Dyanmic General Equilibrim Tax Scoring Model [1]

Jason DeBacker[2] Richard W. Evans[3] Evan Magnusson[4]
Kerk L. Phillips[5] Isaac Swift[6]

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[2]Middle Tennessee State University, Department of Economics and Finance, BAS N306, Murfreesboro, TN 37132, (615) 898-2528,jason.debacker@mtsu.edu.

[3]Brigham Young University, Department of Economics, 167 FOB, Provo, Utah 84602, (801) 422-8303, revans@byu.edu.

[4]Brigham Young University, Department of Economics, 163 FOB, Provo, Utah 84602, evanmag42@gmail.com.

[5]Brigham Young University, Department of Economics, 166 FOB, Provo, Utah 84602, (801) 422-5928, kerk_phillips@byu.edu.

[6]Brigham Young University, Department of Economics, 163 FOB, Provo, Utah 84602, isaacdswift@gmail.com.

Abstract

An open source dynamic scoring model for U.S. federal tax policy...

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Households

1.1 Demographics

A measure $\omega_{1,t}$ of individuals with heterogeneous working ability $e \in \mathcal{E} \subset \mathbb{R}_{++}$ is born in each period t and live for $E+S$ periods, with $S \geq 4$.[1] The population of age-s individuals in any period t is $\omega_{s,t}$. Households are termed "youth" and out of the market during ages $1 \leq s \leq E$. The households enter the workforce and economy in period $E+1$ and remain in the workforce until they unexpectedly die or live until age $s = E+S$.[2] The population of agents of each age in each period $\omega_{s,t}$ evolves according to the following function,

$$\begin{aligned} \omega_{1,t+1} &= \sum_{s=1}^{E+S} f_s \omega_{s,t} \quad \forall t \\ \omega_{s+1,t+1} &= (1 + i_s - \rho_s)\omega_{s,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{1.1}$$

where $f_s \geq 0$ is an age-specific fertility rate, i_s is an age-specific immigration rate, ρ_s is an age specific mortality hazard rate,[3] and $1 + i_s - \rho_s$ is constrained to be nonnegative. The total population in the economy N_t at any period is simply the sum of individuals in the economy, the population growth rate in any period t from the previous period $t-1$ is $g_{n,t}$, $\tilde{N}_t$ is the working age population, and $\tilde{g}_{n,t}$ is the working age population growth rate in any period t from the previous period $t-1$.

$$N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{1.2}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{1.3}$$

[1] Theoretically, the model exposition of the model works without loss of generality for $S \geq 3$. However, because we are calibrating the ages outside of the economy to be one-fourth of S (e.g., ages 21 to 100 in the economy, and ages 1 to 20 outside of the economy), we need S to be at least 4.

[2] We model the population with households age $s \leq E$ outside of the workforce and economy in order most closely match the empirical population dynamics.

[3] The parameter ρ_s is the probability that a household of age s dies before age $s+1$.

$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{1.4}$$

$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{1.5}$$

1.2 Households

The consumer's maximization problem is:

$$\begin{aligned}
&U_{j,s,t} = \sum_{u=0}^{E+S-s} \beta^u \left[\prod_{v=s-1}^{s+u-1} (1-\rho_v)\right] u\left(c_{j,s+u,t+u}, n_{j,s+u,t+u}, b_{j,s+u+1,t+u+1}\right) \\
&\text{where} \quad \rho_{s-1} = 0 \\
&\text{and} \quad u\left(c_{j,s,t}, n_{j,s,t}, b_{j,s+1,t+1}\right) = \frac{\left(c_{j,s,t}\right)^{1-\sigma} - 1}{1-\sigma} ... \\
&\qquad + e^{g_y t(1-\sigma)} \chi^n_s \left(b\left[1 - \left(\frac{n_{j,s,t}}{\tilde{l}}\right)^v\right]^{\frac{1}{v}} + k\right) + \rho_s \chi^b \frac{\left(b_{j,s+1,t+1}\right)^{1-\sigma} - 1}{1-\sigma} \\
&\text{and} \quad c_{j,s,t} = \prod_{i=1}^{I} \left(c_{i,j,s,t} - \bar{c}_{i,s}\right)^{\alpha_i}; \quad \sum_{i=1}^{I} \alpha_i = 1 \\
&\qquad \forall j,t \quad \text{and}\ E+1 \leq s \leq E+S
\end{aligned} \tag{1.6}$$

They maximize subject to the following budget constraint.

$$\begin{aligned}
&\sum_{i=1}^{I} p_{i,t} c_{i,j,s,t} + b_{j,s+1,t+1} \leq (1+r_t)\, b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \frac{BQ_{j,t}}{\lambda_j \tilde{N}_t} - T_{j,s,t} \\
&\qquad \text{where} \quad b_{j,s,1} = 0 \\
&\qquad\qquad \text{for} \quad E+1 \leq s \leq E+S \quad \forall j,t
\end{aligned} \tag{1.7}$$

We set up a Lagrangian and solve by taking derivatives with respect to $\{c_{i,j,s,t}, n_{j,s,t+u}, b_{j,s,t+1}\}$ for all i, j, s and t.

With respect to each consumption good i:

$$\begin{aligned}
&\frac{\partial U}{\partial c_{i,j,s+u,t+u}} = \beta^u \left[\prod_{v=s-1}^{s+u-1} (1-\rho_v)\right] \left[\prod_{i=1}^{I} \left(c_{i,j,s,t} - \bar{c}_{i,s}\right)^{\alpha_i}\right]^{-\sigma} \alpha_i \left(c_{i,j,s,t} - \bar{c}_{i,s}\right)^{\alpha_i - 1} \\
&\qquad - \lambda_{t+u}\left(p_{i,t+u} + \frac{\partial T_{j,s+u,t+u}}{\partial c_{i,j,s+u,t+u}}\right) = 0
\end{aligned} \tag{1.8}$$

With respect to labor:

$$\frac{\partial U}{\partial n_{j,s+u,t+u}} = \beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right] e^{g_y(t+u)(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}$$
$$- \lambda_{t+u}\left(w_{+u}e_{j,s+u} - \frac{\partial T_{j,s+u,t+u}}{\partial n_{j,s+u,t+u}}\right) = 0 \tag{1.9}$$

With respect to savings:

$$\frac{\partial U}{\partial b_{j,s+u+1,t+u+1}} = \beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right] \rho_s\chi^b\big(b_{j,s+U+1,t+U+1}\big)^{-\sigma}$$
$$- \lambda_{t+u} - \lambda_{t+u+1}\left(1+r_{t+u+1} - \frac{\partial T_{j,s+U+1,t+U+1}}{\partial b_{j,s+U+1,t+u+1}}\right) = 0 \tag{1.10}$$

We can solve each of these for λ_{t+u} to get the following.

$$\lambda_{t+u} = \frac{\beta^u\left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right]\left[\prod_{i=1}^{I}\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_i}\right]^{-\sigma}\alpha_i\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_i-1}}{p_{i,t+u}+\frac{\partial T_{j,s+u,t+u}}{\partial c_{i,j,s+u,t+u}}}$$

$$\lambda_{t+u} = \frac{\beta^u\left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right] e^{g_y(t+u)(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}}{w_{+u}e_{j,s+u}-\frac{\partial T_{j,s+u,t+u}}{\partial n_{j,s+u,t+u}}}$$

$$\lambda_{t+u} = \beta^u \left[\prod_{v=s-1}^{s+u-1}(1-\rho_v)\right] \rho_s\chi^b\big(b_{j,s+U+1,t+U+1}\big)^{-\sigma} - \lambda_{t+u+1}\left(1+r_{t+u+1} - \frac{\partial T_{j,s+U+1,t+U+1}}{\partial b_{j,s+U+1,t+u+1}}\right)$$

These then reduce to the following $I+1$ Euler equations for each j, s and t:

Marginal utility of consumption for each good i compared to the marginal utility of labor:

$$\frac{\left[\prod_{i=1}^{I}\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_i}\right]^{-\sigma}\alpha_i\left(c_{i,j,s,t}-\bar{c}_{i,s}\right)^{\alpha_i-1}}{p_{i,t+u}+\frac{\partial T_{j,s+u,t+u}}{\partial c_{i,j,s+u,t+u}}}$$
$$= \frac{e^{g_y(t+u)(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}}{w_{t+u}e_{j,s+u}-\frac{\partial T_{j,s+u,t+u}}{\partial n_{j,s+u,t+u}}} \tag{1.11}$$

Intertemporal Euler equation for savings, including the utility effects of bequests:

$$
\frac{e^{g_y(t+u)(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u,t+u}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}}{w_{t+u}e_{j,s+u}-\frac{\partial T_{j,s+u,t+u}}{\partial n_{j,s+u,t+u}}} = \rho_s\chi^b\left(b_{j,s+U+1,t+U+1}\right)^{-\sigma}
$$

$$
-\frac{\beta(1-\rho_{s+u})e^{g_y(t+u+1)(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s+u+1,t+u+1}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s+u+1,t+u+1}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}}{w_{t+u+1}e_{j,s+u+1}-\frac{\partial T_{j,s+u+1,t+u+1}}{\partial n_{j,s+u+1,t+u+1}}}\times
$$

$$
\left(1+r_{t+u+1}-\frac{\partial T_{j,s+U+1,t+U+1}}{\partial b_{j,s+U+1,t+u+1}}\right) \tag{1.12}
$$

An Euler equation that compares marginal utilities of two arbitrary goods (n & m) is given below.

$$
\frac{\alpha_n\left(c_{n,j,s,t}-\bar{c}_{n,s}\right)^{\alpha_n-1}}{p_{n,t+u}+\frac{\partial T_{j,s,t}}{\partial c_{n,j,s,t}}} = \frac{\alpha_m\left(c_{m,j,s,t}-\bar{c}_{m,s}\right)^{\alpha_m-1}}{p_{m,t+u}+\frac{\partial T_{j,s,t}}{\partial c_{m,j,s,t}}} \tag{1.13}
$$

We can use this equation for $m \in \{1,2,...,I\}$ solving for $c_{m,j,s,t}-\bar{c}_{m,s}$.

$$
c_{m,j,s,t}-\bar{c}_{m,s} = \left[\left(c_{n,j,s,t}-\bar{c}_{n,s}\right)^{1-\alpha_n}\frac{\Gamma_m}{\Gamma_n}\right]^{\frac{1}{1-\alpha_m}} \tag{1.14}
$$

$$
\Gamma_m \equiv \frac{\alpha_m}{p_{m,t}+\frac{\partial T_{j,s,t}}{\partial c_{m,j,s,t}}}
$$

We can solve the household's problem fairly rapidly, if the values of $\frac{\partial T_{j,s,t}}{\partial c_{i,j,s,t}}$ are just constants, as they would be with a typical sales tax.

- First, given $\{p_{i,t}\}_{i=1}^{I}$ use Euler equation (1.11) to find the value for $c_{1,j,s,t}$.
- Then use equation (1.14) to get the rest of the $c_{i,j,s,t}$'s.

Chapter 2

Firms

2.1 Supply Side Model Components: Variables and Parameters

2.1.1 Variables

There are six unique state variables, all of which are exogenous. Note that r_s is directly determined by i_s and the tax rate on interest income.

Table 2.1: State Variables

Variable	Description
K_s^C	Capital stock at the beginning of period s
$K_{s-1}^{\tau C}$	Tax basis of the capital stock at the beginning of period s
B_s^C	Debt at the beginning of period s
p_s^C	Price of corporate good in period s
w_s	Wage rate in period s
i_s	Nominal interest rate in period s
r_s	After tax nominal rate of return in period s

Note in $K_{s-1}^{\tau C}$ the $s-1$ subscript - depreciation in period s will be based on this and investment in period s.

There are 14 control variables, although all of these are trivial after the determination of I_s^C and EL_s^C.

2.1.2 Parameters

The model has 19 parameters. Of these, 7 relate to the firm's production function, 2 to economic growth, 2 to firm financial policy, and 10 to tax policy.

Table 2.2: Control Variables

Variable	Description
I_s^C	Firm investment in period s
EL_s^C	Firm effective labor demand in period s
X_s^C	Corp goods produced
K_{s+1}^C	Firm's capital stock at the end of period s (beginning of period $s+1$)
$EARN_s^C$	Corp earnbefore deprec, corp taxes, and adjust costs, but after property taxes in period s
DIV_s^C	Corporate dividends in period s
TE_s^C	Total corporate income taxes in period s
Φ_s^C	Investment adjustment costs in period s
B_{s+1}^C	Corp debt at the end of period s (start of period $s+1$)
$K_s^{\tau C}$	Tax basis of capital under corp income tax at end of period s (beginning of $s+1$)
VN_s^C	New equity issued by the corp sector in period s
V_s^C	Firm value in period s
q_s^C	Marginal q (change in firm value per dollar of investment)
Q_s^C	Average Q

Table 2.3: Model Parameters

Parameter	Description
Production Function	
γ_C	Capital weighting in CES production function
ϵ_C	Elasticity of substitution of capital for labor in CES production function
δ^C	Rate of economic depreciation on capital stock in the corporate sector
β^C	Scaling parameter for quadratic investment adjustment costs
μ_C	Steady-state investment rate
Economic Growth	
n	Rate of population growth (exogenous)
g	Rate of productivity growth (exogenous)
Financial Policy	
ζ^C	Fraction of earnings paid out in dividends
b^C	Debt/Capital ratio
Tax Policy	
τ_s^b	Corporate business income tax rate
$\delta_s^{\tau C}$	Rate of tax depreciation on corporate capital
τ_s^{pC}	Property tax rate on corporate capital
τ_s^i	Individual income tax rate on interest income
τ_s^g	Individual income tax rate on capital gains
f_e	Dummy variable for full expensing of investment
f_i	Dummy variable for deductibility of corporate interest paid
f_p	Dummy variable for deductibility of repayment of principle on loans
f_b	Dummy variable for inclusion of proceeds of loan in corp income tax base
f_d	Dummy variable for deductibility of depreciation expenses

2.2 Necessary equations

To solve the model, we want to get the optimal choices of labor and investment demand by firms. Labor demand is determined through an intratemporal trade off between the costs and benefits of labor. The necessary condition for the optimal choice of labor is:

$$p_s^C \frac{\partial F(K_s^C, EL_s^C)}{\partial EL_s^C} = w_s \tag{2.1}$$

Investment is more complicated, as it presents an intertemporal tradeoff between the costs of investment today and the benefits of a higher capital stock tomorrow. Once we have investment, all other endogenous variables follow from various accounting identities and assumptions on financial policies.

To derive the necessary conditions for investment, we need to first solve for the value of the firm as a function of the state variables (noted above) and the choice of investment. We do this by substituting in the various accounting identities to our equation for firm value.

Begin with the asset market equilibrium condition that the after-tax returns on all assets must be equalized if households simultaneously hold equity and bonds (and there is no aggregate uncertainty). The after-tax, nominal return on holding bonds is:

$$r_s = (1 - \tau_s^i) i_s, \tag{2.2}$$

Where i_s is the nominal interest rate on bonds. Thus the return on holding corporate equity must equal r_s in equilibrium:

$$r_s = (1 - \tau_s^i) i_s = \frac{(1 - \tau_s^d) DIV_s^C + (1 - \tau_s^g)(V_{s+1}^C - V_s^C - VN_s^C)}{V_s^C}, \tag{2.3}$$

where the first part of the numerator is the dividend returns from holding shares of the corporation and the second part are the capital gains returns from holding corporate equity, which are diluted by the issuance of new shares, VN_s^C.

We can rearrange this equation 2.3 to solve for V_{s+1}^C:

$$\begin{aligned} V_{s+1}^C &= \frac{V_s^C (1 - \tau_s^i) i_s - (1 - \tau_s^d) DIV_s^C}{(1 - \tau_s^g)} + V_s^C + VN_s^C \\ &= V_s^C \underbrace{\left(1 + \frac{(1 - \tau_s^i) i_s}{(1 - \tau_s^g)}\right)}_{\text{Let this be } 1+\theta_s} + VN_s^C - \frac{(1 - \tau_s^d)}{(1 - \tau_s^g)} DIV_s^C \end{aligned} \tag{2.4}$$

Now we can solve this for V_s^c by repeatedly substituting for V_{s+1}^C and applying the transversality condition ($\lim_{T \to \infty} \prod_{t=1}^{T} (1 + \theta_t) V_T^C = 0$):

$$
\begin{aligned}
&V_s^C = \frac{V_{s+1}^C}{(1+\theta_s)} - \frac{VN_s^C}{(1+\theta_s)} + \frac{\left(\frac{1-\tau_s^d}{1-\tau_s^g}\right) DIV_s^C}{(1+\theta_s)} \\
\implies &V_s^C = \frac{V_{s+2}^C}{(1+\theta_s)(1+\theta_{s+1})} - \frac{VN_{s+1}^C}{(1+\theta_s)(1+\theta_{s+1})} + \frac{\left(\frac{1-\tau_s^d}{1-\tau_s^g}\right) DIV_{s+1}^C}{(1+\theta_s)(1+\theta_{s+1})} - \frac{VN_s^C}{(1+\theta_s)} + \frac{\left(\frac{1-\tau_s^d}{1-\tau_s^g}\right) DIV_s^C}{(1+\theta_s)} \\
\implies &V_s^C = \frac{V_{s+3}^C}{(1+\theta_s)(1+\theta_{s+1})(1+\theta_{s+2})} - \frac{VN_{s+2}^C}{(1+\theta_s)(1+\theta_{s+1})(1+\theta_{s+2})} + \frac{\left(\frac{1-\tau_s^d}{1-\tau_s^g}\right) DIV_{s+2}^C}{(1+\theta_s)(1+\theta_{s+1})(1+\theta_{s+2})} \\
&\quad - \frac{VN_{s+1}^C}{(1+\theta_s)(1+\theta_{s+1})} + \frac{\left(\frac{1-\tau_s^d}{1-\tau_s^g}\right) DIV_{s+1}^C}{(1+\theta_s)(1+\theta_{s+1})} - \frac{VN_s^C}{(1+\theta_s)} + \frac{\left(\frac{1-\tau_s^d}{1-\tau_s^g}\right) DIV_s^C}{(1+\theta_s)} \\
&\text{and so on...} \\
\implies &V_s^C = \underbrace{\prod_{\nu=s}^{\infty}\left(\frac{1}{1+\theta\nu}\right) V_\infty^C}_{=0 \text{ by transversality condition}} - \sum_{u=s}^{\infty}\prod_{\nu=s}^{u}\left(\frac{1}{1+\theta\nu}\right)\left[VN_u^C - \left(\frac{1-\tau_s^d}{1-\tau_s^g}\right) DIV_u^C\right] \\
\implies &V_s^C = \sum_{u=s}^{\infty}\prod_{\nu=s}^{u}\left(\frac{1}{1+\theta\nu}\right)\left[\left(\frac{1-\tau_s^d}{1-\tau_s^g}\right) DIV_u^C - VN_u^C\right]
\end{aligned} \tag{2.5}
$$

Now we have firm value as a functions discounted, after-tax value of dividends, less the discounted value of new shares issuance, which dilutes the value of the shares held at time s. We continue working towards writing firm value as a function of the state variable and the choice of investment and labor demand. First, we solve for VN_u^C. New shares issued in period u are given by the cash flow identity:

$$EARN_u^C + BN_u^C + VN_u^C = DIV_u^C + I_u^C(1+\Phi_u^C) + TE_u^C, \tag{2.6}$$

where $EARN_u^C$ are earnings before depreciation, corporate income taxes, and adjustment costs, but after property taxes; BN_u^C are new bond issues, I_u^C is investment, Phi_u^C are adjustment costs, and TE_u^C are total corporate income taxes (all in period u). These variable are determined as follows:

Earnings are determined by an accounting identity and the corporate production function.

$$EARN_u^C = p_u^C X_u^C - w_u EL_u^C - i_u B_u^C - \tau_u^{PC} K_u^C, \tag{2.7}$$

where output, X_u^C, is determined by a constant elasticity of substitution production function:

$$F(K_u^C, EL_u^C) = X_u^C = \left[(\gamma_C)^{1/\epsilon_C}(K_u^C)^{(\epsilon^C-1)/\epsilon_C} + (1-\gamma_C)^{1/\epsilon_C}(EL_u^C)^{(\epsilon_C-1)/\epsilon_C}\right]^{(\epsilon_C/(\epsilon_C-1))} \tag{2.8}$$

New debt issues are solved for by the assumption of a constant debt-to-capital ratio (and the law of motion for the capital stock):

$$BN_u^C = B_{u+1}^C - B_u^C \text{ and } B_u^C = b^C K_u^C \text{ by assumption} \tag{2.9}$$

The law of motion of the capital stock is given by:

$$K_{u+1}^C = (1-\delta^C)K_u^C + I_u^C \tag{2.10}$$

Adjustment costs are assumed to be a quadratic function of deviations from the steady-state investment rate:

$$\Phi_u^C = \frac{p_u^C \left(\frac{\beta^C}{2}\right)\left(\frac{I_u^C}{K_u^C} - \mu^C\right)^2}{\left(\frac{I_u^C}{K_u^C}\right)} \tag{2.11}$$

Corporate income taxes:

$$\begin{aligned} TE_u^C =& \tau_u^b \left[p_u^C X_u^C - w_u EL_u^C - f_e I_u^C - \Phi_u^c I_u^C - f_i i_u B_u^C - f_p \delta^C b^C K_u^C + f_b b^C I_u^C - f_d \delta^{\tau C} K_u^{\tau C} - \tau_u^{pC} K_u^C\right] \\ &+ \underbrace{\tau_u^{icC} I_u^C}_{\text{not in DZ (2013), added to account for investment tax credits as policy}}, \end{aligned} \tag{2.12}$$

Note that we are assuming that investment may or may not be deductible (depending upon the dummy variable f_e), but that investment adjustment costs are always deductible (i.e., they are not preceded by f_e). Under a pre-pay consumption tax system, investments are not deductible. Whether or not adjustment costs are deductible under a pre-pay consumption tax depends upon what you think these costs derive from. For example, if adjustment costs are from retraining employees to use new equipment, then these costs may be deductible under a consumption tax system (pre or post-pay) because they would likely be in the form of wage/labor costs. It's not clear how best to handle this and I believe the notion in Zodrow and Diamond (2013) is inconsistent on this point.
where the tax basis of the capital stock evolves according to:

$$K_{u+1}^{\tau C} = (1-\delta^{\tau C})(K_u^{\tau C} + (1-f_e)I_u^C) \tag{2.13}$$

Note how we form the law of motion for the tax basis. Zodrow and Diamond (2013) do not specify this, but the above formulation accounts for the fact that investment in year t receives a depreciation deduction in year t.[1] We can think about modifying this so that you get no deduction in the year the investment is made, which may or may not be more consistent with the "time to build" built into the law of motion for the physical capital stock.

[1] What is actually used is a "half year rule", where you deduct the value of investment based on the assumption that it was put in place half-way through the year (so you get one half the annual deprecation rate on this new investment).

Dividends are determined by the assumption that dividends are a constant fraction of pre-tax earnings.

$$DIV_u^C = \zeta^C(EARN_u^C - TE_u^C - p_u^C\delta^C K_u^C) \tag{2.14}$$

We will return to how investment, I_u^C, is determined, but first let us write the value of the firm as function of the states and the control variables I_s^C and EL_s^C. Substituting Equations 2.6 - 2.14 into Equation 2.5 (and letting $\Omega_u^C = 1 - \zeta^C + \zeta^C\left(\frac{1-\tau_u^d}{1-\tau_u^g}\right) = \left[\zeta^C(1-\tau_u^d) + (1-\zeta^C)(1-\tau_u^g)\right]/(1-\tau_u^g))$ we get:

$$\begin{aligned}
V_s^C = & \sum_{u=s}^{\infty}\prod_{\nu=s}^{u}\left(\frac{1}{1+\theta\nu}\right)(1-\tau_u^b)\Omega_u^C(p_u^C X_u^C - w_s EL_s^C) \\
& - K_s^C\left\{(1-\tau_u^b)\Omega_u^C\tau_u^{pC} + (1-f_i\tau_u^i)i_u\Omega_u^C b^C - \delta^C(p_u^C - b^C - \Omega_u^C(p_u^C - f_p\tau_u^b b^C))\right\} \\
& - I_u^C\left\{1 - b^C + \Omega_u^C f_b\tau_u^b b^C - \Omega_u^C f_e\tau_u^b + (1-\Omega_u^C\tau_u^b)\Phi_u^C\right\} \\
& - \Omega_u^C f_d\tau_u^b\delta^{\tau C}K_u^{\tau C}
\end{aligned} \tag{2.15}$$

Note that $K_u^{\tau C}$ tracks depreciation deductions in in all periods $u = s, ..., \infty$. Future depreciation deductions on the tax basis of the capital stock in existence at time u do not affect investment decisions at time u (or forward) since the tax basis is predetermined.[2] However, future depreciation deductions for investments made at time u do affect investment decisions (since they lower the after-tax cost of investments). Therefore it's useful to distinguish between old and new capital.

The time u value of future depreciation deductions on the capital stock existing at the beginning of period u, $K_{u-1}^{\tau C}$, is given by:

$$\begin{aligned}
f_d Z_u^C K_{u-1}^{\tau C} &= \sum_{j=u}^{\infty}\prod_{\nu=u}^{j}\left(\frac{1}{1+\theta_\nu)}\right) f_d\Omega_j^c\tau_j^b\delta^{\tau C}(1-\delta^{\tau C})^{j-u}K_u^{\tau C} \\
&= f_d K_{u-1}^{\tau C}\underbrace{\sum_{j=u}^{\infty}\prod_{\nu=u}^{j}\left(\frac{1}{1+\theta_\nu)}\right) f_d\Omega_j^c\tau_j^b\delta^{\tau C}(1-\delta^{\tau C})^{j-u}}_{Z_u^C}
\end{aligned} \tag{2.16}$$

We can derive the time u value of future depreciation deductions on investments made at time u, $I_u^{\tau C}$, similarly. These are given by $f_d(1-f_e)Z_u^C I_u^C$.

Thus we can rewrite Equation 2.15 as:

[2]Note that if there were financial frictions (e.g. a borrowing constraint or costly external finance), then investment would be dependent on cash flow and would then be affected by changes in the value of deductions for the existing capital basis.

$$
\begin{aligned}
V_s^C = &\sum_{u=s}^{\infty}\prod_{\nu=s}^{u}\left(\frac{1}{1+\theta\nu}\right)(1-\tau_u^b)\Omega_u^C(p_u^C X_u^C - w_s E L_s^C) \\
&- K_s^C\left\{(1-\tau_u^b)\Omega_u^C\tau_u^{pC} + (1-f_i\tau_u^i)i_u\Omega_u^C b^C - \delta^C(p_u^C - b^C - \Omega_u^C(p_u^C - f_p\tau_u^b b^C))\right\} \\
&- I_u^C\left\{1 - b^C + \Omega_u^C f_b\tau_u^b b^C - \Omega_u^C f_e\tau_u^b - f_d(1-f_e)Z_u^C + (1-\Omega_u^C\tau_u^b)\Phi_u^C\right\} \\
&+ f_d Z_s^C K_{s-1}^{\tau C}
\end{aligned}
\tag{2.17}
$$

The Lagrangian to the firm's problem at time s can be written as:

$$
\begin{aligned}
\mathcal{L}_s = \max_{\{I_u^c, K_{u+1}^C\}_{u=s}^{\infty}} &\sum_{u=s}^{\infty}\prod_{\nu=s}^{u}\left(\frac{1}{1+\theta\nu}\right)(1-\tau_u^b)\Omega_u^C(p_u^C X_u^C - w_s E L_s^C) \\
&- K_s^C\left\{(1-\tau_u^b)\Omega_u^C\tau_u^{pC} + (1-f_i\tau_u^i)i_u\Omega_u^C b^C - \delta^C(p_u^C - b^C - \Omega_u^C(p_u^C - f_p\tau_u^b b^C))\right\} \\
&- I_u^C\left\{1 - b^C + \Omega_u^C f_b\tau_u^b b^C - \Omega_u^C f_e\tau_u^b - f_d(1-f_e)Z_u^C + (1-\Omega_u^C\tau_u^b)\Phi_u^C\right\} \\
&+ f_d Z_s^C K_{s-1}^{\tau C} + q_u^C((1-\delta^C)K_u^C + I_u^C - K_{u+1}^C)
\end{aligned}
\tag{2.18}
$$

The FOCs w.r.t. investment are given by:

$$
\begin{aligned}
&\frac{\partial \mathcal{L}_s}{\partial I_u^C} = -\left\{1 - b^C + \Omega_u^C f_b\tau_u^b b^C - \Omega_u^C f_e\tau_u^b - f_d(1-f_e)Z_u^C + (1-\Omega_u^C\tau_u^b)\Phi_u^C\right\} - I_u^C(1-\Omega_u^C\tau_u^b)\frac{\partial \Phi_u^C}{\partial I_u^C} + q_u^C = \\
&\Longrightarrow q_u^C = 1 - b^C + \Omega_u^C f_b\tau_u^b b^C - \Omega_u^C f_e\tau_u^b - f_d(1-f_e)Z_u^C + (1-\Omega_u^C\tau_u^b)\Phi_u^C + I_u^C(1-\Omega_u^C\tau_u^b)\frac{\partial \Phi_u^C}{\partial I_u^C} \\
&\Longrightarrow q_u^C = 1 - b^C - \Omega_u^C\tau_u^b(f_e - f_b b^C) - f_d(1-f_e)Z_u^C + (1-\Omega_u^C\tau_u^b)\Phi_u^C + I_u^C(1-\Omega_u^C\tau_u^b)\frac{\partial \Phi_u^C}{\partial I_u^C}
\end{aligned}
\tag{2.19}
$$

$$
q_u^C = 1 - b^C - \Omega_u^C\tau_u^b(f_e - f_b b^C) - f_d(1-f_e)Z_u^C + (1-\Omega_u^C\tau_u^b)\Phi_u^C + I_u^C(1-\Omega_u^C\tau_u^b)\frac{\partial \Phi_u^C}{\partial I_u^C}
\tag{2.20}
$$

q_u^C is Tobin's q or the marginal change in firm value for a dollar of investment (which is to say it's the shadow price of investment). The FOC for investment says that the firm invests until the marginal benefit (the LHS of Equation 2.20) is equal to the marginal cost of investment (the RHS of Equation 2.20). The cost of investment in the absence of taxes and frictions is equal to 1 (the first term on the RHS of Equation 2.20) since investment goods are the numeraire. The second term reflects the reduction in the cost of debt due to debt financing. The third term on the RHS of Equation 2.20 is the change in the cost of capital due to debt being included or excluded from corporate income taxes. The fourth term reflects the reduction in the cost of debt due to depreciation deductions. The last term reflects the costs of capital that are due to adjustment costs (net of the expensing of adjustment costs for tax purposes).

The FOCs w.r.t. capital one period ahead are given by:

$$\frac{\partial \mathcal{L}_s}{\partial K^C_{u+1}} = \prod_{\nu=s}^{u}\left(\frac{1}{1+\theta\nu}\right)\left[-q^C_u\right]$$
$$+\prod_{\nu=s}^{u+1}\left(\frac{1}{1+\theta\nu}\right)\Bigg[(1-\delta^C)q^C_{u+1} + p^C_{u+1}\frac{\partial X^C_{u+1}}{\partial K^C_{u+1}} - \{(1-\tau^b_{u+1})\Omega^C_{u+1}\tau^{pC}_{u+1} + (1-f_i\tau^i_{u+1})i_{u+1}\Omega^C_{u+1}b^C - \delta^C(p^C_{u+}$$
$$\implies q^C_u = \left(\frac{1}{1+\theta_{u+1}}\right)$$
$$\Bigg[(1-\delta^C)q^C_{u+1} + p^C_{u+1}\frac{\partial X^C_{u+1}}{\partial K^C_{u+1}} - \{(1-\tau^b_{u+1})\Omega^C_{u+1}\tau^{pC}_{u+1} + (1-f_i\tau^i_{u+1})i_u\Omega^C_{u+1}b^C - \delta^C(p^C_{u+1} - b^C - \Omega^C_{u+1}(p^C_{u+}$$
$$(2.21)$$

We should be able to solve for I^C_u, K^C_{u+1}, and q^C_u with Equations 2.10, 2.20, and 2.21. We can then use q^C_u to solve for V^C_u as we show next.

As Hayashi (1982) shows, with a constant returns to scale production function and quadratic adjustment costs, we can device that marginal q is equal to average q. Note that in our case, we must make an adjustment for the value of depreciation deductions on the tax basis of the capital stock. Relation between marginal q, q^C_u, and average q, Q^C_u:

$$q^C_u = \frac{[V^C_u - f_d Z^C_u K^{\tau C}_{u-1}]}{K^C_u} \text{ and } Q^C_u = \frac{V^C_u}{K^C_u} \tag{2.22}$$

2.3 Solving the model

We'll solve the model in two steps. First, we solve for the steady state prices and allocations. Next, we iterate backwards solving for prices and allocations along the transition path to the steady state.

2.3.1 Solving for the steady state

On the supply side (with one sector), we have to solve for the factor prices, $\bar{i}$ and $barw$ (the price of output $\bar{p}^C$ is normalized to one), the shadow price of capital, $\bar{q}^C$, and the allocations $\bar{EL}^C$, $\bar{K}^C$, $\bar{I}^C$. From these all the other variables follow trivially.

Start by solving for the steady-states of Equations 2.20 and 2.21. Equation 2.20 becomes:

$$\bar{q}^C = \underbrace{1 - b^C - \bar{\Omega}^C\bar{\tau}^b(f_e - f_b b^C) - f_d(1-f_e)\bar{Z}^C}_{\text{function of only parameters}} \tag{2.23}$$

This yields the solution to $\bar{q}^C$.

Next, consider the steady-state of Equation 2.21:

$$\bar{q}^C = \frac{1}{1+\bar{\theta}}\left[(1-\delta^C)\bar{q}^C + \frac{\partial \bar{X}^C}{\partial \bar{K}^C} - \{(1-\bar{\tau}^b)\bar{\Omega}^C\bar{\tau}^{pC} + (1-f_i\bar{\tau}^i)\bar{i}\bar{\Omega}^C b^C - \delta^C(1-b^C-\bar{\Omega}^C(1-f_p\bar{\tau}^b b^C))\}\right] \tag{2.24}$$

We can rearrange this and solve for the steady-state marginal product of capital in sector C:

$$\frac{\partial \bar{X}^C}{\partial \bar{K}^C} = (\bar{\theta}+\delta^C)\bar{q}^C + (1-\bar{\tau}^b)\bar{\Omega}^C\bar{\tau}^{pC} + (1-f_i\bar{\tau}^i)\bar{i}\bar{\Omega}^C b^C - \delta^C(1-b^C-\bar{\Omega}^C(1-f_p\bar{\tau}^b b^C)) \tag{2.25}$$

Notice that given Equation 2.23, the RHS to the above equation is function of parameters and the steady state nominal interest rate, $\bar{i}$. The LHS of the equation is a function of $\bar{K}^C$ and $\bar{EL}^C$.

I think we can use the following to identify the SS values of the variables of interest:

i. $\bar{i}$ will be determined by the SS of the household's Euler equations (I think this can be done as described in the HH sol'n method)

ii. $\bar{w}$ will be determined by the SS of the household's FOCs for labor supply ((I think this can be done as described in the HH sol'n method)

iii. $\bar{q}^C$ is determined by Equation 2.23

iv. $\bar{EL}^C$ is determined by the SS version of Equation 2.1, plus $\bar{w}$

v. $\bar{K}^C$ is determined by Equation 2.25 and $\bar{i}$

vi. $\bar{I}^C$ is then solved for using the steady state law of motion for capital $\implies \bar{I}^C = \delta^C\bar{K}^C$

In solving for $\bar{EL}^C$ and $\bar{K}^C$, note that we'll have use the MPK and the MPL simultaneously. Given our production function, we have:

$$\frac{\partial X_u^C}{\partial K_u^C} = \left[(\gamma_C)^{1/\epsilon_C}(K_u^C)^{(\epsilon_C-1)/\epsilon_C} + (1-\gamma_C)^{1/\epsilon_C}(EL_u^C)^{(\epsilon_C-1)/\epsilon_C}\right]^{1/(\epsilon_C-1)}(\gamma_C)^{1/\epsilon_C}(K_u^C)^{-1/\epsilon_C} \tag{2.26}$$

and

$$\frac{\partial X_u^C}{\partial EL_u^C} = \left[(\gamma_C)^{1/\epsilon_C}(K_u^C)^{(\epsilon_C-1)/\epsilon_C} + (1-\gamma_C)^{1/\epsilon_C}(EL_u^C)^{(\epsilon_C-1)/\epsilon_C}\right]^{1/(\epsilon_C-1)}(1-\gamma_C)^{1/\epsilon_C}(EL_u^C)^{-1/\epsilon_C} \tag{2.27}$$

We know that, at an optimum, the marginal revenue product of labor equals the wage rate, and the marginal revenue product of capital equals a function of the interest rate, marginal q, and the model parameters. Call this function $g(i_u, q_u^C, q_{u-1}^C, \Theta)$. We thus have $p_u^C \frac{\partial X_u^C}{\partial EL_u^C} = w_u$ and $p_u^C \frac{\partial X_u^C}{\partial K_u^C} = g(i_u, q_u^C, q_{u-1}^C, \Theta)$. Dividing these two equations, we have:

$$\begin{aligned} \frac{\frac{\partial X_u^C}{\partial K_u^C}}{\frac{\partial X_u^C}{\partial EL_u^C}} &= \frac{(\gamma_C)^{1/\epsilon_C}(K_u^C)^{-1/\epsilon_C}}{(1-\gamma_C)^{1/\epsilon_C}(EL_u^C)^{-1/\epsilon_C}} - = \frac{g(i_u, q_u^C, q_{u-1}^C, \Theta)}{w_u} \\ \implies \frac{K_u^C}{EL_u^C} &= \frac{(1-\gamma_C)}{\gamma_C}\left(\frac{w_u}{g(i_u, q_u^C, q_{u-1}^C, \Theta)}\right)^{\epsilon_C} \end{aligned} \tag{2.28}$$

We can use the SS version of Equation 2.28 to solve for capital as function of labor (and $\bar{q}, \bar{i}, \bar{w}$), and then use that in the SS version of Equation 2.27 to solve for labor as s function of $\bar{q}, \bar{i}, \bar{w}$. We then go back to the SS version of Equation 2.28 to get the SS choice of capital as a function of $\bar{q}, \bar{i}, \bar{w}$.

All of the above will work for each sector in a model with any number of sectors (though care has to be taken to include the prices of output and capital in those other sectors, since only one sector's output can be the numeraire).

2.3.2 Solving for the transition path

I believe we can just use the Euler equations to go backwards in time, from the SS back along the transition path to $t = 0$. Assume period T is the SS, The solution would look like the following:

i. Use Equation 2.21 to solve for the for q_{T-1}^C since we have the solution to the RHS of the equation after we've solved for the SS.

ii. Use the law of motion for capital to find: $K_{T-1}^C = \frac{K_T^C - I_{T-1}^C}{(1-\delta^C)} = \frac{\bar{K}^C - I_{T-1}^C}{(1-\delta^C)}$

iii. Use Equation 2.20 and the value of q_{T-1}^C to find I_{T-1}^C (and K_{T-1}^C given the law of motion relationship.

iv. Given w_{T-1} we can use the FOC for labor demand to find EL_{T-1}^C

v. Given i_{T-1} we can use Equation 2.21 to solve for q_{T-2}^C

vi. We then repeat the above steps until we work back to $t = 0$.

2.4 Features model has and those it is lacking

2.5 Roadmap for extensions

Possible order or model extensions:

i. Add industries/goods

Table 2.4: Tax Distortions in DZ Model

Distortion	Accounted for in DZ mo
Amount of investment	Mostly accounted for
Entity form	Not directly accounted f
Location of capital	Not in baseline model.
Type of investment (equip/structures/intangible)	No - just aggregate capit
Bias towards certain industries (because of type of capital or income risk)	No
Bias towards non-risky projects (due to tax loss asymmetry)	No
Bias towards non-risky businesses (due to tax loss asymmetry)	No
Double tax of profits (affects several distortions)	Yes, partially.
Dividend distribution policy	Not really.
Where recognize income (US or abroad)	No
Repatriate income (and when)	No

Table 2.5: Tax Policy Instruments in DZ Model

Instrument	In DZ?	Large macro effects	Likel
Corporate income tax top rate	Yes	Yes	Yes
Corporate income tax rate structure	No	No	Yes
Capital gains tax rate	Yes	Yes	
Dividend income tax rate	Yes	Yes	Yes
Depreciation rate structure	Yes	Yes	
Bonus deprecation/expensing	No	Yes	
Investment tax credits	No	Yes	Yes
General business credits	No	Prob not after above	Yes
Cap/Deny/Index for inflation interest expenses	No	Yes	Yes
Carry back/forward window	No	Maybe	
Inventory accounting rules (LIFO/FIFO)	No	Probably not	Yes
Repatriation holidays	No	Maybe	Yes
International tax system (Territorial vs Worldwide, deferral)	No	Yes	Yes
Consumption tax system (w/ pre and post pay)	Yes	Yes	No

- Try to do this along the lines of Fullerton and Rogers (1993) with composite consumption and production goods. Important additions might include health care and a carbon intensive sector (e.g. utilities, transport). Important goods would be health services, large excise items (gasoline, alcohol, cigarettes), carbon intensive goods (e.g. utilities, transport), food
- Maybe be costly to solve firm problem for many sectors. GE price vector may be large too, but Fullerton and Rogers (1993)) suggest that still just w and r (for each year) that need

ii. Add types of capital

- Try to do this along the lines of Fullerton and Rogers (1993) with composite capital in firm production function, but where capital can change type costlessly.
- Not sure how/if this works in dynamic model where keep track of old capital.

iii. Add profits via some markup

- Want this so have supranormal returns, which are differentially affected by taxes.
- See macro models of the markup, but want to just get markup that is a function of the elasticity of substitution.

iv. Endogenize debt finance.

- Have a cost to bankruptcy - use debt until this cost negates the tax advantage.

v. Endogenize payout policy

- Want to at have dividends respond to dividend tax rate (e.g. by at least being done cash after investment made - since invest a function of div tax rate). This is easy. Harder I think is to fully endogenize so that firm considers the value of dividends to owners after tax vs the value of the dollar inside the firm.

vi. Open economy

- First thing to do here is to have some capital mobility in the model.

vii. Have multiple types of labor (skilled/unskilled) in production function

- Only if have endogenous human capital accumulation.
- Would like to see how taxes on capital affect capital/labor mix to uncover distribution of incidence of taxes by industry and individuals.

Some features that might be interesting, but may not pass a cost-benefit test to adding into the model:

i. Stochastic profitability shocks
    - So can account for loss carry forward/back.
    - Would need to have idiosyncratic shocks over firms in an industry.
    - Think we'd have to solve the firm problem a lot more times and not sure how this interacts in GE with different type of capital etc.
ii. Have a more serious model of income shifting by having some behavior where move profits offshore
    - Not sure how to do even in open economy model.
iii. Endogenize entity choice.
    - Makes firm problem much harder.
    - Not sure how to calibrate elasticities, but Prisinzano and Pearce at OTA might have some estimates to help.
iv. Pollution externalities
    - It'd be cool to be able to do some policy experiments with Pigouvian taxes like ?)

    .
v. Model evasion and avoidance
    - Perhaps just have some elasticity for reported income with respect to the marginal tax rate that scales actual income.
    - DeBacker, Heim, Tran, and Yuskavage can measure with audit data.

Chapter 3

Government

3.1 Overview of Government in the Model

Government will have four functions in our model:

i. The government runs a tax and social security system

   - The tax system will be input by the user and/or determined by the current tax law (the default unless the user supplies changes)

ii. The government makes transfers to households outside of the tax/social security system

iii. The government produces output that contributes to private consumption goods (e.g., education)

iv. The government purchases capital and hires labor to produced a non-rival public good (e.g., national defense)

3.2 Government budgeting

$$D_{t+1} + T_t^{\tau} = (1 + r_t)D_t + T_t^H + G_t^{subs} + G_t^{emp} + I_t^G \tag{3.1}$$

3.2.1 Rule for long-term fiscal stability

Let D_t denote the government's outstanding real debt. T_t is total tax revenue, T_t^H is total household transfers, G_t is government purchases of goods, L_t is the real value of purchases of labor services, and S_t is subsidies to government run firms.

$$D_{t+1} = D_t(1 + r_t) - T_t + T_t^H + G_t + L_t + S_t \tag{3.2}$$

Letting a carat denote the ratio of a variable to GDP, we can rewrite this as follows:

$$(1+g_{Yt})\hat{D}_{t+1} = \hat{D}_t(1+r_t) - \hat{T}_t + \hat{T}_t^H + \hat{G}_t + \hat{L}_t + \hat{S}_t \tag{3.3}$$

We need to adopt a government fiscal rule that determines how our residual expenditure $\hat{G}_t$ evolves over time.

One way is to adopt a balanced budget rule which keeps the debt-to-GDP ratio constant at it's initial value of $\hat{D}_0$.

$$\begin{aligned}(1+g_{Yt})\hat{D}_0 &= \hat{D}_0(1+r_t) - \hat{T}_t + \hat{T}_t^H + \hat{G}_t + \hat{L}_t + \hat{S}_t \\ \hat{G}_t &= \hat{D}_0(g_{Yt} - r_t) + \hat{T}_t - \hat{T}_t^H - \hat{L}_t - \hat{S}_t \end{aligned} \tag{3.4}$$

Another rule is to hold govenrment spending constant and let debt evolve as it will for several period. Then in period T impose fiscal austerity which forces $\hat{G}_t$ to adjust over time so that $\hat{D}_t$ goes to a steady value.

$$\hat{G}_t - \bar{G} = \rho_t(\hat{D}_t - \bar{D}); \quad \rho_t < 0 \tag{3.5}$$

Substituting this into (3.3) gives:

$$\begin{aligned}(1+g_{Yt})\hat{D}_{t+1} &= \hat{D}_t(1+r_t) - \hat{T}_t + \hat{T}_t^H + \rho_t(\hat{D}_t - \bar{D}) + \bar{G} + \hat{L}_t + \hat{S}_t \\ \hat{D}_{t+1} &= \frac{\hat{D}_t(1+r_t) - \hat{T}_t + \hat{T}_t^H + \rho_t(\hat{D}_t - \bar{D}) + \bar{G} + \hat{L}_t + \hat{S}_t}{1+g_{Yt}} \end{aligned} \tag{3.6}$$

Consider the steady state version of this.

$$\begin{aligned}(1+\bar{g}_Y)\bar{D} &= \bar{D}(1+\bar{r}) + \bar{T} - \bar{T}^H + \rho_t(\bar{D} - \bar{D}) + \bar{G} + \bar{L} + \bar{S}_t \\ \bar{G} &= \bar{D}(\bar{g}_Y - \bar{r}) + \bar{T} - \bar{T}^H - \bar{L} - \bar{S} \end{aligned} \tag{3.7}$$

This tells us the long-run value of government spending to GDP that will maintain the debt to GDP target.

In order for (3.6) to be a contraction mapping over $\hat{D}$ and thus converge to a steady state, we must put bounds on ρ_t. Rearranging (3.6) and using (3.7):

$$\begin{aligned}
(1+g_{Yt})\hat{D}_{t+1} &= \hat{D}_t(1+r_t) - \hat{T}_t + \hat{T}^H_t + \rho_t(\hat{D}_t - \bar{D}) + \hat{L}_t + \hat{S}_t \\
&\quad + \bar{D}(\bar{g}_Y - \bar{r}) + \bar{T} - \bar{T}^H - \bar{L} - \bar{S} \\
(1+g_{Yt})\hat{D}_{t+1} &= \hat{D}_t(1+r_t) - \hat{T}_t + \hat{T}^H_t + \rho_t\hat{D}_t - \rho_t\bar{D} + \hat{L}_t + \hat{S}_t \\
&\quad + \bar{g}_Y\bar{D} - \bar{r}\bar{D} + \bar{T} - \bar{T}^H - \bar{L} - \bar{S} \\
(1+g_{Yt})\hat{D}_{t+1} &= \hat{D}_t(1+r_t) + \rho_t(\hat{D}_t - \bar{D}) + (\bar{g}_Y - \bar{r})\bar{D} \\
&\quad - (\hat{T}_t - \bar{T}) + (\hat{T}^H_t - \bar{T}^H) + (\hat{L}_t - \bar{L}) + (\hat{S}_t - \bar{S}) \\
\hat{D}_{t+1} - \bar{D} &= \hat{D}_t\frac{1+r_t}{1+g_{Yt}} + \frac{\rho_t}{1+g_{Yt}}(\hat{D}_t - \bar{D}) + \left(\frac{\bar{g}_Y - \bar{r}}{1+g_{Yt}} - 1\right)\bar{D} \\
&\quad + \frac{-(\hat{T}_t - \bar{T}) + (\hat{T}^H_t - \bar{T}^H) + (\hat{L}_t - \bar{L}) + (\hat{S}_t - \bar{S})}{1+g_{Yt}} \\
\hat{D}_{t+1} - \bar{D} &= \frac{1+r_t+\rho_t}{1+g_{Yt}}(\hat{D}_t - \bar{D}) \\
&\quad + \frac{-(\hat{T}_t - \bar{T}) + (\hat{T}^H_t - \bar{T}^H) + (\hat{L}_t - \bar{L}) + (\hat{S}_t - \bar{S})}{1+g_{Yt}}
\end{aligned} \tag{3.8}$$

We need $\frac{\hat{D}_{t+1}-\bar{D}}{\hat{D}_t-\bar{D}} < 1$ for stability. Equation (3.8) gives:

$$\begin{aligned}
\frac{\hat{D}_{t+1} - \bar{D}}{\hat{D}_t - \bar{D}} = \frac{1+r_t+\rho_t}{1+g_{Yt}} + \frac{-(\hat{T}_t - \bar{T}) + (\hat{T}^H_t - \bar{T}^H) + (\hat{L}_t - \bar{L}) + (\hat{S}_t - \bar{S})}{(1+g_{Yt})(\hat{D}_t - \bar{D})} &< 1 \\
\frac{1+r_t+\rho_t}{1+g_{Yt}} < \frac{(\hat{T}_t - \bar{T}) - (\hat{T}^H_t - \bar{T}^H) - (\hat{L}_t - \bar{L}) - (\hat{S}_t - \bar{S})}{(1+g_{Yt})(\hat{D}_t - \bar{D})} \\
\rho_t < (1+r_t)\frac{(\hat{T}_t - \bar{T}) - (\hat{T}^H_t - \bar{T}^H) - (\hat{L}_t - \bar{L}) - (\hat{S}_t - \bar{S})}{\hat{D}_t - \bar{D}}
\end{aligned} \tag{3.9}$$

3.2.2 Transfer system

We'll need to estimate this. Probably following **?**). Or perhaps the micro simulation model calculates some of these. Or ideally we get something like **?**).

3.3 Government production of private goods

3.4 Government production of public goods

3.5 Steps for adding government to the dynamic model

i. 1 firm

ii. 1 firm + gov't

iii. 2 firms + gov't

iv. tax 2 firms + gov't

v. N firms with taxes + gov't

Chapter 4

Equilibrium

4.1 Market clearing and stationary equilibrium

Labor market clearing requires that aggregate labor demand L_t measured in efficiency units equal the sum of individual efficiency labor supplied $e_{j,s}n_{j,s,t}$. Capital market clearing requires that aggregate capital demand K_t equal the sum of capital investment by households $b_{j,s,t}$. Aggregate consumption C_t is defined as the sum of all individual consumptions, and aggregate investment is defined by the standard $Y = C + I$ constraint as shown in (4.3).

$$L_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j e_{j,s}n_{j,s,t} \quad \forall t \tag{4.1}$$

$$K_t = \sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\omega_{s-1,t-1}\lambda_j b_{j,s,t} \quad \forall t \tag{4.2}$$

$$\begin{aligned} Y_t &= C_t + K_{t+1} - (1-\delta)K_t \quad \forall t \\ \text{where} \quad C_t &\equiv \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j c_{j,s,t} \end{aligned} \tag{4.3}$$

The usual definition of equilibrium would be allocations and prices such that households optimize (**??**), (**??**), and (**??**), firms optimize (**??**) and (**??**), and markets clear (4.1) and (4.2). However, the variables in these characterizing equations are potentially not stationary due to the possible growth rate in the total population $g_{n,t}$ each period coming from the cohort growth rates in (1.1) and from the deterministic growth rate of labor augmenting technological change g_y in (**??**).

Table 4.1: Stationary variable definitions

Sources of growth			Not
$e^{g_y t}$	$\tilde{N}_t$	$e^{g_y t}\tilde{N}_t$	growing[a]
$\hat{c}_{j,s,t} \equiv \frac{c_{j,s,t}}{e^{g_y t}}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{\tilde{N}_t}$	$\hat{Y}_t \equiv \frac{Y_t}{e^{g_y t}\tilde{N}_t}$	$n_{j,s,t}$
$\hat{b}_{j,s,t} \equiv \frac{b_{j,s,t}}{e^{g_y t}}$	$\hat{L}_t \equiv \frac{L_t}{\tilde{N}_t}$	$\hat{K}_t \equiv \frac{K_t}{e^{g_y t}\tilde{N}_t}$	r_t
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$		$\hat{BQ}_{j,t} \equiv \frac{BQ_{j,t}}{e^{g_y t}\tilde{N}_t}$	
$\hat{y}_{j,s,t} \equiv \frac{y_{j,s,t}}{e^{g_y t}}$			
$\hat{T}_{j,s,t} \equiv \frac{T_{j,s,t}}{e^{g_y t}}$			

[a] The interest rate r_t in (**??**) is already stationary because Y_t and K_t grow at the same rate. Individual labor supply $n_{j,s,t}$ is stationary.

Table 4.1 characterizes the stationary versions of the variables of the model in terms of the variables that grow because of labor augmenting technological change, population growth, both, or none. With the definitions in Table 4.1, it can be shown that the equilibrium characterizing equations can be written in stationary form in the following way. The static and intertemporal Euler equations from the individual's optimization problem corresponding to (**??**), (**??**), and (**??**) are the following.

$$
\begin{aligned}
&(\hat{c}_{j,s,t})^{-\sigma}\left(\hat{w}_t e_{j,s} - \frac{\partial \hat{T}_{j,s,t}}{\partial n_{j,s,t}}\right) = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall j,t, \quad \text{and} \quad E+1\leq s\leq E+S \\
&\text{where} \quad \hat{c}_{j,s,t} = (1+r_t)\,\hat{b}_{j,s,t} + \hat{w}_t e_{j,s} n_{j,s,t} + \frac{\hat{BQ}_{j,t}}{\lambda_j} - e^{g_y}\hat{b}_{j,s+1,t+1} - \hat{T}_{j,s,t} \\
&\text{and} \quad \frac{\partial \hat{T}_{j,s,t}}{\partial n_{j,s,t}} = \hat{w}_t e_{j,s}\left[\tau^I\left(F\hat{a}_{j,s,t}\right) + \frac{F\hat{a}_{j,s,t}CD\big[2A(F\hat{a}_{j,s,t})+B\big]}{\big[A(F\hat{a}_{j,s,t})^2 + B(F\hat{a}_{j,s,t}) + C\big]^2} + \tau^P\right] \\
&\text{and} \quad \hat{b}_{j,E+1,t} = 0 \quad \forall j,t
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
&(\hat{c}_{j,s,t})^{-\sigma} = ... \\
&e^{-g_y\sigma}\left(\rho_s\chi^b\big(\hat{b}_{j,s+1,t+1}\big)^{-\sigma} + \beta(1-\rho_s)(\hat{c}_{j,s+1,t+1})^{-\sigma}\left[(1+r_{t+1}) - \frac{\partial T_{j,s+1,t+1}}{\partial b_{j,s+1,t+1}}\right]\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall j,t, \quad \text{and} \quad E+1\leq s\leq E+S-1 \\
&\quad \text{where} \quad \frac{\partial T_{j,s+1,t+1}}{\partial b_{j,s+1,t+1}} = ... \\
&\qquad r_{t+1}\left(\tau^I(F\hat{a}_{j,s+1,t+1}) + \frac{F\hat{a}_{j,s+1,t+1}CD\left[2A(F\hat{a}_{j,s+1,t+1})+B\right]}{\left[A(F\hat{a}_{j,s+1,t+1})^2 + B(F\hat{a}_{j,s+1,t+1}) + C\right]^2}\right)... \\
&\qquad \tau^W(\hat{b}_{j,s+1,t+1}) + \frac{\hat{b}_{j,s+1,t+1}PHM}{\left(H\hat{b}_{j,s+1,t+1}+M\right)^2}
\end{aligned}
\tag{4.5}
$$

$$(\hat{c}_{j,E+S,t})^{-\sigma} = \chi^b e^{-g_y\sigma}(\hat{b}_{j,E+S+1,t+1})^{-\sigma} \quad \forall j,t \tag{4.6}$$

The stationary firm first order conditions for optimal labor and capital demand corresponding to (**??**) and (**??**) are the following.

$$\hat{w}_t = (1-\alpha)\frac{\hat{Y}_t}{\hat{L}_t} \quad \forall t \tag{4.7}$$

$$r_t = \alpha\frac{\hat{Y}_t}{\hat{K}_t} - \delta = \alpha\frac{Y_t}{K_t} - \delta \quad \forall t \tag{??}$$

And the two stationary market clearing conditions corresponding to (4.1) and (4.2)—with the goods market clearing by Walras' Law—are the following.

$$\hat{L}_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\hat{\omega}_{s,t}\lambda_j e_{j,s}n_{j,s,t} \quad \forall t \tag{4.8}$$

$$\hat{K}_t = \frac{1}{1+\tilde{g}_{n,t}}\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\hat{\omega}_{s-1,t-1}\lambda_j\hat{b}_{j,s,t}\right) \quad \forall t \tag{4.9}$$

where $\tilde{g}_{n,t}$ is the growth rate in the working age population between periods $t-1$ and t described in (1.5).

We can now define the stationary steady-state equilibrium for this economy in the following way.

Definition 1 (**Stationary steady-state equilibrium**). A non-autarkic stationary steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability $e_{j,s}$ is defined as constant allocations $n_{j,s,t} = \bar{n}_{j,s}$ and $\hat{b}_{j,s+1,t+1} = \bar{b}_{j,s+1}$ and constant prices $\hat{w}_t = \bar{w}$ and $r_t = \bar{r}$ for all j, s, and t such that the following conditions hold:

i. households optimize according to (4.4), (4.5), and (4.6),

ii. firms optimize according to (4.7) and (**??**),

iii. markets clear according to (4.8) and (4.9), and

iv. the population has reached its stationary steady state distribution $\bar{\omega}_s$ for all ages s, characterized in Appendix **??**.

The steady-state equilibrium is characterized by the system of $2JS$ equations and $2JS$ unknowns $\bar{n}_{j,s}$ and $\bar{b}_{j,s+1}$. Appendix 5.1 details how to solve for the steady-state equilibrium.

Figure 4.1: Stationary steady-state distribution of savings $\log(\bar{\mathbf{\Gamma}})$ for $S = 80$ and $J = 7$

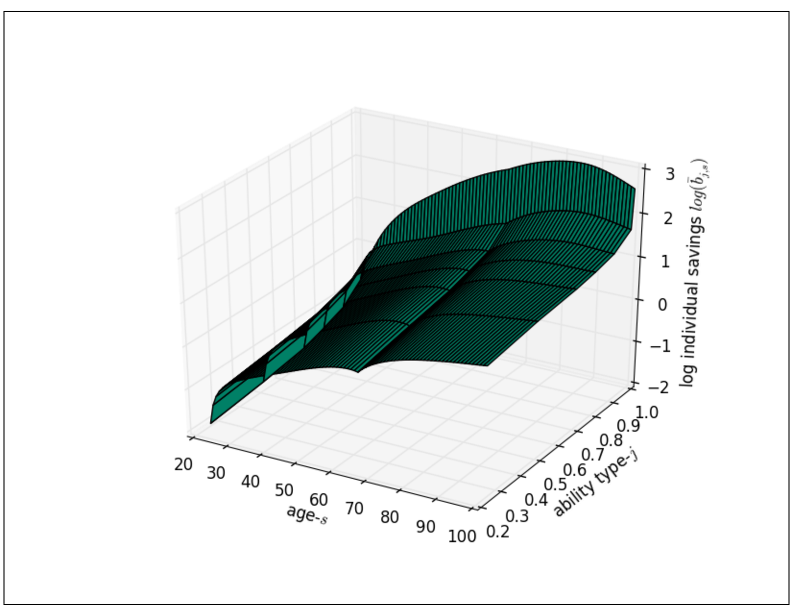


Table 4.2: List of exogenous variables and baseline calibration values

Symbol	Description	Value
$\hat{\boldsymbol{\Gamma}}_1$	Initial distribution of savings	$\bar{\boldsymbol{\Gamma}}$
N_0	Initial population	1
$\{\omega_{s,0}\}_{s=1}^{S}$	Initial population by age	(see App. ??)
$\{f_s\}_{s=1}^{S}$	Fertility rates by age	(see App. ??)
$\{i_s\}_{s=1}^{S}$	Immigration rates by age	(see App. ??)
$\{\rho_s\}_{s=1}^{S}$	Mortality rates by age	(see App. ??)
$\{e_{j,s}\}_{j,s=1}^{J,S}$	Deterministic ability process	(see App. ??)
$\{\lambda_j\}_{j=1}^{J}$	Ability type bin percentages	(see App. ??)
J	Number of ability types	7
S	Maximum periods in economically active household life	80
E	Number of periods of youth economically outside the model	round $\left(\frac{S}{4}\right)$
R	Retirement age (period)	round $\left(\frac{9}{16}S\right)$
$\tilde{l}$	Maximum hours of labor supply	1
β	Discount factor	$(0.96)^{\frac{80}{S}}$
σ	Coefficient of constant relative risk aversion	3
b	Scale parameter in utility of leisure	(see App. ??)
υ	Shape parameter in utility of leisure	(see App. ??)
k	constant parameter in utility of leisure	(see App. ??)
χ_s^n	Disutility of labor level parameter	(see App. ??)
χ^b	Utility of bequests level parameter	(see App. ??)
Z	Level parameter in production function	1
α	Capital share of income	0.35
δ	Capital depreciation rate	$1-(1-0.05)^{\frac{80}{S}}$
g_y	Growth rate of labor augmenting technological progress	$(1+0.03)^{\frac{80}{S}}-1$
A	Coefficient on squared term in $\tau^I(\cdot)$	(see App. ??)
B	Coefficient on linear term in $\tau^I(\cdot)$	(see App. ??)
C	Constant coefficient in $\tau^I(\cdot)$	(see App. ??)
D	Level parameter for $\tau^I(\cdot)$	(see App. ??)
F	Income factor for $\tau^I(\cdot)$	(see App. ??)
τ^P	Payroll tax rate	0.15
$\{\theta^j\}_{j=1}^{J}$	Replacement rate by average income	(see App. ??)
τ^{BQ}	Bequest (estate) tax rate	0
P	Level parameter for $\tau^W(\cdot)$	0
H	Coefficient on linear term in $\tau^W(\cdot)$	0
M	Constant coefficient in $\tau^W(\cdot)$	0
T	Number of periods to steady state	160
ν	Dampening parameter for TPI	0.2

Figure 4.2: Stationary steady-state distribution of individual labor supply $\bar{n}_{j,s}$ for $S = 80$ and $J = 7$

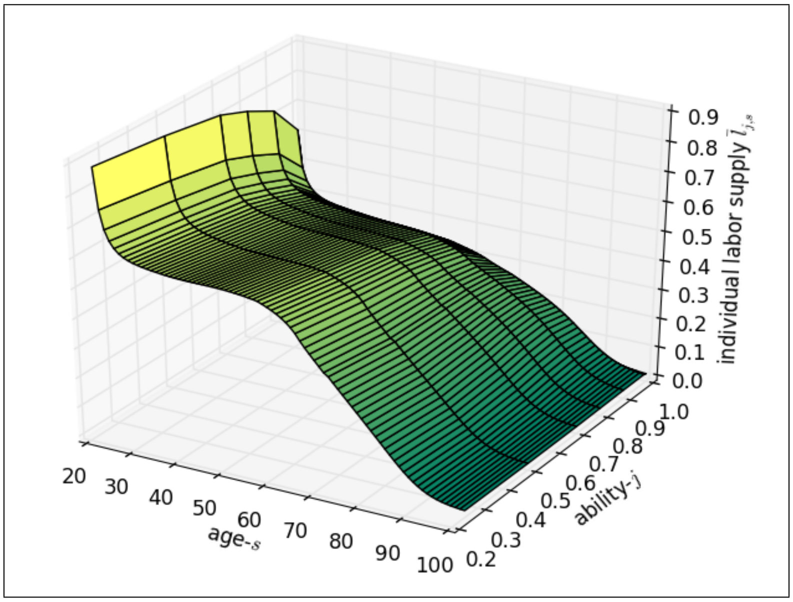
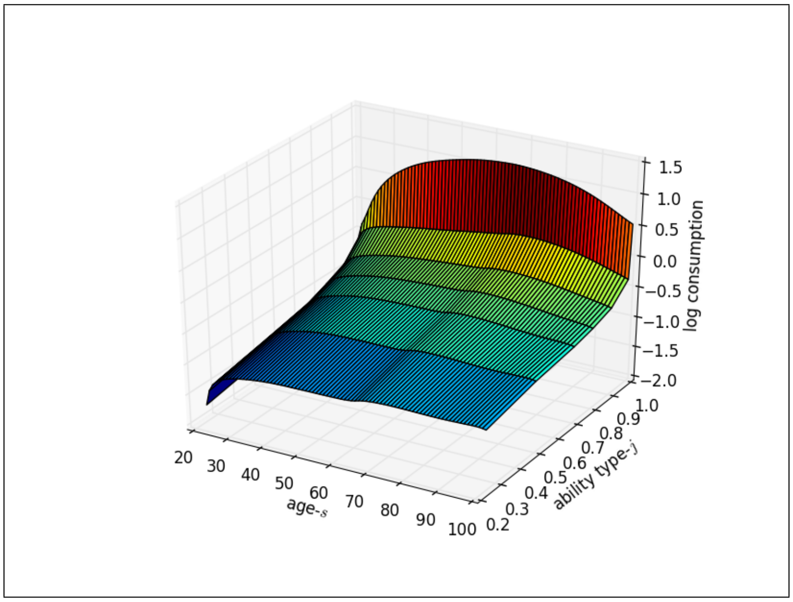


Figure 4.1 shows the stationary steady-state distribution of individual savings $\bar{b}_{j,s}$ in logarithms, Figure 4.2 shows the stationary steady-state distribution of individual labor supply $\bar{n}_{j,s}$, and Figure 4.3 shows the steady-state distribution of consumption $\bar{c}_{j,s}$ in logarithms for a particular calibration of the model described in Table 4.2. Notice from Figure 4.3 the hump-shaped pattern of consumption over the life cycle for each ability type, which is consistent with consumption data. Also note from the comparison of the distribution of savings in Figure 4.1 in comparison to the distribution of consumption in 4.3 that households use savings to smooth out consumption as much as possible, with a sharpe change in savings around retirement $s = R$ and only a small smooth lump in consumption at that age.

The definition of the stationary non-steady-state equilibrium is similar to Definition 1, with the stationary steady-state equilibrium definition being a special case of the stationary non-steady-state equilibrium.

Definition 2 (Stationary non-steady-state equilibrium). A non-autarkic stationary non-steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability $e_{j,s}$ is defined as allocations $n_{j,s,t}$ and

Figure 4.3: Stationary steady-state distribution of consumption $\bar{c}_{j,s}$ for $S = 80$ and $J = 7$



$\hat{b}_{j,s+1,t+1}$ and prices $\hat{w}_t$ and r_t for all j, s, and t such that the following conditions hold:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings, and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\hat{\mathbf{\Gamma}}_{t+u} = \hat{\mathbf{\Gamma}}^e_{t+u} = \Omega^u\left(\hat{\mathbf{\Gamma}}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (4.4), (4.5), and (4.6)

iii. firms optimize according to (4.7) and (**??**), and

iv. markets clear according to (4.8) and (4.9).

Taken together, the household labor-leisure and intended bequest decisions in the last period of life show that the optimal labor supply and optimal intended bequests for age $s = E+S$ are each functions of individual holdings of savings, total bequests received, and the prices in that period $n_{j,E+S,t} = \phi\big(\hat{b}_{j,E+S,t}, \hat{BQ}_{j,t}, \hat{w}_t, r_t\big)$ and $\hat{b}_{j,E+S+1,t+1} = \psi\big(\hat{b}_{j,E+S,t}, \hat{BQ}_{j,t}, \hat{w}_t, r_t\big)$. These two decisions are characterized by final-age version of the static labor supply Euler equation (4.4) and the static intended bequests Euler equation (4.6). Households in their second-to-last period of life in period t have four decisions to make. They must choose how much to work this period $n_{j,E+S-1,t}$ and next period $n_{j,E+S,t+1}$, how much to save this period for next period $\hat{b}_{j,E+S,t+1}$, and how much to bequeath next period $\hat{b}_{j,E+S+1,t+2}$. The optimal responses for this individual are characterized by the $s = E+S-1$ and $s = E+S$ versions of the static Euler equations (4.4), the $s = E+S-1$ version of the intertemporal Euler equation (4.5), and the $s = E+S$ static bequest Euler equation (4.6), respectively.

Optimal savings in the second-to-last period of life $s = E+S-1$ is a function of the current savings as well as the total bequests received and prices in the current period and in the next period $\hat{b}_{j,E+S,t+1} = \psi\big(\hat{b}_{j,E+S-1,t}, \hat{BQ}_{j,t}, \hat{w}_t, r_t, \hat{BQ}_{j,t+1}, \hat{w}_{t+1}, r_{t+1}|\Omega\big)$ given beliefs Ω. As before, the optimal labor supply at age $s = E+S$ is a function of the next period's savings, bequests received, and prices $n_{j,E+S,t+1} = \phi\big(\hat{b}_{j,E+S,t+1}, \hat{BQ}_{j,t+1}, \hat{w}_{t+1}, r_{t+1}\big)$. But the optimal labor supply at age $s = E+S-1$ is a function of the current savings, current bequests received, and the current prices as well as the future bequests received and future prices because of the dependence on the savings decision in that same period $n_{j,E+S-1,t} = \phi\big(\hat{b}_{j,E+S-1,t}, \hat{BQ}_{j,t}, \hat{w}_t, r_t, \hat{BQ}_{j,t+1}, \hat{w}_{t+1}, r_{t+1}|\Omega\big)$ given beliefs Ω. By induction, we can show that the optimal labor supply, savings, and intended bequests functions for any individual with ability j, age s, and in period t is a function of current holdings of savings and the lifetime path of total bequests received and prices given beliefs Ω.

$$n_{j,s,t} = \phi\Big(\hat{b}_{j,s,t}, \big(\hat{BQ}_{j,v}, \hat{w}_v, r_v\big)_{v=t}^{t+S-s}|\Omega\Big) \quad \forall j,s,t \tag{4.10}$$

$$\hat{b}_{j,s+1,t+1} = \psi\Big(\hat{b}_{j,s,t}, \big(\hat{BQ}_{j,v}, \hat{w}_v, r_v\big)_{v=t}^{t+S-s}|\Omega\Big) \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{4.11}$$

If one knows the current distribution of households savings and intended bequests $\hat{\mathbf{\Gamma}}_t$ and has a beliefs function that predicts the law of motion over time for $\hat{\mathbf{\Gamma}}_t$, then one can predict time series for total bequests received $\hat{BQ}_{j,t}$, real wages $\hat{w}_t$ and real interest rates r_t necessary for solving each household's optimal decisions. Characteristic (i) in equilibrium definition 2 that individuals be able to forecast prices with perfect foresight over their lifetimes implies that each individual has correct information and beliefs about all the other individuals optimization problems and information. It also implies that the equilibrium allocations and prices are really just functions of the entire distribution of savings at a particular period, as well as a law of motion for that distribution of savings.

In equilibrium, the steady-state household labor supplies $\bar{n}_{j,s}$ for all j and s, the steady-state savings $\bar{b}_{j,E+S+1}$, the steady-state real wage $\bar{w}$, and the steady-state real rental rate $\bar{r}$ are simply functions of the steady-state distribution of savings $\bar{\Gamma}$. This is clear from the steady-state version of the capital market clearing condition (4.9) and the fact that aggregate labor supply is a function of the sum of exogenous efficiency units of labor in the labor market clearing condition (4.8). And the two firm first order conditions for the real wage $\hat{w}_t$ (4.7) and real rental rate r_t (**??**) are only functions of the stationary aggregate capital stock $\hat{K}_t$ and aggregate labor $\hat{L}_t$.

Figure 4.4: Equilibrium time path of K_t for $S = 80$ and $J = 7$

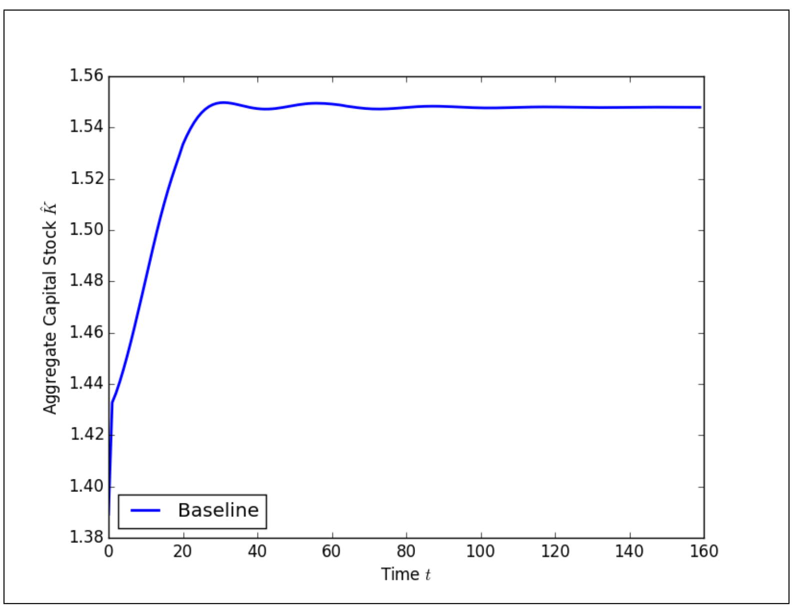
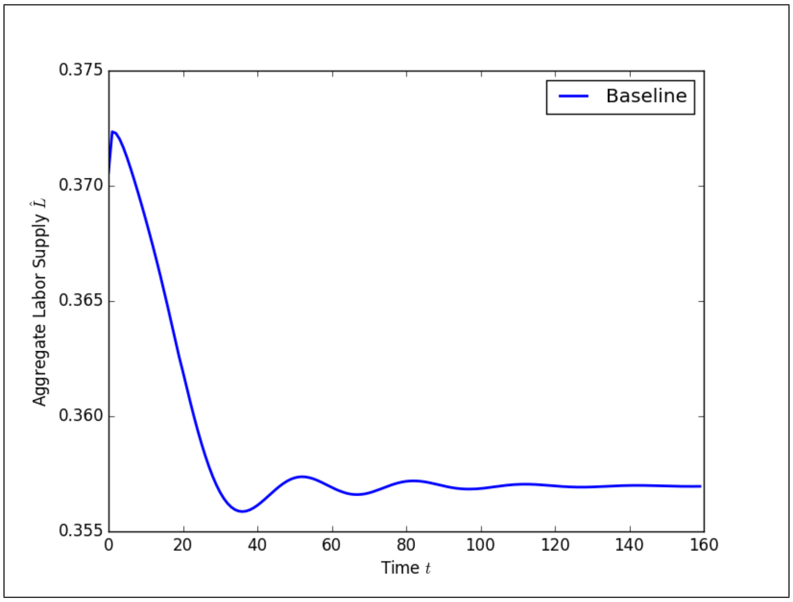


Figure 4.5: Equilibrium time path of L_t for $S = 80$ and $J = 7$



Chapter 5

Numerical Solution

To solve for any stationary non-steady-state equilibrium time path of the economy from an arbitrary current state to the steady state, we follow the time path iteration (TPI) method of Auerbach and Kotlikoff (1987). Appendix 5.2 details how to solve for the non-steady-state equilibrium time path using the TPI method. The approach is to choose an arbitrary time path for the stationary aggregate capital stock $\hat{K}_t$, stationary aggregate labor $\hat{L}_t$, and total bequests received $\hat{BQ}_{j,t}$ for each type j. This initial guess of a path implies arbitrary beliefs that violate the rational expectations requirement. We then solve for households' optimal decisions given the time paths of those variables, which decisions imply new time paths of those variables. We then update the time path as a convex combination of the initial guess and the new implied path. Figure 4.4 shows the equilibrium time path of the aggregate capital stock for the calibration described in Table 4.2 for $T = 160$ periods starting from an initial distribution of savings in which $b_{j,s,1} = \bar{\boldsymbol{\Gamma}}$ for all j and s in the case that no policy experiment takes place. The initial capital stock $\hat{K}_1$ is not at the steady state $\bar{K}$ because the initial population distribution is not at the steady-state.

5.1 Solving for stationary steady-state equilibrium

This section describes the solution method for the stationary steady-state equilibrium described in Definition 1.

1. Use the techniques in Appendix **??** to solve for the steady-state population distribution vector $\bar{\boldsymbol{\omega}}$ of the exogenous population process.

2. Choose an initial guess for the stationary steady-state distribution of capital $\bar{b}_{j,s+1}$ for all j and $s = E+2, E+3, ...E+S+1$ and labor supply $\bar{n}_{j,s}$ for all j and s.

    - A good first guess is a large positive number for all the $\bar{n}_{j,s}$ that is slightly less than $\tilde{l}$ and to choose some small positive number for $\bar{b}_{j,s+1}$ that is small enough to be less than the minimum income that an individual might have $\bar{w}e_{j,s}\bar{n}_{j,s}$.

3. Perform an unconstrained root finder that chooses $\bar{n}_{j,s}$ and $\bar{b}_{j,s+1}$ that solves the $2JS$ stationary steady-state Euler equations.

4. Make sure none of the implied steady-state consumptions $\bar{c}_{j,s}$ is less-than-or-equal-to zero.

   - If one consumption is less-than-or-equal-to zero $\bar{c}_{j,s} \leq 0$, then try different starting values.

5. Make sure that none of the Euler errors is too large in absolute value for interior stationary steady-state values. A steady-state Euler error is the following, which is supposed to be close to zero for all j and s:

$$\frac{\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}}}{\left(\bar{c}_{j,s}\right)^{-\sigma}\left(\bar{w}e_{j,s}-\frac{\partial \bar{T}_{j,s}}{\partial \bar{n}_{j,s}}\right)} - 1 \quad \forall j \quad \text{and} \quad E+1 \leq s \leq E+S \tag{5.1}$$

$$\frac{e^{-g_y\sigma}\left(\rho_s\chi^b\left(\bar{b}_{j,s+1}\right)^{-\sigma}+\beta(1-\rho_s)(\bar{c}_{j,s+1})^{-\sigma}\left[(1+\bar{r})-\frac{\partial \bar{T}_{j,s+1}}{\partial \bar{b}_{j,s+1}}\right]\right)}{\left(\bar{c}_{j,s}\right)^{-\sigma}} - 1 \quad \forall j \quad \text{and} \quad E+1 \leq s \leq E+S-1 \tag{5.2}$$

$$\frac{\chi^b e^{-g_y\sigma}(\bar{b}_{j,E+S+1})^{-\sigma}}{\left(\bar{c}_{j,E+S}\right)^{-\sigma}} - 1 \quad \forall j \tag{5.3}$$

5.2 Solving for stationary non-steady-state equilibrium by time path iteration

This section outlines the benchmark time path iteration (TPI) method of Auerbach and Kotlikoff (1987) for solving the stationary non-steady-state equilibrium transition path of the distribution of savings. TPI finds a fixed point for the transition path of the distribution of capital for a given initial state of the distribution of capital. The idea is that the economy is infinitely lived, even though the agents that make up the economy are not. Rather than recursively solving for equilibrium policy functions by iterating on individual value functions, one must recursively solve for the policy functions by iterating on the entire transition path of the endogenous objects in the economy (see Stokey and Lucas (1989, ch. 17)).

The key assumption is that the economy will reach the steady-state equilibrium described in Definition 1 in a finite number of periods $T < \infty$ regardless of the initial state. Let $\hat{\mathbf{\Gamma}}_t$ represent the distribution of stationary savings at time t.

$$\hat{\mathbf{\Gamma}}_t \equiv \left\{\left\{\hat{b}_{j,s,t}\right\}_{j=1}^{J}\right\}_{s=E+2}^{E+S+1}, \quad \forall t \tag{??}$$

In Section 4.1, we describe how the stationary non-steady-state equilibrium time path of allocations and price is described by functions of the state $\hat{\mathbf{\Gamma}}_t$ and its law of motion. TPI starts the economy at any initial distribution of savings $\hat{\mathbf{\Gamma}}_1$ and solves for its equilibrium time path over T periods to the steady-state distribution $\bar{\mathbf{\Gamma}}_T$.

The first step is to assume an initial transition path for aggregate stationary capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$, aggregate stationary labor $\hat{\boldsymbol{L}}^i = \left\{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\right\}$, and total bequests received $\hat{\boldsymbol{BQ}}_j^i = \{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^i, ...\hat{BQ}_{j,T}^i\}$ for each ability type j such that T is sufficiently large to ensure that $\hat{\mathbf{\Gamma}}_T = \bar{\mathbf{\Gamma}}$, $\hat{K}_T^i(\mathbf{\Gamma}_T)$, $\hat{L}_T^i(\mathbf{\Gamma}_T) = \bar{L}\left(\bar{\mathbf{\Gamma}}\right)$, and $\hat{BQ}_{j,T}^i(\mathbf{\Gamma}_T) = \bar{BQ}_j\left(\bar{\mathbf{\Gamma}}\right)$ for all $t \geq T$. The superscript i is an index for the iteration number. The transition paths for aggregate capital and aggregate labor determine the transition paths for both the real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$ and the real return on investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. The time paths for the total bequests received also figure in each period's budget constraint and are determined by the distribution of savings and intended bequests.

The exact initial distribution of capital in the first period $\hat{\mathbf{\Gamma}}_1$ can be arbitrarily chosen as long as it satisfies the stationary capital market clearing condition (4.9).

$$\hat{K}_1 = \frac{1}{1+\tilde{g}_{n,1}}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\hat{\omega}_{s-1,0}\lambda_j\hat{b}_{j,s,1} \tag{5.1}$$

Simiilarly, each initial value of total bequests received $\hat{BQ}_{j,1}^i$ must be consistent with the initial distribution of capital through the stationary version of (??).

$$\hat{BQ}_{j,1} = \frac{(1+r_1)\lambda_j}{1+\tilde{g}_{n,1}}\sum_{s=E+1}^{E+S}\rho_s\hat{\omega}_{s,0}\hat{b}_{j,s+1,1} \quad \forall j \tag{5.2}$$

However, this is not the case with $\hat{L}_1^i$. Its value will be endogenously determined in the same way the K_2^i is. For this reason, a logical initial guess for the time path of aggregate labor is the steady state in every period $L_t^1 = \bar{L}$ for all $1 \leq t \leq T$.

It is easiest to first choose the initial distribution of savings $\hat{\mathbf{\Gamma}}_1$ and then choose an initial aggregate capital stock $\hat{K}_1^i$ and initial total bequests received $\hat{BQ}_{j,1}^i$ that correspond to that distribution. As mentioned earlier, the only other restrictions on the initial transition paths for aggregate capital, aggregate labor, and total bequests received is that they equal their steady-state levels $\hat{K}_T^i = \bar{K}\left(\bar{\mathbf{\Gamma}}\right)$, $\hat{L}_T^i = \bar{L}\left(\bar{\mathbf{\Gamma}}\right)$, and $\hat{BQ}_{j,T}^i = \bar{BQ}_j\left(\bar{\mathbf{\Gamma}}\right)$ by period T. Evans and Phillips (2014) have shown that the initial guess for the aggregate capital stocks $\hat{K}_t^i$ for periods $1 < t < T$ can take on almost any positive values satisfying the constraints above and still have the time path iteration converge.

Given the initial savings distribution $\hat{\mathbf{\Gamma}}_1$ and the transition paths of aggregate capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$, aggregate labor $\hat{\boldsymbol{L}}^i = \left\{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\right\}$, and total bequests received $\hat{\boldsymbol{BQ}}_j^i = \left\{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^i, ...\hat{BQ}_{j,T}^i\right\}$, as well as the resulting real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$, and real return to savings $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, one can solve for the period-1 optimal labor supply and intended bequests for each type j of $s = E+S$-aged agents in the last period of their lives $n_{j,E+S,1} = \phi_{j,E+S}(\hat{b}_{j,E+S,1}, \hat{BQ}_{j,E+S,1}, \hat{w}_1, r_1)$ and $\hat{b}_{j,E+S+1,2} = \psi_{j,E+S}(\hat{b}_{j,E+S,1}, \hat{BQ}_{j,E+S,1}, \hat{w}_1, r_1)$ using his two $s = E+S$ static Euler equations (4.4) and (4.6).

$$
\begin{aligned}
&(\hat{c}_{j,E+S,1})^{-\sigma}\left(\hat{w}_1^i e_{j,E+S} - \frac{\partial \hat{T}_{j,E+S,1}}{\partial n_{j,E+S,1}}\right) = ... \\
&\qquad \chi^n_{E+S}\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,E+S,1}}{\tilde{l}}\right)^{v-1}\left[1 - \left(\frac{n_{j,E+S,1}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}} \quad \forall j \\
&\text{where} \quad \hat{c}_{j,E+S,1} = ... \\
&\qquad \left(1 + r_1^i\right)\hat{b}_{j,E+S,1} + \hat{w}_1^i e_{j,E+S} n_{j,E+S,1} + \frac{\hat{BQ}_{j,1}}{\lambda_j} - e^{g_y}\hat{b}_{j,E+S+1,2} - \hat{T}_{j,E+S,1} \\
&\text{and} \quad \frac{\partial \hat{T}_{j,E+S,1}}{\partial n_{j,E+S,1}} = ... \\
&\qquad \hat{w}_1^i e_{j,E+S}\left[\tau^I\left(F\hat{a}_{j,E+S,1}\right) + \frac{\hat{a}_{j,E+S,1} CDF\left[2A(F\hat{a}_{j,E+S,1}) + B\right]}{\left[A(F\hat{a}_{j,E+S,1})^2 + B(F\hat{a}_{j,E+S,1}) + C\right]^2} + \tau^P\right]
\end{aligned} \tag{5.3}
$$

$$
(\hat{c}_{j,E+S,1})^{-\sigma} = \chi^b e^{-g_y \sigma}(\hat{b}_{j,E+S+1,2})^{-\sigma} \quad \forall j \tag{5.4}
$$

Note that this is simply two equations (5.3) and (5.4) and two unknowns $n_{j,E+S,1}$ and $\hat{b}_{j,E+S+1,2}$.

We then solve the problem for all j types of $E+S-1$-aged individuals in period $t = 1$, each of which entails labor supply decisions in the current period $n_{j,E+S-1,1}$

and in the next period $n_{j,E+S,2}$, a savings decision in the current period for the next period $\hat{b}_{j,E+S,2}$ and an intended bequest decision in the last period $\hat{b}_{j,E+S+1,3}$. The labor supply decision in the initial period and the savings period in the initial period for the next period for each type j of $E+S-1$-aged individuals are policy functions of the current savings and the total bequests received and prices in this period and the next $\hat{b}_{j,E+S,2} = \psi_{j,E+S-1}(\hat{b}_{j,E+S-1,1}, \{\hat{BQ}_{j,t}, \hat{w}_t, r_t\}_{t=1}^2)$ and $\hat{n}_{j,E+S-1,1} = \phi_{j,E+S-1}(\hat{b}_{j,E+S-1,1}, \{\hat{BQ}_{j,t}, \hat{w}_t, r_t\}_{t=1}^2)$. The labor supply and intended bequests decisions in the next period are simply functions of the savings, total bequests received, and prices in that period $\hat{n}_{j,E+S,2} = \phi_{j,E+S}(\hat{b}_{j,E+S,2}, \hat{BQ}_{j,2}, \hat{w}_2, r_2)$ and $\hat{b}_{j,E+S+1,3} = \psi_{j,E+S}(\hat{b}_{j,E+S,2}, \hat{BQ}_{j,2}, \hat{w}_2, r_2)$. These four functions are characterized by the following versions of equations (4.4), (4.5), and (4.6).

$$(\hat{c}_{j,E+S-1,1})^{-\sigma}\left(\hat{w}_1^i e_{j,E+S-1} - \frac{\partial \hat{T}_{j,E+S-1,1}}{\partial n_{j,E+S-1,1}}\right) = ... \\ \chi^n_{E+S-1}\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,E+S-1,1}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,E+S-1,1}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}} \quad \forall j \tag{5.5}$$

$$(\hat{c}_{j,E+S-1,1})^{-\sigma} = ... \\ e^{-g_y\sigma}\left(\rho_{E+S-1}\chi^b\left(\hat{b}_{j,E+S,2}\right)^{-\sigma} + \beta(1-\rho_{E+S-1})(\hat{c}_{j,E+S,2})^{-\sigma}\left[(1+r_2^i) - \frac{\partial T_{j,E+S,2}}{\partial b_{j,E+S,2}}\right]\right) \\ \forall j \\ \text{where} \quad \frac{\partial T_{j,E+S,2}}{\partial b_{j,E+S,2}} = ... \\ r_2^i\left(\tau^I(F\hat{a}_{j,E+S,2}) + \frac{F\hat{a}_{j,E+S,2}CD\left[2A(F\hat{a}_{j,E+S,2})+B\right]}{\left[A(F\hat{a}_{j,E+S,2})^2 + B(F\hat{a}_{j,E+S,2}) + C\right]^2}\right)... \\ \tau^W(\hat{b}_{j,E+S,2}) + \frac{\hat{b}_{j,E+S,2}PHM}{\left(H\hat{b}_{j,E+S,2}+M\right)^2} \tag{5.6}$$

$$(\hat{c}_{j,E+S,2})^{-\sigma}\left(\hat{w}_2^i e_{j,E+S} - \frac{\partial \hat{T}_{j,E+S,2}}{\partial n_{j,E+S,2}}\right) = ... \\ \chi^n_{E+S}\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,E+S,2}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,E+S,2}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}} \quad \forall j \tag{5.7}$$

$$(\hat{c}_{j,E+S,2})^{-\sigma} = \chi^b e^{-g_y\sigma}(\hat{b}_{j,E+S+1,3})^{-\sigma} \quad \forall j \tag{5.8}$$

Note that this is four equations (5.5), (5.6), (5.7), and (5.8) and four unknowns $n_{j,E+S-1,1}$, $\hat{b}_{j,E+S,2}$, $n_{j,E+S,2}$, and $\hat{b}_{j,E+S+1,3}$.

This process is repeated for every age of household alive in $t=1$ down to the age $s=E+1$ household at time $t=1$. Each of these households j solves the full set of remaining $S-s+1$ labor supply decisions, $S-s$ savings decisions, and one intended bequest decision at the end of life. After the full set of lifetime decisions has been solved for all the households alive at time $t=1$, each ability j household born in period $t\geq 2$ can be solved for, the solution to which is characterized by the following full set of Euler equations analogous to (4.4), (4.5), and (4.6).

$$(\hat{c}_{j,s,t})^{-\sigma}\left(\hat{w}^i_t e_{j,s}-\frac{\partial \hat{T}_{j,s,t}}{\partial n_{j,s,t}}\right)=\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)\right]^{\frac{1-v}{v}} \quad (5.9)$$
$$\forall j \quad \text{and} \quad E+1\leq s\leq E+S \quad \text{and} \quad t\geq 2$$

$$\begin{aligned}(\hat{c}_{j,s,t})^{-\sigma}&=...\\ e^{-g_y\sigma}&\left(\rho_s\chi^b\left(\hat{b}_{j,s+1,t+1}\right)^{-\sigma}+\beta(1-\rho_s)(\hat{c}_{j,s+1,t+1})^{-\sigma}\left[(1+r^i_{t+1})-\frac{\partial T_{j,s+1,t+1}}{\partial b_{j,s+1,t+1}}\right]\right)\end{aligned} \quad (5.10)$$
$$\forall j \quad \text{and} \quad E+1\leq s\leq E+S-1 \quad \text{and} \quad t\geq 2$$

$$(\hat{c}_{j,E+S,t})^{-\sigma}=\chi^b e^{-g_y\sigma}(\hat{b}_{j,E+S+1,t+1})^{-\sigma} \quad \forall j \quad \text{and} \quad t\geq 2 \quad (5.11)$$

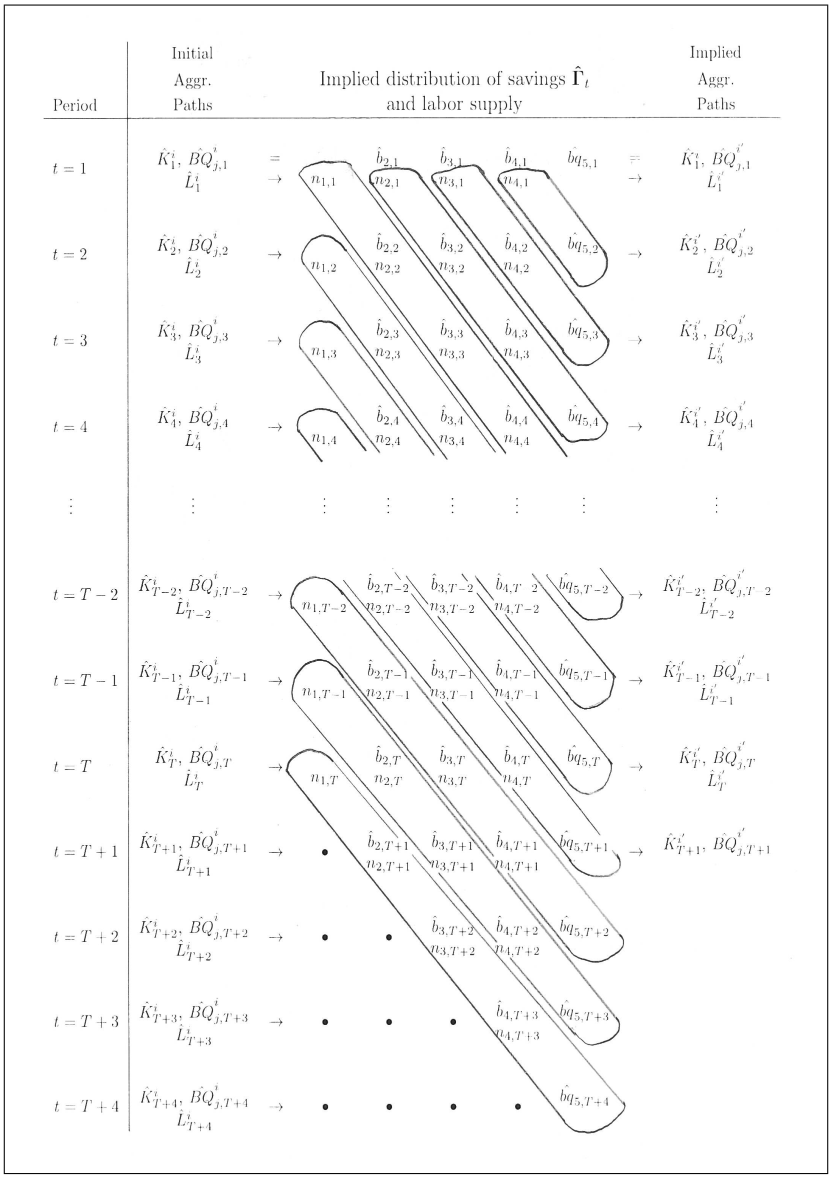
For each household of ability type j entering the economy in period $t\geq 1$, the entire set of $2S$ lifetime decisions is characterized by the $2S$ equations represented in (5.9), (5.10), and (5.11).

We can then solve for the entire lifetime of savings and labor supply decisions for each age $s=1$ individual in periods $t=2,3,...T$. The central part of the schematic diagram in Figure 5.1 shows how this process is done in order to solve for the equilibrium time path of the economy from period $t=1$ to T. Note that for each full lifetime savings and labor supply path solved for an individual born in period $t\geq 2$, we can solve for the aggregate capital stock and total bequests received implied by those savings decisions $\hat{\boldsymbol{K}}^{i'}$ and $\hat{\boldsymbol{BQ}}^{i'}_j$ and aggregate labor implied by those labor supply decisions $\hat{\boldsymbol{L}}^{i'}$.

Once the set of lifetime saving and labor supply decisions has been computed for all individuals alive in $1\leq t\leq T$, we use the household decisions to compute a new implied time path of the aggregate capital stock and aggregate labor. The implied paths of the aggregate capital stock $\hat{\boldsymbol{K}}^{i'}=\{\hat{K}^i_1,\hat{K}^{i'}_2,...\hat{K}^{i'}_T\}$, aggregate labor $\hat{\boldsymbol{L}}^{i'}=\{\hat{L}^i_1,\hat{L}^{i'}_2,...\hat{L}^{i'}_T\}$, and total bequests received $\hat{\boldsymbol{BQ}}^{i'}_j=\{\hat{BQ}^i_{j,1},\hat{BQ}^{i'}_{j,2},...\hat{BQ}^{i'}_{j,T}\}$ in general do not equal the initial guessed paths $\hat{\boldsymbol{K}}^i=\{\hat{K}^i_1,\hat{K}^i_2,...\hat{K}^i_T\}$, $\hat{\boldsymbol{L}}^i=\{\hat{L}^i_1,\hat{L}^i_2,...\hat{L}^i_T\}$, and $\hat{\boldsymbol{BQ}}^i_j=\{\hat{BQ}^i_{j,1},\hat{BQ}^i_{j,2},...\hat{BQ}^i_{j,T}\}$ used to compute the household savings and labor supply decisions $\hat{\boldsymbol{K}}^{i'}\neq\hat{\boldsymbol{K}}^i$, $\hat{\boldsymbol{L}}^{i'}\neq\hat{\boldsymbol{L}}^i$, and $\hat{\boldsymbol{BQ}}^{i'}_j\neq\hat{\boldsymbol{BQ}}^i_j$.

Let $\|\cdot\|$ be a norm on the space of time paths of the aggregate capital stock $\hat{\boldsymbol{K}}\in\mathcal{K}\subset\mathbb{R}^T_{++}$, aggregate labor supply $\hat{\boldsymbol{L}}\in\mathcal{L}\subset\mathbb{R}^T_{++}$, and J paths of total bequests received $\hat{\boldsymbol{BQ}}_j\in\mathcal{B}\subset\mathbb{R}^T_{++}$. Then the fixed point necessary for the equilibrium transition path from Definition 2 has been found when the distance between these

Figure 5.1: Diagram of TPI solution method within each iteration for $S = 4$ and $J = 1$



$J+2$ paths is arbitrarily close to zero.

$$\left\|\left[\hat{\boldsymbol{K}}^{i'},\hat{\boldsymbol{L}}^{i'},\{\hat{\boldsymbol{BQ}}_j^{i'}\}_{j=1}^J\right]-\left[\hat{\boldsymbol{K}}^{i},\hat{\boldsymbol{L}}^{i},\{\hat{\boldsymbol{BQ}}_j^{i}\}_{j=1}^J\right]\right\|\leq\varepsilon\quad\text{for}\quad\varepsilon>0 \tag{5.12}$$

If the fixed point has not been found $\left\|\left[\hat{\boldsymbol{K}}^{i'},\hat{\boldsymbol{L}}^{i'},\{\hat{\boldsymbol{BQ}}_j^{i'}\}_{j=1}^J\right]-\left[\hat{\boldsymbol{K}}^{i},\hat{\boldsymbol{L}}^{i},\{\hat{\boldsymbol{BQ}}_j^{i}\}_{j=1}^J\right]\right\|>\varepsilon$, then new transition paths for the aggregate capital stock and aggregate labor are generated as a convex combination of $\left[\hat{\boldsymbol{K}}^{i'},\hat{\boldsymbol{L}}^{i'},\{\hat{\boldsymbol{BQ}}_j^{i'}\}_{j=1}^J\right]$ and $\left[\hat{\boldsymbol{K}}^{i},\hat{\boldsymbol{L}}^{i},\{\hat{\boldsymbol{BQ}}_j^{i}\}_{j=1}^J\right]$.

$$\begin{aligned}
\hat{\boldsymbol{K}}^{i+1}&=\nu\hat{\boldsymbol{K}}^{i'}+(1-\nu)\hat{\boldsymbol{K}}^{i}\\
\hat{\boldsymbol{L}}^{i+1}&=\nu\hat{\boldsymbol{L}}^{i'}+(1-\nu)\hat{\boldsymbol{L}}^{i}\\
\hat{\boldsymbol{BQ}}_1^{i+1}&=\nu\hat{\boldsymbol{BQ}}_1^{i'}+(1-\nu)\hat{\boldsymbol{BQ}}_1^{i}\qquad\text{for}\quad\nu\in(0,1]\\
&\vdots\\
\hat{\boldsymbol{BQ}}_J^{i+1}&=\nu\hat{\boldsymbol{BQ}}_J^{i'}+(1-\nu)\hat{\boldsymbol{BQ}}_J^{i}
\end{aligned} \tag{5.13}$$

This process is repeated until the initial transition paths for the aggregate capital stock, aggregate labor, and total bequests received are consistent with the transition paths implied by those beliefs and household and firm optimization.

In essence, the TPI method iterates on individual beliefs about the time path of prices represented by a time paths for the aggregate capital stock $\hat{\boldsymbol{K}}^i$, aggregate labor $\hat{\boldsymbol{L}}^i$, and total bequests received $\hat{\boldsymbol{BQ}}_j^i$ until a fixed point in beliefs is found that are consistent with the transition paths implied by optimization based on those beliefs.

The following are the steps for computing a stationary non-steady-state equilibrium time path for the economy.

1. Input all initial parameters. See Table 4.2.

    (a) The value for T at which the non-steady-state transition path should have converged to the steady state should be at least as large as the number of periods it takes the population to reach its steady state $\bar{\boldsymbol{\omega}}$ as described in Appendix **??**.

2. Choose an initial distribution of savings and intended bequests $\hat{\boldsymbol{\Gamma}}_1$ and then calculat the initial state of the stationarized aggregate capital stock $\hat{K}_1$ and total bequests received $\hat{BQ}_{j,1}$ consistent with $\hat{\boldsymbol{\Gamma}}_1$ according to (4.9) and (5.2).

    (a) Note that you must have the population weights from the previous period $\hat{\omega}_{s,0}$ and the growth rate between period 0 and period 1 $\tilde{g}_{n,1}$to calculate $\hat{BQ}_{j,1}$.

3. Conjecture transition paths for the stationarized aggregate capital stock $\hat{\boldsymbol{K}}^1=\{\hat{K}_t^1\}_{t=1}^\infty$, stationarized aggregate labor $\hat{\boldsymbol{L}}^1=\{\hat{L}_t^1\}_{t=1}^\infty$, and total bequests received $\hat{\boldsymbol{BQ}}_j^1=\{\hat{BQ}_{j,t}^1\}_{t=1}^\infty$ where the only requirements are that $\hat{K}_1^i$ and $\hat{BQ}_{j,1}^i$ are functions of the initial distribution of savings $\hat{\boldsymbol{\Gamma}}_1$ for all i is your initial state

and that $\hat{K}_t^i = \bar{K}$, $\hat{L}_t^i = \bar{L}$, and $\hat{BQ}_{j,t}^i = \bar{BQ}_j$ for all $t \geq T$. The conjectured transition paths of the aggregate capital stock $\boldsymbol{\hat{K}}^i$ and aggregate labor $\boldsymbol{\hat{L}}^i$ imply specific transition paths for the real wage $\boldsymbol{\hat{w}}^i = \{\hat{w}_t^i\}_{t=1}^\infty$ and the real interest rate $\boldsymbol{r}^i = \{r_t^i\}_{t=1}^\infty$ through expressions (4.7) and (??).

   (a) An intuitive choice for the time path of aggregate labor is the steady-state in every period $\hat{L}_t^1 = \bar{L}$ for all t.

4. With the conjectured transition paths $\boldsymbol{\hat{w}}^i$, $\boldsymbol{r}^i$, and $\boldsymbol{\hat{BQ}}_j^i$ one can solve for the lifetime policy functions of each household alive at time $1 \leq t \leq T$ using the systems of Euler equations of the form (4.4), (4.5), and (4.6) and following the diagram in Figure 5.1.

5. Use the implied distribution of savings and labor supply in each period (each row of $\hat{b}_{j,s,t}$ and $n_{j,s,t}$ in Figure 5.1) to compute the new implied time paths for the aggregate capital stock $\boldsymbol{\hat{K}}^{i'} = \{\hat{K}_1^i, \hat{K}_2^{i'}, ...\hat{K}_T^{i'}\}$, aggregate labor supply $\boldsymbol{\hat{L}}^{i'} = \{\hat{L}_1^i, \hat{L}_2^{i'}, ...\hat{L}_T^{i'}\}$, and total bequests received $\boldsymbol{\hat{BQ}}_j^{i'} = \{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^{i'}, ...\hat{BQ}_{j,T}^{i'}\}$.

6. Check the distance between the two sets time paths.

$$\left\| \left[\boldsymbol{\hat{K}}^{i'}, \boldsymbol{\hat{L}}^{i'}, \left\{\boldsymbol{\hat{BQ}}_j^{i'}\right\}_{j=1}^J\right] - \left[\boldsymbol{\hat{K}}^{i}, \boldsymbol{\hat{L}}^{i}, \left\{\boldsymbol{\hat{BQ}}_j^{i}\right\}_{j=1}^J\right] \right\|$$

   (a) If the distance between the initial time paths and the implied time paths is less-than-or-equal-to some convergence criterion $\varepsilon > 0$, then the fixed point has been achieved and the equilibrium time path has been found (5.12).

   (b) If the distance between the initial time paths and the implied time paths is greater than some convergence criterion $\|\cdot\| > \varepsilon$, then update the guess for the time paths according to (5.13) and repeat steps (4) through (6) until a fixed point is reached.

Chapter 6

Miscellaneous

6.1 Incorporating Feedbacks with Micro Tax Simulations

Follow this algorthim:

- Period 1
  - Use current IRS public use sample.
  - Run the following within-period routine
    - Do the static tax analysis of this sample, save the results
    - Summarize the public use sample by aggregating into bins over age and earnings ability
    - Use this as a starting point for the dynamic macro model
    - Get values for fundamental interest rates and effective wages for next period
- Period 2
  - Age the public use data demographically by one year.
  - Let wages and interest rates rise by the amounts predicted in the macro model.
  - Rerun the within-period routine
- Iterate over periods until end of forecast period is reached.

6.2 Calibration

6.2.1 Tax Bend Points

We use IRS data which summarizes individual tax returns for 2011 by 19 income categories and 4 filing statuses. For each filing status we fit the mapping from reported income into adjusted gross income (AGI) using a sufficiently high-order polynomial. We then use this function to solve for the income level which corresponds to each of the five bend points in the tax code for each filing type.

Table 6.1: AGI and Income Bend Points

AGI Bend Points

Tax rate	Married Joint	Married Separate	Head of Household	Single
10%	17,400	8700	12,400	8700
15%	70,700	35,350	47,350	35,350
25%	142,700	71,350	122,300	85,650
28%	217,450	108,725	198,050	178,650
33%	388,350	194,175	388,350	388,350

Corresponding Reported Income Bendpoints

Tax rate	Married Joint	Married Separate	Head of Household	Single
0%	5850	91	756	1435
10%	22,932	8591	12,911	9956
15%	75,181	34,592	47,023	36,021
25%	145,866	69,768	120,200	85,244
28%	219,162	106,245	194,176	176,270
33%	386,798	189,674	380,043	381,524

We then fit a bivariate probabililty density function over income and filing type from the data. For each bendpoint we calculate the probability density at that bendpoint and use these as weights in a weighted average over filing types to generate an aggregate bendpoint.

Table 6.2: Aggregated Bend Points

Tax rate	Bend Point
0%	2889
10%	15,116
15%	52,580
25%	114,552
28%	196,201
33%	380,657

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