Chapter 5

S-period Lives, Endogenous Labor, and Heterogeneous Abilities

In this chapter, we augment the model from Section 4 with individuals of each age s having different abilities. The abilities will enter the labor income part of the budget constraint in a way that an individual will earn the average wage plus a premium or a discount based on their different abilities. Age and ability are two of the most important dimensions of individual heterogeneity to have in an economic model.

5.1 Heterogeneous lifetime ability paths

Differences among workers' productivity in terms of ability is one of the key dimensions of heterogeneity to model in a micro-founded macroeconomy. In this chapter, we model this heterogeneity as deterministic lifetime productivity paths to which new cohorts of agents in the model are randomly assigned.[1] In our model, agents' labor income comes from the equilibrium wage and the agent's endogenous quantity of labor supply. In this section, we augment the labor income expression with an individual productivity $e_{j,s}$, where j is the index of the ability type or path of the individual and s is the age of the individual with

[1] Stochastic and persistent idiosyncratic ability is another common way to model income heterogeneity. We treat this in Chapter 6.

that ability path.

$$\text{labor income:} \quad w_t e_{j,s} n_{j,s,t} \quad \forall j,s,t \tag{5.1}$$

In this specification, w_t is an equilibrium wage representing a portion of labor income that is common to all workers. Individual quantity of labor supply is $n_{j,s,t}$, and $e_{j,s}$ represents a labor productivity factor that augments or diminishes the productivity of a worker's labor supply relative to average productivity.

We calibrate deterministic ability paths such that each lifetime income group has a different life-cycle profile of earnings. The distribution on income and wealth are often focal components of macroeconomic models. As such, we use a calibration of deterministic lifetime ability paths from DeBacker et al. (2017b) that can represent U.S. earners in the top 1% of the distribution of lifetime income. Piketty and Saez (2003) show that income and wealth attributable to these households has shown the greatest growth in recent decades. The data come from the U.S. Internal Revenue Services's (IRS) Statistics of Income program (SOI) Continuous Work History Sample (CWHS). DeBacker et al. (2017b) match the SOI data with Social Security Administration (SSA) data on age and Current Population Survey (CPS) data on hours in order to generate a non-top-coded measure of hourly wage.

Figure 5.1: Exogenous life cycle income ability paths $\log(e_{j,s})$ with $S = 80$ and $J = 7$

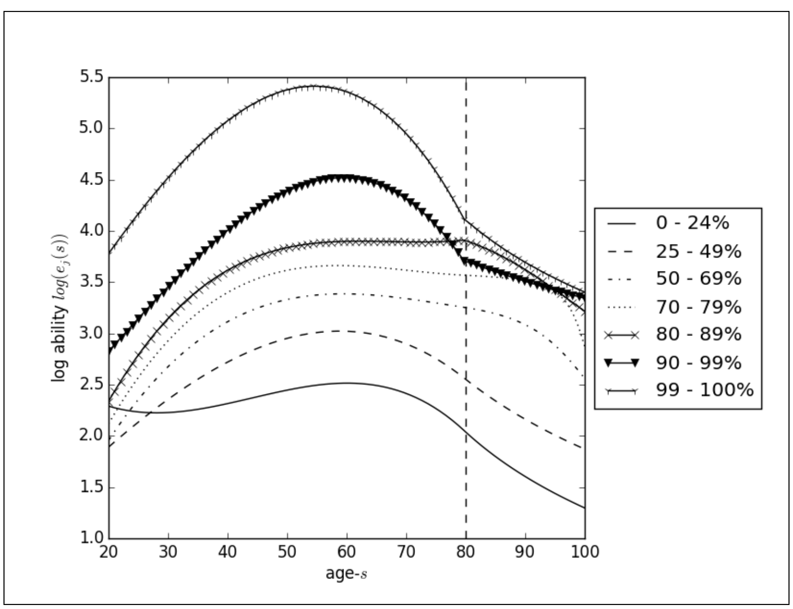


Figure 5.1 shows a calibration for $J = 7$ deterministic lifetime ability paths $e_{j,s}$ corresponding to labor income percentiles $\boldsymbol{\lambda} = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$. Because there are few individuals above age 80 in the data, DeBacker et al. (2017b) extrapolate these estimates for model ages 80-100 using an arctan function.

5.2 Households

Individuals are born at age $s = 1$ and live to age $S \in [3, 80]$. We choose 80 as the upper bound of periods to live so that the minimum amount of time represented by a model period is one year. But this restriction is not important.

At birth, each individual age $s = 1$ is randomly assigned one of J ability groups, indexed by j. Let λ_j represent the fraction of individuals in each ability group, such that $\sum_j \lambda_j = 1$. Note that this implies that the distribution across ability types in each age is given by $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, ...\lambda_J]$. Once an individual is born and assigned to an ability type, he remains that ability type for his entire lifetime. This is deterministic ability heterogeneity. Let $e_{j,s} > 0$ be a matrix of ability-levels such that an individual of ability type j will have lifetime abilities of $[e_{j,1}, e_{j,2}, ...e_{j,S}]$. The household budget constraint is now the following,

$$\begin{aligned} c_{j,s,t} + b_{j,s+1,t+1} = (1 + r_t)b_{j,s,t} + w_t e_{j,s} n_{j,s,t} \quad \forall j, s, t \\ \text{where} \quad b_{j,1,t}, b_{j,S+1,t} = 0 \quad \forall j, t \end{aligned} \tag{5.2}$$

where many of the variables now have j subscripts. The variables with three subscripts (j, s, t) tell you to which ability type j and age s individual the variable belongs and in which period t.

The labor-leisure constraint and the period utility function are similar to equations (4.1) and (4.8), respectively, except the endogenous variables have an additional j subscript.

$$n_{j,s,t} + l_{j,s,t} = \tilde{l} \tag{5.3}$$

$$u\big(c_{j,s,t},n_{j,s,t}\big) = \frac{c_{j,s,t}^{1-\sigma}-1}{1-\sigma} + \chi^n_s b\left[1 - \left(\frac{n_{j,s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1}{\upsilon}} \tag{5.4}$$

We describe the properties and derivation of the elliptical disutility of labor function in (5.4) in Section 4.1. It is an approximation of the standard constant Frisch elasticity disutility of labor supply function, but (5.4) has Inada conditions at both the upper and lower bounds of labor supply. This provides a significant increase in computational tractability with very little cost.

Households choose lifetime consumption $\{c_{j,s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{j,s,t+s-1}\}_{s=1}^S$, and savings $\{b_{j,s+1,t+s}\}_{s=1}^{S-1}$ to maximize discounted lifetime utility,

$$\max_{\{c_{j,s,t+s-1},n_{j,s,t+s-1}\}_{s=1}^S,\{b_{j,s+1,t+s}\}_{s=1}^{S-1}} \sum_{u=0}^{E+S-s}\beta^u u\left(c_{j,s+u,t+u},n_{j,s+u,t+u}\right) \quad \forall j,t \tag{5.5}$$

$$\text{s.t.} \quad c_{j,s,t} + b_{j,s+1,t+1} = (1+r_t)b_{j,s,t} + w_t e_{j,s} n_{j,s,t} \tag{5.2}$$

$$\text{where} \quad u\big(c_{j,s,t},n_{j,s,t}\big) = \frac{c_{j,s,t}^{1-\sigma}-1}{1-\sigma} + \chi^n_s b\left[1 - \left(\frac{n_{j,s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1}{\upsilon}} \tag{5.4}$$

where χ^n_s is a scale parameter that can potentially vary by age s influencing the relative disutility of labor to the utility of consumption. The household's lifetime problem (5.5) can be reduced to choosing S labor supplies $\{n_{j,s,t+s-1}\}_{s=1}^S$ and $S-1$ savings $\{b_{j,s+1,t+s}\}_{s=1}^{S-1}$ by substituting the budget constraints (5.2) in for $c_{j,s,t}$ in each period utility function (5.4) of the lifetime utility function.

The set of optimal lifetime choices for an agent of type j born in period t are characterized by the following S static labor supply Euler equations (5.6), the following $S-1$ dynamic savings Euler equations (5.7), and a budget constraint that binds in all S periods (5.2),

$$\begin{aligned} &w_t e_{j,s} u_1\left(c_{j,s,t+s-1},n_{j,s,t+s-1}\right) = -u_2\left(c_{j,s,t+s-1},n_{j,s,t+s-1}\right) \quad \forall j,t \quad \text{and} \quad s\in\{1,2,...S\} \\ &\Rightarrow \quad w_t e_{j,s}\left(c_{j,s,t}\right)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1-\upsilon}{\upsilon}} \end{aligned} \tag{5.6}$$

$$u_1\left(c_{j,s,t+s-1},n_{j,s,t+s-1}\right) = \beta(1+r_{t+1})u_1\left(c_{j,s+1,t+s},n_{j,s+1,t+s}\right) \quad \forall j,t \quad \text{and} \quad s\in\{1,2,...S-1\}$$
$$\Rightarrow \quad \left(c_{j,s,t}\right)^{-\sigma} = \beta(1+r_{t+1})\left(c_{j,s+1,t+1}\right)^{-\sigma} \tag{5.7}$$

$$c_{j,s,t} + b_{j,s+1,t+1} = (1+r_t)b_{j,s,t} + w_t e_{j,s}n_{j,s,t} \quad \forall j,t \quad \text{and} \quad s\in\{1,2,...S\}, \quad b_{1,t},b_{S+1,t+S}=0 \tag{5.2}$$

where u_1 is the partial derivative of the period utility function with respect to its first argument $c_{j,s,t}$, and u_2 is the partial derivative of the period utility function with respect to its second argument $n_{j,s,t}$. As was demonstrated in detail in Section 3.1, the dynamic Euler equations (5.7) do not include marginal utilities of all future periods because of the principle of optimality and the envelope condition.

Note that these $2S-1$ household decisions are perfectly identified if the household knows what prices will be over its lifetime $\{w_u,r_u\}_{u=t}^{t+S-1}$. As in section 3.1, let the distribution of capital and household beliefs about the evolution of the distribution of capital be characterized by (5.8) and (5.9).

$$\mathbf{\Gamma}_t \equiv \left\{b_{j,s,t}\right\}_{s=2}^{S} \quad \forall j,t \tag{5.8}$$

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u\geq 1 \tag{5.9}$$

5.3 Firms

Firms are characterized exactly as in Section 2.2, with the firm's aggregate capital decision K_t governed by first order condition (5.10) and its aggregate labor decision L_t governed by first order condition (5.11).

$$r_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{5.10}$$
$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{5.11}$$

The per-period depreciation rate of capital is $\delta\in[0,1]$, the capital share of income is $\alpha\in(0,1)$, and total factor productivity is $A>0$.

The only change to note, which is described more carefully in Section 5.4 in the labor market clearing condition (5.12), is that aggregate labor L_t in the production function is now in efficiency units. This means that the aggregate labor L_t used in production is made up of both labor hours $n_{j,s,t}$ and ability $e_{j,s}$.

5.4 Market clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand. The market clearing conditions for this version of the model slightly different from those in the previous sections because we must sum not only over ages s but also over ability types j.

$$L_t = \sum_{s=1}^{S}\sum_{j=1}^{J}\lambda_j e_{j,s} n_{j,s,t} \quad \forall t \tag{5.12}$$

$$K_t = \sum_{s=2}^{S}\sum_{j=1}^{J}\lambda_j b_{j,s,t} \quad \forall t \tag{5.13}$$

$$\begin{aligned} &Y_t = C_t + K_{t+1} - (1-\delta)K_t \quad \forall t \\ &\quad\text{where}\quad C_t \equiv \sum_{s=1}^{S}\sum_{j=1}^{J}\lambda_j c_{j,s,t} \end{aligned} \tag{5.14}$$

The goods market clearing equation (5.14) is redundant by Walras' Law. As noted in Section 5.3, the aggregate labor supply in (5.12) is in efficiency units (i.e., L_t sums over $e_{j,s}n_{j,s,t}$).

It is important to note that the distributional assumptions here are that each age-s cohort has a population of one, which means the total population is S and the population age-s and ability-j is λ_j. Each aggregate variable L_t, K_t, and C_t (and Y_t indirectly) sums over individual decisions multiplied by the individual population measure λ_j.

5.5 Equilibrium

As in previous sections, we give a rough sketch of the equilibrium before providing exact definitions of the functional equilibrium concepts so you can see what the functions look like and understand the exact equilibrium definition more clearly. A rough description of the equilibrium solution to the problem above is the following three points.

i. Households optimize according to equations (5.6) and (5.7).

ii. Firms optimize according to (5.10) and (5.11).

iii. Markets clear according to (5.12) and (5.13).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

The easiest way to understand the equilibrium solution is to substitute the market clearing conditions (5.12) and (5.13) into the firm's optimal conditions (5.10) and (5.11) solve for the equilibrium wage and interest rate as functions of the distribution of capital.

$$w_t\big(\boldsymbol{\Gamma}_t\big): \quad w_t = (1-\alpha)A\left(\frac{\sum_{s=2}^S\sum_{j=1}^J\lambda_j b_{j,s,t}}{\sum_{s=1}^S\sum_{j=1}^J\lambda_j e_{j,s}n_{j,s,t}}\right)^{\alpha} \quad \forall t \tag{5.15}$$

$$r_t\big(\boldsymbol{\Gamma}_t\big): \quad r_t = \alpha A\left(\frac{\sum_{s=1}^S\sum_{j=1}^J\lambda_j e_{j,s}n_{j,s,t}}{\sum_{s=2}^S\sum_{j=1}^J\lambda_j b_{j,s,t}}\right)^{1-\alpha} - \delta \quad \forall t \tag{5.16}$$

It is worth noting here that the equilibrium wage (5.15) and interest rate (5.16) are written as functions of the period-t distribution of savings (wealth) $\boldsymbol{\Gamma}_t$ from (5.8) and are not functions of the period-t distribution of labor supply, which labor distribution shows up in (5.15) and (5.16). This is because, similar to next period savings $b_{j,s+1,t+1}$, current period labor supply $n_{j,s,t}$ must be chosen in period t and is therefore a function of the current state $\boldsymbol{\Gamma}_t$ distribution of savings.

Now (5.15), (5.16), and the budget constraint (5.2) can be substituted into household Euler equations (5.6) and (5.7) to get the following $(2S-1)$-equation system of lifetime characterizing equations for an individual of ability type j. Extended across all time periods, this system completely characterizes the equilibrium.

$$\begin{aligned}
&w_t\big(\boldsymbol{\Gamma}_t\big)e_{j,s}\bigg(\Big[1+r_t\big(\boldsymbol{\Gamma}_t\big)\Big]b_{j,s,t} + w_t\big(\boldsymbol{\Gamma}_t\big)e_{j,s}n_{j,s,t} - b_{j,s+1,t+1}\bigg)^{-\sigma} = ... \\
&\qquad \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s\in\{1,2,...S\} \quad \text{and} \quad \forall j,t
\end{aligned} \tag{5.17}$$

$$\begin{aligned}&\left(\Big[1+r_t\big(\boldsymbol{\Gamma}_t\big)\Big]b_{j,s,t}+w_t\big(\boldsymbol{\Gamma}_t\big)e_{j,s}n_{j,s,t}-b_{j,s+1,t+1}\right)^{-\sigma}= \\ &\quad \beta\Big[1+r_{t+1}\big(\boldsymbol{\Gamma}_{t+1}\big)\Big]\left(\Big[1+r_{t+1}\big(\boldsymbol{\Gamma}_{t+1}\big)\Big]b_{j,s+1,t+1}+w_{t+1}\big(\boldsymbol{\Gamma}_{t+1}\big)e_{j,s}n_{j,s+1,t+1}-b_{j,s+2,t+2}\right)^{-\sigma} \\ &\qquad\quad \text{for}\quad s\in\{1,2,...S-1\}\quad\text{and}\quad\forall j,t\end{aligned} \tag{5.18}$$

The system of S nonlinear static equations (5.17) and $S-1$ nonlinear dynamic equations (5.18) characterizing the lifetime labor supply and savings decisions for each household $\{n_{j,s,t+s-1}\}_{j,s=1}^{J,S}$ and $\{b_{j,s+1,t+s}\}_{j,s=1}^{J,S-1}$ is not identified. Each individual knows the current distribution of capital $\boldsymbol{\Gamma}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\boldsymbol{\Gamma}_{t+1}=\{b_{j,s+1,t+1}\}_{j,s=1}^{J,S-1}$ and a number of subsequent periods for all agents alive in those subsequent periods. We also need to solve for a policy function for the individual $b_{j,s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1}=x_t=\bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 5.1 (Steady-state equilibrium). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents, endogenous labor supply, and deterministic heterogeneous lifetime ability paths is defined as constant allocations of consumption $\{\bar{c}_{j,s}\}_{j,s=1}^{J,S}$, labor supply $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$, and savings $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (5.6) and (5.7),

ii. firms optimize according to (5.10) and (5.11),

iii. markets clear according to (5.12) and (5.13).

The relevant examples of stationary functions in this model are the policy functions for labor and savings. Let the equilibrium policy functions for labor supply be represented by $n_{j,s,t}=\phi_{j,s}\left(\boldsymbol{\Gamma}_t\right)$, and let the equilibrium policy functions for savings be represented by $b_{j,s+1,t+1}=\psi_{j,s}\left(\boldsymbol{\Gamma}_t\right)$. The arguments of the functions (the state) may change overtime causing the labor and savings levels to change over time, but the function of the arguments is constant (stationary) across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state (transition path) equilibrium of the model.

Definition 5.2 (**Non-steady-state functional equilibrium**). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents, endogenous labor supply, and deterministic heterogeneous lifetime ability paths is defined as stationary allocation functions of the state $\left\{n_{j,s,t} = \phi_{j,s}\big(\boldsymbol{\Gamma}_t\big)\right\}_{j,s=1}^{J,S}$, $\left\{b_{j,s+1,t+1} = \psi_{j,s}\big(\boldsymbol{\Gamma}_t\big)\right\}_{j,s=1}^{J,S-1}$ and and stationary price functions $w(\boldsymbol{\Gamma}_t)$ and $r(\boldsymbol{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (2.17), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\boldsymbol{\Gamma}_{t+u} = \boldsymbol{\Gamma}^e_{t+u} = \Omega^u\left(\boldsymbol{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (5.6) and (5.7),

iii. firms optimize according to (5.10) and (5.11),

iv. markets clear according to (5.12) and (5.13).

5.6 Solution Method

In this section we characterize computational approaches to solving for the steady-state equilibrium from Definition 5.1 and the transition path equilibrium from Definition 5.2.

5.6.1 Steady-state equilibrium

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 5.1. The parameters needed for the steady-state solution of this model are $\Big\{J, S, \{\lambda_j\}_{j=1}^J, \beta, \sigma, \tilde{l}, \{e_{j,s}\}_{j,s=1}^{J,S}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta\Big\}$, where S is the number of periods in an individual's life, J is the number of ability groups, $\{\lambda_j\}_{j=1}^J$ is the income percentiles of ability groups, $\Big\{\beta, \sigma, \tilde{l}, \{e_{j,s}\}_{j,s=1}^{J,S}, b, \upsilon, \{\chi^n_s\}_{s=1}^S\Big\}$ are household utility function parameters, and $\{A, \alpha, \delta\}$ are firm production function parameters. These parameters are chosen, calibrated, or estimated outside of the model and are inputs to the solution method.

The steady-state is defined as the solution to the model in which the distributions of individual consumption, labor supply, and savings have settled down and are no longer changing over time. As such, it can be thought of as a long-run solution to the model in which the effects of any shocks or changes from the past no longer have an effect.

$$c_{j,s,t} = \bar{c}_{j,s}, \quad n_{j,s,t} = \bar{n}_{j,s}, \quad b_{j,s,t} = \bar{b}_{j,s} \quad \forall j,s,t \tag{5.19}$$

From the market clearing conditions (5.12) and (5.13) and the firms' first order equations (5.10) and (5.11), the household steady-state conditions imply the following steady-state conditions for prices and aggregate variables.

$$r_t = \bar{r}, \quad w_t = \bar{w}, \quad K_t = \bar{K} \quad L_t = \bar{L} \quad \forall t \tag{5.20}$$

The steady-state is characterized by the steady-state versions of the set of $2S-1$ Euler equations (5.6) and (5.7) over the lifetime of an individual (after substituting in the budget constraint) and the $2S-1$ unknowns $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$,

$$\bar{w}e_{j,s}\Big([1+\bar{r}]\bar{b}_{j,s} + \bar{w}e_{j,s}\bar{n}_{j,s} - \bar{b}_{j,s+1}\Big)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \forall j \quad \text{and} \quad s=\{1,2,...S\} \tag{5.21}$$

$$\Big([1+\bar{r}]\bar{b}_{j,s} + \bar{w}e_{j,s}\bar{n}_{j,s} - \bar{b}_{j,s+1}\Big)^{-\sigma} = \beta(1+\bar{r})\Big([1+\bar{r}]\bar{b}_{j,s+1} + \bar{w}e_{j,s_+1}\bar{n}_{j,s+1} - \bar{b}_{j,s+2}\Big)^{-\sigma} \quad \forall j \quad \text{and} \quad s=\{1,2,...S-1\} \tag{5.22}$$

where both $\bar{w}$ and $\bar{r}$ are functions of the distributions of labor supply and savings as shown in (5.15) and (5.16). This represents a system of $J(2S-1)$ nonlinear dynamic equations and $J(2S-1)$ unknowns.

To solve for the steady-state equilibrium, we use a bisection method on the outer, or general equilibrium, loop to find for $\bar{r}$. At each of iteration through this bisection method, we solve the household problem for the given interest rate and the associated wage rate. Note that the firms' first order conditions allow one to solve for the wage rate as a function of the interest rate and model parameters. The bisection method then finds the fixed point that represents the steady-state equilibrium. This an an interest rate that gives rise to household savings and labor supply decision that produced aggregate amounts of capital and labor which are consistent with that interest rate. The algorithm is the following.

i. Make a guess for the steady-state interest rate, $\bar{r}^i$.

   (a) A guess for the steady-state interest rate $\bar{r}^i$ will imply a value for the steady-state wage $\bar{w}$ from (5.23), which is derived from solving equation (5.10) for the capital labor ratio K/L and substituting it into (5.11).

$$w_t = (1-\alpha)A\left(\frac{\alpha A}{r_t + \delta}\right)^{\frac{\alpha}{1-\alpha}} \quad \forall t \tag{5.23}$$

ii. Given steady-state prices $\bar{r}^i$ and $\bar{w}$, solve for the steady-state household's lifetime decisions $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$ and $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$.

   (a) Given $\bar{r}^i$ and $\bar{w}$, use a root finder to solve for $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$ and $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$ from the $J(2S-1)$ steady-state Euler equations (5.21) and (5.22).

   (b) This solution can be sensitive the initial guess for $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$ and $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$ passed to the root finder.

   (c) If J and S are large, one might want to break this inner loop into J problems of $2S-1$ equations and unknowns each by looping over values of j.

iii. Given solution for optimal household decisions $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$ and $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$ based on the guess for the interest rate $\bar{r}^i$ and the implied wage $\bar{w}$, solve for the aggregate capital $\bar{K}$ and aggregate labor $\bar{L}$ implied by the household solutions and market clearing conditions.

$$\bar{K} = \sum_{s=2}^{S}\sum_{j=1}^{J}\lambda_j \bar{b}_{j,s} \tag{5.24}$$

$$\bar{L} = \sum_{s=1}^{S}\sum_{j=1}^{J}\lambda_j e_{j,s}\bar{n}_{j,s} \tag{5.25}$$

iv. Compute a new value for the interest rate $\bar{r}^{i'}$ using the aggregate capital stock $\bar{K}$ and aggregate labor $\bar{L}$ implied by the household optimization from equations (5.24) and (5.25).

$$\bar{r}^{i'} = \alpha A\left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha} - \delta \tag{5.26}$$

v. Update the guess for the steady-state interest rate $\bar{r}^{i+1}$ until the interest rate implied by household optimization $\bar{r}^{i'}$ equals the initial guess for the interest rate $\bar{r}^{i}$.

(a) The bisection method characterizes the updated guess for the interest rate $\bar{r}^{i+1}$ as a convex combination of the initial guess $\bar{r}^{i}$ and the value implied by household and firm optimization $\bar{r}^{i'}$, where the weight put on the new value $\bar{r}^{i'}$ is given by $\xi \in (0,1]$. The value for ξ must sometimes be small—between 0.05 and 0.2—for certain parameterizations of the model to solve.

$$\bar{r}^{i+1} = \xi\bar{r}^{i'} + (1-\xi)\bar{r}^{i} \quad \text{for} \quad \xi \in (0,1] \tag{5.27}$$

(b) Let $\|\cdot\|$ be a norm on the space of feasible interest rate values r. We often use a sum of squared errors or a maximum absolute error. Check the distance between the initial guess and the implied values as in (5.28). If the distance is less than some tolerance `SS_toler` > 0, then the problem has converged. Otherwise, update the value of the interest rate according to (5.27) and repeat steps (ii) through (v).

$$\texttt{SS\_dist} \equiv \left\|\bar{r}^{i'} - \bar{r}^{i}\right\| \tag{5.28}$$

Define the updating of the interest rate $\bar{r}^{i}$ in step (iv) indexed by i as the "outer loop" of the fixed point solution. Although computationally intensive, the bisection method described above is often the most robust solution method.

Figure 5.2: Steady-state distribution of consumption $\bar{c}_{j,s}$

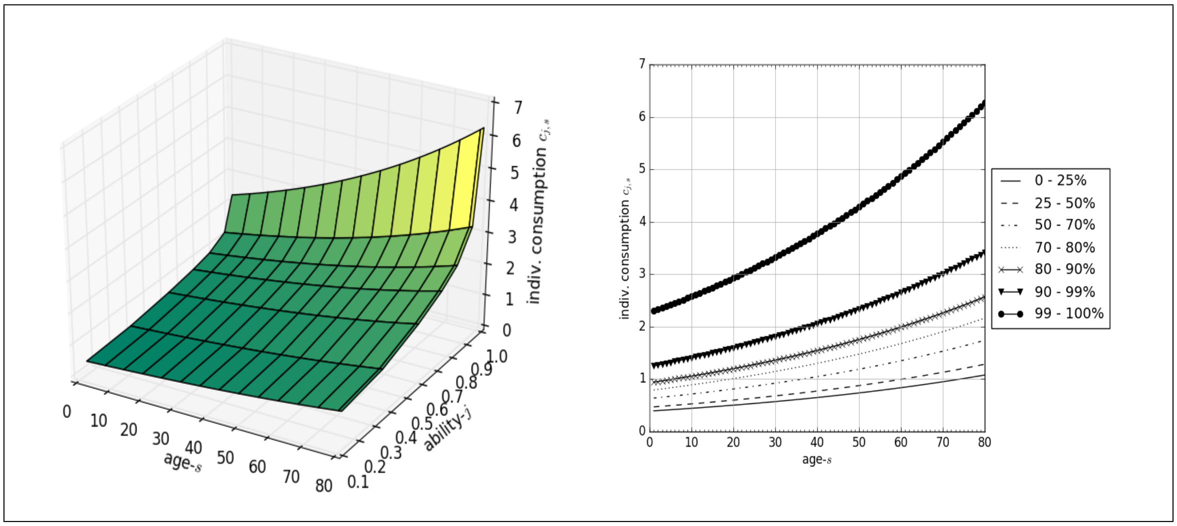


Figure 5.3: Steady-state distribution of labor supply $\bar{n}_{j,s}$

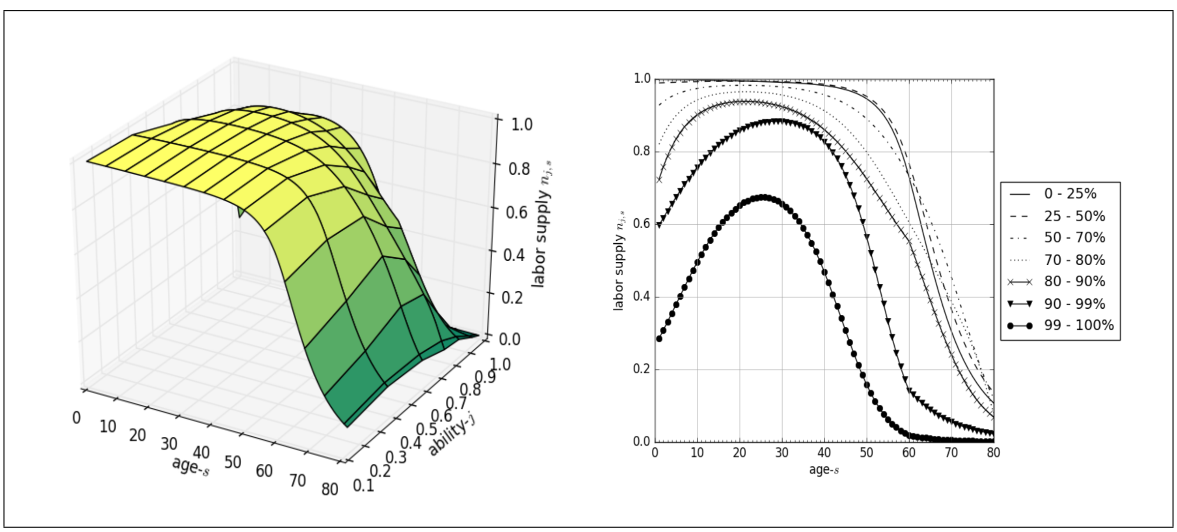


Figure 5.4: Steady-state distribution of savings/wealth $\bar{b}_{j,s}$

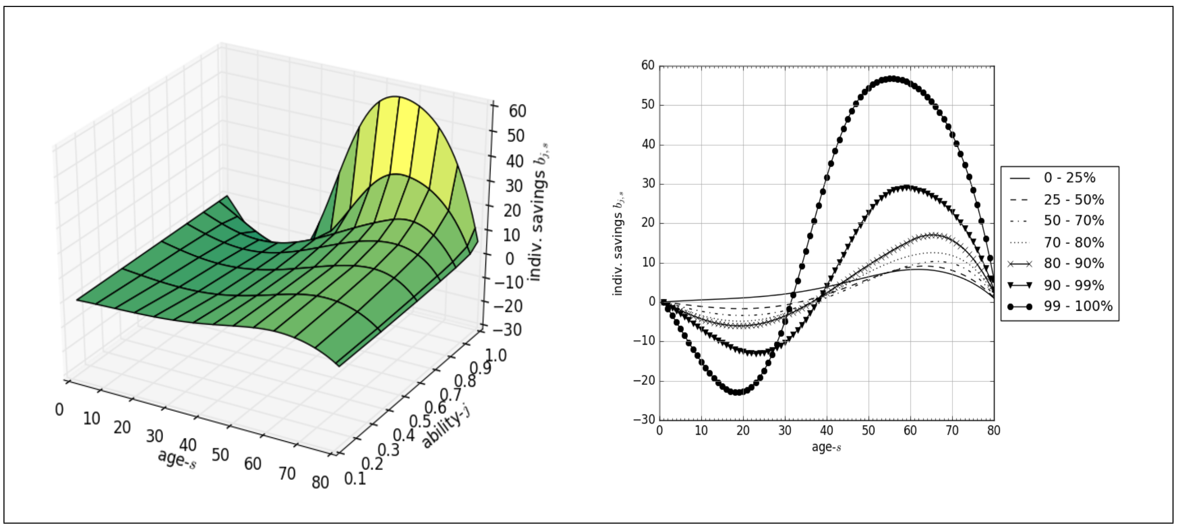


Table 5.1: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Equilibrium error	Value
$\bar{r}$	0.075	Max. absolute savings Euler error	1.78e-15
$\bar{w}$	1.130	Max. absolute labor supply Euler error	7.02e-14
$\bar{K}$	290.632	Max. absolute final period savings $\bar{b}_{j,S+1}$	8.89e-12
$\bar{L}$	59.815	Resource constraint error	-0.576
$\bar{Y}$	104.015		
$\bar{C}$	90.060	Computation time	8 min. 4 sec.

Figures 5.2, 5.3, and 5.4 show the steady-state distributions of individual consumption, labor supply, and savings, respectively, by age s and ability j in an 80-period-lived agent model with parameter values listed above the line in Table 5.3 in Section 5.7. The left side of Table 5.1 gives the resulting steady-state values for the prices and aggregate variables.

As a final note, it is important to make sure that all of the characterizing equations are satisfied in order to verify that the steady-state has been found. In this model, we must check the $J(2S-1)$ Euler errors from the labor supply and savings decisions, the final period savings decision (should be zero), the two firm first order conditions, and the three market clearing conditions (including the goods market clearing condition, which is redundant by Walras' law). The right side of Table 5.1 shows the maximum errors in all these characterizing conditions. Because all Euler errors are smaller than 7.0e-14, the final period individual savings is less than 8.9e-12, and the resource constraint error is smaller than -0.58, we can be confident that we have successfully solved for the steady-state.

5.6.2 Transition path equilibrium

The TPI solution method for the non-steady-state equilibrium transition path for the S-period-lived agent model with endogenous labor and deterministic heterogeneous lifetime ability paths is similar to the method described in Section 4.6.2. The key assumption is that the economy will reach the steady-state equilibrium $\bar{\mathbf{\Gamma}}$ described in Definition 5.1 in a finite number of periods $T < \infty$ regardless of the initial state $\mathbf{\Gamma}_1$.

To solve for the transition path (non-steady-state) equilibrium from Definition 5.2, we must know the parameters from the steady-state problem,

$$\Big\{J, S, \{\lambda_j\}_{j=1}^{J}, \beta, \sigma, \tilde{l}, \{e_{j,s}\}_{j,s=1}^{J,S}, b, \upsilon, \{\chi_s^n\}_{s=1}^{S}, A, \alpha, \delta\Big\}$$

the steady-state solution value of $\bar{r}$, initial distribution of savings $\mathbf{\Gamma}_1$, and TPI parameters $\{T, \xi\}$. Tables 5.1 and 5.3 show a particular calibration of the model and a steady-state solution. The algorithm for solving for the transition path equilibrium by time path iteration (TPI) is the following.

i. Choose a period T upon which and thereafter the economy is assumed to be in the steady state. You must have the guessed time path hit the steady state before individual optimal decisions will hit their steady state.

ii. Guess initial time paths for the real interest rate, $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. Both of these time paths will have to be extended with their respective steady-state values so that they are $T+S-2$ elements long. This is the time-path length that will allow you to solve the lifetime of every individual alive in period T.

iii. Given time path $\boldsymbol{r}^i$, use the firm FOCs, (5.10) and (5.11), to solve for the path of wage rates, $\boldsymbol{w}^i$

iv. Given the paths for interest rates and wage rates, solve for the lifetime labor supply $n_{j,s,t}$ and savings $b_{j,s+1,t+1}$ decisions of all households alive in periods $t=1$ to $t=T$.

   (a) Given the time paths for the interest rate $\boldsymbol{r}^i$ and wage $\boldsymbol{w}^i$ and the period-1 distribution of savings (wealth) $\mathbf{\Gamma}_1$, solve for the lifetime decisions $n_{j,s,t}$, and $b_{j,s,t}$ of each household alive during periods 1 through T. This is done using the method outlined in steps (ii) of the steady-state computational algorithm outlined in Section 5.6.1.

v. Use time path of the distribution of labor supply $n_{j,s,t}$ and savings $b_{j,s,t}$ from households optimal decisions to compute aggregate capital and labor along the time path, $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$ using the market clearing conditions.

vi. Given these paths for aggregate capital and labor, use the firms' FOC for choice of capital to find the newly implied interest rates along the transition path, $\boldsymbol{r}^{i'}$.

vii. Compare the distance between the guess of the time path for interest rates and these rates implied by the solution to the transition path. $\boldsymbol{r}^{i'}$ versus the initial interest rates $\boldsymbol{r}^{i}$.

$$\texttt{dist} = \left\| \boldsymbol{r}^{i'} - \boldsymbol{r}^{i} \right\| \geq 0 \tag{4.33}$$

Let $\|\cdot\|$ be a norm on the space of time paths for the rate interest rate, $\boldsymbol{r}^{i}$. Common norms to use are the L^2 and the L^∞ norms.

(a) If the distance is less than or equal to some tolerance level $\texttt{dist} \leq \texttt{TPI\_toler} > 0$, then the fixed point, and therefore the equilibrium transition path, has been found.

(b) If the distance is greater than some tolerance level, then update the guess for a new set of initial time paths to be a convex combination current initial time paths and the implied time paths.

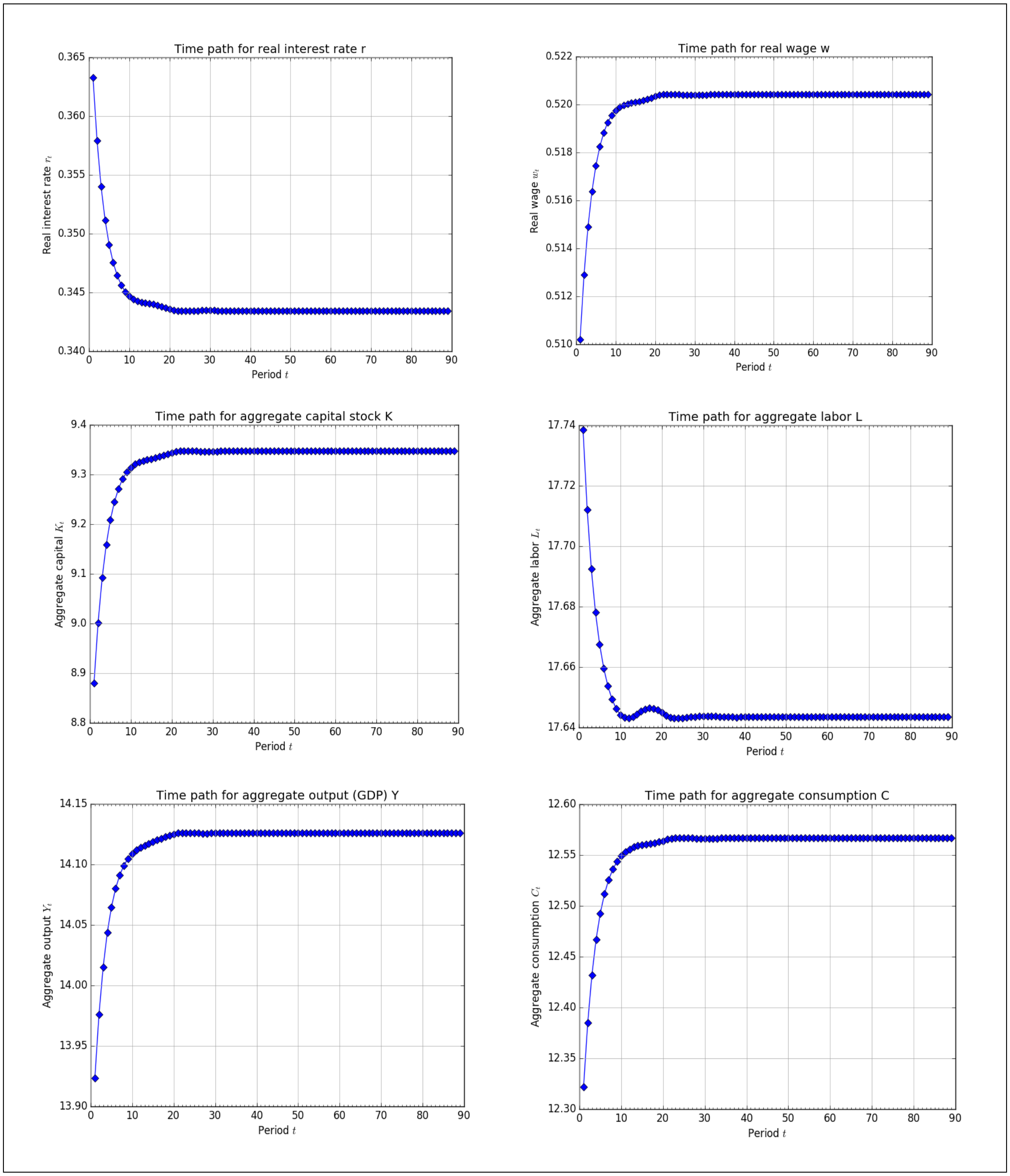
$$\boldsymbol{r}^{i+1} = \xi \boldsymbol{r}^{i'} + (1-\xi)\boldsymbol{r}^{i} \quad \text{for} \quad \xi \in (0,1] \tag{??}$$

Table 5.2: Maximum absolute errors in characterizing equations across transition path

Description	Value
Maximum absolute labor supply Euler error	1.90e-12
Maximum absolute savings Euler error	2.13e-14
Maximum absolute final period savings $\bar{b}_{j,S+1,t}$	1.88e-13
Maximum absolute resource constraint error	1.74e-01

The 6 panels of Figure 5.5 show the equilibrium time paths of the interest rate r_t, wage w_t, and aggregate variables K_t, L_t, Y_t, and C_t. The calibration is the same as that of Section 5.6.1, except that we reduced the number of ages to $S = 20$ and the number of abilities to $J = 3$ with $\boldsymbol{\lambda} = [0.40, 0.35, 0.25]$. We do not show the time paths of individual consumption, labor supply, and savings because they are higher dimensional distributions that would be difficult to show over time. Table 5.2 shows the maximum absolute Euler errors, end-of-life

Figure 5.5: Equilibrium transition paths of prices and aggregate variables



savings, and resource constraint errors across the transition path. All of these should be zero in equilibrium. The fact that none of them is greater than 2.0e-12 in absolute value is evidence that we have successfully solved for the non-steady-state equilibrium transition path of the model.

5.7 Calibration

For the steady-state solution in Section 5.6.1 we used the calibration displayed in Table 5.3. We let agents live for $S = 80$ periods, which implies that each model period of life is one year. The annual discount factor is estimated to be 0.96, so the discount factor per period in the S-period model should be $\beta = 0.96^{80/S} = 0.96$. Let the annual depreciation rate of capital be 0.05. Then the depreciation rate in the S-period model should be $\delta = 1-(1-0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 2.5$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$. For the disutility of labor, let $\chi^n_s = 1$ for all s, and let $[b, \upsilon] = [.501, 1.554]$. We assume $J = 7$ ability types with percentages $\{\lambda_j\}_{j=1}^7 = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$, with a lifetime ability profiles $e_{j,s}$ given in Figure 5.1.

Because the computation of the non-steady-state equilibrium time path of the calibration used for the steady-state would take more than 24 hours, we use a simpler calibration for the solution in Section 5.6.2. We assume that $S = 20$, which implies that each model period equals 4 years. We also assume only three ability types $J = 3$ with $\{\lambda_j\}_{j=1}^3 = [0.40, 0.35, 0.25]$. This changes the value of the discount factor to $\beta = 0.96^{80/S} = 0.849$ and $\delta = 1 - (1 - 0.05)^{80/S} = 0.185$. We also use the method described in Exercise 5.1 to linearly interpolate a 20×3 ability matrix $e_{j,s}$ from the original 80×7 matrix shown in Figure 5.1. All the other parameters are the same as shown in Table 5.3.

5.8 Exercises

Exercise 5.1. Import the comma delimited data file `emat.txt` as a NumPy array `emat`. The file `emat.txt` is data for the 80×7 matrix of ability levels $e_{j,s}$. Each row represents

Table 5.3: Calibrated parameter values for simple endogenous labor model

Parameter	Description	Value
J	Number of lifetime income (ability) groups	7
S	Number of periods in individual life	80
$\{\lambda_j\}_{j=1}^{J}$	Population distribution among ability types	(see Sec. 5.1)
$\{e_{j,s}\}_{j,s=1}^{J,S}$	Individual ability profile factor	(see Fig. 5.1)
β	Per-period discount factor	0.96
σ	Coefficient of relative risk aversion	2.5
$\tilde{l}$	Time endowment per period	1.0
b	Elliptical disutility of labor scale parameter	0.501[a]
υ	Elliptical disutility of labor shape parameter	1.554[a]
$\{\chi^n_s\}_{s=1}^{S}$	Disutility of labor relative scale factor by age	1.0
A	Total factor productivity	1.0
α	Capital share of income	0.35
δ	Per-period depreciation rate of capital	0.05
$\mathbf{\Gamma}_1$	Initial distribution of savings (wealth)	(see Fig. 4.7)
T	Time period in which the model is assumed to hit the steady state	200
ξ	TPI path updating parameter	0.2

[a] The calibration of b and υ is based on matching the marginal disutility of labor supply of a constant Frisch elasticity of labor supply functional form with a Frisch elasticity of 0.8. See Evans and Phillips (2017).

$s = \{1, 2, ...80\}$, and each column represents ability levels $j = \{1, 2, ...7\}$ associated with income percentiles $\boldsymbol{\lambda} = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$. Assume that the population distribution by age is uniform $1/S$. The data in `emat` are scaled such that the average ability is 1 so each value $e_{j,s}$ can be interpreted as the percent premium or discount of the average wage w_t.

$$\frac{1}{S}\sum_{s=1}^{S}\sum_{j=1}^{J}\lambda_j e_{j,s} = 1$$

a. Make a 2D plot of the ability matrix `emat` made up of seven lines showing ability $e_{j,s}$ over the life cycle s for each ability type j. That is, each line on the graph should be a column of the $e_{j,s}$ matrix. Include a legend on the graph that shows "0-24%" for the line for $j = 1$, "25-49%" for the line for $j = 2$, "50-69%" for the line for $j = 3$, "70-79%" for the line for $j = 4$, "80-89%" for the line for $j = 5$, "90-98%" for the line for $j = 6$, and "99-100%" for the line for $j = 7$.

b. Now assume that $S = 10$ (each model year is 8 years) and $J = 3$ with $\{\lambda_j\}_{j=1}^{3} = [0.4, 0.4, 0.3]$. Create a function that uses the original 80×7 `emat` matrix that you imported in the previous part of this exercise as the baseline, and creates a new matrix that is $S_{new} \times J_{new}$, which is 10×3 in this case. Solve for $e_{j,s}$ in the new matrix as a linear interpolation between points on the original matrix.

Exercise 5.2. Using a small version calibration similar to the one that was used for the non-steady-state equilibrium solution in Section 5.6.2, and the steady-state equilibrium Definition 5.1, solve for the steady-state equilibrium values of $\left\{\{\bar{c}_{j,s}\}_{s=1}^{S}\right\}_{j=1}^{J}$, $\left\{\{\bar{n}_{j,s}\}_{s=1}^{S}\right\}_{j=1}^{J}$, $\left\{\{\bar{b}_{j,s}\}_{s=2}^{S}\right\}_{j=1}^{J}$, $\bar{w}$, $\bar{r}$, $\bar{K}$, and $\bar{L}$ numerically. More specifically, let $S = 20$, $J = 2$, $\boldsymbol{\lambda} = [0.6, 0.4]$, $beta = 0.849$, and $\delta = 0.185$. Use the 20×2 $e_{j,s}$ matrix `emat20x2.txt`. And let all other values equal their values from Table 5.3.

Exercise 5.3. Use the calibration from Exercise 5.2 and time path iteration (TPI) to solve for the non-steady state equilibrium transition path of the economy from $\boldsymbol{\Gamma}_1 = 0.95\bar{\boldsymbol{\Gamma}}$ to the steady-state $\bar{\boldsymbol{\Gamma}}$. You must guess a value for T and a time path updating parameter $\xi \in (0, 1)$, but it is likely that $T < 3 * S$ is sufficient. Use an L^2 norm for your distance measure (sum of squared percent deviations), and use a convergence parameter of $\varepsilon = 10^{-9}$. Use a linear or quadratic initial guess for the time path of the real interest rate from the initial state r_1^1 to the steady state r_T^1 at time T.

Exercise 5.4. Plot the equilibrium time paths of the aggregate capital stock $\{K_t\}_{t=1}^{T+10}$ and aggregate labor $\{L_t\}_{t=1}^{T+10}$. How many periods did it take for the economy to get within 0.0001 of the steady-state aggregate capital stock $\bar{K}$? That is, what is the true T?