CS130 - Math review

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Part 1: vectors, dot and cross product

1. Calculate the cosine of the angle between the vectors $\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$.

Let $\overrightarrow{A} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$ and $\overrightarrow{B} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$

$$cos\,\alpha = \frac{\overrightarrow{A} \cdot \overrightarrow{B}}{\|\overrightarrow{A}\| \cdot \|\overrightarrow{B}\|} = \frac{8 + 12 + 0}{\sqrt{2^2 + 4^2 + 4^2} \cdot \sqrt{4^2 + 3^2 + 0^2}} = \frac{20}{\sqrt{36} \cdot \sqrt{25}} = \frac{20}{30} = \frac{2}{3}$$

2. Consider a plane containing the point $P = (1, 3, -1)$, with normal $\mathbf{n} = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$. Let $Q = (5, 6, 7)$ be a point (not on the plane). For each point below, indicate whether the point is on the plane, on the same side of the plane as Q, or on the opposite side of the plane from Q.

1. $(0, 0, 0)$
2. $(-1, 1, 2)$
3. $(1, 4, 0)$
4. $(1, 5, 1)$
5. $(-1, -1, -1)$

Plane equation:

$$\frac{1}{3} * (x - 1) + \frac{2}{3} * (y - 3) - \frac{2}{3} * (z + 1) = 0$$
$$\frac{1}{3} * x + \frac{2}{3} * y - \frac{2}{3} * z = 3$$

1. $(0,0,0)$; Same side of plane as Q
2. $(-1,1,2)$; Same side of plane as Q
3. $(1,4,0)$; On plane
4. $(1,5,1)$; On plane
5. $(-1,-1,-1)$; Same side of plane as Q

3. Calculate the cross product: $\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}4\\5\\6\end{pmatrix}$.

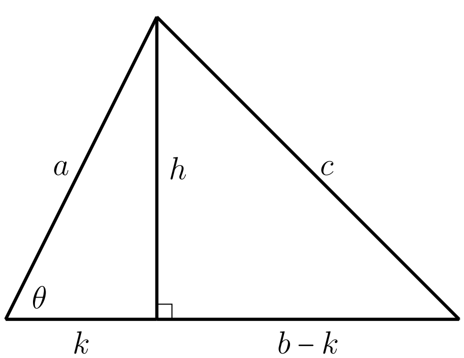
Let $\overrightarrow{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\overrightarrow{B} = \begin{pmatrix}4\\5\\6\end{pmatrix}$

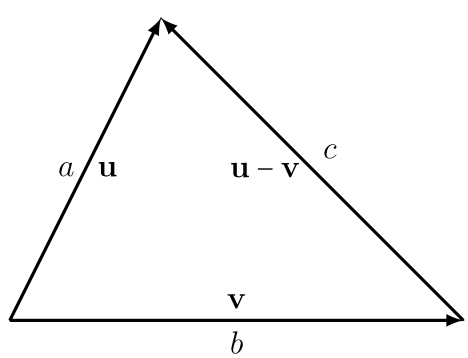
$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = (2*6-3*5)i - (1*6-3*4)j + (1*5-2*4)k = \begin{pmatrix}-3\\6\\-3\end{pmatrix}$$

4. Calculate a vector **w** in the same direction of the vector **u** and that has the same length as the vector **v**.

$$\vec{w} = \frac{\vec{u}}{\|\vec{u}\|} \cdot \|\vec{v}\|$$

5. Consider a triangle with sides **u** (of length a), **v** (of length b), and **u** − **v** (of length c), with the angle between **u** and **v** equal to θ as shown in the two figures below.





(a) Write an identity that relates a, h, and k.

(b) Write an identity that relates c, h, and $b-k$.

(c) Write an identity that relates a, k, and θ.

(d) From the three identities above, eliminate the variables h and k. This should leave you with the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$.

(e) Rewrite the Law of Cosines in terms of the vectors $\mathbf{u}$, $\mathbf{v}$, and θ. a, b, and c should not appear in your expression.

(f) Expand the left hand side. This should result in a term containing $\mathbf{u}\cdot\mathbf{v}$.

(g) Solve for $\mathbf{u}\cdot\mathbf{v}$ and simplify.

(h) Let the components of $\mathbf{u}$ and $\mathbf{v}$ be given as $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Show algebraically, by expanding in terms of the components, that

$$\|\mathbf{u}\times\mathbf{v}\|^2 + (\mathbf{u}\cdot\mathbf{v})^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2.$$

(i) Use your results from part 3 and the expression given in part 4 to solve for $\|\mathbf{u}\times\mathbf{v}\|$. Simplify. You should get an equation that looks similar to the one you got for $\mathbf{u}\cdot\mathbf{v}$.

(a) $a^2 = h^2 + k^2$

(b) $c^2 = h^2 + (b-k)^2 = h^2 + b^2 - 2bk + k^2$

(c) $cos\theta = \frac{k}{a}$

(d) $a^2 - c^2 = 2bk - b^2$ ... (a) - (b)
$c^2 = a^2 + b^2 - 2bk$
Plugging in $k = a * cos\theta$
$c^2 = a^2 + b^2 - 2ab\cos\theta$

(e) $\|u \vec{-} v\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|cos\theta$

(f) LHS: $(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2 * \vec{u}\cdot\vec{v}$

(g) $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|cos\theta$

(h) $LHS : (u_2 * v_3 - u_3 * v_2)^2 + (u_3 * v_1 - u_1 * v_3)^2 + (u_1 * v_2 - u_2 * v_1)^2 + (u_1 * v_1 + u_2 * v_2 + u_3 * v_3)^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)$
$RHS : (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)$
The identity is true because LHS = RHS
Q.E.D

(i) $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2 =$
$\|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2 cos^2\theta =$
$\|\vec{u}\|^2\|\vec{v}\|^2 * (1 - cos^2\theta) = \|\vec{u}\|^2\|\vec{v}\|^2 * sin^2\theta$
$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\| * sin\theta$

6. Given the triangle with vertices $(0,2,-1)$, $(2,0,-1)$, and $(1,0,0)$, calculate the normal of the plane that contains the triangle.

Strategy: Calculate normal of the plane by taking the cross product of two lines part of the plane.
Let P_1 be (0, 2, -1), P_2 be (2, 0, -1), P_3 be (1, 0, 0), $\vec{A} = (P_2 - P_1)$, $\vec{B} = (P_3 - P_1)$
$\vec{A} = (2, -2, 0)$, $\vec{B} = (1, -2, 1)$

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 2 & -2 & 0 \\ 1 & -2 & 1 \end{vmatrix} = (-2*1 - 0*-2)i - (2*1 - 0*1)j + (2*-2 - (-2)*1)k = (-2,-2,-2)$$

Part 2: Matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 7 & 19 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 5 & 2 \\ 1 & -3 \\ -1 & 1 \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

7. Calculate: (a) $\mathbf{A} + \mathbf{B}^T$, (b) $\mathbf{AB}$, and (c) $(\mathbf{AB})^{-1}$.

(a) $B^T = \begin{pmatrix} 5 & 1 & -1 \\ 2 & -3 & 1 \end{pmatrix}$
$A + B^T = \begin{pmatrix} 6 & 3 & 4 \\ 6 & 4 & 20 \end{pmatrix}$
(b) $AB = \begin{pmatrix} 5+2+-5 & 2-6+5 \\ 15+7-19 & 6-21+19 \end{pmatrix} =$
$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$
(c) $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{2*4-1*3}\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix} =$
$\frac{1}{5}\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 4/5 & -1/5 \\ -3/5 & 2/5 \end{pmatrix}$

8. Solve $(\mathbf{AB})\mathbf{x} = \mathbf{u}$ for $\mathbf{x}$.

$x = (AB)^{-1}u = \begin{pmatrix} 4/5 & -1/5 \\ -3/5 & 2/5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} =$
$\begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix}$

9. Show that $\mathbf{AB} \neq \mathbf{BA}$.

AB is a matrix of dimension 2x2, while BA is a matrix of dimension 3x3. By definition, $AB \neq BA$

10. Let $\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Show that $(\mathbf{CD})\mathbf{v} = \mathbf{C}(\mathbf{Dv})$ by calculating $\mathbf{A} = \mathbf{CD}$, $\mathbf{w} = \mathbf{Dv}$, $\mathbf{Av}$, and $\mathbf{Cw}$.

$A = CD = \begin{pmatrix} c_{11} * d_{11} + c_{12} * d_{21} & c_{11} * d_{12} + c_{12} * d_{22} \\ c_{21} * d_{11} + c_{22} * d_{21} & c_{21} * d_{12} + c_{22} * d_{22} \end{pmatrix}$
$w = Dv = \begin{pmatrix} d_{11} * v_1 + d_{12} * v_2 \\ d_{21} * v_1 + d_{22} * v_2 \end{pmatrix}$
$Av = \begin{pmatrix} (c_{11} * d_{11} + c_{12} * d_{21}) * v_1 + (c_{11} * d_{12} + c_{12} * d_{22}) * v_2 \\ (c_{21} * d_{11} + c_{22} * d_{21}) * v_1 + (c_{21} * d_{12} + c_{22} * d_{22}) * v_2 \end{pmatrix}$
$Cw == \begin{pmatrix} (c_{11} * d_{11} + c_{12} * d_{21}) * v_1 + (c_{11} * d_{12} + c_{12} * d_{22}) * v_2 \\ (c_{21} * d_{11} + c_{22} * d_{21}) * v_1 + (c_{21} * d_{12} + c_{22} * d_{22}) * v_2 \end{pmatrix}$
$Av = Cw \iff (CD)v = C(Dv)$
Q.E.D.