SINGULAR VALUE DECOMPOSITION

Revision of Linear Algebra

If $A = A^T$, then A is symmetric.

For an orthogonal matrix, $\boxed{A^{-1} = A^T}$

Suppose the columns of the matrix $\underline{\underline{X}}$ are eigen vectors ie $\underline{\underline{X}} = [\underline{x}_1, \underline{x}_2, \ldots\ldots, x_m]$

If we take the dot product of any two of those Eigen Vectors ie $x_i^T x_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

ORTHONORMAL

– We can decompose a matrix $\underline{\underline{A}} = \underline{\underline{X}}\,\underline{\underline{\Lambda}}\,\underline{\underline{X}}^T$ where $\underline{\underline{X}}$ is a matrix of Eigen Vectors & $\underline{\underline{\Lambda}}$ is a diagonal matrix as shown

$$\underset{m \times m}{\underline{\underline{A}}} = \underbrace{\left[\underset{m\times 1}{\underline{X}_1}\ \underline{x}_2 \ldots\ldots X_m\right]_{m\times m}}_{m \text{ eigen Vectors}} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}_{m\times m}}_{\lambda = \text{Eigen Value}} \begin{bmatrix} \underline{x}_1^T{}_{1\times m} \\ \underline{x}_2^T \\ \vdots \\ \underline{x}_m^T \end{bmatrix}_{m\times m}$$

NOTE: Two underlines denote a MATRIX
One underline denotes a VECTOR.

$$\underline{\underline{A}} = \sum_{i=1}^{m} \underline{x}_i \lambda_i \underline{x}_i^T = \lambda_1 \underline{x}_1 \underline{x}_1^T + \lambda_2 \underline{x}_2 \underline{x}_2^T + \ldots\ldots + \lambda_m \underline{x}_m \underline{x}_m^T$$

NOTE: A matrix A can be written as a number times the outer product of 2 vectors

We can also write
$$\left.\begin{aligned} \underline{\underline{A}}\,\underline{x}_1 &= \lambda_1 \underline{x}_1 \\ \underline{\underline{A}}\,\underline{x}_2 &= \lambda_2 \underline{x}_2 \\ &\vdots \\ \underline{\underline{A}}\,\underline{x}_m &= \lambda_m \underline{x}_m \end{aligned}\right\} \text{Same as above.}$$

Vector Norm:

– For a column vector $\underline{x} \in R^{m\times 1}$, the NORM of $\underline{x}$ is the length of $\underline{x}$ & is found as the EUCLEDIAN distance

$$\|\underline{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_m^2} = \sqrt{\underline{x}^T x} = \text{Length of the vector} = L_2 \text{ NORM.}$$

– If Q is an $m\times m$ ORTHOGONAL matrix: $(Q^T = Q^{-1})$

then $\|\underline{\underline{Q}}\underline{x}\|_2 = \sqrt{(Qx)^T(Qx)} = \sqrt{x^T Q^T Q x}$ $\therefore Q^T = Q^{-1}$

$\Rightarrow Q^TQ = Q^{-1}Q = I$

$$= \sqrt{x^T I x}$$
$$= \sqrt{x^T x} = \|x\|_2$$

Thus Q rotates x, but does not change the length of x.

DIMENSIONAL REDUCTION WITH S.V.D

The Singular Value Decomposition is a MATRIX FACTORISATION process.

Def^n of the S.V.D:

$A \in \mathbb{R}^{m\times n}$ $\quad \boxed{A = U\Sigma V^T}$ where $\quad A_{m\times n} \quad \Sigma_{n\times n} \quad U_{m\times n} \quad V^T_{n\times n}$

$\Sigma \in \mathbb{R}^{n\times n}$ is a diagonal matrix with positive entries (singular values along the diagonal)

$U \in \mathbb{R}^{m\times n}$ has ORTHONORMAL columns

$V \in \mathbb{R}^{n\times n}$ has ORTHONORMAL columns and rows.

$\therefore$ V is an ORTHOGONAL matrix, $V^{-1} = V^T$

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = \underbrace{\begin{bmatrix} \overset{m\times 1}{\underline{u_1}} & \underline{u_2} & \underline{u_3} & \cdots & \underline{u_n} \\ & & & & \end{bmatrix}_{m\times n}}_{\text{Eigen Vectors as Columns}} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}_{n\times n}}_{\text{Singular Values as Diagonal}} \underbrace{\begin{bmatrix} v_1^T \; (1\times n) \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}_{n\times n}}_{\text{Singular Vectors}}$$

$$\underline{\underline{A}} = \sigma_1 . \underline{u_1}\, \underline{v_1}^T + \sigma_2 . \underline{u_2}\, \underline{v_2}^T + \sigma_3 . \underline{u_3}\, \underline{v_3}^T + \cdots\cdots + \sigma_n . \underline{u_n}\, \underline{v_n}^T$$

NOTE: These above are just representations yet. We have not yet proved these.

: Also SVD is an effective method of finding Eigen Vectors if the matrix A is NOT a square matrix.

HOW TO COMPUTE THE S.V.D

We will use the following property: The non-zero singular values of A are the (+ve) square roots of the non-zero Eigen Values of A^TA or AA^T.

— If we can find Eigen Values of A^TA or AA^T, then we can recover U, Σ & V.

Consider $(A^TA)^T = A^TA \Rightarrow$ thus proving that A^TA & AA^T are symmetric matrices. ie $Q^T = Q$ if Q is Symm.

Symmetric Matrices have many useful properties:—

If A is considered as a matrix, then its S.V.D. can be written as $\boxed{A = U\Sigma V^T}$

$$\begin{aligned} A^TA &= (U\Sigma V^T)^T(U\Sigma V^T) \\ &= (V^{T^T}\Sigma^T U^T)(U\Sigma V^T) \\ &= V.\Sigma.\underbrace{U^T.U}.\Sigma.V^T \\ &= V.\Sigma.I.\Sigma V^T \end{aligned}$$

$$\boxed{A^T.A = V.\Sigma^2.V^T}$$

NOTE: Σ is a diagonal matrix, therefore the Transpose is the matrix itself. ie $\Sigma^T = \Sigma$

— Also, by def^n of orthonormal matrix; $U^TU = I$

$$\begin{aligned} AA^T &= (U\Sigma V^T)(U\Sigma V^T)^T \\ &= U.\Sigma V^T.V^{T^T}.\Sigma^T U^T \\ &= U.\Sigma.\underbrace{V^T.V}_{I}.\Sigma U^T \\ &= U.\Sigma.I.\Sigma.U^T \end{aligned}$$

$$\boxed{AA^T = U.\Sigma^2 U^T}$$

NOTE: $\because$ V is orthogonal matrix $\therefore V^T.V = I$

Thus now we need any matrix $A_{m\times n}$ (not necessarily square) because $A^TA = V\Sigma^2V^T$ look like $\equiv X\Lambda X^T$

$\therefore$ start with A, multiply it by its transpose A^T to get (AA^T) & compute the Eigen Values. & Eigen vectors of it, then it simply remains a question of matching / comparing. ie $\Lambda = \Sigma^2$ & $X = V$

Eigen Values Eigen Vector

Thus we can compute Σ & V. And similarly comparing $AA^T = U\Sigma^2U^T \equiv X\Lambda X^T$

Example: $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\substack{m\times n \\ 3\times 1}}$ $\quad \underbrace{A^TA = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [14]}_{\text{Better choice in this Case}}$

we can also consider $\underbrace{AA^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}}_{\text{as compared to this ie } AA^T}$ $\Rightarrow$ Eigen values of this 3x3 is lengthy.

We know that $A^TA = V\Sigma^2V^T$

$$\therefore \quad A^TA = [14] = \underset{V}{[1]}\underset{\Sigma^2}{[\sqrt{14}]^2}\underset{V^T}{[1]}$$

$\therefore V = [1]$ & $\Sigma = \sqrt{14}$ NOTE: $\because V$ has to be 1 $\therefore \Sigma$ is Unique.

We also know $AA^T = U\Sigma^2U^T$. Does it mean we solve the Eigen Values of 3x3 AA^T.

Remember that now V & Σ are known along with the original matrix A.

So, consider $A = U\Sigma V^T$ by defn

Post multiply both sides by V

$$\therefore A\cdot V = U\Sigma \underbrace{V^TV}_{I}$$

$$\therefore AV = U\cancel{\Sigma I}^{\Sigma}$$

Post multiply by Σ^{-1}

$$\therefore AV\cdot\Sigma^{-1} = U\cdot\cancel{\Sigma\Sigma^{-1}}^{I}$$

$$\therefore U = \frac{1}{\Sigma}AV$$

$$= \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}[1]$$

Thus U, Σ & V are known.