Chapter 11

Household taxes

In this chapter, we focus on incorporating a government sector that levies taxes on households and firms and provides transfers to overlapping generations of households based off of the endogenous labor model from Chapter 4. These taxes will be distortionary to the household's decisions, and we will show a few options of how one might model realistic tax law. In this chapter, we will make the simplifying assumption that government revenue and government outlays are equal in every period. However, we show in Chapter 12 how to model a government budget constraint that is not balanced every period.

11.1 Incorporating Tax Functions into DGE Model

Many papers have incorporated tax functions of varying richness into dynamic general equilibrium (DGE) models. When taxes are included in a model with household optimization, the taxes are often distortionary to the household decisions. Further, the existence of taxes in a general equilibrium requires extra assumptions about what a government does with those taxes.

11.1.1 Some preliminary theory on tax rates

In this chapter, we follow the simple and standard convention that tax liability $T^I_{s,t}$ is a function of some function of total household income. However, we use here a richer family of functions than most other studies by assuming that the total tax liability is a function of two types of income—labor income $x_{s,t}$ and capital income $y_{s,t}$.

$$T^I_{s,t} = T^I_{s,t}\big(x_{s,t}, y_{s,t}\big) \quad \forall s,t \tag{11.1}$$

$$x_{s,t} \equiv w_t n_{s,t} \quad \forall s,t \tag{11.2}$$

$$y_{s,t} \equiv r_t b_{s,t} \quad \forall s,t \tag{11.3}$$

Notice that the function in (11.1) is more general than a function of total income $x_{s,t} + y_{s,t}$ because a function of total income is a special case of (11.1). We leave the s and t subscripts on the function on the right-hand-side of (11.1) because one might want to estimate separate functions tax liability functions for different ages and time periods. Further, we assume that government transfers to households X_t are lump sum, equal, and are independent of household decision variables. The household budget constraint is the following,

$$\begin{split} c_{s,t} + b_{s+1,t+1} = (1 + r_t)b_{s,t} + w_t n_{s,t} + X_t - T^I_{s,t} \quad \forall s,t \\ \text{with} \quad b_{1,t}, b_{S+1,t} = 0 \end{split} \tag{11.4}$$

and the two sets of household Euler equations for labor supply and for savings are the following.

$$\left(w_t - \frac{\partial T^I_{s,t}}{\partial n_{s,t}}\right) u_1\left(c_{s,t}, n_{s,t}\right) = -u_2\left(c_{s,t}, n_{s,t}\right) \quad \text{for} \quad \forall t, \quad s \in \{1, 2, ...S\} \tag{11.5}$$

$$u_1\left(c_{s,t}, n_{s,t}\right) = \beta\left(1 + r_{t+1} - \frac{\partial T^I_{s+1,t+1}}{\partial b_{s+1,t+1}}\right) u_1\left(c_{s+1,t+1}, n_{s+1,t+1}\right) \quad \forall t, \quad s \in \{1, 2, ...S - 1\} \tag{11.6}$$

Note the distortionary effect of the household income tax on the Euler equations (11.5) and (11.6). We will derive these equations more carefully in Section 11.2.

We assume that the government's budget is balanced each period.

$$X_t = \frac{1}{S}\sum_{s=1}^{S} T^I_{s,t} \quad \forall t \tag{11.7}$$

Because most marginal tax rates for which we have data are based on some form of measurable income, such as labor income $x_{s,t}$ or capital income $y_{s,t}$, we need to write the marginal tax rates in our Euler equations (11.5) and (11.6) as functions of measurable income variables.

$$\frac{\partial T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{\partial n_{s,t}} = \frac{\partial T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{\partial x_{s,t}}\frac{\partial x_{s,t}}{\partial n_{s,t}} = w_t\frac{\partial T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{\partial x_{s,t}} = w_t\tau^{MTRx}_{s,t}\bigl(x_{s,t},y_{s,t}\bigr) \quad \forall s,t \tag{11.8}$$

$$\frac{\partial T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{\partial b_{s,t}} = \frac{\partial T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{\partial y_{s,t}}\frac{\partial y_{s,t}}{\partial b_{s,t}} = r_t\frac{\partial T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{\partial y_{s,t}} = r_t\tau^{MTRy}_{s,t}\bigl(x_{s,t},y_{s,t}\bigr) \quad \forall s,t \tag{11.9}$$

It is the marginal tax rates with respect to household choice variables on the left-hand-side of (11.8) and (11.9) that we need in our theory equations (11.5) and (11.6). But it is the marginal tax rates and prices on the right-hand-side of (11.8) and (11.9) for which we have data. We call $\frac{\partial T^I}{\partial x}$ on the right-hand-side of (11.8) the marginal tax rate of labor income τ^{MTRx}, and we call $\frac{\partial T^I}{\partial y}$ on the right-hand-side of (11.9) the marginal tax rate of capital income τ^{MTRy}. With these specifications, we can restate the household Euler equations (11.5) and (11.6) in terms of τ^{MTRx} and τ^{MTRy}.

$$w_t\left(1-\tau^{MTRx}_{s,t}\right)u_1\left(c_{s,t},n_{s,t}\right) = -u_2\left(c_{s,t},n_{s,t}\right) \quad \text{for} \quad \forall t, \quad s\in\{1,2,...S\} \tag{11.10}$$

$$u_1\left(c_{s,t},n_{s,t}\right) = \beta\Biggl(1+r_{t+1}\Bigl[1-\tau^{MTRy}_{s+1,t+1}\Bigr]\Biggr)u_1\left(c_{s+1,t+1},n_{s+1,t+1}\right) \quad \forall t, \quad s\in\{1,2,...S-1\} \tag{11.11}$$

The last tax rate that we need to estimate from data is the effective tax rate ($\tau^{ETR}_{s,t}$), which is sometimes called the average tax rate or, more confusingly, the average effective tax rate. This is defined as the total tax liability divided by total income and can also be represented as a function of labor income $x_{s,t}$.

$$\tau^{ETR}_{s,t}\bigl(x_{s,t},y_{s,t}\bigr) \equiv \frac{T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{x_{s,t}+y_{s,t}} \tag{11.12}$$

The effective tax rate is an important unit-free statistic in the tax literature and is widely reported. Because it is a rate (although it has no natural bounds), it will be easier to estimate than the tax liability function $T^I_{s,t}(x_{s,t},y_{s,t})$. For this reason, we restate the tax liability term in the budget constraint as the effective tax rate times total income.

$$\begin{split}c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} + X_t - \tau^{ETR}_{s,t}(x_{s,t},y_{s,t})\bigl(x_{s,t}+y_{s,t}\bigr) \quad \forall s,t \\ \text{with} \quad b_{1,t}, b_{S+1,t} = 0\end{split} \tag{11.13}$$

Note that we are proposing in this chapter to estimate separately the marginal tax rates τ^{MTRx} and τ^{MTRy} from the effective tax rate τ^{ETR}. One could just as easily parameterize a function to fit the effective tax rate τ^{ETR} and then use the characterization of (11.12) to analytically derive the two marginal tax rates τ^{MTRx} and τ^{MTRy} using the expressions (11.8) and (11.9). Deriving the expressions for the marginal tax rates analytically from the effective tax rate would be fine if the function for the effective tax rate were the true function. However, because whatever function we estimate to fit the effective tax rate is simply an approximation, we might miss some features of the marginal tax rates if we do not estimate them separately.

11.1.2 Survey of the literature

Incorporating household income tax law into a DGE model is often a question of what functional forms and what data to use to represent the effective tax rate τ^{ETR} and two marginal tax rates τ^{MTRx} and τ^{MTRy} that appear in equations (11.13), (11.5), and (11.6). Krueger and Ludwig (2016) ask policy relevant questions regarding tax policy and have a rich model comprised of agents with heterogeneous skill-levels, assets, and age. But they model the tax code using linear tax functions. Even models at the frontier of the dynamic of analysis of fiscal policy, such as Nishiyama (2015), impose tax functions that are progressive but do not allow for marginal rates on a particular income source to be a function of other income.

Others have used parameterized tax functions to represent the tax code in general equilibrium models. Fullerton and Rogers (1993) estimate tax rate functions that vary by lifetime income group and age, but their marginal and average rates are not functions of realized income. Zodrow and Diamond (2013) follow a similar methodology. Many of these studies use micro-data to estimate the tax functions. For example, Fullerton and Rogers (1993) use the Panel Study for Income Dynamics to estimate ordinary least squares models that identify the parameters of their tax functions. Guner et al. (2014) use data from the Statistics of Income (SOI) Public Use File to calibrate average and marginal tax rate functions for various definitions of household income, separately for those with different household structures. Other examples of the estimations of flexible tax functions on labor or household income (in the U.S. and across other countries) come from Gouveia and Strauss (1994), Guvenen et al. (2014), and Holter et al. (2014). Nishiyama (2015) uses a version of the Gouveia and Strauss (1994) tax function, but does not condition tax functions on age nor does he allow marginal tax rates to be multivariate functions of the agents' different income sources. Rather, the marginal tax rate on labor income is only a function of labor income and the marginal tax rate on capital income is constant. Nishiyama (2015) uses ordinary least squares to estimate the parameters of his proposed tax functions from data produced by the Congressional Budget Office's microsimulation model.

The approach to estimating marginal tax rate functions $\frac{\partial T^I}{\partial n}$ and $\frac{\partial T^I}{\partial b}$ follows the method of DeBacker et al. (2017). We fit a parameterized functional form to microsimulation data in order to have a smooth and well-behaved, yet rich, representation of the real world tax code that can be used in a DGE model. This method for combining incorporating tax data from rich microsimulation models into a DGE model is a recent innovation.

As a comparison, we describe more deeply the functional form proposed by Gouveia and Strauss (1994), which remains one of the more flexible specifications in the literature, and one of the more widely used functional forms.[1] The Gouveia and Strauss (1994) tax function is given by:

$$T^{I,GS} = \varphi_0[I - (I^{-\varphi_1} + \varphi_2)^{\frac{-1}{\varphi_1}}], \tag{11.14}$$

where $T^{I,GS}$ is total income tax liability and $I = x + y$ is total income. We transform this function to put it in terms of an effective tax rate.

$$\tau^{ETR,GS} = \varphi_0[I - (I^{-\varphi_1} + \varphi_2)^{\frac{-1}{\varphi_1}}]/I. \tag{11.15}$$

[1]See Guvenen et al. (2014) and Nishiyama (2015).

In addition, we use our micro-data from from the`Tax-Calculator` microsimulation model described in Section 11.1.4 to estimate the above ETR specification separately by tax year and age. This is not done in others' work who use the Gouveia and Strauss (1994) functional form, but this will give that specification the best chance to fit the data as closely as our preferred functional form proposed in this paper. We use the same nonlinear least squares estimator to estimate the Gouveia and Strauss (1994) functions.

In Figure 11.1 we compare the Gouveia and Strauss (1994) specification to our specification described in Section 11.1.5. In orange, we show our specification when total income is entirely made up of labor income and in green we show our specification when total income is comprised of 70% labor income and 30% capital income. The Gouveia and Strauss (1994) specification is in purple. We can note a number of differences between these specifications from this picture. First, our specification shows more ability to capture the negative average tax rates at the lower end of the income distribution. Second, given the ability to have this negative intercept, our functional form can show more curvature over lower income ranges, allowing for a better fit to the steep gradient the data show over this range. Finally, by comparing the orange and green lines, one can see the ability of the share parameter to account for the role capital income plays in lowering effective tax rates as total income increases. In particular, our specification allows for the filers portfolio of income (i.e., the shares of total income derives from labor or capital income) at affect this average tax rate, which is a novel contribution of our functional form.

To more precisely test the fit of these specification against each other, Table 11.1 presents the standard errors of the estimates from these two specifications. The table shows the over all fit and also splits these out by age. What we find is that the ratio of polynomials function we propose shows a much better fit to the data, missing the effective tax rate by just under three percentage points on average, compared to an error of 5.5 percentage points in the estimated Gouveia and Strauss (1994) model.

Note that we compare only the effective tax rate functions since Gouveia and Strauss (1994) do not separately estimate marginal tax rate functions, as we do. Instead, they derive the marginal tax rates analytically from their total tax function. The implication, then, is that the marginal rates derives in this way will not fit the data as well as the effective tax

Figure 11.1: Plot of estimated ETR functions: $t = 2017$ and $s = 42$ under current law

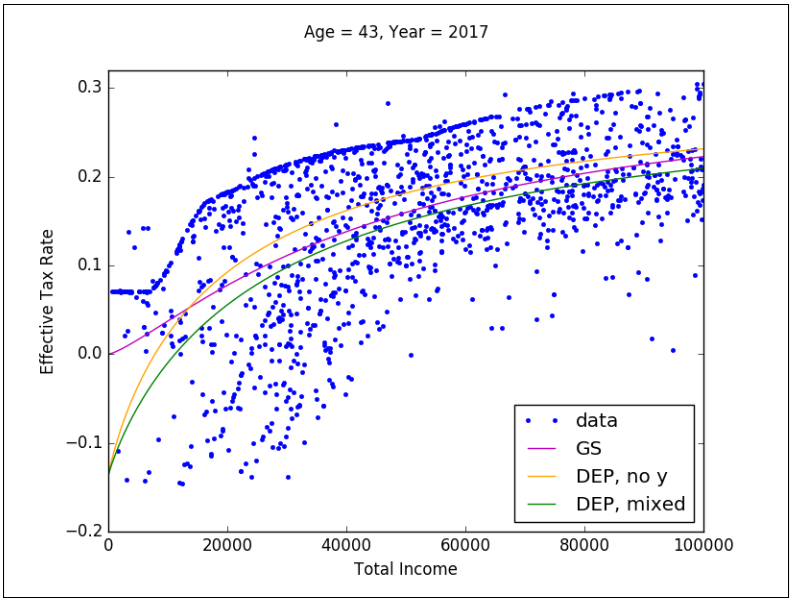


Table 11.1: Standard error of the estimates of ETR functions for age bins in period $t = 2017$

	Age ranges			
	All ages	21 to 54	55 to 65	66 to 80
Ratio of polynomials, ETR	2.98	3.42	1.67	2.22
Gouveia and Strauss (1994), ETR	5.50	6.54	2.57	3.11

rates, which were the target of the estimation. As we note in Section 11.1.1, we eschewed this approach of analytically deriving the marginal tax rates in favor of separately estimating the parameters of the effective and marginal tax rate functions. We have found this allows our model to better capture tax policy that differentially impacts average and marginal rates and to fit the data more closely.

11.1.3 Problem with actual marginal tax rates

Figure 11.2 shows the current implied marginal tax rate (MTR) schedule as a function of household income under U.S. law in October 2016 as well as the respective implied MTRs under the proposals from then-U.S. Presidential Candidates Hilary Clinton and Donald Trump. Note, in all cases, that the lumpiness of the MTR schedules creates a non-convex function. If one were to incorporate functions with this level of real-life richness into Euler equations such as (11.5) and (11.6), it would be very difficult to find the root in just one of those equations. The difficulty is compounded exponentially when we must solve $2S-1$ or more of those equations.

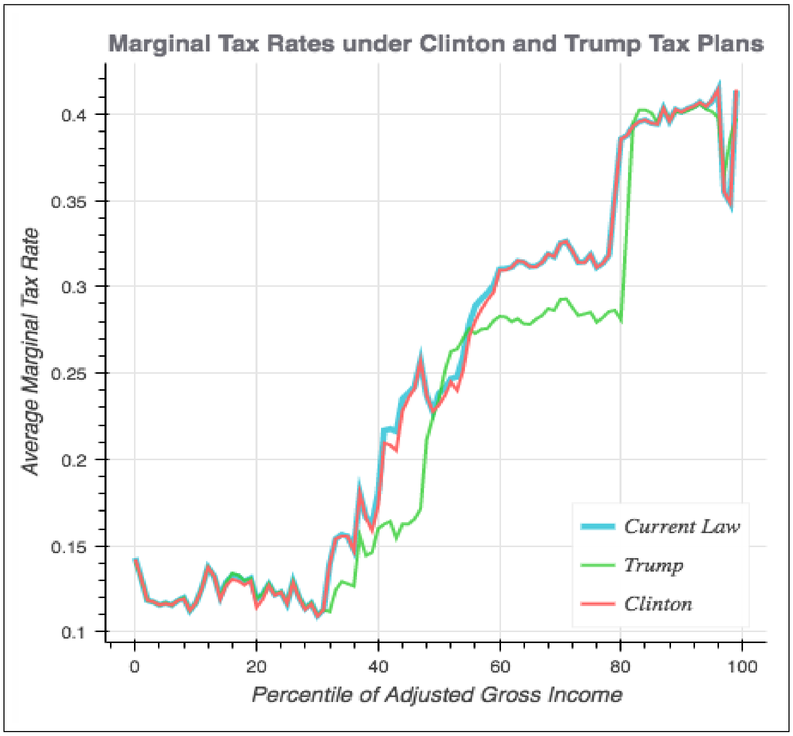
Even if we ignore the high-frequency ups and downs of the implied marginal tax rate schedules, there seems to be flat and even negatively sloped sections of the function for low incomes. This is due to tax credits for low-income individuals that go away as incomes rise. The high-frequency ups and downs come from the many dimensions of heterogeneity that exist in the real-world population on which the tax code may depend. Examples include marital status, filing status, number of children, deduction approach, and types of income.

11.1.4 Marginal tax rates from microsimulation models

The microsimulation model we use is called `Tax-Calculator` and is maintained a group of economists, software developers, and policy analysts.[2] Other than being completely open source, the `Tax-Calculator` is very similar to other tax calculators such as NBER's TaxSim and proprietary models used by think tanks and governmental organizations. For this reason,

[2] The documentation for using `Tax-Calculator` is available at http://taxcalc.readthedocs.org/en/latest/index.html A simple web application that provides an accessible user interface for `Tax-Calculator` is available from the Open Source Policy Center (OSPC) at http://www.ospc.org/taxbrain/. All the source code for the `Tax-Calculator` is freely available at https://github.com/open-source-economics/Tax-Calculator.

Figure 11.2: Implied marginal tax rate (MTR) schedules under Clinton and Trump 2016 proposals versus current law



much of what we say below generalizes if one were to use those other microsimulation models. In this section, we outline the main structure of the `Tax-Calculator` microsimulation model, but encourage the interested reader to follow the links for more detailed documentation.

`Tax-Calculator` uses micro-data on a sample of tax filers from the tax year 2009 Public Use File (PUF) produced by the IRS.[3] These data contain detailed records from the tax returns of about 200,000 tax filers who were selected from the population of filers through a stratified random sample of tax returns. These data come from IRS Form 1040 and a set of the associated forms and schedules. The PUF data are then matched to the Current Population Survey (CPS) to get imputed values for filer demographics such as age, which are not included in the PUF, and to incorporate households from the population of non-filers. The PUF-CPS match includes 219,815 filers.

Since these data are for calendar year 2009, they must be "aged" to be representative of

[3]Technically, the `Tax-Calculator` could use other micro-data as a source, but we choose to use the PUF for the relatively large sample size and the degree of detail provided for various income and deduction items.

the potential tax paying population in the years of interest (e.g. the current year through the end of the budget window). To do this, macroeconomic forecasts of wages, interest rates, GDP, and other variables are used to grow the 2009 values to be representative of the values one might see in subsequent years. Adjustments to the weights applied to each observation in the micro-data are also made. More specifically, weights are adjusted to hit a number of targets in an optimization problem that sets out to minimize the distance between the extrapolated micro-data values and the targets, with a penalty being applied for large changes in the weight individual observations from one year to the next. The targets are comprised of a number of aggregate totals of line items from Form 1040 (and related Schedules) produced by SOI for the years 2010-2013.[4]

Using these micro-data, `Tax-Calculator` is able to determine total tax liability and marginal tax rates by computing the tax reporting that minimizes each filer's total tax liability given the filer's income and deductions items and the parameters describing tax law. The determination of total tax liability from the microsimulation model includes federal income taxes and payroll taxes but currently excludes state income taxes and estate taxes. The output of the microsimulation model includes forecasts of the total tax liability in each year, marginal tax rates in income sources, and items from the filers' tax returns for each of the 219,815 filers in the micro-data. To calculate marginal tax rates on any given income source, the model adds one cent to the income source for each filing unit in the micro-data and then computes the change in tax liability. The change in tax liability divided by the change in income (one cent) yields the marginal tax rate. Population sampling weights are determined through the extrapolation and targeting of the microsimulation model. These weights allow one to calculate population representative results from the model. One can determine changes in tax liability and marginal tax rates across different tax policy options by doing the same simulation where the parameters describing the tax policy are updated to reflect the proposed policy rather than the baseline policy. The baseline policy used by `Tax-Calculator` is a current-law baseline.

To map the output of the microsimulation model, which is based on income reported on

[4]For details on how these data are extrapolated, please see the `Tax Data` program and associated documentation.

tax returns, to the DGE model, where income is defined more broadly, we use the following definitions. In computing the effective tax rates from the microsimulation model, we divided total tax liability by a measure of "adjusted total income". Adjusted total income is defined as total income (Form 1040, line 22) plus tax-exempt interest income, IRA distributions, pension income, and Social Security benefits (Form 1040, lines 8b, 15a, 16a, and 20a, respectively). We consider adjusted total income from the microsimulation model to be the counterpart of total income in the DGE model. Total income in the DGE model is the sum of capital and labor income.

We define labor income as earned income, which is the sum of wages and salaries (Form 1040, line 7) and self-employment income (Form 1040 lines 12 and 18) from the micro-data. Capital income is defined as a residual.[5]

To get the marginal tax rate on composite income amounts (e.g., labor income that is the sum of wage and self-employment income), we take a weighted average that accounts for negative income amounts. In particular, to we calculate the weighted average marginal tax rate on composite of n income sources as:

$$MTR_{composite} = \frac{\sum_{i=1}^{n} MTR_n * abs(Income_n)}{\sum_{i=1}^{n} abs(Income_n)} \tag{11.16}$$

When we look at the raw output from the microsimulation model, we find that there are several observations with extreme values for their effective tax rate. Since this is a ratio, such outliers are possible, for example when the denominator, adjusted total income, is very small. We omit such outliers by making the following restrictions on the raw output of the microsimulation model. First, we exclude observations with an effective tax rate greater than 1.5 times the highest statutory marginal tax rate. Second, we exclude observations where the effective tax rate is less than the lowest statutory marginal tax rate on income minus the maximum phase-in rate for the Earned Income Tax Credit (EITC). Third, we drop observations with marginal tax rates in excess of 99% or below the negative of the highest EITC rate (i.e., -45% under current law). These exclusions limit the influence of those with

[5] This is not an ideal definition of capital income, since it includes transfers between filers (e.g., alimony payments) and from the government (e.g., unemployment insurance), but we have chosen this definition in order to ensure that all of total income is classified as either capital or labor income.

extreme values for their marginal tax rate, which are few and usually result from the income of the filer being right at a kink in the tax schedule. Finally, since total income cannot be negative in the DGE model we use, we drop observations from the microsimulation model where adjusted total income is less than \$5.[6]

Because the tax rates are estimated as functions of income levels in the micro-data, we have to adjust the model income units to match the units of the micro-data. To do this, we find the *factor* such that *factor* times average steady-state model income equals the mean income in the final year of the micro-data.

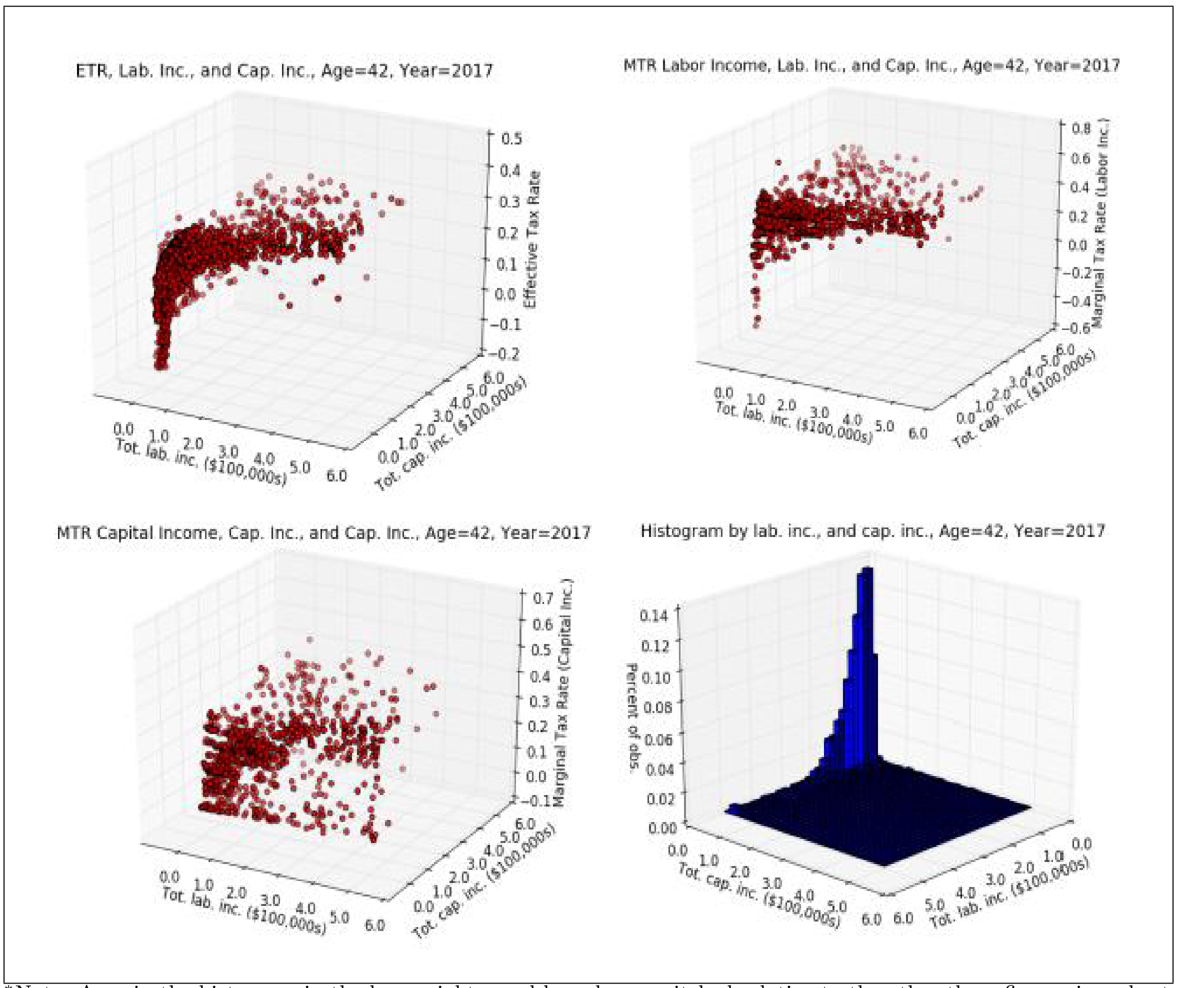
$$factor\sum_{s=1}^{S}\left(\bar{w}\bar{n}_s + \bar{r}\bar{b}_s\right) = (\text{data avg. income}) \tag{11.17}$$

To be precise, the income levels in the model, x and y, must be multiplied by this factor when they are used in the effective tax rate functions, marginal tax rate of labor income functions, and marginal tax rate of capital income functions of the form in Equation (11.18).

Figure 11.3 shows scatter plots of effective tax rates τ^{ETR}, marginal tax rates on labor income τ^{MTRx}, marginal tax rates on capital income τ^{MTRy}, and a histogram of the data points from the `Tax-Calculator` microsimulation model, each plotted as a function of labor income and capital income for all 42-year-olds in the year 2017. The data we use in the `Tax-Calculator` come from the 2009 IRS Public Use File and a statistical match of the Current Population Survey (CPS) demographic data. Labor and capital income are truncated at \$600,000 in order to more clearly see the shape of the data in spite of the long right tail of the income distribution. Although there is noise in the data, effective tax rates are generally increasing in both labor and capital income at a decreasing rate from some slightly negative level to an asymptote around 30 percent. This regular shape in effective tax rates is observed for all ages in all years of the budget window, 2017-2026.

[6]We choose \$5 rather than \$0 to provided additional assurance that small income values are not driving large ETRs.

Figure 11.3: Scatter plot of ETR, MTRx, MTRy, and histogram as functions of labor income and capital income from microsimulation model: $t = 2017$ and $s = 42$ under current law



*Note: Axes in the histogram in the lower-right panel have been switched relative to the other three figures in order to see the distribution more clearly.

11.1.5 Estimating ETR and MTR functions

Because of the regularity in the shape of the effective tax rates as shown in Figure 11.3, we choose to fit a smooth functional form to these data that is able to parsimoniously fit this shape while also being flexible enough to adjust to a wide range of tax policy changes. Our functional form, shown as a general tax rate function in (11.18), is a Cobb-Douglas aggregator of two ratios of polynomials in labor income x and capital income y. We use the same functional form for the effective and marginal tax rate functions. Important properties of this functional form are that it produces this bivariate negative exponential shape, is monotonically increasing in both labor income and capital income, and that it allows for negative tax rates. In order to capture variation in taxes by filer age and model year, we estimate functions for each model age and every year of the budget-window that the microsimulation model captures. In this way, we are able to map more of the heterogeneity from the microsimulation into the macro model than can be explicitly incorporated into a DGE model.

As an example, filing status is correlated with age and income. Thus, although the DGE model we use does not explicitly account for filing status, we are able to capture some of the effects of filing status on tax rates by having age and income dependent functions for effective and marginal tax rates. As another example, investment portfolio decisions differ over the life cycle and these are difficult to model in detail in a DGE model. By using age-dependent tax functions, we are able to capture some of the differentials in tax treatment across different assets (e.g. rates on dividends versus capital gains, tax-preferred retirement savings accounts, certain exemptions for interest income) even if the DGE model does not explicitly model these portfolio decisions.

Finally, consider that many macroeconomic models assume a single composite consumption good. Some of this composite good affects tax liability, such as the consumption of charitable contributions or housing. To the extent that the fraction of the composite good that comes from such consumption varies over a household's income and age, these tax functions will capture that, since they are fitted using microeconomic data that includes information on these tax-relevant forms of consumption.

Let x be total labor income, $x \equiv \hat{w}_t n_{s,t}$, and let y be total capital income, $y \equiv r_t \hat{b}_{s,t}$. We then write our tax rate functions as follows.

$$\begin{split}
\tau(x,y) = &\Bigl[\tau(x) + shift_x\Bigr]^\phi\Bigl[\tau(y) + shift_y\Bigr]^{1-\phi} + shift \\
&\text{where}\quad \tau(x) \equiv (max_x - min_x)\left(\frac{Ax^2 + Bx}{Ax^2 + Bx + 1}\right) + min_x \\
&\text{and}\quad \tau(y) \equiv (max_y - min_y)\left(\frac{Cy^2 + Dy}{Cy^2 + Dy + 1}\right) + min_y \\
&\text{where}\quad A, B, C, D, max_x, max_y, shift_x, shift_y > 0 \quad\text{and}\quad \phi\in[0,1] \\
&\text{and}\quad max_x > min_x \quad\text{and}\quad max_y > min_y
\end{split} \tag{11.18}$$

Note that we let $\tau(x,y)$ represent the effective and marginal rate functions, $\tau^{ETR}(x,y)$, $\tau^{MTRx}(x,y)$ and $\tau^{MTRy}(x,y)$. We assume the same functional form for each of these functions. The parameters values will, in general, differ across the different functions (effective and marginal rate functions) and by age, s, and tax year, t. We drop the subscripts for age and year from the above exposition for clarity.

By assuming each tax function takes the same form, we are breaking the analytical link between the effective tax rate function and the marginal rate functions as we discussed at the end of Section 11.1.1. In particular, one could assume an effective tax rate function and then use the analytical derivative of that to find the marginal tax rate function. However, we've found it useful to separately estimate the marginal and average rate functions. One reason is that we want the tax functions to be able to capture policy changes that have differential effects on marginal and average rates. For example, and relevant to the policy experiment we present below, a change in the standard deduction for tax payers would have a direct effect on their average tax rates. But it will have secondary effect on marginal rates as well, as some filers will find themselves in different tax brackets after the policy change. These are smaller and second order effects. When tax functions are are fit to the new policy, in this case a lower standard deduction, we want them to be able to represent this differential impact on the marginal and average tax rates. The second reason is related to the first. As the additional flexibility allows us to model specific aspects of tax policy more closely, it also allows us to better fit the parameterized tax functions to the data.

The key building blocks of the functional form in equation (11.18) are the $\tau(x)$ and $\tau(y)$ univariate functions. The ratio of polynomials in the $\tau(x)$ function $\frac{Ax^2+Bx}{Ax^2+Bx+1}$ with positive coefficients $A, B > 0$ and positive support for labor income $x > 0$ creates a negative-exponential-shaped function that is bounded between 0 and 1, and the curvature is governed by the ratio of quadratic polynomials. The multiplicative scalar term $(max_x - min_x)$ on the ratio of polynomials and the addition of min_x at the end of $\tau(x)$ expands the range of the univariate negative-exponential-shaped function to $\tau(x) \in [min_x, max_x]$. The $\tau(y)$ function is an analogous univariate negative-exponential-shaped function in capital income y, such that $\tau(y) \in [min_y, max_y]$.

The respective $shift_x$ and $shift_y$ parameters in the functional form (11.18) are analogous to the additive constants in a Stone-Geary utility function. These constants ensure that the two sums $\tau(x) + shift_x$ and $\tau(y) + shift_y$ are both strictly positive. They allow for negative tax rates in the $\tau(\cdot)$ functions despite the requirement that the arguments inside the brackets be strictly positive. The general $shift$ parameter outside of the Cobb-Douglas brackets can then shift the tax rate function so that it can accommodate negative tax rates. The Cobb-Douglas share parameter $\phi \in [0, 1]$ controls the shape of the function between the two univariate functions $\tau(x)$ and $\tau(y)$.

This functional form for tax rates delivers flexible parametric functions that can fit the tax rate data shown in Figure 11.3 as well as a wide variety of policy reforms. Further, these functional forms are monotonically increasing in both labor income x and capital income y. This characteristic of monotonicity in x and y is essential for guaranteeing convex budget sets and thus uniqueness of solutions to the household Euler equations. The assumption of monotonicity does not appear to be a strong one when viewing the tax rate data shown in Figure 11.3. While it does limit the potential tax systems to which one could apply our methodology, tax policies that do not satisfy this assumption would result in non-convex budget sets and thus require non-standard DGE model solutions methods and would not guarantee a unique equilibrium. The 12 parameters of our tax rate functional form from (11.18) are summarized in Table 11.2.

We estimate a transformation of the τ^{ETR}, τ^{MTRx}, and τ^{MTRy} tax rate functions described in functional form equation (11.18) for each age s of the primary filer and time period t in our

Table 11.2: Description of tax rate function $\tau(x,y)$ parameters

Symbol	Description
A	Coefficient on squared labor income term x^2 in $\tau(x)$
B	Coefficient on labor income term x in $\tau(x)$
C	Coefficient on squared capital income term y^2 in $\tau(y)$
D	Coefficient on capital income term y in $\tau(y)$
max_x	Maximum tax rate on labor income x given $y=0$
min_x	Minimum tax rate on labor income x given $y=0$
max_y	Maximum tax rate on capital income y given $x=0$
min_y	Minimum tax rate on capital income y given $x=0$
$shift_x$	shifter $>\lvert min_x\rvert$ ensures that $\tau(x)+shift_x>0$ despite potentially negative values for $\tau(x)$
$shift_y$	shifter $>\lvert min_y\rvert$ ensures that $\tau(y)+shift_y>0$ despite potentially negative values for $\tau(y)$
$shift$	shifter (can be negative) allows for support of $\tau(x,y)$ to include negative tax rates
ϕ	Cobb-Douglas share parameter between 0 and 1

data and budget window, respectively (2,400 separate specifications). We transform these functions so that the labor income, x, and capital income, y, variables in the polynomials are transformed to percent deviations from their respective means. This helps with the scale of the variables in the optimization routine. The transformed ETR and MTR functions are estimated using a constrained, weighted, non-linear least squares estimator. The weighting in this estimator come from the weights assigned to the filers in the microsimulation model.

Let $\boldsymbol{\theta}_{s,t} = (A, B, C, D, max_x, min_x, max_y, min_y, shift_x, shift_y, shift, \phi)$ be the full vector of 12 parameters of the tax function for a particular age of filers in a particular year. We first directly specify min_x as the minimum tax rate in the data for age-s and period-t individuals for capital income close to 0 ($\$0 < y < \$3,000$) and min_y as the minimum tax rate for labor income close to 0 ($\$0 < x < \$3,000$). We then set $shift_x = |min_x| + \varepsilon$ and $shift_y = |min_y| + \varepsilon$ so that the respective arguments in the brackets of (11.18) are strictly positive. Let $\bar{\boldsymbol{\theta}}_{s,t} = \{min_x, min_y, shift_x, shift_y\}$ be the set of parameters we take directly from the data in this way.

We then estimate eight remaining parameters $\tilde{\boldsymbol{\theta}}_{s,t} = (A, B, C, D, max_x, max_y, shift, \phi)$

using the following nonlinear weighted least squares criterion,

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{s,t} = \tilde{\boldsymbol{\theta}}_{s,t} : \quad &\min_{\tilde{\boldsymbol{\theta}}_{s,t}} \sum_{i=1}^{N}\Bigl[\tau_i - \tau_{s,t}\bigl(x_i, y_i|\tilde{\boldsymbol{\theta}}_{s,t}, \bar{\boldsymbol{\theta}}_{s,t}\bigr)\Bigr]^2 w_i, \\ &\text{subject to} \quad A, B, C, D, max_x, max_y > 0, \\ &\text{and} \quad max_x \geq min_x, \quad \text{and} \quad max_y \geq min_y \quad \text{and} \quad \phi \in [0,1]\end{aligned} \tag{11.19}$$

where τ_i is the effective (or marginal) tax rate for observation i from the microsimulation output, $\tau_{s,t}(x_i, y_i|\tilde{\boldsymbol{\theta}}_{s,t}, \bar{\boldsymbol{\theta}}_{s,t})$ is the predicted average (or marginal) tax rate for filing-unit i with x_i labor income and y_i capital income given parameters $\boldsymbol{\theta}_{s,t}$, and w_i is the CPS sampling weight of this observation. The number N is the total number of observations from the microsimulation output for age s and year t. Figure 11.4 shows the typical fit of an estimated tax function $\tau_{s,t}\bigl(x, y|\hat{\boldsymbol{\theta}}_{s,t}\bigr)$ to the data. The data in Figure 11.4 are the same age $s = 42$ and year $t = 2017$ as the data Figure 11.3.

The underlying data can limit the number of tax functions that can be estimated. For example, we use the age of the primary filer from the PUF-CPS match to be equivalent to the age of the DGE model household. The DGE model we use allows for individuals up to age 100, however the data contain few primary filers with age above age 80. Because we cannot reliably estimate tax functions for $s > 80$, we apply the tax function estimates for 80 year-olds to those with model ages 81 to 100. In the case certain ages below age 80 have too few observations to enable precise estimation of the model parameters, we use a linear interpolation method to find the values for those ages $21 \leq s < 80$ that cannot be precisely estimated.[7]

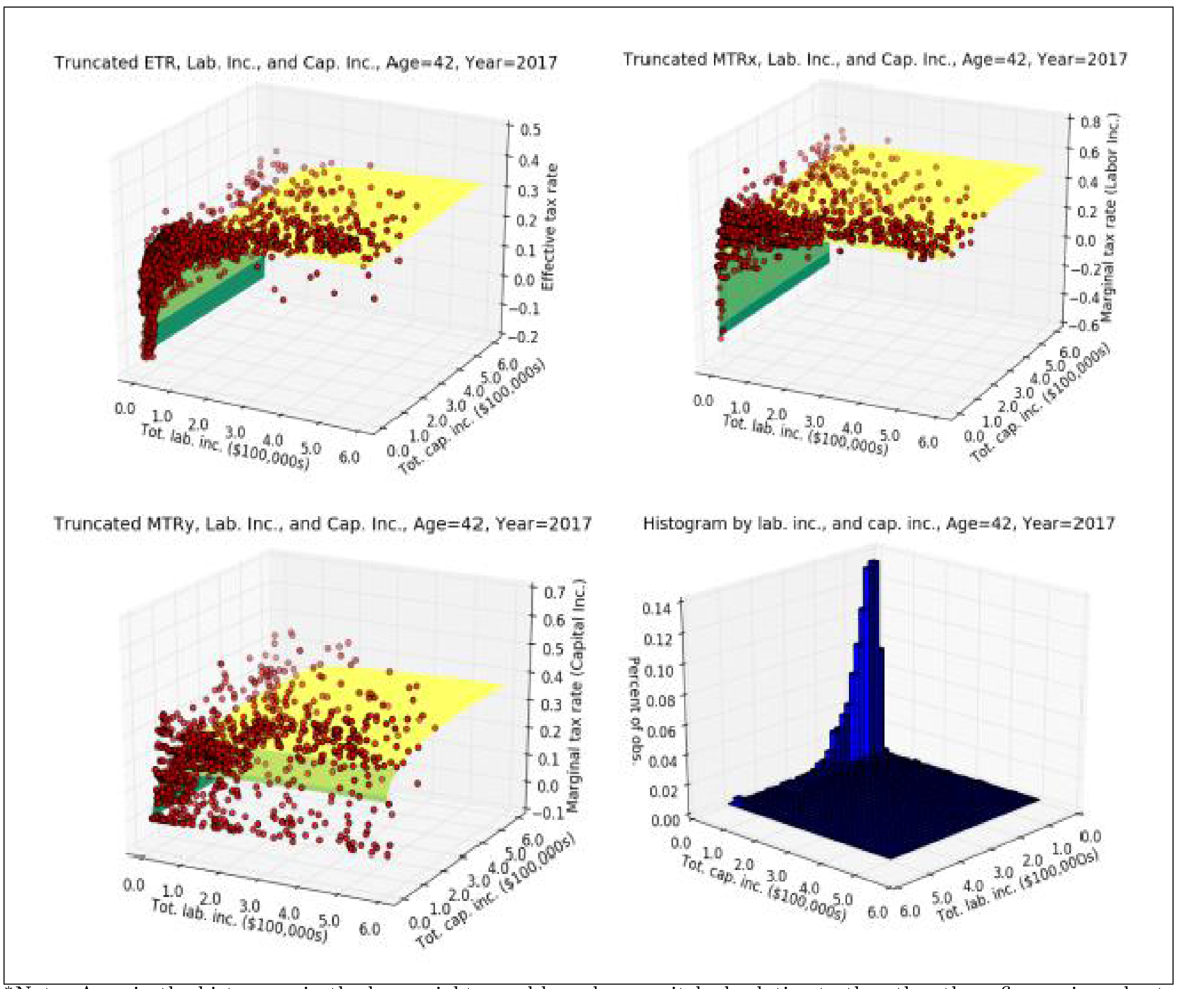
Figure 11.4 shows the estimated function surfaces for tax rate functions for the effective tax rate τ^{ETR}, marginal tax rate on labor income τ^{MTRx}, and marginal tax rate on capital income τ^{MTRy} data shown in Figure 11.3 for age $s = 42$ individuals in period $t = 2017$ under the current law. And the estimated parameters and the corresponding function surface change whenever any of the many policy levers in the microsimulation model that generate the tax rate data are adjusted. The total tax liability function is simply the effective tax

[7] We use two criterion to determine whether the function should be interpolated. First, we require a minimum number of observations of filers of that age and in that tax year. Second, we require that that sum of squared errors meet a pre-defined threshold.

rate function times total income $\tau(x,y)(x+y)$, which is simply a transformation of equation (11.12).

$$\begin{aligned} T^I_{s,t}(x,y) &\equiv ETR_{s,t}(x,y)\big(x+y\big) \\ &= \left(\Big[\tau_{s,t}(x) + shift_{x,s,t}\Big]^{\phi}_{s,t}\Big[\tau_{s,t}(y) + shift_{y,s,t}\Big]^{1-\phi_{s,t}} + shift_{s,t}\right)\big(x+y\big) \end{aligned} \tag{11.20}$$

Figure 11.4: Estimated tax rate functions of ETR, MTRx, MTRy, and histogram as functions of labor income and capital income from microsimulation model: $t = 2017$ and $s = 42$ under current law



*Note: Axes in the histogram in the lower-right panel have been switched relative to the other three figures in order to see the distribution.

As we describe above, each rate function $(\tau^{ETR}, \tau^{MTRx}, \tau^{MTRy})$ varies by age, s, and tax year t. This means a large number of parameters must be estimated. In particular, using our illustrative example with the model of DeBacker et al. (2017b), we will need to fit 12 parameters for each of three tax rate functions for each age (21 to 100) during each of the 10 years of the budget window with the estimated functions in the last year of the budget window assumed to be permanent. The microsimulation model we use, `Tax-Calculator` is able to provide marginal and average tax rates for 10-years forward from the present.[8] The DGE model is solved from the current period forward through the steady-state. The steady-state is generally arrived at well beyond a time horizon of 10 years. Thus we allow variation the in the rate functions only over this 10-year budget window and fix the parameters of the rate functions to the last year of the window for years $t \geq 10$. Thus, in our illustrative example, there are 2,400 tax rate functions comprised of 28,800 parameters.

Because we allow these many functions of labor income and capital income to be independently estimated for each tax rate type, age, and year, we can capture many of the characteristics and discrete variation in the tax code while still preserving the smoothness and monotonicity of the tax functions within each type, age, and year. This monotonicity and smoothness is sufficient to guarantee uniqueness and tractability of the computational solution of the household Euler equations. Allowing for different tax rate functions by age and time period also implicitly incorporates heterogeneity in the data in dimensions that we cannot model in the DGE model, such as broader income items, deductions items, credits, and filing unit structure. The effect of such heterogeneity on tax burdens will affect the effective tax rate functions we fit to the output of the microsimulation model.

Table 11.3: Average values of ϕ for ETR, MTRx, and MTRy for age bins in period $t = 2017$

	Age ranges			
	21 to 54	55 to 65	66 to 80	All years
ETR	0.66	0.28	0.38	0.44
$MTRx$	0.89	0.31	0.23	0.48
$MTRy$	0.77	0.25	0.14	0.43

* Note: Even though agents in the OG model live until age 100, the tax data was too sparse to estimate functions for ages greater than 80. For ages 81 to 100, we simply assumed the age 80 estimated tax functional forms.

It is difficult to show all the estimated tax functions for every age and period in the budget

[8]This is the standard timeframe considered by policy analysts analyzing the effects of tax policy on the federal budget.

window. But Table 11.3 gives a description of the estimated values of the ϕ parameter. This parameter ϕ in the tax function (11.18) governs how important the interaction is between labor income and capital income for determining tax rates. The further interior is ϕ (away from 0 or 1), the more important it is to model tax rates as functions of both labor income and capital income. And the closer ϕ is to 1, the more important is labor income for determining tax rates.

Two key results jump out from Table 11.3. First, it is clear that the interaction between labor income and capital income is significant at all ages for determining effective tax rates ETR, marginal tax rates on labor income $MTRx$, and marginal tax rates on capital income $MTRx$. The last column of Table 11.3 shows the average *phi* value for all ages in the data to be around 0.45 for all three tax rate types. This suggests that models that use univariate tax functions of any type of income miss important information and incentives present in the tax code.

A second result from Table 11.3 is that the relative importance of labor income in determining tax rates varies over the life cycle in similar ways for each tax rate type (τ^{ETR}, τ^{MTRx}, and τ^{MTRy}). The first three columns of each row of Table 11.3 show that labor income is most important for determining tax rates between the ages of 21 and 54 and that capital income is most important for determining tax rates between the ages of 55 and 65. For marginal tax rates capital income continues to be the most important determinant after age 65, but capital income and labor income are equally important determinants of the effective tax rate ETR after age 65. This suggests that models that use tax functions that do not vary with age also miss some important information and incentives present in the tax code.

11.2 Households

A unit measure of identical individuals are born each period and live for S periods. The budget constraint faced by these households in each period is similar to the on presented in Chapter 4. However, households in this model pay taxes on some function of their total income $T^I_{s,t}$ and receive transfers from the government X_t each period.

$$
\begin{split}
c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} + X_t - T^I_{s,t} \quad \forall s,t \\
\text{with} \quad b_{1,t}, b_{S+1,t} = 0
\end{split}
\tag{11.4}
$$

We specify taxes and transfers in (11.4) such that if a household has a positive tax liability to the government, then T^I in positive ($T^I > 0$). And if the household receives a transfer from the government, then X_t is positive ($X_t > 0$).

For simplicity, we assume that transfers X_t are not a function of household labor supply $n_{s,t}$, wealth $b_{s,t}$, or savings $b_{s+1,t+1}$. We also assume that the tax levied on households $T^I_{s,t}$ is only a function of factors that influence current period income, namely $n_{s,t}$ and $b_{s,t}$, but is not a function of factors like savings $b_{s+1,t+1}$ that influence income in the next period. But these assumptions could be easily relaxed.

Households choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{s,t+s-1}\}_{s=1}^S$, and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints,

$$
\max_{\{c_{s,t+s-1},n_{s,t+s-1}\}_{s=1}^S,\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{s=1}^S \beta^{s-1} u(c_{s,t+s-1}, n_{s,t+s-1}) \tag{11.21}
$$

$$
\text{s.t.} \quad c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} + X_t - T^I_{s,t} \tag{11.4}
$$

$$
\text{where} \quad u(c_{s,t}, n_{s,t}) = \frac{c_{s,t}^{1-\sigma} - 1}{1-\sigma} + \chi^n_s b \left[1 - \left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} \tag{11.22}
$$

where $u(c_{s,t}, n_{s,t})$ is the period utility function with elliptical disutility of labor from Chapter 4, and χ^n_s is a scale parameter that can potentially vary by age s influencing the relative disutility of labor to the utility of consumption. The household's lifetime problem (11.21) can be reduced to choosing S labor supplies $\{n_{s,t+s-1}\}_{s=1}^S$ and $S-1$ savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ by substituting the budget constraints (11.4) in for $c_{s,t}$ in each period utility function (11.22) of the lifetime utility function.

The set of optimal lifetime choices for an agent born in period t are characterized by the following S static labor supply Euler equations (11.5), the following $S-1$ dynamic savings Euler equations (11.6), and a budget constraint that binds in all S periods (11.4),

$$\begin{aligned} &\left(w_t - \frac{\partial T^I_{s,t}}{\partial n_{s,t}}\right) u_1\left(c_{s,t+s-1}, n_{s,t+s-1}\right) = -u_2\left(c_{s+1,t+s}, n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S\} \\ &\Rightarrow \quad \left(w_t - \frac{\partial T^I_{s,t}}{\partial n_{s,t}}\right)\left(c_{s,t}\right)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \end{aligned} \tag{11.5}$$

$$\begin{aligned} &u_1\left(c_{s,t+s-1}, n_{s,t+s-1}\right) = \beta\left(1 + r_{t+1} - \frac{\partial T^I_{s+1,t+1}}{\partial b_{s+1,t+1}}\right) u_1\left(c_{s+1,t+s}, n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S-1\} \\ &\Rightarrow \quad \left(c_{s,t}\right)^{-\sigma} = \beta\left(1 + r_{t+1} - \frac{\partial T^I_{s+1,t+1}}{\partial b_{s+1,t+1}}\right)\left(c_{s+1,t+1}\right)^{-\sigma} \end{aligned} \tag{11.6}$$

$$c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} + X_t - T^I_{s,t} \quad \text{for} \quad s\in\{1,2,...S\}, \quad b_{1,t}, b_{S+1,t+S} = 0 \tag{11.4}$$

where u_1 is the partial derivative of the period utility function with respect to its first argument $c_{s,t}$, and u_2 is the partial derivative of the period utility function with respect to its second argument $n_{s,t}$. As was demonstrated in detail in Section 3.1, the dynamic Euler equations (11.6) do not include marginal utilities of all future periods because of the principle of optimality and the envelope condition.

The distortionary effect of taxation is evidenced in these two behavioral equations (11.5) and (11.6) by the new terms in parentheses which include the respective partial derivatives of $\frac{\partial T^I}{\partial n}$ and $\frac{\partial T^I}{\partial b}$. These are the marginal tax rates with respect to labor supply and with respect to savings, respectively. The simple transformation of multiplying each of these derivatives by the wage w_t and interest rate r_{t+1}, respectively, will transform them into the marginal tax rate with respect to labor income and the marginal tax rate with respect to capital income.

Note that these $2S-1$ household decisions are perfectly identified if the household knows what prices will be over its lifetime $\{w_u, r_u\}_{u=t}^{t+S-1}$. As in Section 3.1, let the distribution of capital and household beliefs about the evolution of the distribution of capital be characterized by (11.23) and (11.24).

$$\mathbf{\Gamma}_t \equiv \left\{b_{s,t}\right\}_{s=2}^{S} \quad \forall t \tag{11.23}$$

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{11.24}$$

11.3 Firms

Firms are characterized exactly as in Section 2.2, with the firm's aggregate capital decision K_t governed by first order condition (11.25) and its aggregate labor decision L_t governed by first order condition (11.26).

$$r_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{11.25}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{11.26}$$

The per-period depreciation rate of capital is $\delta \in [0,1]$, the capital share of income is $\alpha \in (0,1)$, and total factor productivity is $A > 0$.

11.4 Government balanced budget

The government is assumed to run a balanced budget in each period, returning tax receipts to households through the lump sum transfers X_t. Given this balanced budget rule, we can write the amount of transfers per household as the following,

$$X_t = \frac{R_t}{S}, \tag{11.27}$$

where R_t are total tax receipts and are given by,

$$R_t = \sum_{s=1}^{S} T^I_{s,t} \tag{11.28}$$

Total tax payments, $T^I_{s,t}$ are given by Equation (11.20).

11.5 Market Clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$L_t = \sum_{s=1}^{S} n_{s,t} \quad \forall t \tag{11.29}$$

$$K_t = \sum_{i=2}^{S} b_{s,t} \quad \forall t \tag{11.30}$$

$$\begin{aligned} &Y_t = C_t + I_t \quad \forall t \\ &\quad \text{where} \quad I_t \equiv K_{t+1} - (1-\delta)K_t \end{aligned} \tag{11.31}$$

The goods market clearing equation (11.31) is redundant by Walras' Law. The market clearing conditions for this version of the model are nearly equivalent to the three conditions described in Section 4.4.

11.6 Equilibrium

The equilibrium solution to the problem above satisfies three general conditions.

i. Households optimize according to equations (11.5) and (11.6).

ii. Firms optimize according to (11.25) and (11.26).

iii. Markets clear according to (11.29) and (11.30).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

The easiest way to understand the equilibrium solution is to substitute the market clearing conditions (11.29) and (11.30) into the firm's optimal conditions (11.25) and (11.26) solve for the equilibrium wage and interest rate as functions of the distribution of capital.

$$w_t(\mathbf{\Gamma}_t): \quad w_t = (1-\alpha)A\left(\frac{\sum_{s=2}^S b_{s,t}}{\sum_{s=1}^S n_{s,t}}\right)^\alpha \quad \forall t \tag{11.32}$$

$$r_t(\mathbf{\Gamma}_t): \quad r_t = \alpha A\left(\frac{\sum_{s=1}^S n_{s,t}}{\sum_{s=2}^S b_{s,t}}\right)^{1-\alpha} - \delta \quad \forall t \tag{11.33}$$

It is worth noting here that the equilibrium wage (11.32) and interest rate (11.33) are written as functions of the period-t distribution of savings (wealth) $\mathbf{\Gamma}_t$ from (11.23) and are not functions of the period-t distribution of labor supply, which labor distribution shows up in (11.32) and (11.33). This is because, similar to next period savings $b_{s+1,t+1}$, current period labor supply $n_{s,t}$ must be chosen in period t and is therefore a function of the current state $\mathbf{\Gamma}_t$ distribution of savings.

Now (11.32), (11.33), and the budget constraint (11.4) can be substituted into household Euler equations (11.5) and (11.6) to get the following $(2S-1)$-equation system. Extended across all time periods, this system completely characterizes the equilibrium.

$$\begin{aligned}&\left(w_t(\mathbf{\Gamma}_t) - \frac{\partial T^I_{s,t}(\mathbf{\Gamma}_t)}{\partial n_{s,t}}\right)\left(w_t(\mathbf{\Gamma}_t)n_{s,t} + \big[1+r_t(\mathbf{\Gamma}_t)\big]b_{s,t} + X_t - T^I_{s,t}(\mathbf{\Gamma}_t) - b_{s+1,t+1}\right)^{-\sigma} = ... \\ &\qquad \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s\in\{1,2,...S\} \quad \text{and} \quad \forall t\end{aligned} \tag{11.34}$$

$$\begin{aligned}&\left(w_t(\mathbf{\Gamma}_t)n_{s,t} + \Big[1+r_t(\mathbf{\Gamma}_t)\Big]b_{s,t} + X_t - T^I_{s,t}(\mathbf{\Gamma}_t) - b_{s+1,t+1}\right)^{-\sigma} = \\ &\quad \beta\left[1 + r_{t+1}(\mathbf{\Gamma}_{t+1}) - \frac{\partial T^I_{s+1,t+1}(\mathbf{\Gamma}_{t+1})}{\partial b_{s,t}}\right]\times ... \\ &\quad \left(w_{t+1}(\mathbf{\Gamma}_{t+1})n_{s+1,t+1} + \Big[1+r_{t+1}(\mathbf{\Gamma}_{t+1})\Big]b_{s+1,t+1} + X_{t+1} - T^I_{s+1,t+1}(\mathbf{\Gamma}_{t+1}) - b_{s+2,t+2}\right)^{-\sigma} \\ &\quad \text{for} \quad s\in\{1,2,...S-1\} \quad \text{and} \quad \forall t\end{aligned} \tag{11.35}$$

The system of S nonlinear static equations (11.34) and $S-1$ nonlinear dynamic equations (11.35) characterizing the lifetime labor supply and savings decisions for each household

$\{n_{s,t+s-1}\}_{s=1}^{S}$ and $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ is not identified. Each individual knows the current distribution of capital $\mathbf{\Gamma}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\mathbf{\Gamma}_{t+1} = \{\{b_{s+1,t+1}\}_{s=1}^{S-1}\}$ and a number of subsequent periods for all agents alive in those subsequent periods. We also need to solve for a policy function for the individual $b_{s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 11.1 (Steady-state equilibrium). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^{S}$, labor supply $\{\bar{n}_s\}_{s=1}^{S}$, and savings $\{\bar{b}_s\}_{s=2}^{S}$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (11.5) and (11.6),

ii. firms optimize according to (11.25) and (11.26),

iii. markets clear according to (11.29) and (11.30).

The relevant examples of stationary functions in this model are the policy functions for labor and savings. Let the equilibrium policy functions for labor supply be represented by $n_{s,t} = \phi_s\left(\mathbf{\Gamma}_t\right)$, and let the equilibrium policy functions for savings be represented by $b_{s+1,t+1} = \psi_s\left(\mathbf{\Gamma}_t\right)$. The arguments of the functions (the state) may change overtime causing the labor and savings levels to change over time, but the function of the arguments is constant (stationary) across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state (transition path) equilibrium of the model.

Definition 11.2 (Non-steady-state functional equilibrium). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as stationary allocation functions of the state $\left\{n_{s,t} = \phi_s\big(\mathbf{\Gamma}_t\big)\right\}_{s=1}^{S}$, $\left\{b_{s+1,t+1} = \psi_s\big(\mathbf{\Gamma}_t\big)\right\}_{s=1}^{S-1}$ and and stationary price functions $w(\mathbf{\Gamma}_t)$ and $r(\mathbf{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (11.24), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\mathbf{\Gamma}_{t+u} = \mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (11.5) and (11.6),

iii. firms optimize according to (11.25) and (11.26),

iv. markets clear according to (11.29) and (11.30).

11.7 Solution Method

In this section we characterize computational approaches to solving for the steady-state equilibrium from Definition 11.1 and the transition path equilibrium from Definition 11.2.

11.7.1 Steady-state equilibrium

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 11.1. The parameters needed for the steady-state solution of this model are $\left\{S, \beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta, \boldsymbol{\theta}_{s,t}\right\}$, where S is the number of periods in an individual's life, $\left\{\beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S\right\}$ are household utility function parameters, $\{A, \alpha, \delta\}$ are firm production function parameters, and $\boldsymbol{\theta}_{s,t}$ is the vector of estimated tax function parameters described in Section 11.1.5. These parameters are chosen, calibrated, or estimated outside of the model and are inputs to the solution method.

The steady-state is defined as the solution to the model in which the distributions of individual consumption, labor supply, and savings have settled down and are no longer changing over time. As such, it can be thought of as a long-run solution to the model in which the effects of any shocks or changes from the past no longer have an effect.

$$n_{s,t} = \bar{n}_s, \quad b_{s,t} = \bar{b}_s \quad \forall s, t \tag{11.36}$$

From the market clearing conditions (11.29) and (11.30) and the firms' first order equations (11.25) and (11.26), the household steady-state conditions imply the following steady-state conditions for prices and aggregate variables.

$$r_t = \bar{r}, \quad w_t = \bar{w}, \quad K_t = \bar{K} \quad L_t = \bar{L} \quad \forall t \tag{11.37}$$

The steady-state is characterized by the steady-state versions of the set of $2S-1$ Euler equations (11.5) and (11.6) over the lifetime of an individual (after substituting in the budget constraint) and the $2S-1$ unknowns $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$,

$$\bar{w}\Big([1+\bar{r}]\bar{b}_s + \bar{w}\bar{n}_s + \bar{X} - \bar{T}_s^I - \bar{b}_{s+1}\Big)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s=\{1,2,...S\} \tag{11.38}$$

$$\Big([1+\bar{r}]\bar{b}_s + \bar{w}\bar{n}_s + \bar{X} - \bar{T}_s^I - \bar{b}_{s+1}\Big)^{-\sigma} = \beta(1+\bar{r})\Big([1+\bar{r}]\bar{b}_{s+1} + \bar{w}\bar{n}_{s+1} + \bar{X} - \bar{T}_{s+1}^I - \bar{b}_{s+2}\Big)^{-\sigma} \quad \text{for} \quad s=\{1,2,...S-1\} \tag{11.39}$$

where both $\bar{w}$ and $\bar{r}$ are functions of the distribution of labor supply and savings as shown in (11.32) and (11.33).

There are several approaches to solving this system of equations. We take an approach here that will continue to work for models with additional complexity.[9] This approach is to use a multivariate root finder that chooses the $2S-1$ steady-state variables $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$ simultaneously to solve the zeros of the $2S-1$ Euler equations (11.38) and (11.39). The tradeoffs between labor supply and savings in each period can create difficulties for root finding algorithms and thus this method can be sensitive to starting values and model parameterizations.

The outer loop of the steady-state solution method is to iterate on guesses for the steady-state interest rate $\bar{r}$, household transfers $\bar{X}$, and the $factor$ that scales model income to the income from the data for the estimated tax functions. We iterate these outer-loop guesses until the firm's capital demand equation (11.25), the government balanced budget constraint (11.27), and the factor equation (11.17) are satisfied. One could also choose the wage $\bar{w}$ or

[9] In particular, we are careful to choose a solution method here that will work for models with endogenous bequests for which the savings at the end of life are endogenous.

the steady-state capital-labor ratio $\bar{K}/\bar{L}$ instead of the interest rate $\bar{r}$ as the outer-loop choice variable. We prefer the interest rate $\bar{r}$ as the outer-loop choice variable because it is usually easier to make an initial guess that is closer to the true value given that it usually lies between 0 and 1. The steady-state solution method algorithm is detailed below.

i. Make a guess for the steady-state interest rate $\bar{r}^i$, household transfer $\bar{X}^i$, and the $factor^i$ for scaling income in the tax functions.

   (a) A guess for the steady-state interest rate $\bar{r}^i$ will imply a value for the steady-state wage $\bar{w}$ from (11.40), which is derived from solving equation (11.25) for the capital labor ratio K/L and substituting it into (11.26).

$$w_t = (1-\alpha)A\left(\frac{\alpha A}{r_t + \delta}\right)^{\frac{\alpha}{1-\alpha}} \quad \forall t \tag{11.40}$$

   (b) An initial guess for the factor is usually large given that model units tend to be at least a few orders of magnitude smaller than data income units.

ii. Given steady-state prices $\bar{r}^i$ and $\bar{w}$, transfers $\bar{X}^i$, and $factor^i$, solve for the steady-state household's lifetime decisions $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$.

   (a) Given $\bar{r}^i$, $\bar{w}$, $\bar{X}^i$, and $factor^i$, use a root finder to solve for $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$ from the $2S-1$ steady-state Euler equations (11.38) and (11.39).

   (b) This solution can be sensitive the initial guess for $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$ passed to the root finder.

iii. Given solution for optimal household decisions $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$ based on the guess for the interest rate $\bar{r}^i$ and the implied wage $\bar{w}$, solve for the aggregate capital $\bar{K}$ and aggregate labor $\bar{L}$ implied by the household solutions and market clearing conditions.

$$\bar{K} = \sum_{s=2}^{S} \bar{b}_s \tag{11.41}$$

$$\bar{L} = \sum_{s=1}^{S} \bar{n}_s \tag{11.42}$$

iv. Compute a new value for the interest rate $\bar{r}^{i'}$ using the aggregate capital stock $\bar{K}$ and aggregate labor $\bar{L}$ implied by the household optimization from equations (11.41) and (11.42). Compute a new value for household transfers $\bar{X}^{i'}$ using the government balanced budget constraint (11.27). And compute a new value for the $factor^{i'}$ using the equation that defines the factor (11.17).

$$\bar{r}^{i'} = \alpha A\left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha} - \delta \tag{11.43}$$

$$\bar{X}^{i'} = \sum_{s=1}^{S} T^I_{s,t} \tag{11.44}$$

$$factor^{i'} = \frac{\text{(data avg. income)}}{\sum_{s=1}^{S}\left(\bar{w}\bar{n}_s + \bar{r}\bar{b}_s\right)} \tag{11.45}$$

v. Update the guess for the steady-state interest rate $\bar{r}^{i+1}$, transfers $\bar{X}^{i+1}$, and $factor^{i+1}$ until the implied values $\{\bar{r}^{i'}, \bar{X}^{i'}, factor^{i'}\}$ equal the initial guesses $\{\bar{r}^{i}, \bar{X}^{i}, factor^{i}\}$

(a) The bisection method characterizes the updated guess for the outer-loop steady-state variables $\{\bar{r}^{i+1}, \bar{X}^{i+1}, factor^{i+1}\}$ as a convex combination of the initial guess $\{\bar{r}^{i}, \bar{X}^{i}, factor^{i}\}$ and the value implied by household and firm optimization $\{\bar{r}^{i'}, \bar{X}^{i'}, factor^{i'}\}$, where the weight put on the new value $\{\bar{r}^{i'}, \bar{X}^{i'}, factor^{i'}\}$ is given by $\xi \in (0,1]$. The value for ξ must sometimes be small—between 0.05 and 0.2—for certain parameterizations of the model to solve.

$$\left[\bar{r}^{i+1}, \bar{X}^{i+1}, factor^{i+1}\right] = \xi\left[\bar{r}^{i'}, \bar{X}^{i'}, factor^{i'}\right] + (1-\xi)\left[\bar{r}^{i}, \bar{X}^{i}, factor^{i}\right] \quad \text{for} \quad \xi \in (0,1] \tag{11.46}$$

(b) Let $\|\cdot\|$ be a norm on the space of feasible interest rate values r. We often use a sum of squared errors or a maximum absolute error. Check the distance between the initial guess and the implied values as in (11.47). If the distance is less than some tolerance `SS_toler` > 0, then the problem has converged. Otherwise, update the value of the interest rate according to (11.46) and repeat steps (ii) through (v).

$$\texttt{SS\_dist} \equiv \left\|\left[\bar{r}^{i'}, \bar{X}^{i'}, factor^{i'}\right] - \left[\bar{r}^{i}, \bar{X}^{i}, factor^{i}\right]\right\| \tag{11.47}$$

11.7.2 Transition path equilibrium

The TPI solution method for the non-steady-state equilibrium transition path for the S-period-lived agent model with endogenous labor is similar to the method described in Section 4.6.2 as well as to the steady-state solution method described in Section 11.7.1. The key assumption is that the economy will reach the steady-state equilibrium $\bar{\boldsymbol{\Gamma}}$ described in Definition 11.1 in a finite number of periods $T < \infty$ regardless of the initial state $\boldsymbol{\Gamma}_1$.

To solve for the transition path (non-steady-state) equilibrium from Definition 11.2, we must know the parameters from the steady-state problem $\Big\{S, \beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta, \boldsymbol{\theta}_{s,t}, factor\Big\}$, the steady-state interest rate $\bar{r}$, initial distribution of savings $\boldsymbol{\Gamma}_1$, and TPI parameters $\{T1, T2, \xi\}$. Note that in the time path equilibrium solution method, we take the $factor$ calculated in the steady-state solution method as a fixed parameter over the time path. Tables **??** and **??** show a particular calibration of the model and corresponding steady-state solution. The algorithm for solving for the transition path equilibrium by time path iteration (TPI) is similar to the steady-state algorithm described in Section 11.7.1.

i. Choose a period $T1$ in which the initial guess for the time path of interest rates $\boldsymbol{r}^i = \{r^i_1, r^i_2, ...r^i_{T1}\}$ and household transfers $\boldsymbol{X}^i = \{X^i_1, X^i_2, ...X^i_{T1}\}$ will arrive at the steady state and stay there. Choose a period $T2$ upon which and thereafter the entire economy is assumed to be in the steady state. You must have the guessed time path hit the steady state before individual optimal decisions will hit their steady state.

ii. Guess the initial time path for the interest rate $\boldsymbol{r}^i = \{r^i_1, r^i_2, ...r^i_{T1}\}$ and household transfers $\boldsymbol{X}^i = \{X^i_1, X^i_2, ...X^i_{T1}\}$.

   (a) The guess for the time path for interest rates $\boldsymbol{r}^i = \{r^i_1, r^i_2, ...r^i_{T1}\}$ implies a time path for wages $\boldsymbol{w} = \{w_1, w_2, ...w_{T1}\}$ using equation (11.40).

   (b) These time paths will have to be extended with their respective steady-state values so that they are each $T2 + S - 1$ elements long. This is the time-path length that enables one to solve the lifetime decisions of every individual alive from period $t = 1$ to $t = T2$.

iii. Given time paths $\boldsymbol{r}^i$, $\boldsymbol{w}$, and $\boldsymbol{X}^i$, solve for the lifetime labor supply $n_{s,t}$, and savings $b_{s+1,t+1}$ decisions of all households alive in periods $t = 1$ to $t = T2$.

(a) The initial old $s = S$ cohort in period $t = 1$ only have one decision to make. They must choose how much to work in the last period of their life $n_{S,1}$. This decision is characterized by one unknown and one equation. The equation is the period-1 version of the labor supply Euler equation (11.5).

$$w_1\Bigl([1+r_1^i]b_{S,1} + w_1 n_{S,1} + X_1^i - T_{S,1}^I\Bigr)^{-\sigma} = \chi_S^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{S,1}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{S,1}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \tag{11.48}$$

(b) Each cohort who is age $1 < s < S$ has an incomplete lifetime of $p = S - s + 1$ periods remaining. Each of these individuals has $2p-1$ variables to solve for. Each has to choose p labor supply decisions $n_{s,t}$ and $p-1$ savings decisions $b_{s+1,t+1}$ using the appropriate system of $2p-1$ Euler equations (11.5) and (11.6), respectively, with the appropriate interest rates, wages, and transfers from time paths $\boldsymbol{r}^i$, $\boldsymbol{w}$, and $\boldsymbol{X}^i$.

(c) Each cohort born in periods 1 through $T2$ has a complete lifetime and has $2S-1$ variables to solve for. Each has to choose S labor supply decisions $n_{s,t}$ and $S-1$ savings decisions $b_{s+1,t+1}$ using the appropriate system of $2S-1$ Euler equations (11.5) and (11.6), respectively, with the appropriate interest rates, wages, and transfers from time paths $\boldsymbol{r}^i$, $\boldsymbol{w}$, $\boldsymbol{X}^i$.

(d) One trick to doing this successfully is to use the previous cohort's solutions as the initial guess for the current cohort's root finder.

iv. Use the time paths of the distribution of labor supply $n_{s,t}$ and savings $b_{s,t}$ from households' optimal decisions given $\boldsymbol{r}^i$, $\boldsymbol{w}$, and $\boldsymbol{X}^i$ to compute time paths for aggregate capital and aggregate labor $\boldsymbol{K} = \{K_1, K_2, ...K_{T2}\}$ and $\boldsymbol{L} = \{L_1, L_2, ...L_{T2}\}$ implied by capital and labor market clearing conditions (11.29) and (11.30).

v. Compute a new time path for interest rates $\boldsymbol{r}^{i'}$ and transfers $\boldsymbol{X}^{i'}$ using household optimal decisions and the time paths of the aggregate capital stock $\bar{K}$ and aggregate labor $\bar{L}$ implied by the household and firm optimization from part (iv) using equations (11.43) and (11.44).

vi. Compare the distance between the new time paths $\left[\boldsymbol{r}^{i'},\boldsymbol{X}^{i'}\right]$ versus the initial guess $\left[\boldsymbol{r}^{i},\boldsymbol{X}^{i}\right]$.

$$\texttt{TPI\_dist} = \left\|\left[\boldsymbol{r}^{i'},\boldsymbol{X}^{i'}\right] - \left[\boldsymbol{r}^{i},\boldsymbol{X}^{i}\right]\right\| \geq 0 \tag{11.49}$$

Let $\|\cdot\|$ be a norm on the space of time paths for the interest rate $\boldsymbol{r}^i$ and transfers $\boldsymbol{X}^i$. Common norms to use are the L^2 and the L^∞ norms.

(a) Let the tolerance level $\texttt{TPI\_toler} > 0$ be some strictly positive number close to zero. If the distance is less than or equal to some tolerance level $\texttt{TPI\_dist} \leq \texttt{TPI\_toler}$, then the fixed point, and therefore the equilibrium transition path, has been found.

(b) If the distance is greater than some tolerance level $\texttt{TPI\_dist} > \texttt{TPI\_toler}$, then update the guess for a new interest rate time path to be a convex combination current initial time path and the implied time path.

$$\left[\boldsymbol{r}^{i+1},\boldsymbol{X}^{i+1}\right] = \xi\left[\boldsymbol{r}^{i'},\boldsymbol{X}^{i'}\right] + (1-\xi)\left[\boldsymbol{r}^{i},\boldsymbol{X}^{i}\right] \quad \text{for} \quad \xi \in (0,1] \tag{11.50}$$

11.8 Calibration

11.9 Exercises