CS130 - Transformations

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Identify what each of the following does to a point in homogeneous coordinates. You may choose from:

- uniform scale by a (identify a)
- non-uniform scale by a, b, c (identify a, b, c)
- translation by a, b, c (identify a, b, c)
- rotation by angle θ about axis a, b, c (identify θ, a, b, c)
- reflections (identify the direction about the reflection is occurring)
- a sequence of the above (specify the operations in the order they are applied)

If the transformation cannot be obtained by applying a sequence of the above, explain why.

1. $\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Non-uniform scale
a=4
b=3
c=2

2. $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Uniform scale
a=0

3. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Uniform scale
a=1

4. $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Reflection over x-axis

5. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Reflection over y = x line

6. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Reflection over y = -x

7. $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Translation by
a=w
b=0
c=0

8. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

Not possible with above transformations; Project each point onto the x-axis

9. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Uniform scale; a=1/2

10. $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Two transformations:
First, Translation
a=w
b=0
c=0
Second, Uniform scale
a=1/2

11. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Rotation
$\theta = pi/2$ (clockwise)
a = x
b = y
c = z

12. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Not possible with above transformations; points are infinitely far away

13. $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Let $\mathbf{v} = \mathbf{Ru}$ be the rotated version of $\mathbf{u}$. Use the dot product $\mathbf{v} \cdot \mathbf{u}$ to show that the angle between $\mathbf{v}$ and $\mathbf{u}$ is θ.

$v = \begin{pmatrix} xcos\theta - ysin\theta \\ xsin\theta + ycos\theta \end{pmatrix}$
$v \cdot u = x^2cos\theta - xysin\theta + xysin\theta + y^2cos\theta = cos\theta(x^2 + y^2)$
$cos\theta = (v \cdot u)/(x^2 + y^2)$
Thus, the angle between $v \cdot u$ is θ

14. Show that $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ for the 2×2 matrix in the previous problem.

$R^T = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$
$R^T * R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

15. Let $\mathbf{R}$ be a 3×3 matrix with columns $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. Show that $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ is equivalent to $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ being unit vectors and mutually orthogonal.

$R^T = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

$R = \begin{pmatrix} u & v & w \end{pmatrix}$

$R^T * R = \begin{pmatrix} u^2 & uv & uw \\ vu & v^2 & vw \\ wu & wv & w^2 \end{pmatrix}$

In order for this matrix to be I, u^2, v^2, and w^2 must be equal to 1. That means that $u^2 = 1 * ||u||^2 = 1$, so $||u||$, $||v||$, and $||w||$ must be equal to 1.
In order for the other 6 elements must be equal to 0, the vectors must be mutually orthogonal because $uv = cos\theta * (1 * 1) = 0$
$\theta = \pi/2$.
Same applies for all other vectors pairs (not only uv)
Thus, $R^T * R = I$ is equivalent to u, v, w being unit vectors and mutually orthogonal.
Q.E.D.

16. Let $\mathbf{R}$ be a matrix with $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ and let $\mathbf{y} = \mathbf{R}\mathbf{x}$. Show that $\mathbf{x}$ and $\mathbf{y}$ must have the same length.

As shown in problem 15 above, if $R^T * R = I$, the columns of R are unit vectors. This means that when R is multiplied to x, the length of x will not change.
Thus x and y must have the same length
Q.E.D.

17. Let $\mathbf{R}$ be a matrix with $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ and let $\mathbf{y} = \mathbf{R}\mathbf{x}$ and $\mathbf{v} = \mathbf{R}\mathbf{u}$. Show that $\mathbf{u} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{v}$ so that dot products are preserved.

Because x and y are the same length and u and v are the same length from problem 16, the dot product of opposite pairs must be equal.
Q.E.D.

18. Given the results of the previous two problems, explain why angles must also be preserved.

When the magnitudes of the vectors are the same and the dot products are the same, the cos of the angle between them must also be the same. $(cos\theta = (u \cdot v)/(||u|| * ||v||))$
Q.E.D.