14/05/2024

UNIT-II

Continuous probability Distributions And Joint probability Distributions.

Exponential distribution

This is a continuous probability distribution, if the random variable 'X' is defined for all +ve values of 'x', then the pdf of exponential distribution is defined as:

$$\boxed{f(x) = \alpha e^{-\alpha x}}, \text{ where } \alpha > 0$$

'α' is a parameter and 'x' ≥ 0

Mean, variance of Exponential distribution:

Mean:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \alpha \cdot x e^{-\alpha x} dx$$

$$= \alpha \left[\int_{0}^{\infty} x \cdot e^{-\alpha x} dx \right]$$

$$= \alpha \left[x \left(-\frac{1}{\alpha} e^{-\alpha x}\right) - 1 \left(\frac{1}{\alpha^2} e^{-\alpha x}\right) \right]_0^{\infty}$$

$$= \alpha \left[0 - 0 - \left(0 - \frac{1}{\alpha^2}\right) \right]$$

$$= \frac{\alpha}{\alpha^2} = \frac{1}{\alpha}$$

$$\boxed{E(X) = \frac{1}{\alpha}}$$

Variance:

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \alpha \int_{-\infty}^{\infty} x^2 e^{-\alpha x} dx = \alpha \int_{0}^{\infty} x^2 e^{-\alpha x} dx$$

$$= \alpha \left[x^2 \left(\frac{e^{-\alpha x}}{-\alpha} \right) - 2x \left(\frac{e^{-\alpha x}}{\alpha^2} \right) + 2 \left(\frac{-e^{-\alpha x}}{\alpha^3} \right) \right]_0^{\infty}$$

$$= \alpha \left[0 - 0 - 0 - \left(0 - 0 + \left(\frac{-2}{\alpha^3} \right) \right) \right]$$

$$= \frac{2}{\alpha^2}$$

$$var(X) = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}$$

$$\boxed{var(X) = \frac{1}{\alpha^2}}$$

* If X is an exponential variate, with mean 5, evaluate
i) $P(0 < X < 1)$
ii) $P(-\infty < X < 10)$
iii) $P(X \leq 0 \text{ or } X \geq 1)$

i) $E(X) = 5 \Rightarrow \boxed{\alpha = \frac{1}{5}}$

$$P(X) = \int_0^1 f(x) dx = \int_0^1 \alpha \cdot e^{-\alpha x} dx$$

$$P(X) = \frac{1}{5} \int_0^1 e^{-x/5} dx$$

$$= \frac{1}{5} \left[\frac{e^{-x/5}}{(-1/5)} \right]_0^1 = \frac{1}{5} \left[\frac{e^{-1/5}}{(-1/5)} - \frac{1}{(-1/5)} \right]$$

$$= 1 - e^{-1/5}$$

$$P(X) = 0.1812$$

ii) $P(-\infty < x < 10)$ but $x \geq 0$

so,

$$\int_0^{10} \alpha e^{-\alpha x} dx = \frac{1}{5} \left[\frac{e^{-x/5}}{(-1/5)} \right]_0^{10} = \frac{1}{5} \left[\frac{e^{-2}}{(-1/5)} - \frac{1}{(-1/5)} \right]$$

$= 1 - e^{-2}$

$P(X) = 0.8646.$

iii) $P(X \le 0 \text{ or } X \ge 1)$

$= P(X \le 0) + P(X \ge 1)$

$= 0 + \int_{1}^{\infty} \alpha e^{-\alpha x}\, dx$

$= \alpha \int_{1}^{\infty} e^{-x/5}\, dx = \frac{1}{5}\left[\frac{e^{-x/5}}{(-1/5)}\right]_1^{\infty} = \frac{1}{5}\left[\frac{0 - e^{-1/5}}{(-1/5)}\right]$

$= \frac{1}{5}\left[5.(e^{-1/5})\right]$

$= e^{-1/5}$

$= 0.8187$

* The length of telephone conversation in a booth has been an exponential distribution & found on an average to be 5 minutes. Find the probability that a random call made from this booth

i) ends less than 5 minutes

ii) ends between 5 & 10 minutes

i) $P(X < 5)$ but $X \ge 0$

$\Rightarrow \int_0^5 \alpha . e^{-\alpha x}\, dx = \alpha \int_0^5 e^{-\alpha x}\, dx$

$E(X) = 5$

$\alpha = \frac{1}{5}$

$\frac{1}{5}\left[\frac{e^{-x/5}}{(-1/5)}\right]_0^5 = 1 - e^{-1} = 0.6321$

ii) $\alpha \int_5^{10} e^{-\alpha x}\, dx = \frac{1}{5}\left[\frac{e^{-x/5}}{(-1/5)}\right]_5^{10} = e^{-1} - e^{-2}$

$= 0.2325$

* The sales per day in a shop are exponentially distributed with average sale amounting to ₹100 and net profit is 8%. Find the probability that the net profit exceeds ₹30 on two consecutive days.

$$E(x) = 100$$

$$\boxed{\alpha = \frac{1}{100}}$$

$$P(x) = \int_{-\infty}^{\infty} f(x)\, dx$$

x = sales per day

Let A = amount of sales per day

8% of total sales = 30 ₹

$$\frac{8}{100} \times A = 30$$

$$A = \frac{3000}{8} = ₹375$$

$$\boxed{A = ₹375}$$

$$f(x) = \alpha \cdot e^{-\alpha x} = \frac{1}{100} e^{-x/100}$$

P(profit exceeds 30₹) = P(amount of sales exceeds 375₹)

$$P(X \geq 375) = \int_{375}^{\infty} \frac{1}{100} e^{-x/100}\, dx$$

$$P(X) = \frac{1}{100} \left(-100 e^{-x/100} \right)_{375}^{\infty}$$

$$= \frac{1}{100} \left(0 + 100\, e^{-375/100} \right)$$

$$= e^{-375/100}$$

$$P(X) = 0.0235$$

for one day $\Rightarrow$ 0.0235

P(A and B) $\Leftarrow$ for consecutive days $\Rightarrow$ 0.0235×0.0235

$= P(A) \cdot P(B)$

$= 0.00055$

* The increase in sales per day in a shop is exponentially distributed with mean ₹ 600. The sales tax is to be levied at the rate of 9%. What is the probability that the sales tax will exceed ₹81 per day.

$$E(X) = 600$$

$$\boxed{\alpha = \frac{1}{600}}$$

$X \rightarrow$ Sales per day

$A \rightarrow$ amount of sales per day

$$\frac{9}{100} \times A = 81$$

$$\boxed{A = ₹900}$$

P(Amount of sales exceeds ₹900)

$$= \int_{900}^{\infty} \frac{1}{600} e^{-x/600} dx$$

$$= \frac{1}{600} \left[-600 e^{-x/600} \right]_{900}^{\infty}$$

$$= e^{-900/600}$$

$$= e^{-3/2}$$

$$= e^{-1.5}$$

$$\boxed{P(X) = 0.2231}$$

15/5/2024

Normal probability Distribution:

This distribution is a continuous probability distribution. The pdf of this distribution is defined as:

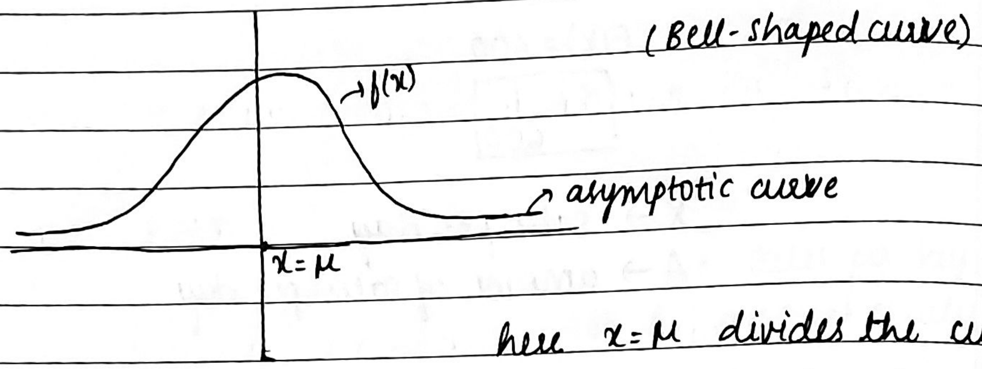
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where, 'μ' is the mean of the distribution

'σ' is the standard deviation.

and $-\infty \leq x \leq \infty$

NOTE: $f(x)$ is an even function, so the graph of $f(x)$ is symmetric about y-axis.



here $x=\mu$ divides the curve into two equal parts.

* Area under the curve = 1

* Standard normal variate $\boxed{z = \frac{x-\mu}{\sigma}}$

$$P(a \leq x \leq b) = \int_a^b y\,dx = \int_a^b f(x)\,dx = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$P(z_1 \leq Z \leq z_2)$: put $z = \frac{x-\mu}{\sigma}$, $z_1 = \frac{a-\mu}{\sigma}$, $z_2 = \frac{b-\mu}{\sigma}$

$$dx = \sigma\,dz$$

$$P(z_1 \leq Z \leq z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}}\,dz$$

* The graph of the normal probability distribution is a bell shaped curve. and the line '$x=\mu$' divides the curve into two equal parts.

* The area under the normal curve = 1

* For normal probability distribution, the mean, median & mode coincide.

* The tails of the normal curve are asymptotic.

Standard normal variate:

we put $\boxed{z = \frac{x-\mu}{\sigma}}$.

so that, for the given normal distribution, with this tranformation,

$\boxed{z = x}$ when $\boxed{\mu = 0}$ & $\boxed{\sigma = 1}$

- Probability that x lies b/w 'a' & 'b' is given by.

$$P(a \le x \le b) = \int_a^b f(x)\,dx$$

$$= \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $\frac{x-\mu}{\sigma} = z$

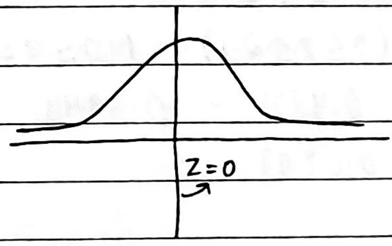
$dx = \sigma\, dz$

let $\boxed{z_1 = \frac{a-\mu}{\sigma}}$ & $\boxed{z_2 = \frac{b-\mu}{\sigma}}$

$$\boxed{P(z_1 \le z \le z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}} dz}$$

Standard Normal curve:

For $z = 0$, $x = \mu$ $\Rightarrow$



The line $z=0$ divides the standard normal curve into 2 parts. The area under the curve $= 1$.

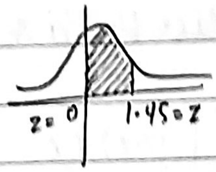
* The standard normal tables give the area b/w $\boxed{z=0 \text{ to } z}$

* For the standard normal distribution of a random variable Z, evaluate :

i) $P(0 \le Z \le 1.45)$
ii) $P(-2.60 \le Z \le 0)$
iii) $P(-3.40 \le Z \le 2.65)$
iv) $P(1.25 \le Z \le 2.1)$
v) $P(Z \ge 1.7)$

From the table

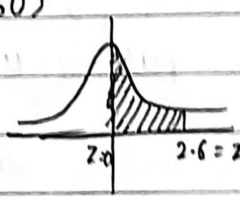
i) 0.42647 $\Rightarrow$ move 1.4 ↓ and 0.05 →



ii) $P(-2.60 \le z \le 0)$ $\Rightarrow$ symmetric curve so, value remains same on either side of $z=0$ line.

$\Rightarrow P(0 \le Z \le 2.60)$

$\Rightarrow$ 0.4953



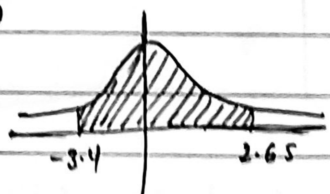
iii) $P(-3.40 \le Z \le 2.65)$

$P(-3.4 \le Z \le 0) + P(0 \le Z \le 2.65)$

$\Rightarrow P(0 \le Z \le 3.4) + P(0 \le Z \le 2.65)$

$= 0.4997 + 0.4960$

$= 0.9957$

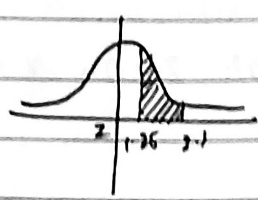


iv) $P(1.25 \le Z \le 2.1)$

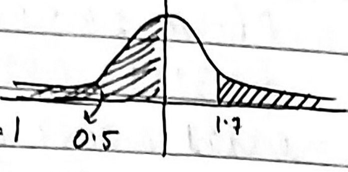
$\Rightarrow P(0 \le Z \le 2.1) - P(0 \le Z \le 1.25)$

$\Rightarrow$ 0.4821 - 0.3944

$\Rightarrow$ 0.0877.



v) $P(Z \geq 1.7)$



$= 0.5 + P(0 \leq Z \leq 1.7) + x = 1$

$x = 0.5 - P(0 \leq Z \leq 1.7)$

$= 0.5 - 0.4554$

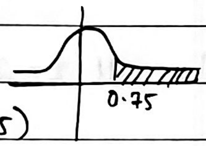
$= 0.0496$

* For the normal distribution with mean = 2 & standard deviation 4, evaluate the following probabilities:

1) $P(X \geq 5)$

$Z = \dfrac{x - \mu}{\sigma} = \dfrac{x-2}{4}$

for $x = 5$, $Z = \dfrac{5-2}{4} = \dfrac{3}{4} = 0.75$



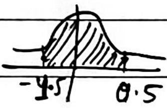
$P(X \geq 5) = P(Z \geq 0.75)$

$= 0.5 - P(0 \leq Z \leq 0.75)$

$= 0.5 - 0.2704$

$= 0.2296$

ii) $P(|X| < 4) \Rightarrow P(-4 < X < 4)$



For $x = -4$, $Z = \dfrac{-4-2}{4} = \dfrac{-3}{2} = -1.5$

$x = 4$, $Z = \dfrac{4-2}{4} = \dfrac{1}{2} = 0.5$

$P(|X| < 4) = P(-1.5 < Z < 0.5)$

$= P(-1.5 < Z < 0) + P(0 \leq Z \leq 0.5)$

$= P(0 < Z < 1.5) + P(0 < Z < 0.5)$

$= 0.4332 + 0.1915$

$= 0.6247$

iii) $P(|X| > 3) \Rightarrow 1 - P(|X| < 3)$

$1 - P(-3 < x < 3)$

for $x = -3 \Rightarrow \dfrac{-5}{4} = Z$; $x = 3 \Rightarrow \dfrac{1}{4} = Z$

$Z = -1.25$ to $Z = 0.25$

$1 - P(-1.25 \le z < 0.25)$

$1 - \{0.3944 + 0.0987\}$

$= \cancel{0.5109}\ 0.5069$

* The life of a certain type of electrical lamps is normally distributed with mean 2040, & standard deviation 60 hours, in a consignment of 3000 lamps, how many would be expected to burn for

i) more than 2150 hours

ii) less than 1950 hours and

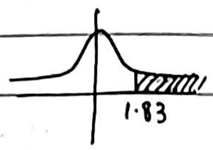
iii) between 1920 & 2160 hours

$\mu = 2040, \quad \sigma = 60$

$$z = \frac{x-\mu}{\sigma} = \frac{x-2040}{60}$$

i) at $x = 2150 \Rightarrow z = \frac{2150-2040}{60} = \frac{11\cancel{0}}{6\cancel{0}} = 1.83$

P (more than 2150)



$P(z \ge 1.83)$

$= 0.5 - P(0 \le z \le 1.83)$

$= 0.5 - 0.4664$

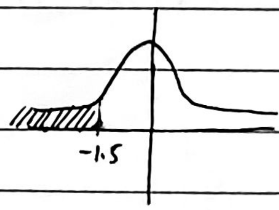
$= 0.0336$

$\Rightarrow$ for 3000 samples $\Rightarrow 3000 \times 0.0336 = 336 \times 3 = 100.8$

ii) less than 1950 hours

$x = 1950 \Rightarrow z = \frac{1950-2040}{60} = -\frac{\cancel{8}90}{60} = -1.5$

$P(z \le -1.5)$



$= 0.5 - P(\cancel{0}-1.5 \le z \le 0)$

$= 0.5 - P(0 \le z \le 1.5)$

$= 0.5 - 0.4332$

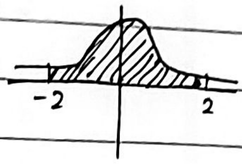
$= 0.0668$

for 3000 lamps $\Rightarrow 3000 \times 0.0668$

$= 200.4$

iii) $x = 1920 \Rightarrow z = \dfrac{1920-2040}{60} = -2$

$x = 2160 \Rightarrow z = \dfrac{2160-2040}{60} = 2$



$P(1920 \le x \le 2160)$

$= P(-2 \le z \le 2)$

$= P(-2 \le z \le 0) + P(0 \le z \le 2)$

$= 2 \cdot P(0 \le z \le 2)$

$= 2(0.4772)$

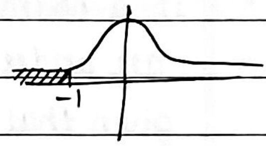
$= 0.9544$

for 3000 lamps $\Rightarrow$ 2863.2

* The marks of 1000 students in an examination follow a normal distribution with mean 70 & standard deviation 5. Find the no. of students whose marks will be
i) less than 65
ii) more than 75
iii) between 65 & 75

$\mu = 70, \ \sigma = 5$

$z = \dfrac{x-\mu}{\sigma} = \dfrac{x-70}{5}$



i) $x = 65 \Rightarrow z = \dfrac{65-70}{5} = -1$

$P(x \le 65) = P(z \le -1)$

$0.5 - P(-1 \le z \le 0) = -P(0 \le z \le 1) + 0.5$

$= -0.3413 + 0.5$

$= 0.1587$

no. of students whose marks is less than 65

$= 1000 \times 0.1587$

≈ 159 (approx)

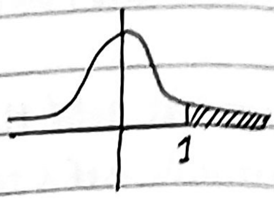
ii) $x = 75 \Rightarrow z = \frac{75-70}{5} = 1$

$P(X \geq 75) = P(Z \geq 1)$

$= 0.5 - P(0 \leq Z \leq 1)$

$= 0.5 - 0.3413$

$= 0.1587$

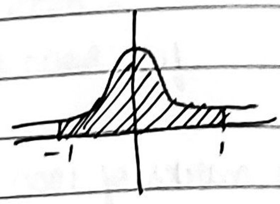


no. of students who got more than 75 $= 1000 \times 0.1587$

≈ 159 (approx).

iii) $x = 65 \Rightarrow z = -1$

$x = 75 \Rightarrow z = 1$



$P(-1 \leq Z \leq 1)$

$= P(-1 \leq Z \leq 0) + P(0 \leq Z \leq 1)$

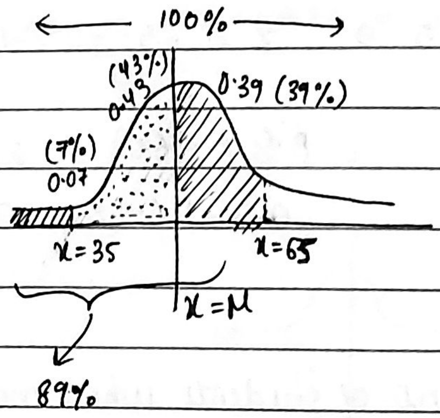
$= 2 \cdot P(0 \leq Z \leq 1)$

$= 2 \cdot 0.3413$

$= 0.6826$

no. of students $= 1000 \times 0.6826 \approx 683$

* In a normal distribution 7% of are under 35 and 89% are under 63. Find the mean and the standard deviation given that $A(1.23) = 0.39$ and $A(01.48) = 0.43$.



$A(1.23) = 0.39 \Rightarrow$ 39 % is at right of $x = \mu$ line so $\Rightarrow$

$Z = +1.23$

$A(1.48) = 0.43 \Rightarrow$ 43% (0.43) is at left of $x = \mu$ line. so $\Rightarrow$

$Z = -1.48$

$z = \dfrac{x - \mu}{\sigma}$

$x = 35 \Rightarrow \quad -1.48 = \dfrac{35 - \mu}{\sigma}$

$-1.48 \cdot \sigma = 35 - \mu \quad — ①$

$x = 63 \Rightarrow \quad 1.23 = \dfrac{63 - \mu}{\sigma}$

$1.23 \cdot \sigma = 63 - \mu \quad — ②$

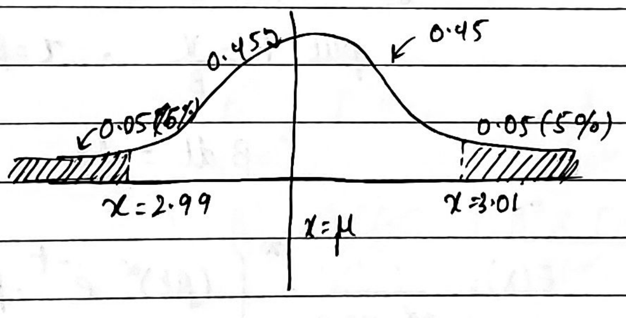
$$\begin{aligned} 1.48\sigma &= -35 + \cancel{\mu} \\ +1.23\sigma &= 63 \cancel{- \mu} \\ \hline 2.71\sigma &= 28 \end{aligned}$$

$\boxed{\sigma = 10.3321}$

$63 - 1.23(10.3321)$

$\boxed{\mu = 50.2915}$

* Steel rods are manufactured to be 3cm in diameter but they are acceptable if they are inside the limits 2.99 cm & 3.01cm. It is observed that 5% are rejected as oversized and 5% are rejected as undersized. Assuming that the diameters are normally distributed, find the standard deviation of the distribution.



$x = 3.01 cm$
$x = 2.99 cm$

for $x = 2.99 \Rightarrow z = -1.65 \Rightarrow$
for $x = 3.01 \Rightarrow z = 1.65$

for $x = 2.99 \Rightarrow \quad \dfrac{2.99 - \mu}{\sigma} = -1.65$

$\boxed{\mu = 3}$
$\boxed{\sigma = 0.006}$

$2.99 - \mu = -1.65\sigma$

for $x = 3.01 cm \Rightarrow \quad \dfrac{3.01 - \mu}{\sigma} = 1.65$

$\Rightarrow 1.65\sigma = 3.01 - \mu$

Gamma Distribution

It is a continuous probability distribution. The pdf of gamma distribution is given by :

$$f(x) = \begin{cases} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$

Mean:

$$\Gamma(n) = \int_0^{\infty} e^{-x} . x^{n-1} dx$$

$$\text{Mean} = E(X)$$

$$= \int_{-\infty}^{\infty} x . f(x) dx$$

$$= \int_0^{\infty} \frac{x . x^{\alpha-1} . e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)} dx$$

$$= \frac{1}{\Gamma(\alpha).\beta^{\alpha}} \int_0^{\infty} x^{\alpha} e^{-x/\beta} dx \quad — ①$$

put $t = \frac{x}{\beta}$, $x = \beta t$ $\begin{smallmatrix} \to 0 \\ \to \infty \end{smallmatrix}$

$\beta dt = dx$

$$E(X) = \frac{1}{\beta^{\alpha}.\Gamma(\alpha)} \int_0^{\infty} (\beta t)^{\alpha} e^{-t} . \beta dt$$

$$= \frac{\cancel{\beta^{\alpha}} . \beta}{\cancel{\beta^{\alpha}}\Gamma(\alpha)} \int_0^{\infty} e^{-t} . t^{\alpha} dt$$

$$= \frac{\beta}{\Gamma(\alpha)} \int_0^{\infty} e^{-t} . t^{\alpha} dt$$

$$= \frac{\beta}{\Gamma(\alpha)} . \Gamma(\alpha+1)$$

classmate

Date

Page

$$= \frac{\beta}{\cancel{\Gamma(\alpha)}} \cdot \alpha \, \Gamma\cancel{(\alpha)}$$

$$\boxed{E(X) = \alpha\beta}$$

Variance:

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_0^\infty x^2 \cdot f(x)\, dx.$$

$$= \int_0^\infty x^2 \cdot \frac{x^{\alpha-1} \cdot e^{-x/\beta}}{\beta^\alpha \cdot \Gamma(\alpha)}\, dx$$

$$= \frac{1}{\beta^\alpha \cdot \Gamma(\alpha)} \int_0^\infty x^{\alpha+1} \cdot e^{-x/\beta}\, dx$$

let $x/\beta = t$

$x = \beta t$ $\quad 0 \to 0,\ \infty \to \infty$

$dx = \beta\, dt$

$$= \frac{1}{\beta^\alpha \, \Gamma(\alpha)} \int_0^\infty (\beta t)^{\alpha+1} \cdot e^{-t} \beta \cdot dt$$

$$= \frac{\beta \cdot \beta^{\alpha+1}}{\beta^\alpha \cdot \Gamma(\alpha)} \int_0^\infty e^{-t} \cdot t^{\alpha+1}\, dt$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha+2) = \frac{\beta^2}{\Gamma(\alpha)} (\alpha+1)\Gamma(\alpha+1)$$

$$= \frac{\beta^2 \cdot \alpha^2 \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} + \frac{\beta^2 \alpha \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}$$

$$E(X^2) = \alpha^2\beta^2 + \alpha\beta^2$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \alpha^2\beta^2 - (\alpha\beta)^2 + \alpha\beta^2$$

$$\boxed{Var(X) = \alpha\beta^2}$$

* The no. of accidents per day on a certain highway is a gamma variate with an average 6 and variance 18. Find the probability that there will be i) more than 8 accidents
ii) Between 5 & 8 accidents.

Mean = 6

$\boxed{\alpha\beta = 6}$

$$\alpha\beta^2 = 18$$

$$6 \cdot \beta = 18$$

$$\boxed{\beta = 3}, \quad \boxed{\alpha = 2}$$

$$\text{Pdf} = f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \quad x > 0.$$

$$f(x) = \frac{1}{9 \cdot \Gamma(2)} \cdot x \cdot e^{-x/3}$$

$$f(x) = \frac{1}{9 \cdot (1)} x e^{-x/3}$$

$$f(x) = \frac{x e^{-x/3}}{9}.$$

i) more than 8 accidents $\Rightarrow$

$$P(X \geq 8) = \int_8^\infty \frac{1}{9} \cdot x e^{-x/3} dx$$

$$= \frac{1}{9}\left[x(-3e^{-x/3}) - 1(9e^{-x/3})\right]_8^\infty$$

$$= \frac{1}{9}\left[0 - 0 - (-24e^{-8/3} - 9e^{-8/3})\right]$$

$$= \frac{33 \cdot e^{-8/3}}{9} = \frac{11e^{-8/3}}{3} =$$

$$P(X > 8) = 0.2547$$

ii) Between 5 & 8 $\Rightarrow$

$$P(5 \leq X \leq 8) = \int_5^8 \frac{x e^{-x/3}}{9} dx$$

$$= \frac{1}{9}\left[x(-3e^{-x/3}) - 1(9e^{-x/3})\right]_5^8$$

$$= \frac{1}{9}\left[-24e^{-8/3} - 9e^{-8/3} + 15e^{-5/3} + 9e^{-5/3}\right]$$

$$= \frac{1}{9}\left[-33e^{-8/3} + 24e^{-5/3}\right]$$

$$= \frac{1}{9}\left[-2.2929 + 4.5330\right]$$

$$= \frac{2.2401}{9} = 0.2489$$

* The demand for a certain item is distributed as a gamma distrib-ution with mean 8 and variance 32. Find the probability that there will be a demand for atleast 10 times.

$\alpha\beta = 8$ $\quad\quad$ $\alpha\beta^2 = 32$

$8 \cdot \beta = 32$

$\boxed{\beta = 4}$ $\quad\quad$ $\alpha = \frac{8}{4} = 2$

$\boxed{\alpha = 2}$

pdf $\Rightarrow$ $f(x) = \frac{1}{4^2\,\Gamma(2)} e^{-x/4} \cdot x, \quad x > 0$

$$f(x) = \frac{1}{16} x e^{-x/4}, \quad x > 0$$

$$P(X \geq 10) = \int_{10}^{\infty} \frac{1}{16} x e^{-x/4}\,dx$$

$$= \frac{1}{16}\left[x(-4e^{-x/4}) - 1(16e^{-x/4})\right]_{10}^{\infty}$$

$$= \frac{1}{16}\left[0 - 0 - (-40e^{-10/4} - 16e^{-10/4})\right]$$

$$= \frac{1}{16}\left(56e^{-10/4}\right)$$

$$\boxed{P(X \geq 10) = 0.2872}$$

* The daily sales of a certain brand of bicycles in a city in excess of 1000 pieces is distributed as the Gamma distribution with $\alpha = 2$ & $\beta = 500$. The city has a daily stock of 1500 pieces

of the brand. Find the probability that the stock is insuffici-ent on a particular day.

$\alpha = 2$, $\beta = 500$

Let , X = sales per day (Random variable)

production / sales in excess of 1000

$\Rightarrow X + 1000$

let $Y = X + 1000$. has a gamma distribution.

$$f(X) = \frac{1}{(500)^2 \Gamma(2)} \cdot x^{2-1} \cdot e^{-x/500}$$

$$f(X) = \frac{1}{(500)^2 \cdot 1} \cdot x \cdot e^{-x/500}$$

stock = 1500

If the stock is insufficient ,

$y > 1500$

$X + 1000 > 1500$

$X > 1500 - 1000$

$X > 500$

$$P(X > 500) = \int_{500}^{\infty} \frac{1}{(500)^2} \cdot x \cdot e^{-x/500} \, dx$$

$$= \frac{1}{(500)^2} \int_{500}^{\infty} x \cdot e^{-x/500} \, dx$$

$$= \frac{1}{(500)^2} \left[x(-500 e^{-x/500}) - 1(500)^2 (e^{-x/500}) \right]_{500}^{\infty}$$

$$= \frac{1}{(500)^2} \left[0 - 0 + (500)^2 e^{-1} + (500)^2 e^{-1} \right]$$

$$= \frac{1}{\cancel{(500)^2}} 2 \cancel{(500)^2} e^{-1}$$

$$= 2e^{-1}$$

$$= 0.7357$$

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* The daily consumption of milk in a town in excess of 30,000 litres is distributed as gamma distribution with $\alpha = 2$, $\beta = 10000$. The town has a daily stock of 40,000 litres. Find the probability that the stock is adequate on a particular day.

$X \rightarrow$ daily consumption of milk

$Y = X + 30000$

$$f(x) = \frac{1}{\beta^{\alpha}\,\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

$$f(x) = \frac{1}{(10000)^2\,\Gamma(2)} x^{1} \cdot e^{-x/10000} = \frac{1}{(10000)^2} x \cdot e^{-x/10000}$$

Stock is adequate when, $y \le 40000$

$x + 30000 \le 40000$

$x \le 10000$

$P(X \le 10000)$

$$= \int_0^{10000} \frac{1}{(10000)^2} x e^{-x/10000}\, dx$$

$$= \frac{1}{(10000)^2}\left[x(-10000 e^{-x/10000}) - 1((10000)^2 e^{-x/10000})\right]_0^{10000}$$

$$= \frac{1}{(10000)^2}\left[-(10000)^2 e^{-1} - (10000)^2 e^{-1} - (0 - (10000)^2)\right]$$

$$= \frac{1}{\cancel{(10000)^2}}\left[\cancel{(10000)^2}\right](1 - 2e^{-1})$$

$$= 1 - 2e^{-1}$$

$$= 0.26424$$

* After the appointment of a new sales manager in a shop, the increase in sales per day in the shop is found as gamma variate with $\alpha = 2$, $\beta = 2000$. What is the probability that the increase in sales tax returns in a randomly chosen day exceeds ₹ 200, given that the sales tax is 5% of sales.

$\alpha = 2$, $\beta = 2000$

$X \rightarrow$ increase in sales per day.

$$f(X) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} e^{-x/\beta} x^{\alpha-1}$$

$$f(x) = \frac{1}{(2000)^2\Gamma(2)} e^{-x/2000} \cdot x^1$$

$$f(x) = \frac{1}{(2000)^2} x \cdot e^{-x/2000}$$

Sales tax $= 5\%$

$A =$ amount of sales

5% of $A = 200$

$$\frac{\cancel{5}A}{100} = \cancel{200}\ 40$$

$$\boxed{A = 4000}$$

P(sales tax increase > 200) = P(amount sales increase > 4000)

$P(X > 4000)$

$$= \int_{4000}^{\infty} \frac{1}{(2000)^2} \cdot x e^{-x/2000} dx$$

$$= \frac{1}{(2000)^2}\left[x(-2000e^{-x/2000}) - 1/((2000)^2 e^{-x/2000})\right]_{4000}^{\infty}$$

$$= \frac{1}{(2000)^2}\left[0 - 0 - (-8000000)e^{-\overset{4000}{x}/2000} + (2000)^2 e^{-2}\right]$$

$$= 2e^{-2} + e^{-2}$$

$$= 3e^{-2}$$

$$= 0.4060$$

Uniform distribution:

Let X be a continuous random variable defined in the interval [a,b]. The pdf of the uniform distribution is defined as

$$f(x) = \begin{cases} \frac{1}{b-a} & , a \le x \le b \\ 0 & , \text{elsewhere} \end{cases}$$

* A random variable X is uniformly distributed over the interval (-4,4). Find $P(X<1)$ and $P(|x-1| \ge \frac{1}{2})$

$$f(x) = \begin{cases} \frac{1}{b-a} & , -4 \le x \le 4 \\ 0 & , \text{elsewhere} \end{cases}$$

$$P(X<1) = \int_{-4}^{1} f(x)\,dx = \left[\frac{x}{8}\right]_{-4}^{1} = \frac{+5}{8}$$

$$P\left(|x-1| \ge \frac{1}{2}\right) = 1 - P\left(|x-1| < \frac{1}{2}\right)$$

$$= 1 - P\left[+\tfrac{1}{2} \le x \le \tfrac{3}{2}\right]$$

$$= 1 - \int_{+1/2}^{3/2} \frac{1}{8}\,dx$$

$$= 1 - \left(\frac{x}{8}\right)_{1/2}^{3/2} = 1 - \frac{3}{16} + \frac{1}{16} = \frac{7}{8}$$

* A random variable X has uniform distribution over the interval (-3,3). Find K for which $P(X>K) = \frac{1}{2}$

$$P(X>K) = \int_{K}^{3} \frac{1}{6}\,dx = \frac{1}{2}$$

$$\frac{1}{6}[x]_K^3 = \frac{1}{2} \Rightarrow \frac{1}{\cancel{6}_3}[3-K] = \frac{1}{\cancel{2}} \Rightarrow \boxed{K=0}$$

21-05-2024

Joint Probability Distributions

Discrete case

Let `x' and `Y' be two discrete random variables defined on the same sample space `S' of a random experiment.

* The joint probability mass function, $P(X,Y)$ is defined for each pair of numbers (x,y) by

$$P(x,y) = P(X=x \text{ and } Y=y)$$

where

i) $P(x,y) \geq 0$

ii) $\sum_{x}\sum_{y} P(x,y) = 1$

Suppose,

$X = \{x_1, x_2, \ldots\ldots, x_m\}$

$Y = \{y_1, y_2, \ldots\ldots, y_n\}$, then

$$\boxed{P(X=x_i \text{ and } Y=y_j) = P_{ij}}$$

for $i = 1, 2, \ldots m$

$j = 1, 2, \ldots, n$

is called the Joint probability distribution of X & Y and the values P_{ij} are represented in the form of 2-way table called Joint probability distribution table and is given by:

X\Y	y_1	y_2	......	y_n	$f(x)$
x_1	P_{11}	P_{12}	......	P_{1n}	$f(x_1)$
x_2	P_{21}	P_{22}	......	P_{2n}	$f(x_2)$
⋮	⋮	⋮	......	⋮	⋮
x_m	P_{m1}	P_{m2}		P_{mn}	$f(x_m)$
$g(y)$	$g(y_1)$	$g(y_2)$	......	$g(y_n)$	

* In the previous table, the last column entries are the row-wise summations and the last row entries are the column-wise summations.

* Here the set $\{f(x_1), f(x_2), \ldots, f(x_m)\}$ are the marginal probabilities of random variable $X = \{x_1, x_2, \ldots, x_m\}$

Similarly,
$\{g(y_1), g(y_2), \ldots, g(y_n)\}$ are the marginal probabilities of random variable $Y = \{y_1, y_2, \ldots, y_n\}$

* The marginal probability distribution of X is

X	x_1	x_2	$\ldots$	x_m
$f(x)$	$f(x_1)$	$f(x_2)$	$\ldots$	$f(x_m)$

similarly,
the marginal probability distribution of Y is

Y	y_1	y_2	$\ldots$	y_n
$f(Y)$	$f(y_1)$	$f(y_2)$	$\ldots$	$f(y_n)$

Independent Random variables:

Two discrete random variables X and Y are said to be independent if:

$$\boxed{p(X,Y) = f(X) \cdot g(Y)}$$

Mean, variance, co-variance & co-relation coefficient.

Mean:

$$E(X) = \sum X \cdot f(X)$$
$$E(Y) = \sum Y \cdot g(Y)$$

Covariance :

$$\text{cov}(X,Y) = E(XY) - \{E(X) \cdot E(Y)\}$$

where

$$E(XY) = \sum_X \sum_Y XY \cdot P(X,Y)$$

* $$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_x \sigma_y}$$

* $$\text{var}(X) = E(X^2) - [E(X)]^2$$
$$\text{var}(Y) = E(Y^2) - [E(Y)]^2$$

* The Joint probability distribution of two random variables is given by the following table

X\Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

Find marginal distribution of X and Y & evaluate cov(X, Y) & $\rho(X,Y)$ and check whether X & Y are independent.

$$f(x) \Rightarrow \{f(x_1), f(x_2), f(x_3)\}$$
$$= \{1/4, 1/2, 1/4\} \Rightarrow \text{row wise summation}$$

$$g(Y) = \{g(y_1), g(y_2), g(y_3)\}$$
$$= \{1/2, 1/3, 1/6\} \Rightarrow \text{column wise summation}$$

Marginal distribution of X

X	2	4	6
f(x)	1/4	1/2	1/4

Marginal distribution of Y

Y	1	3	9
f(Y)	1/2	1/3	1/6

$$COV(X,Y) = E(X,Y) - E(X)\cdot E(Y)$$

$$E(XY) = 2(1)\left(\frac{1}{8}\right) + 2(3)\left(\frac{1}{24}\right) + 2(9)\left(\frac{1}{12}\right) + 4(1)\left(\frac{1}{4}\right)$$

$$+ 4(3)\left(\frac{1}{4}\right) + 4(9)(0) + 6\left(\frac{1}{8}\right) + 6(3)\left(\frac{1}{24}\right)$$

$$+ 6(9)\left(\frac{1}{12}\right)$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{3}{2} + 1 + 3 + \frac{3}{4} + \frac{3}{4} + \frac{9}{2}$$

$$= \frac{1}{2} + 4 + \frac{15}{2} = \frac{16}{2} + 4$$

$$= 8 + 4$$

$$\boxed{E(XY) = 12}$$

$$COV(X,Y) = \frac{12}{1} - E(X)\cdot E(Y).$$

$$E(X) = \sum X\cdot f(X)$$

$$= 2\times\frac{1}{4} + 4\times\frac{1}{2} + 6\times\frac{1}{4}$$

$$= 4$$

$$E(Y) = \sum Y\cdot g(Y)$$

$$= 1\times\frac{1}{2} + 3\times\frac{1}{3} + 9\times\frac{1}{6}$$

$$= 3$$

$$COV(X,Y) = 12 - (3)(4) = 0$$

$$\Downarrow$$

$\rho(X,Y) = 0$ X & Y are independent.

$$\sigma_x = \sqrt{E(X^2) - [E(X)]^2}$$

$$E(X^2) = \sum X^2\cdot f(X)$$

$$= 4\cdot\frac{1}{4} + 16\times\frac{1}{2} + 36\times\frac{1}{4}$$

$$= 18$$

$$\sigma_x = \sqrt{18-16} = \sqrt{2}$$

$\sigma_y = E(y^2) - [E(y)]^2$

$E(y^2) = \sum y_i^2 \, g(y)$

$= 1 \times \frac{1}{2} + 9 \times \frac{1}{3} + 81 \times \frac{1}{6}$

$= 17$

$\sigma_y = \sqrt{17 - 9} = \sqrt{8}$

* Find the Joint probability distribution of X & Y, which are independent random variables with the following respect-ive distributions, also find E(X).

x_i	1	2
$f(x_i)$	0.7	0.3

y_i	-2	5	8
$g(y_i)$	0.3	0.5	0.2

X\Y	-2	5	8	$f(x_i)$
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
$g(y_j)$	0.3	0.5	0.2	

X & Y are independent.

So, $p(x_i, y_i) = f(x_i) \cdot g(y_i)$

$E(X) = \sum X \cdot f(X)$

$= 0.7 + 0.6$

$= 1.3$

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* Find the mean and variance of uniform distribution:

$$Pdf = \begin{cases} \frac{1}{b-a} & ; a \le x \le b \\ 0 & ; \text{elsewhere} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_a^b \frac{x}{b-a}\, dx = \frac{1}{(b-a)} \left[\frac{x^2}{2}\right]_a^b = \frac{1}{(b-a)} \left(\frac{b^2-a^2}{2}\right)$$

$$E(X) = \frac{b+a}{2}$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_a^b \frac{x^2}{(b-a)}\, dx$$

$$= \frac{1}{b-a} \left(\frac{x^3}{3}\right)_a^b$$

$$E(X^2) = \frac{b^2+a^2+ab}{3}$$

$$\text{variance}(X) = \frac{4b^2+4a^2+4ab - 3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2+(-2ab)}{12}$$

$$\text{var}(x) = \frac{(a-b)^2}{12}$$

* On a certain city transport route, buses fly every 30 minutes between 6 a.m & 10 p.m. If a person reaches a bus stop on this route at a random time during this period, what is the probability that he will have to wait atleast 20 minutes.

$$Pdf = \begin{cases} \frac{1}{30-0} & ; 0 \le x \le 30 \\ 0 & ; \text{elsewhere} \end{cases}$$

$P(X \geq 20)$

$$= \int_{20}^{30} \frac{1}{30}\,dt$$

$$= \frac{1}{30}[t]_{20}^{30} = \frac{10}{30} = \frac{1}{3}$$

* Buses arrive at a specificied stop at 15 min intervals starting at 7 a.m i.e., they arrive at 7:00, 7:15, 7:30, 7:45 --- so on. If a passenger arrives at the stop at a random time i.e., uniformly distributed b/w 7 & 7:30a.m. Find the probability that he waits i) less than 5 min for a bus
ii) At least 12 min for a bus

$$\text{Pdf} = \begin{cases} \frac{1}{30-0} & ; 0 \leq x \leq 30 \\ 0 & ; \text{elsewhere} \end{cases}$$

a) $P(X<5) \Rightarrow \int_{10}^{15} \frac{1}{30}\,dt = \frac{1}{30}(t)_{10}^{15} = \frac{1}{6}$ — (I)

between 7:00 & 7:30, he can come at

i) 7:15
ii) 7:30 } If 5 mins late then

he should arrive at

i) 7:10
ii) 7:25

$P(X<5) \Rightarrow \int_{25}^{30} \frac{1}{30}\,dt = \frac{(t)_{25}^{30}}{30} = \frac{1}{6}$ — (II)

$P(X<5) \Rightarrow$ (I) + (II) $= \frac{2}{6} = \frac{1}{3}$

* b) atleast 12 min for a bus.

$$= \int_{0}^{3} \frac{1}{30}\,dt + \int_{15}^{18} \frac{1}{30}\,dt = \frac{1}{10} + \frac{1}{10} = \frac{2}{10} = \frac{1}{5} \Rightarrow P(X \geq 12)$$

already discussed

"Discrete case" ~~for~~ continuous Joint probability distribution.

Let X and Y be two continuous random variables defined on the same sample space 'S', then the pdf of these two random variables

$P(X,Y)$ should satisfy the following two conditions:

1) $P(X,Y) \geq 0$

ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(X,Y)\,dx\,dy = 1$

The marginal distribution of X is given by,

$$f(x) = \int_{-\infty}^{\infty} P(X,Y)\,dy$$

The marginal distribution of Y is given by,

$$g(Y) = \int_{-\infty}^{\infty} P(X,Y)\,dX.$$

Conditional distribution of Y given X $\Rightarrow P(Y/X) = \frac{P(X,Y)}{f(X)}$

conditional distribution of X given Y $\Rightarrow P(X/Y) = \frac{P(X,Y)}{g(Y)}$

Mean:

$$E(X) = \int_{-\infty}^{\infty} X \cdot f(X)\,dX \qquad E(Y) = \int_{-\infty}^{\infty} Y \cdot g(Y)\,dy$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X \cdot Y \cdot P(X,Y)\,dx\,dy$$

* Find the value of 'k' from the joint probability density function of X and Y given by:

$$f(X,Y) = \begin{cases} K(2X+Y) ; & 2<X<6, 0<Y<5 \\ 0 ; & \text{elsewhere.} \end{cases}$$

also find Marginal distributions of X & Y.

Since $f(x,y)$ is pdf of Joint distribution of X & Y:
we have,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$= \int_{0}^{5}\int_{2}^{6} k(2x+y)\,dx\,dy = 1$$

$$= \int_{0}^{5}\left[k\left(\frac{2x^2}{2} + xy\right)\right]_2^6 dy = 1 \quad = \int_{0}^{5} k(32+4y)\,dy = 1$$

$$\Rightarrow k(32)(5)+4\left(\frac{25}{2}\right)$$

$$k = \frac{1}{210}$$

Marginal distribution of X:

$$f(x) = \int_{-\infty}^{\infty} p(x,y)\,\cancel{dx}\,dy = \frac{1}{210}\int_{0}^{5}(2x+y)\,dy$$

$$= \frac{1}{210}\left[2xy + \frac{y^2}{2}\right]_0^5 = \frac{1}{210}\left[10x + \frac{25}{2}\right]$$

Marginal distribution of Y:

$$g(y) = \frac{1}{210}\int_{2}^{6}(2x+y)\,dx = \frac{1}{210}\left[x^2+xy\right]_2^6 = \frac{1}{210}\left[32+4y\right]$$