Lecture 2

Divide-and-conquer, MergeSort, and Big-O notation

Announcements

- Homework!
  - HW1 will be released **Friday**.
  - It is due the following **Friday**.
- See the website for guidelines on homework, including:
  - Collaboration policy
  - Best practices/style guide
    - Will be posted by Friday!

Last time

Philosophy

- Algorithms are awesome and powerful!
- Algorithm designer's question: Can I do better?

Technical content

- Karatsuba integer multiplication
- Example of "Divide and Conquer"
- Not-so-rigorous analysis

Cast

Plucky the pedantic penguin

Lucky the lackadaisical lemur

Ollie the over-achieving ostrich

Siggi the studious stork

Today

- Things we want to know about algorithms:
  - Does it work?
  - Is it efficient?

- We'll start to see how to answer these by looking at some examples of sorting algorithms.
  - InsertionSort
  - MergeSort

SortingHatSort not discussed

The plan

- Part I: Sorting Algorithms
  - InsertionSort: does it work and is it fast?
  - MergeSort: does it work and is it fast?
  - **Skills:**
    - Analyzing correctness of iterative and recursive algorithms.
    - Analyzing running time of recursive algorithms (part 1…more next time!)

- Part II: How do we measure the runtime of an algorithm?
  - Worst-case analysis
  - Asymptotic Analysis

Sorting

- Important primitive
- **For today**, we'll pretend all elements are distinct.

6	4	3	8	1	5	2	7

1	2	3	4	5	6	7	8

I hope everyone did the pre-lecture exercise!

What was the mystery sort algorithm?

1. MergeSort
2. QuickSort
3. InsertionSort
4. BogoSort

```
def mysteryAlgorithmOne(A):
    B = [None for i in range(len(A))]
    for i in range(len(B)):
        if B[i] == None or B[i] > x:
            j = len(B)-1
            while j > i:
                B[j] = B[j-1]
                j -= 1
            B[i] = x
            break
    return B
```

```
def MysteryAlgorithmTwo(A):
    for i in range(1,len(A)):
        current = A[i]
        j = i-1
        while j >= 0 and A[j] > current:
            A[j+1] = A[j]
            j -= 1
        A[j+1] = current
```

Benchmark: insertion sort

We're going to go through this in some detail – it's good practice!

- Say we want to sort:

6	4	3	8	5

- Insert items one at a time.

- How would we actually implement this?

In your pre-lecture exercise…

```
def InsertionSort(A):
   for i in range(1,len(A)):
        current = A[i]
        j = i-1
        while j >= 0 and A[j] > current:
            A[j+1] = A[j]
            j -= 1
        A[j+1] = current
```

InsertionSort

example

6	4	3	8	5

Start by moving A[1] toward the beginning of the list until you find something smaller (or can't go any further):

6	4	3	8	5
4	6	3	8	5

Then move A[2]:

4	6	3	8	5
3	4	6	8	5

Then move A[3]:

3	4	6	8	5
3	4	6	8	5

Then move A[4]:

3	4	6	8	5
3	4	5	6	8

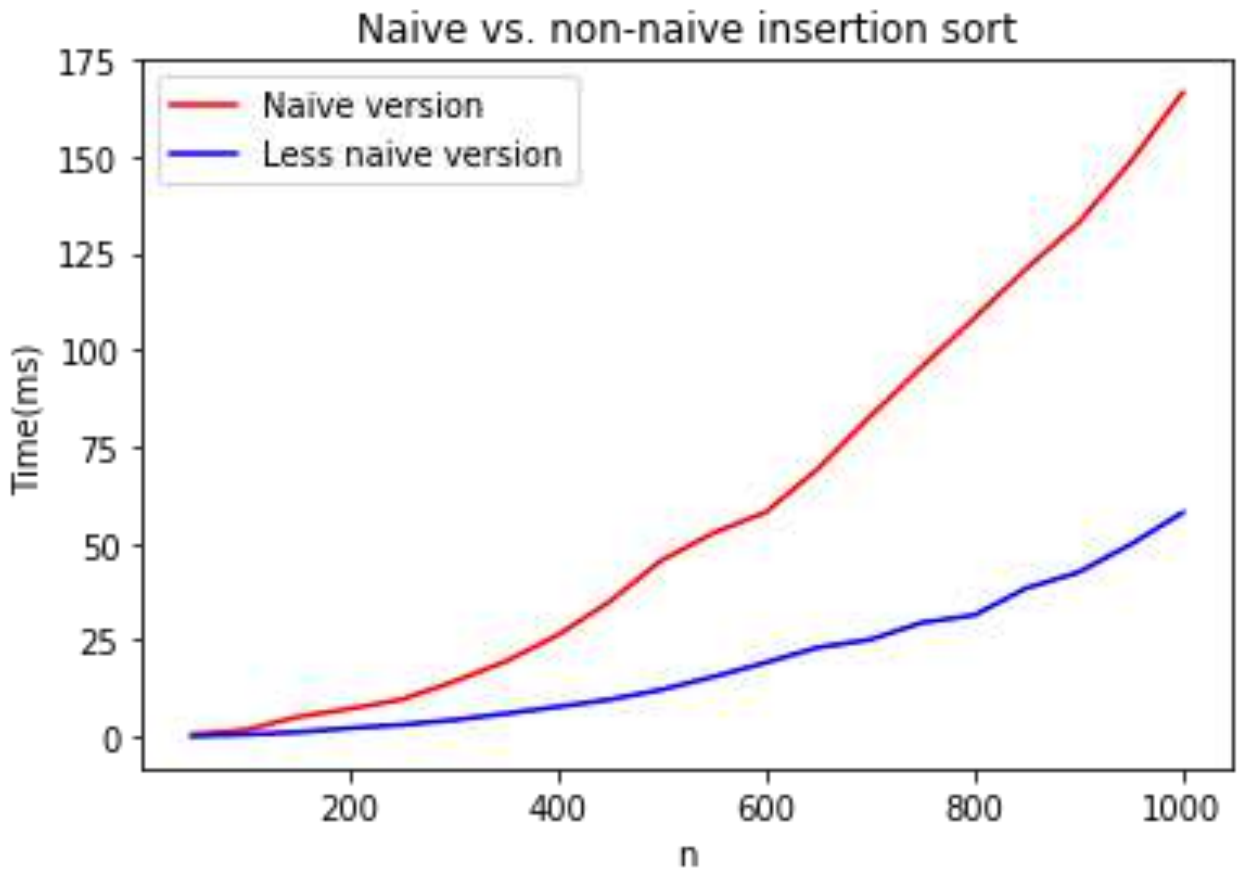
Then we are done!

Insertion Sort

1. Does it work?
2. Is it fast?

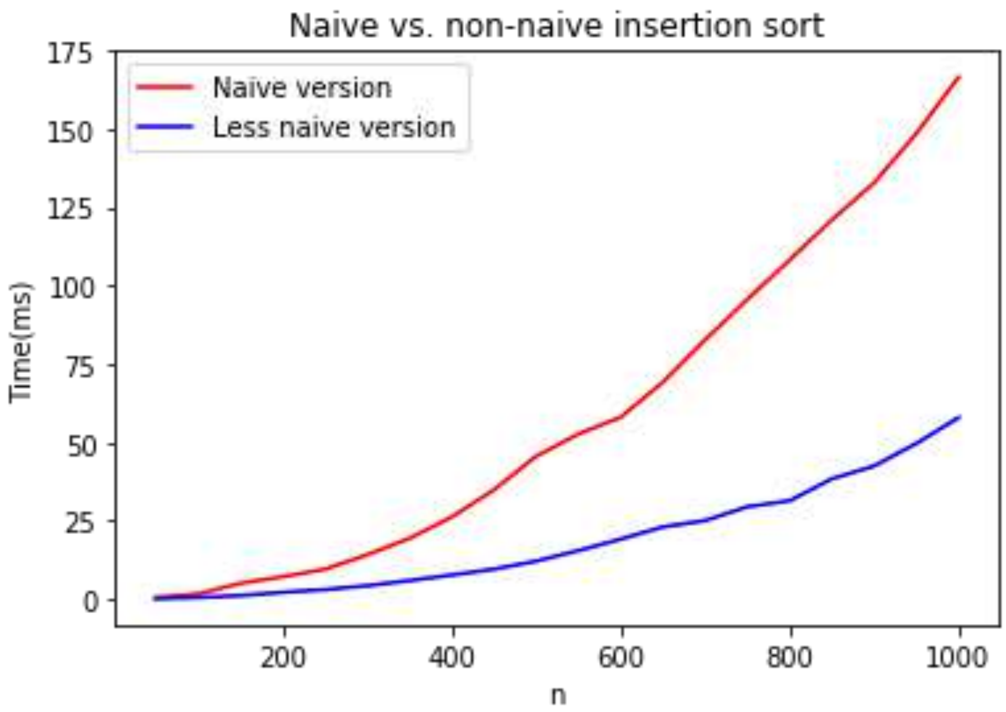
Empirical answers…

- Does it work?
  - You saw it worked on the pre-Lecture exercise.
- Is it fast?
  - IPython notebook lecture2_sorting.ipynb says:



Insertion Sort

1. Does it work?
2. Is it fast?



- The "same" algorithm can be faster or slower depending on the implementation…
- We are interested in how fast the running time scales with n, the size of the input.

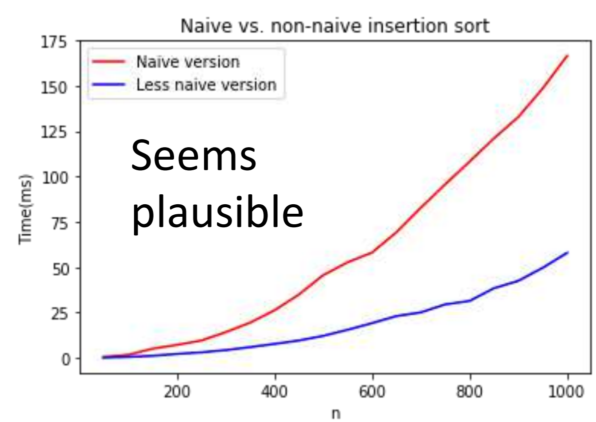
Insertion Sort: running time

```
def InsertionSort(A):
    for i in range(1,len(A)):
        current = A[i]
        j = i-1
        while j >= 0 and A[j] > current:
            A[j+1] = A[j]
            j -= 1
        A[j+1] = current
```

n-1 iterations of the outer loop

In the worst case, about n iterations of this inner loop

Running time scales like n^2



Seems plausible

Insertion Sort

1. Does it work?
2. Is it fast?

- Okay, so it's pretty obvious that it works.

- HOWEVER! In the future it won't be so obvious, so let's take some time now to see how we would prove this rigorously.

Why does this work?

- Say you have a sorted list, | 3 | 4 | 6 | 8 |, and another element | 5 |.
- Insert | 5 | right after the largest thing that's still smaller than | 5 |. (Aka, right after | 4 |).
- Then you get a sorted list: | 3 | 4 | 5 | 6 | 8 |

So just use this logic at every step.

6 4 3 8 5

The first element, [6], makes up a sorted list.

4 6 3 8 5

So correctly inserting 4 into the list [6] means that [4,6] becomes a sorted list.

---

4 6 3 8 5

The first two elements, [4,6], make up a sorted list.

3 4 6 8 5

So correctly inserting 3 into the list [4,6] means that [3,4,6] becomes a sorted list.

---

3 4 6 8 5

The first three elements, [3,4,6], make up a sorted list.

3 4 6 8 5

So correctly inserting 8 into the list [3,4,6] means that [3,4,6,8] becomes a sorted list.

---

3 4 6 8 5

The first four elements, [3,4,6,8], make up a sorted list.

3 4 5 6 8

So correctly inserting 5 into the list [3,4,6,8] means that [3,4,5,6,8] becomes a sorted list.

YAY WE ARE DONE!

This slide skipped in class; for reference only.

Recall: proof by induction

A loop invariant is something that should be true at every iteration.

- Maintain a loop invariant.
- Proceed by induction.

- **Four steps in the proof by induction:**
  - Inductive Hypothesis: The loop invariant holds after the i^{th} iteration.
  - Base case: the loop invariant holds before the 1^{st} iteration.
  - Inductive step: If the loop invariant holds after the i^{th} iteration, then it holds after the $(i+1)^{st}$ iteration
  - Conclusion: If the loop invariant holds after the last iteration, then we win.

Formally: induction

A "loop invariant" is something that we maintain at every iteration of the algorithm.

- Loop invariant(i): `A[:i+1]` is sorted.
- Inductive Hypothesis:
  - The loop invariant(i) holds at the end of the i^{th} iteration (of the outer loop).
- Base case (i=0):
  - Before the algorithm starts, `A[:1]` is sorted. ✓
- Inductive step:
- Conclusion:
  - At the end of the n-1'st iteration (aka, at the end of the algorithm), `A[:n] = A` is sorted.
  - That's what we wanted! ✓

This logic (see Lecture Notes for details)

4	6	3	8	5

The first two elements, [4,6], make up a sorted list.

This was iteration i=2.

3	4	6	8	5

So correctly inserting 3 into the list [4,6] means that [3,4,6] becomes a sorted list.

Aside: proofs by induction

- We're gonna see/do/skip over a lot of them.
- I'm assuming you're comfortable with them from CS103.
  - When you assume…
- If that went by too fast and was confusing:
  - Slides [there's a hidden one with more info]
  - Lecture notes
  - Book
  - Office Hours

Make sure you really understand the argument on the previous slide!

Siggi the Studious Stork

To summarize

InsertionSort is an algorithm that correctly sorts an arbitrary n-element array in time that scales like n^2.

Can we do better?

The plan

- Part I: Sorting Algorithms
  - InsertionSort: does it work and is it fast?
  - MergeSort: does it work and is it fast?

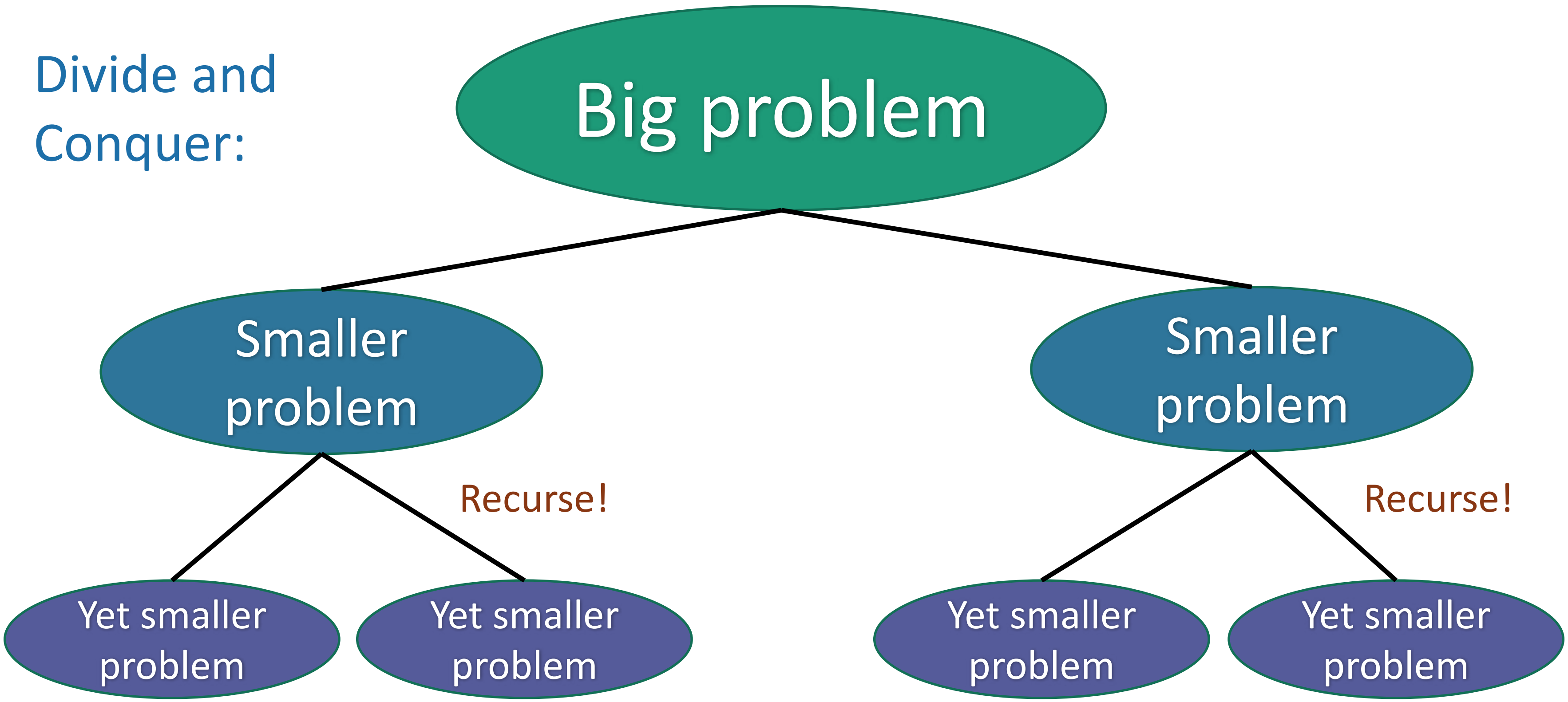
  - **Skills:**
    - Analyzing correctness of iterative and recursive algorithms.
    - Analyzing running time of recursive algorithms (part A)

- Part II: How do we measure the runtime of an algorithm?
  - Worst-case analysis
  - Asymptotic Analysis

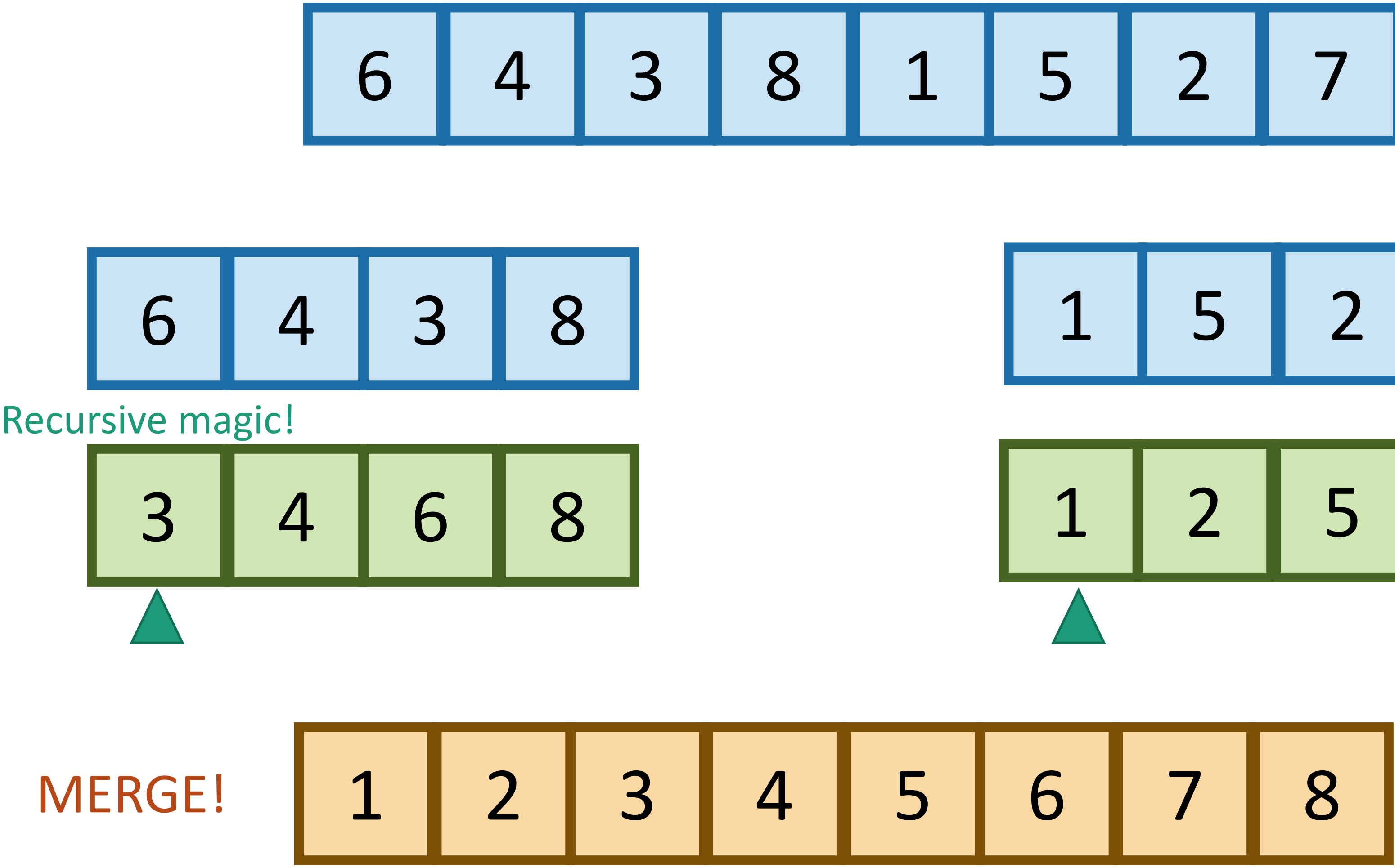
Can we do better?

- MergeSort: a divide-and-conquer approach
- Recall from last time:

Divide and Conquer:



MergeSort



How would you do this in-place?

Ollie the over-achieving Ostrich

Code for the MERGE step is given in the Lecture2 notebook or the Lecture Notes

See Lecture 2 IPython notebook for MergeSort Python Code.

MergeSort Pseudocode

MERGESORT(A):

- n = length(A)
- **if** $n \leq 1$:
  - **return** A

If A has length 1,
It is already sorted!

- L = MERGESORT(A[0 : n/2])

Sort the left half

- R = MERGESORT(A[n/2 : n])

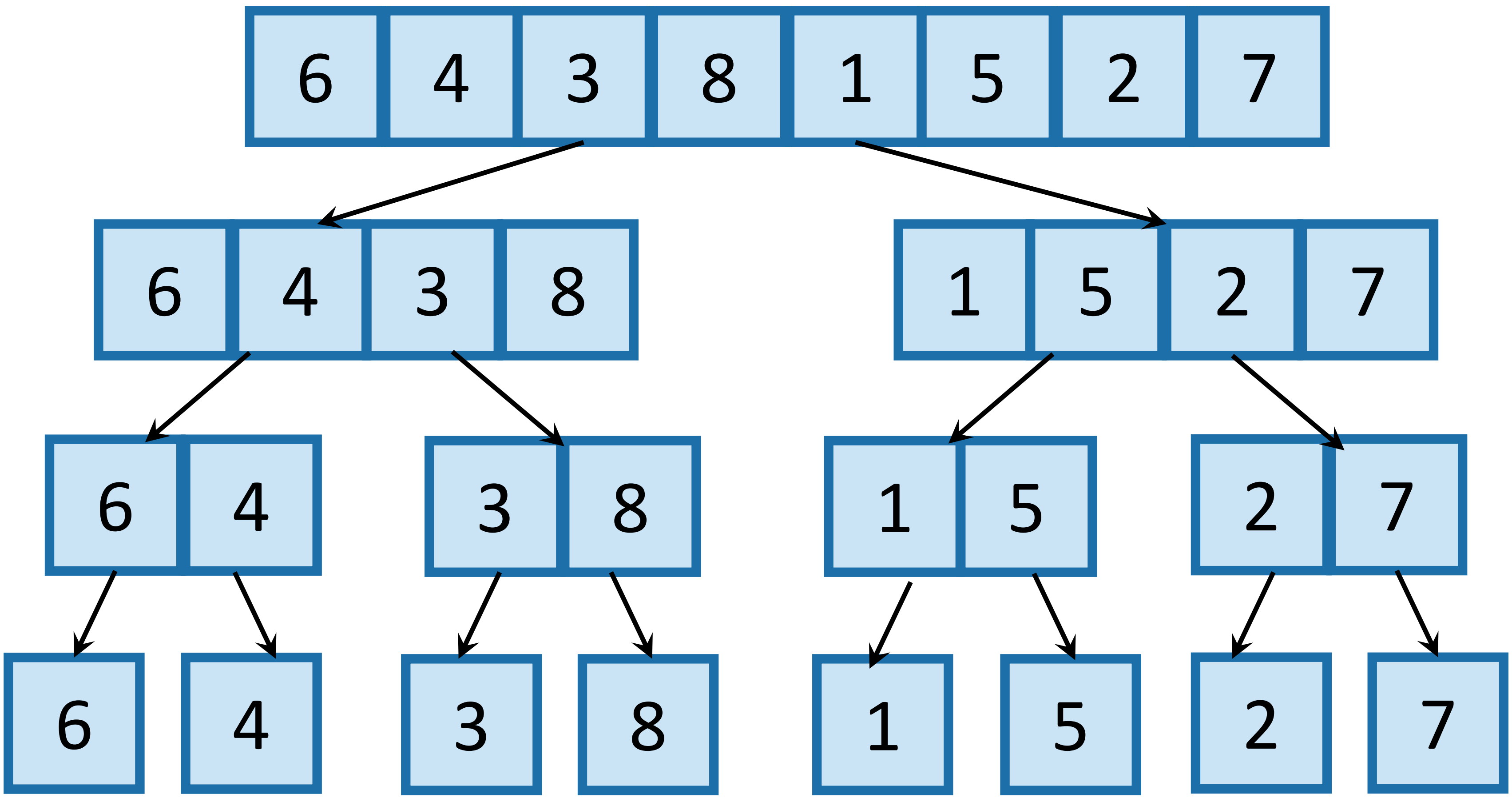
Sort the right half

- **return** MERGE(L,R)

Merge the two halves

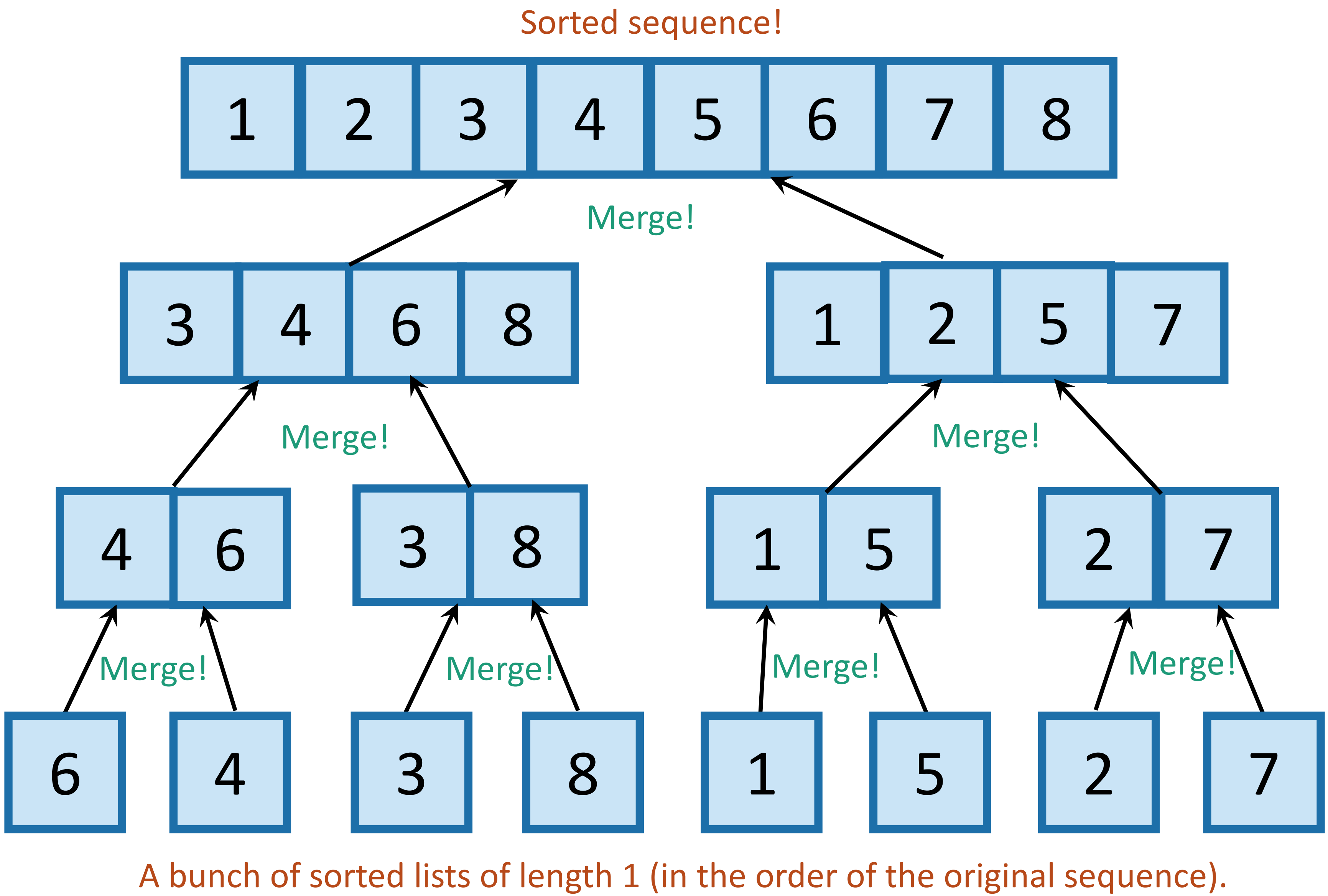
What actually happens?

First, recursively break up the array all the way down to the base cases



This array of length 1 is sorted!

Then, merge them all back up!



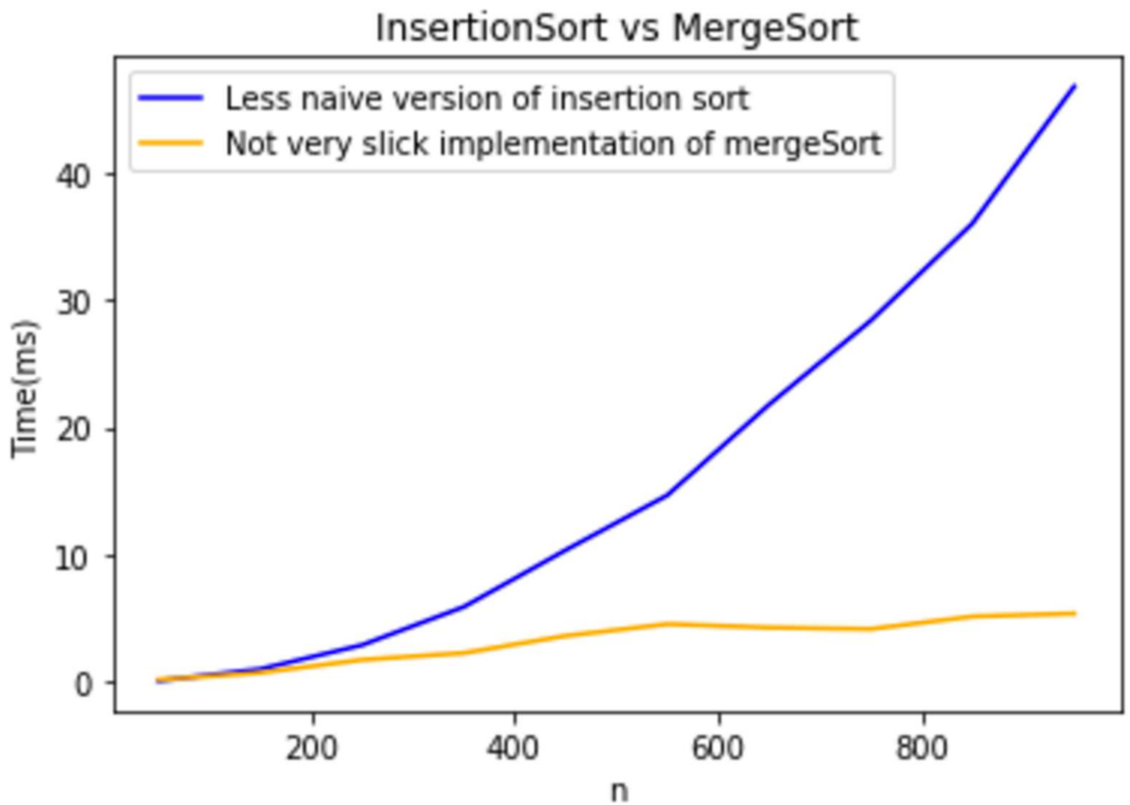
Two questions

1. Does this work?
2. Is it fast?

Empirically:

1. **Seems to.**
2. **Maybe?**

IPython notebook says…



It works

Let's assume $n = 2^t$

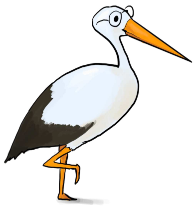
Again we'll use induction.
This time with an invariant
that will remain true after
every recursive call.

- Inductive hypothesis:

"In every recursive call,
MERGESORT returns a sorted array."

- Base case (n=1): a 1-element array is always sorted.
- Inductive step: Suppose that L and R are sorted. Then MERGE(L,R) is sorted.
- Conclusion: "In the top recursive call, MERGESORT returns a sorted array."

- n = length(A)
- **if** $n \leq 1$:
    - **return** A
- L = MERGESORT(A[1 : n/2])
- R = MERGESORT(A[n/2+1 : n])
- **return** MERGE(L,R)



Fill in the inductive step! (Either do it yourself or read it in CLRS!)

It's fast Let's keep assuming $n = 2^t$

<u>CLAIM</u>:

MERGESORT requires at most $11n (\log(n) + 1)$ operations to sort n numbers.

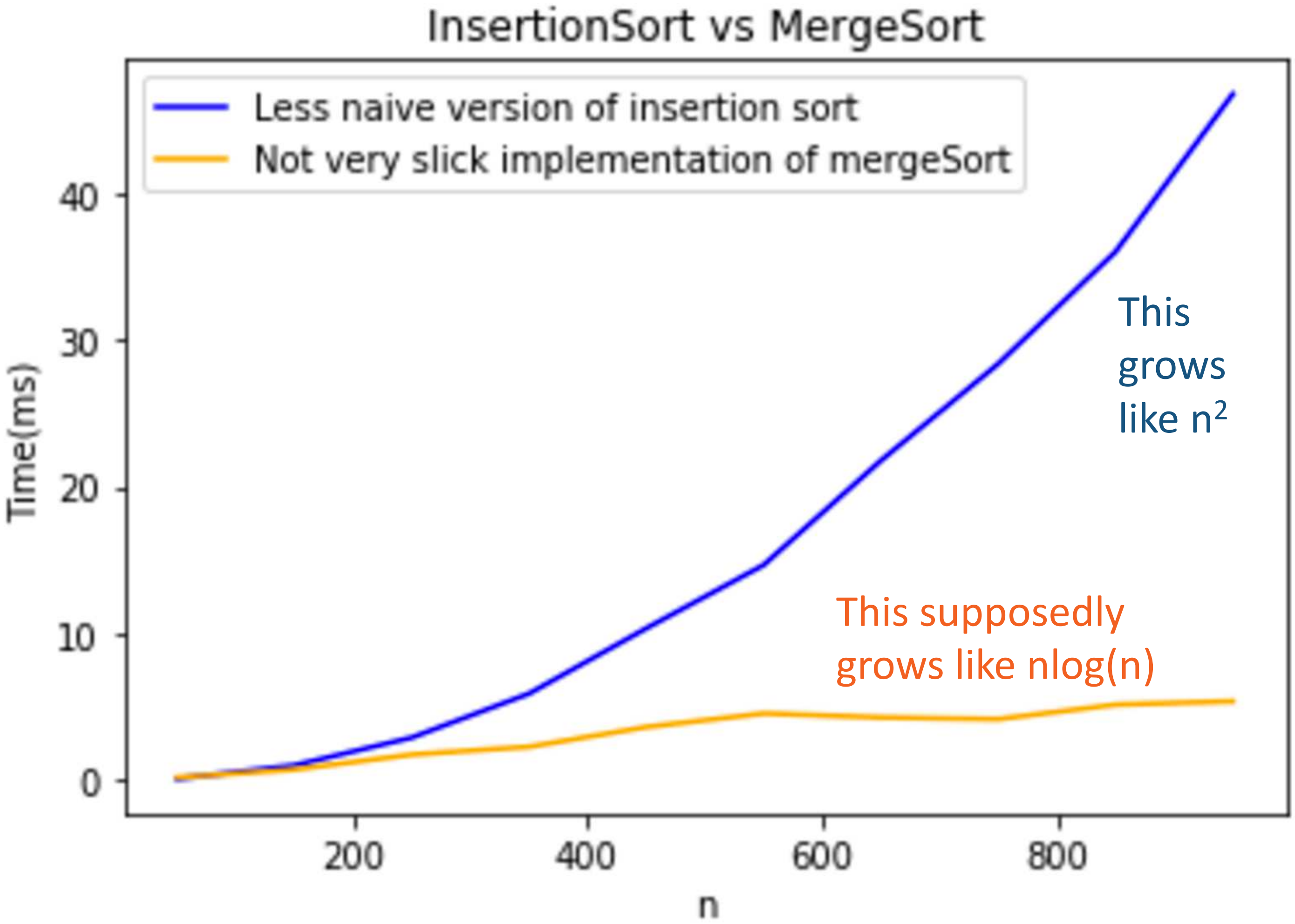
What exactly is an "operation" here?
We're leaving that vague on purpose.
Also I made up the number 11.

How does this compare to InsertionSort?

Scaling like n^2 vs scaling like $n\log(n)$?

[See Lecture 2 Notebook for code]

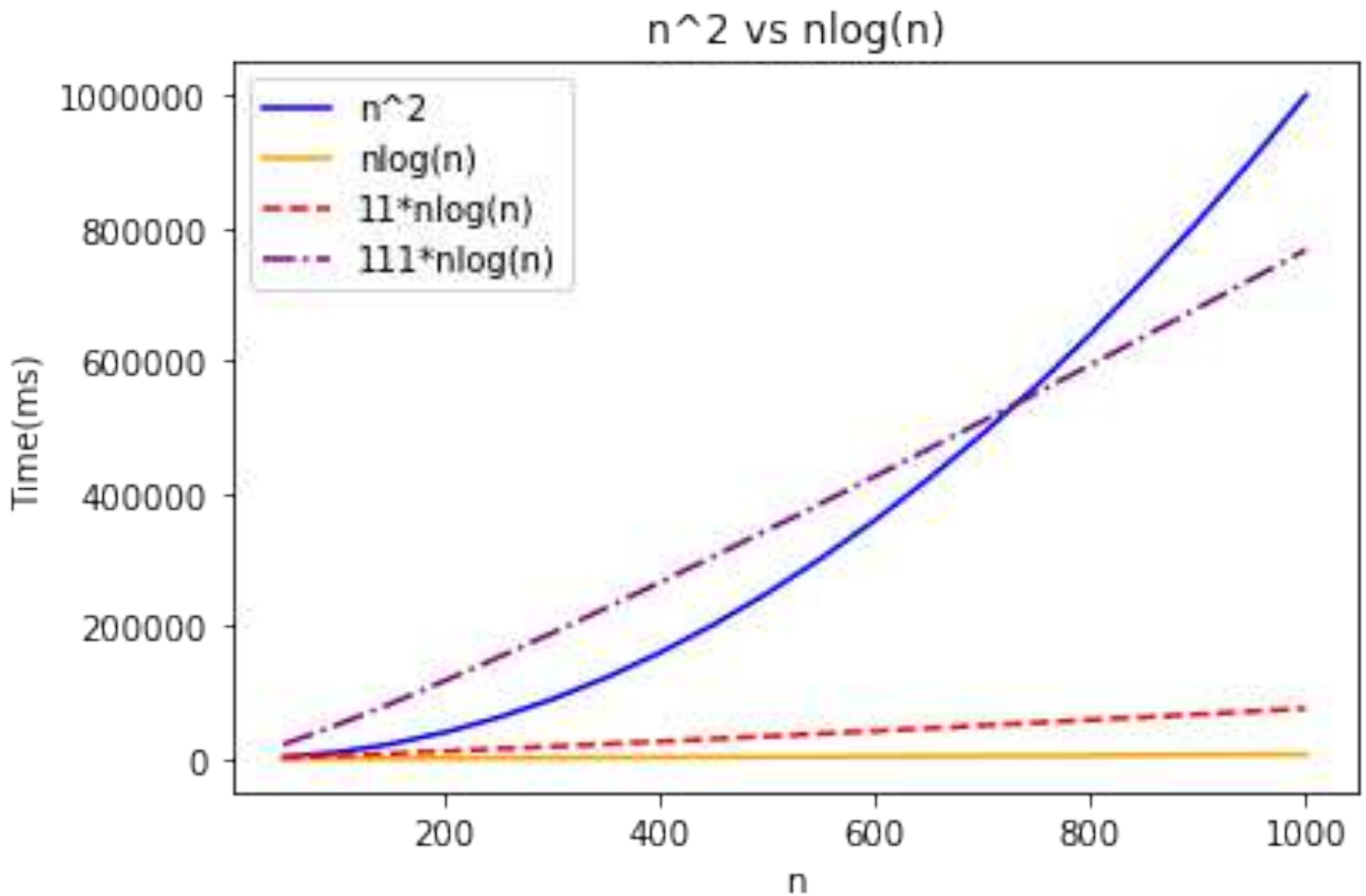
Empirically



This grows like n^2

This supposedly grows like nlog(n)

The constant doesn't matter:
eventually, $n^2 > 111111 \cdot n \log(n)$



Quick log refresher

All logarithms in this course are base 2

- log(n) : how many times do you need to divide n by 2 in order to get down to 1?

32
16
8
4
2
1

log(32) = 5

64
32
16
8
4
2
1

log(64) = 6

log(128) = 7
log(256) = 8
log(512) = 9
.
.
.
log(**number of particles in the universe**) < 280

Moral: log(n) grows very slowly with n.

It's fast!

CLAIM:

MERGESORT requires at most $11n(\log(n) + 1)$ operations to sort n numbers.

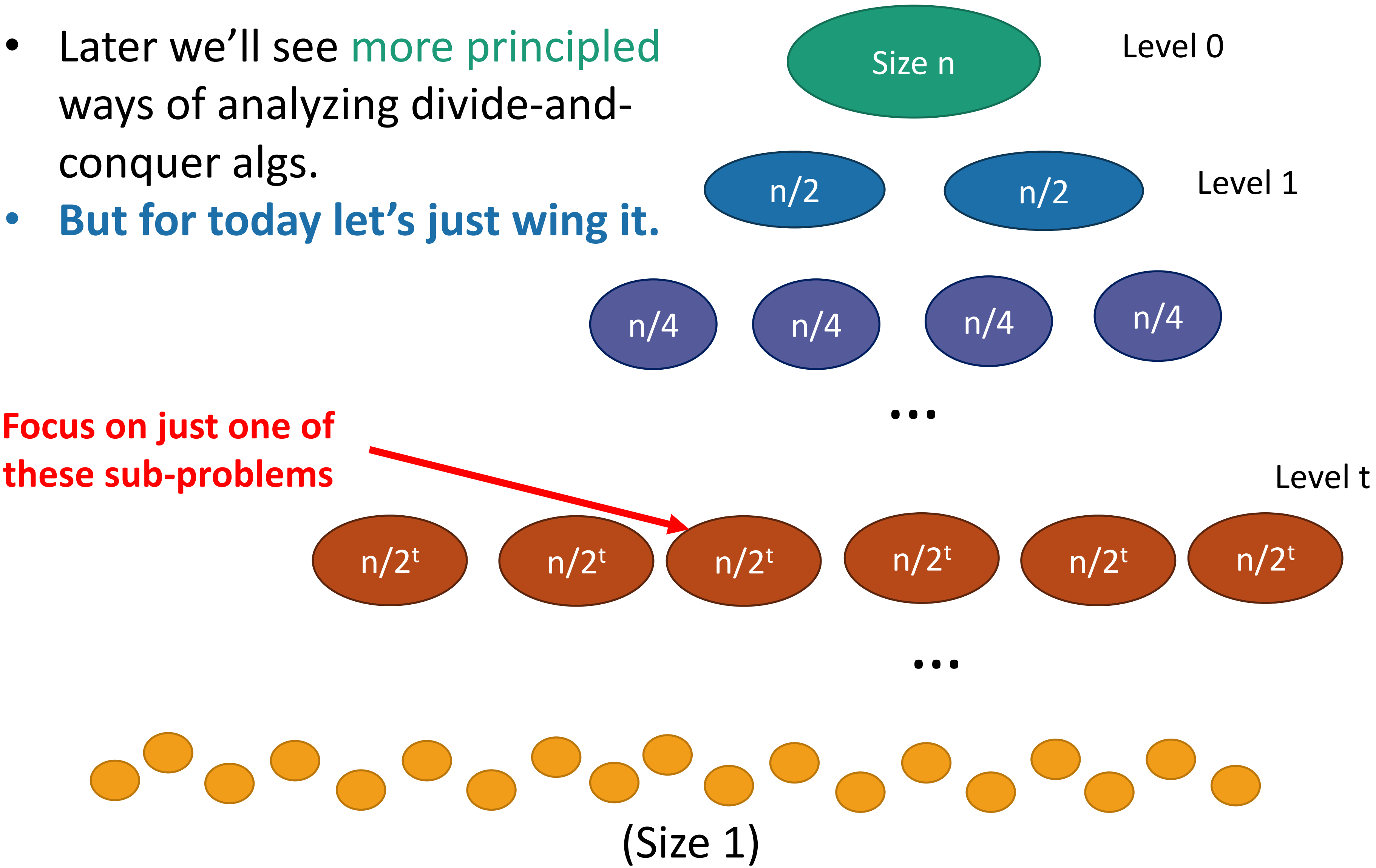
Much faster than InsertionSort for large n!

(No matter how the algorithms are implemented).

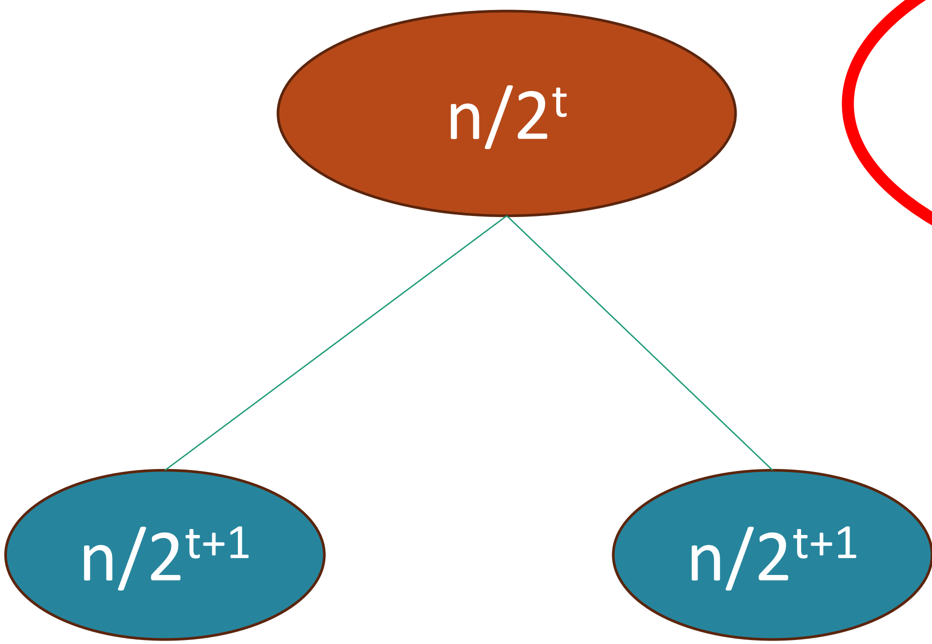
(And no matter what that constant "11" is).

Let’s prove the claim

- Later we’ll see more principled ways of analyzing divide-and-conquer algs.
- **But for today let’s just wing it.**



How much work in this sub-problem?



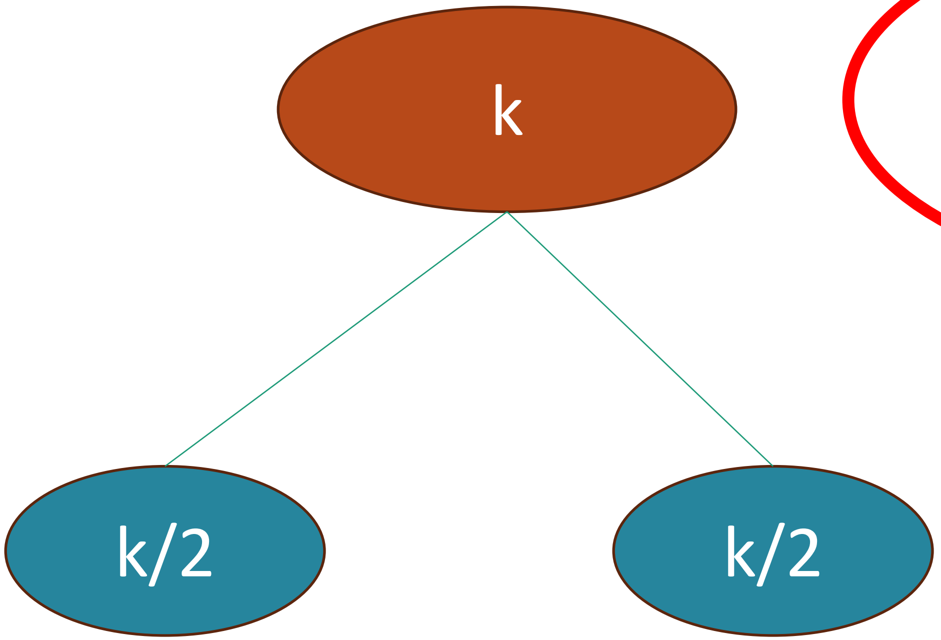
Time spent MERGE-ing the two subproblems

+

Time spent within the two sub-problems

How much work in this sub-problem?

Let $k=n/2^t$...

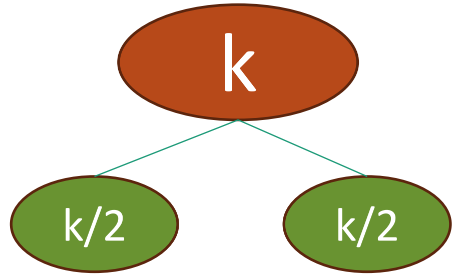


Time spent MERGE-ing the two subproblems

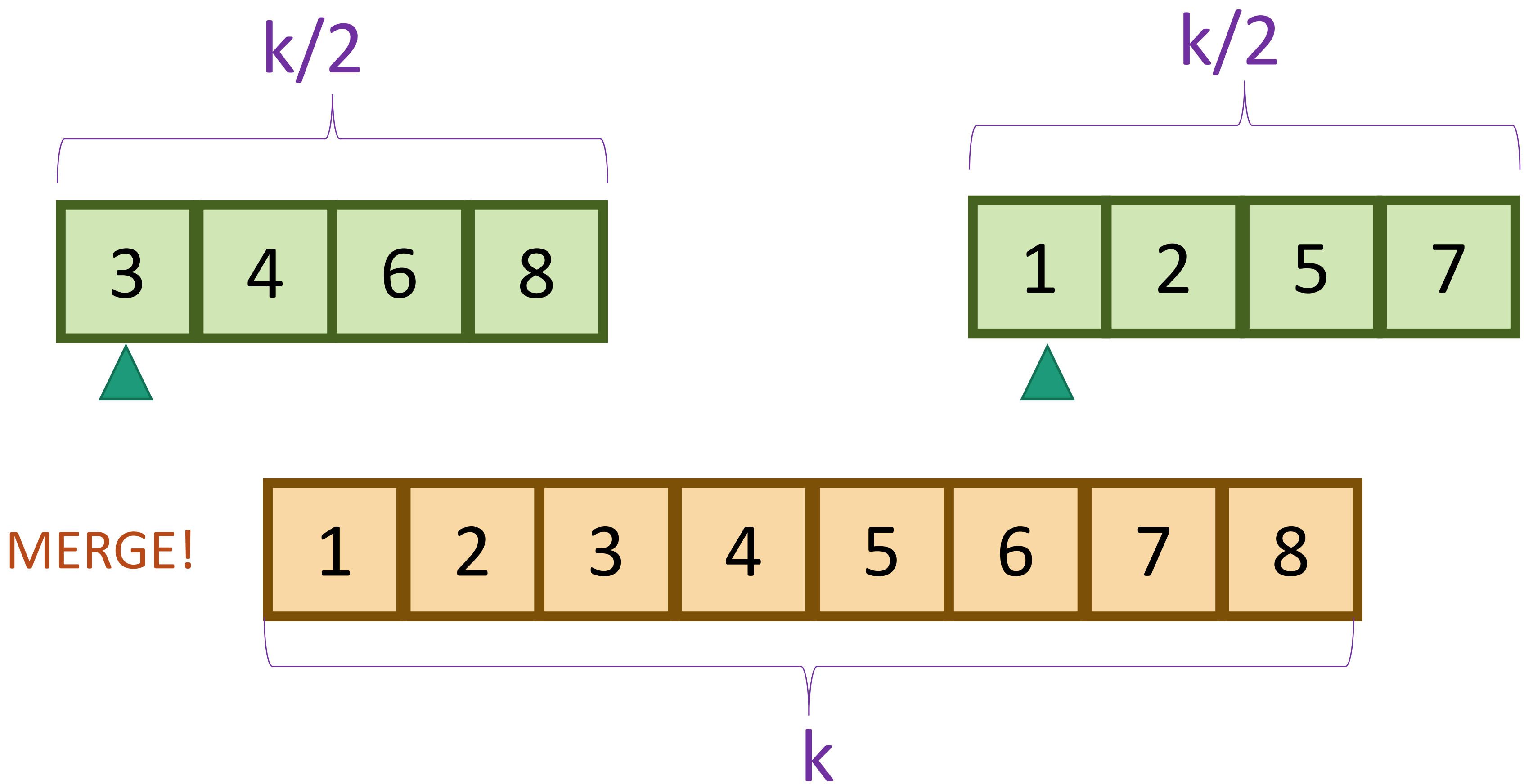
+

Time spent within the two sub-problems

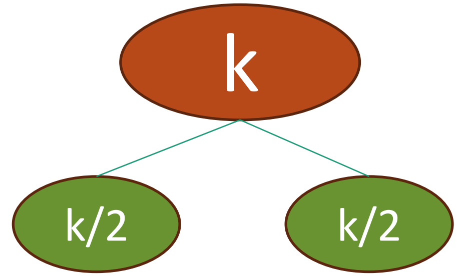
How long does it take to MERGE?



Code for the MERGE step is given in the Lecture2 notebook.



How long does it take to MERGE?



Code for the MERGE step is given in the Lecture2 notebook.

- Time to initialize an array of size k
- Plus the time to initialize three counters
- Plus the time to increment two of those counters k/2 times each
- Plus the time to compare two values at least k times
- Plus the time to copy k values from the existing array to the big array.
- Plus…

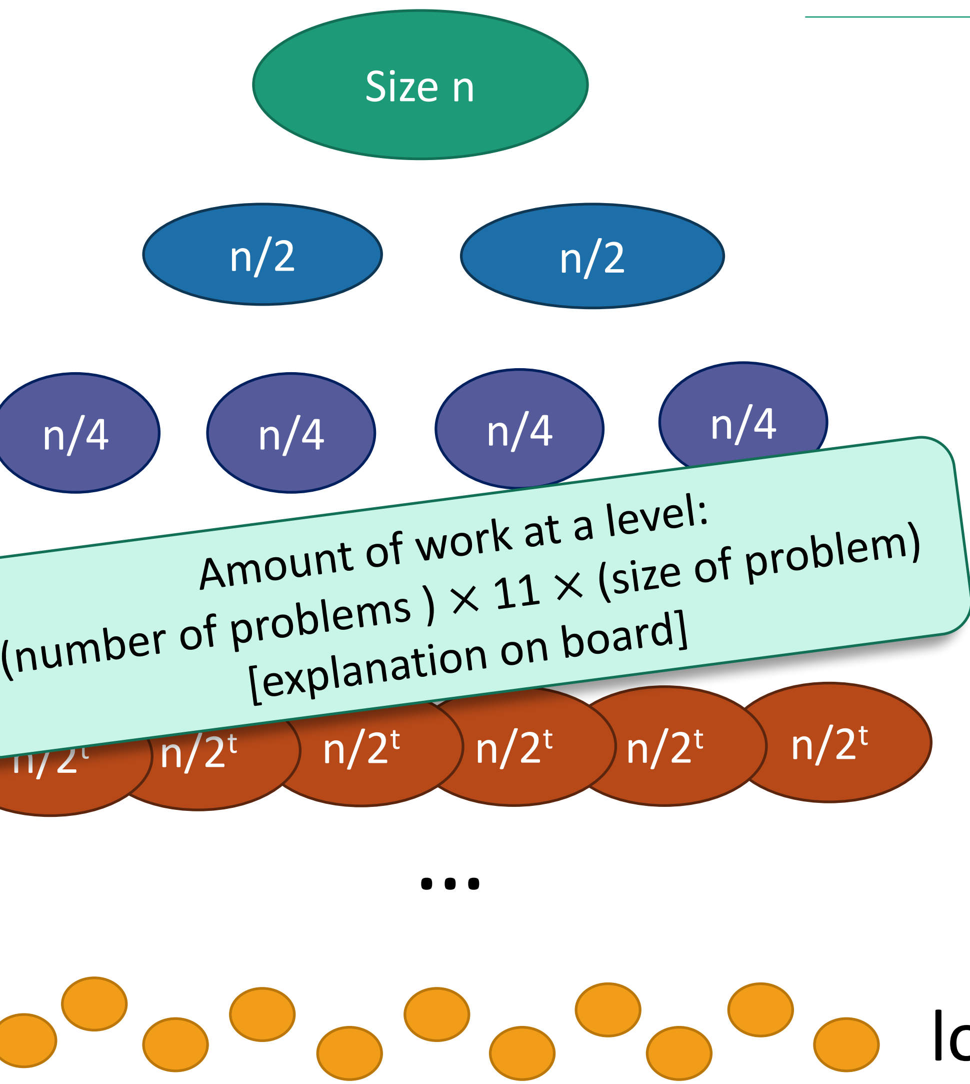
Plucky the Pedantic Penguin

Let's say no more than 11k operations.

There's some justification for this number "11" in the lecture notes, but it's really pretty arbitrary.

Lucky the lackadaisical lemur

Recursion tree



(Size 1)

Level	# problems	Size of each problem	Amount of work at this level
0	1	n	11n
1	2	n/2	11n
2	4	n/4	11n
…			
t	2^t	$n/2^t$	11n
…			
log(n)	n	1	11n

Total runtime…

- 11n steps per level, at every level
- log(n) + 1 levels
- 11n (log(n) + 1) steps total

That was the claim!

A few reasons to be grumpy

- Sorting

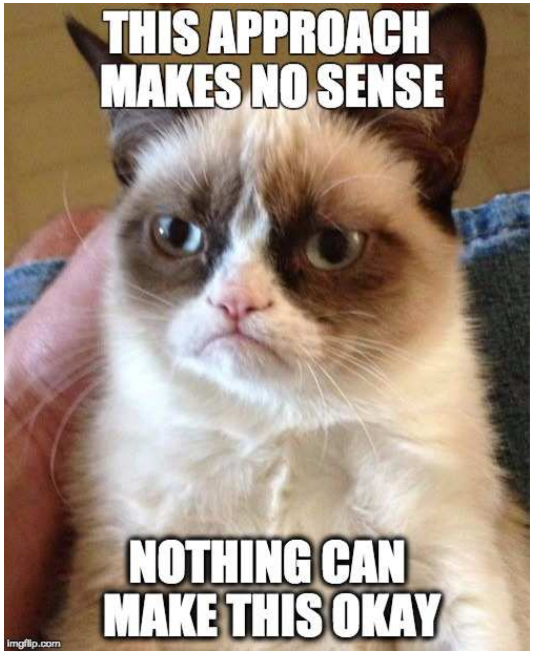
1	2	3	4	5	6	7	8

should take zero steps…

- What's with this 11k bound?
  - You (Mary) made that number "11" up.
  - Different operations don't take the same amount of time.

How we will deal with **grumpiness**

- Take a deep breath…
- Worst case analysis
- Asymptotic notation



The plan

- Part I: Sorting Algorithms
  - InsertionSort: does it work and is it fast?
  - MergeSort: does it work and is it fast?
  - **Skills:**
    - Analyzing correctness of iterative and recursive algorithms.
    - Analyzing running time of recursive algorithms (part A)
- Part II: How do we measure the runtime of an algorithm?
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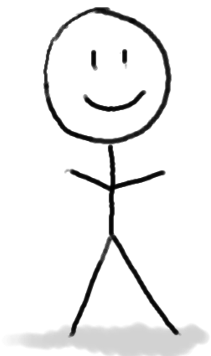
  - Asymptotic Analysis

Worst-case analysis

Sorting a sorted list should be fast!!

1	2	3	4	5	6	7	8

- In this class, we will focus on worst-case analysis



Here is my algorithm!

```
Algorithm:
   Do the thing
   Do the stuff
   Return the answer
```

Algorithm designer

HERE IS AN INPUT!

- Pros: very strong guarantee
- Cons: very strong guarantee

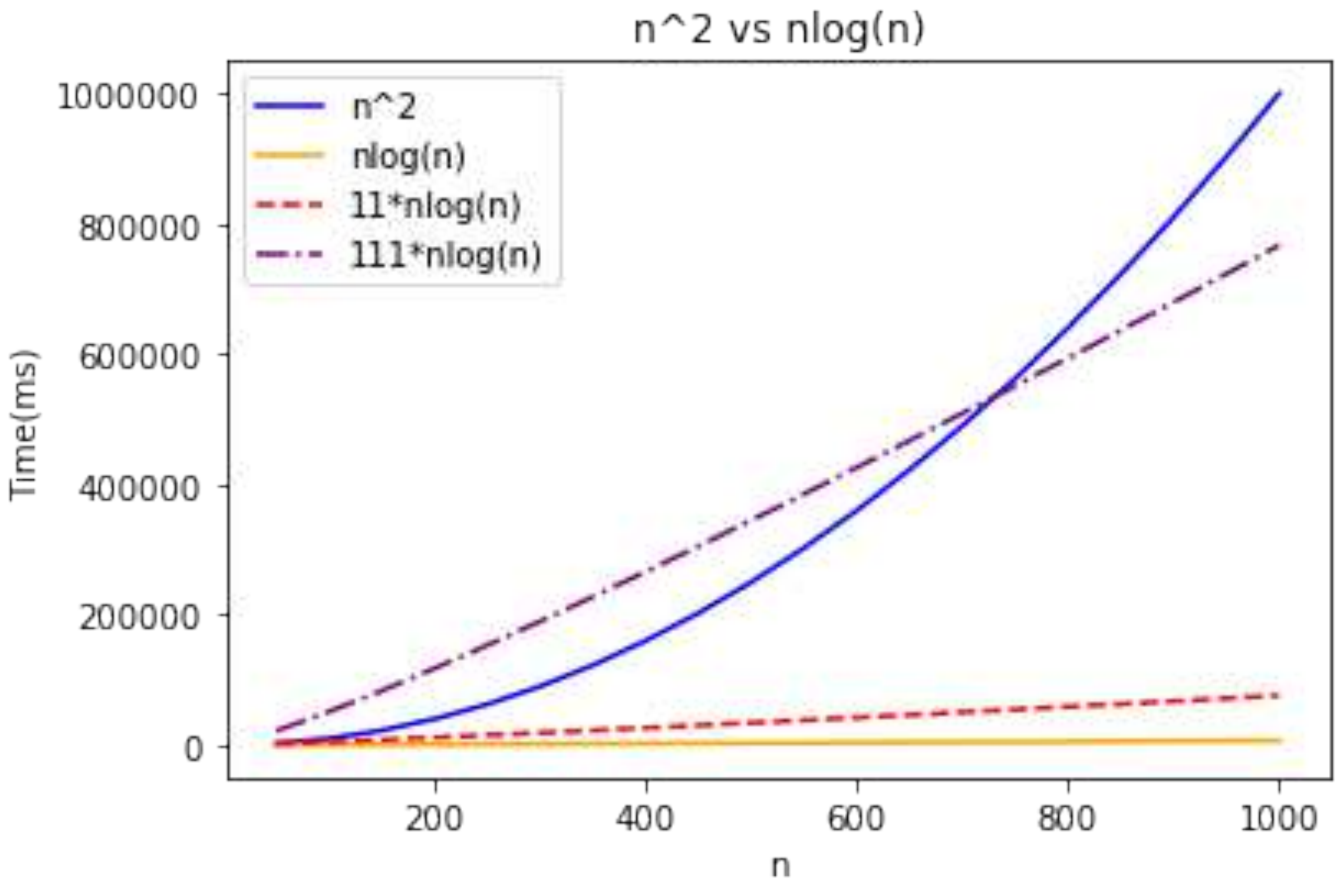
Big-O notation

How long does an operation take? Why are we being so sloppy about that "11"?

- What do we mean when we measure runtime?
  - We probably care about wall time: how long does it take to solve the problem, in seconds or minutes or hours?
- This is heavily dependent on the programming language, architecture, etc.
- These things are very important, but are not the point of this class.
- We want a way to talk about the running time of an algorithm, independent of these considerations.

Main idea:

Focus on how the runtime **scales** with n (the input size).



Asymptotic Analysis

How does the running time scale as n gets large?

One algorithm is "faster" than another if its runtime scales better with the size of the input.

Pros:

- Abstracts away from hardware- and language-specific issues.
- Makes algorithm analysis much more tractable.

Without making Plucky grumpy!

Cons:

- Only makes sense if n is large (compared to the constant factors).

$2^{100000000000000}\, n$ is "better" than n^2 ?!?!

pronounced “big-oh of …” or sometimes “oh of …”

O(…) means an upper bound

- Let T(n), g(n) be functions of positive integers.
  - Think of T(n) as being a runtime: positive and increasing in n.

- We say “T(n) is O(g(n))” if g(n) grows at least as fast as T(n) as n gets large.

- Formally,

$$T(n) = O(g(n))$$

$$\Longleftrightarrow$$

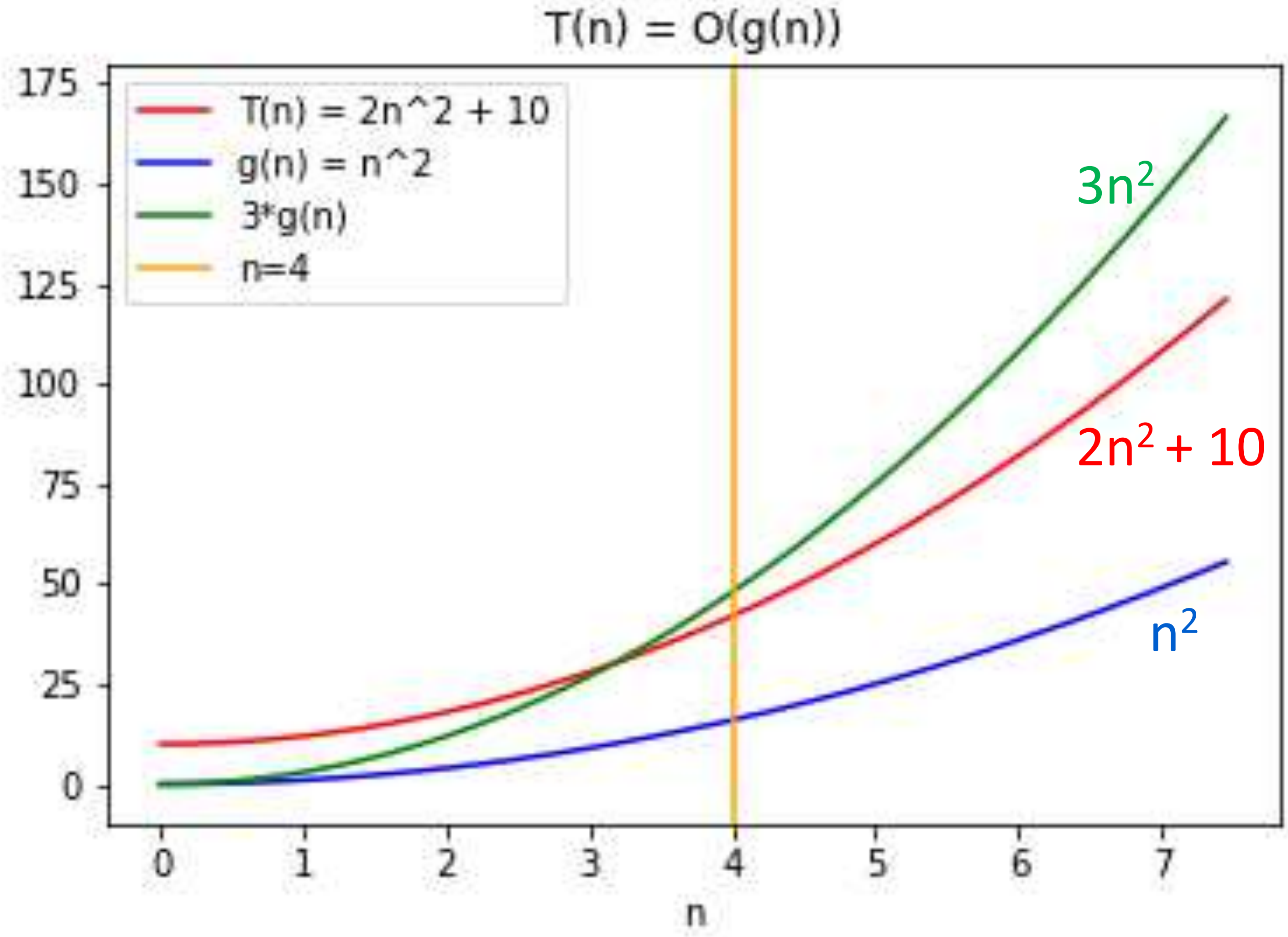
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$

$$0 \leq T(n) \leq c \cdot g(n)$$

Example

$2n^2 + 10 = O(n^2)$

$$T(n) = O\big(g(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot g(n)$$



Example

$2n^2 + 10 = O(n^2)$

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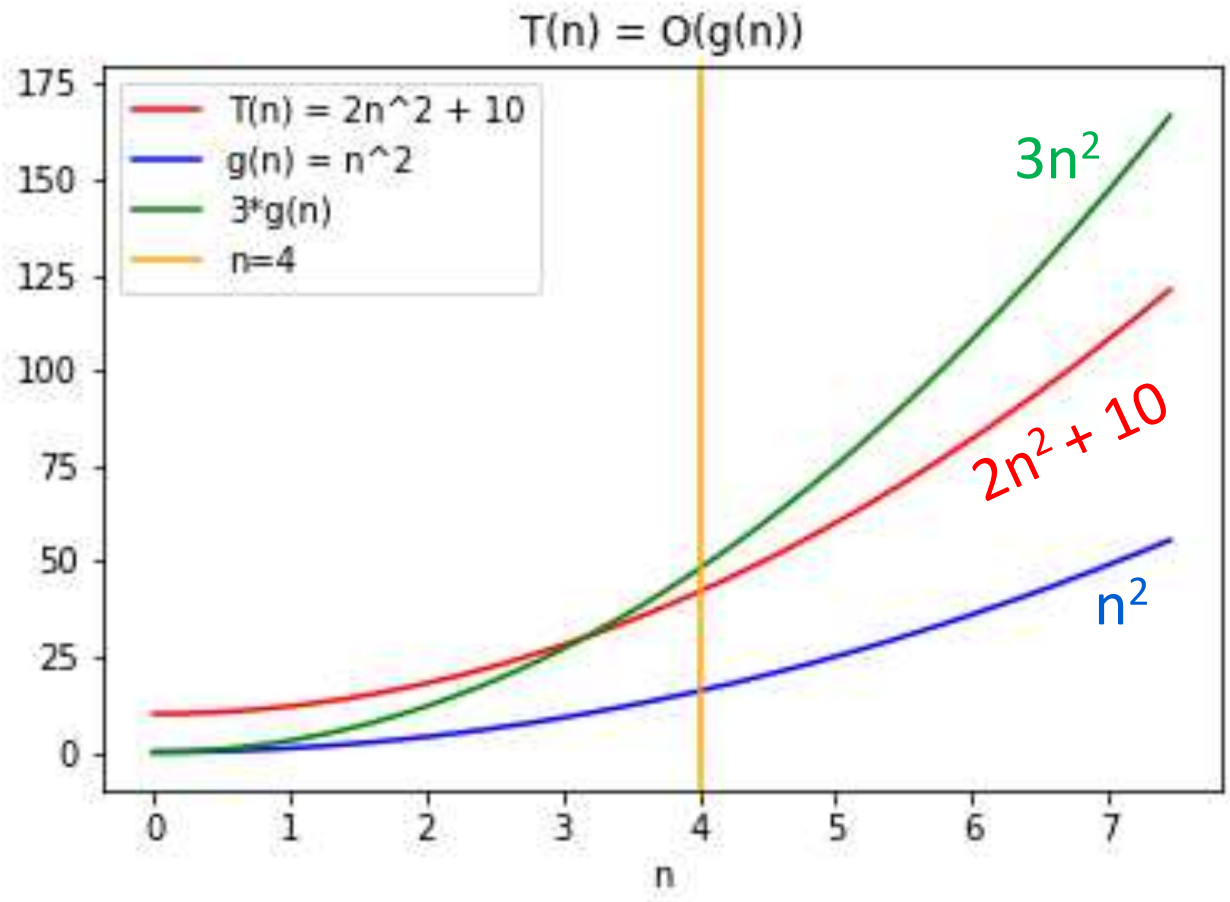


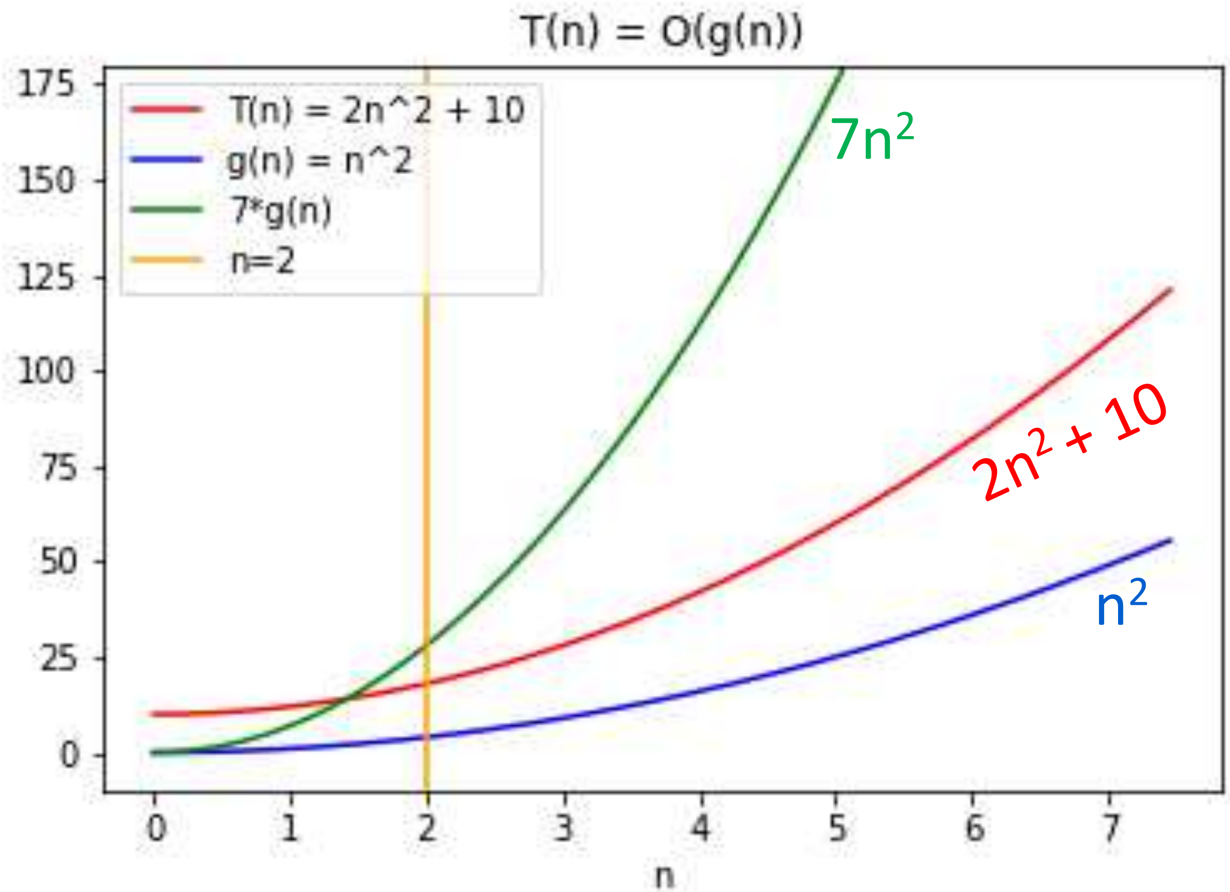
Formally:

- Choose c = 3
- Choose $n_0 = 4$
- Then:

$$\forall n \geq 4,$$
$$0 \leq 2n^2 + 10 \leq 3 \cdot n^2$$

same Example
$2n^2 + 10 = O(n^2)$

$$T(n) = O\big(g(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot g(n)$$



Formally:

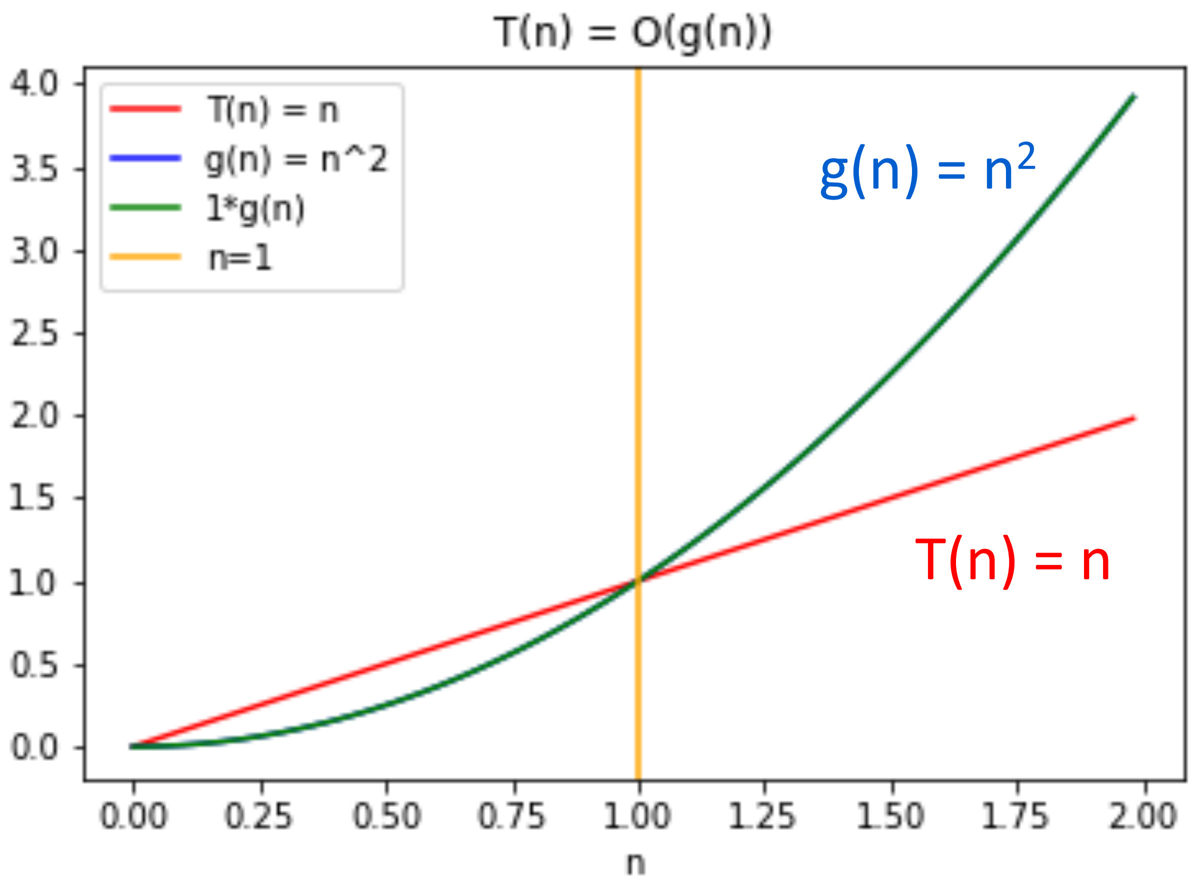
- Choose c = 7
- Choose $n_0 = 2$
- Then:

$$\forall n \geq 2,$$
$$0 \leq 2n^2 + 10 \leq 7 \cdot n^2$$

There is not a "correct" choice of c and n_0

Another example: $n = O(n^2)$

$$T(n) = O\big(g(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot g(n)$$



- Choose c = 1
- Choose $n_0 = 1$
- Then

$$\forall n \geq 1,$$
$$0 \leq n \leq n^2$$

Ω(...) means a lower bound

- We say "T(n) is Ω(g(n))" if g(n) grows at most as fast as T(n) as n gets large.

- Formally,

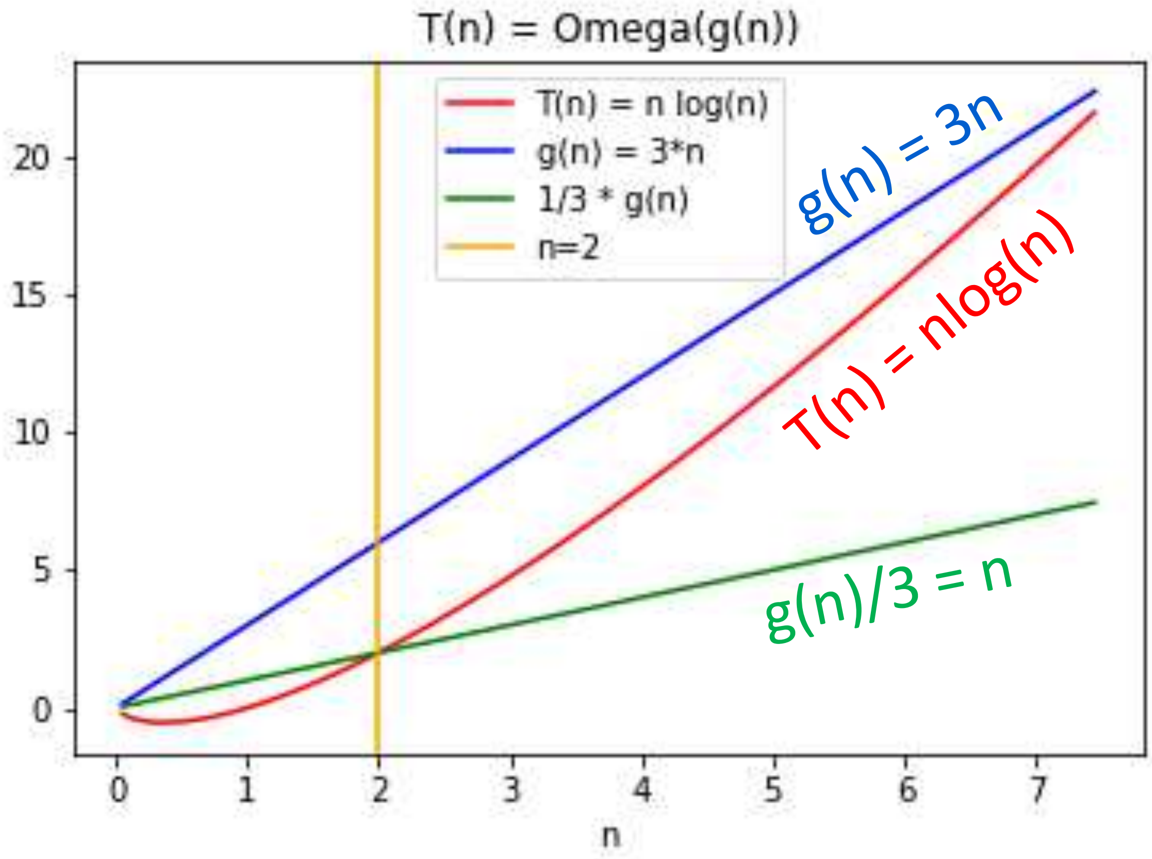
$$T(n) = \Omega\big(g(n)\big)$$
$$\Longleftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq c \cdot g(n) \leq T(n)$$

Switched these!!

Example

$n\log_2(n) = \Omega(3n)$

$$T(n) = \Omega\big(g(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq c \cdot g(n) \leq T(n)$$



- Choose c = 1/3
- Choose n_0 = 3
- Then

$$\forall n \geq 3,$$

$$0 \leq \frac{3n}{3} \leq n\log_2(n)$$

Θ(…) means both!

- We say "T(n) is Θ(g(n))" if:

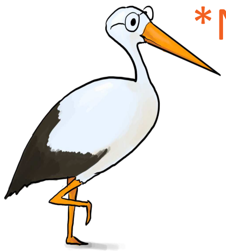
$$T(n) = O(g(n))$$

-AND-

$$T(n) = \Omega(g(n))$$

Some more examples

- All degree k polynomials* are $O(n^k)$
- For any $k \geq 1$, n^k is not $O(n^{k-1})$



*Need some caveat here…what is it?

(On the board if we have time…
if not see the lecture notes!)

Take-away from examples

- To prove $T(n) = O(g(n))$, you have to come up with c and n_0 so that the definition is satisfied.

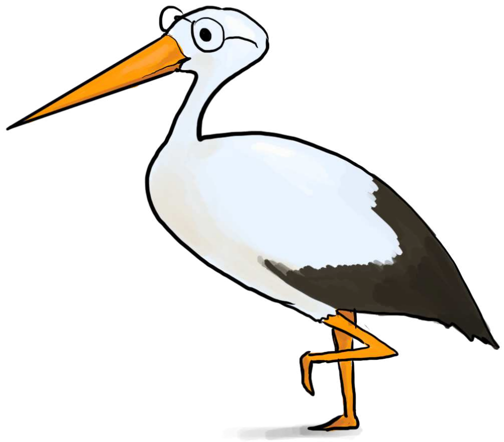
- To prove T(n) is NOT O(g(n)), one way is **proof by contradiction**:
  - Suppose (to get a contradiction) that someone gives you a c and an n_0 so that the definition ***is*** satisfied.
  - Show that this someone must by lying to you by deriving a contradiction.

Yet more examples

- $n^3 + 3n = O(n^3 - n^2)$
- $n^3 + 3n = \Omega(n^3 - n^2)$
- $n^3 + 3n = \Theta(n^3 - n^2)$

- 3^n is **NOT** $O(2^n)$
- $\log(n) = \Omega(\ln(n))$
- $\log(n) = \Theta(\, 2^{\log\log(n)} \,)$

Work through any of these that we don't have time to go through in class!



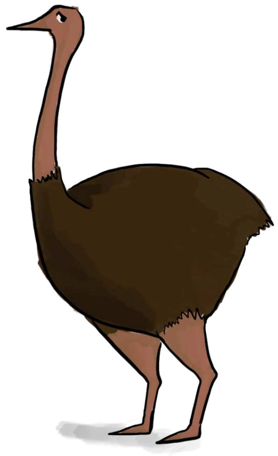
Siggi the Studious Stork

remember that $\log = \log_2$ in this class.

Some brainteasers

- Are there functions f, g so that NEITHER $f = O(g)$ nor $f = \Omega(g)$?
- Are there non-decreasing functions f, g so that the above is true?
- Define the n'th fibonacci number by $F(0) = 1$, $F(1) = 1$, $F(n) = F(n-1) + F(n-2)$ for $n > 2$.
  - 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

  True or false:
  - $F(n) = O(2^n)$
  - $F(n) = \Omega(2^n)$



Ollie the Over-achieving Ostrich

What have we learned?

Asymptotic Notation

- This makes both Plucky and Lucky happy.
  - **Plucky the Pedantic Penguin** is happy because there is a precise definition.
  - **Lucky the Lackadaisical Lemur** is happy because we don’t have to pay close attention to all those pesky constant factors like “11”.
- But we should always be careful not to abuse it.
- In the course, (almost) every algorithm we see will be actually practical, without needing to take $n \geq n_0 = 2^{10000000}$.

The plan

- Part I: Sorting Algorithms
  - InsertionSort: does it work and is it fast?
  - MergeSort: does it work and is it fast?
  - **Skills:**
    - Analyzing correctness of iterative and recursive algorithms.
    - Analyzing running time of recursive algorithms (part A)

- Part II: How do we measure the runtime of an algorithm?
  - Worst-case analysis
  - Asymptotic Analysis

Wrap-Up

Recap

- InsertionSort runs in time **$O(n^2)$**
- MergeSort is a divide-and-conquer algorithm that runs in time **$O(n \log(n))$**

- How do we show an algorithm is correct?
  - Today, we did it by induction
- How do we measure the runtime of an algorithm?
  - Worst-case analysis
  - Asymptotic analysis

Next time

- A more systematic approach to analyzing the runtime of recursive algorithms.

Before next time

- Pre-Lecture Exercise:
  - A few recurrence relations (see website)