Documentation for OG-USA *

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Abstract

This note documents the representative agent version of the OG-USA model that is used in the TaxBrain web application.

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This document outlines the equations and parameterizations used in the solution of the overlapping generations model provided by the Open Source Policy Center, OG-USA.

1 Dynamic General Equilibrium Model

The DGE model is comprised of overlapping generations of homogenous households, perfectly competitive firms, and a government with a balanced budget requirement. A unit measure of identical firms make a static profit maximization decision in which they rent capital and hire labor to maximize profits given a Cobb-Douglas production function. The government levies taxes on households and makes lump sum transfers to households according to a balanced budget constraint.

Households are assumed to live for a maximum of $E + S$ periods. We define an age-s household as being in youth and out of the workforce during ages $1 \leq s \leq E$. We implement this dichotomy of being economically relevant by age in order to more easily match true population dynamics. Households enter the workforce at age $E+1$ and remain in the workforce until they die or until the maximum age $E+S$. Because of mortality risk, households can leave both intentional bequests at the end of life ($s = E + S$) as well as accidental bequests if they die before the maximum age of $E + S$.

Households face deterministic hourly earnings process. Hourly earnings may vary over the lifecycle. Given this exogenous hourly earnings process, households choose their labor supply, savings, and consumption in each year of their lifetime. The only uncertainty in the model is from the mortality risk the households face. The economic environment is one of incomplete markets because the overlapping generations structure prevents households from perfectly smoothing consumption.

1.1 Population dynamics and lifetime earnings profiles

We define $\omega_{s,t}$ as the number of households of age s alive at time t. A unit measure $\omega_{1,t} = 1$ of households are born in each period t and live for up to $E + S$ periods, with $S \geq 4$.[1] Households are termed "youth", and do not participate in market activity during ages $1 \leq s \leq E$. The households enter the workforce and economy in period $E+1$ and remain in the workforce until they unexpectedly die or live until age $s = E+S$.[2] The population of agents of each age in each period $\omega_{s,t}$ evolves according to the following function,

$$\omega_{s+1,t+1} = (1-\rho_s)\omega_{s,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \tag{1}$$

where ρ_s is an age specific mortality hazard rate.[3] The total population in the economy N_t at any period is simply the sum of households in the economy. Given the law of motion for the population and the constant measure of households born in each period, the population growth rate is zero and the population distribution is stationary. This stationary distribution of households over age is shown in Figure 1. We define parameters are defined as:

$$N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{2}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{3}$$

$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{4}$$

$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{5}$$

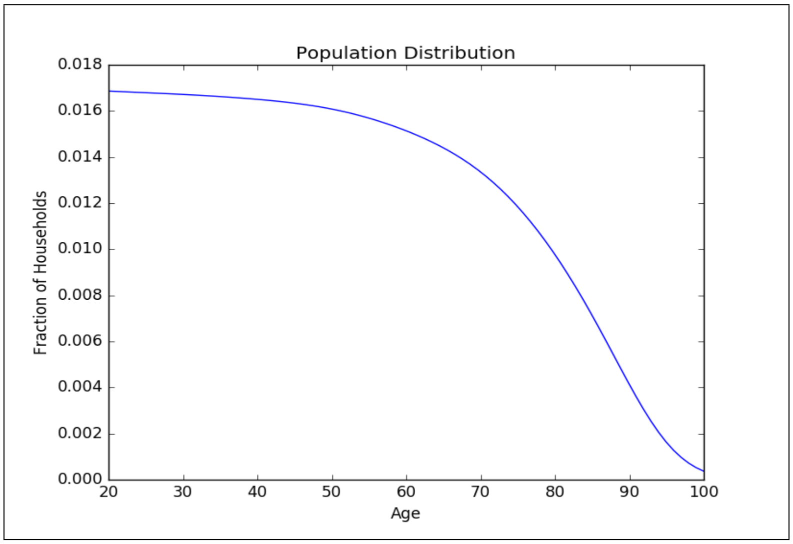
A household's working ability evolves over its working-age lifetime $E+1 \leq s \leq E+S$ according to an age-dependent deterministic process. The population weights

[1]Theoretically, the model works without loss of generality for $S \geq 3$. However, because we are calibrating the ages outside of the economy to be one-fourth of S (e.g., ages 21 to 100 in the economy, and ages 1 to 20 outside of the economy), we need S to be at least 4.

[2]We model the population with households age $s \leq E$ outside of the workforce and economy in order most closely match the empirical population dynamics. Appendix **??** gives more detail on the population process and its calibration.

[3]The parameter ρ_s is the probability that a household of age s dies before age $s+1$.

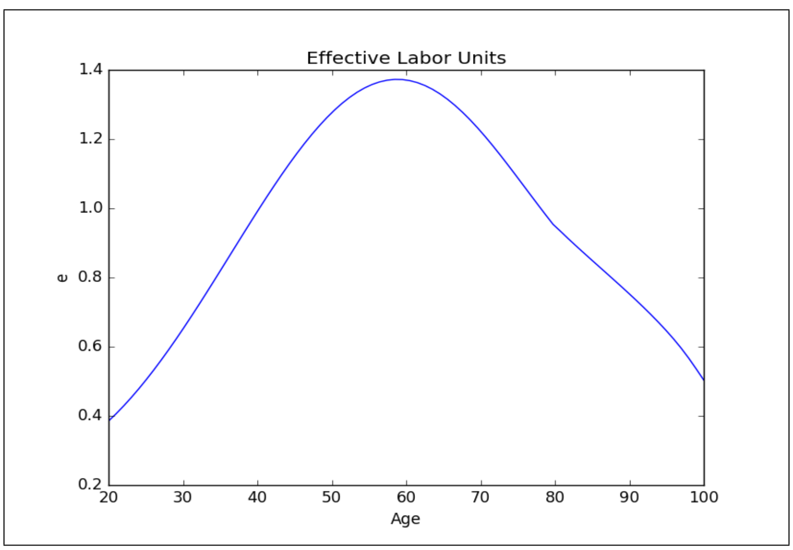
Figure 1: Exogenous population distribution



$\omega_{s,t}$ as well as lifetime earnings are exogenous inputs to the model.

Figure 2 shows the calibrated trajectory of effective labor units (ability), e_s, by age s.[4] The exogenous earnings process is taken from DeBacker et al. (2015).[5]

Figure 2: Exogenous life cycle profile of effective labor units



[4]The units for this figure or normalized effective labor units, where the normalization is that which makes the weighted average effective labor units equal to one. In this way, the model wage will represent the compensation for a single effective labor hour.

[5]We collapse their heterogenous profiles into a single profile for our representative agent.

1.2 Household problem

Households are endowed with a measure of time $\tilde{l}$ in each period t, and they choose how much of that time to allocate between labor $n_{s,t}$ and leisure $l_{s,t}$ in each period. That is, a household's labor and leisure choice is constrained by its total time endowment.

$$n_{s,t} + l_{s,t} = \tilde{l} \tag{6}$$

At time t, all age-s households with ability e_s know the real wage rate, w_t, and know the one-period real net interest rate, r_t, on bond holdings, $b_{s,t}$, that mature at the beginning of period t. They also receive accidental and intentional bequests. They choose how much to consume $c_{s,t}$, how much to save for the next period by loaning capital to firms in the form of a one-period bond $b_{s+1,t+1}$, and how much to work $n_{s,t}$ in order to maximize expected lifetime utility of the following form,

$$\begin{aligned}
U_{s,t} &= \sum_{u=0}^{E+S-s} \beta^u \left[\prod_{v=s}^{s+u-1}(1-\rho_v)\right] u\left(c_{j,s+u,t+u}, n_{j,s+u,t+u}, b_{j,s+u+1,t+u+1}\right) \\
\text{and} \quad u\left(c_{s,t}, n_{s,t}, b_{s+1,t+1}\right) &= \frac{\left(c_{s,t}\right)^{1-\sigma}-1}{1-\sigma}\ldots \\
&\quad + e^{g_y t(1-\sigma)}\chi^n_s\left(b\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} + k\right) + \rho_s\chi^b\frac{\left(b_{s+1,t+1}\right)^{1-\sigma}-1}{1-\sigma} \\
&\qquad \forall t \quad \text{and } E+1 \leq s \leq E+S
\end{aligned} \tag{7}$$

where $\sigma \geq 1$ is the coefficient of relative risk aversion on consumption and on intended (precautionary) bequests, $\beta \in (0,1)$ is the agent's discount factor, and the product term in brackets depreciates the household's discount factor by the cumulative mortality rate. The disutility of labor term in the period utility function looks nonstandard, but is simply the upper right quadrant of an ellipse that closely approximates the standard CRRA utility of leisure functional form.[6] The term χ^n_s is a

[6]Appendix A-1 describes how the elliptical function closely matches the more standard utility of leisure of the form $\frac{(\tilde{l}-n_{s,t})^{1+\theta}}{1+\theta}$. This elliptical utility function forces an interior solution that automatically respects both the upper and lower bound of labor supply, which greatly simplifies the computation of equilibrium. In addition, the elliptical disutility of labor has a Frisch elasticity that asymptotes to a constant rather than increasing to infinity as it does in the CRRA case. For a more in-depth discussion see Evans and Phillips (2015)

constant term that varies by age s influencing the disutility of labor relative to the other arguments in the period utility function,[7] and g_y is a constant growth rate of labor augmenting technological progress, which we explain in Section 1.3.[8]

The last term in (7) incorporates a warm-glow bequest motive in which households value having savings to bequeath to the next generation in the chance they die before the next period. As was mentioned in Section 1.1, households in the model have no income uncertainty because each lifetime earnings path e_s deterministic, model agents thus hold no precautionary savings. Savings is thus motivated by the households preference to smooth consumption over their lifecycle and to provide for intentional and unintentional bequests.

The parameter $\sigma \geq 1$ is the coefficient of relative risk aversion on bequests, and the mortality rate ρ_s appropriately discounts the value of this term.[9] Note that, because of this bequest motive, households in the last period of their lives ($s = S$) will die with positive savings $b > 0$. Also note that the CRRA utility of bequests term prohibits negative wealth holdings in the model, but is not a strong restriction since none of the wealth data for the lifetime income group j and age s cohorts is negative except for the lowest quartile.

The per-period budget constraints for each agent normalized by the price of consumption are the following,

$$\begin{aligned} c_{s,t} + b_{s+1,t+1} \leq (1 + r_t)\, b_{s,t} + w_t e_s n_{s,t} + \frac{BQ_t}{\tilde{N}_t} - T_{s,t} \\ \text{where} \quad b_{E+1,t} = 0 \quad \text{for} \quad E + 1 \leq s \leq E + S \quad \forall t \end{aligned} \tag{8}$$

where $\tilde{N}_t$ is the total working age population at time t defined in (4). Note that the price of consumption is normalized to one, so w_t is the real wage and r_t is the net

[7] DeBacker et al. (2015) calibrate χ^n_s and χ^b_j to match average labor hours by age and some moments of the distribution of wealth.

[8] The term with the growth rate $e^{g_y t(1-\sigma)}$ must be included in the period utility function because consumption and bequests will be growing at rate g_y and this term stationarizes the household Euler equation by making the marginal disutility of labor grow at the same rate as the marginal benefits of consumption and bequests. This is the same balanced growth technique as that used in Mertens and Ravn (2011).

[9] It is necessary for the coefficient of relative risk aversion σ to be the same on both the utility of consumption and the utility of bequests. If not, the resulting Euler equations are not stationarizable.

real interest rate. The term BQ_t represents total bequests from households who died at the end of period $t-1$. $T_{s,t}$ is a function representing net taxes paid, which we specify more fully below in equation (??).

Because the form of the period utility function in (7) ensures that $b_{s,t} > 0$ for all j, s, and t, total bequests will always be positive $BQ_t > 0$ for all j and t.

$$BQ_{t+1} = (1+r_{t+1})\left(\sum_{s=E+1}^{E+S} \rho_s \omega_{s,t} b_{s+1,t+1}\right) \quad \forall t \tag{9}$$

In addition to each the budget constraint in each period, the utility function (7) imposes nonnegative consumption through infinite marginal utility, and the elliptical utility of leisure ensures household labor and leisure must be strictly nonnegative $n_{s,t}, l_{s,t} > 0$. Because household savings or wealth is always strictly positive, the aggregate capital stock is always positive.[10] An interior solution to the household's problem (7) is assured.

The solution to the lifetime maximization problem (7) of the household subject to the per-period budget constraint (8) and the specification of taxes in (??) is a system of $2S$ Euler equations. The S static first order conditions for labor supply $n_{s,t}$ are the following,

$$\begin{aligned}
&(c_{s,t})^{-\sigma}\left(w_t e_s - \frac{\partial T_{s,t}}{\partial n_{s,t}}\right) = e^{g_y t(1-\sigma)}\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \\
&\qquad\qquad \forall t, \quad \text{and} \quad E+1 \leq s \leq E+S \\
&\text{where} \quad c_{s,t} = (1+r_t)\,b_{s,t} + w_t e_s n_{s,t} + \frac{BQ_t}{\tilde{N}_t} - b_{s+1,t+1} - T_{s,t} \\
&\text{and} \quad b_{E+1,t} = 0 \quad \forall t
\end{aligned} \tag{10}$$

where the marginal tax rate with respect to labor supply $\frac{\partial T_{s,t}}{\partial n_{s,t}}$ is described in equation ??.

A household also has $S-1$ dynamic Euler equations that govern his saving deci-

[10] An alternative would be to allow for household borrowing as long as an aggregate capital constraint $K_t > 0$ for all t is satisfied.

sions, $b_{s+1,t+1}$, with the included precautionary bequest saving in case of unexpected death. These are given by:

$$(c_{s,t})^{-\sigma} = \rho_s\chi^b\big(b_{s+1,t+1}\big)^{-\sigma} + \beta(1-\rho_s)(c_{s+1,t+1})^{-\sigma}\left[(1+r_{t+1}) - \frac{\partial T_{s+1,t+1}}{\partial b_{s+1,t+1}}\right] \quad \forall t, \quad \text{and} \quad E+1 \leq s \leq E+S-1 \tag{11}$$

where the marginal tax rate with respect to savings $\frac{\partial T_{s,t}}{\partial b_{s,t}}$ is described in equation **??**. Lastly, Each household also has one static first order condition for the last period of life $s = E+S$, which governs how much to bequeath given that the household will die with certainty. This condition is:

$$(c_{E+S,t})^{-\sigma} = \chi^b(b_{E+S+1,t+1})^{-\sigma} \quad \forall t \tag{12}$$

Define $\hat{\mathbf{\Gamma}}_t$ as the distribution of stationary household savings across households at time t, including the intentional bequests of the oldest cohort.

$$\hat{\mathbf{\Gamma}}_t \equiv \left\{\hat{b}_{s,t}\right\}_{s=E+2}^{E+S+1} \quad \forall t \tag{13}$$

As will be shown in Section 1.5, the state in every period t for the entire equilibrium system described in the stationary, non-steady-state equilibrium characterized in Definition 2 is the stationary distribution of household savings $\hat{\mathbf{\Gamma}}_t$ from (13). Because households must forecast wages, interest rates, and aggregate bequests received in every period in order to solve their optimal decisions and because each of those future variables depends on the entire distribution of savings in the future, we must assume some household beliefs about how the entire distribution will evolve over time. Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\hat{\mathbf{\Gamma}}^{\boldsymbol{e}}_{\boldsymbol{t+u}} = \Omega^u\left(\hat{\mathbf{\Gamma}}_{\boldsymbol{t}}\right) \quad \forall t, \quad u \geq 1 \tag{14}$$

where the e superscript signifies that $\hat{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[11]

1.3 Firm problem

A unit measure of identical, perfectly competitive firms exist in the economy. The representative firm is characterized by the following Cobb-Douglas production technology,

$$Y_t = ZK_t^{\alpha}\left(e^{g_y t}L_t\right)^{1-\alpha} \quad \forall t \tag{15}$$

where Z is the measure of total factor productivity, $\alpha \in (0,1)$ is the capital share of income, g_y is the constant growth rate of labor augmenting technological change, and L_t is aggregate labor measured in efficiency units. The firm uses this technology to produce a homogeneous output which is consumed by households and used in firm investment. The interest rate r_t paid to the owners of capital is the real interest rate net of depreciation. The real wage is w_t. The real profit function of the firm is the following.

$$\text{Real Profits} = ZK_t^{\alpha}\left(e^{g_y t}L_t\right)^{1-\alpha} - (r_t+\delta)K_t - w_t L_t \tag{16}$$

As in the household budget constraint (8), note that the price output has been normalized to one.

Profit maximization results in the real wage, w_t, and the real rental rate of capital r_t being determined by the marginal products of labor and capital, respectively:

$$w_t = (1-\alpha)\frac{Y_t}{L_t} \quad \forall t \tag{17}$$

$$r_t = \alpha\frac{Y_t}{K_t} - \delta \qquad \forall t \tag{18}$$

1.4 Government fiscal policy

The government is represented by a balanced budget constraint. The government collects taxes $\tau_{s,t}(x,y)(x+y)$ from all households and divides total revenues equally

[11] In Section 1.5 we will assume that beliefs are correct (rational expectations) for the stationary non-steady-state equilibrium in Definition 2.

among households in the economy to determine the lump-sum transfer.

$$T_t^H = \frac{1}{\tilde{N}_t}\sum_s \omega_{s,t}\tau_{s,t}(w_t e_s n_{s,t}, r_t b_{s,t})\big(w_t e_s n_{s,t} + r_t b_{s,t}\big) \tag{19}$$

1.5 Market clearing and stationary equilibrium

Labor market clearing requires that aggregate labor demand L_t measured in efficiency units equal the sum of household efficiency labor supplied $e_s n_{s,t}$. Capital market clearing requires that aggregate capital demand K_t equal the sum of capital investment by households $b_{s,t}$. Aggregate consumption C_t is defined as the sum of all household consumptions, and aggregate investment is defined by the resource constraint $Y_t = C_t + I_t$ as shown in (22). That is, the following conditions must hold:

$$L_t = \sum_{s=E+1}^{E+S} \omega_{s,t} e_s n_{s,t} \quad \forall t \tag{20}$$

$$K_t = \sum_{s=E+2}^{E+S+1} \omega_{s-1,t-1} b_{s,t} \quad \forall t \tag{21}$$

$$\begin{aligned} Y_t &= C_t + K_{t+1} - (1-\delta)K_t \quad \forall t \\ &\text{where} \quad C_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} c_{s,t} \end{aligned} \tag{22}$$

An equilibrium would be defined by allocations and prices such that households optimize (10), (11), and (12), firms optimize (17) and (18), and markets clear (20) and (21). However, the variables in the equations characterizing the equilibrium are potentially non-stationary due to the growth rate in the total population $g_{n,t}$ each period coming from the cohort growth rates in (1) and from the deterministic growth rate of labor augmenting technological change g_y in (15).

Table 1 characterizes the stationary versions of the variables of the model in terms of the variables that grow because of labor augmenting technological change, population growth, both, or none. With the definitions in Table 1, it can be shown that the equations characterizing the equilibrium can be written in stationary form in the fol-

Table 1: Stationary variable definitions

Sources of growth			Not
$e^{g_y t}$	$\tilde{N}_t$	$e^{g_y t}\tilde{N}_t$	growing[a]
$\hat{c}_{s,t} \equiv \frac{c_{s,t}}{e^{g_y t}}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{\tilde{N}_t}$	$\hat{Y}_t \equiv \frac{Y_t}{e^{g_y t}\tilde{N}_t}$	$n_{s,t}$
$\hat{b}_{s,t} \equiv \frac{b_{s,t}}{e^{g_y t}}$	$\hat{L}_t \equiv \frac{L_t}{\tilde{N}_t}$	$\hat{K}_t \equiv \frac{K_t}{e^{g_y t}\tilde{N}_t}$	r_t
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$		$\hat{BQ}_t \equiv \frac{BQ_t}{e^{g_y t}\tilde{N}_t}$	
$\hat{y}_{s,t} \equiv \frac{y_{s,t}}{e^{g_y t}}$			
$\hat{T}_{s,t} \equiv \frac{T_{s,t}}{e^{g_y t}}$			

[a] The interest rate r_t in (18) is already stationary because Y_t and K_t grow at the same rate. Household labor supply $n_{s,t}$ is stationary.

lowing way. The static and intertemporal first-order conditions from the household's optimization problem corresponding to (10), (11), and (12) are the following:

$$(\hat{c}_{s,t})^{-\sigma}\left(\hat{w}_t e_s - \frac{\partial \hat{T}_{s,t}}{\partial n_{s,t}}\right) = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}$$
$$\forall t, \quad \text{and} \quad E+1 \leq s \leq E+S \tag{23}$$
$$\text{where} \quad \hat{c}_{s,t} = (1+r_t)\,\hat{b}_{s,t} + \hat{w}_t e_s n_{s,t} + \hat{BQ}_t - e^{g_y}\hat{b}_{s+1,t+1} - \hat{T}_{s,t}$$
$$\text{and} \quad \hat{b}_{E+1,t} = 0 \quad \forall t$$

$$(\hat{c}_{s,t})^{-\sigma} = ...$$
$$e^{-g_y\sigma}\left(\rho_s\chi^b\big(\hat{b}_{s+1,t+1}\big)^{-\sigma} + \beta(1-\rho_s)(\hat{c}_{s+1,t+1})^{-\sigma}\left[(1+r_{t+1}) - \frac{\partial \hat{T}_{s+1,t+1}}{\partial \hat{b}_{s+1,t+1}}\right]\right) \tag{24}$$
$$\forall t, \quad \text{and} \quad E+1 \leq s \leq E+S-1$$

$$(\hat{c}_{E+S,t})^{-\sigma} = \chi^b e^{-g_y\sigma}\big(\hat{b}_{E+S+1,t+1}\big)^{-\sigma} \quad \forall t \tag{25}$$

The stationary firm first order conditions for optimal labor and capital demand corresponding to (17) and (18) are the following.

$$\hat{w}_t = (1-\alpha)\frac{\hat{Y}_t}{\hat{L}_t} \quad \forall t \tag{26}$$

$$r_t = \alpha\frac{\hat{Y}_t}{\hat{K}_t} - \delta = \alpha\frac{Y_t}{K_t} - \delta \quad \forall t \tag{18}$$

And the two stationary market clearing conditions corresponding to (20) and (21)—with the goods market clearing by Walras' Law—are the following.

$$\hat{L}_t = \sum_{s=E+1}^{E+S} \hat{\omega}_{s,t} e_s n_{s,t} \quad \forall t \tag{27}$$

$$\hat{K}_t = \frac{1}{1+\tilde{g}_{n,t}}\left(\sum_{s=E+2}^{E+S+1} \hat{\omega}_{s-1,t-1}\hat{b}_{s,t}\right) \quad \forall t \tag{28}$$

where $\tilde{g}_{n,t}$ is the growth rate in the working age population between periods $t-1$ and t described in (5). It is also important to note the stationary version of the characterization of total bequests BQ_{t+1} from (9) and for the government budget constraint in (19).

$$\hat{BQ}_{t+1} = \frac{(1+r_{t+1})}{1+\tilde{g}_{n,t}}\left(\sum_{s=E+1}^{E+S} \rho_s \hat{\omega}_{s,t}\hat{b}_{s+1,t+1}\right) \quad \forall t \tag{29}$$

$$\hat{T}^H_t = \sum_s \hat{\omega}_{s,t}\hat{T}_{s,t} \tag{30}$$

We can now define the stationary steady-state equilibrium for this economy in the following way.

Definition 1 (**Stationary steady-state equilibrium**)**.** A non-autarkic stationary steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability e_s is defined as constant allocations $n_{s,t} = \bar{n}_s$ and $\hat{b}_{s+1,t+1} = \bar{b}_{s+1}$ and constant prices $\hat{w}_t = \bar{w}$ and $r_t = \bar{r}$ for all s, and t such that the following conditions hold:

i. households optimize according to (23), (24), and (25),

ii. Firms optimize according to (26) and (18),

iii. Markets clear according to (27) and (28), and

iv. The population has reached its stationary steady state distribution $\bar{\omega}_s$ for all ages s, characterized in Appendix **??**.

The steady-state equilibrium is characterized by the system of $2S$ equations and $2S$ unknowns $\bar{n}_s$ and $\bar{b}_{s+1}$. Appendix A-2 details how to solve for the steady-state equilibrium.

The non-steady state equilibrium is characterized by $2ST$ equations and $2ST$ unknowns, where T is the number of periods along the transition path from the current state to the steady state. The definition of the stationary non-steady-state equilibrium is similar to Definition 1, with the stationary steady-state equilibrium definition being a special case of the stationary non-steady-state equilibrium.

Definition 2 (**Stationary non-steady-state equilibrium**). A non-autarkic stationary non-steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability e_s is defined as allocations $n_{s,t}$ and $\hat{b}_{s+1,t+1}$ and prices $\hat{w}_t$ and r_t for all s, and t such that the following conditions hold:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings, and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\hat{\mathbf{\Gamma}}_{t+u} = \hat{\mathbf{\Gamma}}^e_{t+u} = \Omega^u\left(\hat{\mathbf{\Gamma}}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (23), (24), and (25)

iii. Firms optimize according to (26) and (18), and

iv. Markets clear according to (27) and (28).

We describe the methodology to compute the solution to the non-steady-state equilibrium to Appendix A-3. We will use the time path solution to find effects of tax policies on macroeconomic variables over the budget window.

1.6 Calibration

Table 2 shows the calibrated values for the exogenous variables and parameters.

Table 2: List of exogenous variables and baseline calibration values

Symbol	Description	Value
$\hat{\mathbf{\Gamma}}_1$	Initial distribution of savings	$\bar{\mathbf{\Gamma}}(SSdistribution)$
N_0	Initial population	1
$\{\omega_{s,0}\}_{s=1}^{S}$	Initial population by age	$\{\bar{\omega}_{s,0}\}_{s=1}^{S}$ (SS distribution)
$\{\rho_s\}_{s=1}^{S}$	Mortality rates by age	(see App. **??**)
$\{e_s\}_{s=1}^{S}$	Deterministic ability process	(see DeBacker et al., 2015)
S	Maximum periods in economically active household life	80
E	Number of periods of youth economically outside the model	round $\left(\frac{S}{4}\right)$
R	Retirement age (period)	round $\left(\frac{9}{16}S\right)$
$\tilde{l}$	Maximum hours of labor supply	1
β	Discount factor	$(0.96)^{\frac{80}{S}}$
σ	Coefficient of constant relative risk aversion	1.5
b	Scale parameter in utility of leisure	0.573
υ	Shape parameter in utility of leisure	2.856
k	constant parameter in utility of leisure	0.000
χ^n_s	Disutility of labor level parameters	[19.041, 76.623]
χ^b	Utility of bequests level parameters	80
Z	Level parameter in production function	1.0
α	Capital share of income	0.35
δ	Capital depreciation rate	$1-(1-0.05)^{\frac{80}{S}}$
g_y	Growth rate of labor augmenting technological progress	$(1+0.03)^{\frac{80}{S}}-1$
T	Number of periods to steady state	160
ν	Dampening parameter for TPI	0.4

Note that the disutility of labor weight parameter, χ^n_s, takes on 80 values (one for each model age) that increase with age, representing an increasing disutility of labor that is not modeled anywhere else in the utility function. An hour of labor for an older person becomes more costly due to biological reasons related to aging. Such a parametrization helps to fit fact that hours worked decline much more sharply later in life than do hourly earnings.

The parameter representing the utility weight on bequests, χ^b is set to result in a pre-tax SS interest rate of 4.5%.

2 Calibrating Taxes

We interface between OG-USA and OSPC's Tax Calculator, through the total tax function $T_{s,t}$ in the household budget constraint (8) and the marginal tax rate functions that show up in the household's necessary conditions, $\frac{\partial \hat{T}_{s,t}}{\partial \hat{n}_{s,t}}$ and $\frac{\partial \hat{T}_{s+1,t+1}}{\partial \hat{b}_{s+1,t+1}}$. In particular, the tax calculator outputs microdata that can be used to calibrate these functions in a way that is consistent with the tax law parameters entered into the microsimulation model.

2.1 Microsimulation Model: `Tax-Calculator`

The microsimulation model we use is called `Tax-Caculator` and is developed and maintained by a group of researchers at the Open Source Policy Center (OSPC).[12] In this section, we outline the main structure of the `Tax-Calculator` microsimulation model, but encourage the interested reader to follow the links to the more detailed documentation.

`Tax-Calculator` uses microdata on tax filers from the tax year 2009 Public Use Files (PUF) produced by the IRS. These data contain detailed records from the tax returns of about 200,000 tax filers who were selected from the population of filers through a stratified random sample of tax returns. These data come from IRS Form 1040 and a set of the associated forms and schedules. The PUF data are then matched to the Current Population Survey (CPS) to get imputed values for filer demographics such as age, which are not included in the PUF, and to incorporate households from the population of non-filers. The PUF-CPS match includes 219,815 filers.

Since these data are for calendar year 2009, they must be "aged" to be representative of the potential tax paying population in the years of interest (e.g. the current

[12]The documentation for using Tax-Calculator is available at http://taxcalc.readthedocs.org/en/latest/index.html A simple web application that provides an easy user interface for Tax-Calculator is available at http://www.ospc.org/taxbrain/. And all the source code is freely available at https://github.com/open-source-economics/Tax-Calculator.

year through the end of the budget window). To do this, macroeconomic forecasts of wages, interest rates, GDP, and other variables are used to grow the 2009 values to be representative of the values one might see in the years within the budget window. In addition to using macroeconomic variables to extrapolate the 2009 variables, a linear programming algorithm is used to re-weight the observations in each year in order to match target levels in the data such as total income and deduction amounts reported in more recent IRS data.

Using these microdata, `Tax-Calculator` is able to determine total tax liability and marginal tax rates by computing each filer's tax reporting that minimizes the total tax liability subject to the parameters describing the tax policy. Our determination of total tax liability from the microsimulation model includes federal income taxes and payroll taxes but excludes state income taxes and estate taxes.[13] The output of the microsimulation model is forecasts of the total tax liability in each year derived from marginal tax rates, and items from the filers' tax returns for each of the 219,815 filers in the microdata. Population sampling weights are determined through the extrapolation and targeting of the microsimulation model. These weights allow one to calculate population representative results from the model. One can determine changes in tax liability and marginal tax rates by doing the same simulation where the parameters describing the tax policy are updated to reflect the proposed policy rather than the baseline policy. Note that the baseline policy is a current-law baseline.

2.2 Taxes in OG-USA

To calculate the effective tax rates from the microsimulation model, we divided total tax liability by a measure of "adjusted total income". Adjusted total income is defined as total income (Form 1040, line 22) plus tax-exempt interest income, IRA distributions, pension income, and Social Security benefits (Form 1040, lines 8b, 15a, 16a, and 20a, respectively).

We consider adjusted total income from the microsimulation model to be the counterpart of total income in the DGE model. Total income in the DGE model

[13] As the microsimulation model is further developed, we will account for these.

is the sum of capital and labor income. We define labor income as earned income, which is the sum of wages and salaries (Form 1040, line 7) and self-employment income (Form 1040 lines 12 and 18) from the microsimulation model output. Capital income is defined as a residual.[14]

To calculate marginal tax rates on any given income source, we add one cent to that income source for each individual in the micro-data and then use the tax calculator to calculate the change in tax liability. The change in tax liability divided by the change in income (one cent) yields the marginal tax rate. To get the marginal tax rate on composite income amounts (e.g., labor income that is the sum of wage and self-employment income), we take a weighted average that accounts for negative income amounts. We thus have to composite marginal tax rates which we calculate: MTR^l, the marginal rate on labor income (from employed and self-employed work) and MTR^k, the marginal tax rate on capital income (including interest, dividends, capital gains, and pension income). In particular, to we calculate the weighted average marginal tax rate on composite of n income sources as:

$$MTR_{composite} = \frac{\sum_{i=1}^{n} MTR_n * abs(Income_n)}{\sum_{i=1}^{n} abs(Income_n)} \tag{31}$$

When we look at the raw output from the microsimulation model, we find that there are several observations with extreme values for their effective tax rate. Since this is a ratio, such outliers are possible, for example when the denominator, adjusted total income, is very small. We omit such outliers by making the following restrictions on the raw output of the microsimulation model. First, we exclude observations with an effective tax rate greater than 1.5 times the highest statutory marginal tax rate. Second, we exclude observations where the effective tax rate is less than the lowest statutory marginal tax rate on income minus the maximum phase-in rate for the Earned Income Tax Credit (EITC). Third, we drop observations with marginal tax rates in excess of 99% or below the negative of the highest EITC rate (i.e., -45%

[14] This is not an ideal definition of capital income, since it includes transfers between filers (e.g., alimony payments) and from the government (e.g., unemployment insurance), but we have chosen this definition for now in order to ensure that all of total income is classified as either capital or labor income. This accounting will be refined in the future.

under current law). These exclusions limit the influence of those with extreme values for their marginal tax rate, which are few and usually results from the income of the filer being right at a kink in the tax schedule. Finally, since total income cannot be negative in our DGE model, we drop observations from the microsimulation model where adjusted total income is less than \$5.[15]

Our approach to fitting tax functions is simple. This sacrifice in the richness in the model is helpful in ensuring that the DGE model can be simulated with a wide variety of proposals and still yield a solution- something that is helpful when the model is run in an automated fashion through TaxBrain. We use constant tax rates for the effective and marginal taxes the households face. In particular:

$$T_{s,t} = \tau_t^{ETR} * a_{s,t} = \tau_t^{ETR}(e_s w_t n_{s,t} + r_t b_{s,t}) \tag{32}$$

and

$$\frac{\partial \hat{T}_{s,t}}{\partial \hat{n}_{s,t}} = \tau_t^{MTR,l} e_s w_t \tag{33}$$

and

$$\frac{\partial \hat{T}_{s+1,t+1}}{\partial \hat{b}_{s+1,t+1}} = \tau_t^{MTR,k} r_t b_{s,t} \tag{34}$$

Here, τ_t^{ETR} is the effective tax rate on total income in period t, $\tau_t^{MTR,l}$ the marginal tax rate on labor income, and $\tau_t^{MTR,k}$the marginal tax rate on capital income. These constant (over age and income) rates are estimated as the weighted average effective tax rate, marginal tax rate on earned income, and marginal tax rate on capital income from the microsimulation output. We weight by the sampling weights from the data underlying the microsimulation as well as by income. Thus we estimate the average or marginal rate on the average dollar of income. Specifically, we find:

[15]We choose \$5 rather than \$0 to provided additional assurance that small income values are not driving large ETRs.

$$\tau_t^{ETR} = \frac{\sum_{i=1}^{I} ETR_i * wgt_i * income_i}{\sum_{i=1}^{I} wgt_i * income_i}, \tag{35}$$

$$\tau_t^{MTR,l} = \frac{\sum_{i=1}^{I} MTR_i^l * wgt_i * income_i}{\sum_{i=1}^{I} wgt_i * income_i}, \tag{36}$$

$$\tau_t^{MTR,l} = \frac{\sum_{i=1}^{I} MTR_i^l * wgt_i * income_i}{\sum_{i=1}^{I} wgt_i * income_i}, \tag{37}$$

where i represents the filer observation in the microdata, t the calendar year the tax function refers to, τ is the tax rate of interest (with may be either the effective or marginal rates), and wgt are the sampling weights from the underlying microdata used by the tax calculator. We include both federal individual income taxes and payroll taxes in the the effective and marginal tax rates. The measure of income used in these calculations is that of the adjusted total income measure defined above, which maps to the total income in the computational model.

From the tax-calculator, we are able to find the τ_t for each year in the budget window. The DGE model requires a much longer time horizon to reach the SS. Our assumption is that the tax policy in place at the end of the budget window is extended permanently beyond that time.

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APPENDIX

A-1 Derivation of elliptical disutility of labor supply

Evans and Phillips (2015) provide an exposition of the value of using elliptical disutility of labor specification as well as its relative properties to such standard disutility of labor functions such as constant relative risk aversion (CRRA) and constant Frisch elasticity (CFE). A standard specification of additively separable period utility in consumption and labor supply first used in King et al. (1988) is the following,

$$u(c,n) = \frac{c^{1-\sigma}-1}{1-\sigma} + \chi^n \frac{\left(\tilde{l}-n\right)^{1+\theta}}{1+\theta} \tag{A.1.1}$$

where $\sigma \geq 1$ is the coefficient of relative risk aversion on consumption, $\theta \geq 0$ is proportional to the inverse of the Frisch elasticity of labor supply, and $\tilde{l}$ is the time endowment or the maximum labor supply possible. The constant χ^n is a scale parameter influencing the relative disutility of labor to the utility of consumption.

Although labor supply is only defined for $n \in [0, \tilde{l}]$, the marginal utility of leisure at $n = \tilde{l}$ is infinity and is not defined for $n > \tilde{l}$. However, utility of labor in this functional form is defined for $n < 0$. To avoid the well known and significant computational difficulty of computing the solution to the complementary slackness conditions in the Karush, Kuhn, Tucker constrained optimization problem, we impose an approximating utility function that has properties bounding the solution for n away from both $n = \tilde{l}$ and $n = 0$. The upper right quadrant of an ellipse has exactly this property and also has many of the properties of the original utility function. Figure 3 shows how our estimated elliptical utility function compares to the utility of labor from (A.1.1) over the allowed support of n.

The general equation for an ellipse in x and y space with centroid at coordinates (h, k), horizontal radius of a, vertical radius of b, and curvature υ is the following.

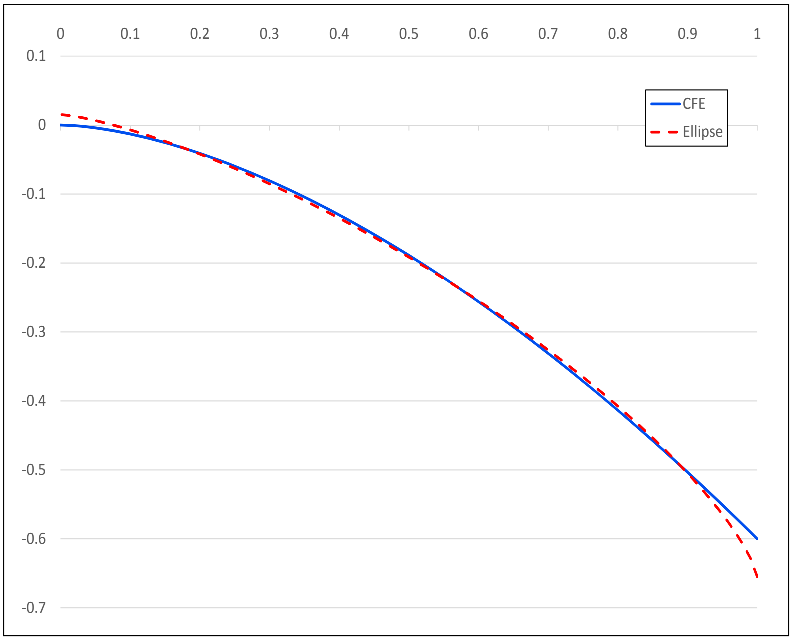
$$\left(\frac{x-h}{a}\right)^{\upsilon} + \left(\frac{y-k}{b}\right)^{\upsilon} = 1 \tag{A.1.2}$$

Figure 4 shows an ellipse with the parameterization $[h, k, a, b, \upsilon] = [1, -1, 1, 2, 2]$.

The graph of the ellipse in the upper-right quadrant of Figure 4 ($x \in [1, 2]$ and $y \in [-1, 1]$) has similar properties to the utility of labor term in (A.1.1). If we let the x variable be labor supply n, the utility of labor supply be $g(n)$, the x-coordinate of the centroid be zero $h = 0$, and the horizontal radius of the ellipse be $a = \tilde{l}$, then the equation for the ellipse corresponding to the standard utility specification is the following.

$$\left(\frac{n}{\tilde{l}}\right)^{\upsilon} + \left(\frac{g-k}{b}\right)^{\upsilon} = 1 \tag{A.1.3}$$

Figure 3: Comparison of standard utility of labor n to elliptical utility



Solving the equation for g as a function of n, we get the following.

$$g(n) = b\left[1 - \left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} + k \tag{A.1.4}$$

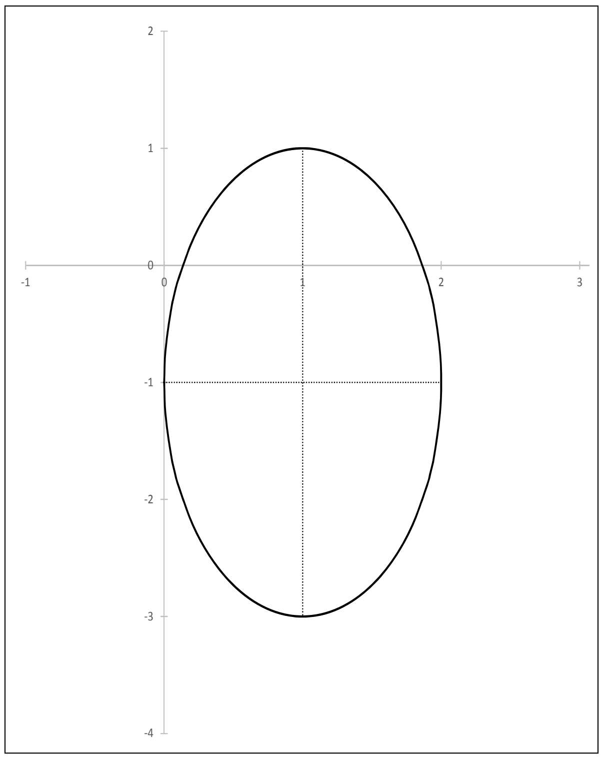
The υ parameter acts like a constant elasticity of substitution, and the parameter b is a shape parameter similar to χ^n in (A.1.1).

We use the upper-right quadrant of the elliptical utility function because the utility of n is strictly decreasing on $n \in (0, \tilde{l})$, because the slope of the utility function goes to negative infinity as n approaches its maximum of $\tilde{l}$ and because the slope of the utility function goes to zero as n approaches its minimum of 0. This creates interior solutions for all optimal labor supply choices $n^* \in (0, \tilde{l})$. Although it is more realistic to allow optimal labor supply to sometimes be zero, the complexity and dimensionality of our model requires this approximating assumption to render the solution method tractable.

Figure 3 shows how closely the estimated elliptical utility function matches the original utility of labor function in (A.1.1) with a Frisch elasticity of 0.4[16] . We choose the ellipse parameters b, k, and υ to best match the points on the original utility of labor function for $n \in [0, 0.9]$. We minimize the sum of absolute errors for 101 evenly spaced points on this domain. The estimated values of the parameters for the elliptical utility shown in Figure 3 and represented in equation (A.1.4) are $[b, k, \upsilon] = [0.573, 0.000, 2.856]$.

[16] See Chetty et al. (2011), Keane and Rogerson (2012) and Peterman (2014) for discussion of this choice.

Figure 4: Ellipse with $[h, k, a, b, \upsilon] = [1, -1, 1, 2, 2]$



A-2 Solving for stationary steady-state equilibrium

This section describes the solution method for the stationary steady-state equilibrium described in Definition 1.

1. Computer the steady-state population distribution vector $\bar{\boldsymbol{\omega}}$ of the exogenous population process.
2. Choose an initial guess for the stationary steady-state distribution of capital $\bar{b}_{s+1}$ for all $s = E+2, E+3, ...E+S+1$ and labor supply $\bar{n}_s$ for all s.
   - A good first guess is a large positive number for all the $\bar{n}_s$ that is slightly less than $\tilde{l}$ and to choose some small positive number for $\bar{b}_{s+1}$ that is small enough to be less than the minimum income that a household might have $\bar{w}e_s\bar{n}_s$.
3. Perform an unconstrained root finder that chooses $\bar{n}_s$ and $\bar{b}_{s+1}$ that solves the $2S$ stationary steady-state Euler equations.
4. Make sure none of the implied steady-state consumptions $\bar{c}_s$ is less-than-or-equal-to zero.
   - If one consumption is less-than-or-equal-to zero $\bar{c}_s \leq 0$, then try different starting values.
5. Make sure that none of the Euler errors is too large in absolute value for interior stationary steady-state values. A steady-state Euler error is the following, which is supposed to be close to zero for all s:

$$\frac{\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}}{(\bar{c}_s)^{-\sigma}\left(\bar{w}e_s - \frac{\partial\bar{T}_s}{\partial\bar{n}_s}\right)} - 1 \qquad \forall \quad E+1 \leq s \leq E+S \tag{A.2.1}$$

$$\frac{e^{-g_y\sigma}\left(\rho_s\chi^b\left(\bar{b}_{s+1}\right)^{-\sigma} + \beta(1-\rho_s)(\bar{c}_{s+1})^{-\sigma}\left[(1+\bar{r}) - \frac{\partial\bar{T}_{s+1}}{\partial\bar{b}_{s+1}}\right]\right)}{(\bar{c}_s)^{-\sigma}} - 1 \qquad \forall \quad E+1 \leq s \leq E+S-1 \tag{A.2.2}$$

$$\frac{\chi^b e^{-g_y\sigma}(\bar{b}_{E+S+1})^{-\sigma}}{(\bar{c}_{E+S})^{-\sigma}} - 1 \tag{A.2.3}$$

A-3 Solving for stationary non-steady-state equilibrium by time path iteration

This section describes the solution to the non-steady-state transition path equilibrium of the model and outlines the benchmark time path iteration (TPI) method of Auerbach and Kotlikoff (1987) for solving the stationary non-steady-state equilibrium transition path of the distribution of savings.

Taken together, the household labor-leisure and intended bequest decisions in the last period of life show that the optimal labor supply and optimal intended bequests for age $s = E + S$ are each functions of household savings, total bequests received, and the prices in that period: $n_{E+S,t} = \phi\big(\hat{b}_{E+S,t}, \hat{BQ}_t, \hat{w}_t, r_t\big)$ and $\hat{b}_{E+S+1,t+1} = \psi\big(\hat{b}_{j,E+S,t}, \hat{BQ}_t, \hat{w}_t, r_t\big)$. These two decisions are characterized by final-age version of the static labor supply Euler equation (23) and the static intended bequests Euler equation (25). households in their second-to-last period of life in period t have four decisions to make. They must choose how much to work this period, $n_{E+S-1,t}$, and next period, $n_{E+S,t+1}$, how much to save this period for next period, $\hat{b}_{E+S,t+1}$, and how much to bequeath next period, $\hat{b}_{E+S+1,t+2}$. The optimal responses for this household are characterized by the $s = E + S - 1$ and $s = E + S$ versions of the static Euler equations (23), the $s = E + S - 1$ version of the intertemporal Euler equation (24), and the $s = E + S$ static bequest Euler equation (25), respectively.

Optimal savings in the second-to-last period of life $s = E + S - 1$ is a function of the current savings as well as the total bequests received and prices in the current period and in the next period $\hat{b}_{E+S,t+1} = \psi\big(\hat{b}_{E+S-1,t}, \hat{BQ}_t, \hat{w}_t, r_t, \hat{BQ}_{t+1}, \hat{w}_{t+1}, r_{t+1}|\Omega\big)$ given beliefs Ω. As before, the optimal labor supply at age $s = E + S$ is a function of the next period's savings, bequests received, and prices.

$$n_{E+S,t+1} = \phi\big(\hat{b}_{E+S,t+1}, \hat{BQ}_{t+1}, \hat{w}_{t+1}, r_{t+1}\big)$$

But the optimal labor supply at age $s = E + S - 1$ is a function of the current savings, current bequests received, and the current prices as well as the future bequests received and future prices because of the dependence on the savings decision in that same period $n_{E+S-1,t} = \phi\big(\hat{b}_{E+S-1,t}, \hat{BQ}_t, \hat{w}_t, r_t, \hat{BQ}_{t+1}, \hat{w}_{t+1}, r_{t+1}|\Omega\big)$ given beliefs Ω. By induction, we can show that the optimal labor supply, savings, and intended bequests functions for any household of age s and in period t is a function of current holdings of savings and the lifetime path of total bequests received and prices given beliefs Ω.

$$n_{s,t} = \phi\Big(\hat{b}_{s,t}, \big(\hat{BQ}_v, \hat{w}_v, r_v\big)_{v=t}^{t+S-s}|\Omega\Big) \quad \forall s,t \tag{A.3.1}$$

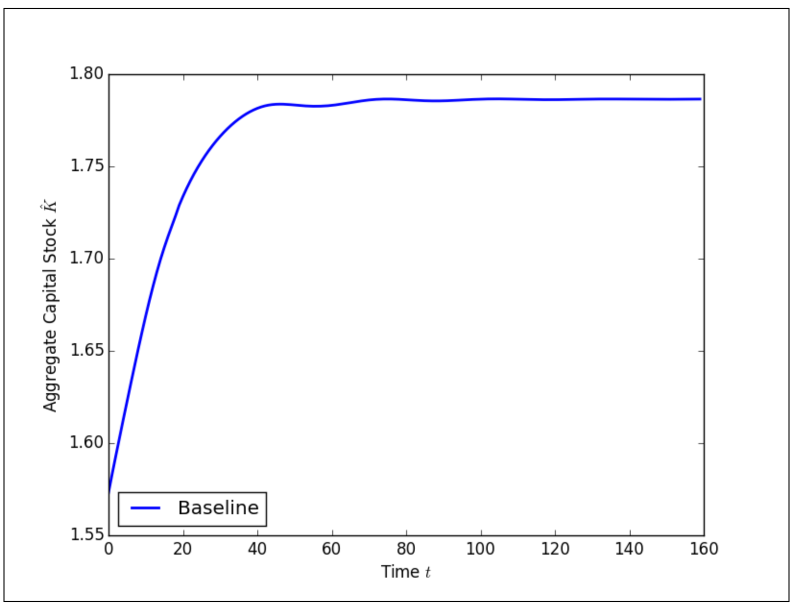
$$\hat{b}_{s+1,t+1} = \psi\Big(\hat{b}_{s,t}, \big(\hat{BQ}_v, \hat{w}_v, r_v\big)_{v=t}^{t+S-s}|\Omega\Big) \quad \forall t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{A.3.2}$$

If one knows the current distribution of households savings and intended bequests, $\hat{\boldsymbol{\Gamma}}_t$, and beliefs about $\hat{\boldsymbol{\Gamma}}_t$, then one can predict time series for total bequests received

$\hat{BQ}_t$, real wages $\hat{w}_t$ and real interest rates r_t necessary for solving each household's optimal decisions. Characteristic (i) in equilibrium definition 2 implies that households be able to forecast prices with perfect foresight over their lifetimes implies that each household has correct information and beliefs about all the other households optimization problems and information. It also implies that the equilibrium allocations and prices are really just functions of the entire distribution of savings at a particular period, as well as a law of motion for that distribution of savings.

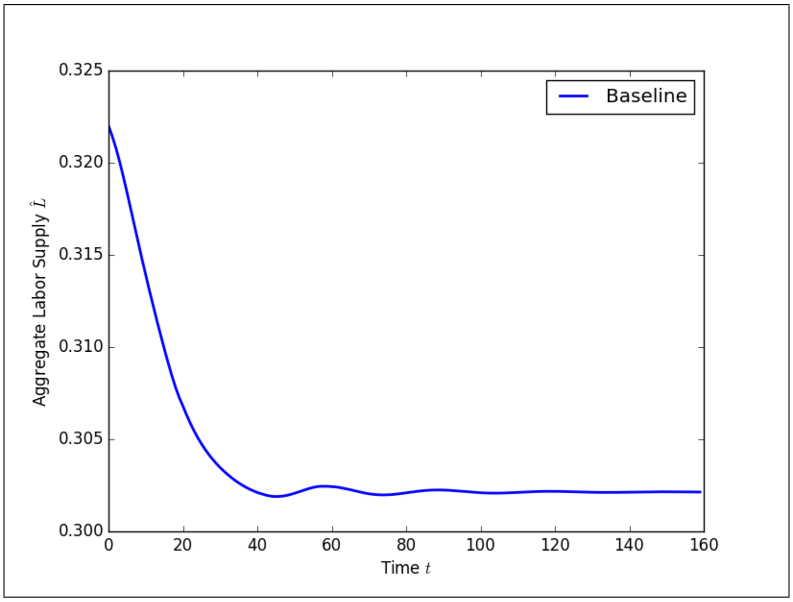
In equilibrium, the steady-state household labor supply, $\bar{n}_s$, for all and s, the steady-state savings, $\bar{b}_{E+S+1}$, the steady-state real wage, $\bar{w}$, and the steady-state real rental rate, $\bar{r}$, are simply functions of the steady-state distribution of savings $\bar{\Gamma}$. This is clear from the steady-state version of the capital market clearing condition (28) and the fact that aggregate labor supply is a function of the sum of exogenous efficiency units of labor in the labor market clearing condition (27). The two firm first order conditions for the real wage $\hat{w}_t$ (26) and real rental rate r_t (18) are only functions of the stationary aggregate capital stock $\hat{K}_t$ and aggregate labor $\hat{L}_t$.

Figure 5: Equilibrium time path of K_t for $S = 80$ in baseline model



To solve for any stationary non-steady-state equilibrium time path of the economy from an arbitrary current state to the steady state, we follow the time path iteration (TPI) method of Auerbach and Kotlikoff (1987). The approach is to choose an arbitrary time path for the stationary aggregate capital stock $\hat{K}_t$, stationary aggregate labor $\hat{L}_t$, and total bequests received $\hat{BQ}_t$. This initial guess of a path implies arbitrary beliefs that violate the rational expectations requirement. We then solve for households' optimal decisions given the time paths of those variables, which decisions imply new time paths of those variables. We then update the time path as a

Figure 6: Equilibrium time path of L_t for $S = 80$ in baseline model



convex combination of the initial guess and the new implied path. Figures 5 and 6 show the equilibrium time paths of the aggregate capital stock and aggregate labor, respectively, for the calibration described in Table 2 for $T = 160$ periods starting from an initial distribution of savings in which $b_{s,1} = \bar{\mathbf{\Gamma}}$ for all s in the case that no policy experiment takes place. The initial capital stock $\hat{K}_1$ is not at the steady state $\bar{K}$ because the initial population distribution is not at the steady-state.

The computational approach to solving for the non-steady-state transition path equilibrium is the time path iteration (TPI) method of Auerbach and Kotlikoff (1987). TPI finds a fixed point for the transition path of the distribution of capital for a given initial state of the distribution of capital. The idea is that the economy is infinitely lived, even though the agents that make up the economy are not. Rather than recursively solving for equilibrium policy functions by iterating on household value functions, one must recursively solve for the policy functions by iterating on the entire transition path of the endogenous objects in the economy (see Stokey and Lucas (1989, ch. 17)).

The key assumption is that the economy will reach the steady-state equilibrium described in Definition 1 in a finite number of periods $T < \infty$ regardless of the initial state. Let $\hat{\mathbf{\Gamma}}_t$ represent the distribution of stationary savings at time t.

$$\hat{\mathbf{\Gamma}}_t \equiv \left\{\hat{b}_{s,t}\right\}_{s=E+2}^{E+S+1}, \quad \forall t \tag{13}$$

In Section 1.5, we describe how the stationary non-steady-state equilibrium time path of allocations and price is characterized by functions of the state $\hat{\mathbf{\Gamma}}_t$ and its law of motion. TPI starts the economy at any initial distribution of savings $\hat{\mathbf{\Gamma}}_1$ and solves for its equilibrium time path over T periods to the steady-state distribution $\bar{\mathbf{\Gamma}}_T$.

The first step is to assume an initial transition path for aggregate stationary capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$, aggregate stationary labor $\hat{\boldsymbol{L}}^i = \left\{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\right\}$, and total bequests received $\hat{\boldsymbol{BQ}}^i = \{\hat{BQ}_1^i, \hat{BQ}_2^i, ...\hat{BQ}_T^i\}$ such that T is sufficiently large to ensure that $\hat{\boldsymbol{\Gamma}}_T = \bar{\boldsymbol{\Gamma}}$, $\hat{K}_T^i\left(\boldsymbol{\Gamma}_T\right)$, $\hat{L}_T^i\left(\boldsymbol{\Gamma}_T\right) = \bar{L}\left(\bar{\boldsymbol{\Gamma}}\right)$, and $\hat{BQ}_T^i\left(\boldsymbol{\Gamma}_T\right) = \bar{BQ}\left(\bar{\boldsymbol{\Gamma}}\right)$ for all $t \geq T$. The superscript i is an index for the iteration number. The transition paths for aggregate capital and aggregate labor determine the transition paths for both the real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$ and the real return on investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. The time paths for the total bequests received also figure in each period's budget constraint and are determined by the distribution of savings and intended bequests.

The exact initial distribution of capital in the first period $\hat{\boldsymbol{\Gamma}}_1$ can be arbitrarily chosen as long as it satisfies the stationary capital market clearing condition (28).

$$\hat{K}_1 = \frac{1}{1+\tilde{g}_{n,1}} \sum_{s=E+2}^{E+S+1} \hat{\omega}_{s-1,0}\hat{b}_{s,1} \tag{A.3.3}$$

Similarly, each initial value of total bequests received $\hat{BQ}_1^i$ must be consistent with the initial distribution of capital through the stationary version of (9).

$$\hat{BQ}_1 = \frac{(1+r_1)}{1+\tilde{g}_{n,1}} \sum_{s=E+1}^{E+S} \rho_s\hat{\omega}_{s,0}\hat{b}_{s+1,1} \tag{A.3.4}$$

However, this is not the case with $\hat{L}_1^i$. Its value will be endogenously determined in the same way the K_2^i is. For this reason, a logical initial guess for the time path of aggregate labor is the steady state in every period $L_t^1 = \bar{L}$ for all $1 \leq t \leq T$.

It is easiest to first choose the initial distribution of savings $\hat{\boldsymbol{\Gamma}}_1$ and then choose an initial aggregate capital stock $\hat{K}_1^i$ and initial total bequests received $\hat{BQ}_{j,1}^i$ that correspond to that distribution. As mentioned earlier, the only other restrictions on the initial transition paths for aggregate capital, aggregate labor, and total bequests received is that they equal their steady-state levels $\hat{K}_T^i = \bar{K}\left(\bar{\boldsymbol{\Gamma}}\right)$, $\hat{L}_T^i = \bar{L}\left(\bar{\boldsymbol{\Gamma}}\right)$, and $\hat{BQ}_T^i = \bar{BQ}\left(\bar{\boldsymbol{\Gamma}}\right)$ by period T. Evans and Phillips (2014) have shown that the initial guess for the aggregate capital stocks $\hat{K}_t^i$ for periods $1 < t < T$ can take on almost any positive values satisfying the constraints above and still have the time path iteration converge.

Given the initial savings distribution $\hat{\boldsymbol{\Gamma}}_1$ and the transition paths of aggregate capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$, aggregate labor $\hat{\boldsymbol{L}}^i = \left\{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\right\}$, and total bequests received $\hat{\boldsymbol{BQ}}_j^i = \left\{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^i, ...\hat{BQ}_{j,T}^i\right\}$, as well as the resulting real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$, and real return to savings $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, one can solve for the period-1 optimal labor supply and intended bequests for $s = E+S$-aged agents in the last period of their lives $n_{E+S,1} = \phi_{E+S}(\hat{b}_{E+S,1}, \hat{BQ}_{E+S,1}, \hat{w}_1, r_1)$ and $\hat{b}_{E+S+1,2} = \psi_{E+S}(\hat{b}_{E+S,1}, \hat{BQ}_{E+S,1}, \hat{w}_1, r_1)$ using his two $s = E+S$ static Euler equations (23) and (25).

$$
\begin{aligned}
(\hat{c}_{E+S,1})^{-\sigma}\left(\hat{w}^i_1 e_{E+S} - \frac{\partial \hat{T}_{E+S,1}}{\partial n_{E+S,1}}\right) &= ... \\
&\chi^n_{E+S}\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{E+S,1}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{E+S,1}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \\
\text{where}\quad \hat{c}_{E+S,1} &= ... \\
&\left(1+r^i_1\right)\hat{b}_{E+S,1} + \hat{w}^i_1 e_{E+S} n_{E+S,1} + \hat{BQ}_1 - e^{g_y}\hat{b}_{E+S+1,2} - \hat{T}_{E+S,1}
\end{aligned} \tag{A.3.5}
$$

$$
(\hat{c}_{E+S,1})^{-\sigma} = \chi^b e^{-g_y\sigma}(\hat{b}_{E+S+1,2})^{-\sigma} \tag{A.3.6}
$$

Note that this is simply two equations (A.3.5) and (A.3.6) and two unknowns $n_{E+S,1}$ and $\hat{b}_{E+S+1,2}$.

We then solve the problem for all $E+S-1$-aged households in period $t=1$, each of which entails labor supply decisions in the current period $n_{E+S-1,1}$ and in the next period $n_{E+S,2}$, a savings decision in the current period for the next period $\hat{b}_{E+S,2}$ and an intended bequest decision in the last period $\hat{b}_{E+S+1,3}$. The labor supply decision in the initial period and the savings period in the initial period for the next period for each $E+S-1$-aged households are policy functions of the current savings and the total bequests received and prices in this period and the next $\hat{b}_{E+S,2} = \psi_{E+S-1}(\hat{b}_{E+S-1,1}, \{\hat{BQ}_t, \hat{w}_t, r_t\}_{t=1}^2)$ and $\hat{n}_{E+S-1,1} = \phi_{E+S-1}(\hat{b}_{E+S-1,1}, \{\hat{BQ}_t, \hat{w}_t, r_t\}_{t=1}^2)$. The labor supply and intended bequests decisions in the next period are simply functions of the savings, total bequests received, and prices in that period $\hat{n}_{E+S,2} = \phi_{E+S}(\hat{b}_{E+S,2}, \hat{BQ}_2, \hat{w}_2, r_2)$ and $\hat{b}_{E+S+1,3} = \psi_{E+S}(\hat{b}_{E+S,2}, \hat{BQ}_2, \hat{w}_2, r_2)$. These four functions are characterized by the following versions of equations (23), (24), and (25).

$$
\begin{aligned}
(\hat{c}_{E+S-1,1})^{-\sigma}\left(\hat{w}^i_1 e_{E+S-1} - \frac{\partial \hat{T}_{E+S-1,1}}{\partial n_{E+S-1,1}}\right) &= ... \\
&\chi^n_{E+S-1}\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{E+S-1,1}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{E+S-1,1}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}
\end{aligned} \tag{A.3.7}
$$

$$
\begin{aligned}
&(\hat{c}_{E+S-1,1})^{-\sigma} = ... \\
&e^{-g_y\sigma}\left(\rho_{E+S-1}\chi^b\left(\hat{b}_{E+S,2}\right)^{-\sigma} + \beta(1-\rho_{E+S-1})(\hat{c}_{E+S,2})^{-\sigma}\left[(1+r^i_2) - \frac{\partial \hat{T}_{E+S,2}}{\partial b_{E+S,2}}\right]\right)
\end{aligned} \tag{A.3.8}
$$

$$
\begin{aligned}
(\hat{c}_{E+S,2})^{-\sigma}\left(\hat{w}^i_2 e_{E+S} - \frac{\partial \hat{T}_{E+S,2}}{\partial n_{E+S,2}}\right) &= ... \\
&\chi^n_{E+S}\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{E+S,2}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{E+S,2}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}
\end{aligned} \tag{A.3.9}
$$

$$(\hat{c}_{E+S,2})^{-\sigma} = \chi^b e^{-g_y\sigma}(\hat{b}_{E+S+1,3})^{-\sigma} \tag{A.3.10}$$

Note that this is four equations (A.3.7), (A.3.8), (A.3.9), and (A.3.10) and four unknowns $n_{E+S-1,1}$, $\hat{b}_{E+S,2}$, $n_{E+S,2}$, and $\hat{b}_{E+S+1,3}$.

This process is repeated for every age of household alive in $t = 1$ down to the age $s = E + 1$ household at time $t = 1$. Each of these households solves the full set of remaining $S - s + 1$ labor supply decisions, $S - s$ savings decisions, and one intended bequest decision at the end of life. After the full set of lifetime decisions has been solved for all the households alive at time $t = 1$, each ability j household born in period $t \geq 2$ can be solved for, the solution to which is characterized by the following full set of Euler equations analogous to (23), (24), and (25).

$$(\hat{c}_{s,t})^{-\sigma}\left(\hat{w}_t^i e_s - \frac{\partial \hat{T}_{s,t}}{\partial n_{s,t}}\right) = \chi_s^n \left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \tag{A.3.11}$$
$$\forall \quad E+1 \leq s \leq E+S \quad \text{and} \quad t \geq 2$$

$$(\hat{c}_{s,t})^{-\sigma} = ...$$
$$e^{-g_y\sigma}\left(\rho_s \chi^b \left(\hat{b}_{s+1,t+1}\right)^{-\sigma} + \beta(1-\rho_s)(\hat{c}_{s+1,t+1})^{-\sigma}\left[(1+r_{t+1}^i) - \frac{\partial \hat{T}_{s+1,t+1}}{\partial b_{s+1,t+1}}\right]\right) \tag{A.3.12}$$
$$\forall \quad E+1 \leq s \leq E+S-1 \quad \text{and} \quad t \geq 2$$

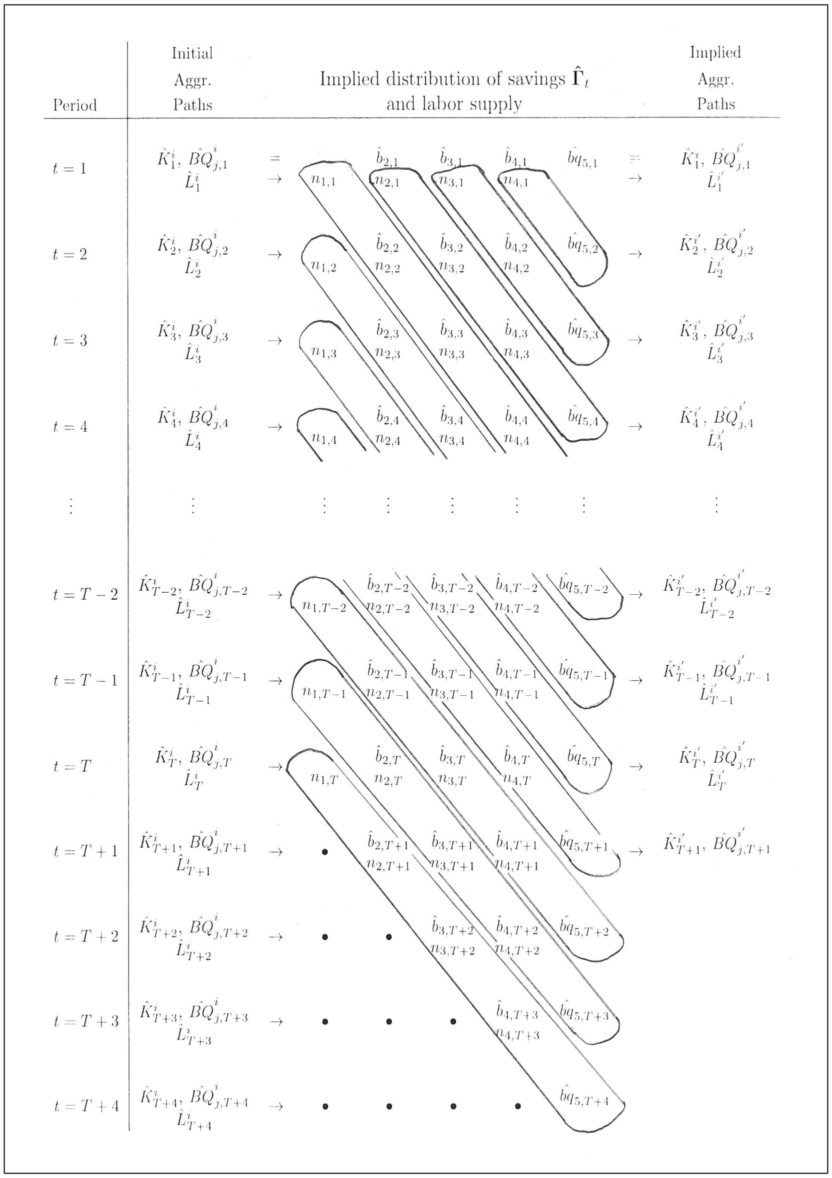
$$(\hat{c}_{E+S,t})^{-\sigma} = \chi^b e^{-g_y\sigma}(\hat{b}_{E+S+1,t+1})^{-\sigma} \quad \forall j \quad t \geq 2 \tag{A.3.13}$$

For each household entering the economy in period $t \geq 1$, the entire set of $2S$ lifetime decisions is characterized by the $2S$ equations represented in (A.3.11), (A.3.12), and (A.3.13).

We can then solve for the entire lifetime of savings and labor supply decisions for each age $s = 1$ household in periods $t = 2, 3, ...T$. The central part of the schematic diagram in Figure 7 shows how this process is done in order to solve for the equilibrium time path of the economy from period $t = 1$ to T. Note that for each full lifetime savings and labor supply path solved for a household born in period $t \geq 2$, we can solve for the aggregate capital stock and total bequests received implied by those savings decisions $\hat{\boldsymbol{K}}^{i'}$ and $\hat{\boldsymbol{BQ}}^{i'}$ and aggregate labor implied by those labor supply decisions $\hat{\boldsymbol{L}}^{i'}$.

Once the set of lifetime saving and labor supply decisions has been computed for all households alive in $1 \leq t \leq T$, we use the household decisions to compute a new implied time path of the aggregate capital stock and aggregate labor. The implied paths of the aggregate capital stock $\hat{\boldsymbol{K}}^{i'} = \{\hat{K}_1^i, \hat{K}_2^{i'}, ...\hat{K}_T^{i'}\}$, aggregate labor $\hat{\boldsymbol{L}}^{i'} = \{\hat{L}_1^i, \hat{L}_2^{i'}, ...\hat{L}_T^{i'}\}$, and total bequests received $\hat{\boldsymbol{BQ}}^{i'} = \{\hat{BQ}_1^i, \hat{BQ}_2^{i'}, ...\hat{BQ}_T^{i'}\}$ in general do not equal the initial guessed paths $\hat{\boldsymbol{K}}^i = \{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\}$, $\hat{\boldsymbol{L}}^i = \{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\}$, and $\hat{\boldsymbol{BQ}}^i = \{\hat{BQ}_1^i, \hat{BQ}_2^i, ...\hat{BQ}_T^i\}$ used to compute the household savings and labor supply decisions $\hat{\boldsymbol{K}}^{i'} \neq \hat{\boldsymbol{K}}^i$, $\hat{\boldsymbol{L}}^{i'} \neq \hat{\boldsymbol{L}}^i$, and $\hat{\boldsymbol{BQ}}^{i'} \neq \hat{\boldsymbol{BQ}}^i$.

Figure 7: Diagram of TPI solution method within each iteration for $S = 4$



Let $\|\cdot\|$ be a norm on the space of time paths of the aggregate capital stock $\hat{\boldsymbol{K}} \in \mathcal{K} \subset \mathbb{R}^T_{++}$, aggregate labor supply $\hat{\boldsymbol{L}} \in \mathcal{L} \subset \mathbb{R}^T_{++}$, and total bequests received $\hat{\boldsymbol{BQ}} \in \mathcal{B} \subset \mathbb{R}^T_{++}$. Then the fixed point necessary for the equilibrium transition path from Definition 2 has been found when the distance between these paths is arbitrarily close to zero.

$$\left\|\left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}, \hat{\boldsymbol{BQ}}^{i'}\right] - \left[\hat{\boldsymbol{K}}^{i}, \hat{\boldsymbol{L}}^{i}, \hat{\boldsymbol{BQ}}^{i}\right]\right\| \leq \varepsilon \quad \text{for} \quad \varepsilon > 0 \tag{A.3.14}$$

If the fixed point has not been found $\left\|\left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}, \hat{\boldsymbol{BQ}}^{i'}\right] - \left[\hat{\boldsymbol{K}}^{i}, \hat{\boldsymbol{L}}^{i}, \hat{\boldsymbol{BQ}}^{i}\right]\right\| > \varepsilon$, then new transition paths for the aggregate capital stock and aggregate labor are generated as a convex combination of $\left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}, \hat{\boldsymbol{BQ}}^{i'}\right]$ and $\left[\hat{\boldsymbol{K}}^{i}, \hat{\boldsymbol{L}}^{i}, \hat{\boldsymbol{BQ}}^{i}\right]$.

$$\begin{aligned} \hat{\boldsymbol{K}}^{i+1} &= \nu\hat{\boldsymbol{K}}^{i'} + (1-\nu)\hat{\boldsymbol{K}}^{i} \\ \hat{\boldsymbol{L}}^{i+1} &= \nu\hat{\boldsymbol{L}}^{i'} + (1-\nu)\hat{\boldsymbol{L}}^{i} \qquad\qquad \text{for} \quad \nu \in (0,1] \\ \hat{\boldsymbol{BQ}}^{i+1} &= \nu\hat{\boldsymbol{BQ}}^{i'} + (1-\nu)\hat{\boldsymbol{BQ}}^{i} \end{aligned} \tag{A.3.15}$$

This process is repeated until the initial transition paths for the aggregate capital stock, aggregate labor, and total bequests received are consistent with the transition paths implied by those beliefs and household and firm optimization.

In essence, the TPI method iterates on household beliefs about the time path of prices represented by a time paths for the aggregate capital stock $\hat{\boldsymbol{K}}^i$, aggregate labor $\hat{\boldsymbol{L}}^i$, and total bequests received $\hat{\boldsymbol{BQ}}^i$ until a fixed point in beliefs is found that are consistent with the transition paths implied by optimization based on those beliefs.

The following are the steps for computing a stationary non-steady-state equilibrium time path for the economy.

1. Input all initial parameters. See Table 2.
   (a) The value for T at which the non-steady-state transition path should have converged to the steady state should be at least as large as the number of periods it takes the population to reach its steady state $\bar{\boldsymbol{\omega}}$ as described in Appendix ??.
2. Choose an initial distribution of savings and intended bequests $\hat{\boldsymbol{\Gamma}}_1$ and then calculate the initial state of the stationarized aggregate capital stock $\hat{K}_1$ and total bequests received $\hat{BQ}_1$ consistent with $\hat{\boldsymbol{\Gamma}}_1$ according to (28) and (A.3.4).
   (a) Note that you must have the population weights from the previous period $\hat{\omega}_{s,0}$ and the growth rate between period 0 and period 1 $\tilde{g}_{n,1}$to calculate $\hat{BQ}_1$.
3. Conjecture transition paths for the stationarized aggregate capital stock $\hat{\boldsymbol{K}}^1 = \{\hat{K}^1_t\}^\infty_{t=1}$, stationarized aggregate labor $\hat{\boldsymbol{L}}^1 = \{\hat{L}^1_t\}^\infty_{t=1}$, and total bequests received $\hat{\boldsymbol{BQ}}^1_j = \{\hat{BQ}^1_t\}^\infty_{t=1}$ where the only requirements are that $\hat{K}^i_1$ and $\hat{BQ}^i_1$ are functions of the initial distribution of savings $\hat{\boldsymbol{\Gamma}}_1$ for all i is your initial

state and that $\hat{K}^i_t = \bar{K}$, $\hat{L}^i_t = \bar{L}$, and $\hat{BQ}^i_t = \bar{BQ}$ for all $t \geq T$. The conjectured transition paths of the aggregate capital stock $\hat{\boldsymbol{K}}^i$ and aggregate labor $\hat{\boldsymbol{L}}^i$ imply specific transition paths for the real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}^i_t\}_{t=1}^{\infty}$ and the real interest rate $\boldsymbol{r}^i = \{r^i_t\}_{t=1}^{\infty}$ through expressions (26) and (18).

   (a) An intuitive choice for the time path of aggregate labor is the steady-state in every period $\hat{L}^1_t = \bar{L}$ for all t.

4. With the conjectured transition paths $\hat{\boldsymbol{w}}^i$, $\boldsymbol{r}^i$, and $\hat{\boldsymbol{BQ}}^i$ one can solve for the lifetime policy functions of each household alive at time $1 \leq t \leq T$ using the systems of Euler equations of the form (23), (24), and (25) and following the diagram in Figure 7.

5. Use the implied distribution of savings and labor supply in each period (each row of $\hat{b}_{s,t}$ and $n_{s,t}$ in Figure 7) to compute the new implied time paths for the aggregate capital stock $\hat{\boldsymbol{K}}^{i'} = \{\hat{K}^i_1, \hat{K}^{i'}_2, ...\hat{K}^{i'}_T\}$, aggregate labor supply $\hat{\boldsymbol{L}}^{i'} = \{\hat{L}^i_1, \hat{L}^{i'}_2, ...\hat{L}^{i'}_T\}$, and total bequests received $\hat{\boldsymbol{BQ}}^{i'} = \{\hat{BQ}^i_1, \hat{BQ}^{i'}_2, ...\hat{BQ}^{i'}_T\}$.

6. Check the distance between the two sets time paths.

$$\left\| \left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}, \hat{\boldsymbol{BQ}}^{i'}\right] - \left[\hat{\boldsymbol{K}}^{i}, \hat{\boldsymbol{L}}^{i}, \hat{\boldsymbol{BQ}}^{i}\right] \right\|$$

   (a) If the distance between the initial time paths and the implied time paths is less-than-or-equal-to some convergence criterion $\varepsilon > 0$, then the fixed point has been achieved and the equilibrium time path has been found (A.3.14).

   (b) If the distance between the initial time paths and the implied time paths is greater than some convergence criterion $\|\cdot\| > \varepsilon$, then update the guess for the time paths according to (A.3.15) and repeat steps (4) through (6) until a fixed point is reached.