earnings. The parameter representing the utility weight on bequests, χ^b, is set to result in a pre-tax SS interest rate of 4.5%.

2 Calibrating Tax Rates

We calibrate the effective tax rate τ_t from (10) and the two marginal tax rates $\frac{\partial T_{s,t}}{\partial n_{s,t}}$ and $\frac{\partial T_{s+1,t+1}}{\partial b_{s+1,t+1}}$ from (11) and (12), respectively, using tax data from the `Tax-Calculator` micro-

Table 2: List of exogenous variables and baseline calibration values

Symbol	Description	Value
$\hat{\mathbf{\Gamma}}_1$	Initial distribution of savings	$\bar{\mathbf{\Gamma}}(SS distribution)$
N_0	Initial population	1
$\{\omega_{s,0}\}_{s=1}^{S}$	Initial population by age	$\{\bar{\omega}_{s,0}\}_{s=1}^{S}$ (SS distribution)
$\{\rho_s\}_{s=1}^{S}$	Mortality rates by age	(see App. A-1)
$\{e_s\}_{s=1}^{S}$	Deterministic ability process	(see DeBacker et al., 2015)
S	Maximum periods in economically active household life	80
E	Number of periods of youth economically outside the model	round $\left(\frac{S}{4}\right)$
R	Retirement age (period)	round $\left(\frac{9}{16}S\right)$
$\tilde{l}$	Maximum hours of labor supply	1
β	Discount factor	$(0.96)^{\frac{80}{S}}$
σ	Coefficient of constant relative risk aversion	1.5
b	Scale parameter in utility of leisure	0.573
υ	Shape parameter in utility of leisure	2.856
k	constant parameter in utility of leisure	0.000
χ^n_s	Disutility of labor level parameters	[19.041, 76.623]
χ^b	Utility of bequests level parameters	80
Z	Level parameter in production function	1.0
α	Capital share of income	0.35
δ	Capital depreciation rate	$1-(1-0.05)^{\frac{80}{S}}$
g_y	Growth rate of labor augmenting technological progress	$(1+0.03)^{\frac{80}{S}}-1$
T	Number of periods to steady state	160
ν	Dampening parameter for TPI	0.4

simulation model. The `Tax-Calculator` outputs microdata that can be used to calibrate these functions in a way that is consistent with the tax law parameters entered into the microsimulation model.

2.1 Microsimulation Model: `Tax-Calculator`

The microsimulation model we use is called `Tax-Calculator` and is developed and maintained by a group of researchers at the Open Source Policy Center (OSPC).[12] In this section, we outline the main structure of the `Tax-Calculator` microsimulation model, but encourage the interested reader to follow the links to the more detailed documentation.

`Tax-Calculator` uses microdata on tax filers from the tax year 2009 Public Use Files (PUF) produced by the IRS. These data contain detailed records from the tax returns of about 200,000 tax filers who were selected from the population of filers through a stratified random sample of tax returns. These data come from IRS Form 1040 and a set of the associated forms and schedules. The PUF data are then matched to the Current Population Survey (CPS) to get imputed values for filer demographics such as age, which are not included in the PUF, and to incorporate households from the population of non-filers. The PUF-CPS match includes 219,815 filers.

Since these data are for calendar year 2009, they must be "aged" to be representative of the potential tax paying population in the years of interest (e.g. the current year through the end of the budget window). To do this, macroeconomic forecasts of wages, interest rates, GDP, and other variables are used to grow the 2009 values to be representative of the values one might see in the years within the budget window. In addition to using macroeconomic variables to extrapolate the 2009 variables, a linear programming algorithm is used to re-weight the observations in each year in order to match target levels in the data such as total income and deduction amounts reported in more recent IRS data.

[12] The documentation for using `Tax-Calculator` is available at http://taxcalc.readthedocs.org/en/latest/index.html A simple web application that provides an easy user interface for `Tax-Calculator` is available at http://www.ospc.org/taxbrain/. And all the source code is freely available at https://github.com/open-source-economics/Tax-Calculator.

Using these microdata, `Tax-Calculator` is able to determine total tax liability and marginal tax rates by computing each filer's tax reporting that minimizes the total tax liability subject to the parameters describing the tax policy. Our determination of total tax liability from the microsimulation model includes federal income taxes and payroll taxes but excludes state income taxes and estate taxes.[13] The output of the microsimulation model is forecasts of the total tax liability in each year derived from marginal tax rates, and items from the filers' tax returns for each of the 219,815 filers in the microdata. Population sampling weights are determined through the extrapolation and targeting of the microsimulation model. These weights allow one to calculate population representative results from the model. One can determine changes in tax liability and marginal tax rates by doing the same simulation where the parameters describing the tax policy are updated to reflect the proposed policy rather than the baseline policy. Note that the baseline policy is a current-law baseline.

2.2 Taxes in `OG-USA`

To calculate the effective tax rates from the microsimulation model, we divided total tax liability by a measure of "adjusted total income". Adjusted total income is defined as total income (Form 1040, line 22) plus tax-exempt interest income, IRA distributions, pension income, and Social Security benefits (Form 1040, lines 8b, 15a, 16a, and 20a, respectively).

We consider adjusted total income from the microsimulation model to be the counterpart of total income in the DGE model. Total income in the DGE model is the sum of capital and labor income. We define labor income as earned income, which is the sum of wages and salaries (Form 1040, line 7) and self-employment income (Form 1040 lines 12 and 18) from the microsimulation model output. Capital income is defined as a residual.[14]

[13]As the microsimulation model is further developed, we will account for these.

[14]This is not an ideal definition of capital income, since it includes transfers between filers (e.g., alimony payments) and from the government (e.g., unemployment insurance), but we have chosen this definition for now in order to ensure that all of total income is classified as either capital or labor income. This accounting will be refined in the future.

To calculate marginal tax rates on any given income source, we add one cent to that income source for each individual in the micro-data and then use the tax calculator to calculate the change in tax liability. The change in tax liability divided by the change in income (one cent) yields the marginal tax rate. To get the marginal tax rate on composite income amounts (e.g., labor income that is the sum of wage and self-employment income), we take a weighted average that accounts for negative income amounts. We thus have to composite marginal tax rates which we calculate: MTR^l, the marginal rate on labor income (from employed and self-employed work) and MTR^k, the marginal tax rate on capital income (including interest, dividends, capital gains, and pension income). In particular, to we calculate the weighted average marginal tax rate on composite of n income sources as:

$$MTR_{composite} = \frac{\sum_{i=1}^{n} MTR_n * abs(Income_n)}{\sum_{i=1}^{n} abs(Income_n)} \tag{33}$$

When we look at the raw output from the microsimulation model, we find that there are several observations with extreme values for their effective tax rate. Since this is a ratio, such outliers are possible, for example when the denominator, adjusted total income, is very small. We omit such outliers by making the following restrictions on the raw output of the microsimulation model. First, we exclude observations with an effective tax rate greater than 1.5 times the highest statutory marginal tax rate. Second, we exclude observations where the effective tax rate is less than the lowest statutory marginal tax rate on income minus the maximum phase-in rate for the Earned Income Tax Credit (EITC). Third, we drop observations with marginal tax rates in excess of 99% or below the negative of the highest EITC rate (i.e., -45% under current law). These exclusions limit the influence of those with extreme values for their marginal tax rate, which are few and usually results from the income of the filer being right at a kink in the tax schedule. Finally, since total income cannot be negative in our DGE model, we drop observations from the microsimulation model where adjusted total income is less than \$5.[15]

[15] We choose \$5 rather than \$0 to provided additional assurance that small income values are not driving large ETRs.

Our approach to fitting tax functions is simple. This sacrifice in the richness in the model is helpful in ensuring that the DGE model can be simulated with a wide variety of proposals and still yield a solution—something that is helpful when the model is run in an automated fashion through TaxBrain. We use constant tax rates for the effective and marginal tax rates the households face. We characterized the effective tax rate τ_t in equation (10). We characterize the marginal tax rates that appear in the Euler equations (11) and (12) as the derivatives of the net tax liability function from (10).

$$\frac{\partial \hat{T}_{s,t}}{\partial \hat{n}_{s,t}} = \tau_t^{MTR,n} e_s w_t \tag{34}$$

$$\frac{\partial \hat{T}_{s+1,t+1}}{\partial \hat{b}_{s+1,t+1}} = \tau_t^{MTR,b} r_t b_{s,t} \tag{35}$$

In (34), $\tau_t^{MTR,n}$ is the marginal tax rate on labor income, and $\tau_t^{MTR,b}$ in (35) is the marginal tax rate on capital income. These constant (over age and income) rates are estimated as the weighted average effective tax rate, marginal tax rate on earned income, and marginal tax rate on capital income from the microsimulation output. We weight by the sampling weights from the data underlying the microsimulation as well as by income. Thus we estimate the average or marginal rate on the average dollar of income. The three respective tax rates are independently computed from the `Tax-Calculator` microsimulation in the following way,

$$\tau_t^{ETR} = \frac{\sum_{i=1}^{I} ETR_i * wgt_i * income_i}{\sum_{i=1}^{I} wgt_i * income_i}, \tag{36}$$

$$\tau_t^{MTR,n} = \frac{\sum_{i=1}^{I} MTR_i^n * wgt_i * income_i}{\sum_{i=1}^{I} wgt_i * income_i}, \tag{37}$$

$$\tau_t^{MTR,b} = \frac{\sum_{i=1}^{I} MTR_i^b * wgt_i * income_i}{\sum_{i=1}^{I} wgt_i * income_i}, \tag{38}$$

where i represents the filer observation in the microdata, t the calendar year the tax function refers to, τ is the tax rate of interest (with may be either the effective or marginal rates), and wgt are the sampling weights from the underlying microdata used by the tax calculator. We include both federal individual income taxes and payroll taxes in the the effective and marginal tax rates. The measure of income used in these calculations is that of the adjusted total income measure defined above, which maps to the total income in the computational model.

From `Tax-Calculator` , we are able to find the τ_t for each year in the budget window. The DGE model requires a much longer time horizon to reach the SS. Our assumption is that the tax policy in place at the end of the budget window is extended permanently beyond that time.

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APPENDIX

A-1 Characteristics of exogenous population dynamics

In this appendix, we detail how we generate the exogenous population dynamics that are inputs to the model described in Section 1.1. All output, tests, functions, and computation in this chapter are available in the `demographics.py` file.

Figure 3: Correspondence of model timing to data timing for model periods of one year

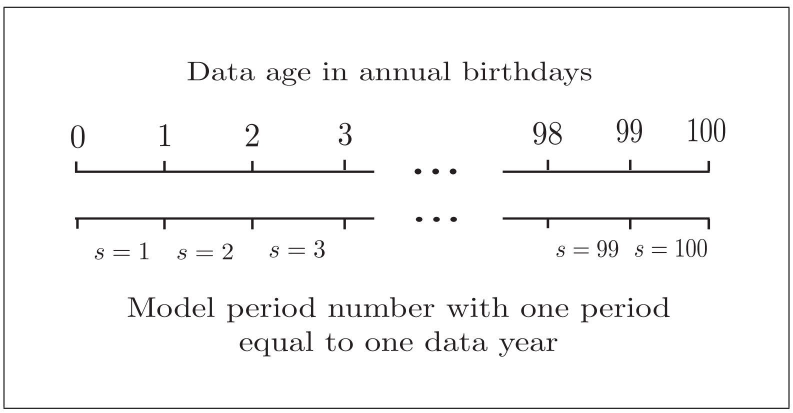


Figure 3 shows the correspondence between model periods and data periods. Period $s = 1$ corresponds to the first year of life between birth and when an individual turns one year old. We use this convention to match our model periods to those in the data.

A-1.1 Nonstationary and stationary population dynamics

We define $\omega_{s,t}$ as the number of households of age s alive at time t. We normalize the population of newborns each period t to $\omega_{1,t} = 1$ and live for up to $E + S$ periods, with $S \geq 4$.[16] Households are termed "youth", and do not participate in market activity during ages $1 \leq s \leq E$. The households enter the workforce and economy in period $E + 1$ and remain in the workforce until they unexpectedly die or live until age $s = E + S$.[17] The population of agents of each age in each period $\omega_{s,t}$ evolves

[16]Theoretically, the model works without loss of generality for $S \geq 3$. However, because we are calibrating the ages outside of the economy to be one-fourth of S (e.g., ages 21 to 100 in the economy, and ages 1 to 20 outside of the economy), we need S to be at least 4.

[17]We model the population with households age $s \leq E$ outside of the workforce and economy in order to most closely match the empirical population dynamics.

according to the following function,

$$\begin{aligned} \omega_{1,t} &= 1 && \forall t \\ \omega_{s+1,t+1} &= (1-\rho_s)\omega_{s,t} && \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{1}$$

where ρ_s is an age specific mortality hazard rate. The parameter ρ_s is the probability that a household of age s dies before age $s+1$. The total population in the economy N_t at any period is simply the sum of households in the economy. Given the law of motion for the population and the constant measure of households born in each period, the population growth rate is zero and the population distribution is stationary. This stationary distribution of households over age is shown in Figure 1. We define parameters are defined as:

$$N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{2}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{3}$$

$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{4}$$

$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{5}$$

We can transform the nonstationary equations in (1) into stationary laws of motion by dividing both sides by the total economically relevant population in the current period $\tilde{N}_t$ and then multiplying the left-hand-side of the equation by $\tilde{N}_{t+1}/\tilde{N}_{t+1}$,

$$\begin{aligned} \hat{\omega}_{1,t+1} &= \frac{1}{\tilde{N}_{t+1}} && \forall t \\ \hat{\omega}_{s+1,t+1} &= \frac{(1-\rho_s)\hat{\omega}_{s,t}}{1+\tilde{g}_{n,t+1}} && \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{A.1.1}$$

where $\hat{\omega}_{s,t}$ is the percent of the total economically relevant population $\tilde{N}_t$ in age cohort s in period t, and $\tilde{g}_{n,t+1}$ is the population growth rate between periods t and $t+1$ defined in (5).[18]

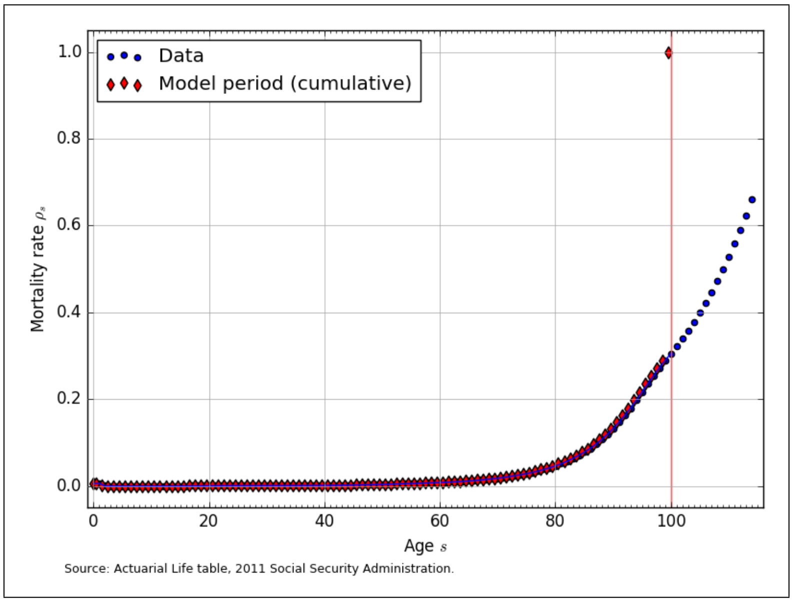
A-1.2 Mortality Rates

The mortality rates in our model ρ_s are a one-period hazard rate and represent the probability of dying within one year, given that an individual is alive at the beginning of period s. We assume that the mortality rates for each age cohort ρ_s are constant across time. Our data for U.S. mortality rates by age come from the Actuarial Life

[18] Note in the specification of the stationary laws of motion (A.1.1) that $\sum_{s=1}^{E+S} \hat{\omega}_{s,t} > 1$ while $\sum_{s=E+1}^{E+S} \hat{\omega}_{s,t} = 1$. This is because in the model we only look at the economically relevant population $\hat{\omega}_{s,t}$ for $E+1 \leq s \leq E+S$.

Tables of the U.S. Social Security Administration (see Bell and Miller, 2015), from which the most recent mortality rate data is for 2011. Figure 4 shows the mortality rate data and the corresponding model-period mortality rates for $E + S = 100$.

Figure 4: Mortality rates by age (ρ_s) for $E + S = 100$



The mortality rates in Figure 4 are a population-weighted average of the male and female mortality rates reported in Bell and Miller (2015). Figure 4 also shows that the data provide mortality rates for ages up to 111-years-old. We truncate the maximum age in years in our model to 100-years old. In addition, we constrain the mortality rate to be 1.0 or 100 percent at the maximum age of 100.

The red diamonds in Figure 4 are the interpolated mortality rates for individuals that live for $E + S = 100$ periods that range between ages 1 and 100. Our mortality rate interpolation function `get_mort()` takes as inputs the total number of periods and the range of data year ages that those periods cover.

A-1.3 Population Steady State and Transition

This model requires information about mortality rates ρ_s in order to solve for the household's problem each period. It also requires the steady-state stationary population distribution $\bar{\omega}_s$ as well as the full transition path of the stationary population distribution $\hat{\omega}_{s,t}$ from the current state to the steady-state. To solve for the steady-state and the transition path of the stationary population distribution, we write the stationary population dynamic equations from (A.1.1) in matrix form.

$$
\begin{bmatrix} \hat{\omega}_{1,t+1} \\ \hat{\omega}_{2,t+1} \\ \hat{\omega}_{2,t+1} \\ \vdots \\ \hat{\omega}_{E+S-1,t+1} \\ \hat{\omega}_{E+S,t+1} \end{bmatrix} = \frac{1}{1+g_{n,t+1}} \times ...
$$

$$
\begin{bmatrix} 1/\tilde{N}_t & 1/\tilde{N}_t & 1/\tilde{N}_t & \dots & 1/\tilde{N}_t & 1/\tilde{N}_t \\ 1-\rho_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1-\rho_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1-\rho_{E+S-1} & 0 \end{bmatrix} \begin{bmatrix} \hat{\omega}_{1,t} \\ \hat{\omega}_{2,t} \\ \hat{\omega}_{2,t} \\ \vdots \\ \hat{\omega}_{E+S-1,t} \\ \hat{\omega}_{E+S,t} \end{bmatrix} \tag{A.1.2}
$$

We can write system (A.1.2) more simply in the following way.

$$
\boldsymbol{\hat{\omega}}_{t+1} = \frac{1}{1+g_{n,t+1}} \boldsymbol{\Omega} \boldsymbol{\hat{\omega}}_t \quad \forall t \tag{A.1.3}
$$

The stationary steady-state population distribution $\boldsymbol{\bar{\omega}}$ is the eigenvector $\boldsymbol{\omega}$ with eigenvalue $(1+\bar{g}_n)$ of the matrix $\boldsymbol{\Omega}$ that satisfies the following version of (A.1.3).

$$
(1+\bar{g}_n)\boldsymbol{\bar{\omega}} = \boldsymbol{\Omega}\boldsymbol{\bar{\omega}} \tag{A.1.4}
$$

Proposition 1. If the age $s=1$ immigration rate is $i_1 > -(1-\rho_0)f_1$ and the other immigration rates are strictly positive $i_s > 0$ for all $s \geq 2$ such that all elements of $\boldsymbol{\Omega}$ are nonnegative, then there exists a unique positive real eigenvector $\boldsymbol{\bar{\omega}}$ of the matrix $\boldsymbol{\Omega}$, and it is a stable equilibrium.

Proof. First, note that the matrix $\boldsymbol{\Omega}$ is square and non-negative. This is enough for a general version of the Perron-Frobenius Theorem to state that a positive real eigenvector exists with a positive real eigenvalue. This is not yet enough for uniqueness. For it to be unique by a version of the Perron-Fobenius Theorem, we need to know that the matrix is irreducible. This can be easily shown. The matrix is of the form

$$
\boldsymbol{\Omega} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & 0 & \dots & 0 & 0 & 0 \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}
$$

Where each * is strictly positive. It is clear to see that taking powers of the matrix causes the sub-diagonal positive elements to be moved down a row and another row of positive entries is added at the top. None of these go to zero since the elements were all non-negative to begin with.

$$\boldsymbol{\Omega}^2 = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}; \quad \boldsymbol{\Omega}^{S+E-1} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

$$\boldsymbol{\Omega}^{S+E} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \end{bmatrix}$$

Existence of an $m \in \mathbb{N}$ such that $(\boldsymbol{\Omega}^{\mathbf{m}})_{ij} \neq 0 \quad (> 0)$ is one of the definitions of an irreducible (primitive) matrix. It is equivalent to saying that the directed graph associated with the matrix is strongly connected. Now the Perron-Frobenius Theorem for irreducible matrices gives us that the equilibrium vector is unique.

We also know from that theorem that the eigenvalue associated with the positive real eigenvector will be real and positive. This eigenvalue, p, is the Perron eigenvalue and it is the steady state population growth rate of the model. By the PF Theorem for irreducible matrices, $|\lambda_i| \leq p$ for all eigenvalues λ_i and there will be exactly h eigenvalues that are equal, where h is the period of the matrix. Since our matrix $\boldsymbol{\Omega}$ is aperiodic, the steady state growth rate is the unique largest eigenvalue in magnitude. This implies that almost all initial vectors will converge to this eigenvector under iteration. □

For a full treatment and proof of the Perron-Frobenius Theorem, see Suzumura (1983). Because the population growth process is exogenous to the model, we calibrate it to annual age data for age years $s = 1$ to $s = 100$. Figure 1 shows the steady-state population distribution $\bar{\boldsymbol{\omega}}$.

A-2 Derivation of elliptical disutility of labor supply

Evans and Phillips (2015) provide an exposition of the value of using elliptical disutility of labor specification as well as its relative properties to such standard disutility of labor functions such as constant relative risk aversion (CRRA) and constant Frisch elasticity (CFE). A standard specification of additively separable period utility in consumption and labor supply similar to one used in King et al. (1988) is the following,

$$u(c,n) = \frac{c^{1-\sigma}-1}{1-\sigma} - \chi^n \frac{(n)^{1+\theta}}{1+\theta} \tag{A.2.1}$$

where $\sigma \geq 1$ is the coefficient of relative risk aversion on consumption and $\theta \geq 0$ is proportional to the inverse of the Frisch elasticity of labor supply. The constant χ^n is a scale parameter influencing the relative disutility of labor to the utility of consumption.

Although labor supply is only defined for $n \in [0, \tilde{l}]$—where $\tilde{l}$ is the time endowment or the maximum labor supply possible—the disutility of labor function in (A.2.1) is defined for values of n greater than $\tilde{l}$ and less than 0. Further, for $n < 0$, the marginal utility of labor is positive. To avoid the well known and significant computational difficulty of computing the solution to the complementary slackness conditions in the Karush, Kuhn, Tucker constrained optimization problem, we impose an approximating utility function that has properties bounding the solution for n away from both $n = \tilde{l}$ and $n = 0$. The upper right quadrant of an ellipse has exactly this property and also has many of the properties of the original utility function. Figure 5 shows how our estimated elliptical utility function compares to the utility of labor from (A.2.1) over the allowed support of n.

The general equation for an ellipse in x and y space with centroid at coordinates (h, k), horizontal radius of a, vertical radius of b, and curvature υ is the following.

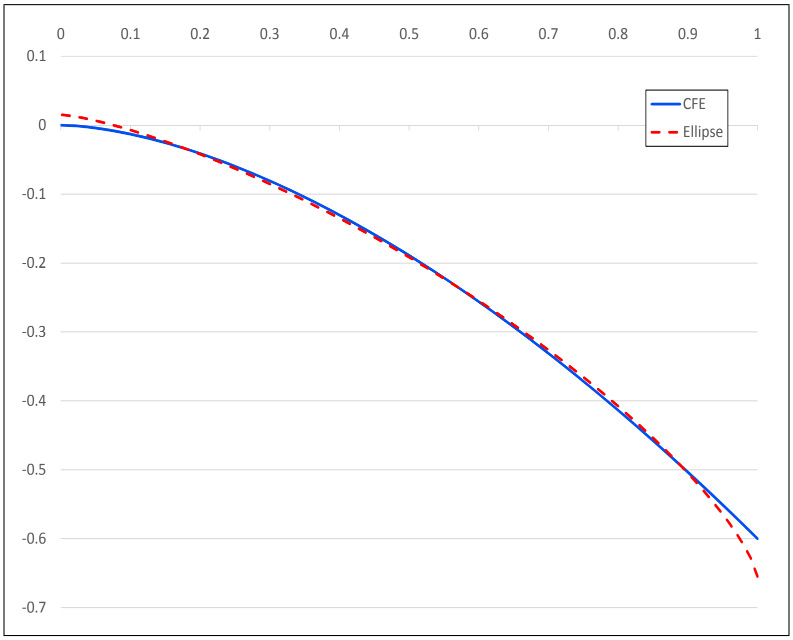
$$\left(\frac{x-h}{a}\right)^{\upsilon} + \left(\frac{y-k}{b}\right)^{\upsilon} = 1 \tag{A.2.2}$$

Figure 6 shows an ellipse with the parameterization $[h, k, a, b, \upsilon] = [1, -1, 1, 2, 2]$.

The graph of the ellipse in the upper-right quadrant of Figure 6 ($x \in [1, 2]$ and $y \in [-1, 1]$) has similar properties to the utility of labor term in (A.2.1). If we let the x variable be labor supply n, the utility of labor supply be $g(n)$, the x-coordinate of the centroid be zero $h = 0$, and the horizontal radius of the ellipse be $a = \tilde{l}$, then the equation for the ellipse corresponding to the standard utility specification is the following.

$$\left(\frac{n}{\tilde{l}}\right)^{\upsilon} + \left(\frac{g-k}{b}\right)^{\upsilon} = 1 \tag{A.2.3}$$

Figure 5: Comparison of standard utility of labor n to elliptical utility



Solving the equation for g as a function of n, we get the following.

$$g(n) = b\left[1 - \left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} + k \tag{A.2.4}$$

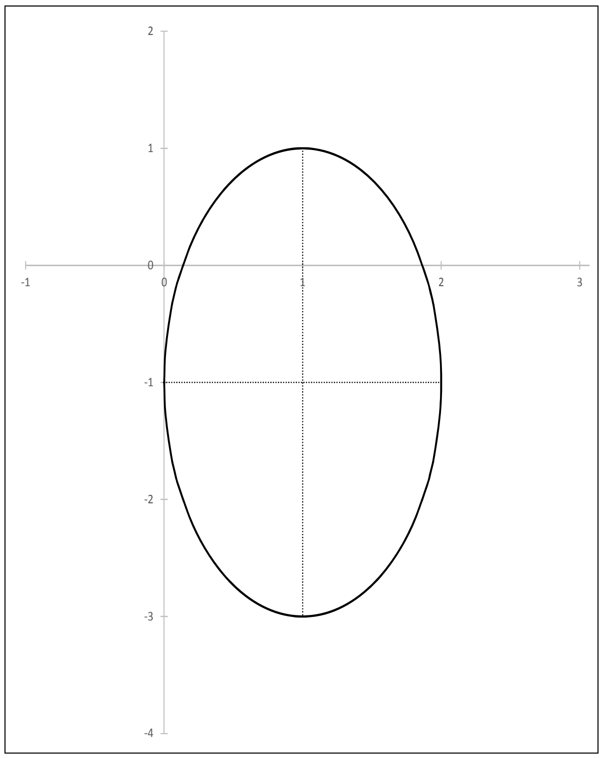
The υ parameter acts like a constant elasticity of substitution, and the parameter b is a shape parameter similar to χ^n in (A.2.1).

We use the upper-right quadrant of the elliptical utility function because the utility of n is strictly decreasing on $n \in (0, \tilde{l})$, because the slope of the utility function goes to negative infinity as n approaches its maximum of $\tilde{l}$ and because the slope of the utility function goes to zero as n approaches its minimum of 0. This creates interior solutions for all optimal labor supply choices $n^* \in (0, \tilde{l})$. Although it is more realistic to allow optimal labor supply to sometimes be zero, the complexity and dimensionality of our model requires this approximating assumption to render the solution method tractable.

Figure 5 shows how closely the estimated elliptical utility function matches the original utility of labor function in (A.2.1) with a Frisch elasticity of 0.4.[19] We choose the ellipse parameters b, k, and υ to best match the points on the original utility of labor function for $n \in [0, 0.9]$. We minimize the sum of absolute errors for 101 evenly spaced points on this domain. The estimated values of the parameters for the elliptical utility shown in Figure 5 and represented in equation (A.2.4) are $[b, k, \upsilon] = [0.573, 0.000, 2.856]$.

[19]See Chetty et al. (2011), Keane and Rogerson (2012) and Peterman (2014) for discussion of this choice.

Figure 6: Ellipse with $[h, k, a, b, \upsilon] = [1, -1, 1, 2, 2]$



A-3 Solving for stationary steady-state equilibrium

This section describes the solution method for the stationary steady-state equilibrium described in Definition 1. The steady-state is characterized by $2S$ equations and $2S$ unknowns. However, because some of the other equations cannot be solved for analytically and substituted into the Euler equations, we must take a two-stage approach to the equilibrium solution. We first make a guess at steady-state wage $\bar{w}$, interest rate $\bar{r}$, and lump-sum transfer $\bar{T}^H$. Then, given those three aggregate variables, we can solve for the second-stage household decisions of steady-state savings $\bar{b}_s$ and labor supply $\bar{n}_s$.

1. Use the techniques in Appendix A-1 to solve for the steady-state population distribution vector $\bar{\boldsymbol{\omega}}$ of the exogenous population process.

2. Choose an initial guess for the values of the steady-state wage $\bar{w}$, interest rate $\bar{r}$, and lump-sum transfer $\bar{T}^H$.

3. Given guesses for $\bar{w}$, $\bar{r}$, and $\bar{T}^H$, solve for the steady-state household savings $\bar{b}_s$ and labor supply $\bar{n}_s$ decisions using $2S$ equations (24), (25).

   - A good first guess for $\bar{b}_s$ and $\bar{n}_s$ is a number close to but less than $\tilde{l}$ for all the $\bar{n}_s$ and to choose some small positive number for $\bar{b}_s$ that is small enough to be less than the minimum income that an individual might have $\bar{w}e_s\bar{n}_s$.
   - Make sure that all of the $2S$ Euler errors is sufficiently close to zero to constitute a solution.

4. Given the solutions $\bar{b}_s$ and $\bar{n}_s$ from step (3), make sure that the three characterizing equations for $\bar{w}$, $\bar{r}$, and $\bar{T}^H$ are solved. These characterizing equations are the zero equations corresponding to the steady-state versions of (27), (19), and (32).

$$\bar{w} - (1-\alpha)\frac{\bar{Y}}{\bar{L}} = 0 \tag{A.3.1}$$

$$\bar{r} - \alpha\frac{\bar{Y}}{\bar{K}} + \delta = 0 \tag{A.3.2}$$

$$\bar{T}^H - \sum_s \bar{\omega}_s \bar{T}_s = 0 \tag{A.3.3}$$

5. Iterate on guesses for outer loop values of $\bar{w}$, $\bar{r}$, and $\bar{T}^H$ until the Euler equations from step (3) and the characterizing equations from step (4) are all solved.

A-4 Solving for stationary non-steady-state equilibrium by time path iteration

This section describes the solution to the non-steady-state transition path equilibrium of the model described in Definition 2 and outlines the time path iteration (TPI) method of Auerbach and Kotlikoff (1987) for solving for this equilibrium. The following are the steps for computing a stationary non-steady-state equilibrium time path for the economy.

1. Input all initial parameters. See Table 2.
   (a) The value for T at which the non-steady-state transition path should have converged to the steady state should be at least as large as the number of periods it takes the population to reach its steady state $\bar{\boldsymbol{\omega}}$ as described in Appendix A-1.
2. Choose an initial distribution of savings and intended bequests $\hat{\boldsymbol{\Gamma}}_1$ and then calculate the initial state of the stationarized aggregate capital stock $\hat{K}_1$ and total bequests received $\hat{BQ}_1$ consistent with $\hat{\boldsymbol{\Gamma}}_1$ according to (29) and (31).
   (a) Note that you must have the population weights from the previous period $\hat{\omega}_{s,0}$ and the growth rate between period 0 and period 1 $\tilde{g}_{n,1}$ to calculate $\hat{BQ}_1$.
3. Conjecture transition paths for the stationarized wage $\hat{\boldsymbol{w}}^1 = \{\hat{w}_t^1\}_{t=1}^{\infty}$, stationarized interest rate $\boldsymbol{r}^1 = \{r_t^1\}_{t=1}^{\infty}$, total bequests received $\hat{\boldsymbol{BQ}}^1 = \{\hat{BQ}_t^1\}_{t=1}^{\infty}$, and the lump-sum transfer from the government $\hat{\boldsymbol{T}}^{\boldsymbol{H,1}} = \{\hat{T}_t^{H,1}\}_{t=1}^{\infty}$. The only requirements are that $\hat{K}_1^i$ and $\hat{BQ}_1^i$ are functions of the initial distribution of savings $\hat{\boldsymbol{\Gamma}}_1$ for all iterations i in your initial state and that the time paths of $\hat{\boldsymbol{w}}^i$, $\boldsymbol{r}^i$, $\hat{\boldsymbol{BQ}}^i$, and $\hat{\boldsymbol{T}}^{\boldsymbol{H,i}}$ equal their respective steady-state values for all $t \geq T$.
   (a) Initial guesses for $\hat{w}_1$ and r_1 can be disciplined a little bit by whether $\hat{K}_1$ is greater than or less than $\bar{K}$. If $\hat{K}_1 > \bar{K}$, then choose $\hat{w}_1 > \bar{w}$ and $r_1 < \bar{r}$. If $\hat{K}_1 < \bar{K}$, then choose $\hat{w}_1 < \bar{w}$ and $r_1 > \bar{r}$.
4. With the conjectured transition paths $\hat{\boldsymbol{w}}^i$, $\boldsymbol{r}^i$, $\hat{\boldsymbol{BQ}}^i$, and $\hat{\boldsymbol{T}}^{\boldsymbol{H,i}}$, one can solve for the lifetime labor and savings decisions for each individual in the model who will be alive between periods $t = 1$ and T. Each individual's lifetime decisions can be solved independently using the systems of $2S$ Euler equations of the form (24), (25), and (26).
   (a) Make sure all the Euler errors for both the savings and labor supply decisions are sufficiently close to zero in order to ensure that the household equilibrium is being solved.
5. Use the implied distribution of savings and labor supply in each period to compute the new implied time paths for the wage $\hat{\boldsymbol{w}}^{i'} = \{\hat{w}_1^i, \hat{w}_2^{i'}, ...\hat{w}_T^{i'}\}$, interest

rate $\boldsymbol{r}^{i'} = \{r_1^i, r_2^{i'}, ...r_T^{i'}\}$, total bequests received $\hat{\boldsymbol{BQ}}^{i'} = \{\hat{BQ}_1^i, \hat{BQ}_2^{i'}, ...\hat{BQ}_T^{i'}\}$, and lump-sum transfer from the government $\hat{\boldsymbol{T}}^{\boldsymbol{H},\boldsymbol{i'}} = \{\hat{T}_1^{H,i'}, \hat{T}_2^{H,i'}, ...\hat{T}_T^{H,i'}\}$.

6. Check the distance between the two sets time paths.

$$\left\| \left[\hat{\boldsymbol{w}}^{i'}, \boldsymbol{r}^{i'}, \hat{\boldsymbol{BQ}}^{i'}, \hat{\boldsymbol{T}}^{\boldsymbol{H},\boldsymbol{i'}}\right] - \left[\hat{\boldsymbol{w}}^{i}, \boldsymbol{r}^{i}, \hat{\boldsymbol{BQ}}^{i}, \hat{\boldsymbol{T}}^{\boldsymbol{H},\boldsymbol{i}}\right] \right\|$$

   (a) If the distance between the initial time paths and the implied time paths is less-than-or-equal-to some convergence criterion $\varepsilon > 0$, then the fixed point has been achieved and the equilibrium time path has been found.

   (b) If the distance between the initial time paths and the implied time paths is greater than some convergence criterion $\|\cdot\| > \varepsilon$, then update the guess for the time paths and repeat steps (4) through (6) until a fixed point is reached.

Figures 7 and 8 show the equilibrium time paths of the aggregate capital stock K_t and aggregate labor supply L_t for the calibration of the model in this paper.

Figure 7: Equilibrium time path of K_t for $S = 80$ in baseline model

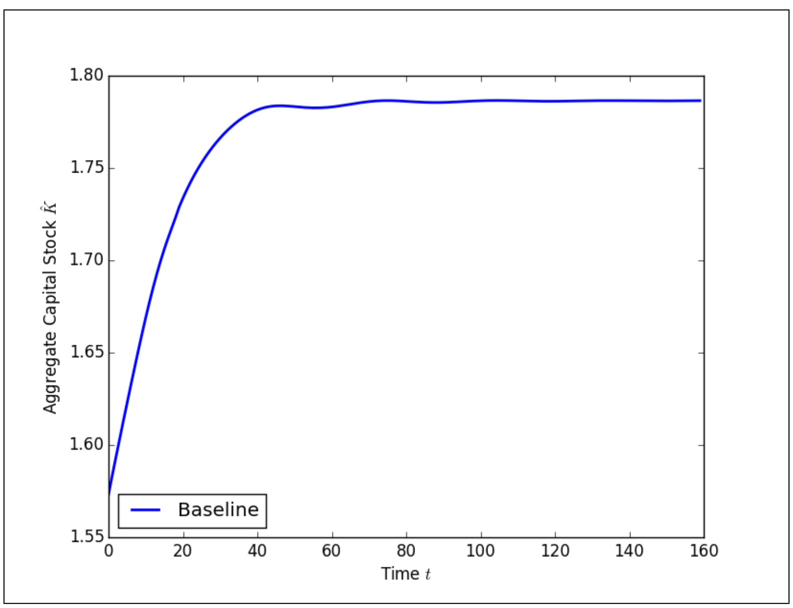


Figure 8: Equilibrium time path of L_t for $S = 80$ in baseline model