Unit-5

Linear Programming Problems (LPP)

→ LP :-

Find the best possible outcome (eg: maximizing profit or minimizing cost) within a set of constraints where the relation b/w the variables and the objective function are linear.

→ Defns.

1) Objective function : funn to be optimized.

2) Constraints : Linear Eqn or inequalities that defines the boundaries of the feasible soln.

3) Decision Variable : The variables that are subjected to constraints that the model uses to make decisions.

4) Feasible Soln : Any combn of decision variables that satisfies all constraints.

5) Optimal Soln : It is the feasible soln that gives best value for the objective funn.

⊛ Applications :-

Manufacturing, Resource allocation, operations management.

→ Problems.

1) A manufacturer produces two types of models, M_1 and M_2. Each M_1 model requires 4 hours of grinding & 2 hours of polishing & each M_2 model requires 2 hours of grinding & 5 hrs of polishing. The manuf has 2 grinders & 3 polishers. Each grinder works for 40 hours a week & polisher 60 hrs a week. Profit on M_1 model is Rs. 3 & M_2 is Rs. 4. How should the manufr allocate his producth capacity to the 2 types of models so that he makes max. profit in a week.

→ x be the no. of products of M_1
y " " " M_2

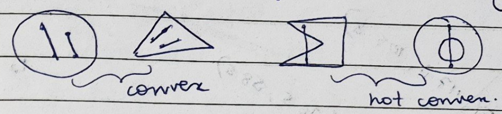
Profit : $z = 3x + 4y$ → be maximised.
(z is called the objective funn).

(x) M_1 : 4 hrs gr, 2 hrs pol
(y) M_2 : 2 hrs gr, 5 hrs pol.

Availabe : 2 gr × 40 hrs = 80 hrs per week.
3 pol × 60 hrs = 180 hrs per week.

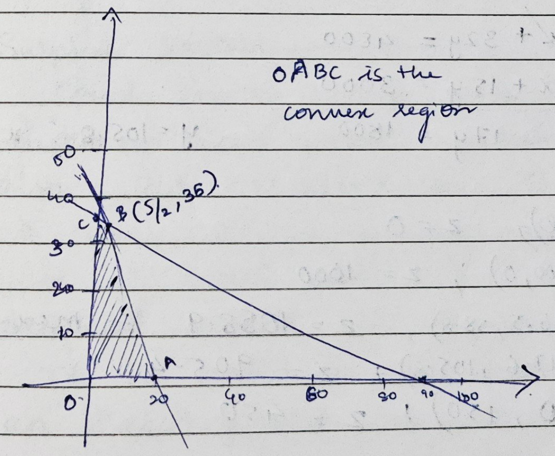
$\{ 4x + 2y \le 80$; $2x + 5y \le 180$; $x, y \ge 0$.

• Graphical Method (for LPP in 2 var)
Working Rule
① Formulate the given problem as an LPP
② Plot the given constraints as equality in X-Y plane and determine the convex region formed by them.
• convex region : If the line joining any 2 pts in it lies completely in the region.



③ Determine the vertices of the convex region & find the value of the objective function at these vertices.
④ The vertex which gives the optimal value of the the fun is the q

Solve the LPP graphically.
Maximize $z = 3x + 4y$ subject to $4x + 2y \le 80$, $2x + 5y \le 180$.



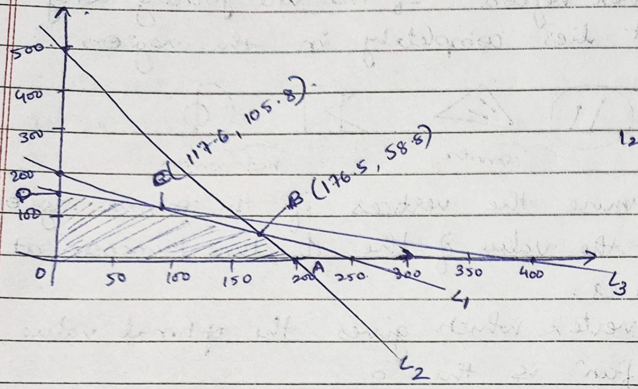
$4x + 2y \le 80$
$x = 0, y = 40$
$y = 0, x = 20$

$2x + 5y \le 180$
$x = 0, y = 36$
$y = 0, x = 90$

B : $-4x + 2y = 80$
$4x + 10y = 360$
$8y = 280$
$y = \frac{280}{8} = 35$

At O : $(0,0)$, $z = 0$
At A : $(20,0)$, $z = 60$
At B : $(5/2, 35)$, $z = 3\times\frac{5}{2} + 35\times 4 = 147.5$ ← max.
At C : $(0, 36)$, $z = 144$

2) Maximize $z = 5x + 3y$, Subject to the Constraints $4x + 5y \le 1000$, $5x + 2y \le 1000$, $3x + 8y \le 1200$, $x, y \ge 0$.



L_1 $4x + 5y = 1000$
$x = 0, y = 200$
$x = 250, y = 0$

L_2 $5x + 2y = 1000$
$x = 0, y = 500$
$x = 200, y = 0$

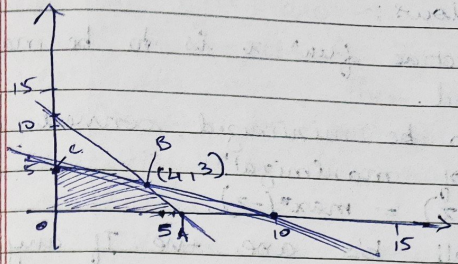
L_3 $3x + 8y = 1200$
$x = 0, y = 150$
$x = 400, y = 0$

B: → $25x + 10y = 5000$
$-8x + 10y = 2000$
$17x = 3000$
$x = 176.5$ $y = 58.8$

C: $12x + 32y = 4800$
$\ominus$ $12x + 15y = 3000$
$17y = 1800$ $y = 105.8$, $x = 117.6$

At O : $(0, 0)$, $z = 0$
At A : $(200, 0)$, $z = 1000$
At B : $(176.5, 58.8)$, $z = 1058.9$ ← max.
At C : $(117.6, 105.8)$, $z = 905.4$
At D : $(0, 150)$, $z = 450$

3) Find the minimum value of $z = 2x - 3y$ under the constraints $x + 2y \le 10$, $2x + y \le 11$, $x, y \ge 0$



$x + 2y = 10$
$x = 0, y = 5$
$y = 0, x = 10$

$2x + y = 11$
$x = 0, y = 11$
$x = 5.5, y = 0$

$2x + 4y = 20$
$\ominus$ $2x + y = 11$
$3y = 9$
$y = 3$
$x = 4$

At O : $(0, 0)$, $z = 0$
At A : $(5.5, 0)$, $z = 11$
At B : $(4, 3)$, $z = -1$
At C : $(0, 5)$, $z = -15$ ← min.

→ Slack Variables. (s_i)

These represent an unusual resource. It is added to less than or equal to constraints in order to get equality of constraints.

→ Surplus Variables.

These represent the amount by which solution values exceed a resource. These are also called -ve slack vars. It is added to $\ge$ constraint to get an equality constraint.

→ Artificial Variable.

It is added to the constraints to get an initial feasible solⁿ to the LPP.

→ Working procedure of SIMPLEX method.

Assuming the existance of an initial basic feasible sol? of an LPP, an optimal solⁿ by SIMPLEX method is found as follows:-

1) Check if objective funⁿ z is to be maximized or minimized.
If it is to be minimized, convert it to a problem of maximizatⁿ.
$$\min(z) = \max(-z)$$

2) Check if all b's are +ve. If any b_i is negative, multiply that Eqⁿ with -1.

3) Express the problem in the std. form. Convert all inequalities of constraints into Eqⁿˢ by introducing slack or surplus variables.

Eg:- $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 + 0s_2 + \dots + 0s_m = b_1$

4) Find an initial basic feasible solⁿ. If there are m Eqⁿˢ involving n unknowns, then assign zero values to any $(n-m)$ variables for finding a solⁿ.

i) Starting with a basic solⁿ for which x_j where $j = 1, 2, \dots, (n-m)$ are all zero, find all s_i's.

ii) If $s_i \geq 0 \ \forall i$, the basic solⁿ is feasible and non-degenerate. If one or more of s_i values are zero, then the solⁿ is degenerate. s_i's are called the basic variables & x_i's are called the non-basic variables.

iii) c_j row denotes the coefficients of the variables in the objective function.
c_B column denotes the co-eff of basic variables in the objective funcⁿ.

5) Apply optimality test.
Compute $C_j = c_j - z_j$, where $z_j = \sum c_B a_{ij}$

If all C_j's are $-$ve, the basic feasible solⁿ is optimal. If even one of the C_j's are +ve, then the current feasible solⁿ is not optimal.

6) Identify the incoming and outgoing variables.
If there are more than one C_j's, then the incoming var is the one that heads the column containing the max. C_j. This then becomes the key column.
If more than one variable has the same max. C_j, then any of the colums can be chose.
Now divide under the b column by the key column & choose the row containing the min positive ratio θ.

→ Problems:-

1) Use Simplex method to maximize $z = 5x_1 + 3x_2$ subject to $x_1 + x_2 \leq 2$, $5x_1 + 2x_2 \leq 10$, $3x_1 + 8x_2 \leq 12$, $x_1, x_2 \geq 0$.

Minimize: $z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to: $x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 2$
$5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 10$
$3x_1 + 8x_2 + 0s_1 + 0s_2 + s_3 = 12$

Initial Solⁿ:- put $x_1, x_2 = 0$
$\Rightarrow s_1 = 2, \ s_2 = 10, \ s_3 = 12$
Since $s_i > 0 \ \forall i$, this is a feasible solⁿ.

Initial	C_j	5	3	0	0	0		
C_B	Basic variables	x_1	x_2	s_1	s_2	s_3	RHS (b)	Ratio (θ)
0	s_1	1	1	1	0	0	2	$2/1 = 2$
0	s_2	5	2	0	1	0	10	$10/5 = 2$
0	s_3	3	8	0	0	1	12	$12/3 = 4$
	$Z_j = \Sigma C_B a_{ij}$	0	0	0	0	0		
	$C_j = c_j - z_j$	5	3	0	0	0		

pivot element

key/pivot column

largest

smallest key/pivot row

$R_2 \to R_2 - 5R_1$, $R_3 \to R_3 - 3R_1$, Change the basic var of pivot row.

First	C_j	5	3	0	0	0		
C_B	Basic variables	x_1	x_2	s_1	s_3	s_3	RHS (b)	Ratio (θ)
5	x_1	1	1	1	0	0	2	
0	s_2	0	-3	-5	1	0	0	
0	s_3	0	5	-3	0	1	6	
	$z_j = \Sigma C_B a_{ij}$	5	5	5	0	0	10	
	$C_j = c_j - z_j$	0	-2	-5	0	0		

all C_j's are -ve & 0 i.e. $C_j \leq 0 \ \forall j$

$\Rightarrow$ We've reached the optimal value ($z = 10$)

$\therefore x_1 = 2, \ x_2 = 0$

2) A firm produces 3 products which are processed on 3 machines. The relevant data is given in the table.

Machine	Time per unit (mins): Prod. A	Prod. B	Prod. C	Machine capacity mins / day
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

The profit per unit for prods A, B & C are ₹. 4, 3, & 6 resp.

Determine the daily no. of units to be manufactured for each product. Assume that all units produced are consumed in the market.

Let the no. of units of A, B & C produced be x_1, x_2 & x_3 resp.

Maximize: $4x_1 + 3x_2 + 6x_3$

Subject to: $2x_1 + 3x_2 + 2x_3 \leq 440$

$4x_1 + 3x_3 \leq 470$

$2x_1 + 5x_2 \leq 430$

$z = 4x_1 + 3x_2 + 6x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to: $2x_1 + 3x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 440$

$4x_1 + 3x_3 + 0s_1 + s_2 + 0s_3 = 470$

$2x_1 + 5x_2 + 0s_1 + 0s_2 + s_3 = 430$

Initial Sol^n :- put, $x_1, x_2, x_3 = 0$

$\Rightarrow s_1 = 440, \ s_2 = 470, \ s_3 = 430$

Since $s_i > 0 \ \forall i$, this is a feasible sol^n.

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Initial C_B	c_j Basic variables	4 x_1	3 x_2	6 x_3	0 s_1	0 s_2	0 s_3	RHS (b)	Ratio (θ)
0	s_1	2	3	2	1	0	0	440	440/2 = 220 ← smallest
0	s_2	4	0	3	0	1	0	470	470/3 = 156 ← smallest
0	s_3	2	5	0	0	0	1	430	430/0 = ∞
	$z_j = \sum C_B a_{ij}$	0	0	0	0	0	0		
	$C_j = c_j - z_j$	4	3	6	0	0	0		

↑ largest

pivot element 3.

$R_2 \to R_2/3$

$R_1 \to R_1 - 2R_2$

First C_B	c_j Basic variables	4	3	6	0	0	0	RHS (b)	Ratio (θ)
0	s_1	-2/3	3	0	1	-2/3	0	380/3	380/9 ← smallest
6	x_3	4/3	0	1	0	1/3	0	470/3	—
0	s_3	2	5	0	0	0	1	430	430/5
	$z_i = \sum C_B a_{ij}$	8	0	6	0	2	0	940	
	$C_j = c_j - z_j$	-4	+3	0	0	-2	0		

↑ pivot

$R_1 \to R_1/3$

Second C_B	c_j Basic va	4	3	6	0	0	0	RHS	Ratio
3	x_2	-2/9	1	0	1/3	-2/9	0	380/9	
6	x_3	4/3	0	1	0	1/3	0	470/3	
0	s_3	28/9	0	0	-5/3	10/9	0	1970/9	
	$z_j = \sum C_B a_{ij}$	-20/3	3	6	-1	-4/3	0		

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3) Solve the LPP by Simplex Method minimize $z = -4x + 2y$.

Subject to to $6x + 2y \le 18$, $3x - 2y \le 6$, $x, y \ge 0$

Maximize $(-z) = z' = 4x - 2y + 0s_1 + 0s_2$.

Subject to: $6x + 2y + s_1 + 0s_2 = 18$

$3x - 2y + 0s_1 + s_2 = 6$

Initial Sol^n: $x, y = 0$

$\Rightarrow s_1 = 18,\ s_2 = 6$

$s_i \ge 0\ \forall\ i$, this is a feasible sol^n.

Initial C_B	c_j Basic variable	4 x	-2 y	0 s_1	0 s_2	RHS (b)	Ratio (θ)
0	s_1	6	2	1	0	18	18/6 = 3
0	s_2	3	-2	0	1	6	6/3 = 2 ← sm
	$z_j = \sum C_B a_{ij}$	0	0	0	0		
	$C_j = c_j - z_j$	+4	+2	0	0		

↑ largest

pivot

2 + 4

$R_2 \to R_2/3$

$R_1 \to R_1 - 6R_2$

First C_B	c_j Basic variab	4 x	2 y	0 s_1	0 s_2	RHS (b)	Ratio (θ)
0	s_1	0	6	1	-2	18	
4	x	1	-2/3	0	1/3	2	
	$z_j = \sum C_B a_{ij}$	4	-8/3	0	4/3		
	$C_j = c_j - z_j$	0	14/3	0	-4/3		

$2 + \frac{8}{3}$ $\frac{14}{3}$ $-6/3 - \frac{2 \times 3}{3} = \frac{-12}{3} = -4$

$R_2 \to R_2/3, \quad R_1 \to R_1 - 6R_2$

First	C_j	4	-2	0	0		
C_B	Basic var	x	y	s_1	s_2	RHS (b)	Ratio (θ)
0	s_1	0	6	1	-2	6	1
4	x	1	-2/3	0	1/3	2	-ve
	$z_j = \Sigma c_B a_{ij}$	4	-8/3	0	4/3	8	
	$C_j = c_j - z_j$	0	2/3	0	-4/3		

pivot

$R_1 \to R_1/6, \quad R_2 \to R_2 + \frac{2}{3}R_1$

Second	C_j	4	-2	0	0		
C_B	Basic var	x	y	S_1	S_2	RHS (b)	Ratio (θ)
-2	y	0	1	1/6	-1/3	1	
4	x	1	0	1/9	1/9	8/3	
	$z_j = \Sigma c_B a_{ij}$	4	-2	1/9	10/9	26/3	
	$C_j = c_j - z_j$	0	0	-1/9	-10/9		

all C_j's are -ve & 0 i.e. $C_j \le 0 \ \forall j$

optimal solⁿ, $z = \frac{26}{3}$

$\therefore x = 8/3, \ y = 1 \Rightarrow Z_{max} = 26/3$

$\therefore Z_{min} = -\frac{26}{3}$

4) Minimize $z = x_1 - 3x_2 + 3x_3$ Subject to
$3x_1 - x_2 + 2x_3 \le 7$, $2x_1 + 4x_2 \ge -12$,
$-4x_1 + 3x_2 + 8x_3 \le 10$, $x_1, x_2, x_3 \ge 0$.

$z = x_1 - 3x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to: $3x_1 - x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 7$
$-2x_1 - 4x_2 + 0s_1 + s_2 + 0s_3 = 12$
$-4x_1 + 3x_2 + 8x_3 + 0s_1 + 0s_2 + s_3 = 10$

Initial Solⁿ: $x_1, x_2, x_3 = 0$
$\Rightarrow s_1 = 7, \ s_2 = 12, \ s_3 = 10$
$s_i \ge 0 \ \forall i$, this is a feasible solⁿ.

Initial	C_j	1	-3	3	0	0	0		
C_B	Basic var	x_1	x_2	x_3	s_1	s_2	s_3	RHS (b)	Ratio (θ)
0	s_1	3	-1	2	1	0	0	7	7/2 = 3.5
0	s_2	-2	-4	0	0	1	0	12	∞
0	s_3	-4	3	8	0	0	1	10	10/8 = 1.25
	$z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0	
	$C_j = c_j - z_j$	1	-3	3	0	0	0		

$R_3 \to R_3/8 \quad R_1 \to R_1 - 2R_3$

First	C_j	1	-3	3	0	0	0		
C_B	Basic var	x_1	x_2	x_3	s_1	s_2	s_3	RHS (b)	Ratio (θ)
0	s_1	4	-7/4	0	1	0	-1/4	4.5	4.5/4 = 9/8
0	s_2	-2	-4	0	0	1	0	12	-ve
3	x_3	-1/2	3/8	1	0	0	1/8	10/8	-ve
	$z_j = \Sigma c_B a_{ij}$	-3/2	9/8	3	0	0	3/8	30/8	
	$C_j = c_j - z_j$	5/2	-33/8	0	0	0	-3/8		

pivot

$R_1 \to R_1/4$ (Make the elements below pivot in other rows 0 by row operⁿ)

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→ Artificial Variables Technique.

This happens w/

→ BIG M Method (Method of Penalties)

• Working Procedure

1) Express the problem in standard form.
2) Add non-negative variables to the LHS of all those constraints which are of ≥ or = type. These are called Artificial Variables and help us obtain an initial basic feasible solⁿ. But their addⁿ causes violation of the corresponding constraints so we don't want them to appear in the final solⁿ. For this purpose, we assign a very large penalty (−M) to these artificial variables in the objective function.
3) Solve the modified LPP by simplex method. 3 cases can arise at every iteration.
i) There is no Artificial variable in the basis & the optimality condⁿ is satisfied. Then the solⁿ is an optimal basic feasible solⁿ.

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ii) There is atleast one artificial variable in the basis at zero level & the optimality condⁿ is satisfied. Then the solⁿ is degenerate optimal basic feasible solⁿ.
iii) The is atleast 1 artificial variable in the basis at non-zero level with +ve value in b-column & the optimality condⁿ is satisfied. Then the problem has no feasible solⁿ. The final is not optimal. since the objective funⁿ contains an unknown quantity (M). This solⁿ is called pseudo-optimal solⁿ.

Remark:
The Artificial variables are only a computational device to get a meaningful solⁿ. Once an AV leaves the basis, it has served its purpose & the column for that variable can be omitted from the simplex table.

→ Problems.

1) Maximize $6x_1 + x_2$ under the constraints $-x_1 + 3x_2 \le 6$, $x_1 - 3x_2 = 6$, $x_1 + x_2 \ge 1$, $x_1, x_2 \ge 0$.

→ In Std. form:

Maximize: $6x_1 + x_2$

$-x_1 + 3x_2 + s_1 + 0s_2 = 6$

$x_1 - 3x_2 = 6$

$x_1 + x_2 + 0s_1 - s_2 = 1$.

Maximize, $Z = 6x_1 + x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

$-x + 3x_2 + s_1 = 6$

$x_1 - 3x_2 + A_1 = 6$

$x_1 + x_2 - s_2 + A_2 = 1$

Initial C_B	C_j / Basic variable	6 x_1	1 x_2	0 s_1	0 s_2	$-M$ A_1	$-M$ A_2	RHS(b)	Ratio(θ)
0	s_1	-1	3	1	0	0	0	6	$6/-1 = \cancel{-6}$
$-M$	A_1	1	-3	0	0	1	0	6	$6/1 = 6$
$-M$	A_2	1	1	0	-1	0	1	1	$1/1 = 1$ small
	$z_j = \Sigma C_B a_{ij}$	$-2M$	$2M$	0	M	$-M$	$-M$	$-7M$	
	$C_j = c_j - z_j$	$6+2M$	$1-2M$	0	$-M$	0	0		

pivot   ↑ large

$R_2 \rightarrow R_2 - R_3$ $R_1 \rightarrow R_1 + R_3$

First C_B	C_j / Basic var	6 x_1	1 x_2	0 s_1	0 s_2	$-M$ A_1	$-M$ A_2	RHS(b)	Ratio(θ)
0	s_1	-1	4	0	-1	0	1	7	$7/-1 = \cancel{-7}$
$-M$	A_1	0	-4	0	1	1	-1	5	$5/1 = 5$
6	x_1	1	1	0	-1	0	1	1	$1/-1 = -ve$
	$z_j = \Sigma C_B a_{ij}$	6	$4M+6$	0	$-M-6$	$-M$	$M+6$	$-5M+6$	
	$C_j = c_j - z_j$	0	$-4M-5$	0	$M+6$	0	$-2M-6$		

↑ large

Second C_B	C_j / Basic var	6 x_1	1 x_2	0 s_1	0 s_2	$-M$ A_1	$-M$ A_2	RHS(b)	Ratio(θ)
0	s_1	-1	0	0	0	1	0	12	$12/0 = \infty$
0	s_2	0	-4	0	1	1	-1	5	$5/-4 = -ve$
6	x_1	1	-3	0	0	1	0	6	$6/-3 = -ve$
	$z_j = \Sigma C_B a_{ij}$	6	-18	0	0	6	0	36	
	$C_j = c_j - z_j$	0	19	0	0	$-M-6$	$-M$		

↑ lar

Ratios have no small +ve values.

∴ No pivot can be found.

Thus, the problem is degenerate.

2) Minimize $2x_1 + x_2$ under the constraints $3x_1 + x_2 = 3$, $4x_1 + 3x_2 \geq 6$, $x_1 + 2x_2 \leq 3$, $x_1, x_2 \geq 0$.

→ Std form

Maximize $z' = -z = -2x_1 - x_2$

$3x_1 + x_2 = 3$

$-4x_1 - 3x_2 \leq -6$

$x_1 + 2x_2 \leq 3$

Maximize, $z' = -2x_1 - x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

$3x_1 + x_2 + A_1 = 3$

$+4x_1 + 3x_2 - s_1 + A_2 = 6$

$x_1 + 2x_2 + s_2 = 3$

Initial C_B	C_j / Basic	-2 x_1	-1 x_2	0 s_1	0 s_2	$-M$ A_1	$-M$ A_2	RHS(b)	Ratio(θ)
$-M$	A_1	3	1	0	0	1	0	3	$3/3 = 1$
$-M$	A_2	4	3	-1	0	0	1	6	$6/4 = 1.5$
0	s_2	1	2	0	1	0	0	3	$3/1 = 3$
	$z_j = \Sigma C_B a_{ij}$	$-7M$	$-4M$	M	0	$-M$	$-M$		
	$C_j = c_j - z_j$	$-2+7M$	$-1+4M$	$-M$	0	0	0		

↑ la

$R_1 \rightarrow R_1/3$

$2 - 1/3$

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$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - R_1$

First	c_j	-2	-1	0	0	-M	-M		
C_B	Basic var	x_1	x_2	s_1	s_2	A_1	A_2	RHS(b)	Ratio(θ)
-2	x_1	1	1/3	0	0	1/3	0	1	3
-M	A_2	0	5/3	-1	0	-4/3	1	2	6/5
0	s_2	0	5/3	0	1	-1/3	0	2	6/5
	$z_j = \Sigma c_B a_{ij}$	-2	$-2/5 - 5M/3$	M	0	$-2/3 + \frac{4M}{3}$	-M		
		0	$-1/3 + \frac{5M}{3}$	-M	0	$\frac{2}{3} - \frac{7M}{3}$	0		

↑

pivot

$\frac{1}{3}$ $\qquad -2/5 + \frac{3}{5} = 1/5$

$R_2 \to R_2 \times 3/5$ $\qquad R \qquad \frac{3}{5} \times \frac{5}{9}$ $\qquad R_1 \to R_1 - \frac{5}{9} R_2$

Second	c_j	-2	-1	0	0	-M	-M		
C_B	Basic var	x_1	x_2	s_1	s_2	A_1	A_2	RHS(b)	Ratio(θ)
-2	x_1	1	0	1/5	0			3/5	
-1	x_2	0	1	-3/5	0	-4/5	3/5	6/5	
0	s_2	0	0	1	1			0	
	$z_j = \Sigma c_B a_{ij}$	-2	-1	1/5	0			-12/5	
	$C_j = c_j - z_j$	0	0	-1/5	0				

$\max z' = -12/5$

$\therefore$ Min $z = 12/5$ at $x_1 = 3/5$
$x_2 = 6/5$

3) Maximize $z = 3x_1 + 2x_2$ subject to $2x_1 + x_2 \leq 2$, $3x_1 + 4x_2 \geq 12$, $x_1, x_2 \geq 0$.

Std LPP :- Maximize : $z = 3x_1 + 2x_2$

Subject to : $2x_1 + x_2 + s_1 + 0s_2 = 2$
$3x_1 + 4x_2 + 0s_1 - s_2 = 12$

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$z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - MA_1$

Subject $2x_1 + x_2 + s_1 + 0s_2 = 2$
$3x_1 + 4x_2 + 0s_1 - s_2 + A_1 = 12$

Initial	c_j	3	2	0	0	-M		
C_B	Basic	x_1	x_2	s_1	s_2	A_1	RHS(b)	Ratio(θ)
0	s_1	2	1	1	0	0	2	2/1 = 1 ←
-M	A_1	3	4	0	-1	1	12	12/4 = 3
	$z_j = \Sigma c_B a_{ij}$	-3M	-4M	0	M	-M	-12M	
	$C_j = c_j - z_j$	3+3M	2+4M	0	-M	0		

pivot $\quad$ key

$R_2 \to R_2 - 4R_1$ $\qquad$ 12 - 8

First	c_j	3	2	0	0	-M		
C_B	Basic var.	x_1	x_2	s_1	s_2	A_1	RHS(b)	Ratio(θ)
2	x_2	2	1	1	0	0	2	
-M	A_1	-5	0	-4	-1	1	4	
	$z_j = \Sigma c_B a_{ij}$	4+5M	2	2+4M	M	-M	4-4M	
	$C_j = c_j - z_j$	-1-5M	0	-2-4M	-M	0		

$C_j \leq 0 \; \forall j$

The objective function will have an artificial var in the sol^n. ⇒ we get PSEUDO optimal sol^n.

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→ Two Phase Method.

- Working procedure

This is another method when we have artificial variables in the std. form of LPP.

- PhaseI.

i) Express the given problem in std form by introducing slack, surplus & artificial vars

ii) Formulate an artificial objective function z^* and this consists of all artificial variables by assigning (-1) cost to each of A.Vs & (0) cost to all other variables.

iii) Maximize z^* subject to the constraints of the original problem using the simplex method. Then 3 cases can arise:-

1) Max $z^* < 0$ & atleast one A-var appears in the optimal basis at a positive level. In this case, the original problem doesn't posses any feasible sol^n & the procedure comes to end.

2) Max $z^* = 0$ & no artificial var appears in the optimal basis at zero level. In this case, a basic feasible sol^n is obtained & we proceed to Phase-II for finding the optimal basic feasible sol^n for the orig. problem

3) Max $z^* = 0$ & atleast one artificial var appears in the optimal basis at zero level

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Here, a feasible sol^n to the auxiliary LPP is also a feasible sol^n to the orig. prob with all A-vals set to zero.

- Phase-II

The basic feasible sol^n found at the end of Phase-I is used as the starting sol^n for the original problem in this phase i.e. the final simplex table of phase-I is taken as the initial simplex table of pase-II & the A-obj fun^n is replaced by the original obj fun^n & then we find the solution.

→ Problems.

1) Use 2-phase method to minimize $z = 7.5x_1 - 3x_2$ subject to $3x_1 - x_2 - x_3 \geq 3$, $x_1 - x_2 + x_3 \geq 2$, $x_1, x_2, x_3 \geq 0$.

→ $3x_1 - x_2 - x_3 - s_1 + A_1 = 3$

$x_1 - x_2 + x_3 - s_2 + A_2 = 2$

Max. $Z^* = -A_1 - A_2$

→ Initial basic feasible sol^n

$x_1, x_2, x_3, s_1, s_2 = 0$

$\Rightarrow A_1 = 3$

$A_2 = 2$

& $z^* = -5$

C_B	c_j / Basic var	x_1 (0)	x_2 (0)	x_3 (0)	s_1 (0)	s_2 (0)	A_1 (-1)	A_2 (-1)	RHS(b)	Ratio(θ)
-1	A_1	3	-1	-1	-1	0	1	0	3	3/3 = 1 ←
-1	A_2	1	-1	1	0	-1	0	1	2	2/1 = 2
	$z_j = \Sigma c_B a_{ij}$	-4	2	0	1	1	-1	-1	-5	
	$C_j = c_j - z_j$	4	-2	0	-1	-1	0	0		

pivot ↑ larg

$R_1 \to R_1/3$ $R_2 \to R_2 - R_1$

C_B	c_j / Basic var	x_1 (0)	x_2 (0)	x_3 (0)	s_1 (0)	s_2 (0)	A_1 (-1)	A_2 (-1)	RHS(b)	Ratio(θ)
0	x_1	1	-1/3	-1/3	-1/3	0	1/3	0	1	-ve
-1	A_2	0	-2/3	4/3	1/3	-1	-1/3	1	1	3/4
	$z_j = \Sigma c_B a_{ij}$	0	2/3	-4/3	-1/3	1	1/3	-1	-1	
	$C_j = c_j - z_j$	0	-2/3	+4/3	-1/3	-1	-4/3	0		

pivot ↑

$R_2 \longrightarrow R_2 \times \frac{3}{4}$ $R_1 \to R_1 + \frac{1}{3} R_2$

C_B	c_j / Basic var	x_1 (0)	x_2 (0)	x_3 (0)	s_1 (0)	s_2 (0)	A_1 (-1)	A_2 (-1)	RHS(b)	Ratio(θ)
0	x_1	1	-1/2	0	-1/4	-1/4	1/4	1/4	5/4	
0	x_3	0	-1/2	1	1/4	-3/4	-1/4	3/4	3/4	
	$z_j = c_B a_{ij}$	0	0	0	0	0	0	0	0	
	$C_j = c_j - z_j$	0	0	0	0	0	-1	-1		

$C_j = 0$, Phase I is complete.

Phase - II

Maximize $z' = -7.5x_1 - 3x_2 + 0x_3 + 0s_1 + 0s_2 + 0A_1 + 0A_2$

st :- $3x_1 - x_2 - x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3$

$x_1 - x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2$

C_B	c_j / Basic var	x_1 (-7.5)	x_2 (+3)	x_3 (0)	s_1 (0)	s_2 (0)	RHS(b)	Ratio(θ)
-7.5	x_1	1	-1/2	0	-1/4	-1/4	5/4	
0	x_3	0	-1/2	1	1/4	-3/4	3/4	
	$z_j = \Sigma c_B a_{ij}$	-7.5	15/4	0	15/8	15/8	-75/8	
	$C_j = c_j - z_j$	0	-3/4	0	-15/8	-15/8		

$x_1 = \frac{5}{4}$, $x_2 = 0$, $x_3 = \frac{3}{4}$

Min $z = \frac{75}{8}$

2) Use two phase method to maximize $Z = 6x + 3y$ subject to $x + y \geq 1$, $2x - y \geq 1$, $3y \leq 2$, $x, y \geq 0$

Max $z^* = -A_1 - A_2$

s.t $x + y - s_1 + A_1 = 1$

$2x - y - s_2 + A_2 = 1$

$3y + s_3 = 2$

Initial basic feasible solⁿ,

$x, y, s_1, s_2 = 0$

$A_1 = 1$, $A_2 = 1$, $s_3 = 2$

& $z^* = -2$

C_B	c_j / Basic var	0 / x	0 / y	0 / s_1	0 / s_2	0 / s_3	-1 / A_1	-1 / A_2	RHS (b)	Ratio (θ)
-1	A_1	1	1	-1	0	0	1	0	1	1
-1	A_2	2	-1	0	-1	0	0	1	1	$1/2 = 0.5$ ←
0	s_3	0	3	0	0	1	0	0	2	0
	$z_j = c_B a_{ij}$	-3	0	1	1	0	-1	-1		
	$C_j = c_j - z_j$	3	0	-1	-1	0	0	0		

pivot ↑ ...

$R_2 \to R_2/2$. $R_1 \to R_1 - R_2$

C_B	c_j / Basic var	0 / x	0 / y	0 / s_1	0 / s_2	0 / s_3	-1 / A_1	-1 / A_2	RHS / (b)	Ratio (θ)
-1	A_1	0	$3/2$	-1	$1/2$	0	1	$-1/2$	$1/2$	$1/3$ ←
0	x	1	$-1/2$	0	$-1/2$	0	0	$1/2$	$1/2$	-ve
0	s_3	0	3	0	0	1	0	0	2	$2/3$
	$z_j = c_B a_{ij}$	0	$-3/2$	1	$-1/2$	0	-1	$1/2$	$-1/2$	
	$C_j = c_j - z_j$	0	$3/2$	-1	$1/2$	0	0	$-3/2$		

pivot ↑

$\frac{1}{2} - \frac{1}{4}$

$R_3 \to R_3 - 3R_1$ $\quad 0 - \frac{1}{2}$. $\quad -\frac{1}{2} + \frac{1}{6}$

$R_1 \to R_1 \times 2/3$ $\qquad R_2 \to R_2 + R_1/2$. $\qquad -\frac{3+1}{6}$

C_B	c_j / Basic var	0 / x	0 / y	0 / s_1	0 / s_2	0 / s_3	-1 / A_1	-1 / A_2	RHS (B)	Ratio (θ)
0	y	0	1	$-2/3$	$+1/3$	0	$2/3$	$-2/3$	$1/3$	
0	x	1	0	$-1/3$	$-1/3$	0	$1/3$	$1/6$	$2/3$	
0	s_3	0	0	2	-1	1	-2	2	1	
		0	0	0	0	0	0	0		

Phase I completed.

Phase-II

Maximize, $z' = 6x + 3y + 0s_1 + 0s_2 + 0s_3 + 0A_1 + 0A_2$

St: $x + y - s_1 + 0s_2 + 0s_3 + A_1 + 0A_2 = 1$

$2x - y + 0s_1 - s_2 + 0s_3 + 0A_1 + A_2 = 1$

$3y + 0s_1 + 0s_2 + s_3 + 0A_1 + 0A_2 = 2$

C_B	c_j / Basic	6 / x	3 / y	0 / s_1	0 / s_2	0 / s_3	RHS (b)	Ratio (θ)
6	x	0	1	$-2/3$	$+1/3$	0	$1/3$	∞
3	y	1	0	$-1/3$	$-1/3$	0	$2/3$	$2/3$ ←
0	s_3	0	0	2	-1	1	1	∞
	$z_j = \Sigma c_B a_{ij}$	3	6	$-7/3$	1	0	4	
	$C_j = c_j - z_j$	3	-3	$7/3$	-1	0		

↑ larg

No solution?

Every LPP has associated with it with the same data and closely related optimal solns. such two problems are said to be dual's of each other, while one of them is called 'Primal' & other is called 'Dual'

If the Primal contains large no. of constraints & a smaller no. of variables, the competitional complexity can be considerably reduced if we connect it to a dual.

& Consider the LPP, Maximize $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to :-

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \end{array}\right\} \text{(I)}$$

To construct a dual problem, the foll^n is done.

i) The maximazation problem in the primal becomes minimz^n problem in dual & vice versa.

ii) ($\le$) type of constraints in primal become ($\ge$) in dual

iii) The coefficients $c_1, c_2, \ldots, c_n$ in primal become $b_1, b_2 \ldots b_m$ in dual

iv) The constants $b_1, b_2, \ldots b_m$ in the constraints of primal become $c_1, c_2 \ldots c_n$ in dual

v) If the primal has n variable & m constraints the dual will have m variable and n constraints.

vi) The variables in both primal & dual are non-negatives.

Dual of (I) is

Minimize, $w = b_1y_1 + b_2y_2 + \cdots + b_my_m$

st:- $a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1$

$\vdots$

$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}x_n \ge c_m$

→ Problems :-

1) Write the dual of the LPP

Minimize $z = 3x_1 - 2x_2 + 4x_3$ subject to :-

$3x_1 + 5x_2 + 4x_3 \ge 7$, $6x_1 + x_2 + 3x_3 \ge 4$,

$7x_1 - 2x_2 - x_3 \le 10$, $x_1 - 2x_2 + 5x_3 \ge 3$

$4x_1 + 7x_2 - 2x_3 \ge 2$, $x_1 x_2 x_3 \ge 0$.

Maximize : $7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

Subject to :-

$3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \le 3$

$5y_1 + 6y_2 + 2y_3 - 2y_4 + 7y_5 \le -2$

$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \le +4$

(*) Note :

For the equality constraints like

$a_{11}x_1 + a_{12}x_2 = b_1$

$\Rightarrow a_{11}x_1 + a_{12}x_2 \ge b_1$

$a_{11}x_1 + a_{12}x_2 \le b_1$

2) Write the dual of LPP :-

Max, $z = 4x_1 + 9x_2 + 2x_3$ subject to

$2x_1 + 3x_2 + 2x_3 \leq 7$,

$3x_1 - 2x_2 + 4x_3 = 5$

constraints :-

$$2x_1 + 3x_2 + 2x_3 \leq 7$$
$$3x_1 - 2x_2 + 4x_3 \leq 5$$
$$-3x_1 + 2x_2 - 4x_3 \leq -5$$

dual :-

Minimize :- $z = 7y_1 + 5y_2 - 5y_3$

st :-
$$2y_1 + 3y_2 - 3y_3 \geq 4$$
$$3y_1 - 2y_2 + 2y_3 \geq 9$$
$$2y_1 + 4y_2 - 4y_3 \geq 2$$