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# Optimal Error Bounds for Cubic Spline Interpolation

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The error bounds considered are of the form $\|f^{(r)} - s^{(r)}\|_\infty \leqslant C_r \|f^{(4)}\|_\infty h^{4-r}$, where $s$ is a cubic spline interpolant of $f \in C^4[a, b]$, matching $f$ in slope or in second derivative at the endpoints of $[a, b]$. By refinement and extension of an earlier (1968) analysis, we obtain constants $C_0$, $C_1$, $C_2$, $C_3$, which are more widely applicable and (except for $C_0$) smaller than heretofore known. It is shown that $C_0$ and $C_1$ cannot be further improved. The argument invokes the Euler spline as extremal function and leads to an error bound for spline interpolation of general odd degree over a uniform mesh.

## 1. INTRODUCTION

In [1], error bounds were derived for cubic spline interpolation over arbitrary partitions. Our pupose here is to investigate the optimality of such bounds. Let

$$\Pi\colon a = x_0 < x_1 < \cdots < x_n = b$$

be a collection of knots, i.e., a mesh, partitioning interval $\Omega = [a, b]$ of the real line $R$. We define

$$\Delta x_i = x_{i+1} - x_i \qquad (i = 0, 1, \ldots, n-1), \tag{1}$$

$$h = \max_i \Delta x_i = \text{mesh gauge}, \tag{2}$$

$$\beta = h/\min_i \Delta x_i = \text{mesh ratio}. \tag{3}$$

If $C^k(\Omega)$ is the space of functions on $\Omega$ to $R$ that are $k$ times continuously differentiable (Here $f \in C^k(*)$ implies not only existence and continuity of $f^{(k)}$ at interior points of (*), but also approach to a finite limit at either frontier of the interval), and if

$$P^m(\Omega, \Pi) \equiv \{p \mid p(x) \text{ is a polynomial of degree } 2m - 1 \text{ in each } (x_i, x_{i+1})\}, \quad (4)$$

then

$$S^2(\Omega, \Pi) \equiv P^2(\Omega, \Pi) \cap C^2(\Omega) \quad (5)$$

constitutes a space of cubic splines. Given any function $f \in C(\Omega)$ appropriately differentiable at $x_0$ and $x_n$, the Type I [Type II] cubic spline interpolant of $f$ is, by definition, the unique element $s \in S^2(\Omega, \Pi)$ satisfying

$$s(x_i) = f(x_i) \qquad (i = 0, 1,..., n) \quad (6)$$

and alternative boundary conditions

$$s'(x_j) = f'(x_j), j = 0, n \qquad [s''(x_j) = f''(x_j), j = 0, n].$$

When speaking of the *convergence* of cubic spline interpolation, we envision a sequence of meshes $\Pi$ such that $h \to 0$. The following results by diverse authors progressively disclosed the rather remarkable approximating capability of cubic splines:

THEOREM 1, [2, 3]. *If $f \in C^2(\Omega)$ and $\beta \to 1$ then $s^{(r)}$ converges uniformly to $f^{(r)}$ for $r = 0, 1, 2$.*

THEOREM 2, [4]. *For $f \in C^4(\Omega)$, the errors in Type I interpolation satisfy*

$$\|(f - s)^{(r)}\|_\infty \leqslant C_r \|f^{(4)}\|_\infty h^{4-r} \qquad (r = 0, 1, 2, 3)$$

*with*

$$C_r = (r + 1)C_{r+1}, r = 0, 1, 2 \text{ and } C_3 = 3 + 6\beta(\beta + 1)^2 (4\beta + 2)/(4\beta + 3).$$

THEOREM 3, [5]. *The constants in Theorem 2 can be replaced by*

$$C_r = C_3, \qquad r = 0, 1, 2, \qquad \text{and} \qquad C_3 = 1 + \beta(1 + \beta^2).$$

THEOREM 4, [1, 6]. *The constants in Theorem 2 can be replaced by*

$$C_0 = \frac{5}{384}, \qquad C_1 = \frac{9 + \sqrt{3}}{216}, \qquad C_2 = \frac{3\beta + 1}{12}, \qquad C_3 = \frac{\beta^2 + 1}{2}.$$

Whereas initially it was proposed that meshes be *asymptotically uniform* ($\beta \to 1$) to bring about convergence, Theorems 2 and 3 made clear that boundedness of the mesh ratio would suffice. Theorem 4 removed even this restriction for $r = 0, 1$; thus, $(s, s') \to (f, f')$ whenever $h \to 0$.

We now state the main result of the present paper.

THEOREM 5. *Let $s \in S^2(\Omega, \Pi)$ be the Type I or Type II interpolant of $f \in C^4(\Omega)$. Then*

$$\|(f - s)^{(r)}\|_\infty \leqslant C_r \|f^{(4)}\|_\infty h^{4-r} \qquad (r = 0, 1, 2, 3), \tag{8}$$

*with*

$$C_0 = 5/384, \qquad C_1 = 1/24, \qquad C_2 = 3/8, \qquad C_3 = (\beta + \beta^{-1})/2. \tag{9}$$

*The constants $C_0$, $C_1$ are optimal in the sense that*

$$C_r = \sup_{f, \Pi} \frac{\|(f - s)^{(r)}\|_\infty}{\|f^{(4)}\|_\infty h^{4-r}} \qquad (r = 0, 1), \tag{10}$$

*where the supremum is taken over all $\Pi$ partitioning $\Omega$ and over all $f \in C^4(\Omega)$ such that $f^{(4)} \not\equiv 0$.*

This represents an advance beyond Theorem 4 in several respects. Three of the constants $C_0$, $C_1$, $C_2$, $C_3$ have been improved and all four transpire as equally true for Type I or Type II interpolation. Most significantly, perhaps, the values $C_0 = 5/384$ and $C_1 = 1/24$ are characterized as best possible. They can be attained (in the limit) with *uniform* mesh.

In Section 2, we argue the validity of (8) and (9). In Section 3 we derive error bounds for general spline interpolation of odd degree, with $\beta = 1$. The development leads to an extremal function which immediately confirms (10) for Type II interpolation.

## 2. DERIVATION OF ERROR BOUNDS

Throughout this section, as before, $f$ represents a given function in $C^4(\Omega)$ and $s$ is its cubic spline interpolant. Also, recalling (4), let $u$ be the unique element of $P^2(\Omega, \Pi) \cap C^1(\Omega)$ that satisfies

$$u(x_i) = f(x_i) \qquad \text{and} \qquad u'(x_i) = f'(x_i) \qquad (i = 0, 1, \ldots, n). \tag{11}$$

We label $u$ the *cubic Hermite interpolant* of $f$, over $\Pi$. In [1], the proof of Theorem 4 was accomplished by combining ,under the triangle inequality, two pointwise error bounds of the form

$$|f^{(r)}(x) - u^{(r)}(x)| \leqslant A_r(x) \|f^{(4)}\|_\infty h^{4-r}, \tag{12}$$

$$|u^{(r)}(x) - s^{(r)}(x)| \leqslant G_r(x) \|f^{(4)}\|_\infty h^{4-r}. \tag{13}$$

Only the latter had to be established, as the functions $A_0, \ldots, A_3$ and their maxima were already available from [7]. The numerical constants of the theorem were calculated according to

$$C_r = \max_{x \in \Omega} A_r(x) + \max_{x \in \Omega} G_r(x), \tag{14}$$

as a matter of expediency. Sharper results would, of course, issue from

$$C_r = \max_{x \in \Omega} [A_r(x) + G_r(x)], \tag{15}$$

if $A_r$ and $G_r$ are not maximized simultaneously.

To explore this possibility and at the same time to show the irrelevance of boundary constraint, we shall reestablish (13). Illustrated in Figs. 1a, and 1b are two fundamental splines $v$ and $w$ that will play an important part in the development. They vanish at all of the meshpoints:

$$v(x_i) = w(x_i) = 0 \qquad (i = 0, 1, \ldots, n) \tag{16}$$

and satisfy either

$$v'(a) = w'(b) = 0; \qquad v''(a) = (-1)^n w''(b) = 1,$$

considered to be Type I conditions, or (17)

$$v''(a) = w''(b) = 0; \qquad v'(a) = (-1)^{n+1} w'(b) = 1, \tag{18}$$

considered to be Type II. There is little difficulty in demonstrating that

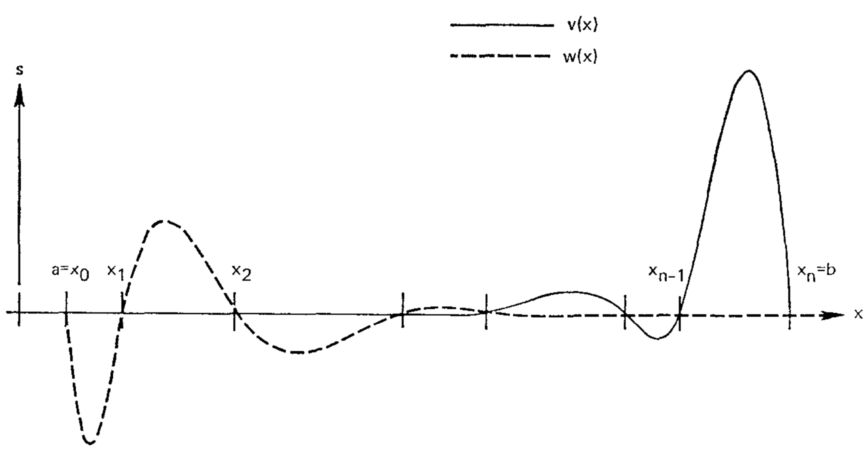


FIG. 1a. Fundamental splines (true plot). Attenuation prevents adequate portrayal of sign behavior.

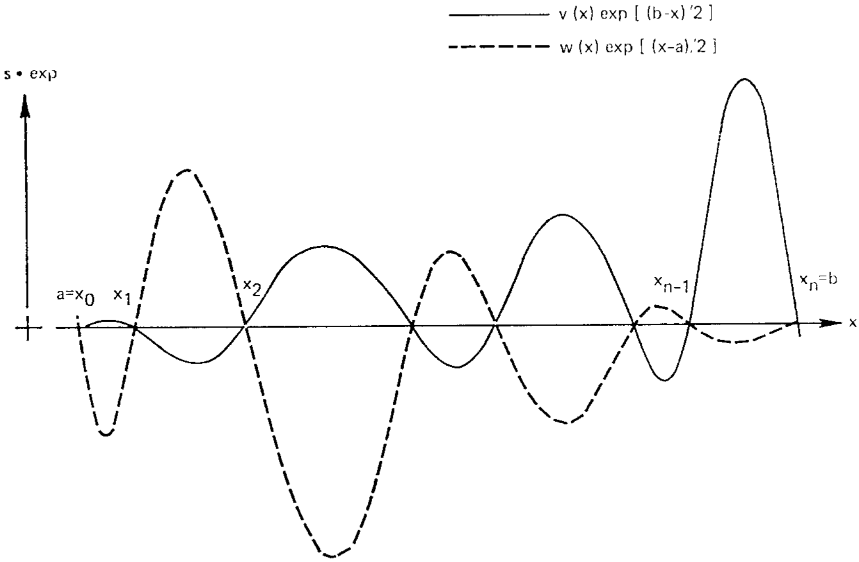


FIG. 1b. Fundamental splines (distorted plot). Exponential weighting gives a better picture.

$v$, for example, must be alternately positive and negative on successive mesh intervals. No cubic spline vanishing at all meshpoints can have a straight line segment except that it vanish identically. So the zeros of $v''$, $v'$, and $v$ are isolated. Let $\xi_1 < \xi_2 < \cdots < \xi_l$ be all the zeros of $v'$ in $(a, b)$. Rolle's theorem assures to $v$, on the one hand, a maximum of $l + 1$ zeros in $[a, b]$ and to $v''$, on the other hand, a zero in each of the intervals $[a, \xi_1)$, $(\xi_1, \xi_2)$,..., $(\xi_{l-1}, \xi_l)$. (Regarding $[a, \xi_1)$, the assurance may come from (18) rather than Rolle's theorem.) As a piecewise linear $v''$ can vanish but once per segment. Thus, $l \leqslant n$, proving the $n + 1$ zeros $x_0$,..., $x_n$ of $v$ to be exhaustive.

A simple formula fixes sign behavior specifically. Through integration by parts we obtain (since $vv'''$ is continuous and $v^{(4)} \equiv 0$ a.e.)

$$\int_a^{x_i} v''(x)^2 \, dx = [v'(x)\, v''(x) - v(x)\, v'''(x)]_a^{x_i} = v'(x_i)\, v''(x_i). \tag{19}$$

At all meshpoints beyond the first, $v'$ and $v''$ are in sign agreement. The foregoing arguments apply with equal force to the $w$-spline, except that sign agreement becomes sign opposition. To summarize (sgn denotes the *signum* function: sgn $x = +1$, $-1$, or 0, accordingly, as $x$ is positive, negative, or zero):

$$\operatorname{sgn} v'(x_i) = \operatorname{sgn} v''(x_i) = (-1)^i \qquad (i = 1,..., n), \tag{20a}$$
$$-\operatorname{sgn} w'(x_i) = \operatorname{sgn} w''(x_i) = (-1)^i \qquad (i = 0,..., n-1). \tag{20b}$$

Let

$$D(x_i) = -w''(x_i)\, v'(x_i) + w'(x_i)\, v''(x_i) \qquad (i = 0,\dots, n). \tag{21}$$

The products $w''(x_i)\, v'(x_i)$ and $w'(x_i)\, v''(x_i)$ are of opposite sign, or else one of them vanishes when $i = 0, n$. In any case

$$| D(x_i)| = | w''(x_i) \| v'(x_i)| + | w'(x_i)\| v''(x_i)|. \tag{22}$$

Our next goal is to place an upper bound on the $L_1$ norm of $v$. Let $e$ be any function of fourth-order differentiability that goes to zero at the limits of some mesh interval $(x_{j-1}, x_j)$. Then

$$\begin{aligned} \int_{x_{j-1}}^{x_j} v(x)\, e^{(4)}(x)\, dx &= \sum_{k=0}^{3} (-1)^k \, [v^{(k)}(x)\, e^{(3-k)}(x)]_{x_{j-1}}^{x_j}, \\ &= [-v'(x)\, e''(x) + v''(x)\, e'(x)]_{x_{j-1}}^{x_j}, \end{aligned} \tag{23}$$

once more through integration by parts. Various formulae for segmental quadrature of $v$ result when $e$ is equated to quartic polynomials. For example, the choices

$$e(x) = [(x - x_{j-1})(x_j - x)\, \Delta x_{j-1}^2 + (x - x_{j-1})^2\, (x_j - x)^2]/24 \tag{24a}$$

and

$$e(x) = (x - x_{j-1})(x_j - x)^2\, (2x_j - 3x_{j-1} + x)/24 \tag{24b}$$

lead to

$$\int_{x_{j-1}}^{x_j} v(x)\, dx = -[v''(x_{j-1}) + v''(x_j)]\, \Delta x_{j-1}^3/24 \tag{25a}$$

and

$$\int_{x_{j-1}}^{x_j} v(x)\, dx = -[v'(x_{j-1}) + v'(x_j)]\, \Delta x_{j-1}^2/4 - v''(x_{j-1})\, \Delta x_{j-1}^3/12, \tag{25b}$$

respectively. (Routine evaluative details are omitted; they become less tedious if one assumes $x_{j-1} = 0$, $x_j = \Delta$.)

Applied to the quadrature of $| v |$ over successive segments, formula (25a) yields

$$\int_a^{x_i} | v(x)|\, dx = \sum_{j=1}^{i} (-1)^j \, [v''(x_{j-1}) + v''(x_j)]\, \Delta x_{j-1}^3/24, \tag{26}$$

taking into account the alternating sign property. As the summands are all positive, an upper bound results when the $\Delta x$'s are replaced by their supremum $h$:

$$\begin{aligned}\int_a^{x_i} |v(x)|\,dx &\leqslant \sum_{j=1}^{i} (-1)^j\,[v''(x_{j-1}) + v''(x_j)]\,h^3/24,\\ &\leqslant [-v''(a) + (-1)^i\,v''(x_i)]\,h^3/24.\end{aligned} \tag{27}$$

And $v''(a) \geqslant 0$ under either Type I or Type II boundary conditions, so

$$\int_a^{x_i} |v(x)|\,dx \leqslant |v''(x_i)|\,h^3/24. \tag{28}$$

Similarly, formula (25b) will yield, instead of (27), the inequality

$$\int_a^{x_i} |v(x)|\,dx \leqslant [-v'(a) + (-1)^i\,v'(x_i)]\,h^2/4 - \sum_{j=0}^{i-1} (-1)^j\,v''(x_j)\,\Delta x_j{}^3/12. \tag{29}$$

Here, the terms under the summation sign as well as $v'$(a) are nonnegative, so

$$\int_a^{x_i} |v(x)|\,dx \leqslant |v'(x_i)|\,h^2/4. \tag{30}$$

Clearly, (28) and (30) are true for the $w$-spline; however, with $(x_i, b)$ as the range of integration:

$$\int_{x_i}^{b} |w(x)|\,dx \leqslant |w''(x_i)|\,h^3/24, \tag{31}$$

$$\int_{x_i}^{b} |w(x)|\,dx \leqslant |w'(x_i)|\,h^2/4. \tag{32}$$

Let us define a pair of kernels $K_1$, $K_2$ as follows.

$$\begin{aligned}K_r(x_i\,;x) &= \frac{w^{(r)}(x_i)\,v(x)}{D(x_i)} \qquad (\text{if } x < x_i)\\ &= \frac{v^{(r)}(x_i)\,w(x)}{D(x_i)} \qquad (\text{if } x > x_i)\end{aligned} \qquad (r = 1, 2) \tag{33}$$

with $D(x_i)$ given by (21). Then, referring to (28) and (31),

$$\begin{aligned}\int_a^b |K_1(x_i\,;x)|\,dx &= \frac{|w'(x_i)|}{|D(x_i)|}\int_a^{x_i} |v(x)|\,dx + \frac{|v'(x_i)|}{|D(x_i)|}\int_{x_i}^{b} |w(x)|\,dx\\ &\leqslant \frac{|w'(x_i)|\,|v''(x_i)| + |v'(x_i)|\,|w''(x_i)|}{|D(x_i)|}\cdot\frac{h^3}{24}.\end{aligned} \tag{34}$$

In light of (22), $h^3/24$ emerges as an upper bound on the $L_1$ norm of $K_1$. Similar treatment of $K_2$, with (30) and (32) the relevant inequalities, leads to $h^2/4$. Thus,

$$\| K_1 \|_1 \leqslant h^3/24; \qquad \| K_2 \|_1 \leqslant h^2/4. \tag{35}$$

We are in a position now to demonstrate a key lemma.

LEMMA 1. *Let $e$ be any function in $C^2(\Omega) \bigcap_{i=1}^n C^4(x_{i-1}, x_i)$ that vanishes at every meshpoint and satisfies homogenous Type I [Type II] boundary conditions*

$$e'(a) = e'(b) = 0 \qquad [e''(a) = e''(b) = 0]. \tag{36}$$

*Then*

$$| e^{(r)}(x_i)| \leqslant \rho_r \| e^{(4)} \|_\infty h^{4-r} \qquad (i = 0,..., n; r = 1, 2), \tag{37}$$

*with*

$$\rho_1 = 1/24; \qquad \rho_2 = 1/4. \tag{38}$$

*Proof.* Assume the $v$ and $w$ splines to be the same in type as $e$. Because $v''e' - v'e''$ is continuous, the limits of integration in formula (23) can be any two (not necessarily consecutive) meshpoints. This, together with the homogenous boundary conditions, justifies

$$\int_a^{x_i} v(x)\, e^{(4)}(x)\, dx = -v'(x_i)\, e''(x_i) + v''(x_i)\, e'(x_i) \tag{39a}$$

and similarly,

$$\int_{x_i}^b w(x)\, e^{(4)}(x)\, dx = w'(x_i)\, e''(x_i) - w''(x_i)\, e'(x_i). \tag{39b}$$

Solving for $e'(x_i)$, $e''(x_i)$,

$$e'(x_i) = \frac{w'(x_i)}{D(x_i)} \int_a^{x_i} v(x)\, e^{(4)}(x)\, dx + \frac{v'(x_i)}{D(x_i)} \int_{x_i}^b w(x)\, e^{(4)}(x)\, dx, \tag{40a}$$

$$e''(x_i) = \frac{w''(x_i)}{D(x_i)} \int_a^{x_i} v(x)\, e^{(4)}(x)\, dx + \frac{v''(x_i)}{D(x_i)} \int_{x_i}^b w(x)\, e^{(4)}(x)\, dx. \tag{40b}$$

More succinctly,

$$e^{(r)}(x_i) = \int_a^b K_r(x_i\, ; x)\, e^{(4)}(x)\, dx \qquad (r = 1, 2). \tag{41}$$

The truth of the lemma is apparent from

$$| e^{(r)}(x_i)| \leqslant \int_a^b | K_r(x_i ; x)| \, | e^{(4)}(x)| \, dx \leqslant \| K_r \|_1 \cdot \| e^{(4)} \|_\infty \tag{42}$$

and inequalities (35). Q.E.D.

We now identify $e$ with the remainder function for spline interpolation, viz.

$$e(x) = f(x) - s(x). \tag{43}$$

The difference $u - s$ becomes the cubic Hermite interpolant of $e$ over $\Pi$. In the interval $[x_{j-1}, x_j]$, it is expressible (cf. [1, pp. 212-213]) as

$$u(x) - s(x) = e'(x_{j-1}) H_3(x) + e'(x_j) H_4(x), \tag{44}$$

where

$$\begin{aligned} H_3(x) &= (x - x_{j-1})(x - x_j)^2/\Delta x_{j-1}^2 ; \\ H_4(x) &= (x - x_j)(x - x_{j-1})^2/\Delta x_{j-1}^2 . \end{aligned} \tag{45}$$

From (44), the triangle inequality, and Lemma 1, there follows

$$| u^{(r)}(x) - s^{(r)}(x)| \leqslant \rho_1 \| e^{(4)} \|_\infty \{| H_3^{(r)}(x)| + | H_4^{(r)}(x)|\}. \tag{46}$$

This is tantamount to (13) since $\| e^{(4)} \|_\infty = \| f^{(4)} \|_\infty$. An appropriate definition of $G_r$ is

$$G_r(x) = \rho_1\{| H_3^{(r)}(x)| + | H_4^{(r)}(x)|\} h^{r-1}. \tag{47}$$

We have gained our primary objective, validation of (13) for both types of boundary condition. The next step will be to compare maximals of $A_r$ and $G_r$. As noted earlier, the functions $A_0(x), \ldots, A_3(x)$ are due to Birkhoff and Priver [7]. On a mesh interval $[x_{j-1}, x_j] = [-1, +1]$ the comparison reads:

Location of Maxima

| | $r = 0$ | 1 | 2 | 3 |
|---|---|---|---|---|
| $A_r$ | 0 | $\pm 1/\sqrt{3}$ | $\pm 1$ | $\pm 1$ |
| $G_r$ | 0 | $\pm 1$ | $\pm 1$ | $\pm 1$. |

Accordingly, only for the case $r = 1$ can one expect to improve the constant $C_r$ by choosing (15) over (14). Write

$$\xi = (x - x_{j-1})/\Delta x_{j-1} \qquad (x \in [x_{j-1}, x_j]). \tag{48}$$

The function $A_1 + G_1$ is symmetric about $\xi = 1/2$ and expressible as

$$\begin{aligned} A_1(x) &+ G_1(x) \\ &= \rho_1\{4\xi^3 - 6\xi^2 + 1\}, \qquad \text{if} \quad 0 \leqslant \xi \leqslant \tfrac{1}{3} \\ &= \rho_1 \left\{4\xi^3 - \frac{129}{4}\xi^2 + \frac{221}{4}\xi - \frac{83}{2} + \frac{33}{2}\xi^{-1} - \frac{13}{4}\xi^{-2} + \frac{1}{4}\xi^{-3}\right\}, \\ &\qquad \text{if} \quad \tfrac{1}{3} < \xi \leqslant \tfrac{1}{2}. \end{aligned} \tag{49}$$

Its graph (Fig. 2) indicates, and the calculus will readily confirm, a maximum value $\rho_1 = 1/24$ at $\xi = 0$. This provides the constant $C_1$ of Theorem 5, in contrast to that of Theorem 4.

Theorem 5 also gives sharper values for $C_2$ and $C_3$ based on a simple lemma about linear interpolation, the proof of which is omitted.

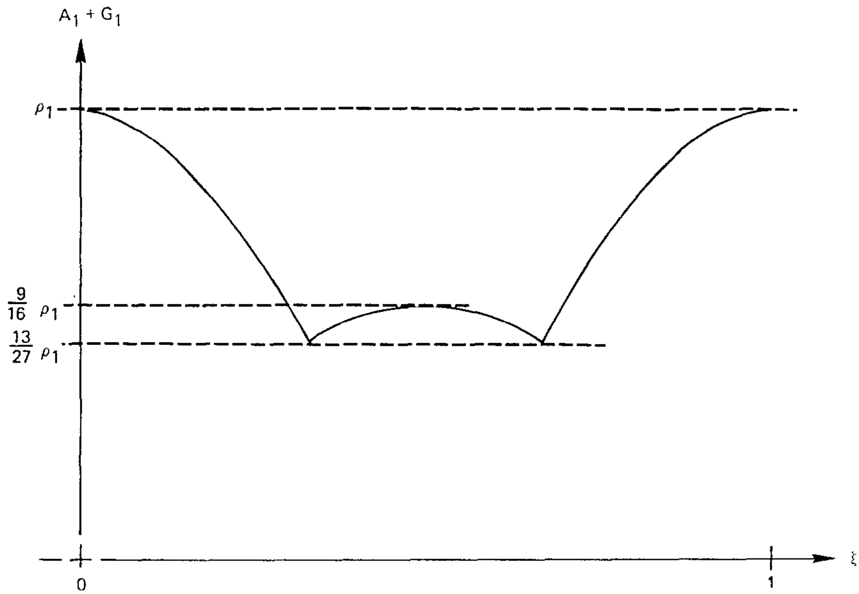


FIG. 2. First derivative error factor $A_1(x) + G_1(x)$, shown as a function of $\xi = (x - x_{j-1})/\Delta x_{j-1}$.

LEMMA 2. *Let*

$$\varphi \in C(\Omega) \bigcap_{i=1}^{n} C^2(x_{i-1}, x_i).$$

*Then*

$$\| \varphi \|_\infty \leqslant \| \boldsymbol{\varphi} \|_\infty + (h^2/8) \| \varphi'' \|_\infty \tag{50}$$

*and*

$$\| \varphi' \|_\infty \leqslant \max_i \{(2/\Delta x_i) \| \boldsymbol{\varphi} \|_\infty + (\Delta x_i/2) \| \varphi'' \|_\infty\}, \tag{51}$$

*where* $\boldsymbol{\varphi} = [\varphi(x_0),..., \varphi(x_n)]^T$.

If $\varphi$ is identified with $e'' = f'' - s''$, the norm $\| \boldsymbol{\varphi} \|_\infty$ cannot exceed $\| e^{(4)} \|_\infty h^2/4$ according to Lemma 1. We substitute this upper bound for $\| \boldsymbol{\varphi} \|_\infty$ in (50) and (51), also replacing both $\| e^{(4)} \|_\infty$ and $\| \varphi'' \|_\infty$ by $\| f^{(4)} \|_\infty$. The resultant inequalities imply that $C_2 = 3/8$ and $C_3 = \max_i (h/\Delta x_i + \Delta x_i/h)/2 = (\beta + \beta^{-1})/2$. The proof of (8) and (9) is complete.

## 3. Odd Degree Spline Interpolation

In this section, we consider polynomial spline interpolation of general odd degree. Theoretical error bounds, derived for arbitrary partitions $\Pi$, are evaluated under the later assumption that $\Pi$ is uniform. As before, let

$$P^m(\Omega, \Pi) \equiv \{p \mid p(x) \text{ is a polynomial of degree } 2m - 1 \text{ in each } (x_i, x_{i+1})\}, \tag{52}$$

whence

$$S^m(\Omega, \Pi) \equiv P^m(\Omega, \Pi) \cap C^{2m-2}(\Omega) \tag{53}$$

is the space of $(2m - 1)$th degree polynomial splines over $\Pi$. The Type II (or Lidstone) interpolant of $f$ is, by definition, the unique element $s \in S^m(\Omega, \Pi)$ satisfying

$$s(x_i) = f(x_i) \qquad (i = 0,..., n) \tag{54}$$

and

$$s^{(2k)}(x_j) = f^{(2k)}(x_j) \qquad (j = 0, n;\ k = 1,..., m - 1). \tag{55}$$

If $\mathscr{P}_x : C^{2m-2}(\Omega) \to S^m(\Omega, \Pi)$ is the projector such that

$$\mathscr{P}_x f = s, \tag{56}$$

then $\mathscr{P}_x$ reproduces any polynomial of degree equal to or less than $2m - 1$ and by Peano's theorem [8, p. 70]

$$f = \mathscr{P}_x f + \int_a^b K(x, t)\, df^{(2m-1)}(t), \tag{57}$$

provided that $f \in C^{2m-1}(\Omega)$ and that $f^{(2m-1)}$ is at least piecewise differentiable on $\Omega$. The Peano kernel is, specifically,

$$\begin{aligned} K(x, t) &= (I - \mathscr{P}_x)[(x - t)_+^{2m-1}/(2m - 1)!] \\ &= (x - t)_+^{2m}/(2m - 1)! - \mathscr{P}_x[(x - t)_+^{2m-1}/(2m - 1)!]. \end{aligned} \tag{58}$$

We observe that $(x-t)_+^{2m-1}$ and $(t-x)_+^{2m-1}$, since they differ by the polynomial $(x-t)^{2m-1}$, must yield the same remainder under $\mathscr{P}_x$ for each value of $t$; hence,

$$\begin{aligned} K(x, t) &= (I - \mathscr{P}_x)[(t-x)_+^{2m-1}/(2m-1)!] \\ &= (t-x)_+^{2m-1}/(2m-1)! - \mathscr{P}_x[(t-x)_+^{2m-1}/(2m-1)!]. \end{aligned} \tag{59}$$

Now, $\mathscr{P}_x[(x-t)_+^{2m-1}]$, for $x$ fixed, is a linear combination of

$$(x - t)_+^{2m-1} \qquad (i = 1,..., n-1) \tag{60}$$

and

$$(x_j - t)_+^{2m-1-2k} \qquad (j = 0, n;\ k = 0,..., m-1), \tag{61}$$

each of which, as a function of $t$, belongs to $S^{2m}(\Omega, \Pi)$. Hence, $\mathscr{P}_x[(x-t)_+^{2m-1}]$ is a polynomial spline of degree $2m-1$ in $t$. Let $\mathscr{P}_t$ be the projector $\mathscr{P}_x$ with $x$ replaced by $t$. We have just observed that

$$\mathscr{P}_t\mathscr{P}_x[(x-t)_+^{2m-1}] = \mathscr{P}_x[(x-t)_+^{2m-1}]. \tag{62}$$

From (58) and (62)

$$K(x, t) = (I - \mathscr{P}_t\mathscr{P}_x)[(x-t)_+^{2m-1}/(2m-1)!]$$

and similarly from (59),

$$K(x, t) = (I - \mathscr{P}_x\mathscr{P}_t)[(t-x)_+^{2m-1}/(2m-1)!].$$

Thus, $K$ is symmetric:

$$K(x, t) = K(t, x).$$

It is now easy to characterize $K(x^*, t)$ for a fixed $x = x^*$. Being a remainder under $\mathscr{P}_t$-interpolation, $K$ vanishes at $t = x_j$, $(j = 0,..., n)$ and its even-order derivatives up to order $2m-2$ vanish at $t = x_0, x_n$. It is a spline of degree $2m-1$ over $\Pi$ with an additional knot at $t = x^*$, where a unit jump in the $(2m-1)$th derivative occurs. From considerations in [9], we deduce

LEMMA 3. *As a function of $t$, for a given $x \in [a, b]$ the Peano kernel $K$ is either identically zero or it changes sign at each knot $t = x_j$ $(j = 1,..., n-1)$ and nowhere else in $(a, b)$.*

For $f \in C^{2m-1}(\Omega)$ the appraisal

$$|f - \mathscr{P}_x f| \leqslant \int_a^b |K(x, t)|\, |f^{(2m)}(t)|\, dt \tag{63}$$

is an immediate consequence of the Peano kernel theorem. Replacing $|f^{(2m)}(t)|$ by its supremum yields

$$|f - \mathscr{P}_x f| \leqslant \int_a^b |K(x, t)|\, dt \cdot \|f^{(2m)}\|_\infty . \tag{64}$$

Note that (57) is equally valid for functions $f \in C^{2m-1}(\Omega)$ whose $2m$th derivative is a step-function with steps at the knots $x = x_i$ ($i = 1,..., n - 1$). Such a function is the spline of *degree* $2m$ over $\Pi$ defined by

$$g(x) = \left[x^{2m} + 2 \sum_{i=1}^{n-1} (-1)^i (x - x_i)_+^{2m}\right] \Big/ (2m)!\,. \tag{65}$$

The step-function $g^{(2m)}(x)$ alternates between $+1$ and $-1$, changing sign at each knot $x_1 ,..., x_{n-1}$. Hence,

$$\left| \int_a^b K(x, t)\, dg^{(2m-1)}(t) \right| = \int_a^b |K(x, t)|\, dt, \tag{66}$$

in view of Lemma 3. But Peano's theorem then implies, for the remainder function $R(x) \equiv g(x) - \mathscr{P}_x g(x)$, the equality

$$|R(x)| = \int_a^b |K(x, t)|\, dt \tag{67}$$

and (63) can be rewritten

$$|f - \mathscr{P}_x f| \leqslant |R(x)| \cdot \|f^{(2m)}\|_\infty . \tag{68}$$

Directly defined, $R(x)$ is the unique element of $C^{2m-2}(\Omega)$ satisfying

$$R^{(2m)}(x) = (-1)^i \qquad (x_i < x < x_{i+1}), \qquad i = 0,..., n - 1, \tag{69}$$

$$R(x_i) = 0 \qquad (i = 0,..., n), \tag{70}$$

$$R^{(2k)}(x_j) = 0 \qquad (j = 0, n;\ k = 1,..., m - 1). \tag{71}$$

From (69)–(71) we shall deduce explicit representations for $R(x)$ in terms of the Euler polynomials. Up to this point, $\Pi$ was an arbitrary partition, but from now on it is assumed *uniform* ($h =$ constant, $\beta = 1$). The transformation

$$T\colon x \to (x - x_0)/h \tag{72}$$

maps $\Omega$ one-to-one onto $[0, n]$. Without loss of generality, we can take

$$\hat{\Pi}\colon x_i = i \qquad (i = 0,..., n)$$

as prototype for the uniform case. Hereafter, a circumflex will be used whenever the reference is to $\hat{\Pi}$; for example: $\hat{\mathscr{P}}_x$, $\hat{K}(x, t)$, $\hat{R}(x)$. If $n = 1$, the interpolation scheme associated with $\hat{\mathscr{P}}_x$ is just *Lidstone*'s [8, p. 28]. The remainder $\hat{R}$ is a polynomial of degree $2m$ that vanishes, along with its derivatives of even order, at both endpoints of [0, 1]. These are properties possessed by the coefficients $E_{2m}(x)$ in the expansion

$$\frac{\cosh(xz - z/2)}{\cosh(z/2)} = \sum_{m=0}^{\infty} E_{2m}(x) \cdot \frac{z^{2m}}{(2m)!}, \tag{73}$$

the so-called *Euler polynomials* [10] of even degree. The left-hand member of (73) proclaims $E_{2m}(x)$ an even function of $(x - 1/2)$, whereas the identity

$$\frac{\cosh(xz - z/2)}{\cosh(z/2)} = \cosh(xz) - \sinh(xz) \tanh(z/2) \tag{74}$$

establishes that $E_{2m}(x)$ has only odd powers of $x$ except for its leading term $x^{2m}$. Consequently,

$$\left.\begin{aligned} E_{2m}^{(2k)}(0) &= E_{2m}^{(2k)}(1) = 0 \\ E_{2m}^{(2k+1)}(0) &= -E_{2m}^{(2k+1)}(1) \end{aligned}\right\} \quad (k = 0,\ldots, m - 1). \tag{75}$$

Since $\hat{R}^{(2m)} \equiv 1$, we infer

$$\hat{R}(x) = E_{2m}(x)/(2m)! \tag{76}$$

The single extremum of $E_{2m}(x)$ on (0, 1) is $E_{2m}(1/2) = 2^{-2m}E_{2m}$, where $E_{2m}$ denotes the $2m$th Euler number, [10, p. 804]. Thus, from (68),

$$\| f - \hat{\mathscr{P}}_x f \|_\infty \leqslant \frac{| E_{2m} |}{2^{2m}(2m)!} \cdot \| f^{(2m)} \|_\infty \tag{77}$$

for Lidstone interpolation on [0, 1].

The general case $n > 1$ yields, in an analogous manner,

$$\hat{R}(x) = (-1)^j \frac{E_{2m}(x - j)}{(2m)!} \qquad (j \leqslant x \leqslant j + 1), \quad j = 0,\ldots, n - 1. \tag{78}$$

The function $\hat{R}$ is a periodic spline, constructed of the same Euler polynomial "arch," alternately positive and negative (cf. [11, 12]). In short, (77) is valid for all $n$. Inverting the transformation (72) we have

THEOREM 6. *Let $s \in S^m(\Omega)$ be the $(2m-1)$th degree, Type II spline interpolant of $f \in C^{2m}(\Omega)$. If $\Pi$ is a uniform partition, then*

$$\|f - s\|_\infty \leqslant \frac{|E_{2m}|}{2^{2m}(2m)!} \cdot \|f^{(2m)}\|_\infty h^{2m}, \tag{79}$$

*where $E_{2m}$ denotes the 2mth Euler number. There can be no sharper inequality valid for all $f \in C^{2m}(\Omega)$.*

The last sentence of the theorem remains to be proved. First, however, let us consider derivatives. The general inequality stemming from (57) is

$$\|f^{(r)} - (\mathscr{P}_x f)^{(r)}\|_\infty \leqslant \mu_r(m)\, \|f^{(2m)}\|_\infty h^{2m-r}, \tag{80}$$

with

$$\mu_r(m) = \max_x \int_0^n |\hat{K}^{(r,0)}(x, t)|\, dt, \qquad (r = 0,..., 2m-1). \tag{81}$$

Suppose that the integral, for given $m$ and $r$, attains its global maximum at $x = x^*$. Any solution $g \in C^{2m-1}(0, n)$ of the differential equation

$$g^{(2m)}(x) = \operatorname{sgn} \hat{K}^{(r,0)}(x^*, x) \tag{82}$$

will be extremal so far as (80) is concerned. In other words, (80) becomes an equality (for the given $m$ and $r$) when $f \equiv g$ and $\mathscr{P}_x = \hat{\mathscr{P}}_x$. Accordingly, the constants $\mu_r(m)$ are best possible relative to the class of all functions $f$ that admit a Peano remainder. Can they be improved by restricting $f$ to the narrower class $C^{2m}(\Omega)$? The answer is "no," as we now attempt to show constructively.

Let $g$ remain a particular solution of (82). The discontinuities of $g^{(2m)}$ are finite in number, occurring at (say) the points $\xi_1,..., \xi_q$ interior to $(0, n)$. For $\epsilon > 0$ and small, the intervals $(\xi_i - \epsilon, \xi_i + \epsilon)$, $i = 1,..., q$, will be disjoint. If $g^{(2m)}$ is redefined in each such interval to vary linearly between $g^{(2m)}(\xi_i - \epsilon)$ and $g^{(2m)}(\xi_i + \epsilon)$, the resulting function, call it $g^{(2m)} + \delta^{(2m)}$, will be continuous, with sup norm still equal to unity. The absolute perturbation $|\delta^{(2m)}|$ consists of identically zero stretches interspersed with triangular "spikes" (Fig. 3c). Each spike has base $2\epsilon$ and altitude equal to one-half the corresponding jump in $g^{(2m)}$ at $\xi_i$, so that

$$\int_0^n |\delta^{(2m)}(t)|\, dt = \tfrac{1}{2}\nu\epsilon, \tag{83}$$

where $\nu$ is the total variation of $g^{(2m)}$. Moreover,

$$\|\delta^{(r)} - (\mathscr{P}_x \delta)^{(r)}\|_\infty \leqslant \max_{x,t} |\hat{K}^{(r,0)}(x, t)| \int_0^n |\delta^{(2m)}(t)|\, dt \tag{84}$$

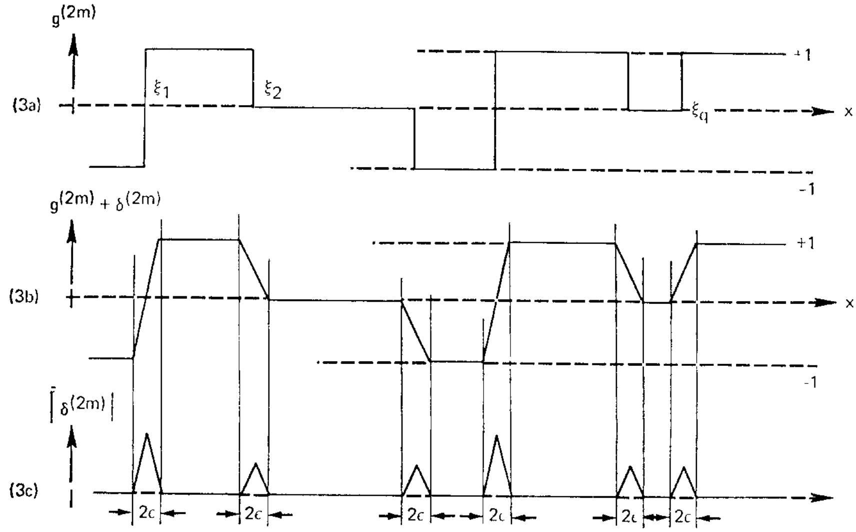


FIG. 3a, b, c. Construction of a near-extremal function $g + \delta$ with continuous $2m$th derivative.

by reason of Peano's theorem, applied to $\delta$. These relations imply the existence of a function $g + \delta$ in $C^{2m}(\hat{\Omega})$, which differs as little from $g$ as may be desired, regarding error of approximation to $r$th derivative. By the same token, since $\|(g + \delta)^{(2m)}\|_\infty = \|g^{(2m)}\|_\infty$, we can choose $\epsilon$ so that (80) comes arbitrarily close to equality for $f \equiv g + \delta$ and $\mathscr{P}_x = \hat{\mathscr{P}}_x$. Thus, the constants $\mu_r(m)$ are best possible relative to the class $C^{2m}(\Omega)$, as Theorem 6 asserted re $\mu_0(m)$.

One further observation will culminate the talk of optimality. On putting $f \equiv \hat{R}$ and $\mathscr{P}_x = \hat{\mathscr{P}}_x$ in (80), we obtain

$$\mu_r(m) \leqslant \| \hat{R}^{(r)} \|_\infty \qquad (r = 0,..., 2m - 1) \tag{85}$$

with equality at least for $r = 0$. It may well be that the Euler spline (78) furnishes $\mu_r(m)$ for all $r < 2m$, but without Lemma 3, the proposition seems difficult to prove. However, in the cubic case ($m = 2$), the maximal amplitudes of $\hat{R}$ and its first derivative are 5/384 and 1/24, respectively. The same numbers, according to Theorem 5, are *lower* bounds on $\mu_0(2)$ and $\mu_1(2)$. So the conjectured proposition is true for $m = 2$, $r < m$. More importantly, by establishing $C_0 = \mu_0(2)$ and $C_1 = \mu_1(2)$ we confirm Theorem 5 with respect to (10). True, the confirmation is not yet total because the present section has dealt exclusively with Type II interpolation, but this shortcoming is easily remedied. If $\mathscr{P}_x$ signifies Type I interpolation, then $\hat{R}$ in (85) should be replaced by $\hat{R} - \hat{\mathscr{P}}_x\hat{R}$. For the cubic case, the bounds on $\mu_r(2)$ remain unchanged numerically since $\hat{\mathscr{P}}_x \hat{R}$ can have but negligible influence when $n$ is large. (If $n$ is odd, the central segment of $-\hat{\mathscr{P}}_x \hat{R}$ is a parabolic arch of height $[32/6^{1/2} \cosh (n \cdot \cosh^{-1} (3/2)^{1/2})]^{-1}$. It diminishes rapidly with increasing $n$.) Thus, optimality persists and Theorem 5 is secure.

Looking beyond cubics, it seems plausible that

$$\| f^{(r)} - (\mathscr{P}_x f)^{(r)} \|_\infty \leqslant \| \hat{R}^{(r)} \|_\infty \| f^{(2m)} \|_\infty h^{2m-r} \qquad (r = 0,..., m-1) \tag{86}$$

generally, *even for nonuniform meshes.* But we have not been able either to mimic or to avoid the sort of matrix analysis proof given in [1]. That analysis is not easily extendable to higher order splines since the "continuity conditions" no longer manifest themselves as "tridiagonal relations." For splines of deficiency $m - 1$, however, one does have the following, which is a consequence of [9]:

THEOREM 7. *Let $f \in C^{2m}(\Omega)$ and let*

$$s \in P^m(\Omega) \cap C^m(\Omega) \tag{87}$$

*be the unique spline of deficiency $m - 1$ associated with $f$ i.e.,*

$$s^{(k)}(x_i) = f^{(k)}(x_i) \qquad (i = 0,..., n;\ k = 0,..., m-2), \tag{88}$$

$$s^{(k)}(x_j) = f^{(k)}(x_j) \qquad (j = 0, n;\ k = 0,..., m-1). \tag{89}$$

*Then*

$$\sup \frac{\| f - s \|_\infty}{\| f^{(2m)} \|_\infty h^{2m}} = \left(\frac{m+3}{m-1}\right) \cdot \frac{2^{-2m}}{(2m)!} \qquad (m > 1), \tag{90}$$

*where the supremum is taken over all partitions $\Pi$ of $\Omega$, and over all $f \in C^{2m}(\Omega)$ such that $f^{(2m)} \not\equiv 0$.*

*Remark.* For quintic splines of deficiency 2 ($m = 3$) the supremum equals 1/15,360. The same number was obtained in [1, p. 210] as an upper bound, but its optimality was not shown.

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