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**CS111 Homework 3**

due Friday, May 7th 11:59 PM

**Problem 1.**

$$\begin{cases} Q_n = 3Q_{n-1} + 10Q_{n-2} \\ Q_0 = 2 \\ Q_1 = 10 \end{cases}$$

The characteristic equation from the given $Q_n$ is:
$x^2 = 3x + 10$
$x^2 - 3x - 10 = 0$
$(x-5)(x+2) = 0$
Thus, the roots to the characteristic equation are:
$x = 5$ and $x = -2$

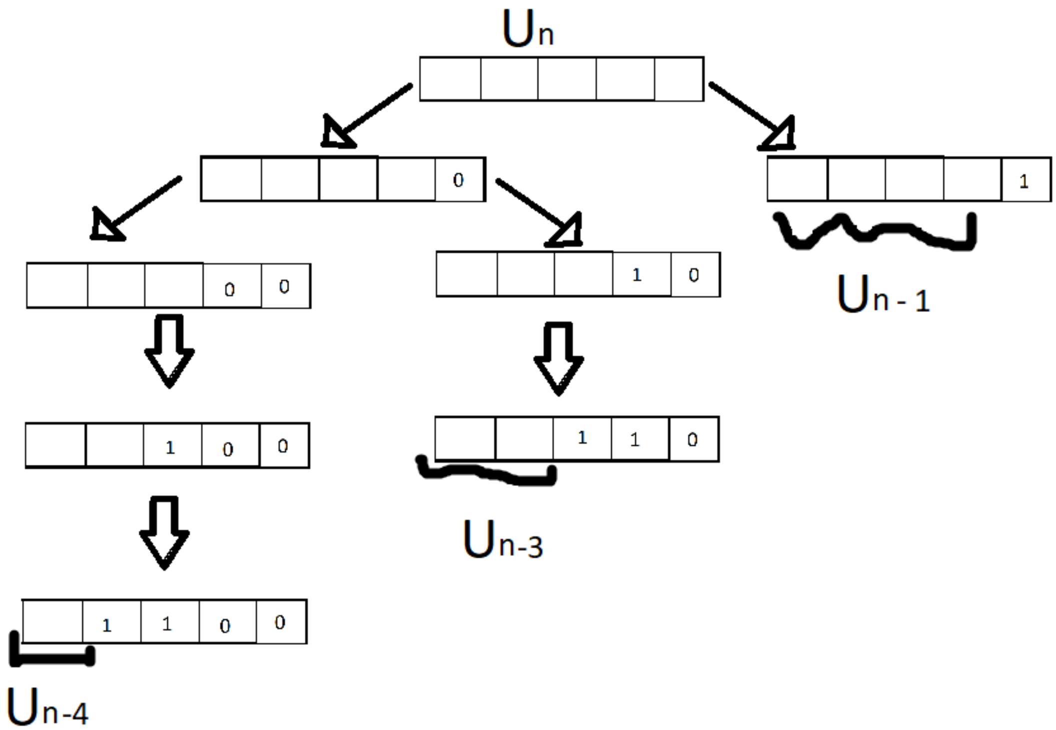
From these roots, we know:
$Q_n = \alpha_1 5^n + \alpha_2(-2)^n$
From the given $Q_0$ and $Q_1$ values:
$Q_0 = \alpha_1 5^0 + \alpha_2(-2)^0 = \alpha_1 + \alpha_2 = 2$
$Q_1 = \alpha_1 5^1 + \alpha_2(-2)^1 = 5\alpha_1 - 2\alpha_2 = 10$
$Q_0 = \alpha_1 + \alpha_2 = 2$
$Q_1 = 5\alpha_1 - 2\alpha_2 = 10$

From $Q_0$, we know $\alpha_1 = 2 - \alpha_2$ ... (1)
Plugging (1) into $Q_1$, we find $5(2 - \alpha_2) - 2\alpha_2 = 10$
$10 - 5\alpha_2 - 2\alpha_2 = 10$
$7\alpha_2 = 0$
$\alpha_2 = 0$
Plugging $\alpha_2 = 0$ into (1), we find $\alpha_1 = 2$.
Plugging $\alpha_1$ and $\alpha_2$ into $Q_n$ ...
$Q_n = 2 * 5^n + 0 * (-2)^n$

$$Q_n = 2 * 5^n$$

**Problem 2.**

No two zeroes distance two apart...
In other words, no string of (0 - *number* - 0) can be possible.
Looking at the right-most branch...




When the string ends with 1, and we know that the previous sequence is valid, we get $U_{n-1}$. This is obvious because there does not exist a zero in this ending string.
If the sequence ends with 0, we have branching options.
We are looking for ending strings that make the previous valid string valid no matter what.

Looking at the left-most branch...
When the sequence ends with 1-0-0, we cannot say $U_{n-3}$ because a valid string before could end with a 0 causing the total string to be invalid.
So we must go down one step to 1-1-0-0, where any valid sequence before that will remain valid when appended, resulting in $U_{n-4}$.

Using this same logic, looking at the middle branch, we cannot use 1-0 because a valid string beforehand could end with a 0, so we arrive at 1-1-0 being another valid ending to a string that ends with 0, being $U_{n-3}$.

Adding these three terms together results in $U_n$.
$U_n = U_{n-1} + U_{n-3} + U_{n-4}$
Because the characteristic equation is degree 4, we need 4 initial conditions, and we should start with 0 because that would require the least amount of calculations.
$U_0 = 1$ because there is one empty string.
$U_1 = 2$ because for length 1, the string can be "0" or "1".
$U_2 = 4$ because there are $2^2$ possible strings = 4.
$U_3 = 6$ because there are $2^3$ possible strings, and 2 of them are invalid, 010 and 000. So $2^3 - 2 = 6$

$$\begin{cases} U_n = U_{n-1} + U_{n-3} + U_{n-4} \\ U_0 = 1 \\ U_1 = 2 \\ U_2 = 4 \\ U_3 = 6 \end{cases}$$

**Problem 3.**

$$\begin{cases} Y_n = 3Y_{n-1} + 9Y_{n-2} + 5Y_{n-3} + 5^n \\ Y_0 = 0 \\ Y_1 = 10 \\ Y_2 = 20 \end{cases}$$

Characteristic equation:
$x^3 = 3x^2 + 9x + 5$
$x^3 - 3x^2 - 9x - 5 = 0$
$(x-5)(x^2 + 2x + 1) = 0$
$(x-5)(x+1)^2 = 0$
$Y_n' = \alpha_1 * 5^n + \alpha_2 * (-1)^n + \alpha_3 * n * (-1)^n$

Because 5 is a root of the characteristic equation, $Y_n$" is of the form...
$Y_n'' = c * n * 5^n$
Plugging into $Y_n$...
$c * n * 5^n = 3 * c * (n-1) * 5^{n-1} + 9 * c * (n-2) * 5^{n-2} + 5 * c * (n-3) * 5^{n-3} + 5^n$
Plugging $n = 3$...
$c * 3 * 125 = 3 * c * 2 * 25 + 9 * c * 1 * 5 + 0 + 125$
$375 * c = 150 * c + 45 * c + 125$
$180 * c = 125$
$c = \frac{25}{36}$
$Y_n'' = \frac{25}{36} * n * 5^n$

Combining $Y_n'$ and $Y_n''$...
$Y_n = \alpha_1 * 5^n + \alpha_2 * (-1)^n + \alpha_3 * n * (-1)^n + \frac{25}{36} * n * 5^n$

Plugging in the initial conditions into $Y_n$...
$Y_0 = \alpha_1 * 5^0 + \alpha_2 * (-1)^0 + \alpha_3 * 0 * (-1)^0 + \frac{25}{36} * 0 * 5^0 = 0$
$\quad \alpha_1 + \alpha_2 = 0$
$\quad \alpha_1 = -\alpha_2$
$Y_1 = \alpha_1 * 5^1 + \alpha_2 * (-1)^1 + \alpha_3 * 1 * (-1)^1 + \frac{25}{36} * 1 * 5^1 = 10$
$\quad 5\alpha_1 - \alpha_2 - \alpha_3 = 10 - \frac{125}{36}$
$\quad 5\alpha_1 + \alpha_1 - \alpha_3 = 10 - \frac{125}{36}$
$\quad 6\alpha_1 - \alpha_3 = 10 - \frac{125}{36}$ ... (1)
$Y_2 = \alpha_1 * 5^2 + \alpha_2 * (-1)^2 + \alpha_3 * 2 * (-1)^2 + \frac{25}{36} * 2 * 5^2 = 20$

$25\alpha_1 + \alpha_2 + 2\alpha_3 = 20 - \frac{1250}{36}$
$25\alpha_1 - \alpha_1 + 2\alpha_3 = 20 - \frac{1250}{36}$
$24\alpha_1 + 2\alpha_3 = 20 - \frac{1250}{36}$ ... (2)
Multiplying (1) by 2 on both sides...
$12\alpha_1 - 2\alpha_3 = 20 - \frac{250}{36}$ ... (3)
Adding (3) to (2) results with:
$36\alpha_1 = 40 - \frac{1500}{36}$
$36\alpha_1 = -\frac{5}{3}$
$\alpha_1 = -\frac{5}{108}$
From $\alpha_1 = -\alpha_2$...
$\alpha_2 = \frac{5}{108}$
Plugging in $\alpha_1$ into (1):
$6 * (-\frac{5}{108}) - \alpha_3 = 10 - \frac{125}{36}$
$-\frac{5}{18} - \alpha_3 = 10 - \frac{125}{36}$
$\alpha_3 = -(10 - \frac{125}{36} + \frac{10}{36})$
$\alpha_3 = -(\frac{360-115}{36})$
$\alpha_3 = -\frac{245}{36}$
Plugging in $\alpha_1$, $\alpha_2$, and $\alpha_3$ into $Y_n$...

$$Y_n = -\frac{5}{108} * 5^n + \frac{5}{108} * (-1)^n - \frac{245}{36} * n * (-1)^n + \frac{25}{36} * n * 5^n$$

**Academic integrity declaration.**
**Anand Mahadevan and Husam Chekfa completed this assignment together, and did not use external websites.**