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Larry J. Kimbell

Glenn W. Harrison

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This paper contains preliminary findings from research work still in progress and should not be quoted without prior approval of the author.

DEPARTMENT OF ECONOMICS
THE UNIVERSITY OF WESTERN ONTARIO
London, Canada
N6A 5C2

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by

Larry J. Kimbell and Glenn W. Harrison

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Kimbell is Associate Professor and Director, UCLA Business Forecasting Project, Graduate School of Management, UCLA; Harrison is Assistant Professor, Department of Economics, University of Western Ontario. We are grateful to Glenn Graves, Edward Leamer, Charles McLure Jr., Herbert Scarf and Henri Thiel for comments. Participants at an NBER Conference on Applied General Equilibrium Analysis (San Diego, 1981), a UCLA Mathematical Economics Workshop, and a System-Wide Economics Colloquium (Perth, 1982) also made valuable suggestions. The usual disclaimer applies. The National Science Foundation and the Reserve Bank of Australia have provided generous research support.

1. INTRODUCTION

Two important and influential themes in applied general equilibrium analysis were initiated by Harberger [1962] [1966] and Scarf [1967][1973], respectively: the use of analytic approximations to problems of differential tax incidence, and the use of numerical computations of general equilibria. Each approach has well-known strengths and weaknesses relative to the other, and both continue to enjoy widespread popularity with economists.[1] The purpose of this paper is to present a consolidation and extension of these two approaches, in the form of two new procedures for the computation of general equilibria.

Section 2 presents our <u>Factor Price Revision Rule</u> (FPRR) for general equilibrium (GE) models. This Rule provides a surprisingly rapid and simple iterative solution algorithm for a wide class of popular GE models. Section 3 presents our <u>Analytic Factor Price Solution</u> (AFPS) to a useful, but restrictive, class of GE models. The AFPS is an exact, algebraic closed-form solution for the GE values of all endogenous variables (i.e., the reduced-form of our structural GE model). No iterations whatsoever are required to solve GE models for which the AFPS applies.

Section 4 provides a formal derivation of the AFPS, demonstrates that it is a special case of the FPRR when the GE model is appropriately restricted, and finally shows that less restrictive GE models do not appear to have a closed-form solution. Section 5 examines the basis for the existence of the AFPS, a <u>System-Wide Separability</u> property of the class of GE models considered. The thrust of Sections 4 and 5 is to examine certain analytical properties of the

AFPS in order to understand (formally and informally) why the FPRR is such a rapid and robust iterative solution algorithm for less restrictive GE models. Many of the exact analytic properties of the AFPS appear to be approximately valid in applications of the FPRR.

Section 6 concludes by discussing the range of applications of our two new solution procedures.

2. THE FACTOR PRICE REVISION RULE

The central thrust of our efforts to develop faster and more intuitive algorithms for applied GE models has been to exploit certain popular and common features of such models. For the class of GE models considered here the only iterative process involves the determination of equilibrium (relative) factor prices. The widespread use of homothetic neoclassical production functions, as extended by fixed-coefficient intermediate input requirements, means that (relative) output prices are analytically determined for given factor prices by cost-minimization and the zero-profit assumption.[2] Finally, closed-form empirical calibration in applied GE models further restricts the choice of functional form of the production function to the Constant Elasticity of Substitution (CES) family.[3]

The critical step in revising factor prices over iterations involves the following Factor Price Revision Rule:

Part 1: $$P_{f,i+1} = P_{f,i}\left[\frac{D_f}{X_f}\right]^{\frac{1}{\sigma}}$$

for $f = 1,2,\ldots,N_f$, where N_f is the number of factors of production, $P_{f,i}$ is the price of factor f at iteration i, D_f is the demand for factor f, X_f is the (perfectly inelastic) aggregate supply of factor f, and σ is a weighted average of the elasticities of substitution in production for all industries;[4] and

$$\underline{\text{Part 2}}: \quad \text{(Renormalize)} \qquad P_{f,i+1} = \frac{P_{f,i+1}}{P_{1,i+1}}$$

for $f = 1,2,\ldots,N_f$. The price of the first factor is taken without loss of generality to be the numeraire, and is simply reset to unity.

Notice that the Factor Price Revision Rule is a simple Walrasian rule that raises the price of a factor in excess demand, lowers the price of a factor in excess supply, and leaves unchanged the price of a factor with own-demand equal to own-supply. The magnitude of the price revision, for a given ratio of demand to supply, is determined by the reciprocal of the elasticity of substitution. This is a <u>crucial discovery that critically influences the speed of convergence</u>. If the elasticity of substitution is very low, say 0.10, then the ratio of demand to supply in market f is raised to the tenth power in determining the price revision factor. If the elasticity of substitution is very high, say 10, then the tenth root of the ratio of demand to supply is the price revision factor. The low elasticity revision rule causes far more "energetic" changes in factor prices than the high elasticity case (for given values of the ratio of demand to supply).

An Appendix (available on request) demonstrates the computational power of our FPRR as applied to several numerical GE models.

3. THE ANALYTIC FACTOR PRICE SOLUTION

For a well-defined class of GE models, to be specified below, the closed-form solution is given by the following Analytic Factor Price Solution:

$$P_f = \left[\frac{K_f}{K_1}\right]^{\frac{1}{\sigma}} \left[\frac{X_f}{X_1}\right]^{-\frac{1}{\sigma}}$$

for $f = 2,\ldots,N_f$, with $P_1 = 1.0$ and

$$K_f = \sum_{g=1}^{N_g} \alpha_g^{\sigma-1} \beta_g^{\sigma} \delta_{f,g}^{\sigma} T_{f,g}^{-\sigma} ,$$

and where N_g is the number of produced goods, α_g is the efficiency parameter in the CES production function for good g, β_g is the distribution parameter for good g in the CES utility function of the (single) consumer, $\delta_{f,g}$ is the distribution parameter of factor f in the CES production function for good g, and $T_{g,f}$ is unity plus the fractional rate of taxation on the use of factor f in industry g. The price of the first factor is taken without loss of generality to be the numeraire.

The above AFPS applies to models that are general with respect to: (i) any number of factors and goods; (ii) any pattern of distribution parameters in the single-level CES production functions or (single) utility function; (iii) any pattern of efficiency parameters in the production function; and (iv) any arbitrary pattern of factor taxes across factors and producing sectors. The exact solution does not apply to models with: (i) more than one private household; (ii) any interindustry (input-output) flows; (iii) elasticities of substitution in production that vary from sector to sector; (iv) an elasticity of substitution in consumption different from the (uniform) elasticity of substitution in production; and (v) government factor demands that are not proportional to aggregate private industry factor demands.

4. THE FACTOR PRICE REVISION RULE AS A GENERAL ALGORITHM

In this section we establish certain relationships between the AFPS and the FPRR. These relationships provide important insights into the remarkable power of the FPRR to solve empirically interesting GE models.

4.1 Derivation of the Analytic Factor Price Solution

The structural form of the GE model is specified in terms of tastes, taxes, technologies and endowments. Our notation is defined in a Glossary presented in Table 1.

Tastes are specified by a CES utility function of one consumer for N_g goods:

$$U = \left[\sum_{g=1}^{N_g} \beta_g Q_g^{\rho} \right]^{\frac{1}{\rho}} .$$

Taxes, $T_{f,g}$, are defined as unity plus the fractional rate of taxation on the use of factor f in industry (or region) g. Industry g therefore pays $T_{g,f} P_f$ for each unit of factor f utilized, but the factor owner receives only P_f.

Technologies are specified by CES production functions for each good g, defined over N_f factors:

$$Q_g = \alpha_g \left[\sum_{f=1}^{N_f} \delta_{f,g} F_{f,g}^{\rho} \right]^{\frac{1}{\rho}}$$

for $g = 1$ to N_g.

Table 1: Glossary of Notation

Symbol	Definition
N_f	Number of factors of production.
N_g	Number of produced goods.
$P_{f,i}$	Price of factor f at iteration i. The iteration subscript may be dropped when it is not needed.
P_g	Price of produced good g.
$F_{f,g}$	Factor intensity--number of units of factor f needed to produce one unit of good g.
$T_{f,g}$	Ad valorem tax rate on the use of factor f in industry g.
Q_g	Aggregate quantity of good g demanded by the (single) consumer.
$\delta_{f,g}$	Distribution parameter for factor f in the CES production function for good g.
α_g	Efficiency parameter for the CES production function for good g.
β_g	Distribution parameter for good g in the CES utility function of the (single) consumer.
σ	Elasticity of substitution for the CES production function of each good and the single utility function.
ρ	$= 1 - \frac{1}{\sigma}$, a useful transform of σ.
X_f	Endowment of factor f, in physical units.
D_f	Demand for factor f.

Endowments are exogenously given as X_f, for $f=1,\ldots,N_f$. Notice the implied restriction that the elasticity of substitution is uniform in technology and tastes. There is only one consumer.

Factor intensities are derived from minimizing total cost, subject to the production of unit output for each good g, yielding:

(1) $$F_{f,g} = \frac{\delta^{\sigma}_{f,g} T^{-\sigma}_{f,g} P^{-\sigma}_{f}}{\alpha_g \left[\sum_{f=1}^{N_f} \delta^{\sigma}_{f,g} T^{1-\sigma}_{f,g} P^{1-\sigma}_{f} \right]^{\frac{1}{\rho}}}$$

Output prices are derived analytically from the requirement that in long-run equilibrium each industry has zero profits, and from constant returns to scale, yielding:

(2a) $$P_g = \sum_{f=1}^{N_f} F_{f,g} T_{f,g} P_f,$$

for $g=1,\ldots,N_g$, or

(2b) $$P_g = \frac{1}{\alpha_g} \left[\sum_{f=1}^{N_f} \delta^{\sigma}_{f,g} T^{1-\sigma}_{f,g} P^{1-\sigma}_{f} \right]^{\frac{1}{1-\sigma}}$$

for $g=1,\ldots,N_g$.

Wealth (or Income) is defined as the sum of the products of endowed factors and factor prices:

(3) $$W = \sum_{f=1}^{N_f} P_f X_f.$$

Output demands are derived by maximizing utility subject to the budget constraint, yielding:

$$(4a)\qquad Q_g = \frac{W\beta_g^{\sigma}P_g^{-\sigma}}{\sum_{g=1}^{N_g}\beta_g^{\sigma}P_g^{1-\sigma}}$$

for $g=1,\ldots,N_g$. Substituting expression (2) for output prices into equation (4a) expresses output demands in terms of factor prices:

$$(4b)\qquad Q_g = \frac{W\alpha_g^{\sigma}\beta_g^{\sigma}\left[\sum_{f=1}^{N_f}\delta_{f,g}^{\sigma}T_{f,g}^{1-\sigma}P_f^{1-\sigma}\right]^{\frac{1}{\rho}}}{\sum_{g=1}^{N_g}\alpha_g^{\sigma-1}\beta_g^{\sigma}\left[\sum_{f=1}^{N_f}\delta_{f,g}^{\sigma}T_{f,g}^{1-\sigma}P_f^{1-\sigma}\right]}$$

for $g=1,\ldots,N_g$.

Producers are instructed to produce exactly the quantities demanded, thereby forcing demand to equal supply in all output markets. Any disequilibrium must therefore appear in factor markets. Private Factor Demands are derived by summing the products of output demands and factor intensities over all private good production. The results are:

$$(5a)\qquad D_f = \sum_{g=1}^{N_g} Q_g F_{f,g}$$

for $f=1,\ldots,N_f$. Government factor demands are derived by exhausting tax revenue on the purchase of factors in proportion to aggregate private factor demands.[4] Total factor demands are then simply some scalar multiple, γD_f, of private factor demands (where γ is greater than or equal to unity). Substituting expression (4b) for Q_g, and expression (1) for $F_{f,g}$ yields the expression for Total Factor Demands:

$$(5b)\qquad D_f = K \cdot K_f \cdot P_f^{-\sigma}$$

for $f=1,\ldots,N_f$, and where

$$K_f = \sum_{g=1}^{N_g} \alpha_g^{\sigma-1} \beta_g^{\sigma} \delta_{f,g}^{\sigma} T_{f,g}^{-\sigma}$$

and

$$K = \frac{\gamma\left[\sum_{f=1}^{N_f} X_f P_f\right]}{\sum_{f=1}^{N_f}\left[\sum_{g=1}^{N_g} \alpha_g^{\sigma-1} \beta_g^{\sigma} \delta_{f,g}^{\sigma} T_{f,g}^{1-\sigma}\right] P_f^{1-\sigma}} .$$

It becomes important in the sequel to recognize that K is the same for all factors, and that K_f is only a function of parameters (i.e., it is <u>not</u> a function of any endogenous variable).

In equilibrium the demand for each factor, D_f, equals the (inelastic) factor supply, X_f:

(6) $$X_f = K \cdot K_f \cdot P_f^{-\sigma}$$

for $f=1,\ldots,N_f$. Let factor 1 be the numeraire and find the ratios of factor demands (or supplies) for non-numeraire factors to factor 1 demand (or supply):

(7) $$\frac{X_f}{X_1} = \frac{K_f}{K_1} \frac{P_f^{-\sigma}}{P_1^{-\sigma}}$$

for $f=2,\ldots,N_f$. Solving (7) for P_f yields:

(8) $$P_f = \left[\frac{K_f}{K_1}\right]^{\frac{1}{\sigma}} \left[\frac{X_f}{X_1}\right]^{-\frac{1}{\sigma}}$$

for $f=2,\ldots,N_f$ (note that $P_1 \equiv 1.0$), and where

$$K_f = \sum_{g=1}^{N_g} \alpha_g^{\sigma-1} \beta_g^{\sigma} \delta_{f,g}^{\sigma} T_{f,g}^{-\sigma} \quad .$$

Equation (8) is the exact algebraic solution to the class of general equilibrium models specified. Substitution of equation (8) for P_f in equations (1) through (4) gives the complete algebraic reduced form of the model; that is, each endogenous variable expressed as explicit algebraic functions of exogenous variables and parameters.

Without essential loss of generality, the full reduced form of the GE model without taxes and with $\alpha_g = 1$ for all $g = 1,\ldots,N_g$ is now presented. The presence of a numeraire complicates notation needlessly, but multiplying the vector of factor prices in (8) by

$$K_1^{\frac{1}{\sigma}} X_1^{-\frac{1}{\sigma}}$$

avoids these problems. This normalization leads to the following simple form for <u>Factor Prices</u>:

$$P_f = K_f^{\frac{1}{\sigma}} X_f^{-\frac{1}{\sigma}} \, ,$$

for $f = 1,\ldots,N_f$, where

$$K_f = \sum_{g=1}^{N_g} \bar{\beta}_g \bar{\delta}_{g,f},$$

$$\bar{\delta}_{g,f} = \delta_{g,f}^{\sigma} \qquad \text{and}$$

$$\bar{\beta}_{g,f} = \beta_{g,f}^{\sigma}.$$

Goods prices are

$$P_g = \left[\sum_{f=1}^{N_f} \bar{\delta}_{g,f} \, K_f^{-\rho} \, X_f^{\rho} \right]^{\frac{1}{1-\sigma}}$$

for $g = 1,\ldots,N_g$. Factor intensities are

$$F_{g,f} = \bar{\delta}_{g,f} \, K_f^{-1} \, X_f \left[\sum_{f=1}^{N_f} \bar{\delta}_{g,f} \, K_f^{-\rho} \, X_f^{\rho} \right]^{-\frac{1}{\rho}}$$

for $f = 1,\ldots,N_f$ and $g = 1,\ldots,N_g$. Wealth is

$$W = \sum_{f=1}^{N_f} K_f^{\frac{1}{\sigma}} X_f^{\rho} \, .$$

Goods demands are

$$Q_g = \bar{\beta}_g \left[\sum_{f=1}^{N_f} \bar{\delta}_{g,f} \, K_f^{-\rho} \, X_f^{\rho} \right]^{\frac{1}{\rho}} ,$$

for $g = 1,\ldots,N_g$, and Factor Demands are trivially $D_f = X_f$ for $f = 1,\ldots,N_f$. An Appendix (available on request) provides a simple computationally recursive statement of the complete GE solution.

4.2 The Analytic Factor Price Solution as a Special Case of the Factor Price Revision Rule

It can be readily shown that the FPRR would solve the above GE model in a single iteration. Substitute expression (5b) for D_f in Part 1 of the FPRR, yielding

$$P_{f,i+1} = P_{f,i} \left[\frac{K \cdot K_f \cdot P_{f,i}^{-\sigma}}{X_f} \right]^{\frac{1}{\sigma}}$$

for $f = 1,\ldots,N_f$. This reduces to

$$P_{f,i+1} = K^{\frac{1}{\sigma}} K_f^{\frac{1}{\sigma}} X_f^{-\frac{1}{\sigma}}$$

for $f = 1,\ldots,N_f$. Normalization, in Part 2 of the FPRR, then gives

$$P_{f,i+1} = \left[\frac{K_f}{K_1}\right]^{\frac{1}{\sigma}} \left[\frac{X_f}{X_1}\right]^{-\frac{1}{\sigma}}$$

for $f = 2,\ldots,N_f$. This, of course, is expression (8) for the exact analytic solution. In other words, after one iteration of the FPRR factor prices are identically equal to the values they would have been had we used the exact AFPS in the first place.

4.3 Two Important Extensions

It would be nice if a closed-form solution could be found for GE models of arbitrary functional form and specification. As noted earlier, the Analytic Factor Price Solution does not appear to generalize to include several features of contemporary, policy-oriented, numerical GE models. We now illustrate two apparent bounds on the existence of such closed-form solutions. On a more positive note, however, we attempt to explain why our iterative Factor Price Revision Rule works so well for more general models that incorporate these "realistic" extensions. An Appendix discusses two additional extensions (viz., intermediate trade and a non-neutral role for government expenditure).

4.3.1 Multiple Private Households

A derivation similar to the above can proceed in the case of multiple (private) households with varying tastes and endowments to the point of an expression for factor demands (analogous to (5b) above). Let the distribution parameters in tastes be indexed by household h (for N_h households) as well as good g, $\beta_{h,g}$, and let household h endowments of factor f be denoted $X_{h,f}$. Reverting to the no-tax case, factor demands then become:

$$D_f = \left[\sum_{h=1}^{N_h} \frac{\left[\sum_{f=1}^{N_f} X_{h,f} P_f \right] \left[\sum_{g=1}^{N_g} \alpha_g^{\sigma-1} \beta_{h,g}^{\sigma} \delta_{f,g}^{\sigma} \right]}{\sum_{f=1}^{N_f} \left[\sum_{g=1}^{N_g} \alpha_g^{\sigma-1} \beta_{h,g}^{\sigma} \delta_{f,g}^{\sigma} \right] P_f^{1-\sigma}} \right] P_f^{-\sigma}$$

for $f = 1, \ldots, N_f$.

With only one consumer the large bracketed expression can be factored into the product $K \cdot K_f$ where K is a function of factor prices but is <u>not specific to factor f</u>, and where K_f is specific to factor f but is <u>not a function of factor prices</u>. Taking the ratio $\frac{D_f}{D_1}$ then eliminates K, so a closed-form solution emerges.

No apparent simplification is available when there are multiple consumers who are differentiated by tastes and/or endowment distribution. The K_f term, which is an interaction between tastes and production function parameters, is weighted by the wealth of households--but these depend on factor prices. This appears to prevent a closed form solution.

However, unless there are strong interactions among endowments, taste parameters and production function distribution parameters, the large bracketed term does not vary across factors very much. An initial, possibly crude, guess at factor prices is apparently sufficient to let the typically stronger relative endowment term

$$\left[\frac{X_f}{X_1} \right]^{-\frac{1}{\sigma}}$$

estimate relative factor prices fairly closely. This argument rationalizes the fact that our FPRR has worked well as an iterative solution algorithm in spite of not giving the exact solution to GE models with multiple households.

4.3.2 Non-uniform Elasticities of Substitution

In the case where elasticities of substitution in production differ from the elasticity of substitution in tastes, a derivation similar to the closed-form analytic solution can proceed to equation (5b), except that the K_f factor becomes

$$K_f = \sum_{g=1}^{N_g} \alpha_g^{\sigma-1} \beta_g^{\mu-1} \delta_{f,g}^{\sigma} T_{f,g}^{-\sigma} P_g^{\sigma-\mu} ,$$

where σ is the (uniform) elasticity of substitution in each sector's production function, μ is the elasticity of substitution in the utility function, and P_g is the price of output in industry g. The K_f expression was previously a function of parameters only; here it is a function of endogenous variables and no closed-form solution is apparent.

Notice that as the elasticities approach each other the last term,

$$P_g^{\sigma-\mu} ,$$

approaches unity and the Analytic Factor Price Solution emerges. Notice also that if output prices are unity (as they might be in the "base case") then this term also disappears. Therefore, in practice, if elasticities do not vary too much or if relative goods prices are close to unity, this term may be relatively small and the Factor Price Revision Rule can give very rapid convergence even though the Rule omits this term.

5. SYSTEM-WIDE SEPARABILITY

The Analytic Factor Price Solution exhibits a property that we will call System-Wide Separability, since it implies that any changes in the tax rates on (non-numeraire) factor f alter only the price of factor f. Equivalently, a change in the tax rates on the numeraire factor 1 change all other factor prices by exactly the same proportion. More formally, since $T_{j,g}$ is not an argument of the formula for $\frac{P_f}{P_k}$,

$$\frac{\partial\left[\frac{P_f}{P_k}\right]}{\partial T_{j,g}} = 0,$$

for f, k, and j not equal, and for all such triples of f, k, and j. This property does not imply that the price of factor f adjusts enough to capitalize fully the tax changes (there can still be excise tax effects via changes in goods prices), but that the burden of increased taxes on one factor produce utterly no relative factor price changes among the other factors. In this Section we provide an explanation for this important and powerful property.

The AFPS leads to equations for factor intensities and output as follows:

$$F_{f,g} = \frac{\delta_{f,g} X_f}{K_f Z}$$

$$Q_g = \beta_g^{\sigma} Z$$

where $$Z = \left[\sum_{f=1}^{N_f} \delta_{f,g}^{\sigma} K_f^{-\rho} X_f^{\rho}\right]^{\frac{1}{\rho}} .$$

The allocation of factor f to the production of good g, $A_{f,g}$, is simply the product of the relevant factor intensity and output:

$$A_{f,g} = F_{f,g}\, Q_g$$
$$= \frac{\delta^{\sigma}_{f,g}\, \beta^{\sigma}_{g}\, X_f}{K_f} .$$

Suppose the endowment of factor i increases, holding constant all other endowments and parameters. The ratio of the new factor allocation to the old factor allocation, for any arbitrary good g, becomes:

$$\frac{A^*_{i,g}}{A_{i,g}} = \frac{X^*_i}{X_i}$$

$$\frac{A^*_{f,g}}{A_{f,g}} = 1 ,$$

for all f not equal to i, and where X^*_i represents the new value of X_i. For all factors other than i the full allocation across all industries is completely invariant to changes in the endowment of factor i. Substituting for $A_{f,g}$ yields the Factor Use Decomposition Formula:

$$\frac{F^*_{f,g}}{F_{f,g}}\, \frac{Q^*_g}{Q_g} = 1 .$$

The ratio of the output of good g after the expansion of the endowment of factor i to the output of good g before the endowment change, $\frac{Q^*_g}{Q_g}$, is called the "expansion effect". It shows the proportion by which the use of factor f in producing good g would increase if output changed but factor prices were constant.[5] The ratio of the factor intensities, $\frac{F^*_{f,g}}{F_{f,g}}$, is called the "substitution effect". It shows how the demand for factor f in producing good g would have changed, given the factor price changes, holding output constant. The Factor Use Decomposition Formula shows that the substitution effect is <u>identically equal to the reciprocal of the expansion effect</u> for all factors other than the one which changed, and for each and every industry. Essentially, the special restrictions involved in our AFPS are sufficient to make the expansion effect <u>identically offset</u> the substitution effect, leaving allocations other than X_i unchanged.[6]

This result also helps to explain a distinctive feature of the AFPS--the System-Wide Separability property. The CES family of production functions is strongly separable; i.e., the marginal rate of substitution between factors f and j is invariant to changes in any other factor i, holding the inputs of f and j constant. An increase in the endowment of factor i, since it does not alter the amounts of f and j used in any production function (in a general equilibrium), leaves unchanged these marginal rates of substitution. The ratio of factor prices f and j is therefore invariant to changes in the endowment of factor i. System-Wide Separability is therefore simply the result of strong separability of CES production functions, combined with the special result that allocations of factors other than i are invariant to changes in the endowment of factor i.

Figure 1 illustrates the results just discussed for the two-good, two-factor case. Assume that the endowment of capital doubles, holding constant the endowment of labour and other parameters. The output of both goods will commonly expand since the production possibility frontier shifts outward for both goods. If

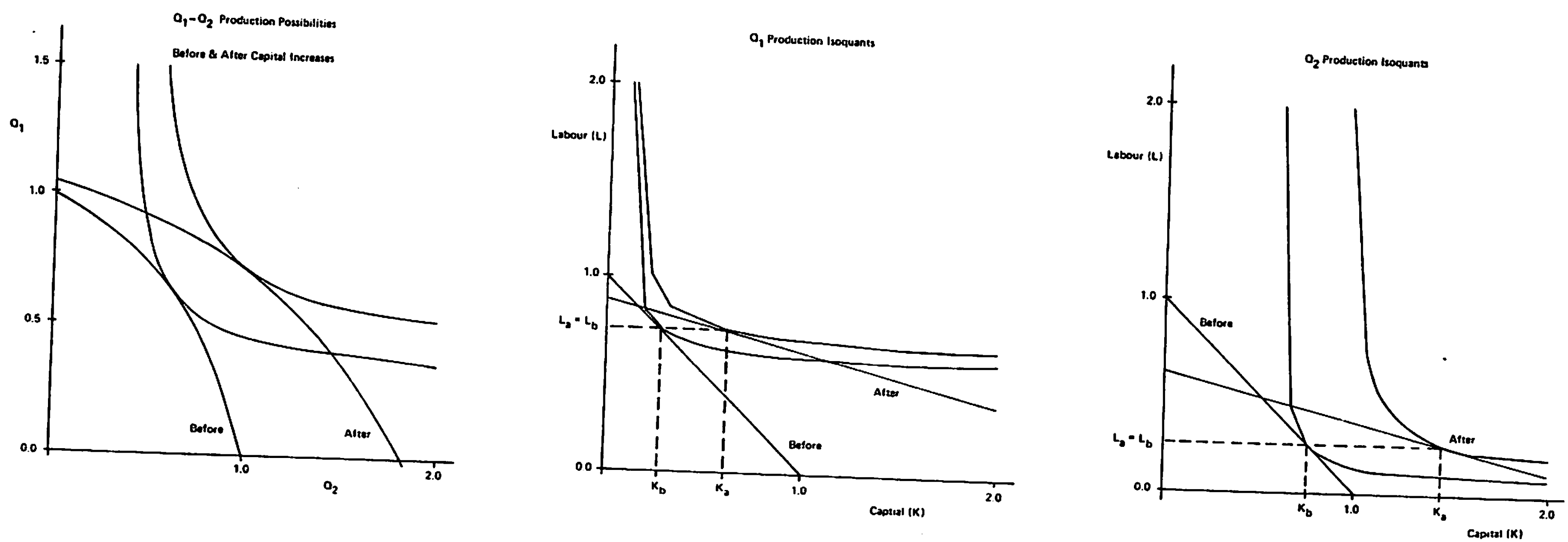

FIGURE 1: Offsetting Expansion and Substitution Effects Before and After a Doubling of the Endowment of Capital.

factor prices did not change, the demand for labour would rise in both industries. The relative factor price of labour must rise, of course, until the aggregate demand equals the (unchanged and inelastic) labour endowment. There could be higher demand in one industry offset by lower demand in the other industry. However, the special restrictions of our model imply that the demand for labour is unchanged in each industry.

Notice in Figure 1 that doubling the amount of capital increases the output of both goods with a larger percentage increase in the output of good 2, which is relatively intensive in capital. The amounts of capital demanded double in both industries and the amount of labour demanded is the same in both industries.

For any general equilibrium model the sum of factor demands across all industries equals the (inelastic) aggregate factor supply, before and after a change in endowments. In other words:

$$\frac{\sum_{g=1}^{N_g} A^*_{f,g}}{\sum_{g=1}^{N_g} A_{f,g}} = \frac{X^*_f}{X_f}$$

for all f. In our special case

$$\frac{A^*_{f,g}}{A_{f,g}} = \frac{X^*_f}{X_f}$$

for all f, g pairs. Thus, what is special about our AFPS is that the expansion effect is exactly offset by the substitution effect <u>in each and every industry</u>.

6. APPLICATIONS

6.1 Application of the Analytic Factor Price Solution

If a particular GE model falls within the requirements for direct application of the AFPS, it obviously constitutes the best way to "find" the solution. Although many applied GE researchers would flinch at these requirements, it must be noted that they are general enough to incorporate many familiar results from the theories of tax incidence and international trade. Kimbell and Harrison [1983; Section 3.3.2] derive several results from the Harberger-Mieszkowski-McLure regional tax incidence literature, and we examine several basic propositions of trade theory below.

International trade theorists have refined our knowledge of the properties of neoclassical GE models; see Jones [1965], Kemp [1969] , and Jones and Schienkman [1977], for example. Jones and Schienkman [1977; p. 911] note that the

> standard Heckscher-Ohlin model of international trade incorporates a set of propositions that reveal essential properties of the two-commodity, two-factor general equilibrium model of production. Of these propositions, four are central:
>
> i) The Heckscher-Ohlin Theorem links the pattern of trade to factor intensities and factor endowments. A country exports that commodity that uses intensively the factor that is relatively cheap prior to trade.
> ii) The Factor-Price Equalization Theorem suggests that free-trading commodities is sufficient to cause factor prices to be equal between countries even if factor supplies cannot cross national boundaries.
> iii) The Stolper-Samuelson Theorem states that an increase in the price of some commodity (through tariff policy, e.g.) must unambiguously raise the real reward to some factor of production.
> iv) The Rybczynski Theorem points out that if prices are kept constant but the endowment of some factor rises, not all outputs can expand. The production of some commodity must fall.
>
> It is convenient to group (i) and (ii) and (iii) and (iv) together. [...The] essence of propositions (i) and (ii) is the link between factor and commodity prices in any particular country.

A direct mapping from factor prices to commodity prices exists for the GE model presented in Section 4.1 and Appendix 2:

$$P_g = \left[\sum_{f=1}^{N_f} \bar{\delta}_{g,f} \, P_f^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

for $g=1,\ldots,N_g$. With a transformation of variables we have the simple matrix mapping $G = WV$ where G is $N_g \times 1$ with typical element $P_g^{1-\sigma}$, W is $N_g \times N_f$ with typical element $\bar{\delta}_{g,f}$, and V is $N_f \times 1$ with typical element $P_f^{1-\sigma}$. If the arbitrary number of goods equals the number of factors (i.e., $N_g = N_f$), and the matrix W is non-singular, the inverse mapping from commodity prices to factor prices also exists: $V = D^{-1}G$. Thus

$$P_f = \left[\sum_{g=1}^{N_g} \bar{\bar{\delta}}_{g,f} \, P_g^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

for $f=1,\ldots,N_f$, and where $\bar{\bar{\delta}}_{g,f}$ is a typical element of D^{-1}. Proof of the Factor Price Equalization Theorem is quite elegant using this mapping.[7] Partition the price vectors for goods and factors in half, referring to two countries:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} (D^{-1})_{11} & 0 \\ 0 & (D^{-1})_{22} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

Trade in goods gives us $G_1 = G_2$, and the assumption of identical technologies gives us $(D^{-1})_{11} = (D^{-1})_{22}$. Hence we have $V_1 = V_2$ and the equalization of factor prices.

These results, of course, are the basic mappings emphasized by Jones and Schienkman. There also exists a simple GE matrix mapping, pertaining to the

Rybczynski Theorem, from physical factor endowments to the output of goods. In fact, one remarkable property of our AFPS in this case is that every endogenous variable can be similarly transformed and represented as a simple matrix mapping of physical endowments (also transformed). Further study of these important and intriguing mappings is beyond the scope of the present paper, but provides a strengthening and generalization of received trade in the sense that: (i) they refer to global and exact GE solutions, rather than local and approximate solutions; (ii) with appropriate partitioning of goods and factors, one may simultaneously obtain comparative static propositions for all relevant regions or nations, rather than for a single region or nation;[8] and (iii) they are based on a complete GE solution, rather than a partial equilibrium solution taking goods prices as given (the "small open economy" assumption of given world prices).

6.2 Application of the Factor Price Revision Rule

Many empirical GE models do not satisfy the requirements for direct application of our AFPS, but can be solved efficiently with our iterative FPRR. We do not claim that any GE model can be solved with the FPRR. Indeed the speed of our algorithm rests on the exploitation of the regular structure that is widely imposed on such models. Accordingly we see our algorithm as complementary to, rather than competitive with, algorithms (such as Scarf's, and its modern variants) that require less structure on the model.[9]

We perceive the value of our FPRR to be derived from the desire in applied work for extensions in the dimensions of the search for general equilibria. These extensions are broadly threefold: the need to adopt alternatives to the assumption of complete factor mobility, the need to study more than two primary factors of production, and the need to undertake systematic sensitivity analyses of simulation results.

6.2.1 Alternatives to Complete Factor Mobility

If we assume that capital and labour are each fully mobile across all sectors of the economy, the essential dimension of a GE search is one (since one factor can be the numeraire). If, however, we wish to assume that one of these factors is specific to each sector, the search dimension is increased to the number of sectors in the model. If we allow twenty sectors (a popular enough number in applied work), we must find twenty relative factor prices. One substantive reason for adopting such an assumption is to focus attention on the "short-run" incidence of policy changes, since factor specificity is widely viewed as characteristic of the short run (see Mayer [1974], Mussa [1974] and Fullerton [1982]). Another popular substantive rationale for adopting specific-factor assumptions rests on the notion of geographic immobility (see McLure [1969] [1971], not to mention the international trade literature in the Heckscher-Ohlin tradition).

Kimbell and Harrison [1983] demonstrate how numerous configurations of factor specificity may be adopted in the empirical calibration of GE models, albeit with concomitant expansions in the dimensions of the search for an equilibrium. If combinations of region-specificity and sector-specificity are of interest[10] the search dimension may easily climb to thirty or more.[11] Fullerton [1982] experienced severe computational cost increases as he introduced a modest number of specific factors in an applied GE model:

> As expected, additional simplex dimensions increase computer cost more than proportionately. However, there seems to be an acceleration in the rate of increase. At first, costs rise by about the square of the number of dimensions, but the addition of the eighth price raises cost by more than the cube. The geometrically increasing costs serve to reinforce the adoption of ad hoc assumptions made earlier to limit the number of dimensions. (p.).

No such difficulties have been experienced with the FPRR.[12]

6.2.2 More Factors of Production

Given the two-factor orientation of the theoretical tradition which initially motivated work on applied GE modelling (especially Shoven and Whalley [1972]), it was natural for later models to retain this feature while extending their models in other important directions. Two additional considerations, however, served to inhibit extensions in the number of factors: (i) existing sectoral value-added data could be readily interpreted as factor payments to Labour and Working Capital, and comparable or consistent data on payments to Skilled or Unskilled Labour, Land, Energy or Water (for example) were not so readily available; and (ii) every additional factor increased by one the search dimension for a GE solution, increasing computational costs dramatically. As certain policy issues requiring the explicit consideration of additional factors loom in importance[13] we may expect improvements in the availability of data. The performance of the FPRR does not deteriorate rapidly as additional factors are incorporated (recall the discussion of performance for the inclusion of specific factors), facilitating the analysis of GE models with more than two primary factors.

6.2.3 Systematic Sensitivity Analysis

The early numerical (or computable) GE models of Scarf [1967] [1973] were modest in size (e.g., two or three producing sectors and two or three primary factors), qualitatively general in the sense of imposing very weak regularity conditions on model structure, and calibrated with hypothetical data. Shoven and Whalley [1972], Shoven [1976] and Whalley [1977] provided the first small, qualitatively restrictive, policy-relevant GE models. Familiar functional forms were chosen for utility and production functions (e.g., single-level CES), techniques for empirical calibration were developed (viz., the notion of a "benchmark equilibrium"), and policy issues familiar from the Harberger [1962] [1966] literature re-examined. More recent developments have extended this class to include "reasonably large" models with twenty or more producing sectors and/or four or more primary factors. Variations on the popular functional forms have also been adopted (e.g., multi-level CES, and LES demand systems), and a wide range of policy issues considered. Fullerton, Henderson and Shoven [1983], Kimbell and Harrison [1983], and Scarf and Shoven [1983] provide surveys of these developments.

The policy-relevance of these models, and their avowedly "empirical" nature, render them open to casual criticism. Most economists are deeply familiar with their underlying neoclassical structure; we are not therefore concerned to defend them from criticisms based on rejection of that structure. On the other hand, criticism based on suspicion of the particular empirical calibration adopted currently leads to non-systematic and/or uninformed debate. The general techniques used to calibrate numerical GE models are discussed in the references given above, and elsewhere.[14] Given, then, that users of numerical GE models are increasingly "informed" as to the various sources of data embodied in their simulations, how is one to systematically identify the robustness of the results for some particular decision? Our response to this important question is to urge a systematic sensitivity analysis of the base case policy simulations. A number of critical dimensions to such analysis may be readily identified from any discussion of the procedures used to calibrate GE models.[15] Perhaps the most important contribution of the FPRR is its ability to cope with the severe computational burden required of sensitivity analyses of complete GE models.

6.3 Joint Application of the Analytic Factor Price Solution and the Factor Price Revision Rule

It is possible to apply both of our solution procedures to certain large GE models in order to improve computational speed. This combination provides an extremely efficient means of solving GE models with 50 or more sectors, 4 or more factors of production, and for which the AFPS does not directly apply. Problems of this dimension arise with increasing frequency, especially in multi-regional and international applications.

The joint application is based on simplifying the "true" GE model to meet the requirements for the AFPS, finding the AFPS, and using that solution as a starting point for applying the FPRR to the "true" model.[16] Why should an approximation provide good starting values, especially considering the restrictions required for the AFPS to hold? Precisely because many of the exact analytic properties of the AFPS are approximately valid in applications of the FPRR, as demonstrated in Sections 4 and 5. That discussion sheds considerable light on the speed of the FPRR as a "stand-alone" algorithm. We are now suggesting a reverse proposition: that it rationalizes the practical use of the AFPS as an approximation to the results of the first few FPRR iterations.

Not only is there an analytical unity of the two procedures (viz., the AFPS being a special case of the FPRR), but a valuable practical unity as well.

The procedure adopted to simplify a GE model to meet the requirements of the AFPS is quite intuitive. If there are multiple households the expenditures of each household are simply aggregated. Variations in elasticities of substitution in production across sectors can be removed by computing a weighted average, with each sector's contribution to total value added used to weight its elasticity. The elasticity of substitution in consumption can then be set equal to this weighted average. Intermediate flows may be ignored altogether,[17] and the factor shares of any governments set equal to aggregate private factor shares. Appendix 1 (available on request) illustrates that the joint application of the AFPS and FPRR can lead to improvements in computational speed (over the FPRR alone) of one-third to two-thirds, depending on the complexity of the original model (i.e., the degree to which it departs from the requirements of the AFPS). These are significant improvements. The process of generating a simplified "synthetic" GE model and computing its benchmark equilibrium implies some fixed (overhead) computational cost to joint application of the AFPS and FPRR. Despite the reduction in variable computational cost that obtains from the joint application, our experience suggests that total computational cost is reduced only when the problem is large (viz., 50 or more sectors and 4 or more factors).

FOOTNOTES

[1] McLure [1975] provides an excellent exposition and survey of analytic extensions and applications of the Harberger model. Scarf and Shoven [1983] collect a number of applications and extensions of the Scarf approach. A number of recent papers use the latter approach to examine policy issues familiar with the earlier Harberger literature: see Fullerton, King, Shoven and Whalley [1981], Fullerton, Henderson and Shoven [1983], and Kimbell and Harrison [1983], for example. Shoven [1976] and Harberger and Bruce [1976] provide an interesting exchange on the relative strengths of the two basic solution approaches. Two alternative perspectives on the general equilibrium solution problem are: (i) approaches that formulate the solution problem in terms of some "equivalent" constrained maximization problem, for which we have known algorithms (see Dixon [1975] [1978] and Ginsburgh and Waelbroeck [1976]); and (ii) approaches that linearize the basic general equilibrium model and then apply known algorithms, such as Euler's method, to the linearized system (see Johansen [1960] and Dixon, Parmenter, Sutton and Vincent [1982; pp. 44-60, Ch. 5]).

[2] See Fullerton, Shoven and Whalley [1978; p. 45] and the earlier references cited there. More general linear activity specifications, such as used by Scarf [1967] [1973], permit multiple outputs but also include output prices in the price vector space as additional dimensions. The linear activity approach does not have to define some goods as factors on an <u>a priori</u> basis, since a given good is a "factor" if it has a negative coefficient in any linear activity. We accept this restriction that factors be stipulated <u>a priori</u> as worth the gain in reduced dimensionality. Similarly, our way of specifying the role of government implies that the government budget constraint is imposed analytically and (unlike the approach of Fullerton, Shoven and Whalley [1978]) does not add another dimension to the search.

[3] Another advantage of the CES form is that cost-minimizing factor input ratios may be derived analytically given factor prices. The continuing popularity of the CES function is illustrated by its use in the "first generation" of applied GE models (e.g., Shoven and Whalley [1972; pp. 290, 301]) as well as the "second generation" (e.g., Fullerton, King, Shoven and Whalley [1981; p. 680]).

[4] The weights adopted, with considerable success in our own empirical work, have been the value-added shares of each sector in aggregate value added. Thus we define

$$\sigma = \sum_{i=1}^{N_g} \left[\left(\frac{v_i}{\sum_{i=1}^{N_g} v_i} \right) \sigma_i \right]$$

where σ_i is the elasticity of substitution adopted for sector i, v_i is the value-added of sector i, and N_g is the number of goods (or sectors). Although these weights are quite intuitive, we have no formal analytic sense of their propriety.

[5]Recall that the class of production functions currently being considered are homothetic with constant returns to scale.

[6]Note that this result is analogous to the familiar result that in the case of Cobb-Douglas production functions cross-price elasticities of factor demand are identically zero, even though they are not zero for the more general CES case.

[7]For all prices of country-specific goods to exist requires that each country produce all goods, ruling out specialization problems.

[8]Kimbell and Harrison [1983; Section 3.2] demonstrate the analytic and empirical nature of this general partitioning procedure. The Heckscher-Ohlin assumption of inter-sectorally mobile factors that are interregionally immobile is a special case of that procedure, which allows for quite general configurations of factor specificity and mobility.

[9]Another way to see this point is to ask the question "what is the best algorithm for GE models?" Clearly the answer to this question depends on the structure of the GE model at hand--for a certain well-defined structure our AFPS trivially dominates any iterative method. We are arguing a similar point, albeit for a less restrictive set of structures, with respect to our FPRR.

[10]In our own applied work we adopt the convention of defining the "short run" as a situation in which some factors are specific to regions and sectors (or blocks of sectors within each region), the "medium run" as a situation in which factors are specific to regions but fully mobile across sectors within each region (the Heckscher-Ohlin case), and the "long run" as a situation of full factor mobility across all sectors and regions.

[11]Model 1 in Appendix 1 (available on request) illustrates an exhaustive configuration for an economy consisting of three regions each producing just ten goods. The GE solution search involves forty-eight relative factor prices.

[12]Contrast, for example, the performance of the FPRR in Kimbell and Harrison [1983; Section 4.1] with Model 1 in Appendix 1. The former model consists of two regions each producing two goods. The latter model, while otherwise qualitatively similar, is much larger. The FPRR works with comparable speed in both models (as measured by the number of iterations required to find a solution under varying circumstances).

[13]An obvious example is the public perception of an "energy crisis" during the 1970's: Borges and Goulder [1983] have extended the Fullerton, Shoven, Whalley [1978] model to include energy; Christensen, Harrison and Kimbell [1982] developed a GE model of the Californian economy with Working Capital, Labour, Energy and Water in order to study the interaction of energy use and water use following the removal of water-pricing distortions.

[14] See Mansur and Whalley [1983], St-Hilaire and Whalley [1980], Piggott and Whalley [1983], and Fullerton, Shoven and Whalley [1978].

[15] For one obvious example, consider the elasticities of substitution in production used to calibrate the CES production functions of each sector. Popular calibration procedure is to employ a vector of point estimates based on a search of the available econometric literature. Assuming that such a vector is available, such estimates are of course accompanied by standard errors. The vector of estimates formed by considering all combinations of estimates within one standard error (say) of the point estimate for each sector provides a continuum of distinctly calibrated GE models whose comparative static (policy) properties need not be identical. Piggott and Whalley [1983; Chs. 4/6] provide a complete exposition of these calibration procedures.

[16] To avoid undue complication we shall only refer to a "true" and a "simplified" model. It should be understood that we are seeking an approximation to the GE solution of a counter-factual version of the true model, by finding the solution of a similarly counter-factual version of the simplified model. There may be certain counter-factual policy changes that are not amenable to representation in such a simplified model (e.g., increasing income taxes on one household type and decreasing them on another). In such cases the joint application of the AFPS and FPRR is of no use: the FPRR must be used alone.

[17] Some improvement in the AFPS approximation may be obtained by computing embodied factor shares in the base case of the "true" model, and employing these values instead of the "true" factor shares.

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APPENDIX 1

Illustrative Computational Experience

To illustrate the computational power of the FPRR, we have generated two basic GE models which are solved under varying circumstances (e.g., ignoring or allowing for intermediate trade).

Although we focus attention here on the performance of the FPRR, the value of the speed implied by knowledge of the AFPS should not be forgotten. An illustration is provided earlier by Figure 1. To trace out Production Possibility Curves such as those illustrated there requires (at least) forty complete GE solutions; in the present case the AFPS was able to generate this frontier about as rapidly as it could be plotted. Despite the precision of purely numerical solutions to GE models, the traditional international trade and tax incidence literature has powerfully demonstrated the insights possible from heuristic constructs such as those presented in Figure 1. Our computational procedures permit a synthesis of these alternative, and complementary, representations of GE solutions. The AFPS also plays an important role in solving more general models, as discussed in Section 6 and illustrated below.

A.1.1 Model 1

Model 1 refers to a hypothetical economy consisting of three regions each producing ten goods (thirty private sector goods in all). Each region has a local government that levies taxes on local production and produces a local public good, and there is one national government that levies taxes on all production and produces a national public good. Each region also has one household. Each of the thirty private sectors in this economy uses four types of primary factor:

(i) a factor that is mobile between all regions and sectors;

(ii) a factor that is mobile between sectors of the region in which the good is produced, but is not able to move to any other region;

(iii) a factor that is specific to the production of a particular good and is able to move between regions; and

(iv) a factor that is both specific to a particular good and a particular region.

Using the procedure for defining specific and immobile factors introduced in Kimbell and Harrison [1983], our hypothetical economy has one factor of the first type, three factors of the second type (one for each region), eleven factors of the third type (one for each private good produced and one for the national public good), and thirty-four factors of the fourth type (one for each private and public good in each region, and one for the national public good). If we define the fully mobile factor to be the numeraire, our search for a GE solution involves forty-eight relative factor prices.

In the base case we have assumed no intermediate trade between sectors or regions. We have also simplified the structure of this economy in the following ways: the CES production functions for each good are calibrated with distribution parameters that take the value of "1" or "0" as the nature of the factor indicates (viz., the distribution parameter $\delta_{g,f}$ for factor f in the production of good g is zero if the factor is unable to be used in that sector, and non-zero otherwise; in the present case "non-zero" simply means "unity"); government taxes are assumed to be levied on each factor in each sector in each region (initially) at an "ad valorem" rate of 5 percent; a uniform elasticity of substitution (in production and consumption) of 1.5 is assumed; and each household is endowed with two units of the fully mobile factor, one unit of each of the regionally-mobile/good-specific factors, two units of the own-region-specific factor that is mobile between goods, and one unit of each of the own-region-specific/good-specific factors. Kimbell and Harrison

[1983; Section 4.1] demonstrate that a qualitatively similar hypothetical economy exhibits a number of familiar and novel substantive properties, despite being empirically spartan.

Table 1 shows the speed of convergence to the general equilibrium set of relative prices in terms of the absolute value of the <u>maximum</u> factor market disequilibria expressed as a percentage of the aggregate endowment of that factor. Convergence was assumed when this criterion was less than 0.1 (i.e., the absolute value of the maximum disequilibrium is less than one-tenth of one percent of aggregate endowment). An alternative measure of convergence, not reported in full here, is the simple <u>average</u> of the absolute values of <u>all</u> factor market disequilibria (each expressed as a percent of the respective factor endowment). In all cases, other than Run 5, reported in Table 1 this measure was less than 1 after Iteration 3 and less than 0.1 after Iteration 4 (in the case of Run 5 this measure was less than 1 and 0.1, respectively, after Iterations 7 and 11).

The first run shown corresponds to the use of the unit vector as an initial guess at equilibrium factor prices. In this case the FPRR was able to find a solution in <u>five</u> iterations. The second run adopts starting values for factor prices that are deliberately disturbed "significantly" away from the (known) equilibrium solution. The sizeable (maximum percent) disequilibrium of 4857.668 (see Iteration 1 for Run 2 in Table 1) indicates what our algorithm had to overcome relative to Run 1. A new solution was nonetheless found in five iterations. Indeed after the shock of the first iteration the FPRR tended to follow the convergence path of Run 1. Run 3 illustrates a common situation--given the solution found in Run 1 (and Run 2), we search for a new solution after some policy perturbation. In this case we increase the tax on the fully mobile factor's use in the production of goods 1 through 5 in each region from 5% to 50%. The FPRR is able to return us to a general equilibrium with five iterations.

TABLE 1: Computational Experience for Model 1

Iteration	PERCENT DISEQUILIBRIUM					
	RUN 1	RUN 2	RUN 3	RUN 4	RUN 5	RUN 6
1	252.841	4857.668	47.481	254.310	5277.982	59.238
2	8.847	9.819	6.869	8.881	53.203	9.402
3	1.716	1.429	0.926	1.742	28.367	1.680
4	0.338	0.216	0.140	1.012	16.172	0.624
5	0.068	0.036	0.030	0.628	9.863	0.341
6				0.409	6.412	0.204
7				0.284	4.417	0.129
8				0.202	3.199	0.085
9				0.148	2.444	
10				0.112	1.968	
11				0.088	1.650	
12					1.427	
13					1.276	
14					1.185	
15					1.101	
16					1.022	
.					.	
.					.	
.					.	
55					0.108	
56					0.103	
57					0.099	

Runs 4, 5 and 6 assume that intermediate trade is allowed between goods and regions (i.e., a full 30-by-30 interregional input-output matrix is assumed). The Leontief Inverse for this exercise was psendo-randomly generated, with the restriction that off-diagonal elements take values between 0.3 and 0.5 and diagonal elements take values between 1 and 1.5 (Runs 1 through 3 essentially assume this to be an identity matrix).

Run 4 adopts the unit vector as an initial guess at factor prices. An equilibrium is found after 11 iterations. Note that the speed of convergence parallels Run 1 for the first three iterations; thereafter a marked slow-down in convergence speed is observed. Run 5 adopts starting values that are deliberately and sizeably different from the equilibrium values. Although a solution is eventually discovered after 57 iterations, the speed of convergence slows to a crawl after the first six or seven iterations. Run 5 illustrates a search for a solution following a policy change (the same change used in Run 3); the starting values for factor prices were the solution values found in Run 4 (and Run 5).

The rapid convergence of Runs 4 and 6, compared to the dismal performance in Run 5, suggest that the FPRR <u>can</u> run into problems if starting values are significantly different from the equilibrium. Comparison of Runs 1 and 2, however, illustrates that these problems do not always arise. It should be noted that the situation depicted in Runs 2 and 5 is largely academic in applied GE research, since the process of benchmark calibration generates "a priori" knowledge of reasonable starting values for realistic counter-factual exercises: the unit vector. Indeed, we have generated a number of parameterizations of Model 1 in which the <u>solution</u> values were significantly different from the unit vector. With the unit vector as an initial guess, the FPRR uniformly found these solution values in less than fifteen iterations. This experience suggests that the unit vector provides a robust starting point for applications of the FPRR in situations in which there is no "a priori" knowledge of a reasonable starting point (such knowledge arises naturally and frequently in practice from the benchmarking process or a previous solution for a "similar" parameterization).

An alternative procedure to accommodate intermediate trade in the choice of starting values (viz., use of the AFPS for a synthetic parameterization) is discussed in Section 6 and, in the context of Model 2, in the next section of this Appendix.

A.1.2 Model 2

Model 2 refers to a small representation of the U.S. economy. Two regions are identified (California and the Rest of the U.S.) and there are ten goods produced in each region (Agriculture, Mining, Construction, Food Processing, Non-Durables, Durables, Transportation and Utilities, Real Estate, Services, and Public Enterprises). A full interregional input-output matrix for 1976 is used. Each region has a single representative private household and a state and local government (exports and imports are not explicitly modelled, and are included with consumer's final demand). For convenience and lack of data, consumers and state and local governments are assumed to buy goods directly only from their own region. Since these industries use intermediate goods from the other region there is, of course, induced demand for the output of sectors in the other region as well. The Federal government demands goods from both regions. Each industry uses two primary substitutable factors: capital, which is mobile across regions, and own-region labour. There are therefore three factors: Capital, California Labour, and Rest of U.S.Labour (the region-specific labour is fully mobile across the sectors within that region).

Table 2 presents measures of convergence speed for Model 2 using the FPRR. In each case a benchmark equilibrium solution was established, and the results show the speed with which a "counter-factual" solution was found. The policy change we studied is a 50% reduction in the California State and Local government tax on capital use in Californian sectors; the substantive results of a similar GE model are discussed in Kimbell and Harrison [1983; Section 4.2]. Run 1 assumes that the elasticity of substitution in production and consumption is set uniformly at 1.5; the FPRR finds a new solution in 13 iterations when the starting values for factor prices are all unity. We shall refer to this as a "cold start" for the FPRR (viz., initial values set to the unit vector). Note that the parameterization of Model 2 in Run 1 violates all but one of the restrictions necessary for the direct applicability of our AFPS: there is more than one private household type (in fact, the two households differ in terms of their expenditure pattern and endowment distribution, since we assume that a household that represents a region owns all of the region-specific labour for that region), we allow for intermediate trade (using "realistic" data), and government factor demands are not proportional to aggregate factor demands (each of the three governments in our model have different factor intensities).

Run 2 allows for non-uniform elasticities of substitution in production across the twenty sectors. These values were generated in a pseudo-random fashion, with the restriction that they lie between 0.5 and 2.500. The scalar value used in the FPRR was the average of these elasticities, with each sector's share in total value added being used to weight its elasticity. The FPRR required 35 iterations in this case, again from a "cold start". An interesting difference in the convergence speeds of Run 1 and Run 2 appears to be quite robust: with uniform elasticities of substitution the rate of convergence

TABLE 2: Computational Experience for Model 2

Iteration	PERCENT DISEQUILIBRIUM			
	RUN 1	RUN 2	RUN 3	RUN 4
1	3.960	19.124	1.024	5.946
2	2.829	13.299	0.739	3.041
3	2.084	9.116	0.491	1.783
4	1.543	6.853	0.318	1.125
5	1.143	5.784	0.227	0.692
6	0.845	5.103	0.163	0.455
7	0.624	4.438	0.117	0.286
8	0.460	4.190	0.097	0.193
9	0.339	3.841		0.159
10	0.250	3.026		0.125
11	0.184	2.702		0.096
12	0.135	2.115		
13	0.099	2.001		
14		1.881		
15		1.769		
16		1.405		
17		1.178		
18		1.015		
19		1.001		
20		0.959		
.		.		
.		.		
.		.		
33		0.141		
34		0.126		
35		0.098		

rapidly settles down to some constant (e.g., the ratio of the criteria values shown in Run 1 of Table 2 for Iterations 3 and 4 is the same as the ratio for Iterations 12 and 13, and indeed for all successive iterations in between; the ratio is 0.74), whereas with non-uniform elasticities it varies. Moreover, the rate of convergence appears to slow down during the middle third of the convergence path (this observation is based on a number of runs comparable to Run 2). A common feature of the convergence paths of Run 1 and Run 2 is that the slowing down of the rate of convergence is associated with a change in sign of at least one of the factor market disequilibria.

In order to provide further perspective on the behavior of our FPRR with non-uniform elasticities of substitution, one hundred sets of this elasticity vector were pseudo-randomly generated (i.e., Run 2 was repeated 99 times, with a "random seed"). In each case the basic model was re-benchmarked and the same counter-factual policy change introduced. The average number of iterations required was 35.2 (Run 2 was selected, in fact, to be representative), the standard deviation was 5.4 iterations, and the extreme bounds were 11 and 94 iterations (these were singular "outsiders", however; the distribution of iterations was nicely Gaussian).

Runs 3 and 4 illustrate the "warm start" procedure discussed in Section 6. A synthetic GE model and parameterization was obtained by simplifying the "true" model and parameterization so that it may be solved directly by our AFPS. The AFPS to this synthetic model is then used as starting values for the FPRR, and the results shown in Table 2. There is clearly a dramatic improvement in the speed of convergence. Run 3 required some 38% fewer iterations than Run 1, and Run 4 some 66% fewer iterations than Run 2. The latter improvement is representative (Run 4 was also selected as representative of one hundred <u>sets</u> of parameters). The improvement in the warm start procedure when compared to the cold start procedure can be traced to two points: (i) the starting value is simply closer to the final solution values (compare the values in Table 2 after the initial iteration), and (ii) the AFPS generally allows the FPRR to <u>begin</u> with the <u>sign pattern</u> of factor market disequilibria that survive until the final iteration. Loosely speaking, the warm start procedure gets the FPRR "over the hump".

Appendix 2

A Simple Recursive Solution

In Section 4.1 we presented the complete reduced form solution of a particular GE structural model. In addition to the assumptions necessary for the AFPS, we shall assume for notational ease that $\alpha_g = 1$ for all $g=1,\ldots,N_g$ and that there are no factor taxes. In this section we present a recursive (from "top-to-bottom") version of the same solution which is much more attractive computationally.

Factor prices are given by

$$P_f = K_f^{\frac{1}{\sigma}} X_f^{-\frac{1}{\sigma}}$$

for $f=1,\ldots,N_f$. Goods prices are then given by

$$P_g = \left[\sum_{f=1}^{N_f} \bar{\delta}_{g,f} \, P_f^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

for $g=1,\ldots,N_g$. Factor intensities are then given by

$$F_{g,f} = \bar{\delta}_{g,f}\left[\frac{P_g}{P_f}\right]^{\sigma}$$

for $f=1,\ldots,N_f$ and $g=1,\ldots,N_g$.

Wealth is then

$$W = \sum_{f=1}^{N_f} P_f \, X_f$$

Goods demands are then given for $g=1,\ldots,N_g$ by

$$Q_g = \frac{W \, \bar{\beta}_g \, P_g^{-\sigma}}{\sum_{g=1}^{N_g} \bar{\beta}_g \, P_g^{1-\sigma}} = \bar{\beta}_g \, P_g^{-\sigma} \; ,$$

since $W = \sum_{g=1}^{N_g} \bar{\beta}_g \, P_g^{1-\sigma}$. Factor allocations are gives as

$$A_{g,f} = F_{g,f} Q_g = \bar{\delta}_{g,f} \bar{\beta}_g P_f^{-\sigma}$$

for $f=1,\ldots,N_f$ and $g=1,\ldots,N_g$. Factor demands are given as

$$D_f = \sum_{g=1}^{N_g} A_{g,f} = \sum_{g=1}^{N_g} F_{g,f} \, Q_g = X_f$$

for $f=1,\ldots,N_f$. Factor shares, $S_{g,f}$, are the fraction of X_f allocated to Q_g production; they are

$$S_{g,f} \equiv \frac{A_{g,f}}{X_f} = \bar{\delta}_{g,f} \, \bar{\beta}_g \left[\frac{K_f}{K_1}\right]^{\frac{1}{\sigma}}$$

for $f=1,\ldots,N_f$ and $g=1,\ldots,N_g$.

Appendix 3

Two Further Extensions to the Analytic Factor Price Solution

In Section 4.3 we examined two important extensions to the restrictive GE model for which the AFPS applies. In this Appendix we consider two further extensions.

A.3.1 Intermediate Trade

A common specification of production in applied GE models is that the production function for each good is Leontief with respect to all inputs (intermediate and primary), but that the primary factor is a composite input produced with a CES technology. The efficient choice of primary inputs is therefore trivially separable from the use of intermediate inputs. If we define $F'_{f,g}$ as the embodied (direct plus indirect) factor intensity of f in the production of g (i.e., the number of units of factor f needed directly and indirectly to produce one unit of good g), we may employ one of the fundamental relations of Input-Output analysis to define

$$(1') \qquad F'_{f,g} = \sum_{k=1}^{N_g} F_{f,k} L_{k,g}$$

for $f=1,\ldots,N_f$, $g=1,\ldots,N_g$, and where L is the <u>standard</u> Leontief Inverse. In matrix notation, L is given as

$$L = (I-A)^{-1}$$

where I is an $N_g \times N_g$ identity matrix and A is the non-negative matrix of direct intermediate requirements (i.e., each column of A shows the direct physical requirements of each row sector output per unit output of the column sector). If we assume that L exists and is non-negative, then we may write it as the following useful matrix expansion:

$$L = I + A + A^2 + A^3 + \cdots.$$

We may further define

$$C = L - I = A + A^2 + A^3 + \cdots ,$$

allowing us to restate (1') as

$$(1'') \quad F'_{f,g} = F_{f,g} + \sum_{k=1}^{N_g} F_{f,k} \, C_{k,g}$$

for $f=1,\ldots,N_f$ and $g=1,\ldots,N_g$.

The original derivation strategy may now proceed with intermediate trade by substituting $F'_{f,g}$ at appropriate places for $F_{f,g}$. Expression (5a) becomes:

$$(5a') \quad D_f = \sum_{g=1}^{N_g} Q_g F_{f,g} + \sum_{g=1}^{N_g} Q_g \left[\sum_{k=1}^{N_g} F_{f,k} \, C_{k,g} \right] .$$

The first expression in (5a') represents "direct" derived demand for factor f, and the second expression represents "indirect" derived demand for factor f. With no intermediate trade the C matrix is null, (5a') reduces immediately to (5a) and the AFPS emerges as before. When C is non-null, however, no apparent simplification of (5a') to "isolate" P_f is possible.

A.3.2 The Role of Government

The AFPS permits completely general specifications of ad valorem factor taxes, which are used commonly in applied general equilibrium taxation analysis to approximate corporate profits taxes, social security taxes, and other tax structures. The expenditure side of fiscal incidence, however, poses a problem for an exact analytic solution. In the special case where the exact algebraic solution holds we assumed that government demands factors in the same proportion as the aggregate of private industry demands. This assumption permits the demand for factors (demanded by private industries and government) to be stated solely as a function of private industry demands.

Assuming that government demands factors based on some rule other than that they be proportional to aggregate private demands (e.g., cost minimization with a balanced budget) leads to an expression for factor demands similar to (5b), except that there is an additional term reflecting government demands. The formal problem for an exact solution lies in the fact that government revenues from ad valorem factor taxes depend on factor prices, inter alia, but factor prices depend, in turn, on government factor demands. This prevents an apparent exact closed-form solution. However, if government demands are relatively small, and/or roughly similar to the demands by the private sector, then an initial rough guess at factor prices will permit very rapid iterations towards the solution since government demands are very much second-order adjustments to the solution obtained by ignoring the difference between private industry demands and total demands.

Our assumption about government factor demands has the same force as assumptions used originally by Harberger to "neutralize" the role of government. Commenting on a presentation of the APPS in Kimbell and Harrison [1983], McLure [1983] notes a similarity between the requirement that we have a single consumer (discussed above) and this assumption on government factor demands:

> It is interesting to note that these limitations on the analytical solution are closely related to those found in the early Harberger literature and force similar simplifying assumptions. In his original article, Harberger was concerned solely with what Musgrave has called incidence on the side of sources of income, that is, with effects on relative factor prices. To simplify analysis he assumed that at the margin redistribution of income within the private sector made no difference for aggregate consumption patterns, by giving everyone the same marginal expenditure pattern. Since he was not concerned with effects on relative product prices and changes in the distribution of income resulting from such changes, he was not concerned that there would be no redistributional effect on the side of uses of income if average private spending patterns were also identical. Whereas Harberger sacrificed examination of the uses side of income in order to simplify his analysis, Kimbell and Harrison are forced to make the same sacrifice in order to obtain an exact analytical solution using their Factor Price Revision Rule. Whereas the uses side could be

salvaged in the earlier analysis by assuming (usually implicitly) a non-unitary income elasticity of demand, Kimbell and Harrison can achieve the same result by using the powerful computational algorithm based on their Factor Price Revision Rule, rather than relying on an analytical solution.

If marginal and public spending patterns diverge, the existence of government keeps the authors from having, in effect, a one-consumer model. But for the analysis of <u>tax</u> incidence it would be quite satisfactory to assume identical marginal public and private expenditure patterns, especially if incidence on the uses side is already being ignored by lumping all consumers together.