Indian Institute of Technology Kanpur

Course

AE-675: Introduction to Finite Element Method

Report On

One-Dimensional Elastic Bar Problem

Submitted to

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Project 1: Bar Problem

Problem statement

Figure 1 shows an elastic bar under traction load and constrained at the ends A and B. Develop a generic finite element code to get the approximate solution to the resulting governing differential equation for the bar shown in Figure 1.

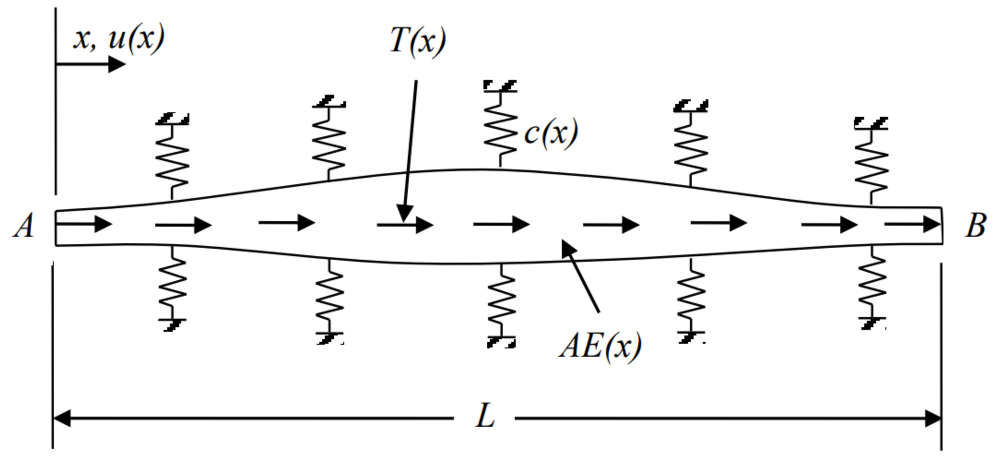


Figure 1: Elastic bar loaded in traction and constrained at ends A and B.

The code should have the following capabilities:

1. Boundary conditions/End Constraints: Both ends can be constrained by specifying

(a) primary variable (Dirichlet/Displacement/Essential),

(b) secondary variable or force (Force/Neumann/Natural) and

(c) springs (Mixed/Robin)

2. The variables $T(x)$, $c(x)$ and $AE(x)$ can vary from a constant to a quadratic function.

3. The length L and the number of elements will be input values. Discretize the domain into given number of elements with equal lengths.

4. There should be a provision to put at least one concentrated load at any given location (excluding the ends).

5. Use of either Lagrange interpolation or hierarchic shape functions up to quartic order should be possible.

6. Postprocessing must be able to represent the primary, secondary and other variables over the domain either continuously or discretely as required.

Produce a detailed report to include the following:

Problem 1. Do the patch test for the following cases: $AE(x) = 1$ and $c(x) = 0$ with $u(x)|x{=}0 = 0$ and $du/dx|\ x{=}1 = 0$. When $T(x) = 1$, use 1, 2, 5, 10 and 100 number of linear and quadratic elements and when $T(x) = x$ use 1, 2, 5, 10 and 100 number of linear, quadratic and cubic elements and superimpose your solutions with the respective exact solutions. Plot the error in the solution. Also plot the derivative of the exact solution and finite element solutions. Discuss the results.

Problem 2. For the problem in Point 1, plot the strain energy of the exact and finite element solution against the number of elements in the mesh for all the cases. Also plot the strain energy of the solution. Discuss the results.

Problem 3. Take $AE(x){=}1$, $c(x){=}1$ and $T(x){=}1$ with $u(x)|x{=}0{=}0$ and $du/dx|x{=}1{=}0$. Obtain the finite element solution with linear, quadratic, cubic and quartic elements respectively for 1, 10, 20, 40, 80 and 100 number of elements. For these cases:

a) Plot the exact and finite element solutions together for these cases.

b) Plot the error in the solution for these cases.

c) Plot the strain energy of the finite element and exact solution as a function of number of elements.

d) Plot the strain energy of the error as a function of number of elements.

e) Plot the log of the relative error in the energy norm versus the log of number of elements.

f)Try to estimate the convergence rate. Discuss the results.

Problem 4. Take $AE(x){=}1$, $c(x){=}0$ and $T(x){=}\sin\pi Lx$ with $AEdudx|x{=}0{=}1\pi$ and $AEdudx|x{=}1{=}kL(\delta L{-}u(L))$ with $kL{=}10$ and $\delta L{=}0$. Then repeat the exercise given a) through f) in Point 3.

Results for Problem 1

Patch test for the following cases:

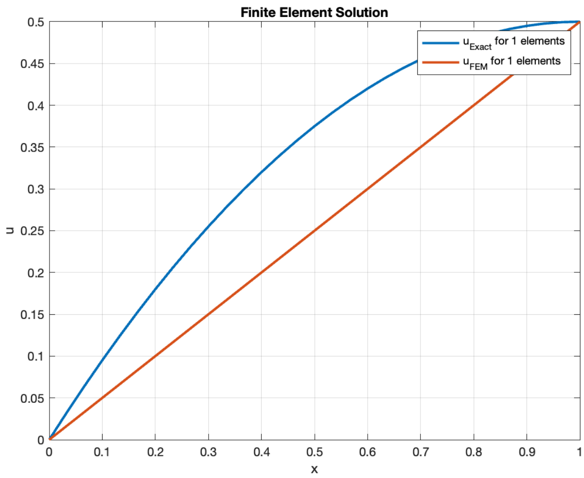
Case 1

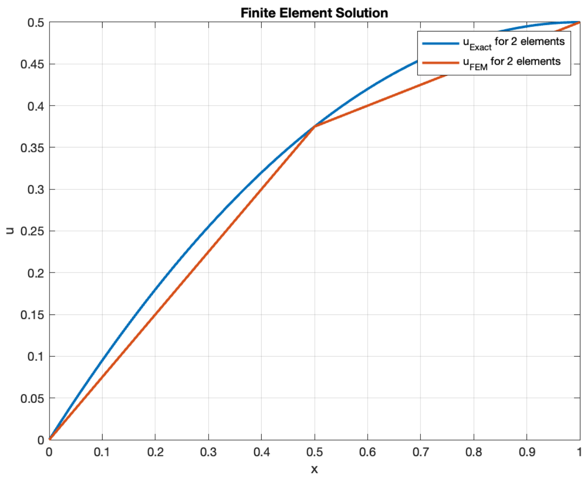
$AE(x) = 1$, $c(x) = 0$, $T(x) = 1$

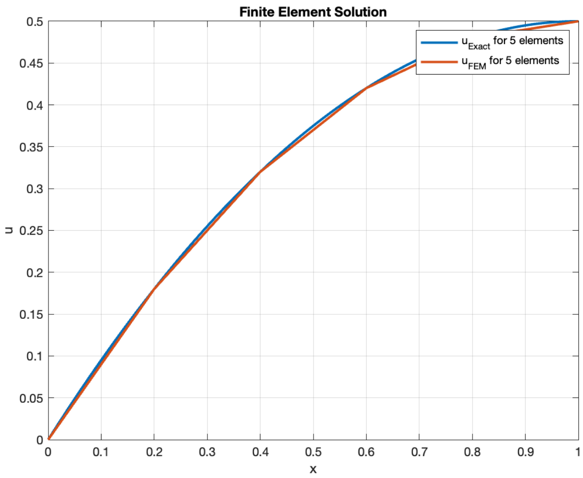
BCs: $u(x)|x{=}0 = 0$, $du/dx|\ x{=}1 = 0$

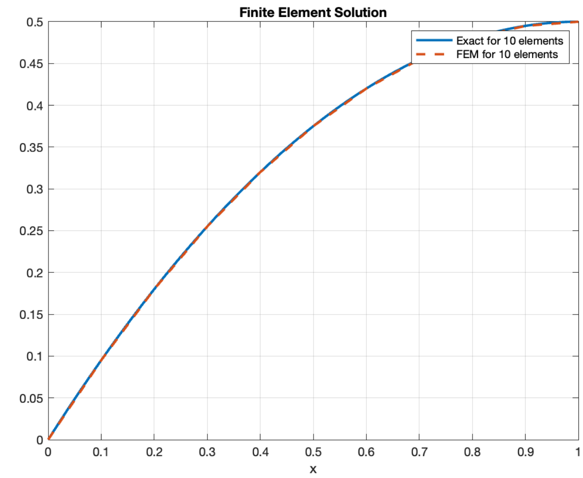
for NELEM 1, 2, 5, 10 and 100

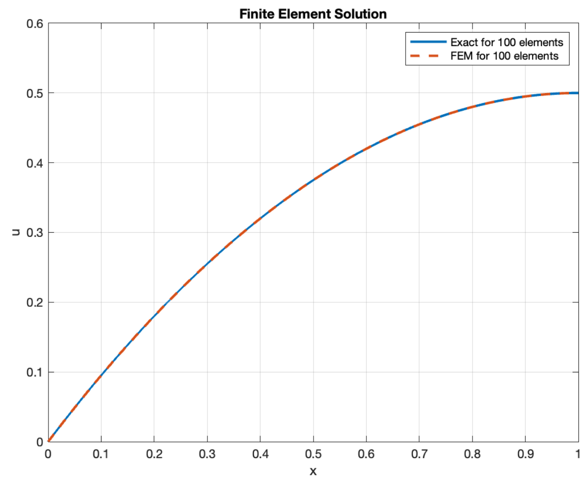
Plots Using Linear Shape Functions:



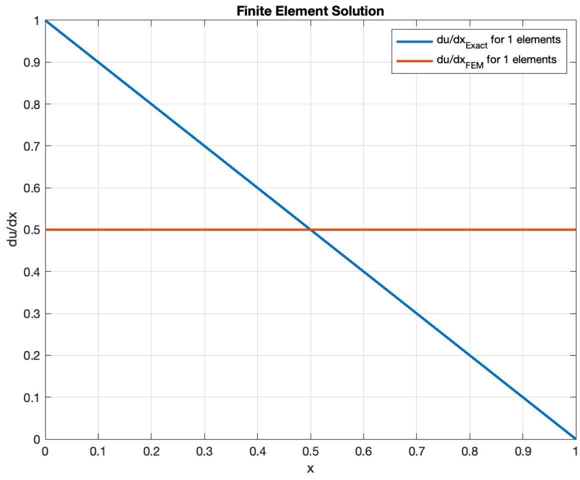


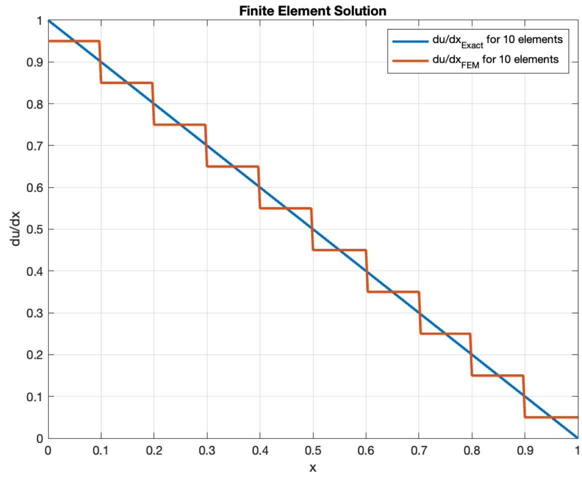


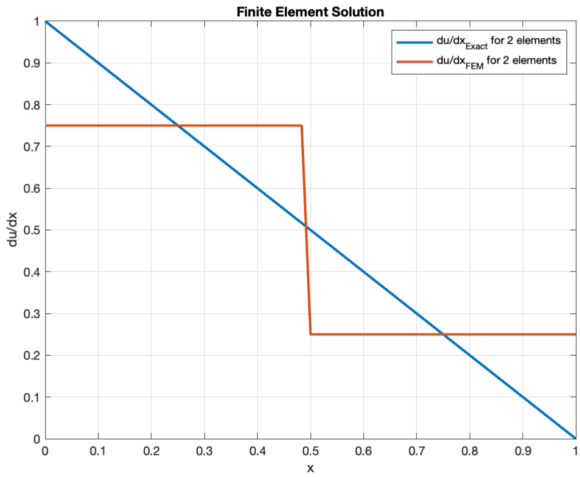


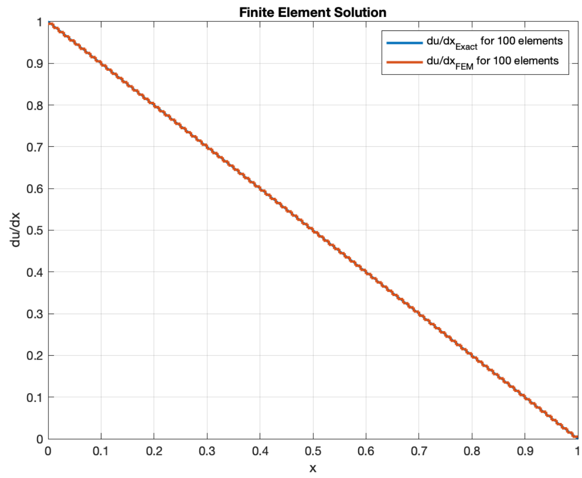


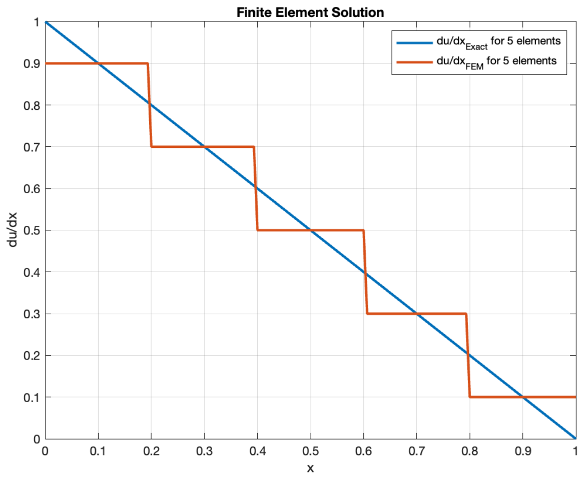
Linear Solution derivative Plots:



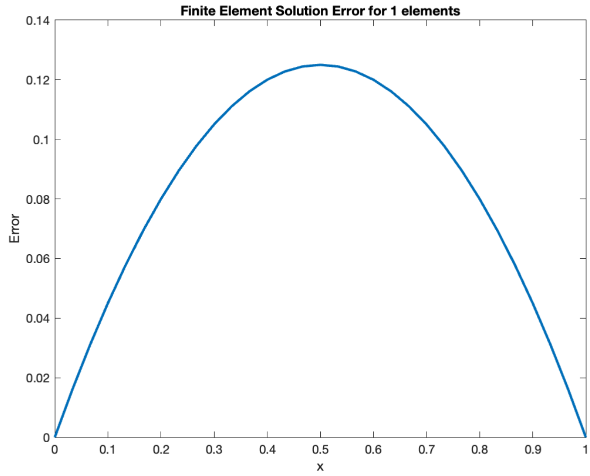


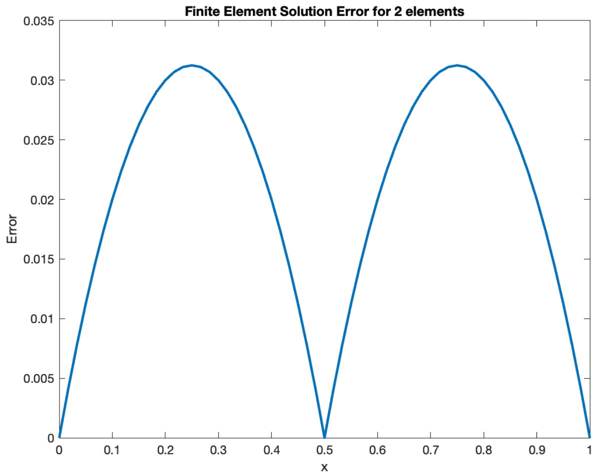


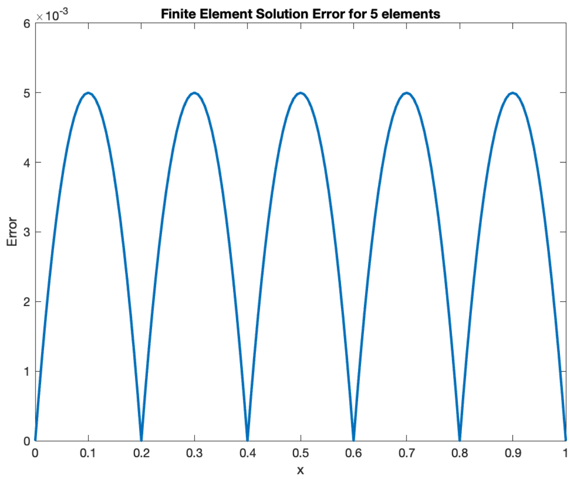


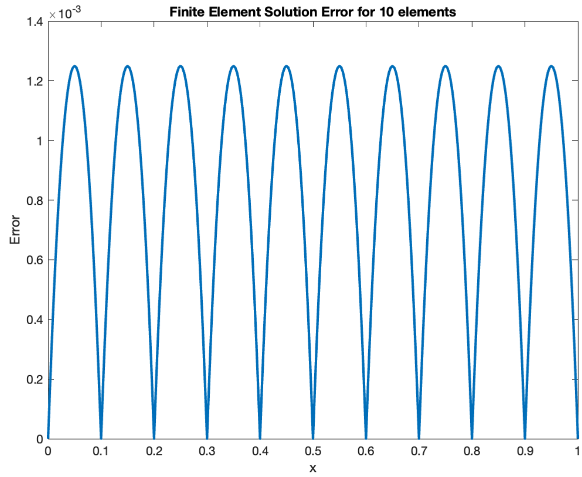


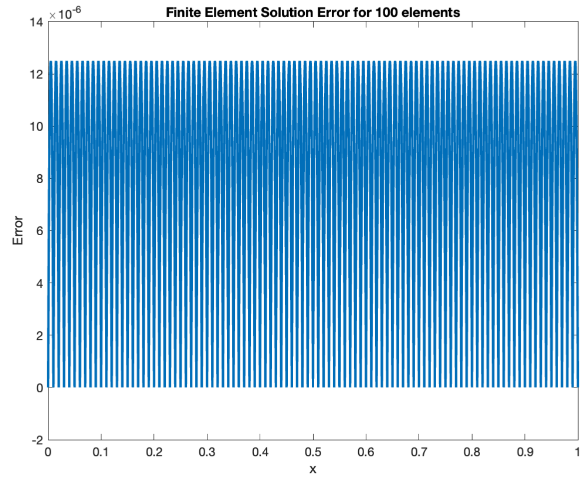
Linear Solution Error Plots:



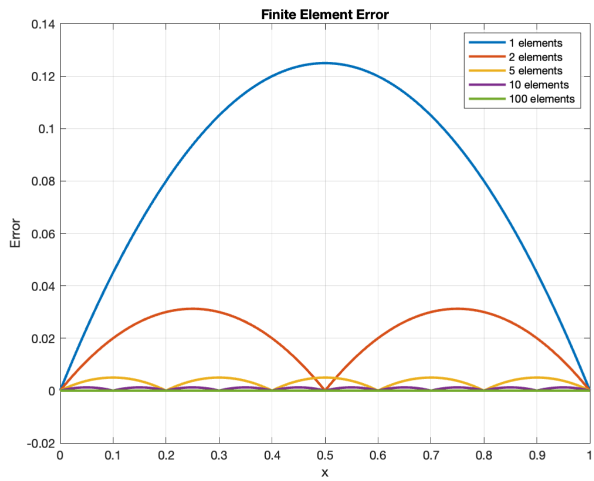




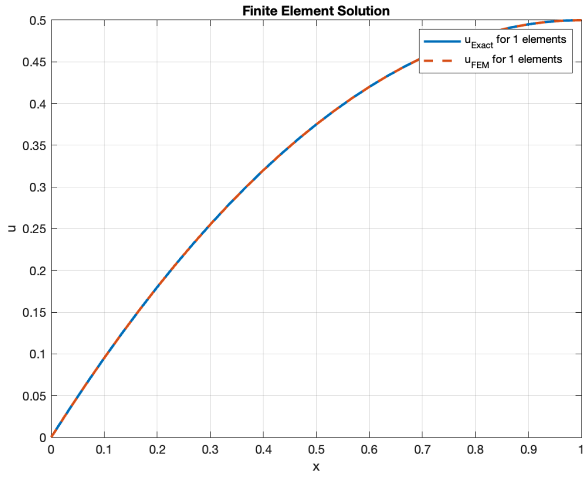


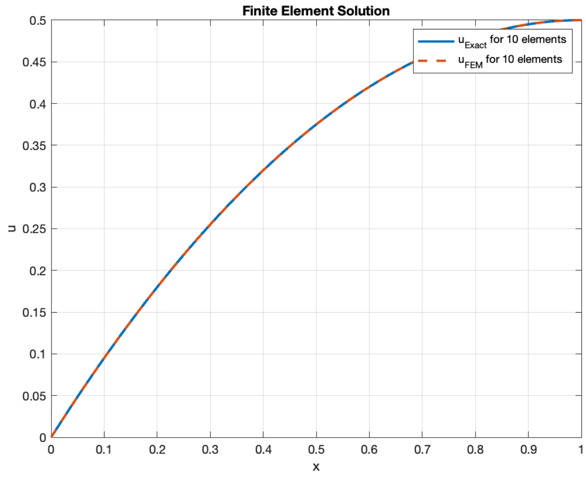


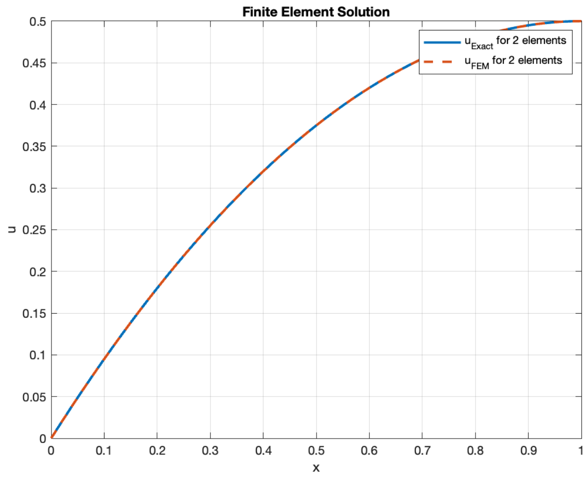
Relative error as per the number of elements

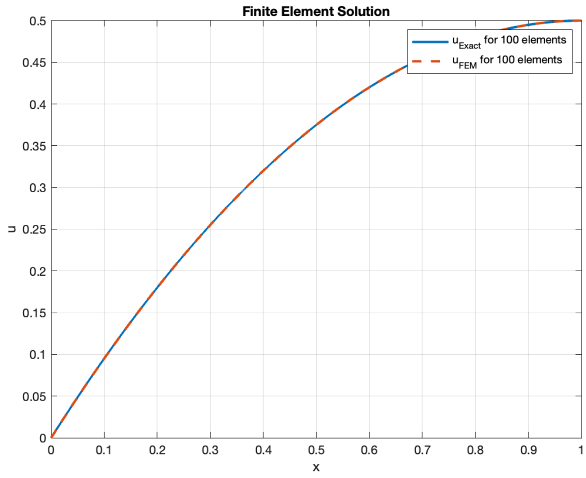


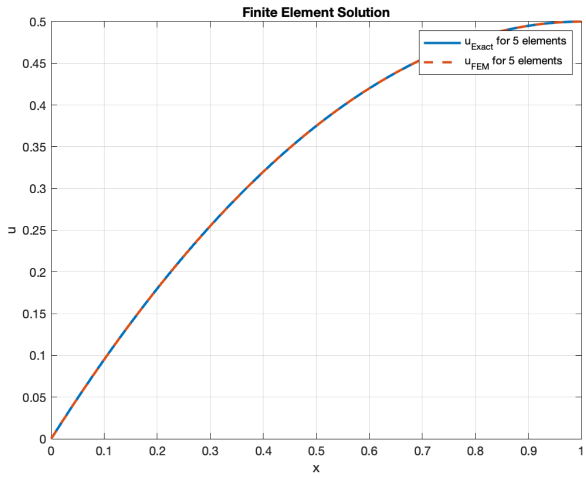
Plot Using Quadratic Shape Functions:



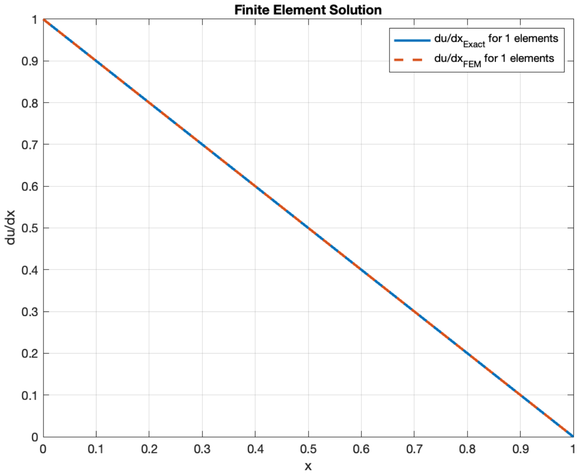


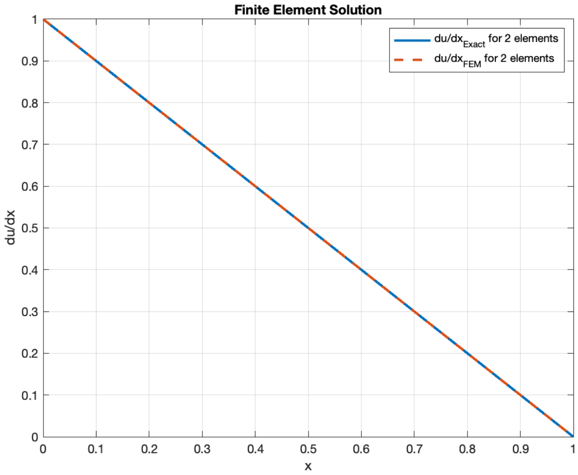


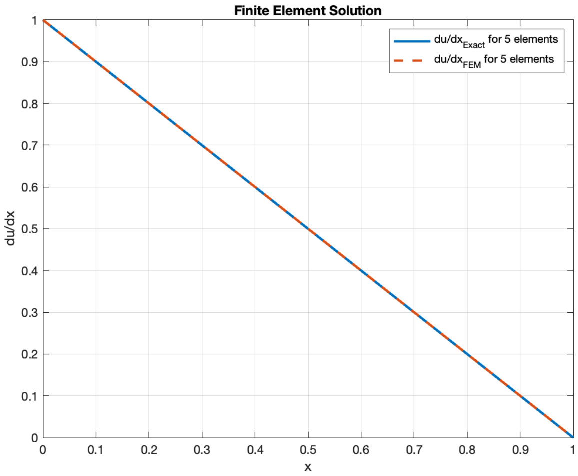


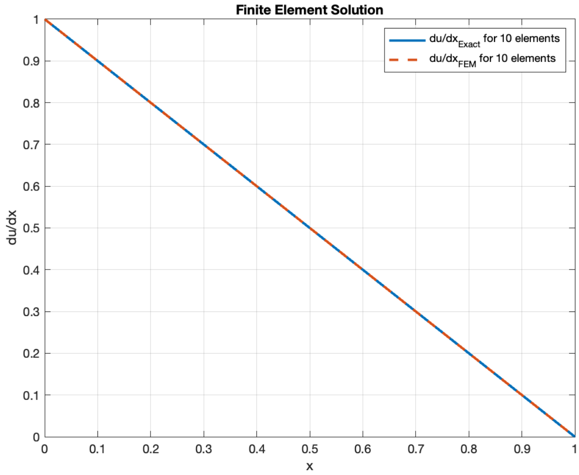


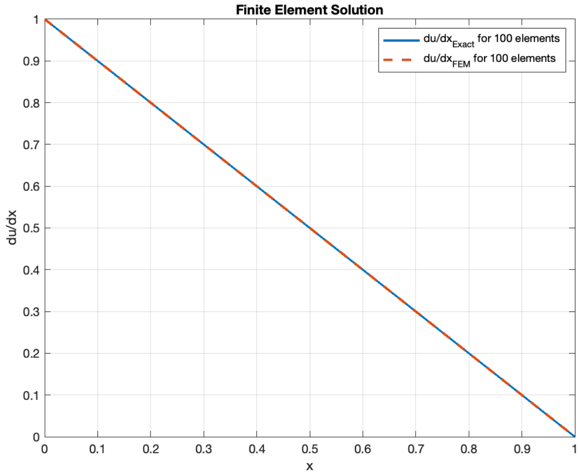
Quadratic Solution derivative Plots:



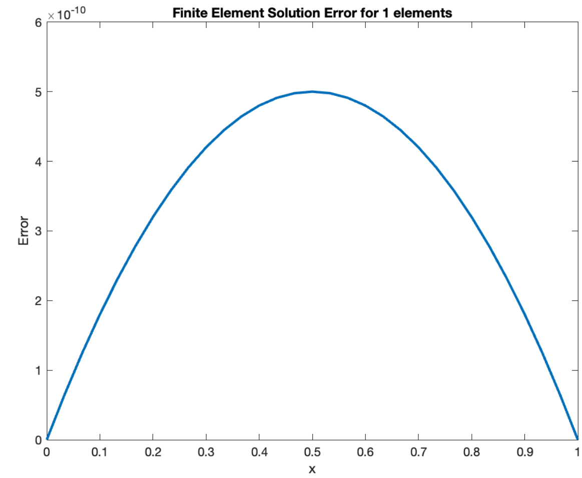


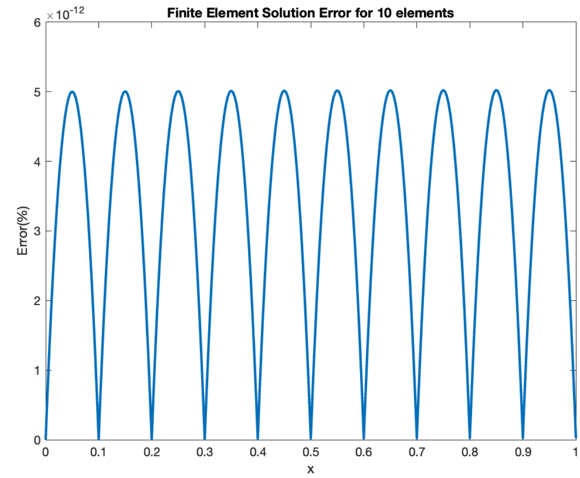


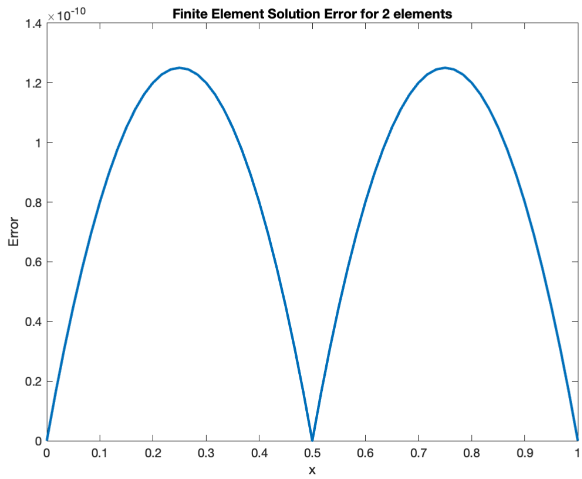


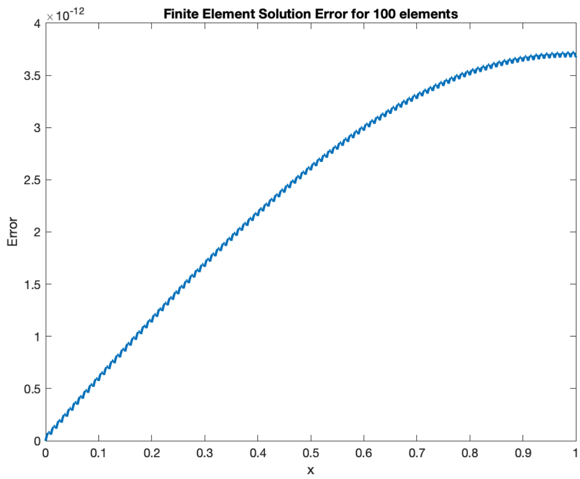


Quadratic solution Error Plots:

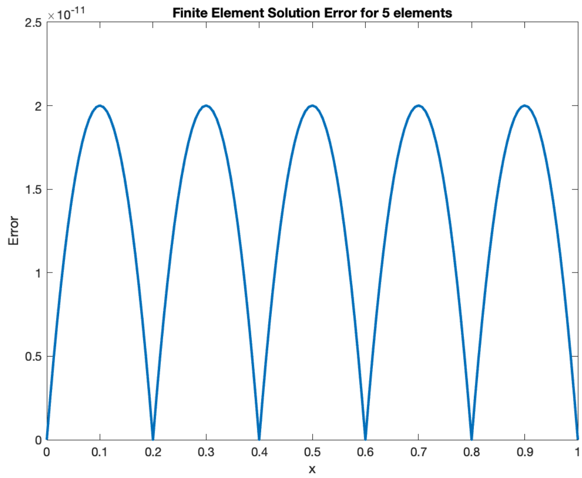


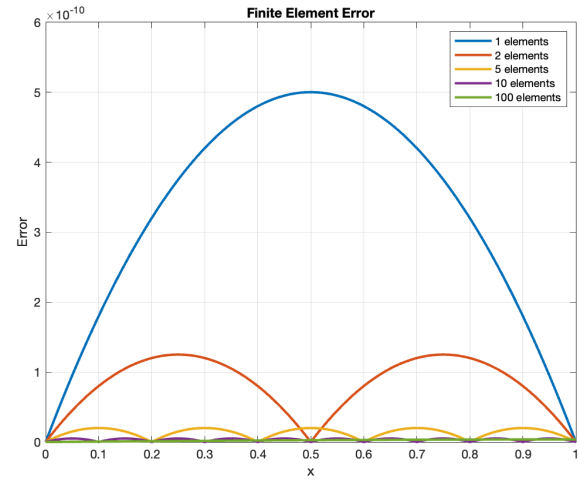






Relative error as per the number of elements





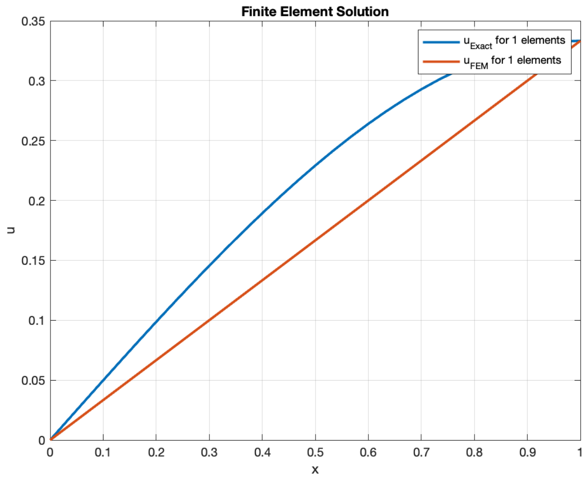
Case 2

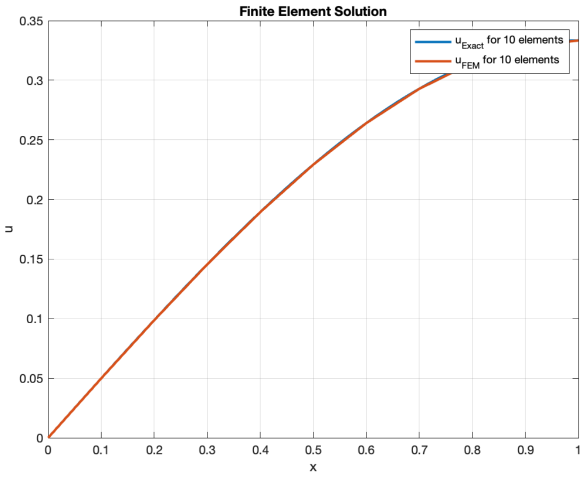
$AE(x) = 1$, $c(x) = 0$, $T(x) = \text{x}$

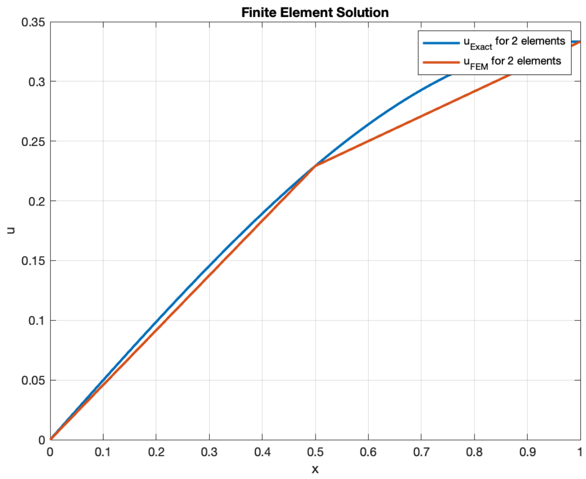
BCs: $u(x)|x{=}0 = 0$, $du/dx|\ x{=}1 = 0$

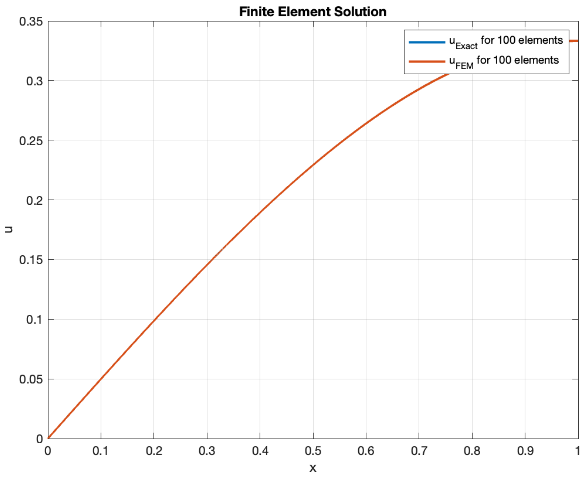
for NELEM 1, 2, 5, 10 and 100

Plots Using Linear Shape Functions:

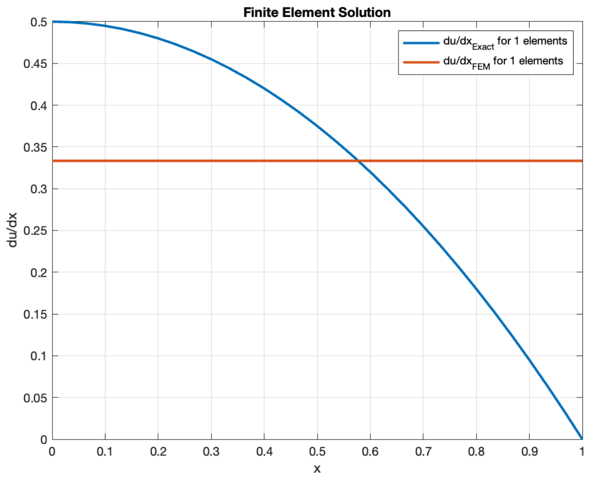


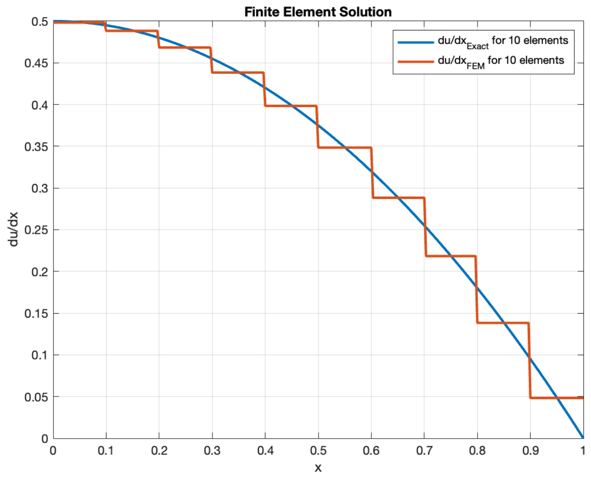


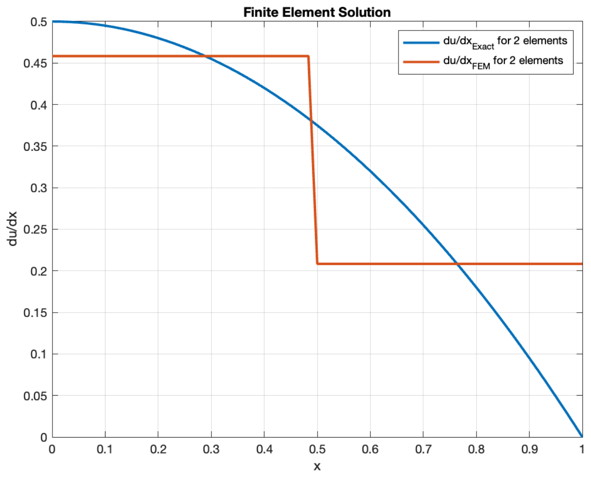


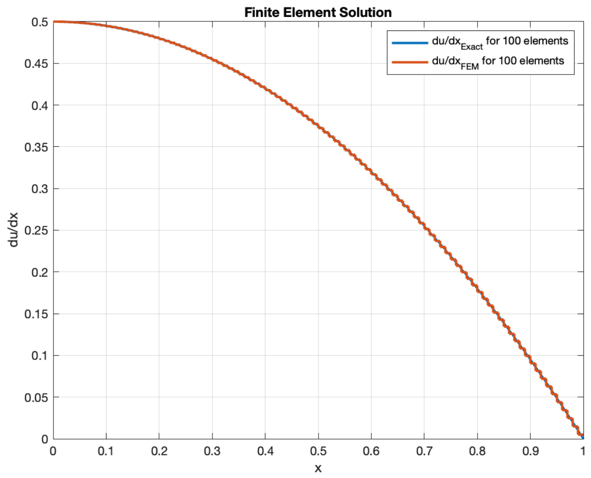


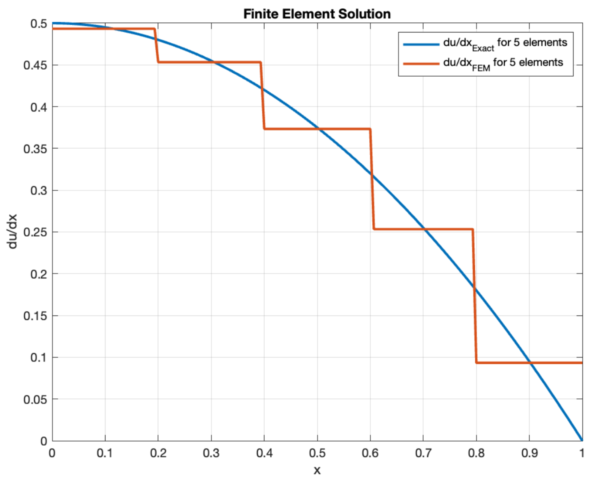
Linear Solution derivative Plots:



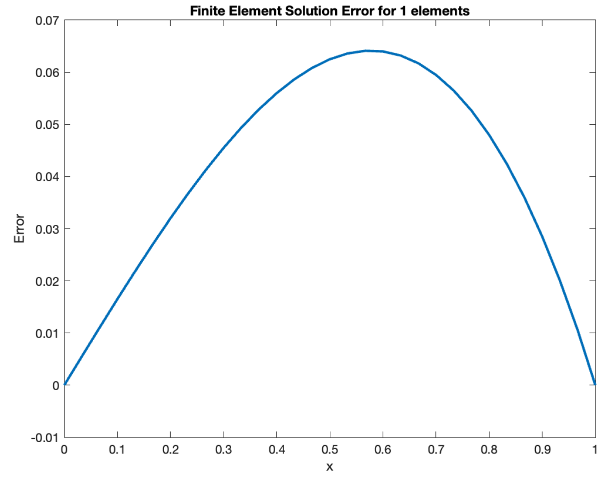


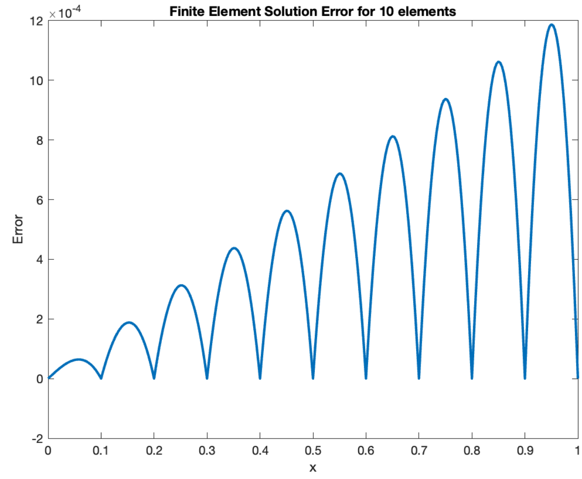


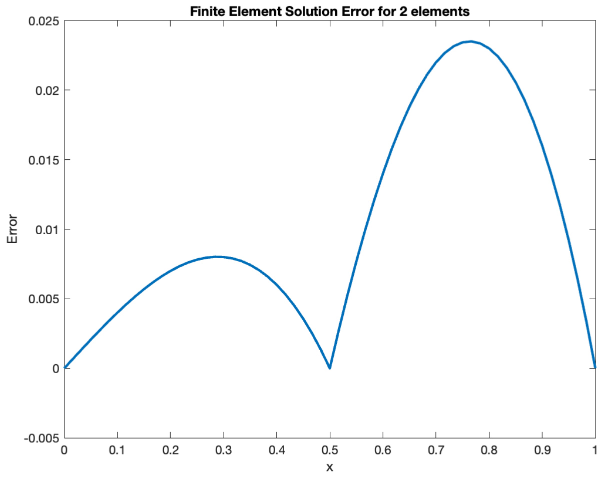


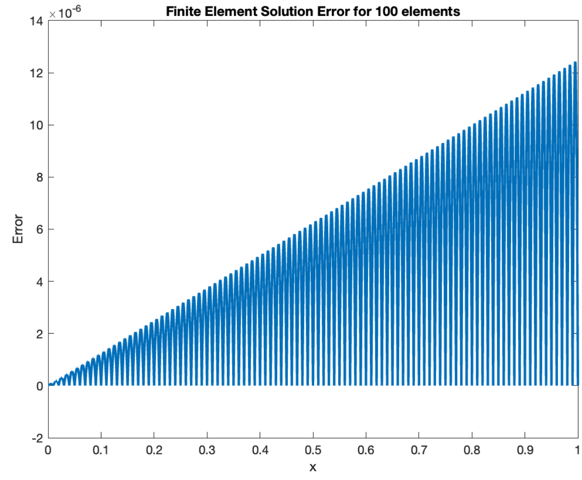


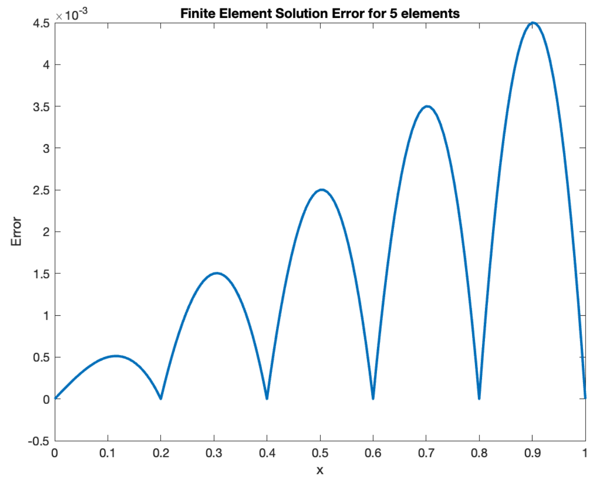
Linear Solution Error Plots:



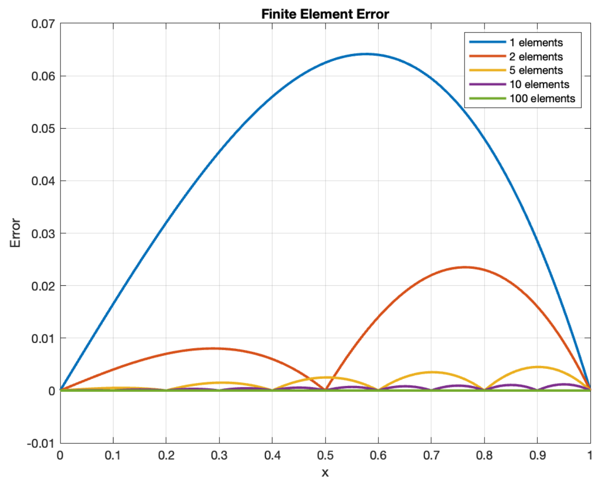




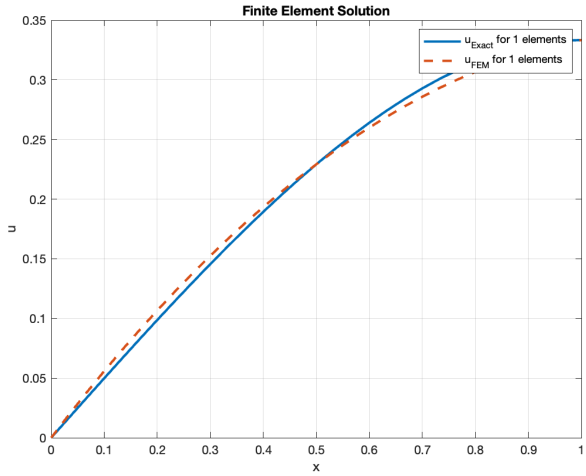


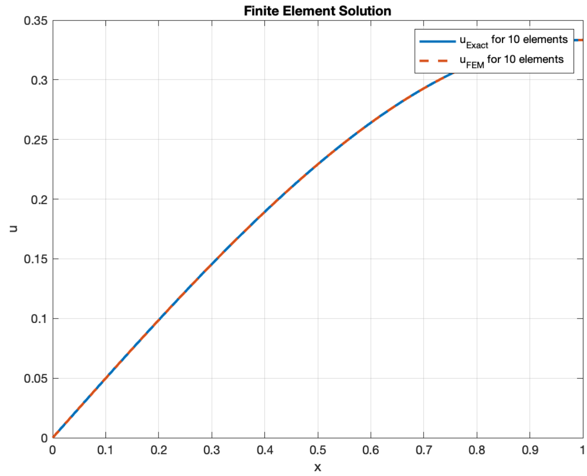


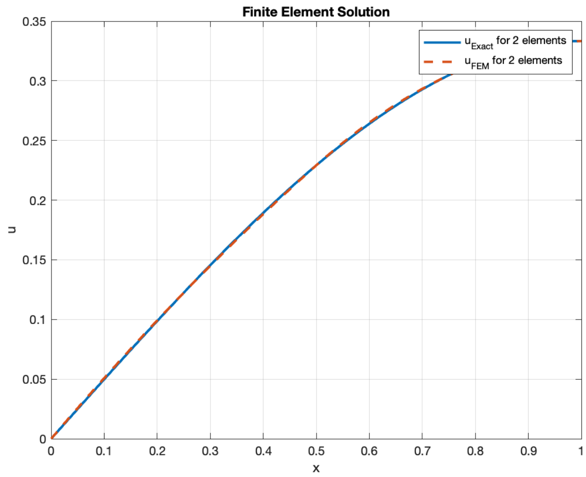
Relative error as per the number of elements

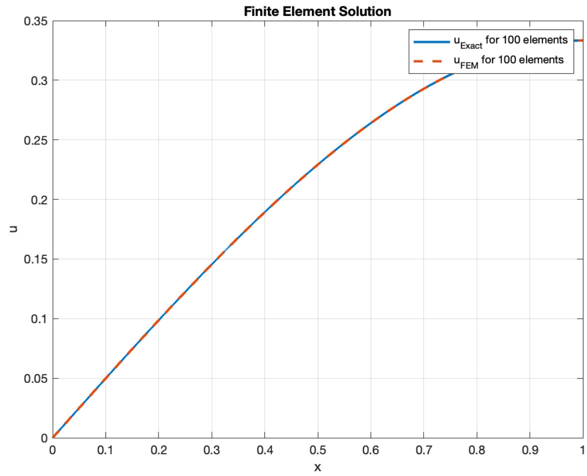


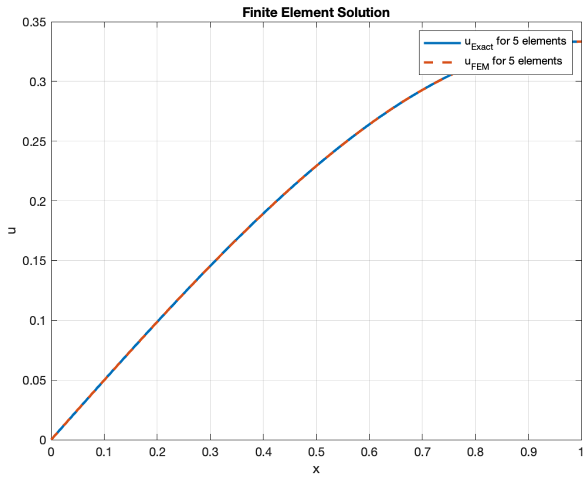
Plot Using Quadratic Shape Functions:



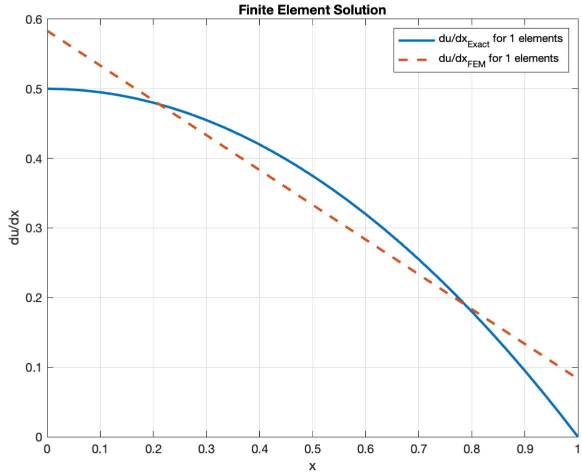


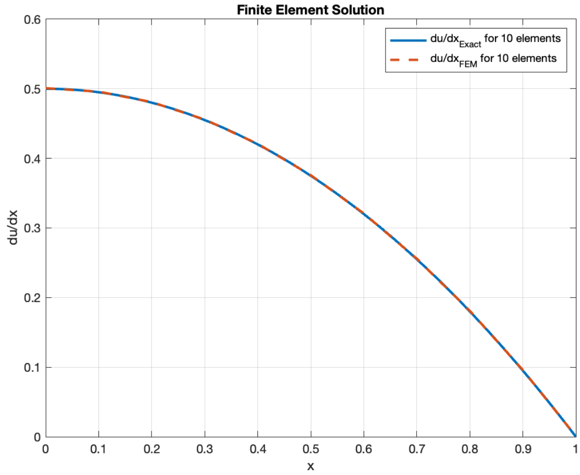


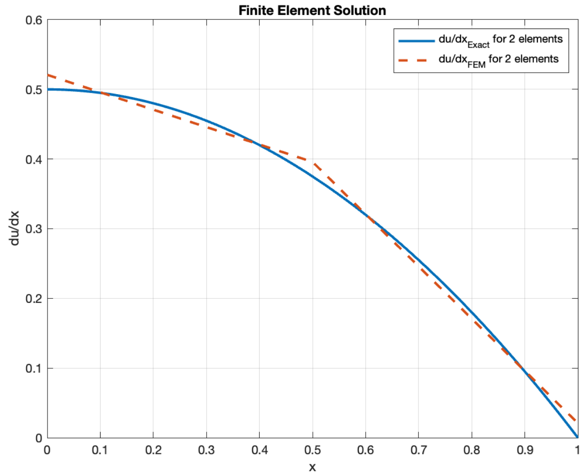


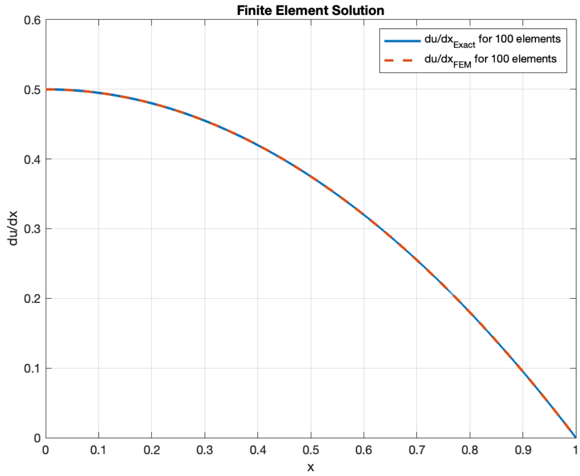


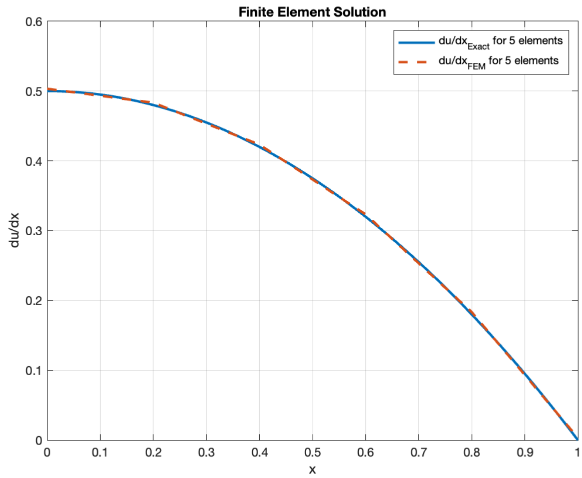
Quadratic Solution derivative Plots:



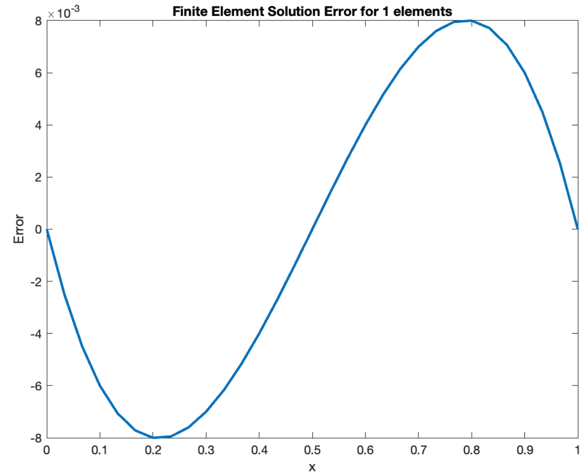


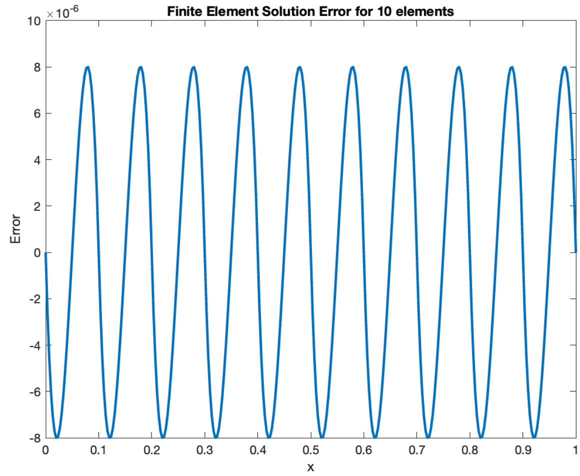


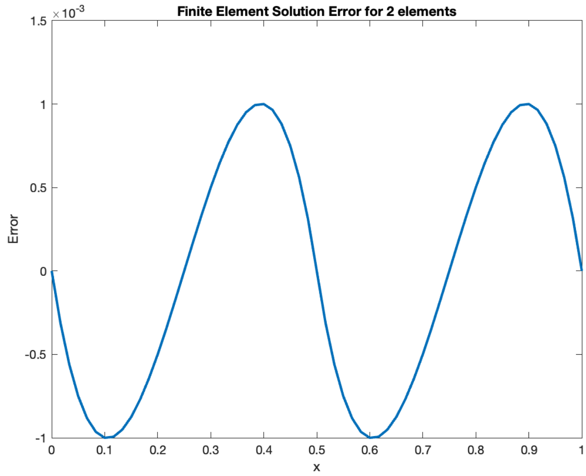


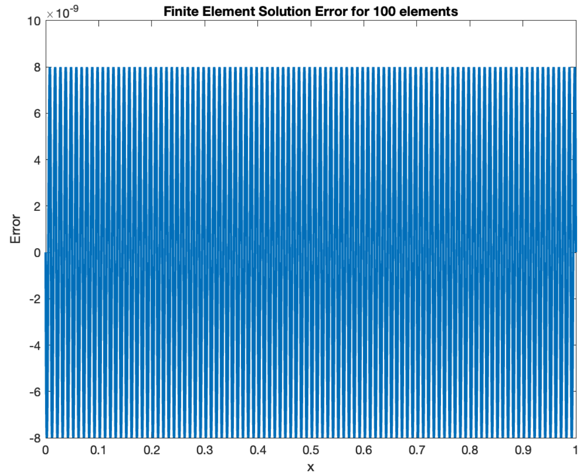


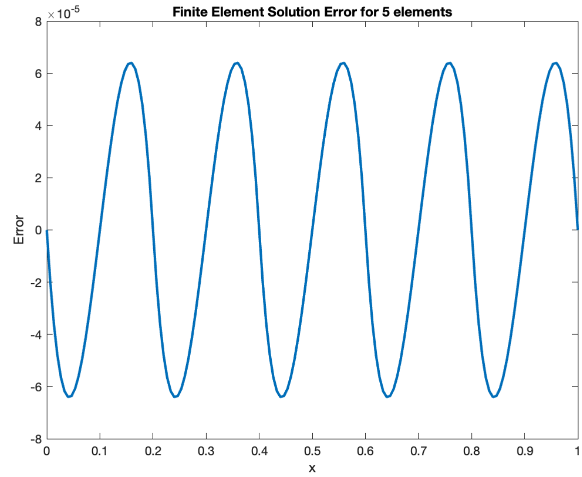
Quadratic solution Error Plots:



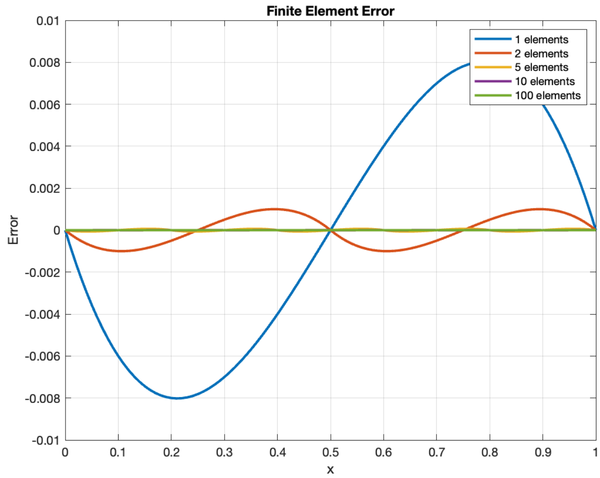




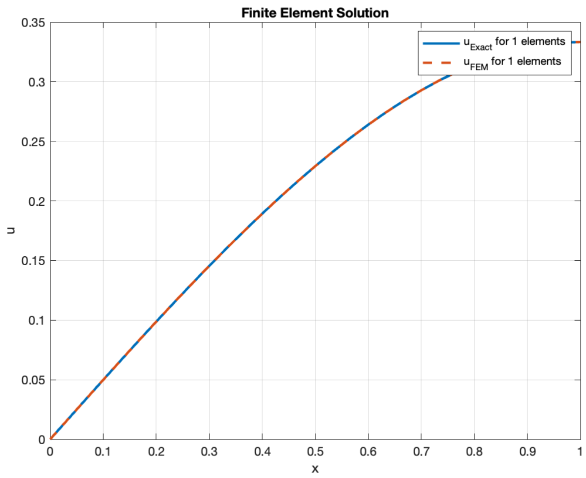


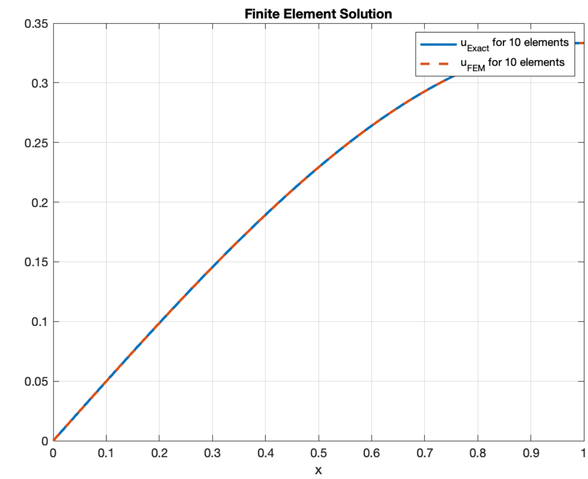


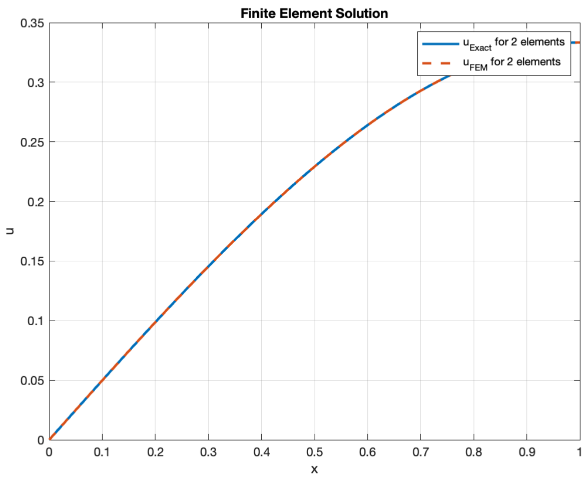
Relative error as per the number of elements

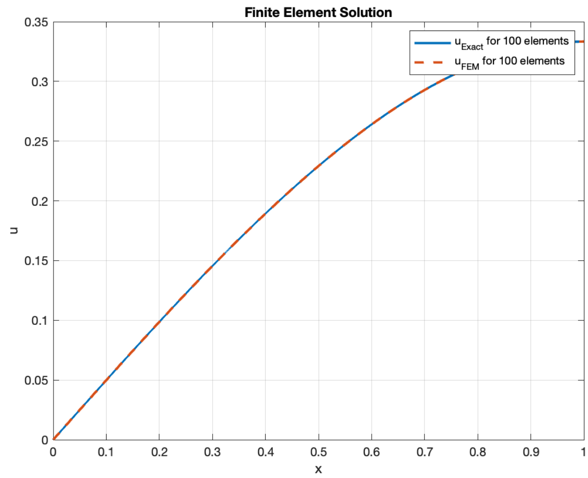


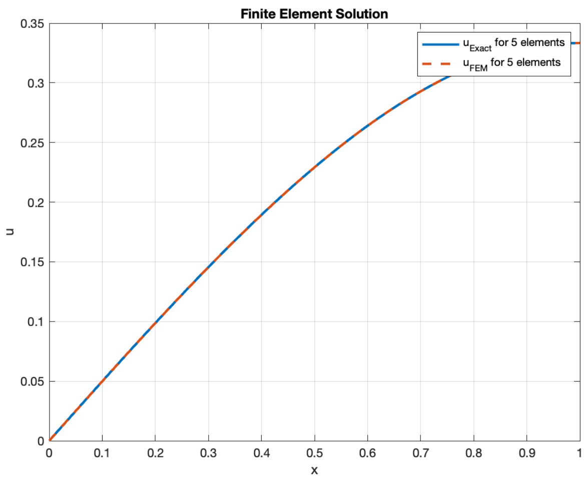
Plot Using Cubic Shape Functions:











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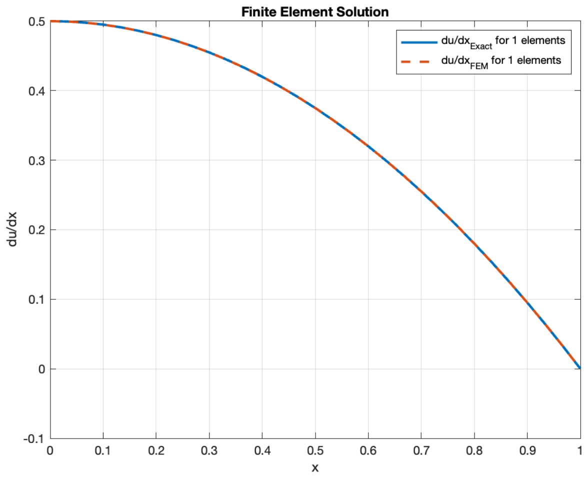
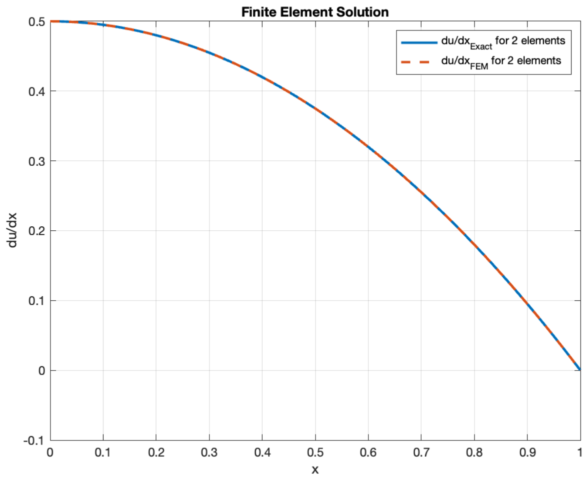
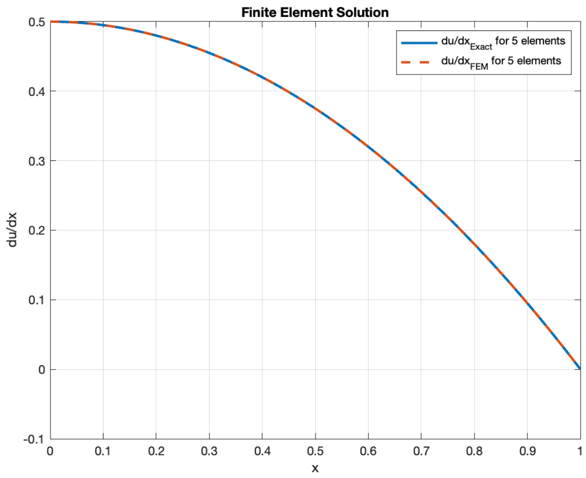
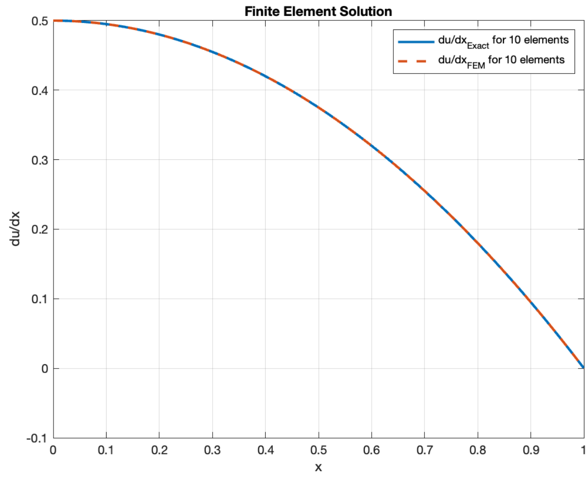
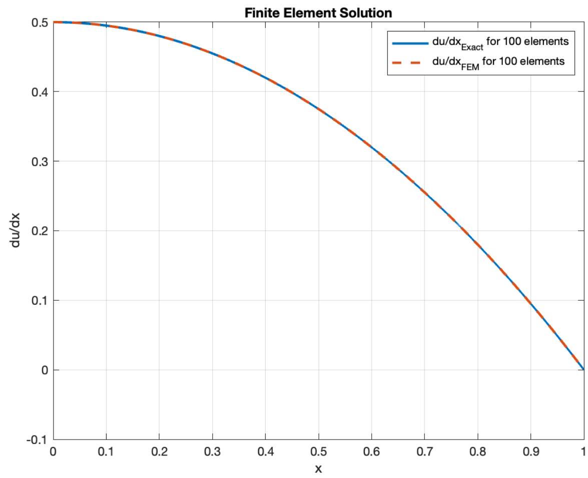
Cubic Solution derivative Plots:



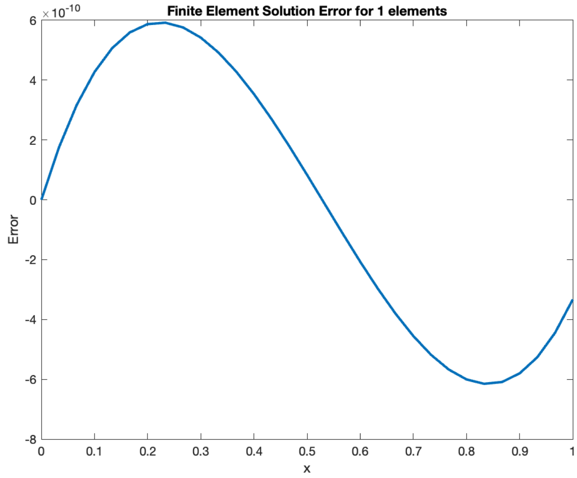


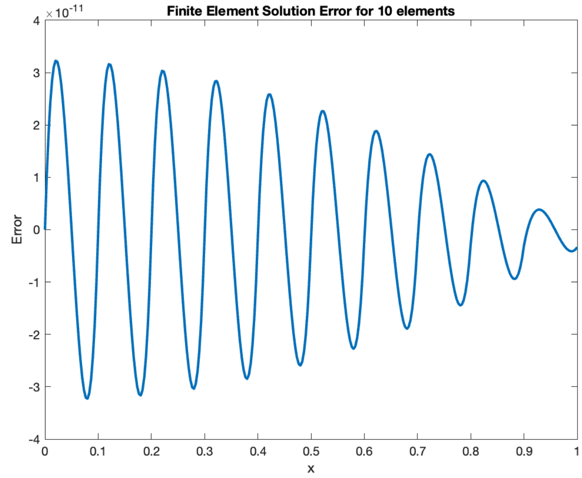


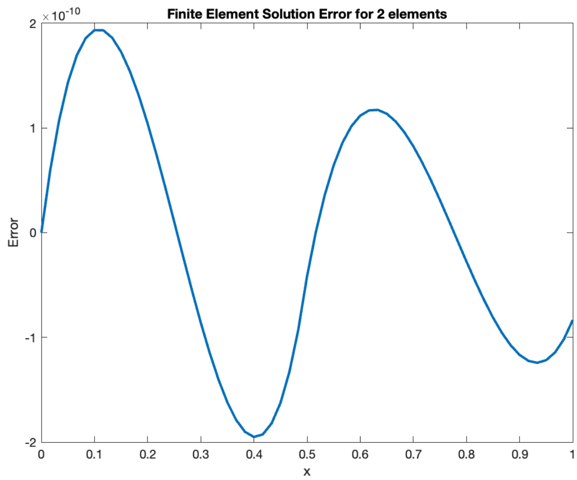


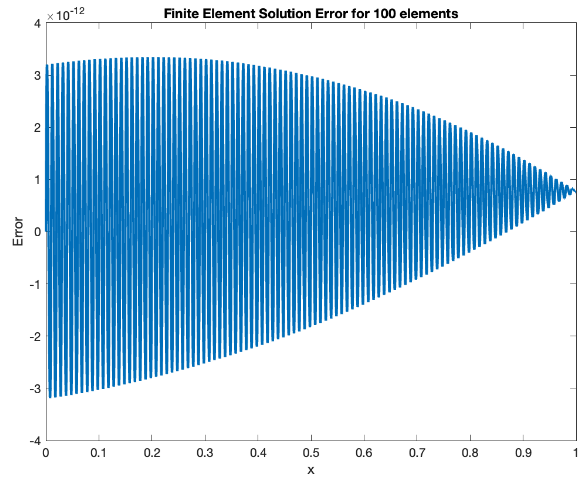


Cubic Solution Error Plots:

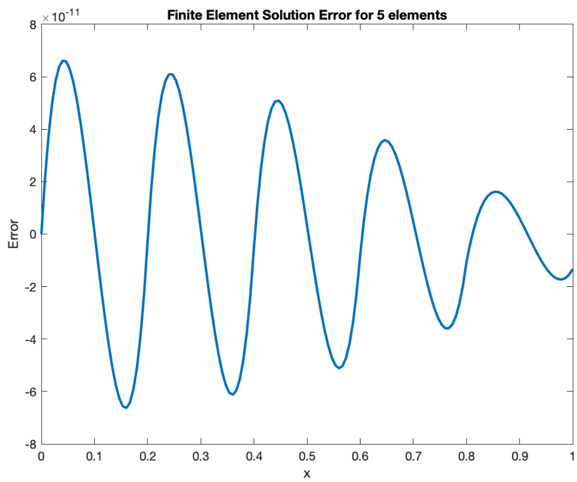


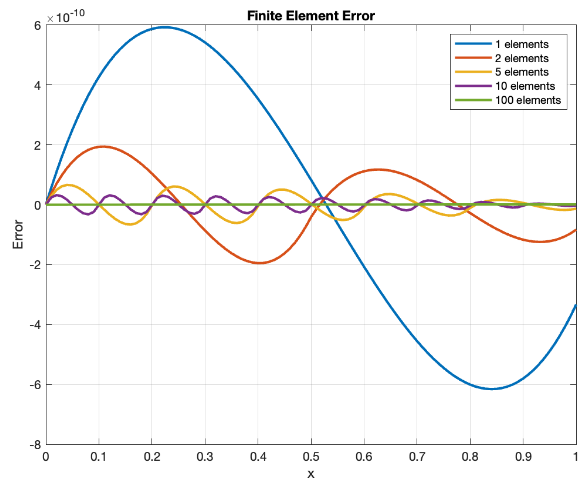






Relative error as per the number of elements





Conclusion of Problem 1

Plots for FEM solution and exact solution

Case 1 (T(x)=1)

The exact solution for this was quadratic, and the curve of this solution is compared with the FEM solutions obtained with different numbers of elements and also with the various shape functions for the patch test.

The plots using linear shape functions illustrate that a single element can accurately match the exact solution at the nodal points or endpoints. However, between these points, the linear interpolation introduces errors in representing the curvature or non-linearity of the exact solution. As the number of elements increases, the number of nodes also increases and the FEM solution matches the exact solution at more points within the domain, indicating improved accuracy and convergence of the FEM solution to the exact solution.

The plots While using quadratic shape functions illustrate that the FEM solution matches the quadratic shape functions at all points which is that on increasing the number of elements there is no change in the curve of the FEM solution.

Case 2 (T(x)=x)

The exact solution for this case was cubic and this was compared with the linear, quadratic, and cubic shape functions.

In the plots using linear shape functions it was seen that the endpoints of the local elements match the exact solution but not in between the nodes and as we increase the number of elements it slowly starts matching with the curve of the exact solution.

Quadratic shape functions also show similar results as linear but they try to match the exact solution curve and the FEM solution matches the exact solution at more points.

Using the cubic shape function the FEM solution matches the curve of the exact solution at all points and on increasing the number of elements there is no significant change, thus the results are verified using a patch test.

Plots for error between exact and FEM solution

For both the cases of the plots of error between exact and FEM solution, we can conclude that error is significantly decreasing with the increase in number of elements.

Also comparing the error plots for linear, quadratic, and cubic shape functions we can say that the error is significantly decreasing with increasing order of shape functions.

Plots for the derivative of exact and FEM solution

The derivative plots compare the derivative of the FEM solution with different numbers of elements to the exact derivative of the solution. It is observed that as the number of elements increases, the derivative of the FEM solution approaches the exact derivative. This trend demonstrates the improved accuracy and convergence of the FEM solution with a higher number of elements.

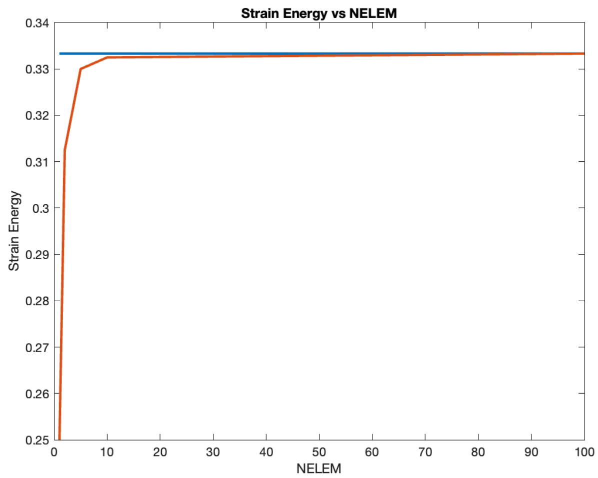
Results of Problem 2

Plot of the strain energy of the exact and finite element solution against the number of elements in the mesh for all the cases in Problem 1.

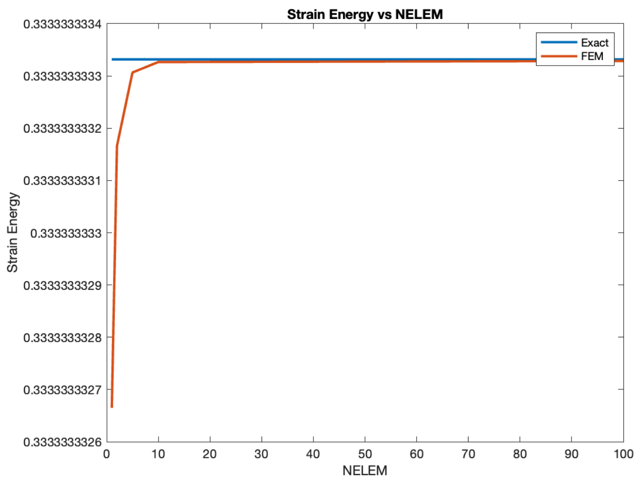
Strain energy Vs Number of elements Plots

For the First case in Problem 1

Plot For Linear solution

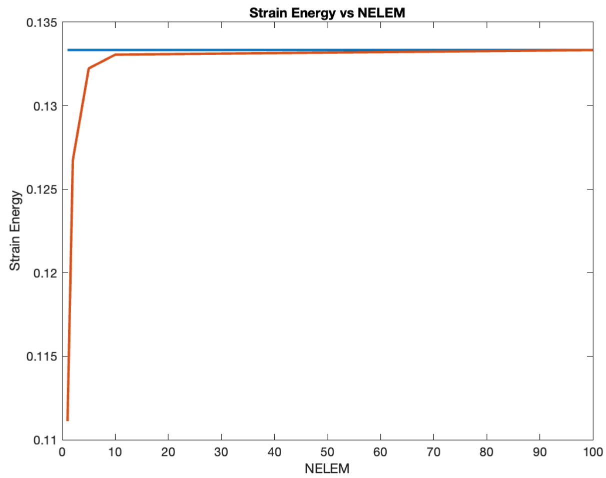


Plot For Quadratic solution

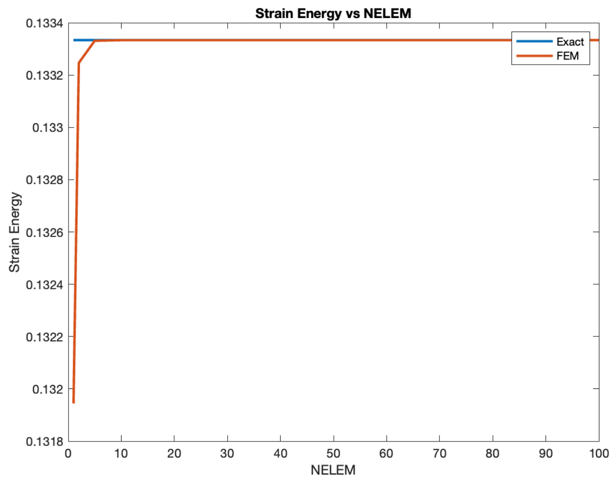


For the Second case in Problem 1

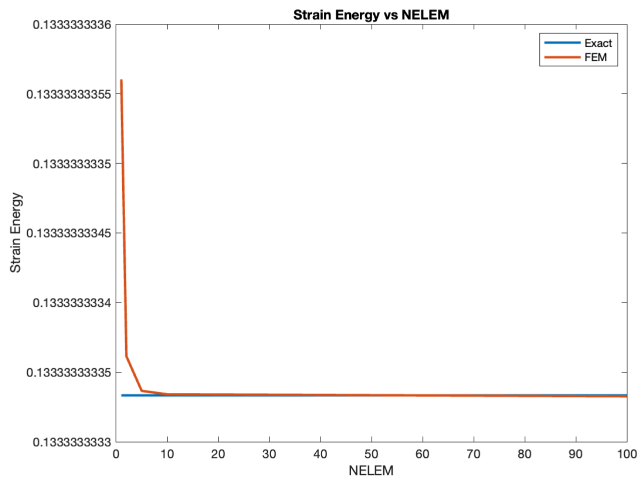
Plot For Linear solution



Plot For Quadratic solution



Plot For Cubic solution



Conclusion of Problem 2

Plots for Strain energy of FEM solution and exact solution

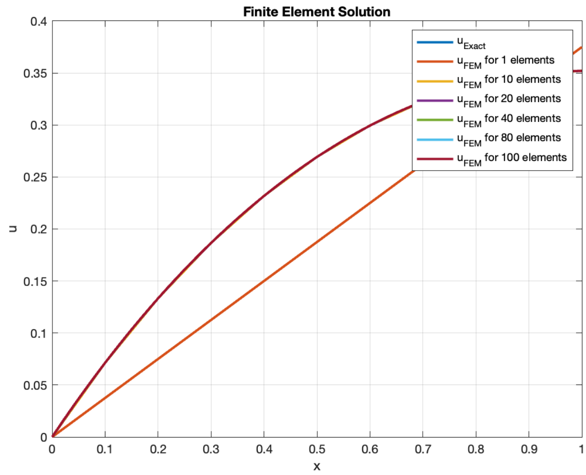
As more elements are used in the finite element method (FEM) solution, the computed strain energy initially increases and eventually converges to the strain energy of the exact solution. This trend demonstrates that increasing the number of elements enhances the accuracy of the FEM solution, bringing it closer to the exact solution. However, after reaching a certain number of elements, further refinement of the mesh has minimal impact on the strain energy value. This indicates that the FEM solution has achieved a stable and accurate representation of the exact solution.

Results of Problem 3

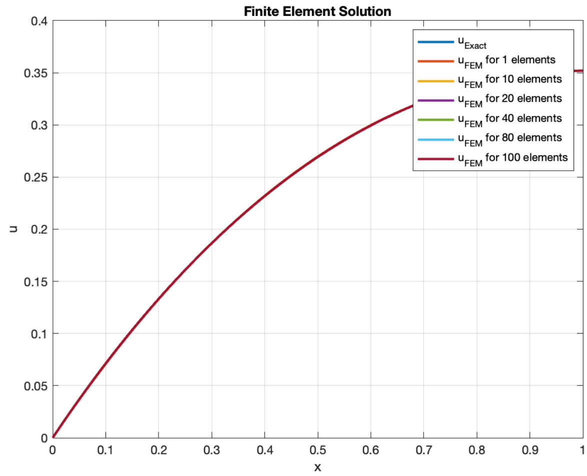
Take $AE(x) = 1$, $c(x) = 1$ and $T(x) = 1$ with $u(x)|x{=}0 = 0$ and $du\ dx|\ x{=}1 = 0$. Obtain the finite element solution with linear, quadratic, cubic and quartic elements respectively for 1, 10, 20, 40, 80 and 100 number of elements.

a) Plot For Exact Solution and finite element solution for different numbers of elements.

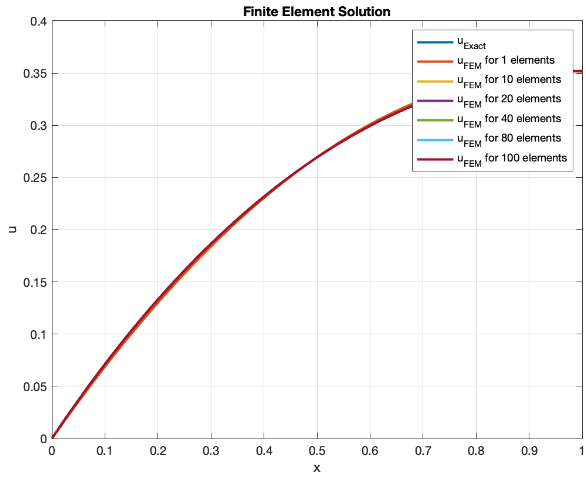
Plot Using Linear Shape Functions:



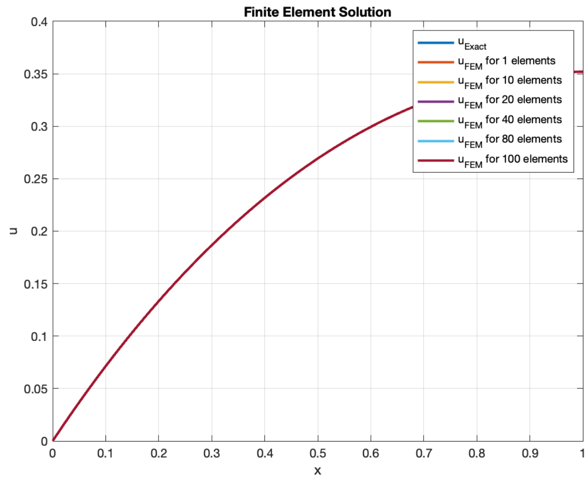
Plot Using Cubic Shape Functions:



Plot Using Quadratic Shape Functions:

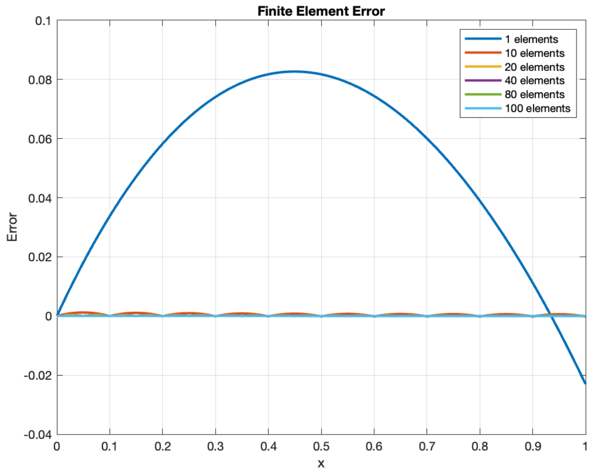


Plot Using Quartic Shape Functions:

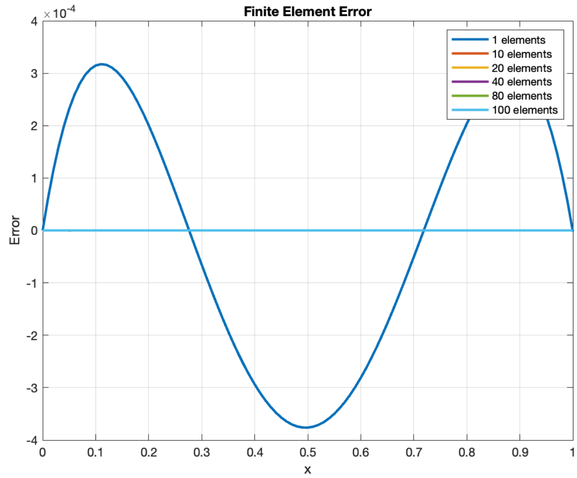


b) Error Plot for different type of element cases

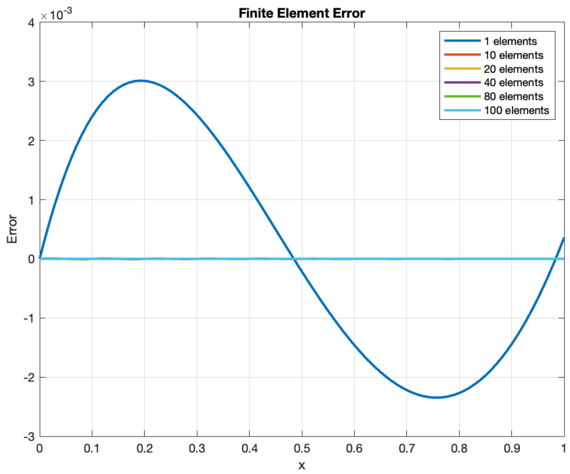
Linear Solution Error Plot:



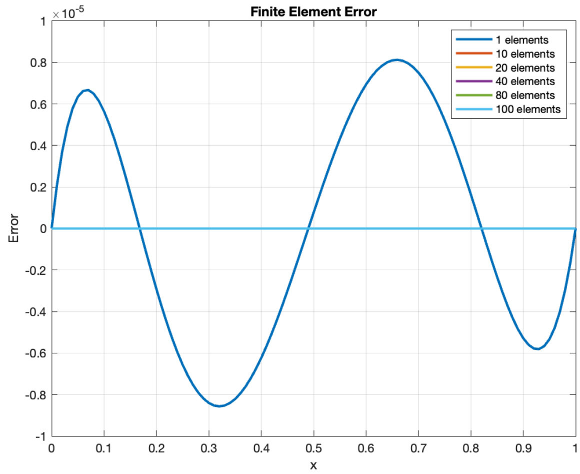
Cubic Solution Error Plot:



Quadratic Solution Error Plot:

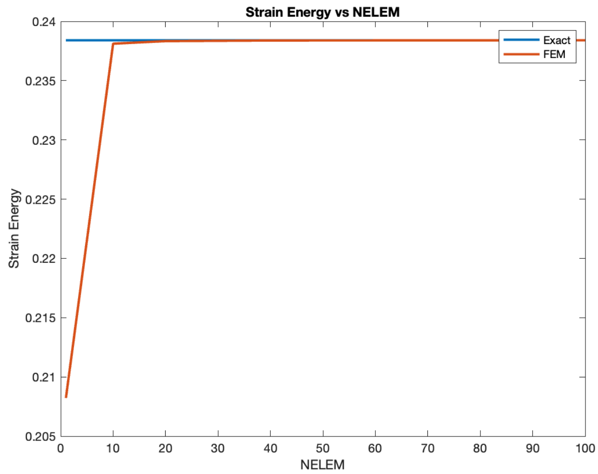


Quartic Solution Error Plot:

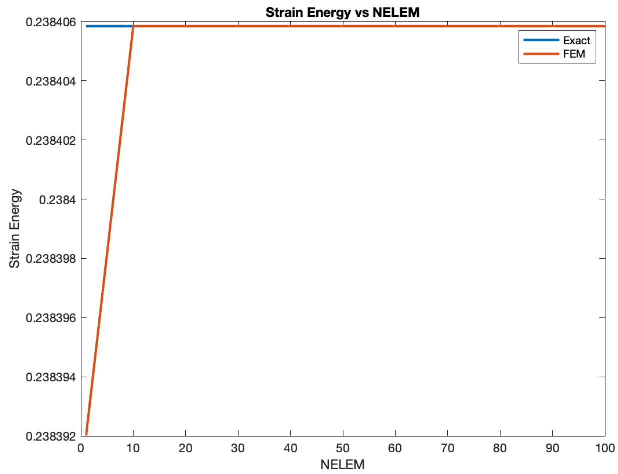


c) Plot the strain energy of the finite element and exact solution as a function of the number of elements.

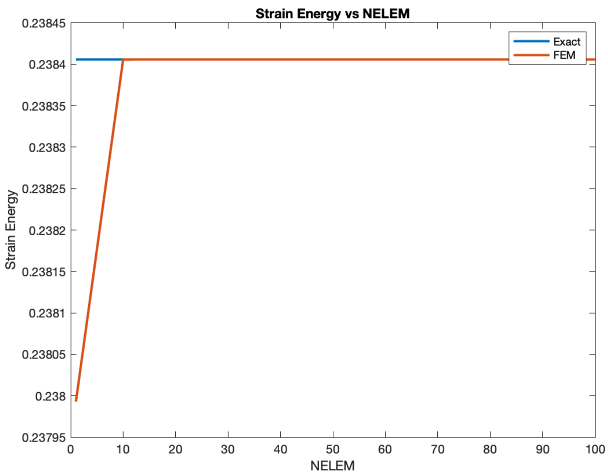
Linear Solution Strain Energy Plot:



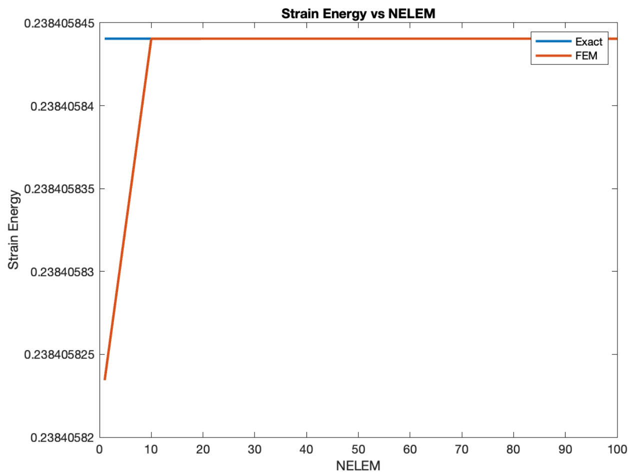
Cubic Solution Strain Energy Plot:



Quadratic Solution Strain EnergyPlot:

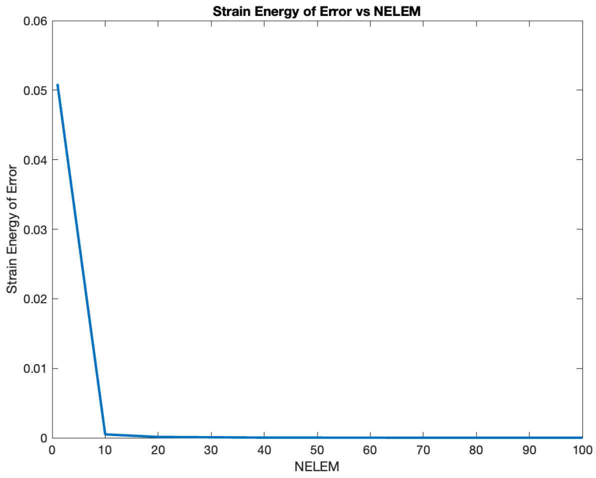


Quartic Solution Strain Energy Plot:

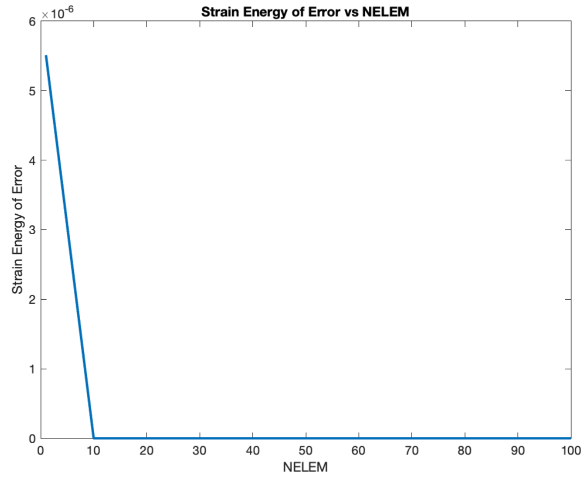


d) Plot of the Strain energy of error vs Number of elements

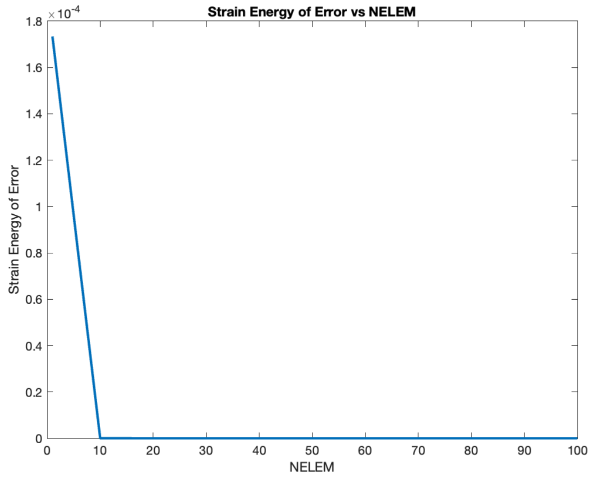
Plot Using Linear Shape Functions:



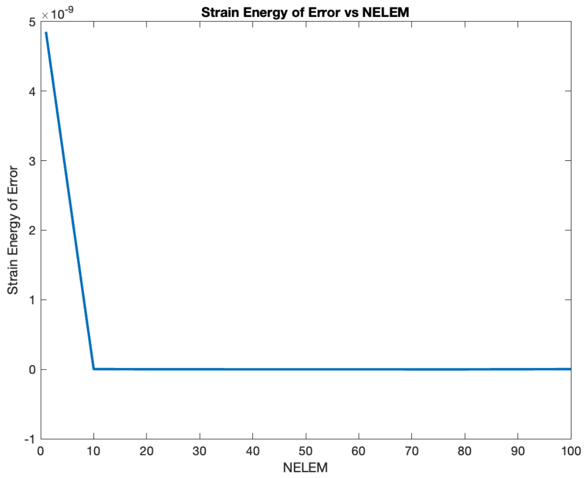
Plot Using Cubic Shape Functions:



Plot Using Quadratic Shape Functions:

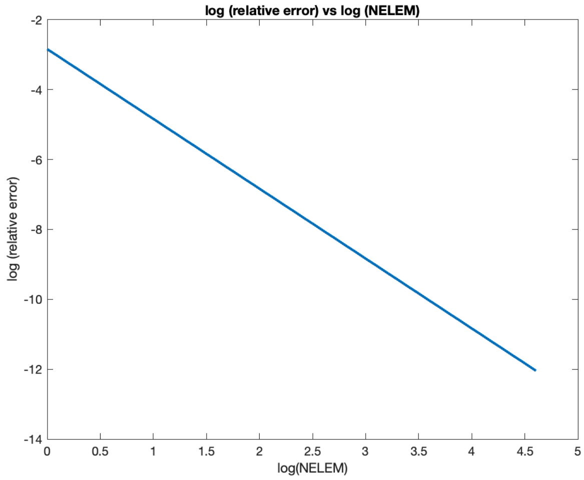


Plot Using Quartic Shape Functions:

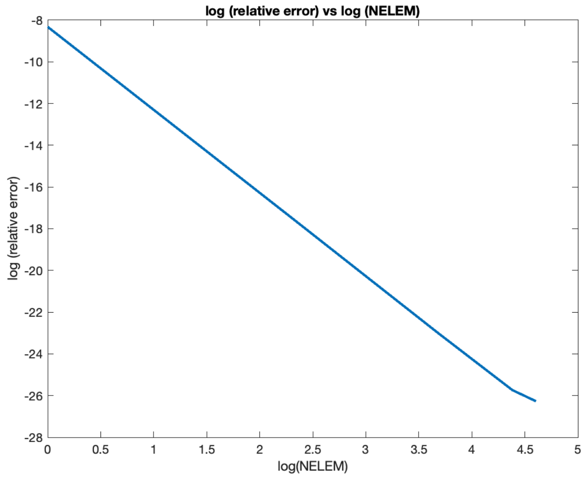


e) Plot the log of the relative error in the energy norm versus the log of number of elements.

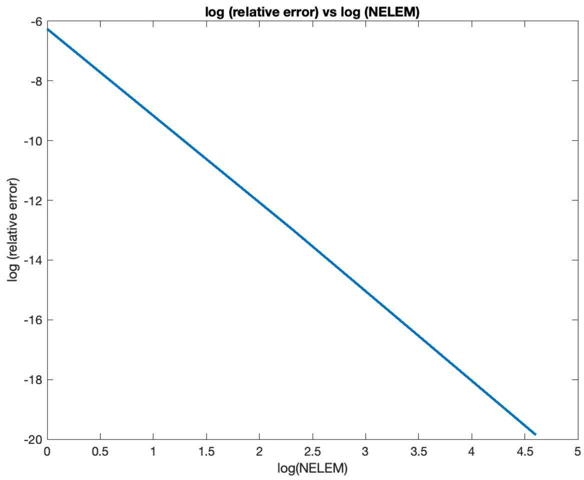
Plot Using Linear Shape Functions:



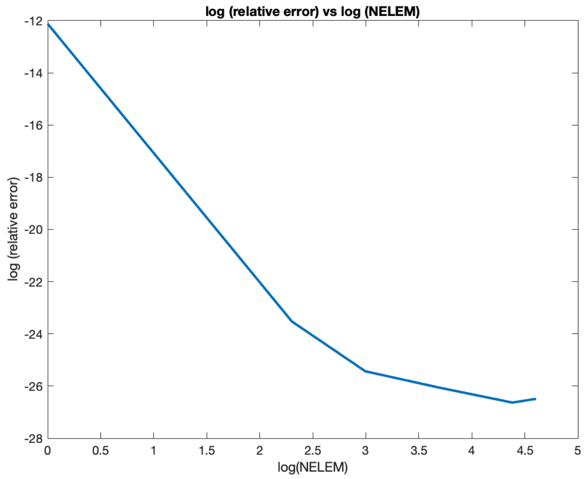
Plot Using Cubic Shape Functions:



Plot Using Quadratic Shape Functions:



Plot Using Quartic Shape Functions:



Conclusion of Problem 3

Plots for FEM solution and exact solution

The exact solution for this problem is cubic, and its curve is compared with the FEM solutions obtained using different numbers of elements and shape functions for patch testing.
With linear and quadratic shape functions, a single element accurately matches the exact solution at the nodal points but introduces errors in representing the curvature or non-linearity between these points. As the number of elements increases, the FEM solution converges to the exact solution, matching it at more points within the domain and indicating improved accuracy and convergence.
Using cubic shape functions, the FEM solution matches the exact solution curve at all points. Increasing the number of elements does not change the curve of the FEM solution, indicating that cubic shape functions accurately represent the exact solution without the need for further refinement.
Quartic shape functions show similar results to cubic which verifies the patch test.

Plots for error between exact and FEM solution

The plots of error between exact and FEM solution we can conclude that error is significantly decreasing with the increase in the number of elements.
Also comparing the error plots for linear, quadratic cubic, and quartic shape functions we can say that the error significantly decreases with increasing order of shape functions with minimum error in the cubic and quartic cases.

Plots for Strain energy of FEM solution and exact solution

As more elements are used in the finite element method (FEM) solution, the computed strain energy initially increases and eventually converges to the strain energy of the exact solution. This trend demonstrates that increasing the number of elements enhances the accuracy of the FEM solution, bringing it closer to the exact solution. However, after reaching a certain number of elements, further refinement of the mesh has minimal impact on the strain energy value. This indicates that the FEM solution has achieved a stable and accurate representation of the exact solution.

Plots for Strain energy of the error for FEM solution and exact solution

From this plot, we can conclude that the strain energy of error decreases with the number of elements and converges to zero.

Estimation of the convergence rate

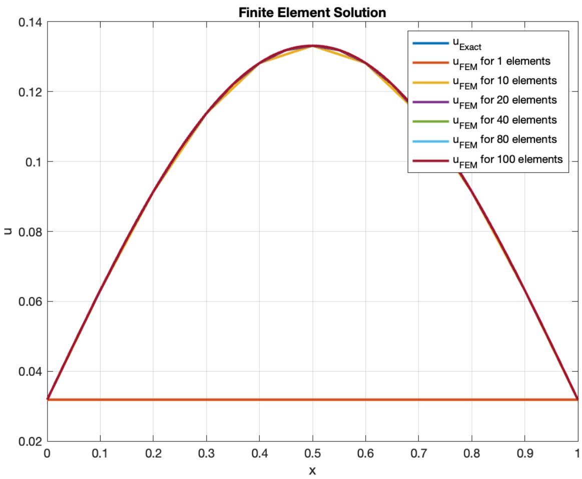
The convergence rate increases with the number of elements as well as an increase in the degree of approximation polynomial function

Results of Problem 4

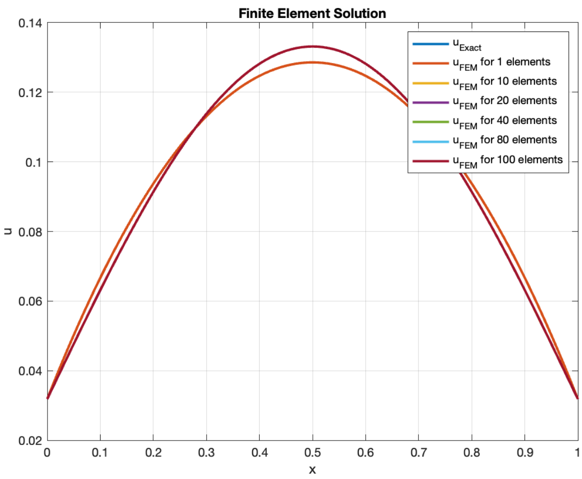
Take $AE(x) = 1$, $c(x) = 0$ and
$T(x) = \sin \pi x/\mathrm{L}$ with
$AE\, du/dx|\ (x{=}0) = 1/\pi$ and
$AE\, du/dx|\ (x{=}1) = kL(\delta L - u(L))$ with $k_{\mathrm{L}} = 10$ and $\delta_L = 0$. Then repeat the exercise given a) through f) in Point 3.

a) Plot For Exact Solution and finite element solution for different numbers of elements.

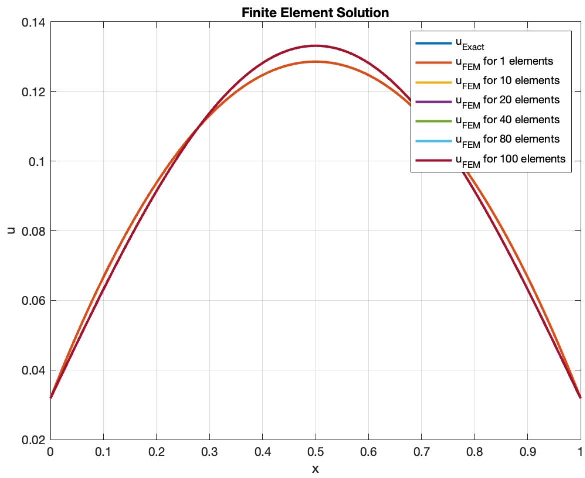
Plot Using Linear Shape Functions:



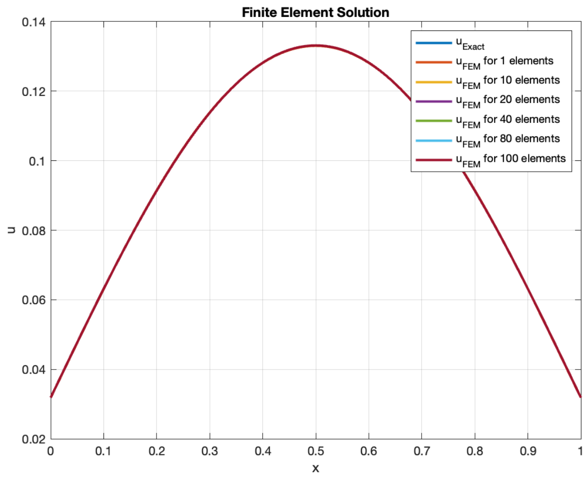
Plot Using Quadratic Shape Functions:



Plot Using Cubic Shape Functions:

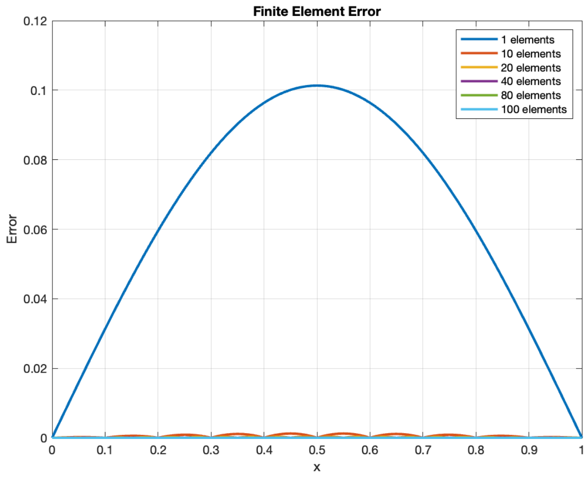


Plot Using Quartic Shape Functions:

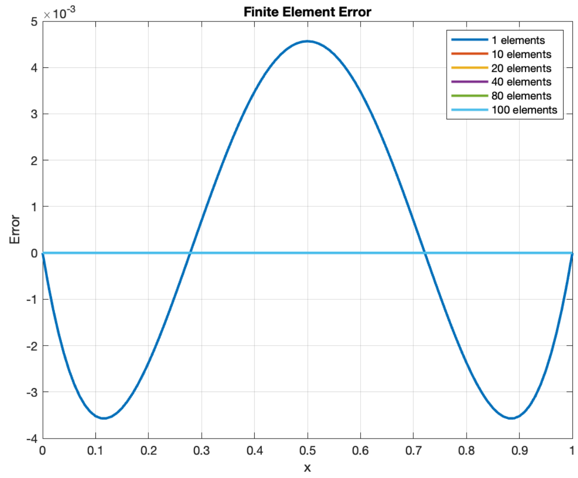


b) Error Plot for different types of element cases

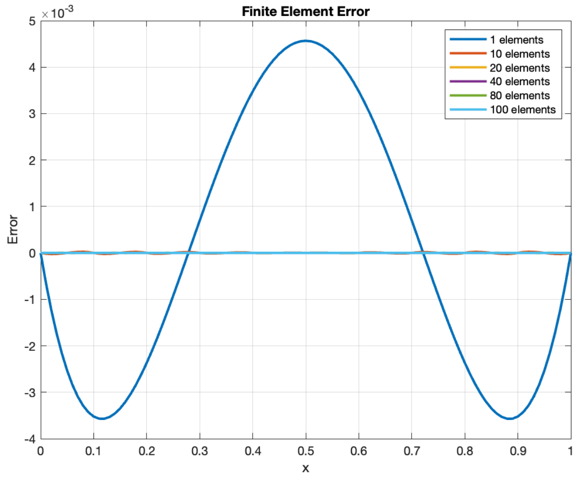
Linear Solution Error Plot:



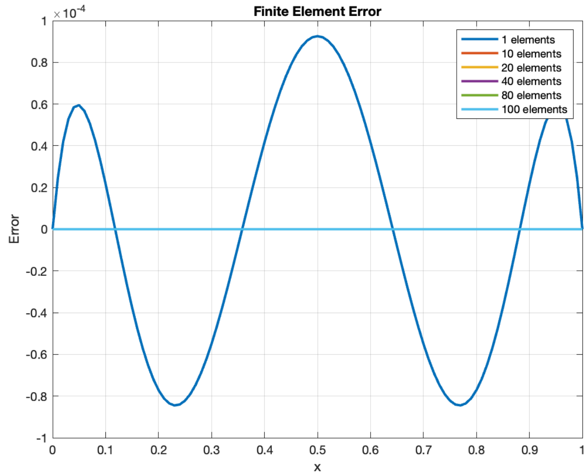
Cubic Solution Error Plot:



Quadratic Solution Error Plot:

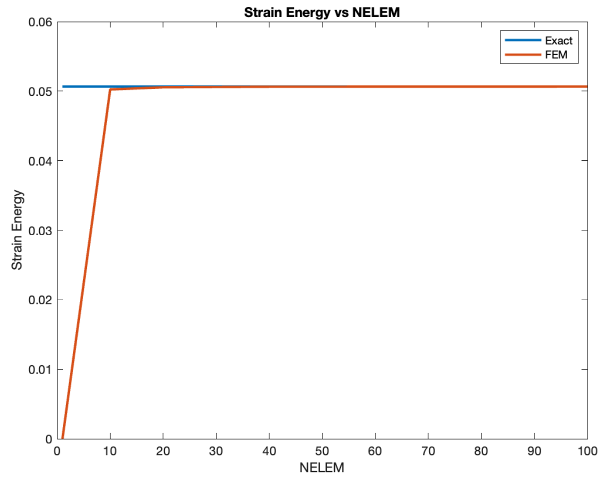


Quartic Solution Error Plot:

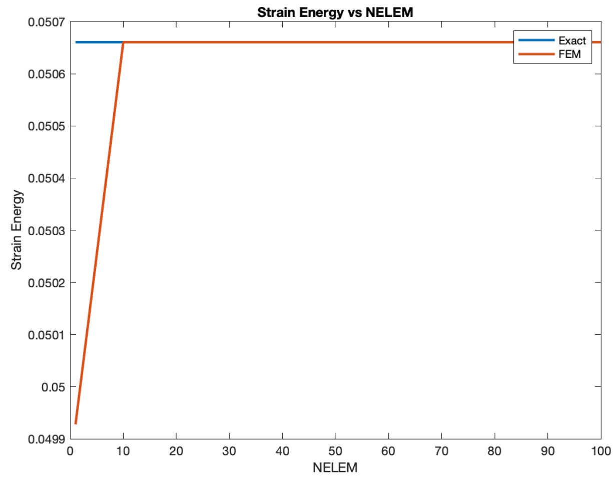


c) Plot the strain energy of the finite element and exact solution as a function of the number of elements.

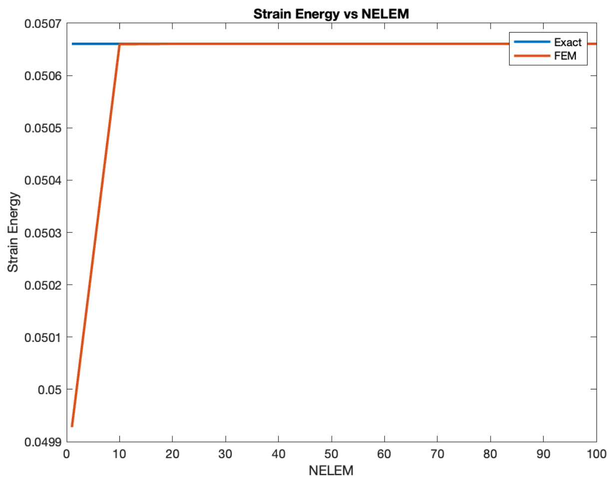
Plot Using Linear Shape Functions:



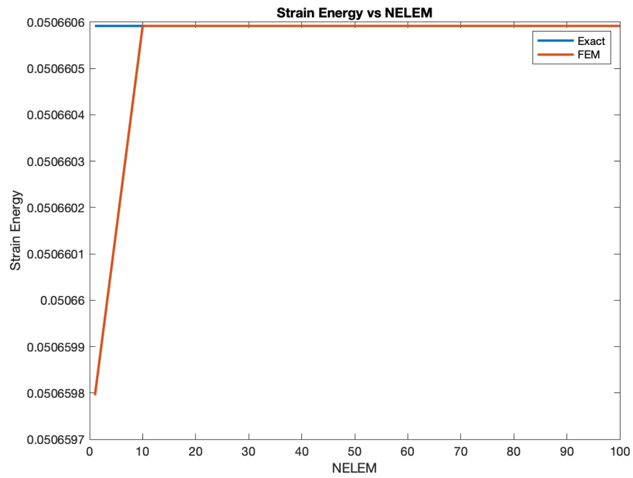
Plot Using Cubic Shape Functions:



Plot Using Quadratic Shape Functions:

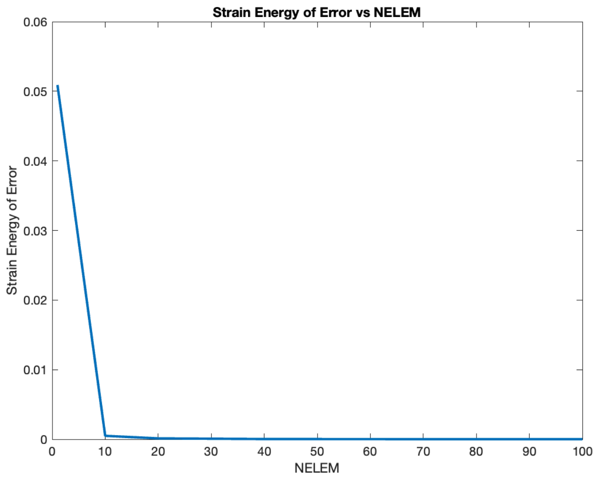


Plot Using Quartic Shape Functions:

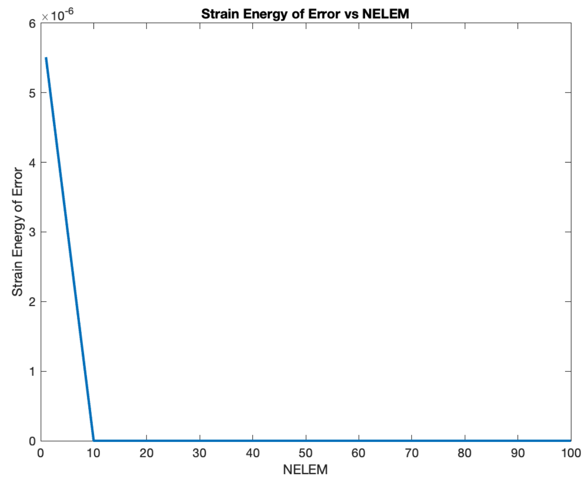


d) Plot of the Strain energy of error vs Number of elements

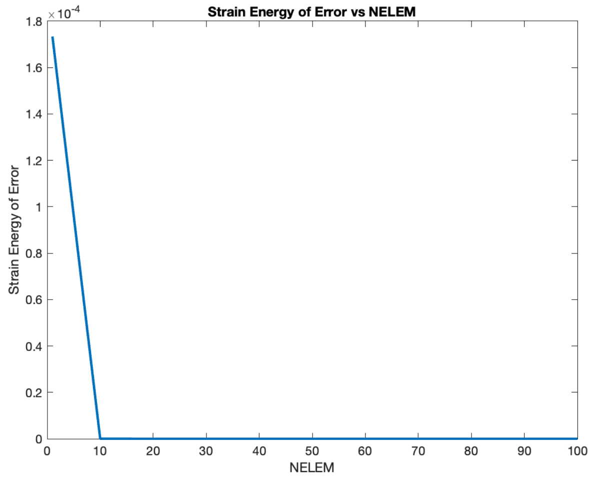
Plot Using Linear Shape Functions:



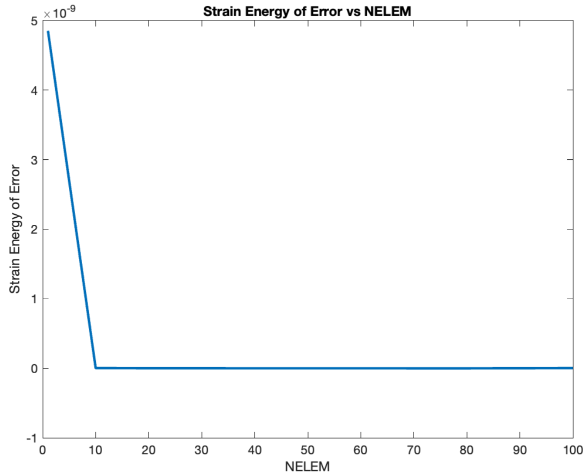
Plot Using Cubic Shape Functions:



Plot Using Quadratic Shape Functions:

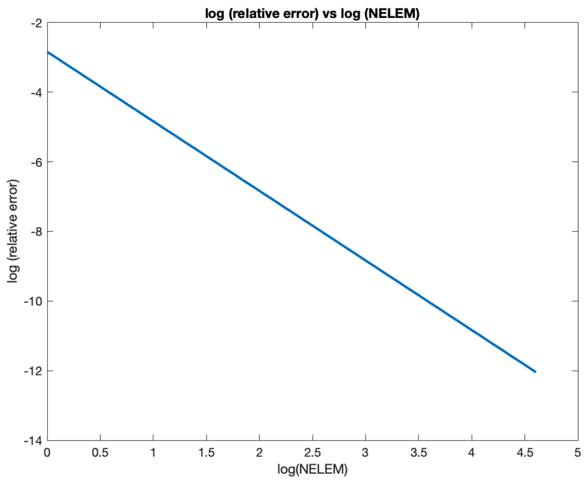


Plot Using Quartic Shape Functions:

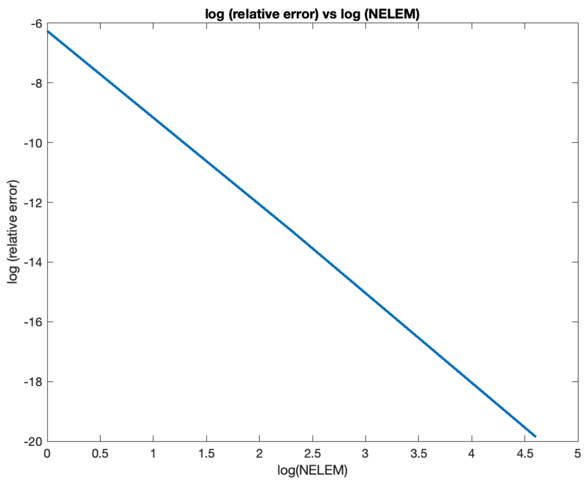


e) Plot the log of the relative error in the energy norm versus the log of number of elements.

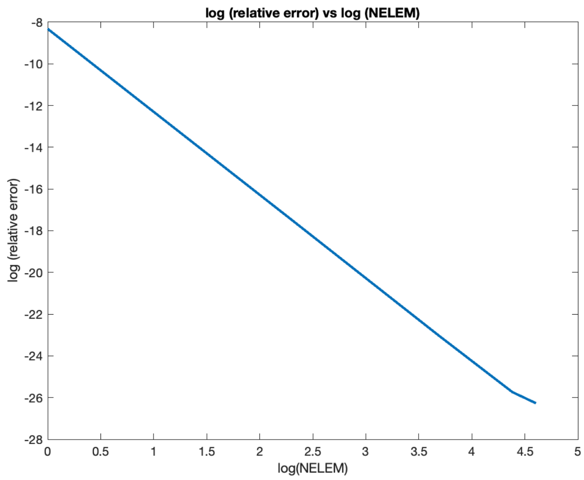
Plot Using Linear Shape Functions:



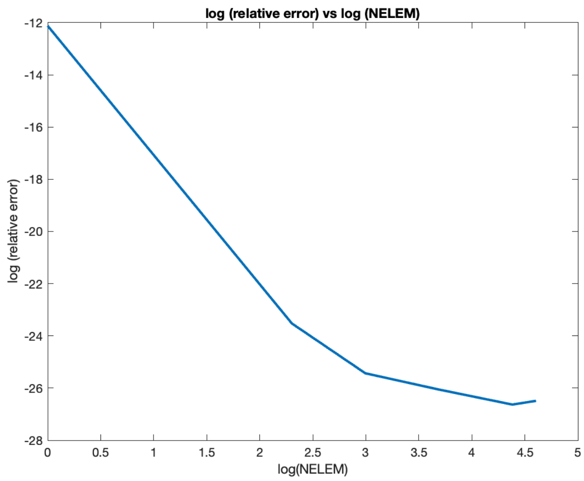
Plot Using Quadratic Shape Functions:



Plot Using Cubic Shape Functions:



Plot Using Quartic Shape Functions:



Conclusion of Problem 4

Plots for FEM solution and exact solution

The exact solution for this problem is sinusoidal, and its curve is compared with the FEM Solutions obtained using different numbers of elements and shape functions for patch testing.
With linear quadratic and cubic shape functions, a single element accurately matches the exact solution at the nodal points but introduces errors in representing the curvature or non-linearity between these points. As the number of elements increases, the FEM solution converges to the exact solution, matching it at more points within the domain and indicating improved accuracy and convergence.
Using quartic shape functions, the FEM solution matches the exact solution curve at all points. Increasing the number of elements does not change the curve of the FEM solution, indicating that quartic shape functions accurately represent the exact solution without the need for further refinement.

Plots for error between exact and FEM solution

From the plots of error between exact and FEM solution, we can conclude that error is significantly decreasing with the increase in the number of elements.
Also comparing the error plots for linear, quadratic cubic, and quartic shape functions we can say that the error significantly decreases with increasing order of shape functions with minimum error in the quartic case.

Plots for Strain energy of FEM solution and exact solution

As more elements are used in the finite element method (FEM) solution, the computed strain energy initially increases and eventually converges to the strain energy of the exact solution. This trend demonstrates that increasing the number of elements enhances the accuracy of the FEM solution, bringing it closer to the exact solution. However, after reaching a certain number of elements, further refinement of the mesh has minimal impact on the strain energy value. This indicates that the FEM solution has achieved a stable and accurate representation of the exact solution.

Plots for Strain energy of the error for FEM solution and exact solution

From this plot, we can conclude that the strain energy of error decreases with the number of elements and converges to zero.

Estimation of the convergence rate

The convergence rate increases with the number of elements as well as an increase in the degree of approximation polynomial function