Fiscal Rule

February 21, 2015

Let D_t denote the government's outstanding real debt. T_t is total tax revenue, T_t^H is total household transfers, G_t is government purchases of goods, L_t is the real value of purchases of labor services, and S_t is subsidies to government run firms.

$$D_{t+1} = D_t(1 + r_t) - T_t + T_t^H + G_t + L_t + S_t \tag{1}$$

Letting a carat denote the ratio of a variable to GDP, we can rewrite this as follows:

$$(1 + g_{Yt})\hat{D}_{t+1} = \hat{D}_t(1 + r_t) - \hat{T}_t + \hat{T}_t^H + \hat{G}_t + \hat{L}_t + \hat{S}_t \tag{2}$$

We need to adopt a government fiscal rule that determines how our residual expenditure $\hat{G}_t$ evolves over time.

One way is to adopt a balanced budget rule which keeps the debt-to-GDP ratio constant at it's initial value of $\hat{D}_0$.

$$\begin{aligned}(1 + g_{Yt})\hat{D}_0 &= \hat{D}_0(1 + r_t) - \hat{T}_t + \hat{T}_t^H + \hat{G}_t + \hat{L}_t + \hat{S}_t \\ \hat{G}_t &= \hat{D}_0(g_{Yt} - r_t) + \hat{T}_t - \hat{T}_t^H - \hat{L}_t - \hat{S}_t \end{aligned} \tag{3}$$

Another rule is to hold govenrment spending constant and let debt evolve as it will for several period. Then in period T impose fiscal austerity which forces $\hat{G}_t$ to adjust over time so that $\hat{D}_t$ goes to a steady value.

$$\hat{G}_t - \bar{G} = \rho_t(\hat{D}_t - \bar{D}); \quad \rho_t < 0 \tag{4}$$

Substituting this into (2) gives:

$$\begin{aligned}(1 + g_{Yt})\hat{D}_{t+1} &= \hat{D}_t(1 + r_t) - \hat{T}_t + \hat{T}_t^H + \rho_t(\hat{D}_t - \bar{D}) + \bar{G} + \hat{L}_t + \hat{S}_t \\ \hat{D}_{t+1} &= \frac{\hat{D}_t(1 + r_t) - \hat{T}_t + \hat{T}_t^H + \rho_t(\hat{D}_t - \bar{D}) + \bar{G} + \hat{L}_t + \hat{S}_t}{1 + g_{Yt}} \end{aligned} \tag{5}$$

Consider the steady state version of this.

$$\begin{aligned}(1 + \bar{g}_Y)\bar{D} &= \bar{D}(1 + \bar{r}) + \bar{T} - \bar{T}^H + \rho_t(\bar{D} - \bar{D}) + \bar{G} + \bar{L} + \bar{S}_t \\ \bar{G} &= \bar{D}(\bar{g}_Y - \bar{r}) + \bar{T} - \bar{T}^H - \bar{L} - \bar{S} \end{aligned} \tag{6}$$

This tells us the long-run value of government spending to GDP that will maintain the debt to GDP target.

In order for (5) to be a contraction mapping over $\hat{D}$ and thus converge to a steady state, we must put bounds on ρ_t. Rearranging (5) and using (6):

$$
\begin{aligned}
(1+g_{Yt})\hat{D}_{t+1} &= \hat{D}_t(1+r_t) - \hat{T}_t + \hat{T}_t^H + \rho_t(\hat{D}_t - \bar{D}) + \hat{L}_t + \hat{S}_t \\
&\quad + \bar{D}(\bar{g}_Y - \bar{r}) + \bar{T} - \bar{T}^H - \bar{L} - \bar{S} \\
(1+g_{Yt})\hat{D}_{t+1} &= \hat{D}_t(1+r_t) - \hat{T}_t + \hat{T}_t^H + \rho_t\hat{D}_t - \rho_t\bar{D} + \hat{L}_t + \hat{S}_t \\
&\quad + \bar{g}_Y\bar{D} - \bar{r}\bar{D} + \bar{T} - \bar{T}^H - \bar{L} - \bar{S} \\
(1+g_{Yt})\hat{D}_{t+1} &= \hat{D}_t(1+r_t) + \rho_t(\hat{D}_t - \bar{D}) + (\bar{g}_Y - \bar{r})\bar{D} \\
&\quad - (\hat{T}_t - \bar{T}) + (\hat{T}_t^H - \bar{T}^H) + (\hat{L}_t - \bar{L}) + (\hat{S}_t - \bar{S}) \\
\hat{D}_{t+1} - \bar{D} &= \hat{D}_t\frac{1+r_t}{1+g_{Yt}} + \frac{\rho_t}{1+g_{Yt}}(\hat{D}_t - \bar{D}) + \left(\frac{\bar{g}_Y - \bar{r}}{1+g_{Yt}} - 1\right)\bar{D} \\
&\quad + \frac{-(\hat{T}_t - \bar{T}) + (\hat{T}_t^H - \bar{T}^H) + (\hat{L}_t - \bar{L}) + (\hat{S}_t - \bar{S})}{1+g_{Yt}} \\
\hat{D}_{t+1} - \bar{D} &= \frac{1+r_t+\rho_t}{1+g_{Yt}}(\hat{D}_t - \bar{D}) \\
&\quad + \frac{-(\hat{T}_t - \bar{T}) + (\hat{T}_t^H - \bar{T}^H) + (\hat{L}_t - \bar{L}) + (\hat{S}_t - \bar{S})}{1+g_{Yt}} \qquad (7)
\end{aligned}
$$

We need $\frac{\hat{D}_{t+1}-\bar{D}}{\hat{D}_t-\bar{D}} < 1$ for stability. Equation (7) gives:

$$
\begin{aligned}
\frac{\hat{D}_{t+1} - \bar{D}}{\hat{D}_t - \bar{D}} = \frac{1+r_t+\rho_t}{1+g_{Yt}} + \frac{-(\hat{T}_t - \bar{T}) + (\hat{T}_t^H - \bar{T}^H) + (\hat{L}_t - \bar{L}) + (\hat{S}_t - \bar{S})}{(1+g_{Yt})(\hat{D}_t - \bar{D})} &< 1 \\
\frac{1+r_t+\rho_t}{1+g_{Yt}} &< \frac{(\hat{T}_t - \bar{T}) - (\hat{T}_t^H - \bar{T}^H) - (\hat{L}_t - \bar{L}) - (\hat{S}_t - \bar{S})}{(1+g_{Yt})(\hat{D}_t - \bar{D})} \\
\rho_t &< (1+r_t)\frac{(\hat{T}_t - \bar{T}) - (\hat{T}_t^H - \bar{T}^H) - (\hat{L}_t - \bar{L}) - (\hat{S}_t - \bar{S})}{\hat{D}_t - \bar{D}} \qquad (8)
\end{aligned}
$$