The Calibration of CES Production Functions☆

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Abstract

The CES production function is increasingly prominent in macroeconomics and growth economics. This paper distinguishes between different uses of "normalized" CES functions, an approach that has become popular in the literature. The results of Klump and La Grandville (2000) provide a simple way to calibrate the parameters of the CES production function when the necessary data are available. But some of the other applications of normalized CES production functions are problematic, especially when the approach is said to isolate the theoretical effects of varying the elasticity of substitution.

JEL Classifications: D24, *O40*

Keywords: CES production functions, elasticity of substitution, normalization, calibration

1. Introduction

In recent years, the CES production technology has gained much greater prominence in growth economics and macroeconomics. It is the most popular alternative to (and generalization of) the Cobb-Douglas technology, and can be used to address a wider range of issues than Cobb-Douglas. At the same time, it is not always straightforward to justify particular choices for the CES technology parameters, or to examine their implications. Klump and La Grandville

☆I am grateful to Paddy Carter, Olivier de La Grandville, Rosaida Mohd Rasep and an anonymous referee for helpful comments or suggestions, but the usual disclaimer applies.

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(2000) drew attention to this problem, and outlined a procedure for explicitly “normalizing”the production function. Their approach has become popular in the literature, but has sometimes been misused. This paper will distinguish between some of the possible uses of explicit normalization, and seek to clarify when it is useful, and when it may be misleading.

There are several reasons for the increasing popularity of the CES technology. These include the observed time-series and cross-section variation in factor shares in advanced economies, and the tendency for empirical studies using single-country data or microeconomic data to estimate the elasticity of substitution between capital and labor to be well below unity.[1] Some cross-country studies reject a unitary elasticity of substitution (Duffy and Papageorgiou 2000). In growth models, technologies more flexible than Cobb-Douglas are needed for the consideration of varying factor shares, factor-biased technical change, and appropriate technology (for example, Acemoglu 2003, Caselli 2005 and Caselli and Coleman 2006). A non-unitary elasticity also has implications for fiscal policy, as in the Backus et al. (2008) study of the cross-country relationship between capital-output ratios and corporate tax rates. More widely, theorists often consider the implications of variation in the elasticity of substitution, and it has been argued that the CES technology deserves much greater prominence in short-run macroeconomics.[2] Another sign of the renewed interest in the CES technology is that the *Journal of Macroeconomics* devoted a special issue to this production function in June 2008.

Although the CES production technology seems relatively straightforward, its mathematical simplicity can be deceptive. La Grandville (1989, 2009), Klump

[1] On factor shares, see Blanchard (1997), Bentolila and Saint-Paul (2006), and Aiyar and Dalgaard (2009); on estimates see, for example, Antràs (2004); and for surveys, see Chirinko (2008) and Klump et al. (2008, 2011).

[2] Relevant work by theorists includes Lucas (1990), Laitner (1995) and Turnovsky (2002). The potential importance of CES for the analysis of short-run macroeconomics is emphasized by Cantore et al. (2010).

and La Grandville (2000) and Klump et al. (2007, 2008) have emphasized that the economic interpretation of the CES production technology requires care. In particular, they recommend "normalizing" CES technologies when analyzing the theoretical consequences of variation in the elasticity of substitution. The central argument is that variation in the elasticity of substitution can only be isolated by normalization. Since Klump and La Grandville (2000), in particular, it has often been argued that normalization is an essential device for examining the effects of variation in the elasticity of substitution. The normalized CES production function has since been used in theoretical work by Antony (2010), Cantore and Levine (2011), Growiec (2011), Irmen (2011), Miyagiwa and Papageorgiou (2007), Nakamura (2009), Papageorgiou and Saam (2008) and Wong and Yip (2010), among others. It has been adopted as a framework for empirical analysis in Antony (2008), Cantore et al. (2010), Klump et al. (2007, 2008), León-Ledesma et al. (2010a, 2010b) and Mallick (2008). Much of this work is surveyed in Klump et al. (2011), who provide further references and make clear that the relevant literature is now extensive. Several of the papers cited above develop or reinterpret the idea of normalization, notably Cantore and Levine (2011).

This paper will argue that the benefits of explicit normalization have been exaggerated, especially for theoretical work. It will seek to distinguish between instances where it is useful to normalize a CES production function in the way that Klump and La Grandville (2000) and Klump et al. (2011) recommend, and instances where the idea could be misused. These dangers arise not from the logic of the procedure, but from its starting point — the idea, or assumption, that the effects of variation in the elasticity of substitution can be isolated in a meaningful way. As we will see, attempts to achieve this in practice will encounter some deep conceptual problems.

2. Normalization

It may seem odd that there is any normalization issue to raise at all. Perhaps the easiest way to demonstrate the underlying problem is to imagine a productivity comparison between two firms, with production functions $AF(K, L)$ and $BG(K, L)$ respectively. At first glance, the parameters A and B enter the production functions symmetrically and have the same interpretation, as TFP parameters. But since the production technologies differ, a direct comparison of the relative magnitudes of A and B has limited economic meaning. The two are not on the same scale, and the mathematical symmetry is misleading about the economic content of the comparison.

This is a simple illustration of a more general problem, which emerges especially clearly in the CES case. If the elasticity of substitution is allowed to vary, this is rather like moving from one function $F(K, L)$ to another, $G(K, L)$. This raises the issue of whether other technology parameters will retain the same economic interpretation as before, and what it means, in economic terms, to vary the elasticity of substitution while holding other parameters "constant". Different proposals for normalizing the CES technology are different proposals about what, exactly, should be held constant as the elasticity of substitution is varied. Cantore and Levine (2011) provide an especially useful discussion of the problem, emphasizing that technology parameters should be 'dimensionless constants', which are independent of the choice of units. But this, by itself, is not enough to ensure that variation in one parameter leaves the interpretation of others unchanged, as we will see below.

For simplicity, the discussion throughout will assume that there are just two inputs, capital and labor, and constant returns to scale. Most researchers who adopt CES use the standard (ACMS) form, due to Arrow et al. (1961):

$$Y = A\left(bK^{\rho} + (1-b)L^{\rho}\right)^{\frac{1}{\rho}}$$

where Y, K and L are output, capital and labor respectively, and where the elasticity of substitution $\sigma = 1/(1-\rho)$. Much of the discussion that follows centres on assumptions about the TFP parameter A, and the distribution parameter b, for which the admissible range is $0 < b < 1$. La Grandville (1989) and some later authors argue that, when varying the elasticity of substitution, both A and b should be considered functions of the elasticity of substitution. They derive explicit relationships that can be used to normalize the function as the elasticity of substitution varies.

The easiest interpretation of normalization is to view the inputs of capital and labor as index numbers, so that each could be measured relative to arbitrarily-chosen benchmark values. We can then write the CES production function in the "calibrated share form" of Rutherford (1995):

$$Y = Y_0 \left(\pi_0 \left(\frac{K}{K_0} \right)^{\rho} + (1 - \pi_0) \left(\frac{L}{L_0} \right)^{\rho} \right)^{\frac{1}{\rho}} \tag{1}$$

where the parameter π_0 corresponds to the capital share that arises at a benchmark capital-labor ratio K_0/L_0 and output per worker level Y_0/L_0, under perfect competition and marginal productivity factor pricing. The ACMS form can then be seen as normalizing the function so that the distribution parameter b is the capital share that arises when the capital-labor ratio K_0/L_0 is unity. In this sense, normalization is inescapable. The argument given for making the normalization *explicit* is that, in a theoretical analysis, it helps to isolate variation in the elasticity of substitution, independently of variation in other parameters. A tacit assumption, which may be mistaken, is that this isolation is possible.

It is well known that the distribution parameter cannot be defined independently of the units of measurement of capital and labor, but the problem is deeper than this. If we want to study the effect of varying the substitution parameter ρ, the problem is that the function can be normalized using any benchmark capital-labor ratio, and this arbitrary choice will influence how the production surface is reshaped by changing the elasticity of substitution. For example, with the ACMS form, an increase in the elasticity of substitution changes the shape of the isoquants, but the old and new isoquants are tangential at a capital-labor ratio of unity (Kamien and Schwartz 1968, p.12). Not least because we might often interpret the capital-labor ratio as an index number, there is no reason to privilege this capital-labor ratio over others, and therefore no inherent reason to prefer one normalization of the CES production function to another. The calibrated share form (1) at least has the advantage of making the normalization explicit, and implies that the tangency of the old and new isoquants will occur at the benchmark capital-labor ratio K_0/L_0 rather than a capital-labor ratio of unity.

There is considerable room for misunderstanding here, however. In economics, the term 'normalization' is usually used in contexts where the formal properties of a system or model are invariant to the choice of a particular parameter or quantity. A good example arises in trade theory, in the analysis of a two-sector model of a small open economy. A theorist analyzing such a model will typically normalize the price of one good relative to the other. Precisely because many of the properties of the model are invariant to the level of the relative price, the price can be chosen arbitrarily. Alternatively, the units in which the goods are measured can be chosen arbitrarily. All such choices — or at least, all strictly positive and finite numbers — are equally valid, even if some are more convenient than others.

The proposal to normalize the CES production function is not so innocuous. As discussed below, the choice of a benchmark capital-labour ratio will determine how productivity responds to an increase in the elasticity of substitution. If the economy is close to the benchmark position, the change in the elasticity will have little effect on productivity. But this raises an immediate conceptual problem. There is no sense in which, on prior grounds, one choice of normalization or benchmark capital-labour ratio can be seen as more defensible or natural than another. Put differently, there is no obvious sense in which the appropriate normalization point is a fixed property of a given economy; it has no existence, independent of the choice of an observer. Why does this matter? It implies that there is no way to determine the magnitude of the effect of a change in the elasticity of substitution, and sometimes, even the sign will be uncertain. Compared to most instances of normalization in economics, we have half of the idea — that the level of a specific quantity or parameter is arbitrary and can be freely chosen — but not the other half, which is that the formal properties of the model, such as comparative statics, should be invariant to the choice. As we will see below, this poses considerable problems for interpreting theoretical results which seek to make general statements about the effects of varying the elasticity of substitution.

3. The uses of normalization

Before criticizing the idea in more detail, this section discusses some benefits of normalization. Consider the problem faced by a researcher studying the transitional dynamics of a growth model which includes a CES production function, written in the ACMS form. How should the researcher choose the distribution parameter b? Conventionally, it is interpreted as the capital share that would arise when the elasticity of substitution is unity, but this is not much use if the researcher is primarily interested in other cases. More generally, we have seen that the ACMS distribution parameter can be interpreted as the capital share that will arise when the capital-labor ratio is unity. But this relationship is not much help, since the researcher will rarely be able to gauge a sensible magnitude for the capital share at that point.

When the researcher has multiple observations on factor shares and factor ratios, the distribution and substitution parameters can be estimated from the data using standard methods. Since the distribution and substitution parameters are treated as fixed constants to be estimated from the data, no issue of normalization arises. Alternatively, if the researcher has just one observation on the capital share, corresponding to particular (known) values of the capital-output ratio and output per worker, then the following expressions can be used to calibrate the TFP and distribution parameters in the ACMS form:

$$A = ((1-\pi_0)(Y_0/L_0)^{\rho} + \pi_0(Y_0/K_0)^{\rho})^{\frac{1}{\rho}} \tag{2}$$

$$b = \pi_0(Y_0/K_0)^{\rho}A^{-\rho} \tag{3}$$

These expressions are equivalent to alternative forms in Klump and La Grandville (2000) and Klump and Saam (2008), and also imply the calibrated share form

(1). In this light, normalizing the function is no more than using data to pin down key parameters, and is closer to calibration than normalization.[3] The use of the normalized form is a helpful short-cut, but not much more.

In empirical work and policy simulations, the normalized form of the CES production function can sometimes be useful, although the gains are sometimes modest, compared to the claims made for them. Often, the main benefit is to eliminate a line of program code which calculates the distribution parameter from an observed factor share. A more substantial gain can be found in León-Ledesma et al. (2010), where normalization circumvents the need to estimate a distribution parameter, and instead requires the estimated technology to be consistent (at least on average) with an observed value for the the capital share. In its simplest form, this procedure requires the extra assumptions of marginal productivity factor pricing and profit maximization. From a strictly econometric point of view, the gains in their proposed method are likely to arise not chiefly from normalization, but from imposing a parameter rather than estimating it, where the restriction on the parameter is made possible by the extra assumptions.

This paper argues that other uses of normalization risk conceptual problems. This is particularly so, when the normalized form is intended to isolate the effects of variation in the elasticity of substitution. For example, a researcher might want to use simulations to study the transitional dynamics of a neoclassical growth model for the CES case. The problem is how to simulate the growth model for distinct values of the elasticity of substitution, and compare outcomes, in a way that isolates the effect of a changing elasticity. The traditional approach has been to hold the distribution parameter fixed and vary the elasticity of substitution. This will imply different factor shares apply at any given capital-output ratio, and different levels of output per worker apply at any

[3]For more on the relation between normalization and calibration, see Cantore and Levine (2011).

given capital-labor ratio. More fundamentally, however, we have already seen that the economic interpretation of holding the distribution parameter fixed is not straightforward. As indicated above, there are infinitely many ways of normalizing the CES production function, corresponding to the choice of K_0/L_0. The precise choice affects how the production surface is reshaped by variation in the elasticity of substitution.

With this in mind, La Grandville (1989, 2009), Klump and La Grandville (2000) and Klump and Preissler (2000) argue that the choices of the elasticity of substitution, the TFP parameter and the distribution parameter are best seen as interdependent. If the researcher simulating a growth model varies the elasticity of substitution, they should also vary the TFP and distribution parameters. Their recommendation is to express the TFP parameter and the distribution parameter as functions of the elasticity of substitution so that, as the elasticity is varied, the production function always yields the same output per worker and marginal rate of technical substitution at a specific capital-labor ratio. Put differently, the procedure forces production surfaces that differ in the elasticity of substitution to be tangent to one another along a particular ray $K = k_0 L$ where k_0 is a baseline capital-labor ratio. The resulting "normalized" production function can then be written in a number of ways, with equation (8) in Klump and La Grandville (2000) as one of the simplest:

$$\frac{Y}{L} = \frac{Y_0}{K_0}\left(\frac{\pi_0}{\pi}\right)^{\frac{1}{\rho}}\frac{K}{L} \tag{4}$$

using the same notation as before. Note that, if the economy is at its baseline position, then output per worker is invariant to the value of the elasticity of substitution. This point will play a role in the subsequent argument.

4. The misuses of normalization

Explicit normalization can sometimes be a useful step, not least for the reasons explained in Cantore and Levine (2011): we would like our technology parameters to be dimensionless constants, and hence independent of the choice of units. But the literature has often gone well beyond this, to argue that explicit normalization brings major gains for theorists, and is essential to the analysis of changes in the elasticity of substitution. Klump and La Grandville (2000) clearly intend that the normalized CES function should allow comparison between economies that differ "only" in the elasticity of substitution. To what extent is such an aim feasible? It does not seem conceptually straightforward: since the original production function has three parameters, it would be surprising if the variation across technologies could be reduced to a one-dimensional parameter space. In fact, normalization does not achieve this. It replaces the choice of the distribution parameter with an arbitrary choice of a benchmark capital-labor ratio, a quantity that is even harder to interpret.

It may help to see this replacement in action, with a specific example. Klump and Saam (2008) study how the rate of convergence to the steady-state in a growth model is influenced by the elasticity of substitution. In their Table 1, they report a range of values for the convergence rate, which vary not only with the elasticity of substitution but also with the "baseline capital intensity" k_0. They write (p. 258) that "The effect a given rise in the elasticity of substitution has on the speed of convergence depends on the relative magnitude of baseline and steady state capital intensity". The last column of their Table 1 makes clear that a higher elasticity of substitution either lowers or raises the rate of convergence, depending on the baseline capital-labor ratio. Since the latter is necessarily arbitrary — it has no independent existence, and is chosen by the researcher — the economic meaning of this variation is unclear. As far as the effect of the changing elasticity is concerned, even its direction is uncertain.

This is far from an isolated example. Wong and Yip (2010) provide a theoretical analysis of indeterminacy in a model with a production externality, a classic topic in business cycle analysis. They find that indeterminacy holds when the steady-state level of capital is below the baseline level, but not when above it. But this should make us pause: the baseline level has no independent existence, and so the dependence of the results on the economy's position relative to the baseline makes those results hard to interpret.

One element of this argument bears emphasis. A particular source of confusion in the literature is the idea that some choices for the baseline capital-labor ratio are somehow more natural than others, or more likely to be appropriate or relevant. If the baseline had some independent existence, its influence on theoretical results would be much less problematic. In fact, its choice is arbitrary. It is determined by the researcher, rather than a fixed characteristic of the economy. If we think of the baseline simply as that point where output per worker is unaffected by a change in the elasticity of substitution, there is no reason to believe that some candidates for the baseline are necessarily more relevant than others, or that a good choice of baseline has to have some historical or future relevance. After all, if the production surface is somehow reshaped, we might expect the techniques currently in use to be among the most affected, rather than the least, which is what the existing literature has tended to assume.

If this argument is granted, the use of normalization does not eliminate the multi-dimensional nature of the parameter space, and may risk obscuring it. To be more specific, the normalization procedure of Klump and La Grandville (2000) is little help in developing comparative static results for changes in the elasticity of substitution. In their paper, they proved two theorems which essentially state that, for two economies with constant-returns CES production functions "differing only in the elasticity of substitution", and sharing common values for the initial capital-labor ratio, population growth and investment rate, the economy with a higher elasticity of substitution will have a higher level of per capita income at other capital-labor ratios, and will have a higher capital intensity and income per capita in the long-run steady-state. La Grandville (2009) presents simplified proofs of the same results, drawing on the mathematical properties of general means.

Some have taken these results to suggest that the elasticity of substitution can be reinterpreted as an index of technology: as economies become more advanced, perhaps their elasticity of substitution increases, and with it their productivity. If true, this would be a surprising and important claim. It is not guaranteed that labor productivity should be monotonically increasing (or decreasing) in the elasticity of substitution. Arrow et al. (1961) pointed out that the elasticity of substitution may vary with the level of development, but there is no reason to expect productivity to respond in a straightforward way to such a change. Neither is there any reason to expect that a rising elasticity of substitution can be the "engine of growth" implied by Klump and La Grandville (2000, p. 287) or that it would have the powerful benefits for productivity emphasized by La Grandville (2009, chapter 5).

To see why those claims may be misplaced, first consider the simpler case of two countries that differ in their Cobb-Douglas production functions. In country A, output is equal to $Y = AK^{\theta}L^{1-\theta}$ and in country B, output is equal to $Y = BK^{\mu}L^{1-\mu}$ where $\mu \neq \theta$. A quick calculation will show that the two production surfaces intersect: for some capital-labor ratios, output per worker will be higher in country A, and for others, it will be higher in country B. It is easy to show that there will be a threshold value for the capital-labor ratio, with the identity of the more productive country changing as this threshold is crossed. This brief example shows that productivity is not uniformly increasing in the output-capital elasticity.[4]

To illustrate the dangers of normalization, we can take this argument a little further. In the Cobb-Douglas case, output per worker will be invariant to the output-capital elasticity when the capital-labor ratio is unity. An advocate of explicit normalization might then say: why not generalize this in terms of an explicit baseline point, that may differ from unity? A simple argument indicates that the Cobb-Douglas production function could be normalized as

[4]This is not to say that other comparative static results are ruled out. In the Cobb-Douglas version of the Solow model, for example, it is possible to derive the effect of the output-capital elasticity on steady-state output per capita, and on the convergence rate in the vicinity of the steady-state. What the example shows, however, is that simple relationships between productivity levels and specific technology parameters do not always exist.

$$\frac{Y}{L} = y_0 \left(\frac{k}{k_0}\right)^{\theta}$$

where y is output per worker, k is capital per worker, and y_0 and k_0 are these same quantities at a baseline point.[5] In the terminology of some papers on normalization, we appear to have defined a 'family' of Cobb-Douglas production functions without an explicit role for TFP, and which appear to differ only in the output-capital elasticity θ. Yet if we attempt to use the above expression for comparative statics, it indicates that output per worker will either be increasing or decreasing in the output-capital elasticity, depending purely on where the capital-labor ratio stands relative to an arbitrarily chosen baseline point. This is mathematically 'correct' but clearly makes little sense. The observer, in fixing a baseline point, can decide to make the output-capital elasticity either an engine of growth, or a drag on it. This shows the dangers of normalization for comparative statics. The process of normalizing the function obscures the key point, which is that attempting to vary one technology parameter (here, θ) will change the interpretation of another (here, the level of TFP). Making the normalization explicit does not overcome this fundamental objection.

The Cobb-Douglas case is useful because it helps to clarify one reason that confusions over normalization persist. Some of the relevant papers seem animated by the idea that there 'must' be a way of deriving comparative static results for the elasticity of substitution, in a way that isolates it from other aspects of the technology. Yet as we can see, even in a case as simple as Cobb-Douglas, it may not be possible to isolate certain effects of a single parameter in this way. Perhaps wisely, few have tried to make the case that an increasing output-capital elasticity is a general engine of growth.[6] To make progress on this

[5] It is interesting to note that this idea pre-dates the use of normalization for CES, as briefly discussed in Cantore and Levine (2011); see that paper for references.

[6] It is true that, by using a growth accounting perspective and holding the growth rate of the capital-labor ratio fixed, an increase in the elasticity would raise the growth rate for countries converging to a steady-state from below — but not for those where capital-labor ratios are

issue, it is likely that we would need to stop treating the production technology as a primitive, and to start thinking instead about the underlying techniques. This point will be discussed later in the paper.

With the above discussion in mind, it is clear that we may not be able to derive a monotonic relationship between productivity and the elasticity of substitution that is meaningful, contrary to the cited interpretation of Theorem 1 in Klump and La Grandville (2000). And examining that theorem more closely raises significant problems of interpretation. A higher elasticity of substitution is said to ensure that a country will have a higher level of productivity "at any stage in its development" (at any capital-labor ratio). But the normalization procedure implies, by construction, that the productivity level is unchanged at the benchmark capital-labor ratio used to normalize the production function as the elasticity of substitution varies. Moreover, precisely because the different production surfaces are tangent at the benchmark capital-labor ratio, the posited increase in productivity will be very small in the vicinity of the benchmark capital-labor ratio. This is clear, for example, from figure 1 in Klump and La Grandville (2000).

These points immediately call into question the ambitious interpretation of the elasticity of substitution as a useful index of productivity or the level of technology. The problem is that, when the elasticity increases, it does not raise productivity at all capital-labor ratios; and, more seriously, it raises productivity by varying amounts depending on a benchmark capital-labor ratio that is arbitrarily chosen. In other words, we cannot even say that productivity is nondecreasing in the elasticity of substitution, or that a higher elasticity allows an economy to make better use of its capital stock. The position is more complicated. At the benchmark capital-labor ratio, productivity is invariant to the elasticity of substitution, and the magnitude of the effect on productivity at

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falling over time, as may occur under convergence from above. And if the elasticity increases, the new production surface will lie below the old for some combinations of capital and labor, which is an unattractive way to model technical change.

other capital-labor ratios will depend on a researcher's chosen normalization.

Hence, it would be hard to use this index to quantify the associated effect on productivity. To clarify just how odd the interpretation becomes, we can make the discussion more concrete by translating it into a slightly different context. Here is an imaginary conversation between a company engineer and the company's manager:

Engineer: I have changed our production process so that it now has a higher elasticity of substitution.

Manager: Excellent. I understand that this will lower our unit costs, at any given ratio of inputs?

Engineer: Not exactly. It will lower them at all input ratios except one, and in the vicinity of that exception, the reduction in unit costs will be modest.

Manager: And what is the input ratio at which unit costs remain the same as before?

Engineer: That depends, and in principle, it could be any of them.

The same point can be made in a more formal way, with a direct application to growth economics. La Grandville (2009, p. 130) derives the elasticity of output per worker with respect to the elasticity of substitution σ, and presents a table showing this output elasticity for a range of capital-labor ratios (denoted k), where the production function is normalized in the standard ACMS way. These elasticities are calculated by assuming that the benchmark capital share is 0.30.[7] Some of these numbers are presented in Table 1, which shows how the elasticity varies across different capital-labor ratios. Note that output per worker is invariant to the elasticity of substitution at the capital-labor ratio of unity, consistent with the previous discussion.

The difficulty is how to interpret these numbers. Say that the production function was instead normalized at a benchmark capital-labor ratio of 10. Then

[7] The heading to La Grandville's Table 5.3 indicates the benchmark capital share to be 0.33, but this seems to be a typo.

Table 1: La Grandville (2009) results

k	$\sigma = 0.8$	$\sigma = 0.9$	$\sigma = 1$	$\sigma = 1.1$	$\sigma = 1.2$
0.5	0.066	0.057	0.050	0.045	0.041
1	0	0	0	0	0
2	0.060	0.055	0.050	0.047	0.043
10	0.579	0.574	0.557	0.533	0.506
50	1.430	1.562	1.607	1.585	1.521

Table 2: Results under an alternative normalization

k	$\sigma = 0.8$	$\sigma = 0.9$	$\sigma = 1$	$\sigma = 1.1$	$\sigma = 1.2$
0.5	1.351	1.131	0.942	0.791	0.672
1	0.783	0.658	0.557	0.477	0.413
2	0.372	0.316	0.272	0.237	0.210
10	0	0	0	0	0
50	0.301	0.287	0.272	0.257	0.242

similar calculations (see the appendix) lead to the numbers in Table 2, which are markedly different to those in Table 1. This in itself is not a surprise: it reflects the variation in the technology parameters that arises when normalizing the CES function to yield a specific capital share at a benchmark capital-labor ratio. The conceptual problem is that there seems to be no intrinsic reason to prefer the numbers in Table 1 to those in Table 2, or indeed the infinitely many other tables that could be generated by alternative choices for the benchmark capital-labor ratio. In that case, the numbers reported in Table 1 do not seem especially meaningful.

Translating the normalization problem into these concrete terms helps to clarify the central argument of this paper. It is true that the CES production function is always normalized, either implicitly or explicitly. But when analyzing changes in the elasticity of substitution, even an explicit normalization ultimately fails to resolve the underlying difficulty. When a production technology changes form, the economic interpretation of all the technology parameters is likely to be altered. As a result, it becomes hard to interpret comparative static results for a parameter such as the elasticity of substitution, whether holding the distribution parameter fixed, or allowing it to vary in the way suggested by Klump and La Grandville (2000). It is also clear, not least by comparing Table 1 and Table 2, that the interpretation of the elasticity of substitution as a productivity index is less than straightforward.

The conceptual problems of normalization seem especially serious when theorists seek to contrast outcomes associated with the different production surfaces. To be clear, the issue raised here does not concern the logical or mathematical consistency of the proofs given in Klump and La Grandville (2000) and La Grandville (2009), but rather their economic interpretation. In particular, it is not clear what it means to say that these alternative economies are "differing only by their elasticity of substitution". The assumption that such a statement is unambiguous has been promoted by the literature on normalization, but it carries significant risks.

In interpreting such theorems, precision in language is everything. Given the discussion above, an appropriate restatement of Klump and La Grandville's Theorem 1 would be "If the parameters of the CES production functions of two different economies are calibrated so that both functions yield the same capital share at a specific capital-output ratio, then the economy with the higher elasticity of substitution will be predicted to have higher output per worker at all other values of the capital-labor ratio". This claim seems internally consistent, but it is a claim about what happens when CES production functions are calibrated to match specific quantities. It does not tell us what happens to an economy that somehow becomes more flexible in substituting capital for labor. It does not imply that the elasticity of substitution can be interpreted as an index of productivity, or that economies with a higher elasticity of substitution are more productive, or that we should expect higher levels of economic development to be associated with higher elasticities of substitution. Instead, the theorem tells us how production surfaces are reshaped by changes in the elasticity of substitution, when the old and new surfaces are constrained (calibrated) to match certain quantities at a particular benchmark point.

This makes clear the fundamental argument of this paper. Without an explicit account of how and why the production surface has changed, it is unclear why the old and new surfaces should be constrained at one point rather than another. And without such an account, any choice of normalization is both arbitrary and consequential: as we have seen above, a variety of theoretical results will not be invariant to the choice of the benchmark point. Hence, contrary to various statements in the literature, normalization does not allow theorists to isolate changes in the elasticity of substitution for the purpose of theoretical analysis. The effects of those changes depend on the benchmark point, and since the choice of this point is inherently arbitrary, so are the conclusions.[8]

One possible lesson is that, if a researcher thinks that the elasticity of substitution may vary, or that it can be an engine of growth, the analysis will probably need to be located within an explicit structural model in which the production technology is endogenously determined. Such a model could then relate the parameters of the technology to a set of invariant ("deep") structural parameters; the elasticity of substitution is no longer a deep parameter. In that case, because the appropriate variation of the endogenous technology parameters can be deduced from the structure of the model, the need for an arbitrary normalization does not pose the same conceptual problems. Moreover, the comparative statics can be worked out for parameters that are genuinely "deep".

In the literature, there are two main approaches to treating the aggregate elasticity of substitution as endogenous. As La Grandville (2009) emphasizes, one is to consider a multi-sector economy: for example, in the 2×2 model of trade theory, with two sectors and two factors, factor prices are invariant to

[8] It is sometimes implied that, since normalization is inescapable, making the normalization explicit must necessarily resolve the conceptual difficulty. We have already seen that this argument does not work.

factor endowments while the economy remains incompletely specialized. This implies that the aggregate elasticity of substitution is infinite. Jones (1965, 2008) presents results for closed and open economies that allow the aggregate elasticity of substitution to be influenced by the indirect substitution that takes place when consumers switch demand between commodities that vary in factor intensity; see also Miyagiwa (2008). All these results, however, follow from defining the aggregate elasticity as relating changes in ratios of factor usage to a given change in factor price ratios. Hence, the quantity in question is implied by the general equilibrium outcomes that arise from the interaction of multiple sectors. The aggregate elasticity does not correspond to a parameter in a single aggregate production function.[9] In this case, the normalization issue is not relevant, and nor is it useful to take the aggregate elasticity of substitution as an index of overall productivity. Instead, aggregate productivity will depend on the sectoral productivity levels and factor endowments, via the allocations of factors across sectors. Nevertheless, analysis of the aggregate elasticity may be of interest, not least because it may evolve as an economy develops (see, for example, Miyagiwa and Papageorgiou 2007).

The second approach is more directly relevant to the current paper. Arguably, the misuses of normalization arise because the production technology, and its associated parameters, are mistakenly treated as primitives or 'deep'. The true primitives here are the distinct techniques that, combined and optimally selected, constitute the production technology.[10] From this perspective, the problems of normalization arise because a change in the elasticity of substitution cannot be analyzed in a meaningful way without specifying what has happened at the level of individual techniques and their optimal combination.

[9] Put differently, in most cases, it is not possible to write down a closed form for an aggregate production technology that is independent of factor allocations across sectors. What the literature sometimes calls a "technology" is really a production possibility frontier.

[10] The implications of this were emphasized in Atkinson and Stiglitz (1969), a paper that remains under-acknowledged in the mainstream literatures on growth and productivity.

We should redefine the problem in terms of the true primitives — the underlying techniques — and then analyze systematic changes in the set of available techniques, rather than treating the elasticity of substitution as a deep parameter. Certain sets of changes might lead to productivity levels that are unchanged at some factor combinations and altered at others, but the diversity in outcomes will depend on the nature of the changes and not on an arbitrary normalization. The desirability of this is the fundamental difference between the position of this paper, and the current uses of normalization in the theoretical literature.

To implement such an approach, the production function has to be modeled explicitly as an aggregate of a set of techniques. This has been undertaken in Jones (2005) and Growiec (2008, 2011), building on an earlier idea due to Kortum (1997). Each technique is characterized by an n-tuple of unit factor productivities, one per factor of production. The production function corresponds to the convex hull of these "local" techniques. If the unit factor productivities are drawn from independent Pareto distributions then, under some additional restrictions, the production function becomes Cobb-Douglas as the number of draws becomes infinite (Jones 2005). Growiec (2008, 2011) has generalized this approach — not requiring an infinite number of draws — and shown that, if the distributions are Weibull then, again with some further restrictions, it is possible to obtain a CES production function (Growiec 2008, 2011).

From the point of view of this paper, the contribution of Growiec (2011) is especially noteworthy. This is because his formulation is designed to lead to a CES production function that is explicitly normalized. It does not, however, resolve the issue of how to carry out comparative statics based on the elasticity of substitution. His framework implies that the elasticity of substitution is a function of the elasticity for 'local' production techniques — which may be zero, as in the case where each individual technique is Leontief — and the (common) shape parameter of the Weibull distributions. Hence, if the elasticity of substitution is to vary, there must be changes in the elasticity of substitution within the local production techniques, or in the shape of the distributions of unit factor productivities. It is not clear how these changes would arise, and in the absence of a theory of them, the implications for how their effect should be studied remain unclear. Developing such an account is unlikely to be trivial. But, as with the multi-sector approach, the more detailed model does suggest how variation in the aggregate elasticity of substitution could arise in a specification defined in terms of primitives. This approach also suggests that the notion of the elasticity of substitution as a "deep" parameter is the true source of confusion, and may have to be abandoned.

5. Conclusions

The above discussion of normalizing CES production functions can be summarized as follows. The various recent papers on this topic have rightly drawn attention to the potential importance of CES technologies. They have also helped to show that, when a researcher studies or calibrates a model using a CES technology, holding the distribution parameter fixed while varying the elasticity of substitution is not innocuous. Proceeding in this way would imply variation in the capital share that applies at a particular capital-output ratio. When data are available on the capital share at a particular capital-output ratio, it would make sense to calibrate the CES production function in a way that remains consistent with the data, as the elasticity of substitution is varied. In particular, this provides the most natural way to calibrate the distribution parameter, in the way suggested by Klump and La Grandville (2000). Their procedure acknowledges that, if one technology parameter varies, the interpretation of others is likely to vary. These are all useful contributions to our understanding of CES technologies, which should have a significant impact on the future development of the literature.

There is a danger, however, that some of the subsequent uses of normalization have moved too far beyond this. In particular, normalizing CES technologies does not reduce the dimension of the underlying parameter space without raising significant problems of interpretation. The normalization approach is not enough to allow a meaningful comparison of economies that differ "only" in the elasticity of substitution, nor to permit the claim that labor productivity is uniformly increasing in the elasticity of substitution, nor that increases in the elasticity will inevitably form an engine of growth. Contrary to some claims, normalization does not allow the effects of variation in the elasticity to be isolated.

A final conclusion is that, if theoretical claims are to be made in a model where the elasticity of substitution can vary, ideally these claims should be made within a structural model in which production techniques, rather than aggregate technologies, are treated as the relevant primitives. That might allow comparative statics to be carried out with respect to genuinely structural parameters, avoiding the need for an arbitrary normalization. The use of a structural model defined in terms of primitives seems the only way to eliminate the problem rightly identified by the normalization literature, which is that changing the value of one parameter within a given production technology will typically alter the economic interpretation of others.

6. Appendix

Writing the CES production form in a normalized, per capita form, we have:

$$\frac{Y}{L} = \frac{Y_0}{L_0}\left[\pi_0\left(\frac{k}{k_0}\right)^{\rho} + (1-\pi_0)\right]^{1/\rho} \tag{5}$$

where the elasticity of substitution $\sigma = \frac{1}{1-\rho}$ and k_0 denotes the benchmark capital-labor ratio. This implies that the elasticity of output per person with respect to the elasticity of substitution is equal to:

$$e_{y,\sigma}(k,\sigma;\pi_0,k_0) = \frac{\partial y}{\partial \sigma}\frac{\sigma}{y} = \frac{\partial \log y}{\partial \rho}\frac{d\rho}{d\sigma}\sigma \tag{6}$$

$$= \frac{1}{\sigma\rho}\left\{-\frac{1}{\rho}\log\left[\pi_0\left(\frac{k}{k_0}\right)^{\rho} + (1-\pi_0)\right] + \frac{\pi_0(\frac{k}{k_0})^{\rho}\log k}{\pi_0(\frac{k}{k_0})^{\rho} + (1-\pi_0)}\right\} \tag{7}$$

This corresponds to equation (17) in La Grandville (2009, p. 130), apart from a minor typo in his version: the final bracket in the printed version of that equation should appear at the end of the formula, rather than earlier. The figures he reports in Table 5.3, some of which are shown in Table 1 above, appear to be based on the correct formula, assuming a benchmark capital share of 0.30.

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