Adding Multiple Goods/Production Sectors to the OLG Model

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The Big Picture

What we are working towards is having two representative firms (one who faces corporate tax treatment and the other with non-corporate tax treatment) for each of M production industries. Each firm will produce an unique output that is part of the household's consumption bundle. There will exist a firm in each sector (corporate/noncorporate) and each industry in equilibrium because households will have preferences such that they want to consume a strictly positive amount of the good from each sector and industry. The equilibrium shares of the households' composite good will vary as the prices of those goods vary. The prices of these goods varies as factor prices and taxes change, which impact production sectors/industries differentially.

What are are doing is unique in that we have many production industries *and* have forward looking, truly dynamically optimizing firms. Zodrow and Diamond (2013) have dynamic firms, but only four representative firms. Fullerton and Rogers (1993) have many production industries, but static firms. The CORTAX model has dynamic firms with a medium number of representative firms (2-3 per country in a model of maybe 9 countries). These papers/models represent our main sources of inspiration and are the starting point for the model we are trying to build.

I think the key to making this model feasible is a structure like in Fullerton and Rogers (1993), where factor prices can be used to determine all other prices in the model. In particular, given r and w, their model allows one to derive the price of capital, the price of producer outputs, the price of individual consumption goods, and the price of the composite consumption good. With all these prices, they then start with the consumers problem and figure out labor supply and demand for each consumption good. The demand for consumption goods can then be put in terms of demand for producer goods. The demand for producer goods (output) then implies the amount of capital and labor the firm will employ. This means all the endogenous prices and quantities in the household and production sectors fall out of the factor prices r and w. Other methods would involve guessing a lot more prices than just r and w (e.g., guessing prices for each of the $Mx2$ production outputs).

We want to structure our model in the way Fullerton and Rogers (1993) do in terms of how all the within period endogenous variables unravel, but will have the firm as a dynamic optimizer (as in Zodrow and Diamond (2013) and CORTAX). We'll also want to have firms with economic profits so that 1) they have profits to shift to other tax jurisdictions and 2) we can analyze the impact of taxes on normal and supranormal returns. Of the above papers, only the CORTAX model has economic profits.

Starting point

This document assumes that you have written code that computes the steady-state equilibrium and transition path for an OLG model with households who live for S periods and can be one of J types (where the difference between types is in the amount of effective labor units they can provide). Households choose labor supply, savings, and consumption. Bequests are left when a household dies and they get a warm glow utility effect from this (at least for intentional bequests). I'll be assuming the disutility of labor function is a CRRA function, but one can easily adapt this to the case of an elliptical utility function. A single, representative firm rents capital and labor to produce output with a constant returns to scale, Cobb Douglas production function. The codes solves for the model's steady state using a root finder (probably

Scipy's `fsolve`) to simultaneously solve $S \times J \times 2$ equations (household FOCs) for the $S \times J \times 2$ unknowns ($n_{j,s}, b_{j,s}$ - these then determine $c_{j,s}$). This solution results in Euler errors that are very small (e.g. 1e-10) and satisfies the aggregate resource constraint: $Y = C + I$ (where $I = \delta K$ in the SS). These conditions are satisfied in both the SS and along the transition path.

Step 1: Modifying the SS solution algorithm

The first step is to take the code for solving for the steady state and adjust it just slightly so that it is setup to add the additional pieces in the next sections. The "one big fsolve" method used in your code is not robust to different initial values and becomes difficult to work with when multiple firms are added. What we'll do instead is make a guess at the factor prices, r and w. The SS algorithm will look like:

1. Make an initial guess at $\bar{r}$ and $\bar{w}$
2. Taking $\bar{r}$ and $\bar{w}$ as given, solve the household's problem:
   - For each j type:
     (a) Make an initial guess at the household's optimal savings and labor supply decisions, $b_{j,s}$, $n_{j,s}$.
     (b) Use a root finder (e.g. `fsolve`) to determine the optimal allocations given $\bar{r}$ and $\bar{w}$.
3. Aggregate over J and S to determine aggregate supply of labor and capital (where savings=capital), K, L
   - Remember to find L as the aggregate amount of effective labor units supplied, so you not only want to sum over J and S, but weight by the number of effective labor units each type/age supplies.
4. Use the fact that supply=demand in eq'm and plug the aggregate factor supplies into the firm's problem

5. Using the Firm's FOC for capital demand, find the interest rate implied by these factor supplies. Call this interest rate r_{new}. $r_{new} = MPK(K, L) - \delta$.

6. Using the Firm's FOC for labor demand, find the wage rate implied by these factor supplies. Call this interest rate w_{new}. $w_{new} = MPL(K, L)$.

7. Take differences between the guess at $\bar{r}$ and $\bar{w}$.

8. Use a root finder to determine the eq'm $\bar{r}$ and $\bar{w}$ (i.e. it'll find the $\bar{r}$ and $\bar{w}$ where $r_{new} - \bar{r} = 0$ and $w_{new} - \bar{w} = 0$).

So in this algorithm, there is an outer `fsolve`, solving for r and w. Within that, there is a loop over the J types and an `fsolve` at each iteration of that loop (each solving $S \times 2$ equations).

You'll want to be sure to have separate functions for the inner and outer loops. E.g. a `SS_solve` function that takes the parameters and initial guesses of r and w as inputs and a `hh_solve` that takes relevant parameters and the $b_{j,s}, n_{j,s}$ as inputs. The `hh_solve` function will be within the for loop within the `SS_solve` function. Make sure that all the functions called with in the `hh_solve` function are compatible with the dimensions of the inputs (which will only be $S \times 1$ as opposed to $S \times J$ from the previous method.

Note one trick that I've found really helps with the solution is to adjust your initial guesses for the household problem $(b_{j,s}, n_{j,s})$ for each type j. In particular, assuming that ability changes monotonically along the J dimension, then use the solution to the household problem from $j - 1$ as the initial guess to solve the problem for the household of type j.

To check that this all works as expected, makes sure:

1. Euler errors are very small

2. The aggregate resource constraint is satisfied ($Y = C + \delta K$ in the SS).

3. You get the same equilibrium from this algorithm as with the previous "one big fsolve" method (i.e., $b_{j,s}, n_{j,s}, \bar{r}, \bar{w}$ are the same as before).

One can do this same method along the time path. But I'm thinking that we build up only the SS solution for now, then do the time path once we've got more of the multiple firm problem fleshed out and working in the the SS solution.

Step 2: Make the production function a more general CES production function

The initial set up, firms have a Cobb-Douglas production function. To allow for the model user to more easily change the elasticity of substitution between capital and labor. Let's write the production function as a more general CES production function. In particular, let the production function be given by:

$$X_t = F(A_t, K_t, EL_t) = A_t \left[(\gamma)^{1/\epsilon}(K_t)^{(\epsilon-1)/\epsilon} + (1-\gamma)^{1/\epsilon} EL_t^{(\epsilon-1)/\epsilon}\right]^{(\epsilon/(\epsilon-1))}, \quad (1)$$

where A_t is total factor productivity, EL is effective labor units (same as our L in the current write up of the firm problem - we'll change the notation so that we can keep track of both L, total hours worked, and EL total effective labor units worked) and the parameters γ and ϵ are the share share parameter and the elasticity of substitution parameters. Note the labor augmenting technological growth. Also note that we've update the notation so that X_t denotes the output of the firm in period t. As we expand the model, Y will represent aggregate household income and X will represent firm output.

The marginal products of capital and labor are thus given by:

$$\begin{aligned} MPK_t = \frac{\partial X_t}{\partial K_t} &= A_t \left(\gamma^{\frac{1}{\epsilon}} K_t^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma)^{\frac{1}{\epsilon}} EL_t^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{1}{\epsilon-1}} \gamma^{\frac{1}{\epsilon}} K_t^{\frac{-1}{\epsilon}} \\ &= A_t^{\frac{\epsilon-1}{\epsilon}} \left(X_t \frac{\gamma}{K_t}\right)^{\frac{1}{\epsilon}} \end{aligned} \quad (2)$$

$$
\begin{aligned}
MPL_t = \frac{\partial X_t}{\partial EL_t} &= A_t\left(\gamma^{\frac{1}{\epsilon}}K^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma)^{\frac{1}{\epsilon}}EL_t^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{1}{\epsilon-1}}(1-\gamma)^{\frac{1}{\epsilon}}EL_t^{\frac{-1}{\epsilon}}e^{g_y t} \\
&= A_t^{\frac{\epsilon-1}{\epsilon}}\left(X_t\frac{1-\gamma}{EL_t}\right)^{\frac{1}{\epsilon}}
\end{aligned} \tag{3}
$$

We thus need to go into the code and edit the equations for the MPK and MPL to use those above. These equations should be in just two functions in the code - one that determines the equilibrium wage rate and one that determines the equilibrium interest rate. You will also need to update the production function in the function that determines firm output. In addition, you'll need to change the notation for output from Y to X and for aggregate effective labor from L to EL.

Note that when $\epsilon = 1$, the function becomes the Cobb-Douglas Production Function. We should probably put an "if" statement in the code such that if $\epsilon = 1$ then the production function is $X_t = F(A_t, K_t, EL_t) = A_t K_t^{\gamma} EL_t^{(1-\gamma)}$. Marginal products don't have to change, but when $\epsilon = 1$, the production function is not defined.

Go ahead and set $\gamma = 0.36$ and $\epsilon = 0.6$ and solve the model. Check that:

1. Euler errors are very small

2. The aggregate resource constraint is satisfied ($X = C + \delta K$ in the SS).

Step 3: Adding a second static firm

In this step, we will add a representative firm for a second production industry. This comes with several substantive changes and so we'll just add one additional firm at this point, not making it a general M-firm problem until later steps.

A few remarks about the economy with multiple firms. The multiple firms will produce differentiated output. These outputs contribute to distinct consumption goods and these various consumption goods go into a composite consumption good consumed each household. In addition, the output of the firms will contribute to the capital stock. The capital stock for each representative firm is made up of a different mix out output from the various industries.

In the remainder of this section, we'll work through how we begin with our guesses of the factor prices (r and w) and work through the producer and consumer problems. I'll lay out the theory first, then discuss implementation into the existing code.

The exposition here only deals with the SS solution, so the "bars" on variables will be implicit rather than me typing them. We'll adapt this to the time path solution in a future step.

Theory

The household's optimization problem

Consumers maximize the present discounted value of utility from consumption a composite consumption good, $\tilde{c}$, leisure, $\tilde{l} - n$, and from bequests:

$$
\begin{aligned}
U_{j,s} &= \sum_{s=1}^{S} \beta^u u\left(\tilde{c}_{j,s}, n_{j,s}, b_{j,S+1}\right) \\
\text{where} \quad u\left(\tilde{c}_{j,s}, n_{j,s}, b_{j,S+1}\right) &= \frac{\left(c_{j,s}\right)^{1-\sigma} - 1}{1-\sigma} ... \\
&+ \chi^n \left(\frac{\left(\tilde{l} - n_{j,s}\right)^{1-\nu} - 1}{1-\nu} \right) + \chi^b \frac{\left(b_{j,S+1,}\right)^{1-\sigma} - 1}{1-\sigma} \\
&\forall j, 1 \leq s \leq S
\end{aligned}
\tag{4}
$$

Note that this formulation is written without mortality risk and with a warm glow motive for intentional bequests. χ^n and χ^b are the utility weights on the disutility of labor and the warm glow bequest motive, respectively. The household chooses the optimal sequence of $\tilde{c}_{j,s}$, $n_{j,s}$, and $b_{j,s}$ to maximize lifetime utility subject to the per period budget constraint:

$$\sum_{i=1}^{I} p_i \bar{c}_{i,s} + \tilde{p}_s \tilde{c}_s + b_{j,s+1} \leq (1+r)\, b_{j,s} + w_t e_j n_{j,s} + \frac{BQ_j}{\lambda_j \tilde{N}}$$
$$\text{where} \quad b_{j,1} = 0 \tag{5}$$
$$\text{for} \quad 1 \leq s \leq S$$

Prices for individual consumption goods are given by p_i, whereas $\tilde{p}$ is the price of the composite consumption good. The parameters $\bar{c}_{i,s}$ are the minimum consumption amounts for good i. BQ_j are aggregate bequests from those of type j, which are divided equally between the living households of type j. $\tilde{N}$ is the total population, which can be normalized to one for the SS analysis. Total bequests are given by:

$$BQ_j = \sum_{J} \lambda_j b_{j,S+1} \tilde{N} \tag{6}$$

The first order necessary conditions that must be satisfied in the households optimization problem are:

$$\frac{\partial U}{\partial \tilde{b}_{j,s+1}} = \frac{\tilde{c}_{j,s}^{-\sigma}}{\tilde{p}_s} - \beta(1+r)\frac{\tilde{c}_{j,s+1}^{-\sigma}}{\tilde{p}_{s+1}} = 0, \forall s, j \tag{7}$$

$$\frac{\partial U}{\partial n_{j,s}} = \chi_s^n \left(\tilde{l} - n_{j,s}\right)^{-\nu} - \frac{w e_j}{\tilde{p}_s} \tilde{c}_{j,s}^{-\sigma} = 0, \forall s, j \tag{8}$$

$$\frac{\partial U}{\partial b_{j,S+1}} = \frac{\tilde{c}_{j,S}^{-\sigma}}{\tilde{p}_S} - \beta(1+r)\chi^b b_{j,S+1}^{-\sigma}, \forall j \tag{9}$$

The composite consumption good is make up of the individual consumption goods, with the amounts determined by the consumer's consumption subutility function. We assume a Stone-Geary utility function here, with the composite consumption good defined as:

$$\tilde{c}_{j,s} = \prod_{i=1}^{I} \left(c_{i,j,s} - \bar{c}_{i,s}\right)^{\alpha_i}, \tag{10}$$

where i denotes the particular consumption good. $\bar{c}_{i,s}$ are the minimum consumption amounts for good i. The composite consumption good is the composite of "discretionary" consumption on all the goods (consumption above the minimum amounts). The α_i parameters are the share parameters and define the share of discretionary consumption spending (called the "supernumerary expenditure) that goes to each good i. What this utility function is modeling is that there some basic requirements for sustenance. For example, you need a certain amount of calories to live, giving you a minimum food expenditure, but you might choose to go above that. This specification has a couple nice properties as far as our model is concerned. First, it helps to get a more realistic tax incidence since it'll have the rich and poor spending different shares of their income on different goods (without resorting to preferences that depend upon ability type j). Second, it'll give us more realistic responses of savings to interest rates. Typically these models have responses that are much stronger than we see in the data. The minimum consumption shares help to temper that because some may be close to those thresholds and therefore still have a high marginal utility of consumption for the composite good.

The consumer chooses $c_{i,j,s}$ to maximize Equation 10 subject to the budget constraint:

$$\sum_{i=1}^{I} p_i(c_{i,j,s} - \bar{c}_{i,s}) = \tilde{p}_s \tilde{c}_{j,s} \tag{11}$$

where p_i is the gross of tax price of good i at time t and $\tilde{p}_s$ is the gross of tax price of the the discretionary component of the composite consumption good consumed by those of age s at time t. Maximization of **??** subject to 11 yields:

$$\mathcal{L} = \max_{\{c_{i,j,s}\}_{i=1}^{I}} \prod_{i=1}^{I} \left(c_{i,j,s} - \bar{c}_{i,s}\right)^{\alpha_{i,s}} + \lambda \left(\tilde{p}_s \tilde{c}_{j,s} - \sum_{i=1}^{I} p_i(c_{i,j,s} - \bar{c}_{i,s})\right) \tag{12}$$

Which as I FOCs (for each j, s, t):

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{i,j,s}} &= \frac{\alpha_{i,s}\prod_{i=1}^{I}\left(c_{i,j,s}-\bar{c}_{i,s}\right)^{\alpha_{i,s}}}{\left(c_{i,j,s}-\bar{c}_{i,s}\right)} - \lambda p_i = 0, \forall\ i \\
\implies & \frac{\alpha_{i,s}\prod_{i=1}^{I}\left(c_{i,j,s}-\bar{c}_{i,s}\right)^{\alpha_{i,s}}}{\left(c_{i,j,s}-\bar{c}_{i,s}\right)} = \lambda p_i, \forall\ i \\
\implies & \frac{\alpha_{i,s}\prod_{i=1}^{I}\left(c_{i,j,s}-\bar{c}_{i,s}\right)^{\alpha_{i,s}}}{p_i\left(c_{i,j,s}-\bar{c}_{i,s}\right)} = \lambda, \forall\ i \\
\implies & \frac{\alpha_{i,s}}{p_i\left(c_{i,j,s}-\bar{c}_{i,s}\right)} = \frac{\alpha_{k,s}}{p_k\left(c_{k,j,s}-\bar{c}_{k,s}\right)}, \forall\ i,k \\
\implies & c_{i,j,s} = \frac{\alpha_{i,s}p_k\left(c_{k,j,s}-\bar{c}_{k,s}\right)}{\alpha_{k,s}p_i} + \bar{c}_{i,s}\forall i,k
\end{aligned} \tag{13}
$$

Now substitute the last line of 13 into the budget constraint (Equation 11):

$$
\begin{aligned}
&\tilde{p}_s\tilde{c}_{j,s} = \sum_{i=1}^{I} p_i\left(c_{i,j,s}-\bar{c}_{i,s}\right) \\
&\implies \tilde{p}_s\tilde{c}_{j,s} = \sum_{i=1}^{I} p_i\left[\frac{\alpha_{i,s}p_k\left(c_{k,j,s}-\bar{c}_{k,s}\right)}{\alpha_{k,s}p_i} + \bar{c}_{i,s} - \bar{c}_{i,s}\right] \\
&\implies \tilde{p}_s\tilde{c}_{j,s} = \sum_{i=1}^{I}\left[\frac{\alpha_{i,s}p_k\left(c_{k,s}-\bar{c}_{k,s}\right)}{\alpha_{k,s}}\right] \\
&\implies \tilde{p}_s\tilde{c}_{j,s} = \frac{p_k\left(c_{k,j,s}-\bar{c}_{k,s}\right)}{\alpha_{k,s}}\underbrace{\sum_{i=1}^{I}\alpha_{i,s}}_{=1} \\
&\implies \tilde{p}_s\tilde{c}_{j,s} = \frac{p_k\left(c_{k,j,s}-\bar{c}_{k,s}\right)}{\alpha_{k,s}} \\
&\implies \frac{p_k\left(c_{k,j,s}-\bar{c}_{k,s}\right)}{\alpha_{k,s}} = \tilde{p}_s\tilde{c}_{j,s} \\
&\implies c_{k,j,s} = \frac{\alpha_{k,s}\tilde{p}_s\tilde{c}_{j,s}}{p_k} + \bar{c}_{k,s}, \forall\ k
\end{aligned} \tag{14}
$$

Thus, total consumption of each good i, $c_{i,j,s}$, is given by the the amount of minimum consumption plus the share of total expenditures remaining after making the minimum expenditures on all goods (this is called the "supernumerary" expenditure). We derive the prices of the age s composite consumption good in period t, $\tilde{p}_s$ by using the demand for good i provided in Equation 14 in the function defining aggregate discretionary consumption, Equation 10:

$$\begin{aligned}
&\tilde{c}_{j,s} = \prod_{i=1}^{I}(c_{i,j,s} - \bar{c}_{i,s})^{\alpha_{i,s}} \\
&\implies \tilde{c}_{j,s} = \prod_{i=1}^{I}\left(\frac{\alpha_{i,s}\tilde{p}_s\tilde{c}_{j,s}}{p_i} + \bar{c}_{i,s} - \bar{c}_{i,s}\right)^{\alpha_{i,s}} \\
&\implies \tilde{c}_{j,s} = \prod_{i=1}^{I}\left(\frac{\alpha_{i,s}\tilde{p}_s\tilde{c}_{j,s}}{p_i}\right)^{\alpha_{i,s}} \\
&\implies \tilde{c}_{j,s} = \tilde{p}_s\tilde{c}_{j,s}\prod_{i=1}^{I}\left(\frac{\alpha_{i,s}}{p_i}\right)^{\alpha_{i,s}} \\
&\implies \frac{\tilde{p}_s\tilde{c}_{j,s}}{\tilde{c}_{j,s}} = \prod_{i=1}^{I}\left(\frac{p_i}{\alpha_{i,s}}\right)^{\alpha_{i,s}} \\
&\implies \tilde{p}_s = \prod_{i=1}^{I}\left(\frac{p_i}{\alpha_{i,s}}\right)^{\alpha_{i,s}}
\end{aligned} \tag{15}$$

This composite good price is then used in the household's intertemporal optimization problem described in Equation 4. With the parameters and endogenous variables, we then use 14 to find the $c_{i,j,s}$.

The firm's optimization problem

Each industry is represented by a competitive firm with a constant returns to scale (CRS) CES production function. We will assume here that each industry output becomes a unique consumption good. That is if there two production industries, the industry one produces output used for c_1 and industry two produces output used for c_2. Thus we will denote each industry with the subscript i, which corresponds to the consumption good they produce. We'll relax this in the future. Also, at this point we'll assume that capital can be made from output from either sector. One unit of output from each sector can be used to produce one unit of capital which can be used by either sector.[1] Because of the CRS and competitive assumptions, firms earn zero profits in equilibrium. The capital and labor market equilibrium also imply that the wage and rental rates are the same across industry. We thus have three equations that define the firm's problem, two from the firm's first order conditions for labor and capital demand, and the third from the zero profit condition. These are:

$$r = p_i * MPK(K_i, EL_i) - \delta, \forall i \tag{16}$$

$$w = p_i * MPL(K_i, EL_i), \forall i \tag{17}$$

$$p_i X_i = w * EL_i + (r + \delta) K_i \tag{18}$$

From factor prices to industry output prices

Given the guess of the equilibrium interest rate and wage rate, we can use the first order conditions of the firm and the zero profit condition to determine the price of the output of the firms. In particular, we can use the firm FOCs to find $K(r, w, X)$ and $EL(r, w, X)$.

With CES production, we'll want to solve for $EL(r, w, X)$ and $K(r, w, X)$. These can be solved for using the firm's FOCs and are given by:

$$EL_i(r, w, X_i) = \frac{(1-\gamma)X_i}{\left(\frac{w}{p_i}\right)^{\epsilon} A^{1-\epsilon}} \tag{19}$$

$$K_i(r, w, X_i) = \frac{\gamma X_i}{\left(\frac{(r+\delta)}{p_i}\right)^{\epsilon} A^{1-\epsilon}} \tag{20}$$

It'll also be useful to write factor demands as derived from the production function and FOCs together. We'll use this when we derive the demand for capital that is

[1] It may be helpful here to think about a financial intermediary that is implicitly sitting between the household and the firm. This intermediary takes the dollars from the household and transforms them into capital for the firms.

required to produce a given amount of output.[2]

$$K_i = \frac{X_i}{A}\left[\gamma^{\frac{1}{\epsilon}} + (1-\gamma)^{\frac{1}{\epsilon}}\left(\frac{r+\delta}{w}\right)^{\epsilon-1}\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{1-\epsilon}} \tag{21}$$

and

$$EL_i = K_i\left(\frac{(1-\gamma)}{\gamma}\right)\left(\frac{(r+\delta)}{w}\right)^{\epsilon} \tag{22}$$

Plugging these factor demands from equations 19 and 20 into the zero profit condition, we get:

$$\begin{aligned}
p_i X_i &= wEL_i + (r+\delta)K_i \\
p_i X_i &= w\frac{(1-\gamma)X_i}{\left(\frac{w}{p_i}\right)^{\epsilon} A^{1-\epsilon}} + (r+\delta)\frac{\gamma X_i}{\left(\frac{(r+\delta)}{p_i}\right)^{\epsilon} A^{1-\epsilon}} \\
\implies p_i &= \left[(1-\gamma)\left(\frac{w}{A}\right)^{1-\epsilon} + \gamma\left(\frac{(r+\delta)}{A}\right)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}
\end{aligned} \tag{23}$$

From prices of individual consumption goods to the price of the composite consumption good

We've got the prices of individual consumption goods from the zero profit condition of the firm's problem. As noted above, the price of the composite consumption good can be derived from the prices of individual consumption goods and the solution to the consumer's subutility maximization problem. The yields:

$$\tilde{p}_s = \prod_{i=1}^{I}\left(\frac{p_i^c}{\alpha_{i,s}}\right)^{\alpha_{i,s}} \tag{24}$$

It is this composite price that enters the household's intertemporal optimization problem where is chooses the amount of discretionary consumption, labor supply, and

[2] I can't point to the theory as to why we need this, but without using this version the model won't solve. I think it might have to do with not having enough unique equations if we use the first derivation of the demand for capital only - if we do that, we never end up using the equation describing the production function).

savings for each period. The budget constraint will contain a term for the cost of require consumption and discretionary consumption.

Finding total demand for the output from industry i

Given the CRS production function, we need to find total output to determine the demands for capital and labor by the firm. To find the total demand for output, we'll use the resource constraint. In particular, the demand for output from each sector is determined by the demand for output from that sector for consumption and investment.

From the solution to the household's problem, we have the demands for each consumption good, c_i. We'll let the aggregate demand for consumption goods from industry i (summing over S and J), be given by C_i. Total demand for output from industry i is the sum of the demands from consumption and investment.

To find the demands for investment, not that in the SS, $I_i = \delta K_i$. Recall that from Equation 20, we can write the demand for capital as a function of output. We can use this to find the total demand for output from each industry:

$$\begin{aligned} X_i &= C_i + \delta K_i \\ X_i &= C_i + \delta \left(\frac{\gamma X_i}{(r+\delta)^{\epsilon} A^{1-\epsilon}} \right) \\ \implies X_i &= \frac{C_i}{1 - \frac{\delta\gamma}{(r+\delta)^{\epsilon} A^{1-\epsilon}}} \end{aligned} \tag{25}$$

With the demand for output from each industry, r, and w, we can solve for each industry's factor demands from equations 19 and 20.

Closing up the model - finding an equilibrium

The SS equilibrium will be defined by prices and allocations such that the above equations are all satisfied and markets clear. Walra's Law says that we need only check for market clearing in two of the three markets.[3] The market clearing conditions with multiple firms become:

$$\sum_i K_i = \sum_J \sum_S b_{j,s} \tag{26}$$

and

$$\sum_i EL_i = \sum_J \sum_S e_{j,s} * n_{j,s} \tag{27}$$

Computation

To compute the solution to the SS of the model with two firms, we'll build off the algorithm set out in step one. New steps/functions are highlighted in red:

1. Make an initial guess at $\bar{r}$ and $\bar{w}$
2. Use r and w and Equations 23 and 24 to solve for the price of consumption goods 1 and 2 and the composite good price.
3. Taking $\bar{r}$, $\bar{w}$, p_1, p_2, and $\tilde{p}$ as given, solve the household's problem:
    - For each j type:
        (a) Make an initial guess at the household's optimal savings and labor supply decisions, $b_{j,s}$, $n_{j,s}$.
        (b) Use a root finder (e.g. `fsolve`) to determine the optimal allocations given $\bar{r}$ and $\bar{w}$.
4. Aggregate over J and S to determine aggregate supply of labor and capital (where savings=capital), and aggregate consumption of each good; K, EL, C_1, C_2.
    - Remember to find EL as the aggregate amount of effective labor units supplied, so you not only want to sum over J and S, but weight by the number of effective labor units each type/age supplies.

[3] Note that we have a goods market, a capital market, and a labor market.

5. Use the aggregate demands for each of the two consumption goods and Equation 25 to solve for the output from each industry i.

6. Use Equations 22 and 21 to solve for the factor demands from each industry.

7. Find the aggregate capital stock demanded: $K = K_1 + K_2$.

8. Find the aggregate effective labor demanded: $EL = EL_1 + EL_2$.

9. Take differences between aggregate amounts supplied and demanded.

10. Use a root finder to determine the eq'm $\bar{r}$ and $\bar{w}$ (i.e. it'll find the $\bar{r}$ and $\bar{w}$ where $K_{demand} - K_{supply} = 0$ and $EL_{demand} - EL_{supply} = 0$).

You might start by setting $\bar{c}_1 = \bar{c}_2 = 0$ and $\alpha_1 = \alpha_2 = (1 - \alpha_1) = 0.5$ (note that the α's have to sum to one). Once you solve the model with this parameterization, try changing these parameters to make sure everything works out. Note that you won't want to set the minimum consumption amounts too high since that may result in the consumer not being able to afford positive amounts of the composite consumption good.

To check that this all works as expected, makes sure:

1. Euler errors are very small

2. The aggregate resource constraint is satisfied ($X_i = C_i + \delta K_i$ for each industry i in the SS).

3. If you set $\alpha_1 = 1$ (so $\alpha_2 = 0$), that you get the same solution as with the one good/firm problem.

Step 4: Making the firm's problem dynamic

In this step, we'll make the firm's problem dynamic. This means firms own capital, not rent it from the household. This doesn't in itself have a large effect on the equations governing the firm in the steady state, but it will certainly change the solution along the time path and requires us to redefine the "zero profit" condition of the firm. One major effect here is that the capital market clearing condition will be replaced by an asset market clearing condition. In particular, that the household demand for equities equal the value of the firms in the economy.

The problem of a firm (sequence problem)

Let's begin by defining the problem of a firm so that we understand it's optimization problem and how that will fit into our general equilibrium model.

Let's consider a simple model where the firm hires labor and purchases capital to maximize profits. The firm is infinitely lived, and so it maximizing the discounted value of the stream of future profits. Let the per period profits be given by $\pi(K_t, L_t, K_{t+1}) = p_t F(K_t, L_t) - w_t L_t - p_t^k (K_{t+1} - (1-\delta)K_t)$.[4] The notation is as follows:

- The subscript t refers to the period.
- K is the capital stock owned/used by the firm
- L are the units of labor hired by the firm
- $F(K, L)$ is the firm's production function
- w is the wage rate, the price of labor
- p is the price of output
- p^k is the price of capital
- δ is the rate at which the capital stock depreciates

We have a transition equation for the dynamic variable, K_t, which tells us how the capital stock evolves over time: $K_{t+1} = (1-\delta)K_t + I_t$, where I_t is gross investment. Thus another way to write the per period profit function is: $\pi(K_t, L_t, K_{t+1}) =$

[4] You often just see revenue minus labor costs as the profit function in these dynamic problems, but this notation makes the exposition of the problem more clear and doesn't change the substance.

$p_tF(K_t, L_t) - w_tL_t - p_t^kI_t$. We can normalize one of these prices to one by dividing through by that price. We'll make the price of output the numeraire, dividing through by p_t yields: $\pi(K_t, L_t, K_{t+1}) = F(K_t, L_t) - \frac{w_t}{p_t}L_t - \frac{p_t^k}{p_t}I_t$. In an economy with one good, the produced good is the same as capital and so $p_t^k = p_t$. Let $\tilde{w}_t = \frac{w_t}{p_t}$, be the real wage (the number of units of output paid for a unit of labor). We can thus write the profit function as: $\pi(K_t, L_t, K_{t+1}) = F(K_t, L_t) - \tilde{w}_tL_t - I_t$.

The problem of the firm is the maximize the discounted present value of these profits. Let the the discount rate of the firm be $\beta = \frac{1}{1+r}$. That is, the firm discounts profits based on the return it can get on a risk free asset, given by r. In principle r can change from year to year, but let's just write the discount rate as β to simply the notation here (we'll add the complexity in a bit). The problem of the firm is thus:

$$\max_{\{L_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \pi(K_t, L_t, K_{t+1}) \tag{28}$$

Note that you can also write this optimization problem as one where the firm chooses investment:

$$\begin{gathered}\max_{\{L_t, I_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \pi(K_t, L_t, I_t) \\ \text{subject to: } K_{t+1} = (1-\delta)K_t + I_t\end{gathered} \tag{29}$$

In either case, the household is choosing labor supply today and the amount of capital to start the next period with (either directly through the choice of K_{t+1} or indirectly through the choice of I_t. Their will be an infinite number of first order conditions - a set of which are for the choice of labor and a set of which are for the choice of capital/investment. In particular, for the choice of labor, we have:

$$\begin{aligned}&\frac{\partial \pi(K_t, L_t, K_{t+1})}{\partial L_t} = 0, \forall\ t \\ \implies &\frac{\partial F(K_t, L_t)}{\partial L_t} = \tilde{w}_t, \forall\ t\end{aligned} \tag{30}$$

and for the choice of capital we have:

$$
\begin{aligned}
&\frac{\partial \pi(K_t, L_t, K_{t+1})}{\partial K_{t+1}} + \frac{\partial \pi(K_{t+1}, L_{t+1}, K_{t+2})}{\partial K_{t+1}} = 0, \forall\ t \\
\implies &1 = \beta \left[\frac{\partial F(K_{t+1}, L_{t+1})}{\partial K_{t+1}} + 1 - \delta\right], \forall\ t \\
&\text{Noting that } \beta = \frac{1}{1+r}, \text{ we have:} \\
&r + \delta = \frac{\partial F(K_{t+1}, L_{t+1})}{\partial K_{t+1}}, \forall\ t
\end{aligned}
\tag{31}
$$

You can see that these conditions look very similar to that of the static firm. Firms hire labor up until the point where the marginal product of labor (which is the marginal benefit of labor) equals the wage rate (which is the marginal cost of labor). Firms purchase capital up to the point where the marginal product of capital (which is the marginal benefit of having capital) is equal to the marginal cost (given by the user cost of capital - the sum of the opportunity cost of not earning rate r on the money invested in capital and the cost of depreciation).

Firm Sequence Problem Exercises

1. Write out the Lagrangian describing the firm's constrained optimization problem of Equation 29. What are the first order necessary conditions?

2. Assume a Cobb-Douglas production function. Solve for the steady state level of capital and labor as functions of the parameters, $\bar{r}$, and $\bar{w}$ in the partial equilibrium model (i.e., only consider the firm).

The problem of the firm (Bellman equation)

It'll often be useful to write the problem of the firm as a Bellman equation, rather than as a sequence problem. This means we are writing the firm's problem as a functional equation. A functional equation is an equation which specifies a function implicitly. In this case, that implicit function is the value function - or the maximum of the firms objective problem. We'll denote the value function with $V(\cdot)$ and the solution to the Bellman equation is the maximum firm value, which is the maximum net present value of firm profits. The value function for the firm is written as:

$$V(K;r,w) = \max_{K',L} \pi(K,L,K') + \beta V(K';r',w') \tag{32}$$

A few notes on this equation:

- I'm using w as the real wage so I don't have to type $\tilde{w}$.
- Since the time horizon is infinite, time subscripts are omitted. At any period, there are an infinite number of periods ahead for the firm to plan for.
- "Primed" variables denote one period ahead variables. So K is the firm's capital stock in the current period and K' is it's capital stock in the next period.
- The arguments in the $V(\cdot)$ functions are the "state variables". These variables include all the relevant information the firm needs to know about it's situation or the aggregate economy in order to make a decision.
- $V(\cdot)$ appears on both sides of the equality. It is thus implicitly defined by this equation (hence the Bellman Equation being termed a functional equation).
- With this new notation, the transition equation becomes: $K' = (1-\delta)K + I$
- We are solving this functional equation for 3 functions:
    1. A policy function for L: $L(K;r,w)$, which tell us how much labor the firm hires given the state variables K, r, w
    2. A policy function for K': $K'(K;r,w)$, which tell us how much capital the firm purchases given the state variables K, r, w
    3. A value function, $V(K;r,w)$, which tell us the value of the firm given the state variables K, r, w

If we take the first order conditions for labor and capital, respectively, we get:

$$\begin{aligned}\frac{\partial V(K;r,w)}{\partial L} &= \frac{\partial \pi(K,L,K')}{\partial L} = 0 \\ \implies \frac{\partial F(K,L)}{\partial L} &= w\end{aligned} \tag{33}$$

and

$$\begin{aligned}\frac{\partial V(K;r,w)}{\partial K'} &= \frac{\partial \pi(K,L,K')}{\partial K'} + \beta\frac{\partial V(K';r',w')}{\partial K'} = 0 \\ \implies 1 &= \beta\frac{\partial V(K';r',w')}{\partial K'}\end{aligned} \tag{34}$$

Note that we don't know $\frac{\partial V(K';r',w')}{\partial K'}$. $V(K;r,w)$ is a function we are solving for and to find it, we need to know the optimal choices of K'. But the FOC for capital is saying that we need $V(K;r,w)$ to solve for the choice of capital! What can we do? We are going to use something called the Envelope Condition. Let's apply, then explain it in words.

First, take the derivative of $V(K;r,w)$ with respect to K:

$$\begin{aligned}\frac{\partial V(K;r,w)}{\partial K} &= \frac{\partial \pi(K,L,K')}{\partial K} + \frac{\partial \pi(K,L,K')}{\partial K'}\frac{\partial K'}{\partial K} + \beta\frac{\partial V(K';r',w')}{\partial K'}\frac{\partial K'}{\partial K} \\ \implies \frac{\partial V(K;r,w)}{\partial K} &= \frac{\partial \pi(K,L,K')}{\partial K} + \underbrace{\left(\frac{\partial \pi(K,L,K')}{\partial K'} + \beta\frac{\partial V(K';r',w')}{\partial K'}\right)}_{=0\text{, by FOC for } K'}\frac{\partial K'}{\partial K} \\ \implies \frac{\partial V(K;r,w)}{\partial K} &= \frac{\partial \pi(K,L,K')}{\partial K} = \frac{\partial F(K,L)}{\partial K} + 1 - \delta \\ &\text{iterating one period ahead, we have:} \\ \frac{\partial V(K';r',w')}{\partial K'} &= \frac{\partial \pi(K',L',K'')}{\partial K'} = \frac{\partial F(K',L')}{\partial K'} + 1 - \delta\end{aligned} \tag{35}$$

So we know what $\frac{\partial V(K';r',w')}{\partial K'}$ is in terms of variables and parameters. The Envelope Condition (or Envelope Theorem) that helps us determine this is a result of what's called the "principle of optimality". This is the idea that once we have optimized along the entire path (i.e., the firm is choosing the optimal K' in all periods), then the change in those policy functions (the functions determining K' as a function of the state variables (K, r, w) is zero. So if we change a state variable (in this case K') the effect on the value function is only the direct effect of the change in K' on next period profits - the effect of the change in K' on K'', K''', etc. are all zero due to the principle of optimality.

Using this Envelope Condition, we can rewrite the FOC for K' as:

$$\begin{aligned}
&\frac{\partial V(K;r,w)}{\partial K'} = \frac{\partial \pi(K,L,K')}{\partial K'} + \beta\frac{\partial V(K';r',w')}{\partial K'} = 0 \\
\implies &1 = \beta\frac{\partial V(K';r',w')}{\partial K'} \\
\implies &1 = \beta\left[\frac{\partial F(K',L')}{\partial K'} + 1 - \delta\right] \\
&\text{or, noting that } \beta = \frac{1}{1+r}: \\
\implies &r + delta = \frac{\partial F(K',L')}{\partial K'}
\end{aligned} \tag{36}$$

So the FOCs are the same as in the sequence problem, which they out to be (since it is just a different way to state the same problem).

Finding the price of producer output in the steady state

In the dynamic firm problem, households hold shares in firms, earning a rate of return that depends on the dividends distributed by the firm and the change in firm value (i.e., capital gains). For households to hold shares in all firms, they need to have the same rate of return. Specifically, the rate of return of a share of the firm is given by:

$$r_t = \frac{DIV_t + (V_{t+1} - V_t - VN_t)}{V_t}, \tag{37}$$

where V_t is the value of the firm at time t, DIV are dividend distributions, and VN are new equity issues. Dividends are the flows distributed to the shareholders. They come from the profits the firms earns as well as from external financing - in this case new equity issues - that can be used to fund investment or dividend distributions. Using our notation above, $DIV_t = \pi(K_t, L_t, K_{t+1}) + VN_t$. We'll restrict $VN > 0$ because, while in reality firms can buy back shares, in practice the IRS typically treats recurrent share repurchases as dividend distributions. In the steady state, this becomes:

$$\bar{r} = \frac{\overline{DIV} + (\bar{V} - \bar{V} - \bar{VN})}{\bar{V}} = \frac{\overline{DIV} - \bar{VN}}{\bar{V}} \tag{38}$$

In this simple case, without taxes or costly external finance, the amount of new equity issues are indeterminate. That is, the firm can issue a dollar of new equity and use that to pay a dollar of dividends. The shareholders are indifferent - they lose one dollar from the dilution of the shares they hold due to the new equity issued, but gain one dollar of dividends. When we add financial frictions (e.g. a dollar of new equity issued doesn't bring one dollar into the firm) and taxes (specifically when we have dividend taxes that are at least as high as capital gains taxes) then firms will never distribute dividend and issue equity together. Since the sum/difference between DIV and VN is indeterminate, we'll just make the assumption that firms never distribute dividends and issue equity at the same time (which is what will hold in the more general specification of the model).

If firms never distribute dividend and issue new equity at the same time, then $\overline{DIV} = 0$ *or* $\bar{VN} = 0$. If $\overline{DIV} = 0$ and $\bar{VN} > 0$, then this would imply the steady state rate of return on equity in the firm is negative, which can't be an equilibrium. Thus it must be the case that $\overline{DIV} > 0$ and $\bar{VN} = 0$. So we can write the steady state return on equity as:

$$\bar{r} = \frac{\overline{DIV}}{\bar{V}} \tag{39}$$

We can rewrite this in terms of the value of the firm:

$$\bar{V} = \frac{\overline{DIV}}{\bar{r}} \tag{40}$$

Dividends are given by:

$$DIV_t = p_t X_t - w_t EL_t - p_t^k I_t, \tag{41}$$

where p_t^k is the price of capital and I_t is investment, which is given by the law of motion for capital:

$$I_t = K_{t+1} - (1-\delta)K_t \tag{42}$$

In the steady state, we have:

$$\overline{DIV} = \bar{p}\bar{X} - \bar{w}\bar{EL} - \bar{p}^k\delta\bar{K}, \tag{43}$$

Note that with constant returns to scale, the the factor demands, $K(r, w, p^k, X)$ and $EL(r, w, p^k, X)$, are proportional to output, X. Letting $k = \frac{K}{X}$ and $l = \frac{EL}{X}$ we can rewrite the SS equation for dividends:

$$\begin{aligned}
\overline{DIV} &= \bar{p}\bar{X} - \bar{w}\bar{EL} - \bar{p}^k\delta\bar{K} \\
&= \bar{p}\bar{X} - \bar{w}\bar{EL}(\bar{r}, \bar{w}, \bar{p}^k, \bar{X}) - \bar{p}^k\delta\bar{K}(\bar{r}, \bar{w}, \bar{p}^k, \bar{X}) \\
&= \bar{p}\bar{X} - \bar{w}\bar{X}\bar{l}(\bar{r}, \bar{w}, \bar{p}^k) - \bar{p}^k\bar{X}\delta\bar{k}(\bar{r}, \bar{w}, \bar{p}^k) \\
&= \bar{X}\left(\bar{p} - \bar{w}\bar{l}(\bar{r}, \bar{w}, \bar{p}^k) - \bar{p}^k\delta\bar{k}(\bar{r}, \bar{w}, \bar{p}^k)\right)
\end{aligned} \tag{44}$$

Hayashi (1982) shows that with quadratic adjustment costs (which is what we'll use - here they are zero), the marginal change in firm value for a change in the capital stock (called Tobin's q and given by $\frac{\partial V_t}{\partial K_t}$) is equivalent to value of the firm per unit of the capital stock (called average q and given by $\frac{V_t}{K_t}$). We thus have:

$$\frac{\partial V_t}{\partial K_t} = \frac{V_t}{K_t} \tag{45}$$

From the FOC for capital one period ahead, we have:

$$
\begin{aligned}
p_t^k &= \frac{1}{1+r_t}\frac{\partial V(Z_{t+1},K_{t+1})}{\partial K_{t+1}} \\
p_t^k &= \frac{1}{1+r_t}\left(p_{t+1}MPK_{t+1} + (1-\delta)p_{t+1}^k\right),
\end{aligned}
\tag{46}
$$

Where the second line follows from the Envelope Condition, which gives the value for Tobin's q;

$$
\frac{\partial V(Z_{t+1},K_{t+1})}{\partial K_{t+1}} = \left(p_{t+1}MPK_{t+1} + (1-\delta)p_{t+1}^k\right) \tag{47}
$$

Using Equation 47 in Equation 45 we have:

$$
\left(p_t MPK_t + (1-\delta)p_t^k\right) = \frac{V_t}{K_t} \tag{48}
$$

We can then use the steady state version of this equality in Equation 49 to find:

$$
\begin{aligned}
\bar{V} &= \frac{\overline{DIV}}{\bar{r}} \\
\implies \left(\bar{p}\overline{MPK} + (1-\delta)\bar{p}^k\right)\bar{K} &= \frac{\overline{DIV}}{\bar{r}}
\end{aligned}
\tag{49}
$$

Substituting in for $\overline{DIV}$ with Equation 44 we have:

$$
\left(\bar{p}\overline{MPK} + (1-\delta)\bar{p}^k\right)\bar{K} = \frac{\bar{X}\left(\bar{p} - \bar{w}\bar{l}(\bar{r},\bar{w},\bar{p}^k) - \bar{p}^k\bar{k}(\bar{r},\bar{w},\bar{p}^k)\right)}{\bar{r}} \tag{50}
$$

Dividing both sides by $\bar{X}$ we get:

$$
\frac{\bar{p}\overline{MPK}\times\bar{K}}{\bar{X}} + (1-\delta)\bar{p}^k\frac{\bar{K}}{\bar{X}} = \frac{\left(\bar{p} - \bar{w}\bar{l}(\bar{r},\bar{w},\bar{p}^k) - \bar{p}^k\bar{k}(\bar{r},\bar{w},\bar{p}^k)\right)}{\bar{r}} \tag{51}
$$

With constant returns to scale, we can write the left hand side of the above as a function of just parameters, $\bar{r}$, $\bar{w}$, $\bar{p}$, and $\bar{p}^k$.

Note that there is one an equation like 51 for each industry, m. The price of capital is determined by the price of output, $\bar{p}$, and the fixed-coefficient matrix Ξ.

In particular, if we let $\boldsymbol{p_t}$ be the $1 \times M$ vector of output prices, we can determine the capital prices as:

$$\underbrace{\boldsymbol{p_t^k}}_{1\times M} = \underbrace{\boldsymbol{p_t}}_{1\times M} \times \underbrace{\Xi}_{M\times M} \tag{52}$$

We can see that $\frac{\bar{K}}{\bar{X}}$ is only a function of $\bar{r}$, $\bar{w}$, and $\bar{p}^k$ from by using the firms FOCs and production function to solve for $\bar{K}(\bar{r}, \bar{w}, \bar{p}^k, \bar{X})$.

$$\begin{aligned}
\text{FOC for } K \implies & MPK_t = \frac{p_t^k}{p_t}(r_t + \delta) \\
\implies & Z_t^{\frac{\epsilon-1}{\epsilon}} \left(X_t \frac{\gamma}{K_t}\right)^{\frac{1}{\epsilon}} = \frac{p_t^k}{p_t}(r_t + \delta) \\
\implies & \left(X_t \frac{\gamma}{K_t}\right)^{\frac{1}{\epsilon}} = Z_t^{\frac{1-\epsilon}{\epsilon}} \frac{p_t^k}{p_t}(r_t + \delta) \\
\implies & \left(X_t \frac{\gamma}{K_t}\right) = Z_t^{1-\epsilon} \left(\frac{p_t^k}{p_t}(r_t + \delta)\right)^{\epsilon} \\
\implies & K_t = \frac{X_t \gamma}{Z_t^{1-\epsilon}\left(\frac{p_t^k}{p_t}(r_t + \delta)\right)^{\epsilon}}
\end{aligned} \tag{53}$$

With this, we can see that $\frac{K_t}{X_t}$ is not a function of X_t. In particular, with CES production we have:

$$\frac{K_t}{X_t} = \frac{\gamma}{Z_t^{1-\epsilon}\left(\frac{p_t^k}{p_t}(r_t + \delta)\right)^{\epsilon}} \tag{54}$$

We can do the same for EL_t:

$$
\begin{aligned}
\text{FOC for } EL \implies MPL_t &= \frac{w_t}{p_t} \\
\implies \left(e^{g_y t} Z_t\right)^{\frac{\epsilon-1}{\epsilon}} \left(X_t \frac{(1-\gamma)}{EL_t}\right)^{\frac{1}{\epsilon}} &= \frac{w_t}{p_t} \\
\implies \left(X_t \frac{(1-\gamma)}{EL_t}\right)^{\frac{1}{\epsilon}} &= \left(e^{g_y t} Z_t\right)^{\frac{1-\epsilon}{\epsilon}} \frac{w_t}{p_t} \\
\implies \left(X_t \frac{(1-\gamma)}{EL_t}\right) &= \left(e^{g_y t} Z_t\right)^{1-\epsilon} \left(\frac{w_t}{p_t}\right)^{\epsilon} \\
EL_t &= \frac{X_t(1-\gamma)}{\left(e^{g_y t} Z_t\right)^{1-\epsilon} \left(\frac{w_t}{p_t}\right)^{\epsilon}}
\end{aligned} \tag{55}
$$

We can take these results to rewrite Equation 51:

$$
\begin{aligned}
\frac{\bar{p}\overline{MPK} \times \bar{K}}{\bar{X}} + (1-\delta)\bar{p}^k \frac{\bar{K}}{\bar{X}} &= \frac{\left(\bar{p} - \bar{w}\bar{l}(\bar{r}, \bar{w}, \bar{p}^k) - \bar{p}^k \bar{k}(\bar{r}, \bar{w}, \bar{p}^k)\right)}{\bar{r}} \\
\implies \frac{\bar{p}^k(1+\bar{r})\gamma}{\bar{Z}^{1-\epsilon}\left(\frac{\bar{p}^k}{\bar{p}}(\bar{r}+\delta)\right)^{\epsilon}} &= \frac{\left(\bar{p} - \bar{w}\bar{l}(\bar{r}, \bar{w}, \bar{p}^k) - \bar{p}^k \bar{k}(\bar{r}, \bar{w}, \bar{p}^k)\right)}{\bar{r}} \\
\implies \frac{\bar{p}^k(1+\bar{r})\gamma}{\bar{Z}^{1-\epsilon}\left(\frac{\bar{p}^k}{\bar{p}}(\bar{r}+\delta)\right)^{\epsilon}} &= \frac{\bar{p} - \bar{w}\frac{(1-\gamma)}{\left(e^{g_y}\bar{Z}\right)^{1-\epsilon}\left(\frac{\bar{w}}{\bar{p}}\right)^{\epsilon}} - \bar{p}^k \frac{\gamma}{\bar{Z}^{1-\epsilon}\left(\frac{\bar{p}^k}{p_k}(\bar{r}+\delta)\right)^{\epsilon}}}{\bar{r}} \\
\implies \bar{r}\bar{p}^k(1+\bar{r})\gamma = p^{1-\epsilon}\left(\bar{p}^k\right)^{\epsilon} \bar{Z}^{1-\epsilon}(\bar{r}+\delta)^{\epsilon} &- \frac{\bar{w}^{1-\epsilon}(1-\gamma)\bar{p}^k(\bar{r}+\delta)^{\epsilon}}{\left(e^{g_y}\right)^{1-\epsilon}} - p^k\gamma
\end{aligned} \tag{56}
$$

In the end, to solve for prices in the steady state ($\bar{p}^k$ and $\bar{p}$) we have to solve $2 \times M$ equations for the $2 \times M$ unknowns.[5] The equations will need are:

1. Equation 56 (for each m): $\bar{r}\bar{p}^k_m(1+\bar{r})\gamma_m = p^{1-\epsilon_m}\left(\bar{p}^k_m\right)^{\epsilon_m} \bar{Z}^{1-\epsilon_m}(\bar{r}+\delta_m)^{\epsilon_m} - \frac{\bar{w}^{1-\epsilon}(1-\gamma_m)\bar{p}^k_m(\bar{r}+\delta_m)^{\epsilon_m}}{\left(e^{g_y}\right)^{1-\epsilon_m}} - p^k_m\gamma_m$ (M equations)

2. Equation 52: $\bar{\boldsymbol{p}}^{\boldsymbol{k}} = \bar{\boldsymbol{p}} \times \Xi$ (M equations)

[5] With Cobb-Douglas production functions, these will be all linear equations and thus more easily solved. They don't appear to be linear in $\bar{p}$ and $\bar{p}^k$ with CES production, making the solution to the system a big more computationally intensive.

Note that the in non-steady state solution we'll be working backwards from the steady state, which will help to pin down investmet, I_t. Otherwise the solution will look much like the above, where we find what dividends much be in period t to give the shareholder their require rate of return, r_t. We'll discuss the solution along the time path in the next step. Thus in the dynamic context, the required rate of return is like the zero profit condition. If the required rate of return owning equities is higher than that on holding bonds, then we firms much have earnings that given them an above market rate of return. Taxes on the firm will be passed through by higher prices and, in general equilibrium, lower rates of return and wages.

Changes to the household's problem

Finding the consumer's demand for consumption will be the same as the static problem. And the consumer's problem will look the same. However, now savings, b, will be interpreted as the value of shares of the firms owned. Since all firms will give the rate of return, households will hold a diversified portfolio of firms.

Finding total demand for output from industry m

Same as in the static problem.

Finding factor demands

Same as in the static problem, updated to use the FOCs from the static firm's problem:

$$MPK_t = \frac{p_t^k}{p_t}(r_t + \delta) \tag{57}$$

and

$$MPL_t = \frac{w_t}{p_t} \tag{58}$$

Market clearing

There are three markets - the labor market, the asset market, and the goods market. By Walras' Law we only need to check two and we'll use the labor and capital markets:

$$\sum_{M} V_m = \sum_{J} \sum_{S} b_{j,s} \tag{59}$$

This says that the total value of assets held by the households must equal the total value of the firms in which they own shares. The condition for the labor market is:

$$\sum_{M} EL_m = \sum_{J} \sum_{S} e_{j,s} * n_{j,s} \tag{60}$$

Future steps

4. Expand to M firms
5. Add solution for firms along the time path
6. Add simple taxes (div, cap gain, corp inc tax on accouting profits)
7. Add endogenous financial policy (it's at this step that we'll add the feature that households invest their savings in two assets - bonds and equities - which have potentially different returns).
8. Add more complex taxes (parameters for various consumption tax/income tax systems, invest tax credits, tax depreciation)
9. Allow consumer subutility preferences to be age dependent.
10. Add government that purchases capital and labor to make public good. Don't need to change consumer utility function to account for this, but we could.
11. Add government production firm
12. Add a fixed factor of production so that there are economic profits (this will necessitate a transfer of profits back to the household) (????)

13. Add a noncorporate sector
14. Add income shifting. This involves adding multinational firms.
15. Add government debt????