last sum to $n-1$. Similarly, there are F_{n-2} sequences in the second class. Since the classes don't overlap, and every sequence lies in one of them, the recurrence relation follows.

This recurrence, together with the initial values $F_1 = 1$ and $F_2 = 2$, determines F_n for all n.

We can extend the definition of F_n to $n = 0$ in a natural way: the empty sequence is the only one with sum 0, so $F_0 = 1$. It is possible to define F_n for negative n so that the recurrence is satisfied, but there is no natural counting interpretation of these numbers.

Standard arguments give the formula

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

Since $\alpha = (1+\sqrt{5})/2 = 1.618\ldots > 1$ and $\beta = (1-\sqrt{5})/2 = -0.618\ldots > -1$, we see that F_n is the nearest integer to $(1/\sqrt{5})((1+\sqrt{5})/2)^{n+1}$. In particular, F_{n+1}/F_n tends to the limit $(1+\sqrt{5})/2$ as $n \to \infty$.

Note that $\alpha - 1 = -\beta$ is the *golden ratio* τ, the point of division of a unit interval with the property that the ratio of the larger part to the whole is equal to the ratio of the smaller part to the larger.

2 The Fibonacci successor function

In this section we define a function on the positive integers which has the property that it maps each Fibonacci number to the next. It depends on the following property of Fibonacci numbers:

Theorem 2.1 *Every positive integer n has a unique expression in the form*

$$n = F_{i_1} + F_{i_2} + \cdots + F_{i_k},$$

where $i_{j+1} \geq i_j + 2$ for $j = 1, \ldots, k-1$; in other words, as the sum of a set of Fibonacci numbers with no two consecutive.

We call the expression in the theorem the *Fibonacci representation* of n.

To prove this, we use a simple fact about Fibonacci numbers, easily demonstrated by induction:

Fibonacci notes

Peter J. Cameron and Dima G. Fon-Der-Flaass

Abstract

These notes put on record part of the contents of a conversation the first author had with John Conway in November 1996, concerning some remarkable properties of the Fibonacci numbers discovered by Clark Kimberling [2] and by Conway himself. Some of these properties are special cases of much more general results, while others are specific to the Fibonacci sequence; some are proved, while others are merely observation (as far as we know). The first four sections are purely expository. The last two sections on numeration systems are the work of the second author. We am grateful to members of the Combinatorics Study Group at QMW, especially Julian Gilbey, for discussions and help with the details.

1 Fibonacci numbers

The *Fibonacci number* F_n, for positive integer n, can be defined as the number of ways of writing n as the sum of a sequence of terms, each equal to 1 or 2. So, for example, 4 can be expressed in any of the forms

$$2+2 = 2+1+1 = 1+2+1 = 1+1+2 = 1+1+1+1,$$

so $F_4 = 5$.

The most important property of the Fibonacci numbers is that they satisfy the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 3$. For consider all the sequences of 1s and 2s with sum n. Divide the sequences into two classes according to whether the last term is 1 or 2. There are F_{n-1} sequences in the first class, since the terms except the