An Open Source Macroeconomic Model for Dynamic Scoring of Tax Policy *

Jason DeBacker† Richard W. Evans‡ Evan Magnusson§
Kerk L. Phillips¶ Isaac Swift‖

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Abstract

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†Middle Tennessee State University, Department of Economics and Finance, BAS N306, Murfreesboro, TN 37132, (615) 898-2528,jason.debacker@mtsu.edu.

‡Brigham Young University, Department of Economics, 167 FOB, Provo, Utah 84602, (801) 422-8303, revans@byu.edu.

§Brigham Young University, Department of Economics, 163 FOB, Provo, Utah 84602, evanmag42@gmail.com.

¶Brigham Young University, Department of Economics, 166 FOB, Provo, Utah 84602, (801) 422-5928, kerk_phillips@byu.edu.

‖Brigham Young University, Department of Economics, 163 FOB, Provo, Utah 84602, isaacdswift@gmail.com.

1 Introduction

Put introduction here.

2 Model with Endogenous Labor

This is the basic OLG model in which households live $E + S$ periods. We term a household as being youth and out of the workforce in periods $1 \leq s \leq E$. They enter the workforce at age $E+1$ and remain in the workforce until they die or until age S.

When households are born at age $s = 1$, they are randomly assigned to one of J ability types. Households remain deterministically in their assigned ability type throughout their lives. The ability process is calibrated to match the wage distribution by age in the United States, and labor is endogenously supplied by individuals.

Individuals have both accidental and intended bequests. The production side of the economy is characterized by a unit measure of identical, perfectly competitive firms. We have also included a richer model of exogenous population growth, asymmetric ability bins, and accidental and intended bequests.

2.1 Individual problem

A measure $\omega_{1,t}$ of individuals with heterogeneous working ability $e \in \mathcal{E} \subset \mathbb{R}_{++}$ is born in each period t and live for $E + S$ periods, with $S \geq 3$. The population of age-s individuals in any period t is $\omega_{s,t}$. Households are termed "youth" and out of the market during ages $1 \leq s \leq E$. The households enter the workforce and economy in period $E+1$ and remain in the workforce until they unexpectedly die or live until age $s = E + S$.[1] The population of agents of each age in each period $\omega_{s,t}$ evolves

[1]We model the population with households age $s \leq E$ outside of the workforce and economy in order to get the actual population dynamics correct. Appendix A-1 gives more detail on the population process and its calibration.

according to the following function,

$$
\begin{aligned}
\omega_{1,t+1} &= \sum_{s=1}^{E+S} f_s\omega_{s,t} \quad \forall t \\
\omega_{s+1,t+1} &= (1 + i_s - \rho_s)\omega_{s,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1
\end{aligned}
\tag{1}
$$

where $f_s \geq 0$ is an age-specific fertility rate, i_s is an age-specific immigration rate, ρ_s is an age specific mortality hazard rate,[2] and $1 + i_s - \rho_s$ is constrained to be nonnegative. The total population in the economy N_t at any period is simply the sum of individuals in the economy, the population growth rate in any period t from the previous period $t-1$ is $g_{n,t}$, $\tilde{N}_t$ is the working age population, and $\tilde{g}_{n,t}$ is the working age population growth rate in any period t from the previous period $t-1$.[3]

$$
N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{2}
$$

$$
g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{3}
$$

$$
\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{4}
$$

$$
\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{5}
$$

A household's working ability evolves over his working-age lifetime $E+1 \leq s \leq E+S$ according to an age-dependent deterministic process. At birth, a set fraction λ_j of the $\omega_{1,t}$ measure of new agents are randomly assigned to each of the J ability types indexed by $j = 1, 2, ...J$ such that $\sum_{j=1}^{J}\lambda_j = 1$. Once ability type is determined, that measure $\lambda_j\omega_{s,t}$ of individuals' ability evolves deterministically according to $e_{j,s}$. The process for the evolution of the population weights $\omega_{s,t}$ is an exogenous input to the model. We calibrate the matrix of lifetime ability paths $e_{j,s}$ for all types j using CPS hourly wage by age distribution data.[4]

[2]The parameter ρ_s is the probability that a household of age s dies before age $s+1$.

[3]Appendix A-1 describes in detail the exogenous population dynamics.

[4]Appendix A-2 gives a detailed description of the calibration of the deterministic ability process

Individuals are endowed with a measure of time $\tilde{l}$ in each period t, and they choose each period how much of that time to allocate between labor $n_{j,s,t}$ and leisure $l_{j,s,t}$.

$$n_{j,s,t} + l_{j,s,t} = \tilde{l} \tag{6}$$

At time t, all generation-s agents with ability $e_{j,s}$ know the real wage rate w_t and know the one-period real net interest rate r_t on bond holdings $b_{j,s,t}$ that mature at the beginning of period t. They also receive accidental and intentional bequestsIn each period t, age-s agents with working ability $e_{j,s}$ choose how much to consume $c_{j,s,t}$, how much to save for the next period by loaning capital to firms in the form of a one-period bond $b_{j,s+1,t+1}$, and how much to work $n_{j,s,t}$ in order to maximize expected lifetime utility of the following form,

$$\begin{aligned}
U_{j,s,t} = \sum_{u=0}^{E+S-s}\beta^u\left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right]u\left(c_{j,s+u,t+u},n_{j,s+u,t+u}\right) \quad \text{where} \quad \rho_{s-1}=0 \\
\text{and} \quad u\left(c_{j,s,t},n_{j,s,t}\right) = \frac{\left(c_{j,s,t}\right)^{1-\sigma}-1}{1-\sigma} + \chi_n e^{g_y t(1-\sigma)}\frac{\left(\tilde{l}-n_{j,s,t}\right)^{1-\eta}}{1-\eta} \\
\forall j,t \quad \text{and}\; E+1\leq s<E+S \\
\text{and} \quad u\left(c_{j,s,t},n_{j,s,t}\right) = \frac{\left(c_{j,s,t}\right)^{1-\sigma}-1}{1-\sigma} + \chi_n e^{g_y t(1-\sigma)}\frac{\left(\tilde{l}-n_{j,s,t}\right)^{1-\eta}}{1-\eta} + ... \\
\chi_b\frac{\left(bq_{j,s+1,t+1}\right)^{1-\sigma}-1}{1-\sigma} \quad \forall j,t \quad \text{and} \quad s=E+S
\end{aligned} \tag{7}$$

where $\sigma \geq 1$ is the coefficient of relative risk aversion on consumption, $\eta \geq 1$ is proportional to the Frisch elasticity of labor supply, $\beta \in (0,1)$ is the agent's discount factor, and the term in brackets depreciates the household's discount factor by the cummulative mortality rate.[5] The term χ_n is a constant term influencing the disutility of labor, and g_y is a constant growth rate of labor augmenting technological progress, which we explain in Section 2.2.[6] The term χ_b is a constant term influencing the

by age s and type j, as well as alternative specifications and calibrations.

[5]The initial period mortality rate ρ_{s-1} is forced to be zero because the household knows it has survived to that point.

[6]The term with the growth rate $e^{g_y t(1-\sigma)}$ must be included in the period utility function because consumption and bequests will be growing at rate g_y and this term stationarizes the household Euler equation by making the marginal disutility of labor grow at the same rate as the marginal benefits

marginal utility of intentional bequests $bq_{j,s+1,t+1}$ relative to the marginal utility of consumption.

Because agents are born without any bonds maturing and because they purchase no bonds in the last period of life $s = E + S$, the per-period budget constraints for each agent normalized by the price of consumption are the following,

$$c_{j,s,t} + b_{j,s+1,t+1} \leq w_t e_{j,s} n_{j,s,t} + \frac{BQ_{j,t}}{\lambda_j \tilde{N}_t} \quad \text{for} \quad s = E+1 \quad \forall j,t \tag{8}$$

$$\begin{aligned} c_{j,s,t} + b_{j,s+1,t+1} \leq (1+r_t)\, b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \frac{BQ_{j,t}}{\lambda_j \tilde{N}_t} \\ \text{for} \quad E+2 \leq s \leq E+S-1 \quad \forall j,t \end{aligned} \tag{9}$$

$$c_{j,s,t} + bq_{j,s+1,t+1} \leq (1+r_t)\, b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \frac{BQ_{j,t}}{\lambda_j \tilde{N}_t} \quad \text{for} \quad s = E+S \quad \forall j,t \tag{10}$$

where $\tilde{N}_t$ is the total working age population at time t defined in (4) and $\lambda_j \tilde{N}_t$ is the number of the total working households of type j. Note that the price of consumption is normalized to one, so w_t is the real wage and r_t is the real net interest rate. The term $BQ_{j,t}$ represents the ability-j age-s household's total bequests received at the end of the last period $t-1$ from both accidental bequests of those who died unexpectedly and intended bequests from those who died at age $s = E + S$.

We first assume the symmetry that individual accidental bequests $(1+r_{t+1})b_{j,s+1,t+1}$ for $s < E+S$ can be both positive and negative. This means that households that die before age $s = E + S$ can bequeath either savings or debt. However, the functional form for the utility of intended bequests in period $s = E + S$ in (7) ensures that intended bequests will always be positive $bq_{j,E+S+1,t} > 0$ for all j and t. We also impose the constraint, similar to the aggregate capital stock, that aggregate bequests be nonnegative $BQ_{j,t} \geq 0$ for all j and t.

$$BQ_{j,t+1} = (1+r_{t+1})\lambda_j \left(\omega_{E+S,t} bq_{j,E+S+1,t+1} + \sum_{s=E+1}^{E+S-1} \rho_s \omega_{s,t} b_{j,s+1,t+1} \right) \quad \forall j,t \tag{11}$$

of consumption and bequests.

We assume the total accidental and intentional bequests from ability-j households are equally divided among surviving type-j households.[7]

In addition to the budget constraints in each period, the utility function imposes nonnegative consumption through inifinite marginal utility and individual labor and leisure must be nonnegative $n_{j,s,t}, l_{j,s,t} \geq 0$. We allow the possibility for individual agents to borrow $b_{j,s,t} < 0$ for some j and s in period t. However, the borrowing must satisfy a series of individual feasibility constraints as well as a strict constraint that the aggregate capital stock be positive $K_t > 0$ in every period t.[8]

We next describe the Euler equations that govern the choices of consumption $c_{j,s,t}$, savings $b_{j,s+1,t+1}$, and labor supply $n_{j,s,t}$ by household of age s and ability $e_{j,s}$ in each period t as well as their choice of intended bequests $bq_{j,E+S+1,t}$ at age $s = E + S$. We work backward from the last period of life $s = E + S$. For households that reach age $s = E + S$, they do not save for themselves. They simply choose how much to work $n_{j,E+S,t}$ and how much to bequeath to their posterity $bq_{j,E+S+1,t+1}$. The household's final-period maximization problem is given by the following.

$$\begin{aligned} &\max_{n_{j,E+S,t},bq_{j,E+S+1,t+1}} \frac{(c_{j,E+S,t})^{1-\sigma} - 1}{1-\sigma} + \chi_n e^{g_y t(1-\sigma)} \frac{(\tilde{l} - n_{j,E+S,t})^{1-\eta}}{1-\eta} + \chi_b \frac{(bq_{j,E+S+1,t+1})^{1-\sigma} - 1}{1-\sigma} \\ &\text{s.t.} \quad c_{j,E+S,t} = (1+r_t)\, b_{j,E+S,t} + w_t e_{j,E+S} n_{j,E+S,t} + \frac{BQ_{j,t}}{\lambda_j \tilde{N}_t} - bq_{j,E+S+1,t+1} \quad \forall j,t \end{aligned} \tag{12}$$

Because $u(c)$ is monotonically increasing in c, the $s = E + S$ consumption part of the maximization problem (12) is simply to choose the maximum amount of consumption possible. The household trivially consumes all of its income in the last period of life. However, the household must choose labor and bequests to balance its benefits in extra consumption with its costs in disutility for labor and its benefits giving for

[7]Another allocation rule at the other extreme would be to equally divide all bequests among all surviving households. An intermediate rule would be some kind of distribution of bequests with most going to ones own type and a declining proportion going to the other types.

[8]We describe these constraints in detail in Appendix A-3.

bequests.

$$c_{j,E+S,t} = (1+r_t)\,b_{j,E+S,t} + w_t e_{j,E+S} n_{j,E+S,t} + \frac{BQ_{j,t}}{\lambda_j \tilde{N}_t} - bq_{j,E+S+1,t+1} \quad \forall j,t \tag{13}$$

$$w_t e_{j,E+S}\big(c_{j,E+S,t}\big)^{-\sigma} = \chi_n e^{g_y t(1-\sigma)}(\tilde{l} - n_{j,E+S,t})^{-\eta} \quad \forall j,t \tag{14}$$

$$\big(c_{j,E+S,t}\big)^{-\sigma} = \chi_b\big(bq_{j,E+S+1,t+1}\big)^{-\sigma} \quad \forall j,t \tag{15}$$

A household in his second-to-last period of life $s = E+S-1$ must choose how much to consume and how much to save for the last period of life $b_{j,E+S,t+1}$ as well as how much to work in the current period $n_{j,E+S-1,t}$ and how much to work in the final period $n_{j,E+S,t+1}$ as well as how much to bequeath in the final period $bq_{j,E+S+1,t+2}$. The $s = E+S-1$ household optimization problem is governed by two static first order conditions (16) and (18) for labor $n_{j,E+S-1,t}$ and $n_{j,E+S,t+1}$, an intertemporal Euler equation (17) for the savings decision $b_{j,E+S,t+1}$, and a static Euler equation (19) for bequests in the final period $bq_{j,E+S+1,t+2}$.

$$\begin{aligned} &w_t e_{j,E+S-1}\big(c_{j,E+S-1,t}\big)^{-\sigma} = \chi_n e^{g_y t(1-\sigma)}(\tilde{l} - n_{j,E+S-1,t})^{-\eta} \quad \forall j,t \\ &\quad\text{where } c_{j,E+S-1,t} = (1+r_t)\,b_{j,E+S-1,t} + w_t e_{j,E+S-1} n_{j,E+S-1,t} + ... \\ &\qquad\qquad \frac{BQ_{j,t}}{\lambda_j \tilde{N}_t} - b_{j,E+S,t+1} \end{aligned} \tag{16}$$

$$\big(c_{j,E+S-1,t}\big)^{-\sigma} = \beta(1-\rho_{E+S-1})(1+r_{t+1})\big(c_{j,E+S,t+1}\big)^{-\sigma} \quad \forall j,t \tag{17}$$

$$\begin{aligned} &w_{t+1} e_{j,E+S}\big(c_{j,E+S,t+1}\big)^{-\sigma} = \chi_n e^{g_y (t+1)(1-\sigma)}(\tilde{l} - n_{j,E+S,t+1})^{-\eta} \quad \forall j,t \\ &\quad\text{where } c_{j,E+S,t+1} = (1+r_{t+1})\,b_{j,E+S,t+1} + w_{t+1} e_{j,E+S} n_{j,E+S,t+1} + ... \\ &\qquad\qquad \frac{BQ_{j,t+1}}{\lambda_j \tilde{N}_{t+1}} - bq_{j,E+S+1,t+2} \end{aligned} \tag{18}$$

$$\big(c_{j,E+S,t+1}\big)^{-\sigma} = \chi_b\big(bq_{j,E+S+1,t+2}\big)^{-\sigma} \quad \forall j,t \tag{19}$$

In general for a household of type j, maximizing (7) with respect to (8), (9), (10), and the implied individual and aggregate borrowing constraints gives the following set of $S-1$ intertemporal Euler equations for savings $b_{j,s+1,t+1}$ in (21), S static first order conditions characterizing lifetime labor supply $n_{j,s,t}$ in (20), and one static first order condition characterizing intended bequests $bq_{j,E+S+1,t+1}$ in age $s=E+S$ in (22).

$$
\begin{aligned}
&w_t e_{j,s}\big(c_{j,s,t}\big)^{-\sigma} = \chi_n e^{g_y t(1-\sigma)}(\tilde{l}-n_{j,s,t})^{-\eta} \\
&\qquad\qquad \forall j,t \quad \text{and} \quad E+1\leq s\leq E+S \quad \text{with} \quad b_{j,E+1,t},b_{j,E+S+1,t}=0 \\
&\quad \text{where} \quad c_{j,s,t} = (1+r_t)\,b_{j,s,t} + w_t e_{j,s}n_{j,s,t} + \frac{BQ_{j,t}}{\lambda_j\tilde{N}_t} - b_{j,s+1,t+1} \\
&\qquad\qquad \text{for} \quad E+1\leq s\leq E+S-1 \\
&\quad \text{and} \quad c_{j,s,t} = (1+r_t)\,b_{j,s,t} + w_t e_{j,s}n_{j,s,t} + \frac{BQ_{j,t}}{\lambda_j\tilde{N}_t} - bq_{j,s+1,t+1} \\
&\qquad\qquad \text{for} \quad s=E+S
\end{aligned}
\tag{20}
$$

$$\big(c_{j,s,t}\big)^{-\sigma} = \beta(1-\rho_s)(1+r_{t+1})\big(c_{j,s+1,t+1}\big)^{-\sigma} \quad \forall j,t \quad \text{and} \quad E+1\leq s\leq E+S-1 \tag{21}$$

$$\big(c_{j,E+S,t}\big)^{-\sigma} = \chi_b\big(bq_{j,E+S+1,t+1}\big)^{-\sigma} \quad \forall j,t \tag{22}$$

2.2 Firm problem

A unit measure of identical, perfectly competitive firms exist in this economy. The representative firm is characterized by the following Cobb-Douglas production technology,

$$Y_t = AK_t^\alpha\left(e^{g_y t}L_t\right)^{1-\alpha} \quad \forall t \tag{23}$$

where A is a constant level effect on the technology process, $\alpha\in(0,1)$ is the capital share of income, g_y is the constant growth rate of labor augmenting technological change, and L_t is measured in efficiency units of labor. The interest rate r_t in the cost function is a net real interest rate because depreciation δ is paid by the firms. The real wage is w_t. The real profit function of the firm is the following.

$$\text{Real Profits} = AK_t^\alpha\left(e^{g_y t}L_t\right)^{1-\alpha} - (r_t+\delta)K_t - w_tL_t \tag{24}$$

As in the budget constraints (8), (9), and (10), note that the price of the good has been normalized to one.

Profit maximization results in the real wage w_t and the real rental rate of capital r_t

being determined by the marginal products of labor and capital, respectively.

$$w_t = (1-\alpha)\frac{Y_t}{L_t} \quad \forall t \tag{25}$$

$$r_t = \alpha\frac{Y_t}{K_t} - \delta \qquad \forall t \tag{26}$$

2.3 Market clearing and stationary equilibrium

Labor market clearing requires that aggregate labor demand L_t measured in efficiency units equal the sum of individual efficiency labor supplied $e_{j,s}n_{j,s,t}$. Capital market clearing requires that aggregate capital demand K_t equal the sum of capital investment by households $b_{j,s,t}$. Aggregate consumption C_t is defined as the sum of all individual consumptions, and aggregate investment is defined by the standard $Y = C + I$ constraint as shown in (29).

$$L_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j e_{j,s}n_{j,s,t} \quad \forall t \tag{27}$$

$$K_t = \sum_{s=E+2}^{E+S}\sum_{j=1}^{J}\omega_{s-1,t-1}\lambda_j b_{j,s,t} + \sum_{j=1}^{J}\omega_{E+S,t-1}\lambda_j bq_{j,E+S+1,t} \quad \forall t \tag{28}$$

$$\begin{aligned} &Y_t = C_t + K_{t+1} - (1-\delta)K_t \quad \forall t \\ &\quad\text{where}\quad C_t \equiv \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j c_{j,s,t} \end{aligned} \tag{29}$$

The usual definition of equilibrium would be allocations and prices such that households optimize (20), (21), and (22), firms optimize (25) and (26), and markets clear (27) and (28). However, the variables in these characterizing equations are potentially not stationary due to the possible growth rate in the total population $g_{n,t}$ each period coming from the cohort growth rates in (1) and from the deterministic growth rate of labor augmenting technological change g_y in (23).

Define the following stationary versions of the variables of the model in Table 1 in which the variables are represented in per-capita terms and in which the growth rate from labor augmenting technical change has been removed.

With the definitions in Table 1, it can be shown that the equilibrium characterizing equations can be written in stationary form in the following way. The static and intertemporal

Table 1: Stationary variable definitions

Sources of growth			Not
$e^{g_y t}$	$\tilde{N}_t$	$e^{g_y t}\tilde{N}_t$	growing[a]
$\hat{c}_{j,s,t} \equiv \frac{c_{j,s,t}}{e^{g_y t}}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{\tilde{N}_t}$	$\hat{Y}_t \equiv \frac{Y_t}{e^{g_y t}\tilde{N}_t}$	$n_{j,s,t}$
$\hat{b}_{j,s,t} \equiv \frac{b_{j,s,t}}{e^{g_y t}}$	$\hat{L}_t \equiv \frac{L_t}{\tilde{N}_t}$	$\hat{K}_t \equiv \frac{K_t}{e^{g_y t}\tilde{N}_t}$	r_t
$\hat{bq}_{j,s,t} \equiv \frac{bq_{j,s,t}}{e^{g_y t}}$		$\hat{BQ}_{j,t} \equiv \frac{BQ_{j,t}}{e^{g_y t}\tilde{N}_t}$	
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$			

[a] The interest rate r_t in (26) is already stationary because Y_t and K_t grow at the same rate. Individual labor supply $n_{j,s,t}$ is stationary.

Euler equations from the individual's optimization problem corresponding to (20), (21), and (22) are the following.

$$\begin{aligned}
&\hat{w}_t e_{j,s}\big(\hat{c}_{j,s,t}\big)^{-\sigma} = \chi_n(\tilde{l} - n_{j,s,t})^{-\eta} \\
&\qquad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \quad \text{with} \quad \hat{b}_{j,E+1,t}, \hat{b}_{j,E+S+1,t} = 0 \\
&\text{where} \quad \hat{c}_{j,s,t} = (1+r_t)\,\hat{b}_{j,s,t} + \hat{w}_t e_{j,s} n_{j,s,t} + \frac{\hat{BQ}_{j,t}}{\lambda_j} - e^{g_y}\hat{b}_{j,s+1,t+1} \\
&\qquad \text{for} \quad E+1 \leq s \leq E+S-1 \\
&\text{and} \quad \hat{c}_{j,s,t} = (1+r_t)\,\hat{b}_{j,s,t} + \hat{w}_t e_{j,s} n_{j,s,t} + \frac{\hat{BQ}_{j,t}}{\lambda_j} - e^{g_y}\hat{bq}_{j,s+1,t+1} \\
&\qquad \text{for} \quad s = E+S
\end{aligned} \tag{30}$$

$$\big(\hat{c}_{j,s,t}\big)^{-\sigma} = \beta(1-\rho_s)(1+r_{t+1})e^{-\sigma g_y}\big(\hat{c}_{j,s+1,t+1}\big)^{-\sigma} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S-1 \tag{31}$$

$$\big(\hat{c}_{j,E+S,t}\big)^{-\sigma} = \chi_b e^{-\sigma g_y}\big(\hat{bq}_{j,E+S+1,t+1}\big)^{-\sigma} \quad \forall j,t \tag{32}$$

The stationary firm first order conditions for optimal labor and capital demand corresponding to (25) and (26) are the following.

$$\hat{w}_t = (1-\alpha)\frac{\hat{Y}_t}{\hat{L}_t} \quad \forall t \tag{33}$$

$$r_t = \alpha\frac{\hat{Y}_t}{\hat{K}_t} - \delta = \alpha\frac{Y_t}{K_t} - \delta \quad \forall t \tag{26}$$

And the two stationary market clearing conditions corresponding to (27) and (28)—with the goods market clearing by Walras' Law—are the following.

$$\hat{L}_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\hat{\omega}_{s,t}\lambda_j e_{j,s}n_{j,s,t} \quad \forall t \tag{34}$$

$$\hat{K}_t = \frac{1}{1+\tilde{g}_{n,t}}\left(\sum_{s=E+2}^{E+S}\sum_{j=1}^{J}\hat{\omega}_{s-1,t-1}\lambda_j\hat{b}_{j,s,t} + \sum_{j=1}^{J}\hat{\omega}_{E+S,t-1}\lambda_j\hat{bq}_{j,E+S+1,t}\right) \quad \forall t \tag{35}$$

where $\tilde{g}_{n,t}$ is the growth rate in the working age population between periods $t-1$ and t described in (5).

We can now define the stationary steady-state equilibrium for this economy in the following way.

Definition 1 (Stationary steady-state equilibrium). A non-autarkic stationary steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability $e_{j,s}$ is defined as constant allocations $\hat{b}_{j,s+1,t+1} = \bar{b}_{j,s+1}$ and $\hat{n}_{j,s,t} = \bar{n}_{j,s}$ and constant prices $\hat{w}_t = \bar{w}$ and $\hat{r}_t = \bar{r}$ for all j, s, and t such that the following conditions hold:

i. households optimize according to (30), and (31),

ii. firms optimize according to (33) and (26),

iii. markets clear according to (34) and (35), and

iv. the population has reached its stationary steady state distribution $\bar{\omega}_s$ for all ages s, characterized in Appendix A-1.

The steady-state equilibrium is characterized by the system of $J(2S-1)$ equations and $J(2S-1)$ unknowns $\bar{n}_{j,s}$ and $\bar{b}_{j,s+1}$ along with the individual borrowing constraints and aggregate borrowing constraint described in Appendix A-3.

$$\bar{w}e_{j,s}\Big[(1+\bar{r})\,\bar{b}_{j,s}+\bar{w}e_{j,s}\bar{n}_{j,s}-e^{g_y}\bar{b}_{j,s+1}\Big]^{-\sigma}=\chi(\tilde{l}-\bar{n}_{j,s})^{-\eta}$$
$$\forall j \quad \text{and} \quad 1\leq s\leq S \quad \text{with} \quad \bar{b}_{j,1},\bar{b}_{j,S+1}=0 \tag{36}$$

$$\Big[(1+\bar{r})\,\bar{b}_{j,s}+\bar{w}e_{j,s}\bar{n}_{j,s}-e^{g_y}\bar{b}_{j,s+1}\Big]^{-\sigma}=...$$
$$\beta(1+\bar{r})e^{-\sigma g_y}\Big[(1+\bar{r})\,\bar{b}_{j,s+1}+\bar{w}e_{j,s+1}\bar{n}_{j,s+1}-e^{g_y}\bar{b}_{j,s+2}\Big]^{-\sigma} \tag{37}$$
$$\forall j \quad \text{and} \quad 1\leq s\leq S-1 \quad \text{with} \quad \bar{b}_{j,1},\bar{b}_{j,S+1}=0$$

Define $\hat{\mathbf{\Gamma}}_t$ as the distribution of stationary savings across individuals at time t.

$$\hat{\mathbf{\Gamma}}_t\equiv\{\hat{b}_{j,s,t}\}_{j=1,s=2}^{J,S}\quad\forall t \tag{38}$$

In equilibrium, the steady-state individual labor supplies $\bar{n}_{j,s}$ for all j and s, the steady-state real wage $\bar{w}$, and the steady-state real rental rate $\bar{r}$ are simply functions of the steady-state distribution of savings $\bar{\Gamma}$. This is clear from the steady-state version of the capital market clearing condition (35) and the fact that aggregate labor supply is a function of the sum of exogenous efficiency units of labor in the labor market clearing condition (34). And the two firm first order conditions for the real wage $\hat{w}_t$ (33) and real rental rate r_t (26) are only functions of the aggregate capital stock $\hat{K}_t$ and aggregate labor $\hat{L}_t$. Appendix A-4 details how to solve for the steady-state equilibrium.

Figure 1 shows the stationary steady-state distribution of individual savings $\bar{\mathbf{\Gamma}}$ and Figure 2 shows the stationary steady-state distribution of individual labor supply $\bar{n}_{j,s}$ for a particular calibration of the model described in Table 2. Notice ....

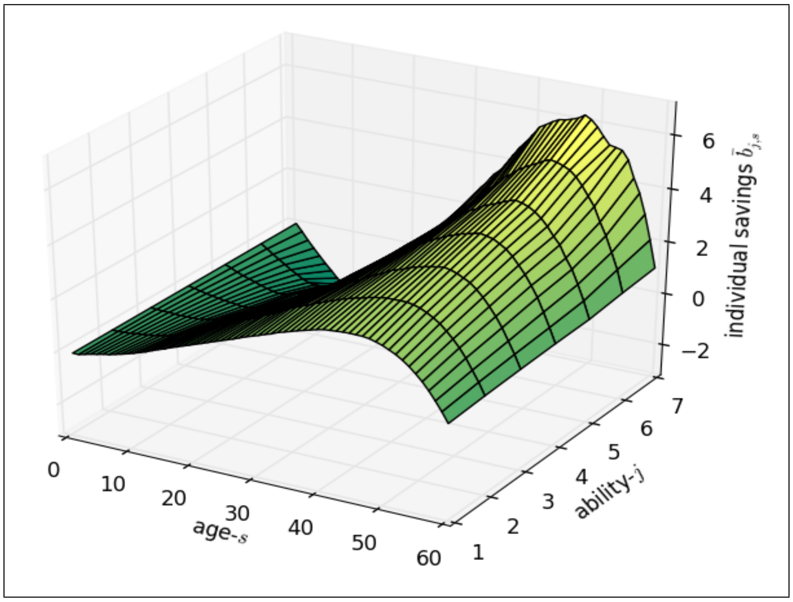
The definition of the stationary non-steady-state equilibrium is similar to Definition 1, with the stationary steady-state equilibrium definition being a special case of the stationary non-steady-state equilibrium.

Table 2: List of exogenous variables and calibration values

Symbol	Description	Value
$\hat{\mathbf{\Gamma}}_1$	Initial distribution of savings	$0.9\bar{\mathbf{\Gamma}}$
$\{\omega_{s,1}\}_{s=1}^{S}$	Initial population by age	(see App. A-1)
$\{f_s\}_{s=1}^{S}$	Fertility rates by age	(see App. A-1)
$\{i_s\}_{s=1}^{S}$	Immigration rates by age	(see App. A-1)
$\{\rho_s\}_{s=1}^{S}$	Mortality rates by age	(see App. A-1)
$\{e_{j,s}\}_{j,s=1}^{J,S}$	Deterministic ability process	(see App. A-2)
S	Periods in individual life	60
J	Number of ability types	7
$\tilde{l}$	Maximum hours of labor supply	1
β	Discount factor	$(0.96)^{\frac{60}{S}}$
σ	Coefficient of constant relative risk aversion	3
χ	Disutility of labor level parameter	1
η	Proportional to elasticity of labor supply	2.5
A	Level parameter in production function	1
α	Capital share of income in production function	0.35
δ	Capital depreciation rate	$1-(1-0.05)^{\frac{60}{S}}$
g_y	Growth rate of labor augmenting technological progress	$(1+0.03)^{\frac{60}{S}}-1$
T	Number of periods to steady state	120
ν	Dampening parameter for TPI	0.2

Note: Maybe put sources here.

Figure 1: Stationary steady-state distribution of savings $\bar{\Gamma}$ for $S = 60$ and $J = 7$



Definition 2 (**Stationary non-steady-state equilibrium**). A non-autarkic stationary non-steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability $e_{j,s}$ is defined as allocations $n_{j,s,t}$ and $\hat{b}_{j,s+1,t+1}$ and prices $\hat{w}_t$ and r_t for all j, s, and t such that the following conditions hold:

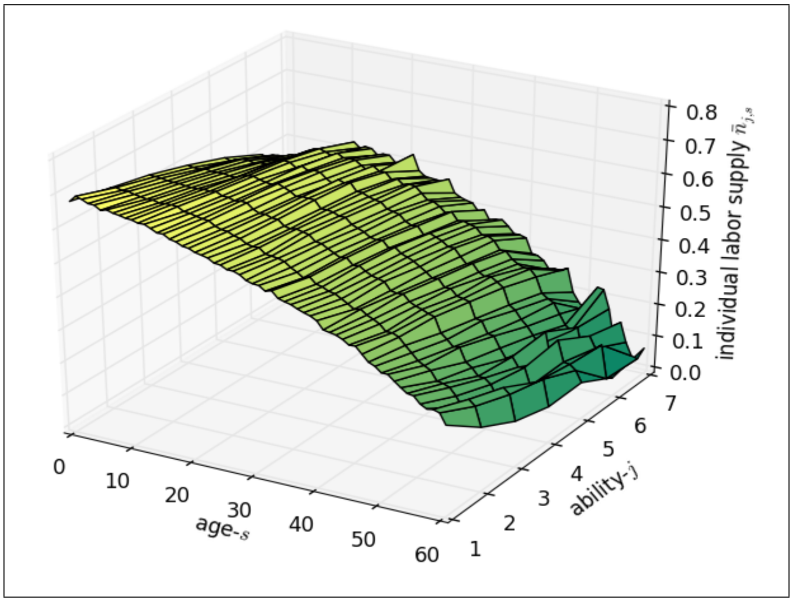
i. households optimize according to (30), and (31),

ii. firms optimize according to (33) and (26), and

iii. markets clear according to (34) and (35).

The household labor-leisure decision in the last period of life shows that the optimal labor supply for age $s = S$ is a function of individual holdings of savings and the prices in that period $n_{j,S,t} = \phi\big(\hat{b}_{j,S,t}, \hat{w}_t, r_t\big)$. This decision is characterized by final-age version of thet static Euler equation (30). Households in their second-to-last period of life in period t have three decisions to make. They must choose how much to work this period $n_{j,S-1,t}$ and next $n_{j,S,t+1}$ and how much to save this period for next period $\hat{b}_{j,S,t+1}$. The optimal responses for this individual are characterized by the $s = S-1$ and $s = S$ versions of the static Euler equations (30) and the $s = S-1$ version of the intertemporal Euler equation (31), respectively.

Figure 2: Stationary steady-state distribution of individual labor supply $\bar{n}_{j,s}$ for $S = 60$ and $J = 7$



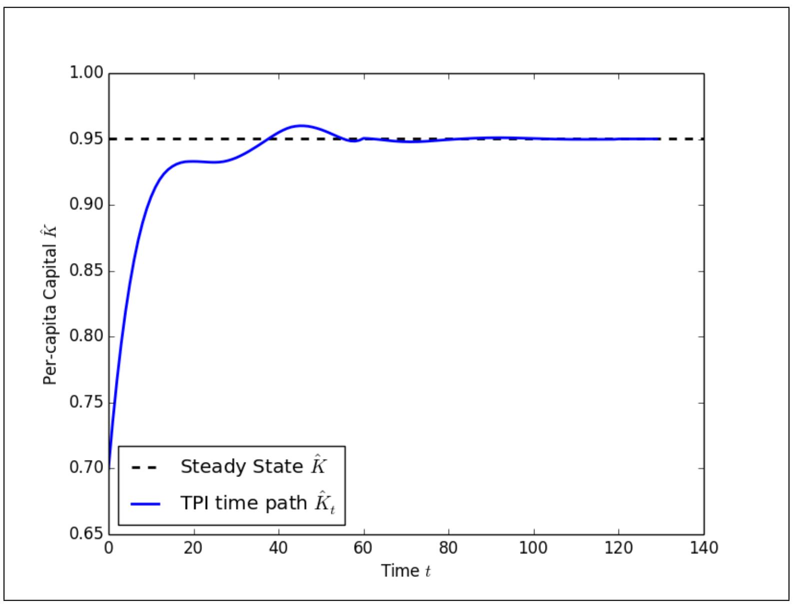
Optimal savings in the second-to-last period of life $s = S - 1$ is a function of the current savings and the prices in the current period and in the next period $\hat{b}_{j,S,t+1} = \psi\big(\hat{b}_{j,S-1,t}, \hat{w}_t, r_t, \hat{w}_{t+1}, r_{t+1}\big)$. As before, the optimal labor supply at age $s = S$ is a function of the next period's savings and prices $n_{j,S,t+1} = \phi\big(\hat{b}_{j,S,t+1}, \hat{w}_{t+1}, r_{t+1}\big)$. But the optimal labor supply at age $s = S - 1$ is a function of the current savings and the current prices as well as the future prices because of the dependence on the savings decision in that same period $n_{j,S-1,t} = \phi\big(\hat{b}_{j,S-1,t}, \hat{w}_t, r_t, \hat{w}_{t+1}, r_{t+1}\big)$. By induction, we can show that the optimal labor supply and savings functions for any individual with ability j, age s, and in period t is a function of current holdings of savings and the lifetime path of prices.

$$n_{j,s,t} = \phi\Big(\hat{b}_{j,s,t}, (\hat{w}_v, r_v)_{v=t}^{t+S-s}\Big) \quad \forall j,s,t \tag{39}$$

$$\hat{b}_{j,s+1,t+1} = \psi\Big(\hat{b}_{j,s,t}, (\hat{w}_v, r_v)_{v=t}^{t+S-s}\Big) \quad \forall j,t \quad \text{and} \quad 1 \leq s \leq S-1 \tag{40}$$

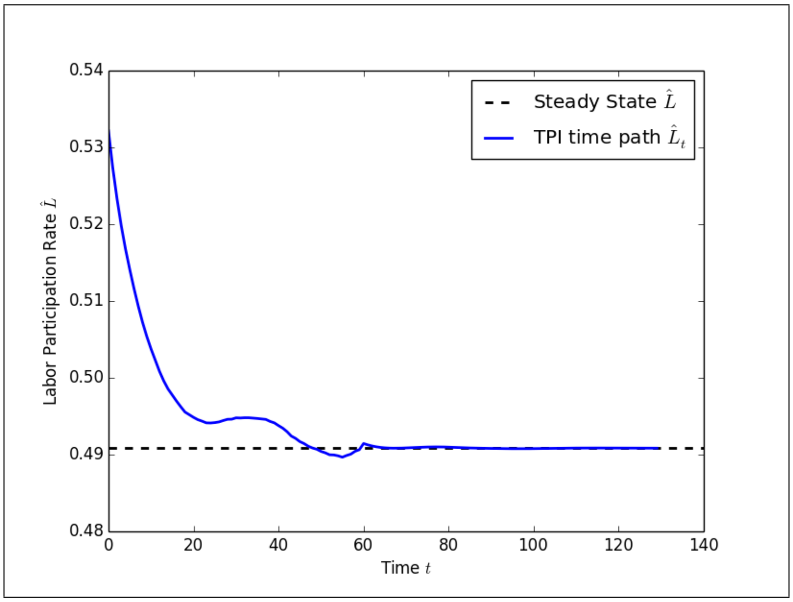
Each optimal saving decision for each household requires knowledge of at least todays prices and tomorrows prices and at most S periods of prices. In equilibrium, one can see that the prices $(\hat{w}_t, r_t)$ in each period t are functions of the entire distribution of savings $\mathbf{\Gamma}_t$. The requirement that individuals be able to forecast prices with perfect foresight over their lifetimes implies that each individual has correct information and beliefs about all the other individuals optimization problems and information. It also implies that the equilibrium allocations and prices are really just functions of the entire distribution of savings at a particular period, as well as a law of motion for that distribution of savings.

Figure 3: Equilibrium time path of K_t for $S = 60$ and $J = 7$



To solve for any non-steady-state equilibrium time path of the economy from an arbitrary current state to the steady state, we follow the time path iteration (TPI) method of Auerbach and Kotlikoff (1987). Appendix A-5 details how to solve for the non-steady-state equilibrium time path using the TPI method. Figure 3 shows the equilibrium time path of the aggregate capital stock for the calibration used in Figure 3 for $T = 80$ periods starting from an initial distribution of savings in which $b_{j,s,1} = (0.9\bar{K})/[(S-1)J]$ for all j and s. We used $\nu = 0.2$ as our time-path updating dampening parameter (see Equation (A.5.8) in Appendix A-5.)

Figure 4: Equilibrium time path of L_t for $S = 60$ and $J = 7$



APPENDIX

A-1 Characteristics of exogenous population growth assumptions

In this appendix, we describe in detail the exogenous population growth assumptions in the model and their implications. In Section 2.1, we define the laws of motion for the population of each cohort $\omega_{s,t}$ to be the following.

$$
\begin{aligned}
\omega_{1,t+1} &= \sum_{s=1}^{E+S} f_s\omega_{s,t} \quad \forall t \\
\omega_{s+1,t+1} &= (1 + i_s - \rho_s)\omega_{s,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1
\end{aligned} \tag{1}
$$

We can transform the nonstationary equations in (1) into stationary laws of motion by dividing both sides by the total populations N_t and N_{t+1} in both periods,

$$
\begin{aligned}
\hat{\omega}_{1,t+1} &= \frac{\sum_{s=1}^{E+S} f_s\hat{\omega}_{s,t}}{1+g_{n,t+1}} \quad \forall t \\
\hat{\omega}_{s+1,t+1} &= \frac{(1+\phi_s-\rho_s)\hat{\omega}_{s,t}}{1+g_{n,t+1}} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1
\end{aligned} \tag{A.1.1}
$$

where $\hat{\omega}_{s,t}$ is the percent of the total population in age cohort s and the population growth rate $g_{n,t+1}$ between periods t and $t+1$ is defined in (3),

$$
\begin{bmatrix} \hat{\omega}_{1,t+1} \\ \hat{\omega}_{2,t+1} \\ \hat{\omega}_{2,t+1} \\ \vdots \\ \hat{\omega}_{E+S-1,t+1} \\ \hat{\omega}_{E+S,t+1} \end{bmatrix} = \frac{1}{1+g_{n,t+1}} \begin{bmatrix} f_1 & f_2 & f_3 & \dots & f_{E+S-1} & f_{E+S} \\ 1+i_1-\rho_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1+i_2-\rho_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1+i_3-\rho_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1+i_{E+S-1}-\rho_{E+S-1} & 0 \end{bmatrix} \begin{bmatrix} \hat{\omega}_{1,t} \\ \hat{\omega}_{2,t} \\ \hat{\omega}_{2,t} \\ \vdots \\ \hat{\omega}_{E+S-1,t} \\ \hat{\omega}_{E+S,t} \end{bmatrix} \tag{A.1.2}
$$

where we restrict $1 + i_s - \rho_s \geq 0$ for all s.

We write (A.1.2) in matrix notation as the following.

$$
\boldsymbol{\hat{\omega}}_{t+1} = \frac{1}{1+g_{n,t+1}}\boldsymbol{\Omega}\boldsymbol{\hat{\omega}}_t \quad \forall t \tag{A.1.3}
$$

The stationary steady state population distribution $\boldsymbol{\bar{\omega}}$ is the eigenvector $\boldsymbol{\omega}$ with eigenvalue $(1 + \bar{g}_n)$ of the matrix $\boldsymbol{\Omega}$ that satisfies the following version of (A.1.3).

$$
(1+\bar{g}_n)\boldsymbol{\bar{\omega}} = \boldsymbol{\Omega}\boldsymbol{\bar{\omega}} \tag{A.1.4}
$$

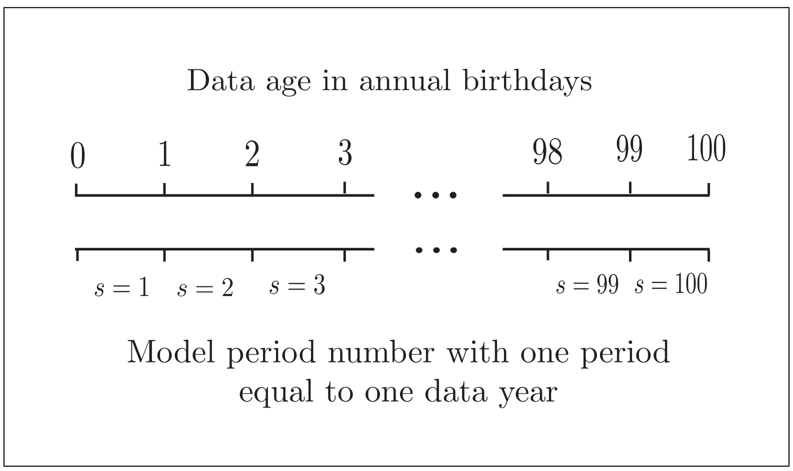
TODO:

- We need to show the conditions under which the matrix $\boldsymbol{\Omega}$ has only one eigenvector associated with one positive eigen value with no complex part.

- Another approach is to simply simulate the problem from the initial population distribution $\boldsymbol{\omega}_0$ and what the steady state $\bar{\boldsymbol{\omega}}$ is and how many periods it takes to get there.
  - We can use the number of periods to arrive at the steady state as a lower bound for T in the time path iteration algorithm.

Because the population growth process is exogenous to the model, we calibrate it to annual age data for age years $s = 1$ to $s = 100$. As is shown in Figure 5, period $s = 1$ corresponds to the first year of life between birth and when an individual turns one year old.

Figure 5: Correspondence of model timing to data timing for model periods of one year



Our initial population distribution $\{\omega_{s,1}\}_{s=1}^{100}$ in Figure 6 comes from Census Bureau (2014) population estimates for both sexes for 2013. The fertility rates $\{f_s\}_{s=1}^{100}$ in Figure 7 come from **?**, Table 1. The mortality rates $\{\rho_s\}_{s=1}^{99}$ in Figure 8 come from the 2010 death probabilities in Social Security Administration (2010). We enforce a strict mortality rate of $\rho_{100} = 1$ in our model.

The immigration rates $\{i_s\}_{s=1}^{99}$ in Figure 9 are essentially residuals. We take total population for two consecutive years N_t and N_{t+1} and the population distribution by age in both of those years $\boldsymbol{\omega}_t$ and $\boldsymbol{\omega}_{t+1}$from the Census Bureau (2014) data. We then deduce the immigration rates $\{i_s\}_{s=1}^{99}$ using equation (A.1.1). We do this for three consecutive sets of years, so that our calibrated immigration rates by age are the average of our three years of deduced rates from the data for each age.

Figure 10 shows the predicted time path of the total population N_t given $\omega_{s,1}$ f_s, i_s, and ρ_s. Notice that the population approaches a constant growth rate. This is a

Figure 6: Initial population distribution $\omega_{s,1}$ by year, $1 \leq s \leq 100$

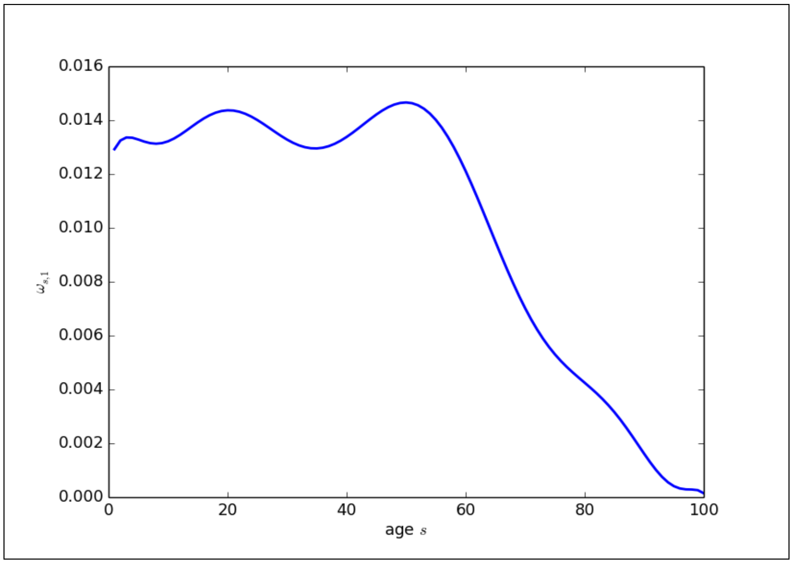


Figure 7: Fertility rates f_s by year, $1 \leq s \leq 100$

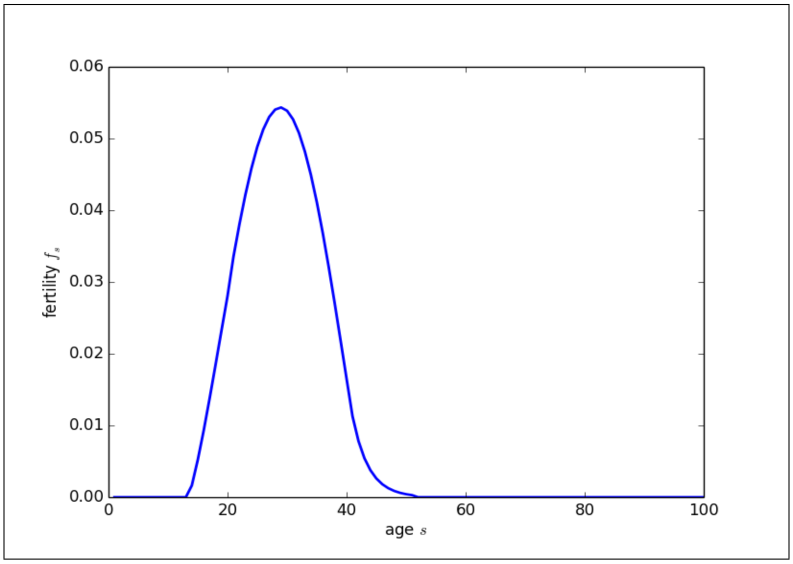


Figure 8: Mortality rates ρ_s by year, $1 \leq s \leq 100$

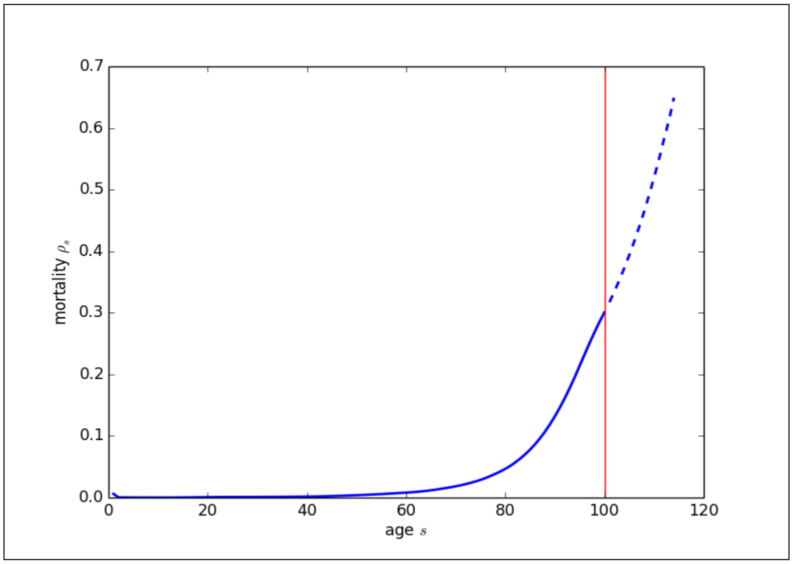
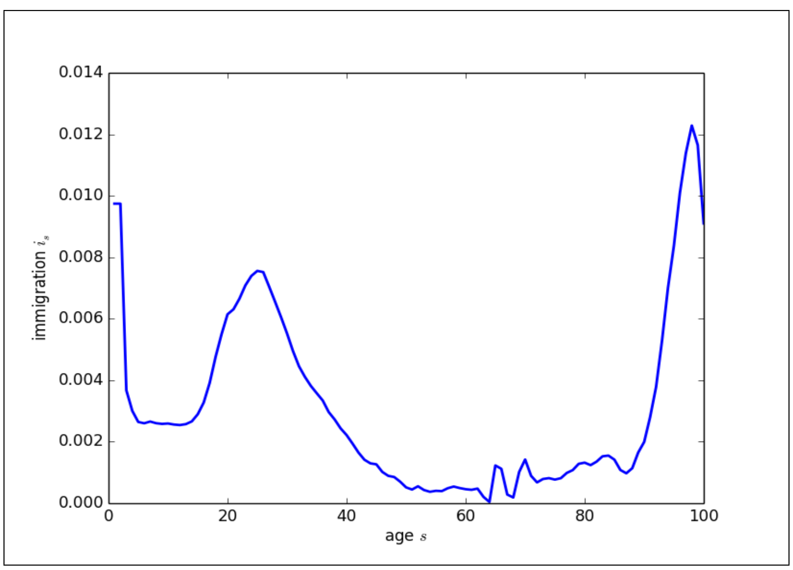


Figure 9: Immigration rates i_s by year, $1 \leq s \leq 100$



result of the stationary population percent distribution $\bar{\boldsymbol{\omega}}$ eventually being reached. Figure 11 shows the steady-state population percent distribution by age $\bar{\boldsymbol{\omega}}$.

Figure 10: Forecast time path of population growth rate $g_{n,t}$

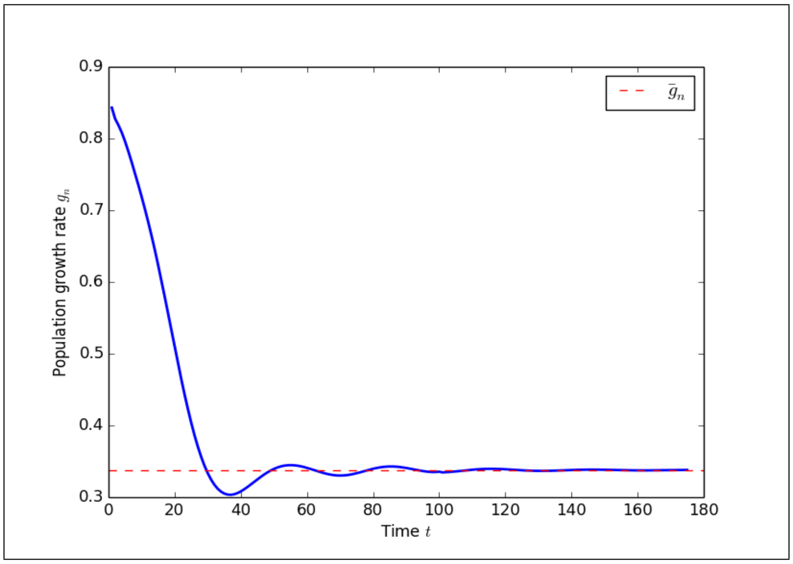
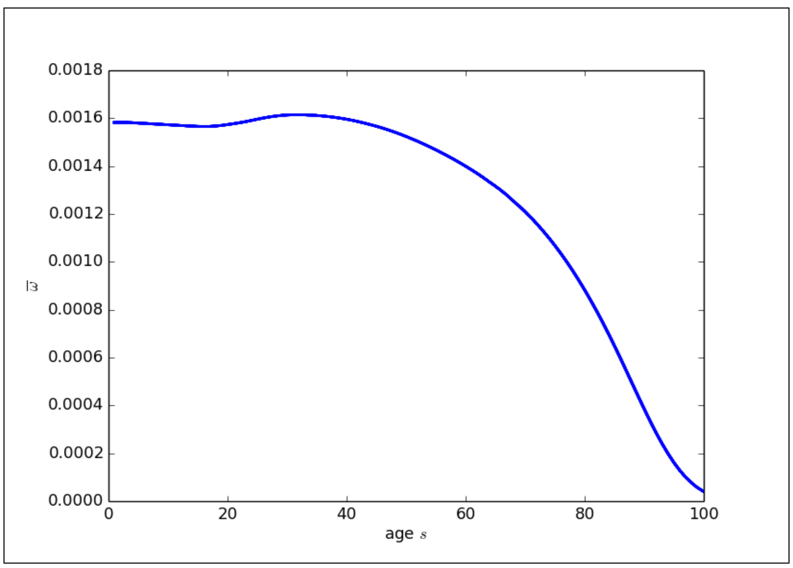


Figure 11: Steady-state population percent distribution by age $\bar{\omega}$

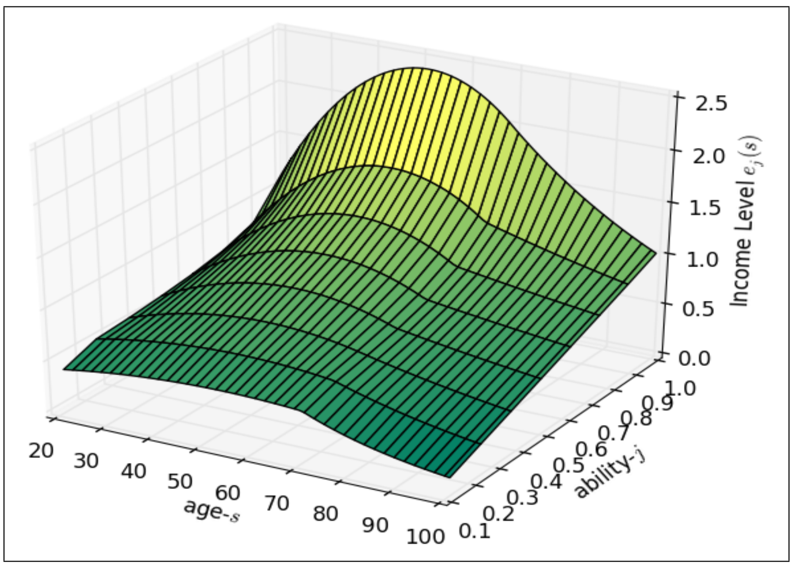


A-2 Calibration of ability process

The calibration of the ability process $e_{j,s}$ is as follows. First, the ability types themselves must be calibrated. For each age group $s \in S$, the hourly wage rates are sorted into J percentile groups. The ability type for each percentile group is the mean wage for the percentile group, normalized by the average wage of all individuals in the data set.

The data used to calibrate the ability types were obtained from the Current Population Survey.[9] A polynomial is fit to the wage data for individuals age 20 to 70. For lack of hourly wage data from the elderly, wages for individuals age 71 to 100 are extrapolated using a simple exponential function. This was done for each percentile group. Due to a limited number of observations in the survey who included their hourly wage, data was taken from the months of January, February, March, April, and May 2014. Population weights were also used to obtain the correct percentile groups of individuals. The income levels for the J ability types were then calculated for each month, and then an average was taken of the five calibrations of the ability types in order to produce a final calibration to be used in the model. Figure 12 shows this income distribution across age and ability type.

Figure 12: Distribution of Income where $S = 80$ and $J = 7$



In this paper, individuals are assigned ability types at the beginning of their life, and cannot change types later on.

[9]U.S. Census Bureau, Dataferret, Current Population Survey, 2014. The variables $PRTAGE$, $PTERNHLY$, and $PWCMPWGT$ were used for the age, hourly wage rate, and population weight of individuals, respectively.

A-3 Stationary constraints on individual borrowing

As described in Section 2.1, individuals are allowed to borrow $b_{j,s,t}$ for some j and s in period t. However, two constraints must hold. First, the individual must be able to pay back the balance with interest r_{t+1} in the next period without driving consumption in the next period $c_{j,s+1,t+1}$ to be nonpositive. Let $\tilde{b}_{j,s,t}$ be the minimum stationary value of savings in a period.

$$\hat{b}_{j,s,t} \geq \tilde{b}_{j,s,t} \quad \forall j, s, t \tag{A.3.1}$$

Rearranging the stationary versions of the bugdet constraints in (8), (9), and (10) and using backward induction gives the following expressions for $\tilde{b}_{j,s,t}$,

$$\begin{aligned}
\tilde{b}_{j,S,t} &= \frac{\tilde{c} - \hat{w}_t e_{j,S}\tilde{l}}{1 + r_t} \\
\tilde{b}_{j,S-1,t-1} &= \frac{\tilde{c} + e^{g_y}\tilde{b}_{j,S,t} - \hat{w}_{t-1}e_{j,S-1}\tilde{l}}{1 + r_{t-1}} \\
&\vdots \\
\tilde{b}_{j,2,t-S+2} &= \frac{\tilde{c} + e^{g_y}\tilde{b}_{j,3,t-S+3} - \hat{w}_{t-S+2}e_{j,2}\tilde{l}}{1 + r_{t-S+2}}
\end{aligned} \tag{A.3.2}$$

where $\tilde{c} > 0$ is some minimum amount of stationary consumption and $\tilde{l}$ is the maximum amount an individual can work from the time constraint (6). With endogenous labor supply $n_{j,s,t}$, it is less likely that the individual borrowing constraints ever bind. This is because the disutility of labor increases exponentially according to $\eta > 1$ in the period utility function (7).

In addition to the individual borrowing constraint (A.3.1), a strict aggregate borrowing constraint must be met. That is, the stationary aggregate capital stock must be strictly positive.

$$\hat{K}_t > 0 \quad \forall t \tag{A.3.3}$$

A-4 Solving for stationary steady-state equilibrium

This section describes the solution method for the stationary steady-state equilibrium described in Definition 1.

1. Use the techniques in Appendix A-1 to solve for the steady-state population distribution vector $\bar{\boldsymbol{\omega}}$ of the exogenous population process.
2. Choose an initial guess for the stationary steady-state distribution of capital $\bar{b}_{j,s+1}$ for all j and $s = 1, 2, ...S-1$ and labor supply $\bar{n}_{j,s}$ for all j and s.
   - A good first guess is a large positive number for all the $\bar{n}_{j,s}$ that is slightly less than $\tilde{l}$ and to choose some small positive number that is small enough to be less than the minimum income that an individual might have $\bar{w}e_{j,s}\bar{n}_{j,s}$.
3. Perform a constrained root finder that chooses $\bar{b}_{j,s+1}$ and $\bar{n}_{j,s}$ that solves the $J(2S-1)$ stationary steady-state Euler equations (36) and (37).
4. Make sure none of the implied steady-state consumptions $\bar{c}_{j,s}$ is less-than-or-equal-to zero.
   - If one consumption is less-than-or-equal-to zero $\bar{c}_{j,s} \leq 0$, then try different starting values.
5. Make sure that none of the Euler errors is too large in absolute value for interior stationary steady-state values. A steady-state Euler error is the following, which is supposed to be close to zero for all j and $s = 1, 2, ...S-1$:

$$\frac{\beta\left(1+\bar{r}\right)\left(\bar{c}_{j,s+1}\right)^{-\sigma}}{\left(\bar{c}_{j,s}\right)^{-\sigma}} - 1 \tag{A.4.1}$$

6. Make sure that the unconstrained solution satisfies the individual borrowing constraints in (A.3.1) and (A.3.2).
   - If any individual's borrowing constraint is not satisfied using the unconstrained root finding operation, rerun the root finding operation in step (3) as a constrained minimization problem with the borrowing constraints imposed on those individuals.
   - Repeat steps (3) through (6) until all the individual borrowing constraints are met.
7. Make sure that the solution satisfies the aggregate borrowing constraint (A.3.3).
   - If it does not, what is the least distortionary upward adjustment to individual steady-state savings $\bar{b}_{j,s+1}$?

A-5 Solving for stationary non-steady-state equilibrium by time path iteration

This section outlines the benchmark time path iteration (TPI) method of Auerbach and Kotlikoff (1987) for solving the stationary non-steady-state equilibrium transition path of the distribution of savings. TPI finds a fixed point for the transition path of the distribution of capital for a given initial state of the distribution of capital. The idea is that the economy is infinitely lived, even though the agents that make up the economy are not. Rather than recursively solving for equilibrium policy functions by iterating on individual value functions, one must recursively solve for the policy functions by iterating on the entire transition path of the endogenous objects in the economy (see Stokey and Lucas (1989, ch. 17)).

The key assumption is that the economy will reach the steady-state equilibrium described in Definition 1 in a finite number of periods $T < \infty$ regardless of the initial state. Let $\hat{\mathbf{\Gamma}}_t$ represent the distribution of savings at time t.

$$\hat{\mathbf{\Gamma}}_t \equiv \{\hat{b}_{j,s,t}\}_{j=1,s=1}^{J,S} \quad \forall t \tag{38}$$

In Section 2.3, we describe how the stationary non-steady-state equilibrium time path of allocations and price is described by functions of the state $\hat{\mathbf{\Gamma}}_t$ and its law of motion. TPI starts the economy at any initial distribution of savings $\hat{\mathbf{\Gamma}}_1$ and solves for its equilibrium time path over T periods to the steady-state distribution $\bar{\mathbf{\Gamma}}_T$.

The first step is to assume an initial transition path for aggregate stationary capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$ and aggregate stationary labor $\hat{\boldsymbol{L}}^i = \left\{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\right\}$ such that T is sufficiently large to ensure that $\hat{\mathbf{\Gamma}}_T = \bar{\mathbf{\Gamma}}$, $\hat{K}_T^i\left(\mathbf{\Gamma}_T\right)$, and $\hat{L}_T^i\left(\mathbf{\Gamma}_T\right) = \bar{L}\left(\bar{\mathbf{\Gamma}}\right)$ for all $t \geq T$. The superscript i is an index for the iteration number. The transition paths for aggregate capital and aggregate labor determine the transition paths for both the real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$ and the real return on investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$.

The exact initial distribution of capital in the first period $\hat{\mathbf{\Gamma}}_1$ can be arbitrarily chosen as long as it satisfies $\hat{K}_1^i = \frac{1}{J}\sum_{s=1}^{S}\sum_{j=1}^{J}\hat{\omega}_{s,1}\hat{b}_{j,s,1}$ according to market clearing condition (35). However, this is not the case with $\hat{L}_1^i$. Its value will be endogenously determined in the same way the K_2^i is. For this reason, a logical initial guess for the time path of aggregate labor is the steady state in every period $L_t^1 = \bar{L}$ for all $1 \leq t \leq T$. For the initial guess of the stationary aggregate capital stock in the first period, one could also first choose the initial distribution of savings $\hat{\mathbf{\Gamma}}_1$ and then choose an initial aggregate capital stock $\hat{K}_1^i$ that corresponds to that distribution. As mentioned earlier, the only other restriction on the initial transition path for aggregate capital is that it equal the steady-state level $\hat{K}_T^i = \bar{K}\left(\bar{\mathbf{\Gamma}}\right)$ by period T. Evans and Phillips (2014) have shown that the initial guess for the aggregate capital stocks $\hat{K}_t^i$ for periods $1 < t < T$ can take on almost any positive values.

Given the initial capital distribution $\hat{\mathbf{\Gamma}}_1$ and the transition paths of aggregate capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$ and aggregate labor $\hat{\boldsymbol{L}}^i = \left\{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\right\}$, the real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$, and the real return to savings $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, one can solve for the optimal labor supply for each type j of $s = S$-aged agents in the last period of their lives $n_{j,S,1} = \phi_{j,S}(\hat{b}_{j,S,1}, \hat{w}_1, r_1)$ using his one static Euler equation, which is the $s = S$ version of (30).

$$\hat{w}_1^i e_{j,S}\Big[\left(1+r_1^i\right)\hat{b}_{j,S,1} + \hat{w}_1^i e_{j,S} n_{j,S,1}\Big]^{-\sigma} = \chi(\tilde{l} - n_{j,S,1})^{-\eta} \tag{A.5.1}$$

We then solve the problem for all j types of $S-1$-aged individuals in period $t = 1$, each of which entails labor supply decisions in the current period $n_{j,S-1,1}$ and in the next period $n_{j,S,2}$ and a savings decision in the current period for the next period $b_{j,S,2}$. The labor supply decision in the initial period and the savings period in the initial period for the next period for each type j of $S-1$-aged individuals are policy functions of the current savings and the prices in this period and the next $\hat{b}_{j,S,2} = \psi_{j,S-1}(\hat{b}_{j,S-1,1}, \{\hat{w}_t, r_t\}_{t=1}^2)$ and $\hat{n}_{j,S-1,1} = \phi_{j,S-1}(\hat{b}_{j,S-1,1}, \{\hat{w}_t, r_t\}_{t=1}^2)$. The labor supply decision in the next period is simply a function of the savings and prices in that period $\hat{n}_{j,S,2} = \phi_{j,S}(\hat{b}_{j,S,2}, \hat{w}_2, r_2)$. These three functions are characterized by the following versions of equations (30) and (31).

$$\begin{aligned} \hat{w}_1^i e_{j,S-1}\Big[\left(1+r_1^i\right)\hat{b}_{j,S-1,1} + \hat{w}_1^i e_{j,S-1} n_{j,S-1,1} - e^{g_y}\hat{b}_{j,S,2}\Big]^{-\sigma} = ... \\ \chi(\tilde{l} - n_{j,S-1,1})^{-\eta} \end{aligned} \tag{A.5.2}$$

$$\hat{w}_2^i e_{j,S}\Big[\left(1+r_2^i\right)\hat{b}_{j,S,2} + \hat{w}_2^i e_{j,S} n_{j,S,2}\Big]^{-\sigma} = \chi(\tilde{l} - n_{j,S,2})^{-\eta} \tag{A.5.3}$$

$$\begin{aligned} \Big[\left(1+r_1^i\right)\hat{b}_{j,S-1,1} + \hat{w}_1^i e_{j,S-1} n_{j,S-1,1} - e^{g_y} b_{j,S,2}\Big]^{-\sigma} = ... \\ \beta(1+r_2^i)e^{-\sigma g_y}\Big[\left(1+r_2^i\right)\hat{b}_{j,S,2} + \hat{w}_2^i e_{j,S} n_{j,S,2}\Big]^{-\sigma} \quad \forall j \end{aligned} \tag{A.5.4}$$

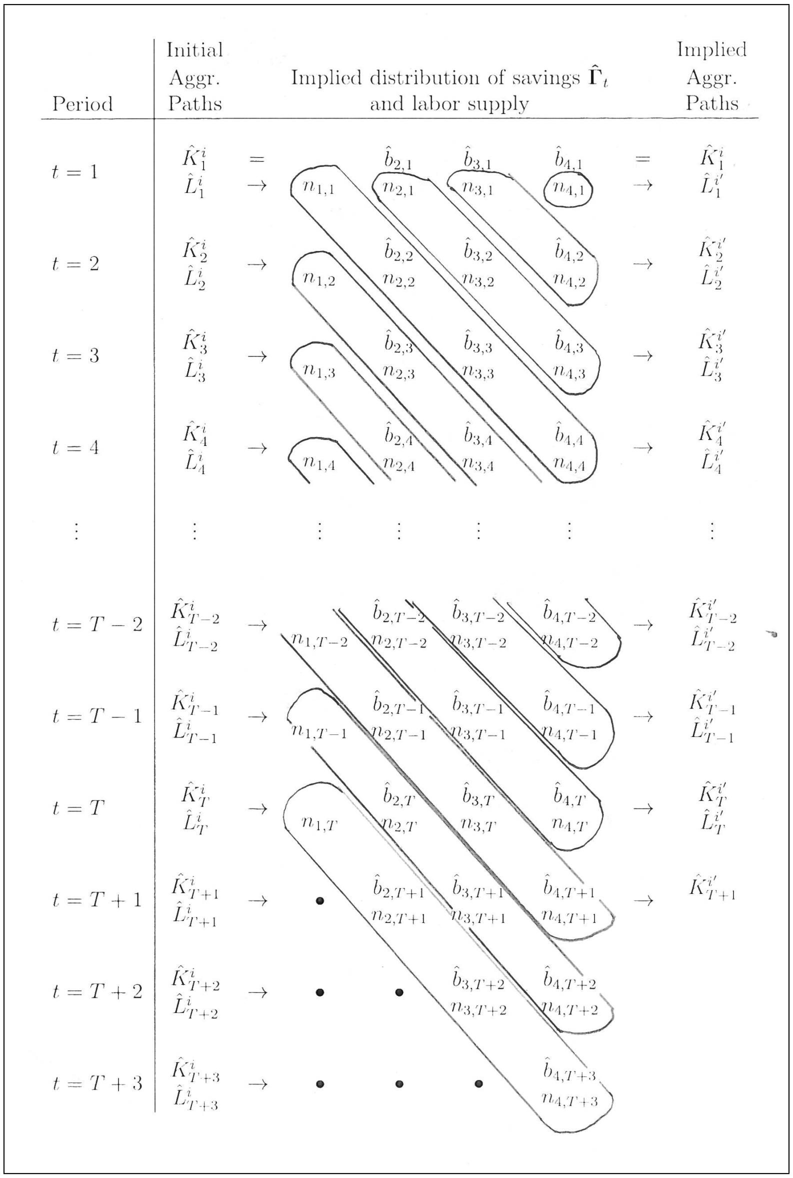
This process is repeated for every age of household alive in $t = 1$ down to the age $s = 1$ household at time $t = 1$. Each of these households j solve the full set of $S-1$ savings decisions and S labor supply decisions characterized by the following full set of Euler equations analogous to (30) and (31).

$$\begin{aligned} \hat{w}_t^i e_{j,s}\Big[\left(1+r_t^i\right)\hat{b}_{j,s,t} + \hat{w}_t^i e_{j,s} n_{j,s,t} - e^{g_y}\hat{b}_{j,s+1,t+1}\Big]^{-\sigma} = \chi(\tilde{l} - n_{j,s,t})^{-\eta} \\ \forall j \quad \text{and} \quad 1 \leq s = t \leq S \quad \text{with} \quad \hat{b}_{j,1,1}, \hat{b}_{j,S+1,S+1} = 0 \end{aligned} \tag{A.5.5}$$

$$\begin{aligned} \Big[\left(1+r_t^i\right)\hat{b}_{j,s,t} + \hat{w}_t^i e_{j,s} n_{j,s,t} - e^{g_y} b_{j,s+1,t+1}\Big]^{-\sigma} = ... \\ \beta(1+r_{t+1}^i)e^{-\sigma g_y}\Big[\left(1+r_{t+1}^i\right)\hat{b}_{j,s+1,t+1} + \hat{w}_{t+1}^i e_{j,s+1} n_{j,s+1,t+1} - e^{g_y}\hat{b}_{j,s+2,t+2}\Big]^{-\sigma} \\ \forall j \quad \text{and} \quad 1 \leq s = t \leq S-1 \quad \text{with} \quad \hat{b}_{j,1,1}, \hat{b}_{j,S+1,S+1} = 0 \end{aligned} \tag{A.5.6}$$

We can then solve for the entire lifetime of savings decisions for each age $s = 1$ individual in periods $t = 2, 3, ...T$. The central part of the schematic diagram in Figure 13 shows how this process is done in order to solve for the equilibrium time path of the economy from period $t = 1$ to T. Note that for each full lifetime savings path solved for an individual born in period $t \geq 2$, we can solve for the aggregate capital stock

Figure 13: Diagram of TPI solution method within each iteration for $S = 4$ and $J = 1$



implied by those savings decisions $\hat{K}_t^{i'} = \frac{1}{J}\sum_{s=1}^S\sum_{j=1}^J \hat{\omega}_{s,t}\hat{b}_{j,s,t}$ and the aggregate labor implied by those labor supply decisions $\hat{L}_t^{i'} = \frac{1}{J}\sum_{s=1}^S\sum_{j=1}^J \hat{\omega}_{s,t}e_{j,s}n_{j,s,t}$.

Once the set of lifetime saving and labor supply decisions has been computed for all individuals alive in $1 \leq t \leq T$, we use the household decisions to compute a new implied time path of the aggregate capital stock and aggregate labor. The implied paths of the aggregate capital stock $\hat{\boldsymbol{K}}^{i'} = \{\hat{K}_1^i, \hat{K}_2^{i'}, ...\hat{K}_T^{i'}\}$ and aggregate labor $\hat{\boldsymbol{L}}^{i'} = \{\hat{L}_1^i, \hat{L}_2^{i'}, ...\hat{L}_T^{i'}\}$ in general do not equal the initial guessed paths $\hat{\boldsymbol{K}}^i = \{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\}$ and $\hat{\boldsymbol{L}}^i = \{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\}$ used to compute the household savings and labor supply decisions $\hat{\boldsymbol{K}}^{i'} \neq \hat{\boldsymbol{K}}^i$ and $\hat{\boldsymbol{L}}^{i'} \neq \hat{\boldsymbol{L}}^i$.

Let $\|\cdot\|$ be a norm on the space of time paths of the aggregate capital stock $\hat{\boldsymbol{K}} \in \mathcal{K} \subset \mathbb{R}_{++}^T$ and aggregate labor supply $\hat{\boldsymbol{L}} \in \mathcal{L} \subset \mathbb{R}_{++}^T$. Then the fixed point necessary for the equilibrium transition path from Definition 2 has been found when the distance between $\hat{\boldsymbol{K}}^{i'}$ and $\hat{\boldsymbol{K}}^i$ is arbitrarily close to zero.

$$\left\|\left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}\right] - \left[\hat{\boldsymbol{K}}^{i}, \hat{\boldsymbol{L}}^{i}\right]\right\| \leq \varepsilon \quad \text{for} \quad \varepsilon > 0 \tag{A.5.7}$$

If the fixed point has not been found $\left\|\left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}\right] - \left[\hat{\boldsymbol{K}}^{i}, \hat{\boldsymbol{L}}^{i}\right]\right\| > \varepsilon$, then new transition paths for the aggregate capital stock and aggregate labor are generated as a convex combination of $\left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}\right]$ and $\left[\hat{\boldsymbol{K}}^{i}, \hat{\boldsymbol{L}}^{i}\right]$.

$$\begin{aligned} \hat{\boldsymbol{K}}^{i+1} &= \nu\hat{\boldsymbol{K}}^{i'} + (1-\nu)\hat{\boldsymbol{K}}^i \\ \hat{\boldsymbol{L}}^{i+1} &= \nu\hat{\boldsymbol{L}}^{i'} + (1-\nu)\hat{\boldsymbol{L}}^i \end{aligned} \quad \text{for} \quad \nu \in (0,1] \tag{A.5.8}$$

This process is repeated until the initial transition paths for the aggregate capital stock and aggregate labor are consistent with the transition paths implied by those beliefs and household and firm optimization.

In essence, the TPI method iterates on individual beliefs about the time path of prices represented by a time paths for the aggregate capital stock $\hat{\boldsymbol{K}}^i$ and aggregate labor $\hat{\boldsymbol{L}}^i$ until a fixed point in beliefs is found that are consistent with the transition paths implied by optimization based on those beliefs.

The following are the steps for computing a stationary non-steady-state equilibrium time path for the economy.

1. Input all initial parameters. See Table 2.

   (a) The value for T at which the non-steady-state transition path should have converged to the steady state should be at least as large as the number of periods it takes the population to reach its steady state $\bar{\boldsymbol{\omega}}$ as described in Appendix A-1.

2. Choose an initial state of the stationarized aggregate capital stock $\hat{K}_1$. Choose an initial distribution of savings $\hat{\boldsymbol{\Gamma}}_1$ consistent with $\hat{K}_1$ according to (35).

3. Conjecture transition paths for the stationarized aggregate capital stock $\hat{\boldsymbol{K}}^1 = \{\hat{K}_t^1\}_{t=1}^\infty$ and stationarized aggregate labor $\hat{\boldsymbol{L}}^1 = \{\hat{L}_t^1\}_{t=1}^\infty$ where the only requirements are that $\hat{K}_1^i = \frac{1}{J}\sum_{s=1}^S\sum_{j=1}^J \hat{\omega}_{s,1}\hat{b}_{j,s,1}$ for all i is your initial state

and that $\hat{K}^i_t = \bar{K}$ and $\hat{L}^i_t = \bar{L}$ for all $t \geq T$. The conjectured transition paths of the aggregate capital stock $\hat{\boldsymbol{K}}^i$ and aggregate labor $\hat{\boldsymbol{L}}^i$ imply specific transition paths for the real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}^i_t\}_{t=1}^{\infty}$ and the real interest rate $\boldsymbol{r}^i = \{r^i_t\}_{t=1}^{\infty}$ through expressions (33) and (26).

   (a) An intuitive choice for the time path of aggregate labor is the steady-state in every period $\hat{L}^1_t = \bar{L}$ for all t.

4. With the conjectured transition paths $\boldsymbol{w}^i$ and $\boldsymbol{r}^i$, one can solve for the lifetime policy functions of each household alive at time $1 \leq t \leq T$ using the systems of Euler equations of the form (A.5.1) through (A.5.6) and following the diagram in Figure 13.

   (a) Make sure that the individual borrowing constraints (A.3.1) are satisfied for each individual in every period.

   (b) Increase any individual savings to the minimum $\tilde{b}_{j,s,t}$ if the borrowing constraint is not satisfied.

5. Use the implied distribution of savings and labor supply in each period (each row of $\hat{b}_{j,s,t}$ and $n_{j,s,t}$ in Figure 13) to compute the new implied time paths for the aggregate capital stock $\hat{\boldsymbol{K}}^{i'} = \{\hat{K}^i_1, \hat{K}^{i'}_2, ...\hat{K}^{i'}_T\}$ and aggregate labor supply $\hat{\boldsymbol{L}}^{i'} = \{\hat{L}^i_1, \hat{L}^{i'}_2, ...\hat{L}^{i'}_T\}$.

   (a) Make sure that the aggregate borrowing constraint (A.3.3) is satisfied in each period t.

   (b) If the aggregate borrowing constraint is not satisfied, increase every individual's savings by the fraction that makes the aggregate capital stock slightly greater than zero.

6. Check the distance between the two sets time paths $\left\| \left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}\right] - \left[\hat{\boldsymbol{K}}^{i}, \hat{\boldsymbol{L}}^{i}\right] \right\|$.

   (a) If the distance between the initial time paths and the implied time paths is less-than-or-equal-to some convergence criterion $\varepsilon > 0$, then the fixed point has been achieved and the equilibrium time path has been found (A.5.7).

   (b) If the distance between the initial time paths and the implied time paths is greater than some convergence criterion $\|\cdot\| > \varepsilon$, then update the guess for the time path of the aggregate capital stock according to (A.5.8) and repeat steps (4) through (6) until a fixed point is reached.

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TECHNICAL APPENDIX

T-1 Comments and Notes

Structures to add to the model and order

1. Add household tax structures
2. Add firm structures
3. Add small open economy feature