Chapter 3

S-period-lived Agent Problem with Exogenous labor Supply

In this chapter, we extend the simple 3-period-lived agent problem from Chapter 2 and generalize it to an $S \in [3, 80]$-period-lived agent problem.

3.1 Households

A unit measure of identical individuals are born each period and live for S periods. Let the age of an individual be indexed by $s = \{1, 2, ...S\}$. In general, an age-s individual faces the same per-period budget constraint (2.1) as in the previous section.

$$c_{s,t} + b_{s+1,t+1} = (1 + r_t)b_{s,t} + w_t n_{s,t} \quad \forall s, t \tag{2.1}$$

We assume the individuals supply a unit of labor inelastically in the first two thirds of life ($s \leq \text{round}(2S/3)$) and are retired during the last third of life ($s > \text{round}(2S/3)$).

$$n_{s,t} = \begin{cases} 1 & \text{if} \quad s \leq \text{round}\left(\frac{2S}{3}\right) \\ 0.2 & \text{if} \quad s > \text{round}\left(\frac{2S}{3}\right) \end{cases} \quad \forall s, t \tag{3.1}$$

Because exogenous labor in (3.1) is not dependent on the time period, we drop the t subscript from labor n_s for the rest of this section. We also assume that households are born with no savings $b_{1,t} = 0$ and that individuals save no income in the last period of their lives $b_{S+1,t} = 0$ for all periods t. Assume that $c_{s,t} \geq 0$ because negative consumption neither has an intuitive interpretation nor is it household utility defined for it. It is the latter condition that will make $c_{s,t} > 0$ in equilibrium.

Let the utility of consumption in each period be defined by the constant relative risk aversion function (2.6) $u(c_{s,t})$ from the previous section, such that $u' > 0$, $u'' < 0$, and $\lim_{c\to 0} u(c) = -\infty$. Individuals choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^{S}$, savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints.

$$
\begin{aligned}
&\max_{\{c_{s,t+s-1}\}_{s=1}^{S},\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{u=0}^{S-s} \beta^u u(c_{s+u,t+u}) \quad \forall s,t \\
&\text{s.t.} \quad c_{s,t} = (1+r_t)b_{s,t} + w_t n_s - b_{s+1,t+1} \quad \forall s,t \\
&\text{and} \quad b_{1,t}, b_{S+1,t} = 0 \quad \forall t \quad \text{and} \quad c_{s,t} \geq 0 \quad \forall s,t
\end{aligned}
\tag{3.2}
$$

The number of variables to choose in the household's optimization problem can be reduced by substituting the budget constraints into the optimization problem (3.2). The optimal choice of how much to save in the each of the first $S-1$ periods of life $b_{s+1,t+1}$ is found by taking the derivative of the lifetime utility function with respect to each of the lifetime savings amounts $\{b_{s+1,t+s+1}\}_{s=1}^{S-1}$ and setting the derivatives equal to zero.

In the last period of life, the household optimally chooses no savings $b_{S+1,t+1} = 0$ for all t because any positive savings only imposes a cost of reduced consumption, and negative savings (borrowing) would impose an automatic default on anyone lending to him. The final period S decision is simple. The individual enters the period with wealth $b_{S,t}$, he knows the interest rate r_t and the wage w_t, and inelatically supplies zero labor $n_{S,t} = 0$. Everything in the budget constraint (2.1) is determined except for $c_{S,t}$. In the final period, the individual simply consumes all his resources.

$$
c_{S,t} = (1+r_t)b_{S,t} + w_t n_S \quad \forall t \tag{3.3}
$$

In the second-to-last period of life $s = S - 1$, the household has a savings decision to make. He enters the period with wealth $b_{S-1,t}$, he knows the current interest rate r_t and the current wage w_t, and he must know or be able to forecast next period's interest rate r_{t+1} and wage w_{t+1}. In this case, the household's lifetime utility function is one equation and one unknown.

$$\max_{b_{S,t+1}} u\Big([1+r_t]b_{S-1,t} + w_t n_{S-1} - b_{S,t+1}\Big) + \beta u\Big([1+r_{t+1}]b_{S,t+1} + w_{t+1} n_S\Big) \tag{3.4}$$

The first order condition, or dynamic Euler equation, for this second-to-last period of life savings decision is the following.

$$u'\Big([1+r_t]b_{S-1,t} + w_t n_{S-1} - b_{S,t+1}\Big) = \beta(1+r_{t+1})u'\Big([1+r_{t+1}]b_{S,t+1} + w_{t+1} n_S\Big) \tag{3.5}$$

The solution for savings $b_{S,t+1}$ in the second-to-last period of life to be returned with interest in the last period of life is characterized by the nonlinear dynamic Euler equation (3.5) and is a function of individual wealth $b_{S-1,t}$, the interest rate r_t, and the wage w_t at the beginning of the second-to-last period of life, as well as the interest rate r_{t+1} and wage w_{t+1} in the last period of life.

$$b_{S,t+1} = \psi_{S-1}\left(b_{S-1,t}, r_t, w_t, r_{t+1}, w_{t+1}\right) \quad \forall t \tag{3.6}$$

Call $\psi_{S-1}(\cdot)$ the policy function for savings $b_{S,t+1}$ in the second-to-last period of life.

In the third-to-last period of life $s = S - 2$, the individual has two remaining lifetime decisions to make. He must choose how much to save in the third-to-last period of life $b_{S-1,t}$ and how much to save in the second-to-last period of life $b_{S,t+1}$. The latter of these two decisions will be characterized by the same function (3.6) that equates (3.5). However, the maximization problem is trickier for the third-to-last period savings $b_{S-1,t}$ because the individual must maximize utility over three periods.

$$\begin{aligned} \max_{b_{S-1,t}} u\Big([1+r_{t-1}]b_{S-2,t-1} + w_{t-1} n_{S-2} - b_{S-1,t}\Big) + ... \\ \beta u\Big([1+r_t]b_{S-1,t} + w_t n_{S-1} - b_{S,t+1}\Big) + \beta^2 u\Big([1+r_{t+1}]b_{S,t+1} + w_{t+1} n_S\Big) \end{aligned} \tag{3.7}$$

It initially looks like the savings $b_{S-1,t}$ only shows up in two places, which should make this derivative very easy. However, we must remember that it is also in the optimal function for the second to last period savings $b_{S,t+1}$ from (3.6). The derivative of (3.7) with respect to $b_{S-1,t}$ and set equal to zero is, therefore,

$$-u'(c_{S-2,t-1}) + \beta\left(1 + r_t - \frac{\partial\psi_{S-1}}{\partial b_{S-1,t}}\right)u'(c_{S-1,t}) + \beta^2(1 + r_{t+1})\frac{\partial\psi_{S-1}}{\partial b_{S-1,t}}u'(c_{S,t+1}) = 0 \quad (3.8)$$

This looks very different from the equation characterizing optimal savings in the second-to-last period (3.4). However, factoring out the partial derivative terms gives the following version of the equation.

$$-u'(c_{S-2,t-1}) + \beta(1 + r_t)u'(c_{S-1,t}) = \beta\frac{\partial\psi_{S-1}}{\partial b_{S-1,t}}\Big[u'(c_{S-1,t}) - \beta(1 + r_{t+1})u'(c_{S,t+1})\Big] \quad (3.9)$$

Notice that the term on the right in brackets is zero from (3.4). This is the envelope theorem or the principle of optimality. It means that the savings decisions in all future periods will be made optimally, so the derivative of that function will be zero with respect to today's savings. The third-to-last period Euler equation in (3.9) reduces to the following due to the envelope theorem.

$$u'(c_{S-2,t-1}) = \beta(1 + r_t)u'(c_{S-1,t}) \quad (3.10)$$

Using the expressions for $c_{S-2,t-1}$ and $c_{S-1,t}$ from the budget constraint (2.1) and the function for second-to-last period savings $b_{S,t+1}$ from (3.6), it is simple to show that the policy function for third-to-last period savings $b_{S-1,t}$ characterized by nonlinear dynamic Euler equation (3.10) is the following.

$$b_{S-1,t} = \psi_{S-2}\left(b_{S-2,t-1}, r_{t-1}, w_{t-1}, r_t, w_t, r_{t+1}, w_{t+1}\right) \quad \forall t \quad (3.11)$$

By backward induction, it is straightforward to show that the $S-1$ savings decisions over an individual's lifetime are characterized by $S-1$ nonlinear dynamic Euler equations of the form,

$$
\begin{aligned}
u'(c_{s,t}) = \beta(1+r_{t+1})u'(c_{s+1,t+1}) \quad \forall t, \quad \text{and} \quad 1 \leq s \leq S-1 \\
\text{and} \quad c_{s,t} = w_t n_s + (1+r_t)b_{s,t} - b_{s+1,t+1} \quad \forall s,t \\
\text{and} \quad b_{1,t}, b_{S-1,t} = 0 \quad \forall t
\end{aligned}
\tag{3.12}
$$

Following the pattern of (3.6) and (3.11), the policy functions for each of the savings decisions is a function of the individual's wealth at the beginning of the period $b_{s,t}$ and the time path of wages and interest rates over the remaining periods of the individual's life.

$$
b_{s+1,t+1} = \psi_s\Big(b_{s,t}, \{r_v\}_{u=t}^{t+S-s}, \{w_u\}_{u=t}^{t+S-s}\Big) \quad \forall t \quad \text{and} \quad 1 \leq s \leq S-1 \tag{3.13}
$$

To summarize the individual's problem, if one knows his initial savings or wealth $b_{s,t}$ and the time path of factor prices over his remaining lifetime, he can solve for all of his optimal savings levels $\{b_{s+1,t+s}\}_{s=1}^{S-1}$.

To conclude the household's problem, we must make an assumption about how the age-s household can forecast the time path of interest rates and wages $\{r_u, w_u\}_{u=t}^{t+S-s}$ over his remaining lifetime. As we will show in Section 3.4, the equilibrium interest rate r_t and wage w_t will be functions of the state vector $\mathbf{\Gamma}_t$, which turns out to be the entire distribution of savings at in period t.

Define $\mathbf{\Gamma}_t$ as the distribution of household savings across households at time t.

$$
\mathbf{\Gamma}_t \equiv \big\{b_{s,t}\big\}_{s=2}^{S} \quad \forall t \tag{3.14}
$$

Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$
\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{2.17}
$$

where the e superscript signifies that $\mathbf{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$

based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[1]

3.2 Firms

The production side of this economy is identical to the one in Section 2.2 with a unit measure of identical, perfectly competitive firms that rent investment capital from individuals for real return r_t and hire labor for real wage w_t. Firms use their total capital K_t and labor L_t to produce output Y_t every period according to a Cobb-Douglas production technology,

$$Y_t = F(K_t, L_t) \equiv AK_t^{\alpha}L_t^{1-\alpha} \quad \text{where} \quad \alpha \in (0,1) \quad \text{and} \quad A > 0. \tag{2.18}$$

The representative firm chooses how much capital to rent and how much labor to hire to maximize profits,

$$\max_{K_t, L_t} AK_t^{\alpha}L_t^{1-\alpha} - (r_t + \delta)K_t - w_t L_t \tag{2.19}$$

where $\delta \in [0,1]$ is the rate of capital depreciation, and the two first order conditions that characterize firm optimization are the following.

$$r_t = \alpha A \left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{2.20}$$

$$w_t = (1-\alpha)A \left(\frac{K_t}{L_t}\right)^{\alpha} \tag{2.21}$$

[1] In Section 3.4 we will assume that beliefs are correct (rational expectations) for the non-steady-state equilibrium in Definition 3.2.

3.3 Market clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$L_t = \sum_{s=1}^{S} n_s \tag{3.15}$$

$$K_t = \sum_{i=2}^{S} b_{s,t} \tag{3.16}$$

$$\begin{aligned} &Y_t = C_t + I_t \\ &\quad\text{where}\quad I_t \equiv K_{t+1} - (1-\delta)K_t \end{aligned} \tag{2.24}$$

The goods market clearing equation (2.24) is redundant by Walras' Law.

3.4 Equilibrium

Before providing exact definitions of the functional equilibrium concepts, we give a rough sketch of the equilibrium, so you can see what the functions look like and understand the exact equilibrium definition more clearly. A rough description of the equilibrium solution to the problem above is the following three points

i. Households optimize according to equations (3.12).

ii. Firms optimize according to (2.20) and (2.21).

iii. Markets clear according to (3.15) and (3.16).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

The easiest way to understand the equilibrium solution is to substitute the market clearing conditions (3.15) and (3.16) into the firm's optimal conditions (2.20) and (2.21) solve for the equilibrium wage and interest rate as functions of the distribution of capital.

$$w_t\big(\boldsymbol{\Gamma}_t\big): \quad w_t = (1-\alpha)A\left(\frac{\sum_{s=2}^S b_{s,t}}{\sum_{s=1}^S n_s}\right)^\alpha \quad \forall t \tag{3.17}$$

$$r_t\big(\boldsymbol{\Gamma}_t\big): \quad r_t = \alpha A\left(\frac{\sum_{s=1}^S n_s}{\sum_{s=2}^S b_{s,t}}\right)^{1-\alpha} - \delta \quad \forall t \tag{3.18}$$

Now (3.17) and (3.18) can be substituted into household Euler equations (3.12) to get the following $(S-1)$-equation system that completely characterizes the equilibrium.

$$\begin{aligned} &u'\Big(w_t\big(\boldsymbol{\Gamma}_t\big)n_s + \Big[1 + r_t\big(\boldsymbol{\Gamma}_t\big)\Big]b_{s,t} - b_{s+1,t+1}\Big) = \\ &\qquad \beta\Big[1 + r_{t+1}\big(\boldsymbol{\Gamma}_{t+1}\big)\Big]u'\Big(w_{t+1}\big(\boldsymbol{\Gamma}_{t+1}\big)n_{s+1} + \Big[1 + r_{t+1}\big(\boldsymbol{\Gamma}_{t+1}\big)\Big]b_{s+1,t+1} - b_{s+2,t+2}\Big) \\ &\qquad\quad \forall t, \quad \text{and} \quad 1 \leq s \leq S-1 \end{aligned} \tag{3.19}$$

The system of $S-1$ nonlinear dynamic equations (3.19) characterizing the the lifetime savings decisions for each household $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ is not identified. Each individual knows the current distribution of capital $\boldsymbol{\Gamma}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\boldsymbol{\Gamma}_{t+1} = \{\{b_{s+1,t+1}\}_{s=1}^{S-1}\}$ for all agents alive next period, and for a policy function for the individual $b_{s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 3.1 (**Steady-state equilibrium**)**.** A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^S$, capital $\{\bar{b}_s\}_{s=2}^S$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (3.12),

ii. firms optimize according to (2.20) and (2.21),

iii. markets clear according to (3.15) and (3.16).

As we saw earlier in this section, the characterizing equations in Definition 3.1 reduce to (3.19). These $S-1$ equations are exactly identified in the steady state. That is, they are $S-1$ equations and $S-1$ unknowns $\{\bar{b}_s\}_{s=2}^S$.

$$
\begin{split}
&u'\Bigl(\bar{w}(\bar{\mathbf{\Gamma}})n_s + \bigl[1+\bar{r}(\bar{\mathbf{\Gamma}})\bigr]\bar{b}_s - \bar{b}_{s+1}\Bigr) = \\
&\qquad \beta\bigl[1+\bar{r}(\bar{\mathbf{\Gamma}})\bigr]u'\Bigl(\bar{w}(\bar{\mathbf{\Gamma}})n_{s+1} + \bigl[1+\bar{r}(\bar{\mathbf{\Gamma}})\bigr]\bar{b}_{s+1} - \bar{b}_{s+2}\Bigr) \\
&\qquad\quad \text{for} \quad 1\leq s\leq S-1
\end{split}
\tag{3.20}
$$

We can solve for steady-state $\{\bar{b}_s\}_{s=2}^S$ by using an unconstrained optimization solver. Then we solve for $\bar{w}$, $\bar{r}$, and $\{\bar{c}_s\}_{s=1}^S$ by substituting $\{\bar{b}_s\}_{s=2}^S$ into the equilibrium firm first order conditions and into the household budget constraints.

In the S-period-lived agent, perfect foresight, OG model described in this section, the state vector can be seen in the system of Euler equations (3.19). What is the smallest set of variables that completely summarize all the information necessary for the three generations of all three generations living at time t to make their consumption and saving decisions? What information do they have at time t that will allow them to make their savings decisions? The state vector of this model in each period is the distribution of capital $\mathbf{\Gamma_t}$.

Definition 3.2 (Non-steady-state functional equilibrium). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents is defined as stationary allocation functions of the state $\bigl\{b_{s+1,t+1} = \psi_s\bigl(\mathbf{\Gamma}_t\bigr)\bigr\}_{s=1}^{S-1}$ and stationary price functions $w(\mathbf{\Gamma}_t)$ and $r(\mathbf{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (2.17), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\mathbf{\Gamma}_{t+u} = \mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u\geq 1$$

ii. households optimize according to (3.19),

iii. firms optimize according to (2.20) and (2.21),

iv. markets clear according to (3.15) and (3.16).

We have already shown how to boil down the characterizing equations in Definition 3.2 to $S-1$ equations (3.19) and $S-1$ unknowns. But we have also seen that those $S-1$ equations are not identified. So how do we solve for these equilibrium functions? The solution to the non-steady-state equilibrium in Definition 3.2 is a fixed point in function space. Choose $S-1$ functions $\{\psi_s\}_{s=1}^{S-1}$ and verify that they satisfy the Euler equations for all points in the state space (all possible values of the state).

3.5 Solution method: time path iteration (TPI)

The solution method is time path iteration (TPI) as described in Section 2.5. The key assumption is that the economy will reach the steady-state equilibrium $\bar{\mathbf{\Gamma}}$ described in Definition 3.1 in a finite number of periods $T < \infty$ regardless of the initial state $\mathbf{\Gamma}_1$.

The first step is to assume a transition path for aggregate capital $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_T^i\}$ such that T is sufficiently large to ensure that $\mathbf{\Gamma}_T = \bar{\mathbf{\Gamma}}$. The superscript i is an index for the iteration number. The transition path for aggregate capital determines the transition path for both the real wage $\boldsymbol{w}^i = \{w_1^i, w_2^i, ...w_T^i\}$ and the real return on investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. The exact initial distribution of capital in the first period $\mathbf{\Gamma}_1$ can be arbitrarily chosen as long as it satisfies $K_1^i = \sum_{s=2}^S b_{s,1}$ according to market clearing condition (3.16). One could also first choose the initial distribution of capital $\mathbf{\Gamma}_1$ and then choose an initial aggregate capital stock K_1^i that corresponds to that distribution. As mentioned earlier, the only other restriction on the initial transition path for aggregate capital is that it equal the steady-state level $K_T^i = \bar{K} = \sum_{s=2}^S \bar{b}_s$ by period T. But the initial guess for the aggregate capital stocks K_t^j for periods $1 < t < T$ can be any level.

Given the initial capital distribution $\mathbf{\Gamma}_1$ and the transition paths of aggregate capital $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_T^i\}$, the real wage $\boldsymbol{w}^i = \{w_1^i, w_2^i, ...w_T^i\}$, and the real return to investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, one can solve for the optimal savings decision for the initial age $s = S-1$ individual for the last period of his life $b_{S,2}$ using his last intertemporal Euler equation similar to (3.5).

$$u'\Big([1+r_1^i]b_{S-1,1} + w_1^i n_{S-1} - b_{S,2}\Big) = \beta\Big(1+r_2^i\Big)u'\Big([1+r_2^i]b_{S,2} + w_2^i n_S\Big) \tag{3.21}$$

Notice that everything in equation (3.21) is known except for the savings decision $b_{S,2}$. This is one equation and one unknown.

The next step is to solve for the remaining lifetime savings decisions for the next oldest individual alive in period $t = 1$. This individual is age $s = S - 2$ and has two remaining savings decisions $b_{S-1,2}$ and $b_{S,3}$. From (3.12), we know that the two equations that characterize these two decisions are the following.

$$u'\Big([1+r_1^i]b_{S-2,1} + w_1^i n_{S-2} - b_{S-1,2}\Big) = \beta(1+r_2^i)u'\Big([1+r_2^i]b_{S-1,2} + w_2^i n_{S-1} - b_{S,3}\Big) \tag{3.22}$$

$$u'\Big([1+r_2^i]b_{S-1,2} + w_2^i n_{S-1} - b_{S,3}\Big) = \beta(1+r_3^i)u'\Big([1+r_3^i]b_{S,3} + w_3^i n_S\Big) \tag{3.23}$$

Euler equations (3.22) and (3.23) represent two equations and two unknowns $b_{S-1,2}$ and $b_{S,3}$. Everything else is known.

We continue solving the remaining lifetime decisions of each individual alive between periods 1 and T. This includes all the individuals who were already alive in period 1 and therefore have fewer than $S - 1$ savings decisions to solve for. It also includes all the individuals born between periods 1 and T for whom we have the full set of $S - 1$ lifetime decisions. Once we have solved for all the individual savings decisions for individuals alive between periods 1 and T, then we have the complete distribution of savings $\{\mathbf{\Gamma}_t\}_{t=1}^T$ for each period between 1 and T. We can use this to compute a new time path of the aggregate capital stock consistent with the individual savings decisions $K_t^{i'} = \sum_{s=2}^S b_{s,t}$ for all $1 \leq t \leq T$. I put a "$'$" on this aggregate capital stock because, in general, $K_t^{i'} \neq K_t^i$. That is, the initial conjectured path of the aggregate capital stock from which the savings decisions were made is not necessarily equal to the path of the aggregate capital stock consistent with those savings decisions.[2]

Let $\|\cdot\|$ be a norm on the space of time paths for the aggregate capital stock. Common

[2] A check here for whether T is large enough is if $K_T^{i'} = \bar{K}$ as well as $K_{T+1}^{i'}$ and $K_{T+2}^{i'}$. If not, then T needs to be larger.

norms to use are the L^2 and the L^∞ norms. Then the fixed point necessary for the equilibrium transition path from Definition 3.2 has been found when the distance between $\boldsymbol{K}^{i'}$ and $\boldsymbol{K}^i$ is arbitrarily close to zero.

$$\left\|\boldsymbol{K}^{i'} - \boldsymbol{K}^i\right\| < \varepsilon \quad \text{for} \quad \varepsilon > 0 \tag{2.34}$$

If the fixed point has not been found $\left\|\boldsymbol{K}^{i'} - \boldsymbol{K}^i\right\| > \varepsilon$, then a new transition path for the aggregate capital stock is generated as a convex combination of $\boldsymbol{K}^{i'}$ and $\boldsymbol{K}^i$.

$$\boldsymbol{K}^{i+1} = \xi \boldsymbol{K}^{i'} + (1-\xi)\boldsymbol{K}^i \quad \text{for} \quad \xi \in (0,1) \tag{2.35}$$

This process is repeated until the initial transition path for the aggregate capital stock is consistent with the transition path implied by those beliefs and household and firm optimization. TPI solves for the equilibrium transition path from Definition 3.2 by finding a fixed point in the time path of the economy.

3.6 Calibration

Use the following parameterization of the model for the problems below. Assume that agents are born at age 21 and die at age 100 (80 years of life). Now your time dependent parameters can be written as functions of S, because each period of the model is $80/S$ years. If the annual discount factor is estimated to be 0.96, then the model period discount factor is $\beta = 0.96^{80/S}$. Assume initially that $S = 80$. Let the annual depreciation rate of capital be 0.05. Then the model period depreciation rate is $\delta = 1 - (1-0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 3$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$.

3.7 Exercises

Exercise 3.1. Using the calibration from Section 3.6, write a Python function named `feasible()` that has the following form,

```
b_cnstr, c_cnstr, K_cnstr = feasible(f_params, bvec_guess)
```

where the inputs are a tuple `f_params = (nvec, A, alpha, delta)`, and a guess for the steady-state savings vector `bvec_guess = np.array([scalar`$_2$`, scalar`$_3$`,...scalar`$_S$`])`. The outputs should be Boolean (`True` or `False`, `1` or `0`) vectors of lengths $S-1$, S, and 1, respectively. `K_cnstr` should be a singleton Boolean that equals `True` if $K \leq 0$ for the given `f_params` and `bvec_guess`. The object `c_cnstr` should be a length-S Boolean vector in which the sth element equals `True` if $c_s \leq 0$ given `f_params` and `bvec_guess`. And `b_cnstr` is a length-$(S-1)$ Boolean vector that denotes which element of `bvec_guess` is likely responsible for any of the consumption nonnegativity constraint violations identified in `c_cnstr`. If the first element of `c_cnstr` is `True`, then the first element of `b_cnstr` is `True`. If the second element of `c_cnstr` is `True`, then both elements of `b_cnstr` are `True`. And if the last element of `c_cnstr` is `True`, then the last element of `b_cnstr` is `True`.

a. Which, if any, of the constraints is violated if you choose an initial guess for steady-state savings of `bvec_guess = np.ones(S-1)`?

b. Which, if any, of the constraints is violated if you choose the following initial guess for steady-state savings?

```
bvec_guess = \
  np.array([-0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2])
```

c. Which, if any, of the constraints is violated if you choose the following initial guess for steady-state savings?

```
bvec_guess = \
```

```
        np.array([-0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1,
                  0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1,
                  0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1])
```

d. What is a principle or a rule that might help you in this problem to choose a good initial guess? That is, what properties should a feasible initial guess have? [Hint: There are upper bounds and lower bounds on all the savings levels $\bar{b}_{s+1}$ that you cannot calculate *ex ante*.]

Exercise 3.2. Use the calibration from Section 3.6 and the steady-state equilibrium Definition 3.1. Write a function named `get_SS()` that has the following form,

```
ss_output = get_SS(params, bvec_guess, SS_graphs)
```

where the inputs are a tuple of the parameters for the model `params = (beta, sigma, nvec, L, A, alpha, delta, SS_tol)`, an initial guess of the steady-state savings `bvec_guess`, and a Boolean `SS_graphs` that generates a figure of the steady-state distribution of consumption and savings if it is set to `True`.

The output object `ss_output` is a Python dictionary with the steady-state solution values for the following endogenous objects.

```
ss_output = {
  'b_ss': b_ss, 'c_ss': c_ss, 'w_ss': w_ss, 'r_ss': r_ss,
  'K_ss': K_ss, 'Y_ss': Y_ss, 'C_ss': C_ss,
  'EulErr_ss': EulErr_ss, 'RCerr_ss': RCerr_ss,
  'ss_time': ss_time}
```

Let `ss_time` be the number of seconds it takes to run your steady-state program. You can time your program by importing the time library. And let the object `EulErr_ss` be a length-$(S-1)$ vector of the Euler errors from the resulting steady-state solution given in difference form $\beta(1+\bar{r})u'(\bar{c}_{s+1}) - u'(\bar{c}_s)$. The object `RCerr_ss` is a resource constraint error which should be close to zero. It is given by $\bar{Y} - \bar{C} - \delta\bar{K}$.

a. Solve numerically for the steady-state equilibrium values of $\{\bar{c}_s\}_{s=1}^S$, $\{\bar{b}_s\}_{s=2}^S$, $\bar{w}$, $\bar{r}$, $\bar{K}$, $\bar{Y}$, $\bar{C}$, the $S-1$ Euler errors and the resource constraint error. List those values. Time your function. How long did it take to compute the steady-state?

b. Generate a figure that shows the steady-state distribution of consumption and savings by age $\{\bar{c}_s\}_{s=1}^S$ and $\{\bar{b}_s\}_{s=2}^S$.

c. What happens to each of these steady-state values if all households retire sooner? That is, what happens if exogenous labor supply becomes the following?

$$n_s = \begin{cases} 1.0 & \text{if} \quad s \leq \text{round}\left(\frac{S}{2}\right) \\ 0.2 & \text{if} \quad s > \text{round}\left(\frac{S}{2}\right) \end{cases}$$

Specifically, how does this change affect each steady-state value $\{\bar{c}_s\}_{s=1}^S$, $\{\bar{b}_s\}_{s=2}^S$, $\bar{w}$, and $\bar{r}$? What is the intuition?

Exercise 3.3. Use time path iteration (TPI) to solve for the non-steady state equilibrium transition path of the economy from Definition 3.2. Use the calibration from Section 3.6 and the steady-state solution computed in Exercise 3.2. Let the initial state of the economy be given by the following distribution of savings,

$$\{b_{s,1}\}_{s=2}^S = \{x(s)\bar{b}_s\}_{s=2}^S \quad \text{where} \quad x(s) = \frac{(1.5-0.87)}{78}(s-2) + 0.87$$

where the function of age $x(s)$ is simply a linear function of age s that equals 0.87 for $s=2$ and equals 1.5 for $s=S=80$. This gives an initial distribution where there is more inequality than in the steady state. The young have less than their steady-state values and the old have more than their steady-state values. You'll have to choose a guess for T and a time path updating parameter $\xi \in (0,1)$, but I can assure you that $T < 320$. Use an L^2 norm for your distance measure (sum of squared percent deviations), and use a convergence parameter of $\varepsilon = 10^{-9}$. Use a linear initial guess for the time path of the aggregate capital stock from the initial state K_1^1 to the steady state K_T^1 at time T.

a. Plot the equilibrium time paths of the aggregate capital stock $\{K_t\}_{t=1}^{T+5}$, wage $\{w_t\}_{t=1}^{T+5}$, and interest rate $\{r_t\}_{t=1}^{T+5}$.

b. Also plot the equilibrium time path for savings of every person age $s = 15$ in every period $\{b_{15,t}\}$. Are there any periods t in which $b_{15,t}$ rises above its steady-state value $\bar{b}_{15}$?

c. How many periods did it take for the economy to get within 0.00001 of the steady-state aggregate capital stock $\bar{K}$? What is the period after which the aggregate capital stock never is again farther than 0.00001 away from the steady-state?