Working through Fullerton + Rogers cons. model ①

Composite consumption good modeled as Stone-Geary function of the individual consumption goods:

$$① \quad \tilde{c}_t = \prod_{i=1}^{N} (c_{it} - b_{it})^{\beta_{it}}$$

N = # cons. goods

b_{it} = min required purchase of good i at age t

β_{it} = share parameter that varies w/ age

→ reflects changing composition of consumption over lifetime

→ NOTE: $\sum_{i=1}^{N} \beta_{it} = 1$

Fullerton + Rogers reference Starrett (1988) and Summers (1981) and say min cons helps w/ more realistic save behavior

→ FR give nice justification for this func form

Consumer maximizes ① s.t. budget constraint:

$$\sum_{i=1}^{N} p_i (c_{it} - b_{it}) = \hat{p}_t \tilde{c}_t$$

→ price of agg cons. good → price varies by age since composition of composite good varies by age

→ agg cons. demand

* Why don't min required purchases figure into the B.C.? i.e. why subtract b_{it} there?

I guess makes sense since $\tilde{c}_t$ above defined as net of min cons and resulting demands do make sense

but do we need to adj cons. budget constraint → e.g. if $c_{it} = b_{it} \ \forall i$ then $\tilde{c}_t = 0$ but still outlay to purchase $c_{it} = b_{it}$, right?

p_i = gross of tax price of good i

→ constant in SS

→ what @ outside?

→ varies over trans path, but not dynamic

So do I do:

$$\cancel{\sum_{i=1}^{N} p_i (c_{it} - b_{it}) = \tilde{p}_t \prod_{i=1}^{N} (c_{it} - b_{it})^{\beta_{it}}}$$

$$\mathcal{L} = \prod_{i=1}^{N} (c_{it} - b_{it})^{\beta_{it}} + \lambda \left(\tilde{p}_t \hat{c}_t - \sum_{i=1}^{N} p_i (c_{it} - b_{it}) \right)$$

$$\frac{\partial \mathcal{L}}{\partial c_{it}} = \beta_{it} \frac{\prod_{i=1}^{N} (c_{it} - b_{it})^{\beta_{it}}}{(c_{it} - b_{it})} - \lambda p_i = 0, \quad \forall i, t$$

$$\Rightarrow \lambda p_i = \frac{\beta_{it}}{c_{it} - b_{it}} \prod_{i=1}^{N} (c_{it} - b_{it})^{\beta_{it}} \qquad \forall i$$

$$\Rightarrow \lambda = \frac{\beta_{it}}{p_i(c_{it} - b_{it})} \prod_{i=1}^{N} (c_{it} - b_{it})^{\beta_{it}} , \quad \forall i$$

$$\Rightarrow \frac{\beta_{it}}{p_i(c_{it} - b_{it})} = \frac{\beta_{jt}}{p_j(c_{jt} - b_{jt})} , \quad \forall i,j$$

$$\Rightarrow c_{it} - b_{it} = \frac{\beta_{it} \; p_j(c_{jt} - b_{jt})}{\beta_{jt} \, p_i}$$

$$\Rightarrow c_{it} = \frac{\beta_{it} \; p_j(c_{jt} - b_{jt})}{\beta_{jt} \, p_i} + b_{it}$$

$$c_{it} = \frac{\beta_{it}}{p_i} \left[\frac{p_j(c_{jt} - b_{jt})}{\beta_{jt}} \right] + b_{it}$$

plug this into: ←

$$\tilde{p}_t \tilde{c}_t = \sum_{i=1}^{N} p_i (c_{it} - b_{it})$$

$$\tilde{p}_t \tilde{c}_t = \sum_{i=1}^{N} \frac{\beta_{it}}{p_i} \left(\frac{p_j(c_{jt} - b_{jt})}{\beta_{jt}} \right)$$

$$\tilde{p}_t \tilde{c}_t = \left(\frac{p_j(c_{jt} - b_{jt})}{\beta_{jt}} \right) \sum_{i=1}^{N} \frac{\beta_{it}}{p_i}$$

$$\frac{p_i(c_{it} - b_{it})}{\beta_{it}} = \prod_{i=1}^{N} (c_{it} - b_{it})^{\beta_{it}} = \tilde{c}_t$$

$$\frac{\tilde{c}_t \, \beta_{it}}{\lambda \, p_i} + b_{it} = c_{it}$$

$$\lambda = \frac{1}{\tilde{p}_t} \; ?$$

note, $\sum_i \beta_{it} = 1$

$$\tilde{p}_t \tilde{c}_t = \sum_{i=1}^{N} p_i c_{it} - \sum_{i=1}^{N} p_i b_{it}$$

$$\tilde{p}_t \tilde{c}_t = \sum_{i=1}^{N} \cancel{p_i} \frac{\beta_{it}}{\cancel{p_i}} \left[\frac{p_j(c_{jt} - b_{jt})}{\beta_{jt}} \right] + p_i b_{it} - \sum_{i=1}^{N} p_i b_{it}$$

$$= \sum_{i} \left[\frac{p_j(c_{jt} - b_{jt})}{\beta_{jt}} \right] \underbrace{\sum_{i=1}^{N} \beta_{it}}_{=1} + \cancel{\sum_{i=1}^{N} p_i b_{it}} \cancel{- \sum_{i=1}^{N} p_i b_{it}}$$

$$\Rightarrow \tilde{p}_t \tilde{c}_t = \frac{p_j(c_{jt} - b_{jt})}{\beta_{jt}} \Rightarrow$$

$$c_{jt} = \frac{\tilde{p}_t \tilde{c}_t \, \beta_{jt}}{p_j} + b_{jt}$$

how determine cons. basket?

→ β_{it} estimated

→ b_{it} estimated

→ $\tilde{C}_t$ from individual optimization

→ $\tilde{P}_t = ?$ aggregate price level? but varies w/ age

— "price of age t composite cons. good"

→ $P_j =$ ~~from producers' problem~~ (?) from fixed coeff (transition matrix) mapping production goods to cons. goods

$$P_j = \sum_{i=1}^{M} P_i^{\circ} z_{ji}$$

deter. earlier — trans

$$c_{it} = b_{it} + \frac{\tilde{P}_t \tilde{C}_t \beta_{it}}{P_i}$$

$$\sum_i c_{it} = \tilde{C}_t = \sum_i b_{it} + \sum_i \frac{\tilde{P}_t \tilde{C}_t \beta_{it}}{P_i}$$

$$\tilde{C}_t = \underbrace{\sum_i b_{it}}_{\tilde{B}_t} + \tilde{P}_t \tilde{C}_t \sum_{i=1}^{N} \frac{\beta_{it}}{P_i}$$

$$\tilde{C}_t - \tilde{B}_t = \tilde{P}_t \tilde{C}_t \sum \frac{\beta_{it}}{P_i}$$

$$\frac{\tilde{C}_t - \hat{B}_t}{\tilde{C}_t \sum \frac{\beta_{it}}{P_i}} = \tilde{P}_t$$

→ could find this given info above

$$\sum_{i=1}^{N} p_i(c_{it} - b_{it}) = \tilde{P}_t \prod_{i=1}^{N} (c_{it} - b_{it})^{\beta_{it}}$$

$$\tilde{C}_t = \prod_{i=1}^{N} \left(\frac{\tilde{P}_t \tilde{C}_t \beta_{it}}{P_i} + \cancel{b_{it}} - \cancel{b_{it}} \right)^{\beta_{it}}$$

$$\cancel{\tilde{C}_t} = \tilde{P}_t \cancel{\tilde{C}_t} \prod_{i=1}^{N} \left(\frac{\beta_{it}}{P_i} \right)^{\beta_{it}} \Rightarrow \frac{1}{\tilde{P}_t} = \prod_{i=1}^{N} \left(\frac{\beta_{it}}{P_i} \right)^{\beta_{it}}$$

✓

$$\Rightarrow \tilde{P}_t = \prod_{i=1}^{N} \left(\frac{P_i}{\beta_{it}} \right)^{\beta_{it}}$$

Consumer prefs over ~~produced~~ production goods

(3) $$\tilde{Q}_j = \left[\gamma_j^{1/\varepsilon_3} Q_j^{c\,(\varepsilon_3-1)/\varepsilon_3} + (1-\gamma_j)^{1/\varepsilon_3} \left(Q_j^{nc}\right)^{\frac{(\varepsilon_3-1)}{\varepsilon_3}} \right]^{\varepsilon_3/(\varepsilon_3-1)}$$

total cons. of production good j

Max (3) s.t.: $P_j^c Q_j^c + P_j^{nc} Q_j^{nc} = P_j^Q Q_j$

P_j^c = price of prod good j from corp

$$\Rightarrow \mathcal{L} = \left[\gamma_j^{1/\varepsilon_3} Q_j^{c\,\frac{\varepsilon_3-1}{\varepsilon_3}} + (1-\gamma_j)^{1/\varepsilon_3} \left(Q_j^{nc}\right)^{\frac{\varepsilon_3-1}{\varepsilon_3}} \right]^{\varepsilon_3/(\varepsilon_3-1)}$$

$$+ \lambda\left(P_j^Q Q_j - P_j^c Q_j^c - P_j^{nc} Q_j^{nc} \right)$$

(1) $$\frac{\partial \mathcal{L}}{\partial Q_j^c} = \left(\frac{\varepsilon_3}{\varepsilon_3-1}\right) \left[\gamma_j^{1/\varepsilon_3} Q_j^{c\,\frac{\varepsilon_3-1}{\varepsilon_3}} + (1-\gamma_j)^{1/\varepsilon_3} \left(Q_j^{nc}\right)^{\frac{\varepsilon_3-1}{\varepsilon_3}} \right]^{\frac{\varepsilon_3}{\varepsilon_3-1}-1}$$

$$\left(\frac{\varepsilon_3-1}{\varepsilon_3} \gamma_j^{1/\varepsilon_3} Q_j^{c\,\frac{\varepsilon_3-1}{\varepsilon_3}-1} \right) - \lambda P_j^c = 0$$

(2) $$\frac{\partial \mathcal{L}}{\partial Q_j^{nc}} = \left(\frac{\varepsilon_3}{\varepsilon_3-1}\right) \left[\gamma_j^{1/\varepsilon_3} Q_j^{c\,\frac{\varepsilon_3-1}{\varepsilon_2}} + (1-\gamma_j)^{1/\varepsilon_3} \left(Q_j^{nc}\right)^{\frac{\varepsilon_3-1}{\varepsilon_3}} \right]^{\frac{\varepsilon_3}{\varepsilon_3-1}-1}$$

$$\left[\left(\frac{\varepsilon_3-1}{\varepsilon_3}\right)(1-\gamma_j)^{1/\varepsilon_3} \left(Q_j^{nc}\right)^{\frac{\varepsilon_3-1}{\varepsilon_3}-1} \right] - \lambda P_j^{nc} = 0$$

(3) $$\frac{\partial \mathcal{L}}{\partial \lambda} = P_j^Q Q_j - P_j^c Q_j^c - P_j^{nc} Q_j^{nc} = 0$$

1 can be rewritten as:

$$\left[\gamma_j^{\frac{1}{\varepsilon_3}} Q_j^{c\,\frac{\varepsilon_3-1}{\varepsilon_3}} + (1-\gamma_j)^{\frac{1}{\varepsilon_3}} (Q_j^{nc})^{\frac{\varepsilon_3-1}{\varepsilon_3}}\right]^{\frac{\cancel{\varepsilon_3}-(\cancel{\varepsilon_3}-1)}{\varepsilon_3-1}}$$

$$\left(\gamma^{\frac{1}{\varepsilon_3}} Q_j^{c\,\frac{\cancel{\varepsilon_3}-1-\cancel{\varepsilon_3}}{\varepsilon_3}}\right) = \lambda P_j^c$$

2 as:

$$\left[\gamma_j^{\frac{1}{\varepsilon_3}} Q_j^{c\,\frac{\varepsilon_3-1}{\varepsilon_3}} + (1-\gamma_j)^{\frac{1}{\varepsilon_3}} Q_j^{nc\,\frac{\varepsilon_3-1}{\varepsilon_3}}\right]^{\frac{1}{\varepsilon_3-1}}$$

$$\left[(1-\gamma_j)^{\frac{1}{\varepsilon_3}} (Q_j^{nc})^{\frac{\cancel{\varepsilon_3}-1-\cancel{\varepsilon_3}}{\varepsilon_3}} = \lambda P_j^{nc}\right.$$

$$\frac{1}{2} = \frac{\gamma^{\frac{1}{\varepsilon_3}} Q_j^{c\,-\frac{1}{\varepsilon_3}}}{(1-\gamma)^{\frac{1}{\varepsilon_3}} Q_j^{nc\,-\frac{1}{\varepsilon_3}}} = \frac{\cancel{\lambda} P_j^c}{\cancel{\lambda} P_j^{nc}}$$

$$\Leftrightarrow \frac{\gamma^{\frac{1}{\varepsilon_3}} Q_j^{nc\,\frac{1}{\varepsilon_3}}}{(1-\gamma)^{\frac{1}{\varepsilon_3}} Q_j^{c\,\frac{1}{\varepsilon_3}}} = \frac{P_j^c}{P_j^{nc}}$$

$$\Leftrightarrow \frac{\gamma\, Q_j^{nc}}{(1-\gamma\, Q_j^c} = \left(\frac{P_j^c}{P_j^{nc}}\right)^{\varepsilon_3}$$

$$\Rightarrow Q_j^{nc} = \left(\frac{P_j^c}{P_j^{nc}}\right)^{\varepsilon_3} \cdot \frac{(1-\gamma)}{\gamma} Q_j^c$$

Plug into (3):

$$P_j^Q Q_j = P_j^c Q_j^c + P_j^{nc} \left(\frac{P_j^c}{P_j^{nc}}\right)^{\varepsilon_3} \cdot \frac{(1-\gamma)}{\gamma} Q_j^c$$

$$\gamma \cdot P_j^Q Q_j = \left\{ Q_j^c \left[P_j^c + P_j^{nc\,1-\varepsilon_3} P_j^{c\,\varepsilon_3} \left(\frac{1-\gamma}{\gamma}\right)\right]\right\} \cdot \gamma$$

$$\gamma P_j^Q Q_j = Q_j^c \left[\gamma P_j^c + (1-\gamma) P_j^{c\,\varepsilon_3} P_j^{nc\,1-\varepsilon_3}\right]$$

$$\Rightarrow Q_j^c = \frac{\gamma P_j^Q Q_j}{\left[\gamma P_j^c + (1-\gamma) P_j^{c\,\varepsilon_3} P_j^{nc\,1-\varepsilon_3}\right]}$$

$$\Rightarrow Q_j^c = \frac{\gamma P_j^Q Q_j}{P_j^{c\,\varepsilon_3}\left[\gamma P_j^{c\,1-\varepsilon_3} + (1-\gamma) P_j^{nc\,1-\varepsilon_3}\right]}$$

now find Q_j^{nc} :

$$Q_j^{nc} = \left(\frac{P_j^c}{P_j^{nc}}\right)^{\varepsilon_3} \cdot \left(\frac{1-\gamma}{\gamma}\right) Q_j^c$$

$$= \left(\frac{\cancel{P_j^c}}{P_j^{nc}}\right)^{\cancel{\varepsilon_3}} \frac{(1-\gamma)}{\cancel{\gamma}} \left[\frac{\cancel{\gamma} P_j^Q Q_j}{\cancel{P_j^{c\,\varepsilon_3}}\left[\gamma P_j^{c\,1-\varepsilon_3} + (1-\gamma) P_j^{nc\,1-\varepsilon_3}\right]}\right]$$

$$Q_j^{nc} = \frac{(1-\gamma) P_j^Q Q_j}{P_j^{nc\,\varepsilon_3}\left[\gamma P_j^{c\,1-\varepsilon_3} + (1-\gamma) P_j^{nc\,1-\varepsilon_3}\right]}$$

$$P_j^Q Q_j = P_j^{nc} Q_j^{nc} + P_j^c Q_j^{nc}$$

$$= P_j^{nc} \frac{(1-\gamma) P_j^Q Q_j}{P_j^{nc\,\varepsilon_3}\left[\gamma P_j^{c\,1-\varepsilon_3} + (1-\gamma) P_j^{nc\,1-\varepsilon_3}\right]} + P_j^c \frac{\gamma P_j^Q Q_j}{P_j^{c\,\varepsilon_3}\left[\gamma P_j^{c\,1-\varepsilon_3} + (1-\gamma) P_j^{nc\,1-\varepsilon_3}\right]}$$

$$\Rightarrow P_j^Q =$$

How determine amounts?

$\rightarrow P_j^Q =$ price of composite output
→ use same coeff matrix as below and prices of cons goods?

$\rightarrow Q_j =$ ~~after determine demands for each cons. good, use transition matrix to map how these demands for cons. goods map into demands for production goods~~

~~Q~~ ok

see pg 8

$\rightarrow P_j^{nc} =$ ~~from produce~~ problem? see pg 9 → from firm problem

$\rightarrow P_j^c =$

To determine $P_j^Q \rightarrow$

→ note that $U(x,y)$ is linearly homogenous:

~~U(λx, λ~~ $U(\lambda Q_j^c, \lambda Q_j^{nc}) = \lambda U(Q_j^c, Q_j^{nc})$

~~$U(Q_j^c, Q_j^{nc})$~~ =

→ Because $U(\cdot,\cdot)$ is linearly homogenous, we know indirect utility, V, is homogenous of degree one in Q:

$$V(p_j^c, p_j^{nc}, \lambda Q_j) = \lambda V(p_j^c, p_j^{nc}, Q_j)$$

and V is homogenous of degree -1 in p:

$$V(\lambda p_j^c, \lambda p_j^{nc}, Q_j) = \frac{V(p_j^c, p_j^{nc}, Q_j)}{\lambda}$$

also know that when utility is linearly homogenous; ~~...~~.

$$V(p_j^c, p_j^{nc}, Q_j) = \frac{P_j^Q Q_j}{e(p_j^c, p_j^{nc})}$$, where

$e(p_j^c, p_j^{nc})$ = the min cost for a unit of utility

$$\Rightarrow e(p_j^c, p_j^{nc}) = \frac{P_j^Q Q_j}{V(p_j^c, p_j^{nc}, Q_j)}$$

$$= \frac{Q_j}{\left[\gamma_j^{1/\varepsilon_3}\left(\frac{\gamma P_j^Q Q_j}{p_j^{c\,\varepsilon_3}[\cdot]}\right)^{\frac{\varepsilon_3-1}{\varepsilon_3}} + (1-\gamma_j)^{1/\varepsilon_3}\left(\frac{(1-\gamma)P_j^Q Q_j}{p_j^{nc\,\varepsilon_3}[\cdot]}\right)^{\frac{\varepsilon_3-1}{\varepsilon_3}}\right]^{\frac{\varepsilon_3}{\varepsilon_3-1}}}$$

$$\gamma^{\frac{\varepsilon_3-1}{\varepsilon_3}+\frac{1}{\varepsilon_3}} = \gamma^{\frac{\varepsilon_3}{\varepsilon_3}} = \gamma$$

$$= \frac{Q_j}{\left[\gamma_j^{1/\varepsilon_3}\left(\frac{\gamma P_j^Q Q_j}{[\cdot]}\right)^{\frac{\varepsilon_3-1}{\varepsilon_3}} p_j^{c\,-\varepsilon_3\left(\frac{\varepsilon_3-1}{\varepsilon_3}\right)} + (1-\gamma_j)^{1/\varepsilon_3}\left(\frac{(1-\gamma)P_j^Q Q_j}{[\cdot]}\right)^{\frac{\varepsilon_3-1}{\varepsilon_3}} p_j^{nc\,\ldots}\right]^{\frac{\varepsilon_3}{\varepsilon_3-1}}}$$

$$e(p^{Q}_j, p^{nc}_j, Q_j) \equiv \frac{\cancel{Q_j}}{\frac{p^{Q}_j \cancel{Q_j}}{[\cdot]}\left[\gamma\, p^{c\ 1-\varepsilon_3}_j + (1-\gamma_j)\, p^{nc\ 1-\varepsilon_3}_j\right]^{\varepsilon_3/\varepsilon_3-1}}$$

$$= \frac{[\cdot]}{p^{Q}_j [\cdot]^{\varepsilon_3/\varepsilon_3-1}}$$

$$= \frac{[\cdot]^{1-\frac{\varepsilon_3}{\varepsilon_3-1}}}{p^{Q}_j} = \frac{[\cdot]^{\frac{\varepsilon_3-1-\varepsilon_3}{\varepsilon_3-1}}}{p^{Q}_j} = \frac{[\cdot]^{-\frac{1}{\varepsilon_3-1}}}{p^{Q}_j}$$

Indirect Util:

$$V(p^{c}_j, p^{nc}_j, p^{Q}_j, Q_j) = p^{Q}_j Q_j [\cdot]^{\frac{\varepsilon_3-\varepsilon_3+1}{\varepsilon_3-1}} = p^{Q}_j Q_j [\cdot]^{\frac{1}{\varepsilon_3-1}}$$

$$e(p^{c}_j, p^{nc}_j) = \text{price of comp} = p^{Q}_j = \frac{p^{Q}_j Q_j}{V(p^{c}_j, p^{nc}_j, p^{Q}_j Q_j)}$$

$$= \frac{\cancel{p^{Q}_j} Q_j}{\cancel{p^{Q}_j} Q_j [\cdot]^{\frac{1}{\varepsilon_3-1}}}$$

$$= [\cdot]^{-\frac{1}{\varepsilon_3-1}} = [\cdot]^{\frac{1}{1-\varepsilon_3}}$$

How get P_j^{nc}, P_j^c ?

Zero profit condition?

→ I think yes

→ final price$_j$ = mc_j

where mc_j is the cost of producing one unit of output of good j given optimal mix of capital and labor

→ this a function of r, w

→ I haven't written this for supply side of model, but think I can do similar to what's done in Tax DSGE paper

→ so guess ω, r ⟶ then all other prices fall out

⇒ P_j^{nc}, P_j^c

after aggregates, everything else here is static

→ P_j^{nc}, P_j^c ⇒ P_j^Q by CES utili from goods across ~~goos~~ sectors

→ P_j^Q ⇒ P_i by transition matrix (fixed coeffs)

→ P_i ⇒ $\tilde{P}_t = \prod_{i=1}^{N} \left(\frac{P_i}{\beta_{i,t}}\right)^{\beta_{it}}$

$\tilde{P}_t$ ~~agg price level~~ price of composite cons. good for ~~agg~~ age + indiv

except does seem like need to think @ how consumers forecast how $\tilde{P}_t$ varies over time when make intertemporal cons/save decision → ok, but have whole path of $\tilde{P}_t$ when guess w, r paths

→ ~~[illegible]~~

What about quantities across levels of cons/prod?

$\rightarrow Q^{c}_{jt} + Q^{nc}_{jt} = Q_{jt}$

$\rightarrow Q^{c}_{jt} \Rightarrow c_{it}$ by "trans" matrix

$\rightarrow \sum_t \cancel{Q_{it} = Q_i}$

$\rightarrow \sum_i c_{it} = \tilde{c}_t$ (right?) ~~or not~~ ~~w/ cons~~?

$\rightarrow \sum_t \tilde{c}_t = c$

$c + i + g = y$

(Note Q_{jt} also needs to be mapped into capital (FA's, land, inventories))

invest i

$\cancel{k_{it}}$

$\rightarrow \cancel{k_i} \; \tilde{k}$

and so for gov't cons