Poisson distribution

A discrete random variable is X is said to follow a Poisson distribution if its probability function is given by

$$P(X = x) = \frac{e^{-m}m^x}{x!} \text{ , } if\ x = 0, 1, 2, \dots$$

Where $m = np$ is a parameter of the distribution.

NOTE:

$$\sum P(X = x) = e^{-m}\left\{\frac{m^0}{0!} + \frac{m^1}{1!} + \frac{m^2}{2!} + \cdots .\right\}$$

$$\sum P(X = x) = e^{-m}e^{m} = 1$$

The Poisson distribution is a distribution related to probabilities of the rare events having a large number of independent opportunities for occurrence.

i.e., $(n \to \infty,\ p \to 0)$ & $np = m$ is finite

Example(1)

The number of defective items in a packing manufactured by a good company.

Example(2)

The number of children born blind per year in a large city.

Poisson distribution as limiting case of binomial distribution

Statement:

The limiting form of the binomial distribution as $n \to \infty$ & $p \to 0$ in such a way that $np = m$ (where m is finite) is the Poisson distribution

Proof:

The probability of x success in n trials in a binomial distribution is

$$P(x) = n_{C_x} p^x q^{n-x}$$

$$\Rightarrow P(x) = \frac{n!}{x!\ (n-x)!} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-(x-1))(n-x)!}{x!\ (n-x)!} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-(x-1))}{x!} p^x (1-p)^{n-x}$$

$$P(x) = \frac{np(np-p)(np-2p)\ldots\ldots(np-(x-1)p)}{x!} (1-p)^{n-x}$$

Taking $m = np$,we get

$$P(x) = \frac{m(m-p)(m-2p)\ \ldots\ldots (m-(x-1)p)}{x!}\left(1-\frac{m}{n}\right)^{n-x}$$

$$\left(\because p = \frac{m}{n}\right)$$

Now apply limit as $n \to \infty$ & $p \to 0$ in such a way that $np = m$ is finite.

$$\therefore P(x) = \frac{m^x}{x!}\lim_{n\to\infty}\left(1-\frac{m}{n}\right)^{n-x}$$

$$\therefore P(x) = \frac{m^x}{x!}\lim_{n\to\infty}\left[\frac{\left(1-\frac{m}{n}\right)^{n}}{\left(1-\frac{m}{n}\right)^{x}}\right]$$

$$\therefore P(x) = \frac{m^x}{x!}\lim_{n\to\infty}\left[\frac{\left(1-\frac{m}{n}\right)^{\left(-\frac{n}{m}\right)(-m)}}{1}\right]$$

$$\therefore P(x) = \frac{m^x}{x!}\lim_{n\to\infty}\left[\left(1-\frac{m}{n}\right)^{\left(-\frac{n}{m}\right)}\right]^{(-m)}$$

$\therefore$ $\boxed{P(x) = \frac{m^x}{x!} e^{(-m)}}$ which is a PDF of Poisson distribution.

Mean of Poisson distribution:

The mean of binomial distribution is

$$\mu = np$$

$\Rightarrow$ $\boxed{\mu = m}$ where m is finite

Thus, the parameter m present in the PDF $P(x) = \frac{e^{-m} m^x}{x!}$ is mean of the Poisson distribution.

Hence $\boxed{P(x) = \frac{e^{-\mu} \mu^x}{x!}}$

Variance of Poisson distribution:

The Variance of binomial distribution is

$$\sigma^2 = npq$$

$$\Rightarrow \sigma^2 = np(1-p)$$

$$\Rightarrow \sigma^2 = \mu(1-p) = \mu - \mu p$$

For Poisson distribution, μ is finite p is very small.

$$\therefore \boxed{\sigma^2 \approx \mu}$$

Conclution:

For the Poisson distribution,

$$\boxed{\text{Mean} = \text{Variance} = \mu = np}$$

Problems:

1. The probabilities of a Poisson variate taking the values 3 and 4 are equal. Calculate the probabilities of the variate taking the values 0 and 1.

Soln: Given that

$$P(X=3)=P(X=4)$$

We know that PDF of Poisson distribution is

$$P(X=x)=\frac{e^{-\mu}\mu^x}{x!}$$

$$\therefore \frac{e^{-\mu}\mu^3}{3!}=\frac{e^{-\mu}\mu^4}{4!}$$

$$\Rightarrow \boxed{\mu=\frac{4!}{3!}=4}$$

$$P(X=0)=\frac{e^{-\mu}\mu^0}{0!}=e^{-4}=0.01832$$

$$P(X=1)=\frac{e^{-\mu}\mu^1}{1!}=\frac{e^{-4}4^1}{1!}=0.0733$$

2. A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the probability that on a certain day
   (i) there is no demand
   (ii) demand is refused.

Soln: Given that $\boxed{\mu = 1.5}$

$$\therefore P(X = x) = \frac{e^{-\mu}\mu^{x}}{x!} = \frac{e^{-1.5}(1.5)^{x}}{x!}$$

(i) The probability that there is no demand is

$$P(X = 0) = e^{-1.5} = 0.2231$$

(ii) The probability that some demand is refused is

$$P(X > 2) = 1 - P(X \leq 2)$$

$$\Rightarrow P(X > 2) = 1 - [P(0) + P(1) + P(2)]$$

$$\therefore P(X > 2) = 0.19\ 13$$

3. 2% of the fuses manufactured by a firm are found to be defective. Find the probability that a box containing 200 fuses contains
   (i) no defective fuses
   (ii) 3 or more defective fuses.

Soln: Given that $\boxed{p = \frac{2}{100} = 0.02 \;\; \& \;\; n = 200}$

$$\therefore \mu = np = 4$$

$$\therefore P(X = x) = \frac{e^{-\mu}\mu^x}{x!} = \frac{e^{-4}(4)^x}{x!}$$

(i) $P(X = 0) = \frac{e^{-4}(4)^0}{0!} = e^{-4} = 0.01831$

(ii) $P(X \geq 3) = 1 - P(X < 3)$

$$P(X \geq 3) = 1 - [P(0) + P(1) + P(2)]$$

$$\therefore P(X \geq 3) = 0.76\;\;2$$

4. In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10, use Poisson distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 10,000 packets.

Soln:

$$p = 0.002\,, n = 10\ \ , \mu = np = 0.02$$

i) P(0)=0.9802

Number of packets is

0.9802 X 10,000 = 9802

ii) P(1)=0.0196

Number of packets is

0.0196 X 10,000 = 196

iii) P(2)=0.000196

Number of packets is

0.000196 X 10,000 = 1.96$\approx$ 2

5. Alpha particles are emitted by a radioactive source at an average rate of 5 in a 20 minute interval. Using Poisson distribution, find the probability that there will be
   (i) two emissions
   (ii) at least two emissions in a particular 20 minute interval.

{ Ans: 0.0842, 0.9596}

6. The probability that an individual suffers a bad reaction from a certain injection is 0.002. Determine the probability that out of 1000 individuals
   (i) exactly 3
   (ii) more than 2

   will suffer a bad reaction.

{ Ans: 0.1804, 0.3233}