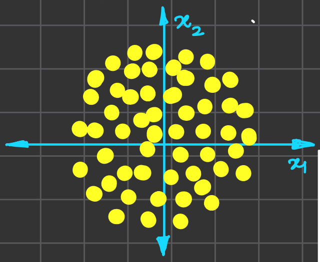
BIVARIATE NORMAL

BIVARATE NORMAL DISTRIBUTION

Assumption: Let x_1 and x_2 two independent normal r.v with $N_1(\mu_1, \sigma_1^2)$ and $N_2(\mu_2, \sigma_2^2)$ respectively



It is clear from the adjacent scatter plot that the two are independent.
i.e. there is NO correlation between them. when x_1 changes, there is very little information about corresponding changes in x_2. $\boxed{\sigma_{12}=0}$ $\boxed{S_{12}=0}$

$$\boxed{f(x_1) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left\{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right\}} \qquad -\infty \le x_1 \le \infty$$

and $$\boxed{f(x_2) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left\{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}} \qquad -\infty \le x_2 \le \infty$$

As x_1 and x_2 are Independent, therefore their JOINT DISTRIBUTION is given as follows $f(A,B) = f(A)\cdot f(B)$

$$f(x_1, x_2) = \prod_{i=1}^{2} f(x_i)$$

$$= f(x_1)\cdot f(x_2)$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}}\, e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} \times \frac{1}{\sqrt{2\pi\sigma_2^2}}\, e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$$

$$= \underbrace{\frac{1}{(2\pi)^{2/2}(\sigma_1^2\sigma_2^2)^{1/2}}}_{\text{normalizing constant}}\, e^{\underbrace{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}_{\text{exponent}}} \qquad \text{—— (1)}$$

(2.) We want to derive the normalizing constant and the Exponent part from the Population parameters mean vector $\underline{\mu}$ and the covariance matrix Σ where

$$\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

Consider the determinant $|\Sigma|$

$$|\Sigma| = \begin{vmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{vmatrix}$$ But due to independency $\sigma_{12} = \sigma_{21} = 0$

$$\therefore |\Sigma| = \begin{vmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{vmatrix} = \sigma_1^2\sigma_2^2$$ This implies $\boxed{|\Sigma| = \sigma_1^2\sigma_2^2}$ or $|\Sigma|^{\frac{1}{2}} = (\sigma_1^2\sigma_2^2)^{\frac{1}{2}}$

Thus, only the constant part of the joint distribution $= \dfrac{1}{(2\pi)^{2/2}(\sigma_1^2\sigma_2^2)^{1/2}} = \dfrac{1}{(2\pi)^{2/2}|\Sigma|^{1/2}}$

NOTE: For a 3-D Normal r.v. $\underline{X}$, we can use analogy and write the expression for the exponential term as below:

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1} \quad \therefore \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}$$

Then the constant component of the JOINT DISTRIBUTION

$$= \frac{1}{(2\pi)^{3/2}(\sigma_1^2 \cdot \sigma_2^2 \cdot \sigma_3^2)^{1/2}} = \frac{1}{(2\pi)^{3/2} \cdot |\Sigma|^{1/2}}$$

NOTE: We can generalize this to a 'p' dimensional Normal distribution. The constant term in the JOINT DISTRIBUTION can be similarly expressed as

$$= \frac{1}{(2\pi)^{p/2} \cdot |\Sigma|^{1/2}}$$

(3) Now, we derive the exponent term of the BIVARIATE case.

$$\exp\left\{ -\frac{1}{2} \left[\left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2 \right] \right.$$

Now, for comparison sake, consider the UNIVARIATE case exponential term:

$$e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = e^{-\frac{1}{2}\left[(x-\mu)\,(\sigma^2)^{-1}(x-\mu)\right]}$$ — Note: μ & σ^2 are SCALARS.

In order to convert this expression to MULTI-VARIATE case, replace $x \to \underline{X}$; $\mu \to \underline{\mu}$ and $\sigma^2 \to \Sigma$ (Matrices)

In the matrix form, the exponent term of a MultiVariate

$$-\frac{1}{2}\left\{(\underline{X} - \underline{\mu})^T \cdot \Sigma^{-1} (\underline{X} - \underline{\mu})\right\}$$

$$\therefore -\frac{1}{2} \underbrace{\begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix}_{1\times2}}_{A} \underbrace{\begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}^{-1}_{2\times2}}_{B} \underbrace{\begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}_{2\times1}}_{C} \quad \text{— (1)}$$

NOTE: that the resultant term of these matrix multiplication is 1x1 $\therefore$ constant term.

* How to calculate Matrix Inverse of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{|A|}\,\text{adjoint}(A)$

where, adjoint is the transpose of the cofactor elements.

$$\Sigma^{-1} = \frac{1}{|\Sigma|} \text{ cofactor of } [\Sigma]$$

$$= \frac{1}{\sigma_1^2 \sigma_2^2} \begin{bmatrix} (-1)^{1+1}\sigma_2^2 & 0 \\ 0 & (-1)^{2+2} \cdot \sigma_1^2 \end{bmatrix}$$

$$\therefore \boxed{\Sigma^{-1} = \frac{1}{\sigma_1^2 \sigma_2^2} \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}} \quad \text{— (B)}$$

$\therefore$ Eqⁿ ① becomes,

$$= \frac{-1}{2}\underbrace{\left[(x_1-\mu_1)\quad(x_2-\mu_2)\right]}_{1\times 2} \cdot \frac{1}{\sigma_1^2\sigma_2^2}\underbrace{\begin{bmatrix}\sigma_2^2 & 0\\ 0 & \sigma_1^2\end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}}_{2\times 1}$$

$$= \frac{-1}{2}\underbrace{\left[(x_1-\mu_1)\quad(x_2-\mu_2)\right]}_{1\times 2} \cdot \frac{1}{\sigma_1^2\sigma_2^2}\underbrace{\begin{bmatrix}\sigma_2^2(x_1-\mu_1)\\ \sigma_1^2(x_2-\mu_2)\end{bmatrix}}_{2\times 1}$$

$$= \frac{-1}{2\sigma_1^2\sigma_2^2}\underbrace{\left[\sigma_2^2(x_1-\mu_1)^2+\sigma_1^2(x_2-\mu_2)^2\right]}_{1\times 1}$$

$$= \frac{-1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right] \text{ ____ QED}$$

Thus the Bivariate Normal Distribution

$$= \frac{1}{(2\pi)^{2/2}|\Sigma|^{1/2}}\exp\left\{\frac{-1}{2}\left[(\underline{x}-\underline{\mu})^T\cdot\Sigma^{-1}\cdot(\underline{x}-\underline{\mu})\right]\right\}$$

✱ Now, consider the case where x_1 & x_2 are CORRELATED. ie $\sigma_{12}\neq 0$

$\therefore$ covariance matrix $\Sigma = \begin{bmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{bmatrix}$

Determinant $|\Sigma| = \begin{vmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{vmatrix} = \sigma_1^2\sigma_2^2-\sigma_{12}^2$ Covariance = (correlation x std. dev)

$$= \sigma_1^2\sigma_2^2-(\rho_{12}\cdot\sigma_1\sigma_2)^2$$

$$= \sigma_1^2\sigma_2^2(1-\rho_{12}^2)$$ where ρ_{12} = correlatⁿ betⁿ x_1 & x_2

Now, $\Sigma^{-1} = \frac{1}{|\Sigma|}\text{adjoint}[\Sigma]$

$$= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}\begin{bmatrix}\sigma_1^2 & -\sigma_{12}\\ -\sigma_{12} & \sigma_2^2\end{bmatrix}$$

$$= \frac{1}{(\sigma_1^2\sigma_2^2-\sigma_{12}^2)}\begin{bmatrix}\sigma_1^2 & -\sigma_{12}\\ -\sigma_{12} & \sigma_2^2\end{bmatrix} \text{ ____ QED}$$

And, the Exponent term of the JOINT DISTRIBUTION

$$= \frac{-1}{2}\left[(\underline{x}-\underline{\mu})^T\cdot\Sigma^{-1}(\underline{x}-\underline{\mu})\right]$$

$$= \frac{-1}{2}\underbrace{\left[x_1-\mu_1\quad x_2-\mu_2\right]}_{1\times 2}\cdot\frac{1}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\underbrace{\begin{bmatrix}\sigma_2^2 & -\sigma_{12}\\ -\sigma_{12} & \sigma_2^2\end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}}_{2\times 1}$$

$$= \frac{-1}{2}\underbrace{\left[x_1-\mu_1\quad x_2-\mu_2\right]}_{1\times 2}\frac{1}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\underbrace{\begin{bmatrix}\sigma_2^2(x_1-\mu_1)-\sigma_{12}(x_2-\mu_2)\\ -\sigma_{12}(x_1-\mu_1)+\sigma_1^2(x_2-\mu_2)\end{bmatrix}}_{2\times 1}$$

$$= \frac{-1}{2(\sigma_1^2\sigma_2^2-\sigma_{12}^2)}\left[\sigma_2^2(x_1-\mu_1)^2-\sigma_{12}(x_1-\mu_1)(x_2-\mu_2)-\sigma_{12}(x_1-\mu_1)(x_2-\mu_2)+\sigma_1^2(x_2-\mu_2)^2\right]$$

$$= \frac{-1}{2(\sigma_1^2\sigma_2^2-\sigma_{12}^2)}\left[\sigma_2^2(x_1-\mu_1)-2\sigma_{12}(x_1-\mu_1)(x_2-\mu_2)+\sigma_1^2(x_2-\mu_2)^2\right]$$

Multiply and divide by $(\sigma_1^2\sigma_2^2)$

correlation $= \rho_{12}$

$$= \frac{-1}{2}\frac{\cancel{\sigma_1^2\sigma_2^2}}{\cancel{\sigma_1^2\sigma_2^2}(1-\rho_{12}^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\frac{\sigma_{12}}{\sigma_1\sigma_2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

Exponent Term $= \dfrac{-1}{2(1-\rho_{12}^2)}\left[\left(\dfrac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho_{12}\left(\dfrac{x_1-\mu_1}{\sigma_1}\right)\left(\dfrac{x_2-\mu_2}{\sigma_2}\right)+\left(\dfrac{x_2-\mu_2}{\sigma_2}\right)^2\right]$

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