Step-by-Step Derivations of a Model for Dyanmic General Equilibrim Tax Scoring with Micro Tax Simulations *

Richard W. Evans† Kerk L. Phillips‡

June 2014
(version 14.06.a)
Preliminary and incomplete; please do not cite.

Abstract

This paper ...

keywords: dynamic general equilibrium, taxation, numerical simulation, computational techniques, simulation modeling.

JEL classifications: C63, C68, E62, H24, H25, H68

*

†Brigham Young University, Department of Economics, 167 FOB, Provo, Utah 84602, (801) 422-8303, revans@byu.edu.

‡Brigham Young University, Department of Economics, 166 FOB, Provo, Utah 84602, kerk_phillips@byu.edu.

1 Introduction

2 Details of the Macro Model

We use a model based initially on that from Evans and Phillips (2014) and incorporate many of the features of Zodrow and Diamond (2013) which we refer to hereafter as the DZ model.

2.1 Baseline Model - Model 1

For our first baseline model we take Evans and Phillips (2014) and add a leisure-labor decision, while removing the switching of ability from period to period. Hence all workers remain the same type throughout their lifetime. Agents live for S periods and exogenously retire in period R. This is a perfect foresight model. Both households and firms exist in a unit measure. All firms are idendical, but households are distinguished by age and ability.

2.1.1 Households

Housholds maximize utility as given in the equation below.

$$U_{ist} = \sum_{u=0}^{S-s} \beta^u u(c_{i,s+u,t+u}, \ell_{i,s+u,t+u}); \text{ where } u(c,\ell) = \frac{c^{1-\gamma}-1}{1-\gamma} - \eta\frac{\ell^{1-\xi}-1}{1-\xi}$$

U_{ist} is the remaining lifetime utility of a household with ability level i of age s in period t. c denotes consumption of goods and ℓ denotes labor supplied to the market.

The household faces the following set of budget constraints.

$$w_t \ell_{ist} n_i \geq c_{ist} + k_{i,s+1,t+1} \text{ for } s = 1, \forall i \tag{2.1}$$

$$w_t \ell_{ist} n_i + (1 + r_t - \delta) k_{ist} \geq c_{ist} + k_{i,s+1,t+1} \text{ for } 1 < s < S, \forall i \tag{2.2}$$

$$w_t \ell_{ist} n_i + (1 + r_t - \delta) k_{ist} \geq c_{ist} \text{ for } s = S, \forall i \tag{2.3}$$

k_{ist} is the holdings of capital by household of type i coming due in period t when the household is age s. w is the wage rate, r denotes the return on savings, n denotes the effective labor productivity of the houshold.

The Euler equations from this maximization problem are given below.

$$c_{ist}^{-\gamma} = \beta c_{i,s+1,t+1}^{-\gamma}(1 + r_{t+1} - \delta) \text{ for } 1 \leq s < S, \forall i \tag{2.4}$$

$$c_{ist}^{-\gamma} w_t = \eta \ell_{ist}^{-\xi}, \forall s, i \tag{2.5}$$

2.1.2 Firms

Firms produce using a Cobb-Douglas production function each period and maximize profits as shown below:

$$\Pi_t = K_t^{\alpha}(e^{gt}L_t)^{1-\alpha} - r_t K_t - w_t L_t$$

The profit maximizing conditions are:

$$r_t = \alpha K_t^{\alpha-1}(e^{gt}L_t)^{1-\alpha} \tag{2.6}$$

$$w_t = (1-\alpha)K_t^{\alpha} e^{(1-\alpha)gt} L_t^{-\alpha} \tag{2.7}$$

2.1.3 Market Clearing

Market-clearing conditions require the following:

$$K_t = \sum_{s=2}^{S}\sum_{i=1}^{I} \phi_i k_{ist} \tag{2.8}$$

$$L_t = \sum_{s=1}^{S}\sum_{i=1}^{I} \phi_i \ell_{ist} \tag{2.9}$$

ϕ_i is the proportion of type i in the total population of workers.

2.1.4 Solution and Simulation

This model can be simulated using either the TPI or AMF method described in Evans and Phillips (2014).

Assuming there are I abilitiy types and S cohorts alive in any period, equations (2.1) through (2.9) define a dynamic system of $4+3IS-S$ equations in the variables: K_t, L_t, w_t, r_t, $\{c_{ist}\}_{s=1}^S$, $\{\ell_{ist}\}_{s=1}^S$ and $\{k_{ist}\}_{s=2}^S$.

The parameters of the model are α, β, δ, γ, ξ, g, $\{n_i\}$ and $\{\phi_i\}$

2.1.5 Stationarizing

We need to divide all growing variables by the cummulative level of technology. This gives the following transformed system in $\hat{K}_t$, L_t, $\hat{w}_t$, r_t, $\{\hat{c}_{ist}\}_{s=1}^S$, $\{\ell_{ist}\}_{s=1}^S$ and $\{\hat{k}_{ist}\}_{s=2}^S$. which is stationary.

$$\hat{w}_t \ell_{ist} n_i \geq \hat{c}_{ist} + (1+g)\hat{k}_{i,s+1,t+1} \tag{2.10}$$

$$\hat{w}_t \ell_{ist} n_i + (1+r_t-\delta)\hat{k}_{ist} \geq \hat{c}_{ist} + (1+g)\hat{k}_{i,s+1,t+1} \tag{2.11}$$

$$\hat{w}_t \ell_{ist} n_i + (1+r_t-\delta)\hat{k}_{ist} \geq \hat{c}_{ist} \tag{2.12}$$

$$\hat{c}_{ist}^{-\gamma} = \beta[(1+g)\hat{c}_{i,s+1,t+1}]^{-\gamma}(1+r_{t+1}-\delta) \tag{2.13}$$

$$\hat{c}_{ist}^{-\gamma}\hat{w}_t = \eta \ell_{ist}^{-\xi} \tag{2.14}$$

$$r_t = \alpha \hat{K}_t^{\alpha-1}(L_t)^{1-\alpha} \tag{2.15}$$

$$\hat{w}_t = (1-\alpha)\hat{K}_t^{\alpha} L_t^{-\alpha} \tag{2.16}$$

$$\hat{K}_t = \sum_{s=2}^{S}\sum_{i=1}^{I}\phi_i \hat{k}_{ist} \tag{2.17}$$

$$L_t = \sum_{s=1}^{S}\sum_{i=1}^{I}\phi_i \ell_{ist} \tag{2.18}$$

2.2 Adding Taxes on the Household - Model 2

2.2.1 Households

The social security payroll tax paid or benefit received is caclualted as follows.

$$\hat{T}^P_{ist} = \begin{cases} \tau_P \hat{w}_t \ell_{ist} n_i & \text{if } w_t \ell_{ist} n_i < \chi_P, s < R \\ \tau_P \chi_P & \text{if } w_t \ell_{ist} n_i \geq \chi_P, s < R \\ -\theta \hat{w}_t n_i & \text{if } s \geq R \end{cases} \tag{2.19}$$

τ_P is the payroll tax rate and χ_P is the payroll tax ceiling.

Income is $\hat{w}_t \ell_{ist} n_i + (r_t - \delta)\hat{k}_{ist}$. Define $D\{\hat{w}\ell n + (r-\delta)b, \Omega\}$ as the exemptions and benefits claimed as a function of income and other variables, Ω. Adjusted gross income is $\hat{X}_{ist} \equiv \hat{w}_t \ell_{ist} n_i + (r_t - \delta)\hat{k}_{ist} - D\{\hat{w}_t \ell_{ist} n_i + (r_t - \delta)\hat{k}_{ist}, \Omega_{ist}\} - \tau_\delta \delta \hat{k}_{ist}$. The final term is a capital depreciation allowance at rate τ_δ. We have fit this D function to the data for 2011 using a polynomial function. Income tax paid is defined as follows.

$$\hat{T}^I_{ist} = \begin{cases} 0 & \text{if } \hat{X}_{ist} < \chi_1 \\ \tau_1(\hat{X}_{ist} - \chi_1) & \text{if } \chi_1 \leq \hat{X}_{ist} < \chi_2 \\ \tau_1\chi_1 + \tau_2(\hat{X}_{ist} - \chi_2) & \text{if } \chi_2 \leq \hat{X}_{ist} < \chi_3 \\ \tau_1\chi_1 + \tau_2(\chi_3 - \chi_2) + \tau_3(\hat{X}_{ist} - \chi_3) & \text{if } \chi_3 \leq \hat{X}_{ist} < \chi_4 \\ \vdots & \\ \tau_1\chi_1 + \tau_2(\chi_3 - \chi_2) + \cdots + \tau_N(\hat{X}_{ist} - \chi_N) & \text{if } \chi_N \leq \hat{X}_{ist} \end{cases} \tag{2.20}$$

τ_i is the marginal tax rate in bracket i, the bend points between brackets are denoted χ_i.

The consumption tax rate is denoted τ_c

The household faces the following set of budget constraints.

$$\hat{c}_{ist} = (1-\tau_c)\left[w_t \ell_{ist} n_i - (1+g)\hat{k}_{i,s+1,t+1} - T^p_{ist} - T^i_{ist}\right] \tag{2.21}$$

$$\hat{c}_{ist} = (1-\tau_c)\left[w_t \ell_{ist} n_i + \hat{(}1 + r_t - \delta)k_{ist} - (1+g)\hat{k}_{i,s+1,t+1} - T^p_{ist} - T^i_{ist}\right] \tag{2.22}$$

$$\hat{c}_{ist} = (1-\tau_c)\left[w_t \ell_{ist} n_i + \hat{(}1 + r_t - \delta)k_{ist} - T^p_{ist} - T^i_{ist}\right] \tag{2.23}$$

2.2.2 Government

Government collects the following amounts of tax revenue each period.

$$\hat{R}_t = \sum_s \sum_i \phi_i \left(\hat{T}^p_{ist} + \hat{T}^i_{ist} + \tfrac{\tau_c}{1-\tau_c} \hat{c}_{ist} \right) \tag{2.24}$$

2.2.3 Solution and Simulation

The model now consists of $5 + 5IS - I$ equations with the addtion of (2.19), (2.20) and (2.24), and the substitution of (2.21) - (2.23) for (2.1) - (2.3).

The variables are $\hat{K}_t$, L_t, $\hat{w}_t$, r_t, $\{\hat{c}_{ist}\}_{s=1}^S$, $\{\ell_{ist}\}_{s=1}^S$, $\{\hat{k}_{ist}\}_{s=2}^S$, $\{\hat{T}^p_{ist}\}_{s=1}^S$, $\{\hat{T}^i_{ist}\}_{s=1}^S$ and $\hat{R}_t$.

The parameters of the model are α, β, δ, γ, ξ, g, $\{n_i\}$, $\{\phi_i\}$, τ_p, χ_p, $\{\tau_n\}_{n=1}^N$, $\{\chi_n\}_{n=1}^N$ and τ_c.

2.3 Adding Taxes on Firms - Model 3

We allow firms to acquire capital by renting it as above, or by accumulating their own capital and paying dividends, or by issuing bonds.

We assume both firms and households pay a percent quadratic capital adjustment cost of $\psi(K_{t+1}) = \frac{\kappa}{2}\left(K_{t+1} - K_t\right)^2$. Both housholds and firms receive a depreciation allowance at the rate τ_δ and an investment credit at the rate $\tau_{\Delta k}$

2.3.1 Households

In addtion to capital (k_{ist}), households now also hold bonds in the amount b_{ist} and equities in the amount q_{ist}. Interest income and dividends are taxed as regular income, but capital gains are taxed separately (T^q).

The household's problem can bewritten in the following recursive form:

$$
\begin{aligned}
&V_s^h(k_{ist}, b_{ist}, q_{ist}) = \\
&\max_{k_{t+1}, b_{t+1}, q_{t+1}, \ell_t} \frac{c_{ist}^{1-\gamma} - 1}{1-\gamma} - \eta \frac{\ell_{ist}^{1-\xi} - 1}{1-\xi} + \beta V_{s+1}^h(k_{i,s+1,t+1}, b_{i,s+1,t+1}, q_{i,s+1,t+1})
\end{aligned}
$$

The typical household budget constraint is:

$$
c_{ist} = (1-\tau_c)\begin{bmatrix} w_t \ell_{ist} n_i + (1 + r_t - \delta) k_{ist} + (1 + i_t) b_{ist} + p_t q_{ist} \\ -\frac{\kappa}{2}\left(k_{i,s+1,t+1} - k_{ist}\right)^2 - k_{i,s+1,t+1} - b_{i,s+1,t+1} - p_t q_{i,s+1,t+1} \\ -T_{ist}^p - T_{ist}^i - T_{ist}^q + \tau_\delta \delta k_{ist} + \tau_{\Delta k}(k_{i,s+1,t+1} - k_{ist}) \end{bmatrix} \tag{2.25}
$$

Income subject to taxation (I_{ist}) and AGI (X_{ist}) are:

$$
I_{ist} = w_t \ell_{ist} n_i + (r_t - \delta) k_{ist} + i_t b_{ist} + \pi_t q_{ist} \tag{2.26}
$$

$$
X_{ist} = I_{ist} - D\{I_{ist}, \Omega_{ist}\} - \tau_\delta \delta k_{ist} \tag{2.27}
$$

With the following tax code formulas.

$$T^P_{ist} = \begin{cases} \tau_P w_t \ell_{ist} n_i & \text{if } w_t \ell_{ist} n_i < \chi_P, s < R \\ \tau_P \chi_P & \text{if } w_t \ell_{ist} n_i \geq \chi_P, s < R \\ -\theta w_t n_i & \text{if } s \geq R \end{cases} \tag{2.28}$$

$$T^I_{ist} = \begin{cases} 0 & \text{if } X_{ist} < \chi_1 \\ \tau_1 (X_{ist} - \chi_1) & \text{if } \chi_1 \leq X_{ist} < \chi_2 \\ \tau_1 \chi_1 + \tau_2 (X_{ist} - \chi_2) & \text{if } \chi_2 \leq X_{ist} < \chi_3 \\ \tau_1 \chi_1 + \tau_2 (\chi_3 - \chi_2) + \tau_3 (X_{ist} - \chi_3) & \text{if } \chi_3 \leq X_{ist} < \chi_4 \\ \vdots & \\ \tau_1 \chi_1 + \tau_2 (\chi_3 - \chi_2) + \cdots + \tau_N (X_{ist} - \chi_N) & \text{if } \chi_N \leq X_{ist} \end{cases} \tag{2.29}$$

$$T^q_{ist} = \tau_q \left(\tfrac{p_t}{p_{t-1}} - 1 \right) q_{ist} \tag{2.30}$$

As above, τ_i is the marginal tax rate in bracket i, the bend points between brackets are denoted χ_i, and the consumption tax rate is denoted τ_c

The first-order conditions from the household's problem are:

$$\begin{aligned} & c_{ist}^{-\gamma} \left[(1 - \tau_c) \left(-1 + \tau_\delta \delta + \tau_{\Delta k} - \kappa \left| k_{i,s+1,t+1} - k_{ist} \right| \right) \right] \\ & + \beta \frac{\partial V^h_{s+1}(k_{i,s+1,t+1}, b_{i,s+1,t+1}, q_{i,s+1,t+1})}{\partial k_{i,s+1,t+1}} = 0 \\ & c_{ist}^{-\gamma} [(1 - \tau_c)(-1)] + \beta \frac{\partial V^h_{s+1}(k_{i,s+1,t+1}, b_{i,s+1,t+1}, q_{i,s+1,t+1})}{\partial b_{i,s+1,t+1}} = 0 \\ & c_{ist}^{-\gamma} [(1 - \tau_c)(-p_t)] + \beta \frac{\partial V^h_{s+1}(k_{i,s+1,t+1}, b_{i,s+1,t+1}, q_{i,s+1,t+1})}{\partial q_{i,s+1,t+1}} = 0 \end{aligned}$$

The envelope conditions are:

$$\begin{aligned}
&\frac{\partial V^h_{s+1}(k_{i,s+1,t+1}, b_{i,s+1,t+1}, q_{i,s+1,t+1})}{\partial k_{i,s+1,t+1}} \\
&= c_{ist}^{-\gamma}(1-\tau_c)\left[1 + (1-\tau_j)r_t + (\tau_\delta - 1)\delta + \tau_{\Delta k}\right] \\
&\frac{\partial V^h_{s+1}(k_{i,s+1,t+1}, b_{i,s+1,t+1}, q_{i,s+1,t+1})}{\partial b_{i,s+1,t+1}} \\
&= c_{ist}^{-\gamma}(1-\tau_c)\left[1 + (1-\tau_j)i_t\right] \\
&\frac{\partial V^h_{s+1}(k_{i,s+1,t+1}, b_{i,s+1,t+1}, q_{i,s+1,t+1})}{\partial q_{i,s+1,t+1}} \\
&= c_{ist}^{-\gamma}(1-\tau_c)\left[p_t - \tau_q(p_t - p_{t-1} + (1-\tau_j)\pi_t)\right]
\end{aligned}$$

The Euler equations are:

$$\begin{aligned}
&c_{ist}^{-\gamma}\left[(1-\tau_c)\left(-1 + \tau_\delta\delta + \tau_{\Delta k} - \kappa\left|k_{i,s+1,t+1} - k_{ist}\right|\right)\right] \\
&= \beta c_{i,s+1,t+1}^{-\gamma}(1-\tau_c)\left[1 + (1-\tau_j)r_{t+1} + (\tau_\delta - 1)\delta + \tau_{\Delta k}\right] \text{ for } E \leq s < S, \forall i
\end{aligned} \tag{2.31}$$

$$c_{ist}^{-\gamma} = \beta c_{i,s+1,t+1}^{-\gamma}\left[1 + (1-\tau_j)i_{t+1}\right] \text{ for } 1 \leq s < S, \forall i \tag{2.32}$$

$$c_{ist}^{-\gamma} = \beta c_{i,s+1,t+1}^{-\gamma}\left[1 + (1-\tau_q)(\tfrac{p_{t+1}}{p_t} - 1) + (1-\tau_j)\tfrac{\pi_t}{p_t}\right] \text{ for } 1 \leq s < S, \forall i \tag{2.33}$$

$$c_{ist}^{-\gamma} w_t(1 - \tau_j - \tau_p) = \eta \ell_{ist}^{-\xi}, \forall s, i \tag{2.34}$$

2.3.2 Firms

The firm's intertemporal profits are now:

$$\Pi_t = \sum_{u=0}^{\infty} d_{ut}\pi_{t+u}$$

where

$$d_{ut} \equiv \begin{cases} 1 & \text{if } u = 0 \\ \prod_{j=1}^{u} \frac{1}{1+i_{t+u+j}} & \text{otherwise} \end{cases}$$

We can write its problem as a dynamic program.

$$V^f(B_t, H_t) = \max \pi_t + \tfrac{1}{1+i_{t+1}} V^f(B_{t+1}, H_{t+1})$$

Profits are defined as:

$$\pi_t = (1-\tau_f)\begin{bmatrix} (K_t+H_t)^\alpha(e^{gt}L_t)^{1-\alpha} - r_tK_t - w_tL_t - (1+i_t)B_t \\ +B_{t+1} + (1-\delta)H_t - H_{t+1} - H_{t+1}\psi\left\{\frac{H_{t+1}}{H_t}\right\} \\ +\tau_\delta\delta H_t + \tau_{\Delta k}(H_{t+1}-H_t) \end{bmatrix} \tag{2.35}$$

We also have the following constraint which indicates that new bonds are used to fincance either capital rentals or expansion of the capital stock:

$$B_{t+1} = K_t + H_{t+1} - \tfrac{\kappa}{2}(H_{t+1}-H_t)^2 - \tau_{\Delta k}(H_{t+1}-H_t)$$

FOCs with respect to K_t, L_t and H_{t+1} are:

$$\begin{aligned} \alpha(K_t+H_t)^{\alpha-1}(e^{gt}L_t)^{1-\alpha} - r_t - 1 + \tfrac{1}{1+i_{t+1}}V_B^F(B_{t+1},H_{t+1}) &= 0 \\ (1-\alpha)(K_t+H_t)^{-\alpha}e^{(1-\alpha)gt}L_t^{-\alpha} - w_t &= 0 \\ 1 - 1 - \kappa\left|H_{t+1}-H_t\right| - \tau_{\Delta k} + \tfrac{1}{1+i_{t+1}}V_H^F(B_{t+1},H_{t+1}) &= 0 \end{aligned}$$

Envelope conditions for H_t and B_t are:

$$\begin{aligned} V_B^F(B_t,H_t) &= -(1+t_{t+1}) \\ V_B^F(B_t,H_t) &= (K_t+H_t)^{\alpha-1}(e^{gt}L_t)^{1-\alpha} + 1 - \delta \end{aligned}$$

Combining the above gives:

$$r_t = (K_t+H_t)^{\alpha-1}(e^{gt}L_t)^{1-\alpha} \tag{2.36}$$

$$w_t = (1-\alpha)(K_t+H_t)^{-\alpha}e^{(1-\alpha)gt}L_t^{-\alpha} \tag{2.37}$$

$$1 + r_{t+1} - \delta = (1+i_{t+1})\left(\kappa\left|H_{t+1}-H_t\right| + \tau_{\Delta k}\right) \tag{2.38}$$

2.3.3 Market Clearing

Market-clearing conditions which determine r_t, w_t, i_t and p_t are:

$$K_t = \sum_{s=2}^{S} \sum_{i=1}^{I} \phi_i k_{ist} \tag{2.39}$$

$$L_t = \sum_{s=1}^{S} \sum_{i=1}^{I} \phi_i \ell_{ist} \tag{2.40}$$

$$B_t = \sum_{s=1}^{S} \sum_{i=1}^{I} \phi_i b_{ist} \tag{2.41}$$

$$1 = \sum_{s=1}^{S} \sum_{i=1}^{I} \phi_i q_{ist} \tag{2.42}$$

2.3.4 Government

Government collects the following amounts of tax revenue each period.

$$\begin{aligned} R_t = \sum_s \sum_i \phi_i \left[T^p_{ist} + T^i_{ist} + T^q_{ist} + \tfrac{\tau_c}{1-\tau_c} c_{ist} \right] - \tau_\delta (H_{t+1} + K_{t+1} - H_t - K_t) \\ - \tau_\delta \delta (H_t + K_t) \end{aligned} \tag{2.43}$$

2.3.5 Solution and Simulation

This version of the model has $9 + 9IS - 3I$ equations defined by replacing (2.21) - (2.23) with (2.25); replacing (2.4) & (2.5) with (2.31) - (2.34); adding (2.26), (2.27), (2.35); replacing (2.6) & (2.7) with (2.36) - (2.38); and replacing (2.24) with (2.43).

The variables are K_t, L_t, w_t, r_t, $\{c_{ist}\}_{s=1}^S$, $\{\ell_{ist}\}_{s=1}^S$, $\{k_{ist}\}_{s=2}^S$, $\{T^p_{ist}\}_{s=1}^S$, $\{T^i_{ist}\}_{s=1}^S$, R_t, $\{b_{ist}\}_{s=2}^S$, $\{q_{ist}\}_{s=2}^S$, $\{I_{ist}\}_{s=1}^S$, $\{X_{ist}\}_{s=1}^S$, H_t, B_t, π_t and i_t.

The parameters of the model are α, β, δ, γ, ξ, g, $\{n_i\}$, $\{\phi_i\}$, τ_p, χ_p, $\{\tau_n\}_{n=1}^N$, $\{\chi_n\}_{n=1}^N$, τ_c, τ_q, τ_δ, $\tau \Delta k$ and κ.

2.3.6 Stationarization

The stationary version of the model uses $\hat{K}_t$, L_t, $\hat{w}_t$, r_t, $\{\hat{c}_{ist}\}_{s=1}^S$, $\{\ell_{ist}\}_{s=1}^S$, $\{\hat{k}_{ist}\}_{s=2}^S$, $\{\hat{T}^p_{ist}\}_{s=1}^S$, $\{\hat{T}^i_{ist}\}_{s=1}^S$, $\hat{R}_t$, $\{\hat{b}_{ist}\}_{s=2}^S$, $\{q_{ist}\}_{s=2}^S$, $\{\hat{I}_{ist}\}_{s=1}^S$, $\{X_{ist}\}_{s=1}^S$, $\hat{H}_t$, $\hat{B}_t$ $\hat{\pi}_t$ and i_t.

$$\hat{I}_{ist} = \hat{w}_t \ell_{ist} n_i + (r_t - \delta)\hat{k}_{ist} + i_t \hat{b}_{ist} + \hat{\pi}_t q_{ist} \tag{2.44}$$

$$\hat{X}_{ist} = \hat{I}_{ist} - D\{\hat{I}_{ist}, \Omega_{ist}\} - \tau_\delta \delta \hat{k}_{ist} \tag{2.45}$$

$$\hat{T}^P_{ist} = \begin{cases} \tau_P \hat{w}_t \ell_{ist} n_i & \text{if } \hat{w}_t \ell_{ist} n_i < \chi_P, s < R \\ \tau_P \chi_P & \text{if } \hat{w}_t \ell_{ist} n_i \geq \chi_P, s < R \\ -\theta \hat{w}_t n_i & \text{if } s \geq R \end{cases} \tag{2.46}$$

$$\hat{T}^I_{ist} = \begin{cases} 0 & \text{if } \hat{X}_{ist} < \chi_1 \\ \tau_1(\hat{X}_{ist} - \chi_1) & \text{if } \chi_1 \leq \hat{X}_{ist} < \chi_2 \\ \tau_1\chi_1 + \tau_2(\hat{X}_{ist} - \chi_2) & \text{if } \chi_2 \leq \hat{X}_{ist} < \chi_3 \\ \tau_1\chi_1 + \tau_2(\chi_3 - \chi_2) + \tau_3(\hat{X}_{ist} - \chi_3) & \text{if } \chi_3 \leq \hat{X}_{ist} < \chi_4 \\ \vdots & \\ \tau_1\chi_1 + \tau_2(\chi_3 - \chi_2) + \cdots + \tau_N(\hat{X}_{ist} - \chi_N) & \text{if } \chi_N \leq \hat{X}_{ist} \end{cases} \tag{2.47}$$

$$\hat{T}^q_{ist} = \tau_q \left(\frac{\hat{p}_t}{\hat{p}_{t-1}} - 1\right) q_{ist} \tag{2.48}$$

$$\hat{c}_{ist} = (1-\tau_c)\begin{bmatrix} \hat{w}_t \ell_{ist} n_i + (1 + r_t - \delta)\hat{k}_{ist} + (1+i_t)\hat{b}_{ist} + \hat{p}_t q_{ist} \\ -\frac{\kappa}{2}\left((1+g)\hat{k}_{i,s+1,t+1} - \hat{k}_{ist}\right)^2 - (1+g)\hat{k}_{i,s+1,t+1} \\ -(1+g)\hat{b}_{i,s+1,t+1} - \hat{p}_t q_{i,s+1,t+1} \\ -\hat{T}^p_{ist} - \hat{T}^i_{ist} - \hat{T}^q_{ist} + \tau_\delta \delta \hat{k}_{ist} + \tau_{\Delta k}\left((1+g)\hat{k}_{i,s+1,t+1} - k_{ist}\right) \end{bmatrix} \tag{2.49}$$

$$\begin{aligned} &\hat{c}_{ist}^{-\gamma}\left[(1-\tau_c)\left(-1 + \tau_\delta\delta + \tau_{\Delta k} - \kappa\left|(1+g)\hat{k}_{i,s+1,t+1} - k_{ist}\right|\right)\right] \\ &= \beta[(1+g)\hat{c}_{i,s+1,t+1}]^{-\gamma}(1-\tau_c)\left[1 + (1-\tau_j)r_{t+1} + (\tau_\delta - 1)\delta + \tau_{\Delta k}\right] \end{aligned} \tag{2.50}$$

$$\hat{c}_{ist}^{-\gamma} = \beta[(1+g)\hat{c}_{i,s+1,t+1}]^{-\gamma}\left[1 + (1-\tau_j)i_{t+1}\right] \tag{2.51}$$

$$\hat{c}_{ist}^{-\gamma} = \beta[(1+g)\hat{c}_{i,s+1,t+1}]^{-\gamma}\left[1 + (1-\tau_q)(\frac{p_{t+1}}{p_t} - 1) + (1-\tau_j)\frac{\pi_t}{p_t}\right] \tag{2.52}$$

$$\hat{c}_{ist}^{-\gamma}\hat{w}_t(1-\tau_j-\tau_p) = \eta \ell_{ist}^{-\xi} \tag{2.53}$$

$$\hat{\pi}_t = (1-\tau_f)\begin{bmatrix} (\hat{K}_t + \hat{H}_t)^\alpha (L_t)^{1-\alpha} - r_t\hat{K}_t - \hat{w}_t L_t - (1+i_t)\hat{B}_t \\ +\hat{B}_{t+1} + (1-\delta)\hat{H}_t - (1+g)\hat{H}_{t+1} - \hat{H}_{t+1}\psi\left\{\frac{(1+g)\hat{H}_{t+1}}{H_t}\right\} \\ +\tau_\delta\delta\hat{H}_t + \tau_{\Delta k}\left((1+g)\hat{H}_{t+1} - \hat{H}_t\right) \end{bmatrix} \tag{2.54}$$

$$\begin{aligned} (1+g)\hat{B}_{t+1} = \hat{K}_t + (1+g)\hat{H}_{t+1} - \frac{\kappa}{2}\left((1+g)\hat{H}_{t+1} - \hat{H}_t\right)^2 \\ -\tau_{\Delta k}\left((1+g)\hat{H}_{t+1} - \hat{H}_t\right) \end{aligned} \tag{2.55}$$

$$r_t = (\hat{K}_t + \hat{H}_t)^{\alpha-1}(L_t)^{1-\alpha} \tag{2.56}$$

$$\hat{w}_t = (1-\alpha)(\hat{K}_t + \hat{H}_t)^{-\alpha} e L_t^{-\alpha} \tag{2.57}$$

$$1 + r_{t+1} - \delta = (1 + i_{t+1}) \left(\kappa \left| 1 + g) \hat{H}_{t+1} - \hat{H}_t \right| + \tau_{\Delta k} \right) \tag{2.58}$$

$$\hat{K}_t = \sum_{s=2}^{S} \sum_{i=1}^{I} \phi_i \hat{k}_{ist} \tag{2.59}$$

$$L_t = \sum_{s=1}^{S} \sum_{i=1}^{I} \phi_i \ell_{ist} \tag{2.60}$$

$$\hat{B}_t = \sum_{s=1}^{S} \sum_{i=1}^{I} \phi_i \hat{b}_{ist} \tag{2.61}$$

$$1 = \sum_{s=1}^{S} \sum_{i=1}^{I} \phi_i q_{ist} \tag{2.62}$$

2.4 Adding Demographics - Model 4

Our next step is to add demographic movements to the model. Rather than keeping the population distribution constant we allow it to evolve naturally over time.

We denote the number of households of type i age s in period t as N_{ist}. We assume that workers all become economically active at age E. For each age cohort a fraction ρ_s survive from period t into $t+1$. For each age cohort a fraction f_s of new children are added to next period's age 1 cohort. The distribution of these children over ability is assumed to be the previously described $\{\phi_i\}_{i=1}^{I}$ This gives us the following exogenous laws of motion for the population.

$$N_{i1,t+1} = \sum_s f_s N_{ist} \phi_i \tag{2.63}$$

$$N_{i,s+1,t+1} = N_{ist} \rho_s \tag{2.64}$$

Since workers become economically active at age E, rather than at age 1, we need to keep track of $I - E + 1$ cohorts' capital stocks, bond holdings and equity holdings. We assume that people in cohorts 1 throgh E do not work or consume (alternatively their conssumption is included in that of their parents until age E).

We define the total working age population in period t as below.

$$N_t = \sum_{s=E}^{S} \sum_{i=1}^{I} N_{ist} \tag{2.65}$$

This model has unintentional bequests since households will die before age S. We assume their holdings of capital, bonds, and equity shares are distributed lump sum at the begining of the next peroid. The amounts of each of these lump-sum transfers is shown below.

$$\hat{T}_t^k = \frac{1}{N_t}\sum_{s=E+1}^{S}\sum_{i=1}^{I} N_{ist}(1-\rho_i)(1+g)\hat{k}_{i,s+1,t+1} \tag{2.66}$$

$$\hat{T}_t^b = \frac{1}{N_t}\sum_{s=E+1}^{S}\sum_{i=1}^{I} N_{ist}(1-\rho_i)(1+g)\hat{b}_{i,s+1,t+1} \tag{2.67}$$

$$\hat{T}_t^q = \frac{1}{N_t}\sum_{s=E+1}^{S}\sum_{i=1}^{I} N_{ist}(1-\rho_i)q_{i,s+1,t+1} \tag{2.68}$$

We add these lump sum transfers to the budget constraint and generate (2.69), whcih we substitute in place of (2.25)

$$\hat{c}_{ist} = (1-\tau_c)\begin{bmatrix} \hat{w}_t\ell_{ist}n_i + (1+r_t-\delta)\hat{k}_{ist} + (1+i_t)\hat{b}_{ist} + \hat{p}_t q_{ist} \\ -\frac{\kappa}{2}\left((1+g)\hat{k}_{i,s+1,t+1} - \hat{k}_{ist}\right)^2 - (1+g)\hat{k}_{i,s+1,t+1} \\ -(1+g)\hat{b}_{i,s+1,t+1} - (1+g)\hat{p}_t q_{i,s+1,t+1} \\ +\tau_\delta\delta k_{ist} + \tau_{\Delta k}(k_{i,s+1,t+1} - k_{ist}) \\ -\hat{T}_{ist}^p - \hat{T}_{ist}^i - \hat{T}_{ist}^q + \hat{T}_t^k + \hat{T}_t^b + \hat{p}_t T_t^q \end{bmatrix} \tag{2.69}$$

We also add additional discounting to the household's problem and rewrite (2.31) - (2.33) as below:

$$\begin{aligned} &\hat{c}_{ist}^{-\gamma}\left[(1-\tau_c)\left(-1+\tau_\delta\delta+\tau_{\Delta k}-\kappa\left|(1+g)\hat{k}_{i,s+1,t+1}-k_{ist}\right|\right)\right] \\ &= \beta\rho_s[(1+g)\hat{c}_{i,s+1,t+1}]^{-\gamma}(1-\tau_c)\left[1+(1-\tau_j)r_{t+1}+(\tau_\delta-1)\delta+\tau_{\Delta k}\right] \text{ for } E \le s < S, \forall i \end{aligned} \tag{2.70}$$

$$\hat{c}_{ist}^{-\gamma} = \beta\rho_s[(1+g)\hat{c}_{i,s+1,t+1}]^{-\gamma}\left[1+(1-\tau_j)i_{t+1}\right] \text{ for } E \le s < S, \forall i \tag{2.71}$$

$$\hat{c}_{ist}^{-\gamma} = \beta\rho_s[(1+g)\hat{c}_{i,s+1,t+1}]^{-\gamma}\left[1+(1-\tau_q)(\tfrac{p_{t+1}}{p_t}-1)+(1-\tau_j)\tfrac{\hat{\pi}_t}{\hat{p}_t}\right] \text{ for } E \le s < S, \forall i \tag{2.72}$$

Market-clearing conditions (2.39) - (2.40) are replaced with the following.

$$\hat{K}_t = \sum_{s=E+1}^{S} \sum_{i=1}^{I} N_{ist} \hat{k}_{ist} \tag{2.73}$$

$$L_t = \sum_{s=E}^{S} \sum_{i=1}^{I} N_{ist} \ell_{ist} \tag{2.74}$$

$$\hat{B}_t = \sum_{s=E+1}^{S} \sum_{i=1}^{I} N_{ist} \hat{b}_{ist} \tag{2.75}$$

$$1 = \sum_{s=E+1}^{S} \sum_{i=1}^{I} N_{ist} q_{ist} \tag{2.76}$$

We have added $4 + IS$ new variables to the model: $\hat{T}_t^k$, $\hat{T}_t^q$, $\hat{T}_t^b$, N_t and $\{N_{ist}\}_{s=1}^{S}$.

We have also eliminated $9(E-1)$ variables: $\{\hat{c}_{ist}\}_{s=1}^{E-1}$, $\{\ell_{ist}\}_{s=1}^{E-1}$, $\{\hat{k}_{ist}\}_{s=1}^{E-1}$, $\{\hat{T}_{ist}^p\}_{s=1}^{E-1}$, $\{\hat{T}_{ist}^i\}_{s=1}^{E-1}$, $\{\hat{b}_{ist}\}_{s=1}^{E-1}$, $\{q_{ist}\}_{s=1}^{E-1}$, $\{\hat{I}_{ist}\}_{s=1}^{E-1}$ and $\{\hat{X}_{ist}\}_{s=1}^{E-1}$.

This gives a total of $13 + 9(S - E + 1)I - 3I + IS$ varaibles.

We have also added the following parameters: $\{\rho_s\}_{s=1}^{S}$ and $\{f_s\}_{s=E}^{S}$

2.4.1 Additional Stationarization

Inducing stationarity in this case requires transforming many variables into their per capita versions. We have defined the population in equation (2.65) and will use this to define the growth of the working age population:

$$\begin{aligned} n_{t+1} &= \frac{N_{t+1}}{N_t} - 1 \\ &= \frac{\sum_{s=E}^{S} \sum_{i=1}^{I} N_{i,s,t+1}}{N_t} - 1 \\ &= \frac{\sum_{s=E}^{S} \sum_{i=1}^{I} N_{ist} \rho_s}{N_t} - 1 \end{aligned}$$

We also define the stationary distribution of individuals as a percentage of the working age population as:

$$\hat{N}_{ist} \equiv \frac{N_{ist}}{N_t}$$

Jointly, these give us:

$$n_{t+1} = \sum_{s=E}^{S} \sum_{i=1}^{I} \hat{N}_{ist}\rho_s - 1 \tag{2.77}$$

We alter (2.63) and (2.64) to get:

$$\hat{N}_{i1,t+1} = \frac{\sum_s f_s \hat{N}_{ist}\phi_i}{1 + n_{t+1}} \tag{2.78}$$

$$\hat{N}_{i,s+1,t+1} = \frac{\hat{N}_{ist}\rho_s}{1 + n_{t+1}} \tag{2.79}$$

We also alter (2.73) - (2.76) to get:

$$\hat{K}_t = \sum_{s=E+1}^{S} \sum_{i=1}^{I} \hat{N}_{ist} k_{ist} \tag{2.80}$$

$$\hat{L}_t = \sum_{s=E}^{S} \sum_{i=1}^{I} \hat{N}_{ist} \ell_{ist} \tag{2.81}$$

$$\hat{B}_t = \sum_{s=E+1}^{S} \sum_{i=1}^{I} \hat{N}_{ist} b_{ist} \tag{2.82}$$

$$1 = \sum_{s=E+1}^{S} \sum_{i=1}^{I} \hat{N}_{ist} q_{ist} \tag{2.83}$$

2.5 *** TO DO ***

- Add immigration - average into the cohort/type bin
- Let mortality and fertility vary by ability type as well as age.
- Allow for trends in these parameters

2.6 Adding Open Economy Interest Rate Responsiveness - Model 5

2.7 Adding Mulitiple Industries - Model 6

2.8 Adding Ability Switching - Model 7

3 Conclusion

TECHNICAL APPENDIX

References

Evans, Richard W. and Kerk L. Phillips, "Linearization about the Current State: A Computational Method for Approximating Nonlinear Policy Functions during Simulation," Technical Report, Brigham Young University Department of Economics 2014.

Zodrow, George R. and John W. Diamond, *Handbook of CGE Modeling - Vol. I*, North Holland,