UNIT-2

Relations.

cartesian product:

Let A and B are two sets, cartesian product of A and B is denoted by '$A \times B$' and is defined as

$$A \times B = \{(a,b) / a \in A, b \in B\}$$

i.e., $A \times B$ is the set of all possible ordered pairs whose first component comes from A and whose second component comes from B.

Example:

If $A = \{1, 2\}$ & $B = \{3, 4\}$

then $A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

$B \times A = \{(3,1), (3,2), (4,1), (4,2)\}$

$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

Binary Relation:

A binary relation or Relation R from $A \to B$ is the subset of $A \times B$ i.e., If $(a,b) \in R$ then aRb

Example:

Let $A = \{1,2,3\}$, $B = \{4,5,6\}$ be the two sets.

Now $A \times B$ is given by $A \times B = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$

i) define a relation R as $xRy \Leftrightarrow y/x \in \mathbb{Z}$

$R = \{(1,4), (1,5), (1,6), (2,4), (2,6), (3,6)\}$

which is a subset of $A \times B$. i.e., $R \subset A \times B$.

ii) define a relation R as $xRy \Leftrightarrow x+y = 7$

$R = \{(1,6), (2,5), (3,4)\}$

$R \subset A \times B$.

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Matrix of a relation:

consider the finite sets $A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $B = \{b_1, b_2, b_3, \ldots, b_n\}$ of order m & n respectively, then $A \times B$ consists of all ordered pairs of the form (a_i, b_j) for $1 \le i \le m$, $1 \le j \le n$.

Let R be the relation from A to B, so that $R \subseteq A \times B$.

Let's define $m \times n$ numbers $[m_{ij}]$ as follows:

$$[m_{ij}] = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

the $m \times n$ matrix form by these m_{ij}s is called "Adjacency Matrix or Boolean matrix" or matrix of the relation R.

Denoted by M_R or $[m_{ij}]$

where $1 \le i \le m$, $1 \le j \le n$

Example:

If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$

$A \times B = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$

$xRy \Leftrightarrow y/x \in \mathbb{Z}$

$R = \{(1,4), (1,5), (1,6), (2,4), (2,6), (3,6)\} \subseteq A \times B$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

(1) consider the set $A = \{0, 1, 2\}$ and $B = \{r, s\}$, the relation $R: A \to B$ is defined as

$R = \{(0, r), (1, s), (2, r)\}$

Determine the matrix of relation.

$$M_R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} r & s \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

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(2) Let $A = \{1, 2\}$ and $B = \{p, q, r, s\}$ and let R be the relation from $A \to B : R$ defined by

$$R = \{(1,q), (1,r), (2,p), (2,q), (2,s)\}$$

Write M_R

$$M_R = \begin{array}{c|c} & \begin{array}{cccc} p & q & r & s \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

(3) Let $A = \{a, b, c\}$ $B = \{0, 1\}$

$R = \{(a,0), (b,0), (c,1)\}$ be a relation from A to B. M_R

$$M_R = \begin{array}{c|c} & \begin{array}{cc} 0 & 1 \end{array} \\ \hline \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

(4) Let $A = \{1, 2, 3, 4\}$ and R be the relation on A defined by aRb iff $a < b$. Write the relation R and also matrix of R.

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

$$M_R = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(5) Determine the Relation R from $R : A \to B$ as described by the following matrix

$$M_R = \begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \hline \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}_{4 \times 3} \end{array}$$

Let $A = \{a, b, c, d\}$
$B = \{x, y, z\}$

$$R = \{(a,x), (a,z), (b,x), (b,y), (c,z), (d,x)\}$$

since M_R is 4×3 matrix, $A = 4$, $B = 3$

★ Determine the relation $R: A \to B$, $M_R = \begin{array}{c|c} & x \ y \ z \ w \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{4\times 4} \end{array}$

$R = \{(a,x), (a,y), (a,z), (b,y), (b,w), (c,x), (c,w), (d,z), (d,w)\}$.

Let $A = \{a, b, c, d\}$
$B = \{x, y, z, w\}$

<u>Digraphs:</u>

On a finite A, let R be the relation, then R can be represented graphically as follows.

Step 1: Draw a small circle for each element of A & label the circle with the corresponding element of A.

These circles are called "vertices".

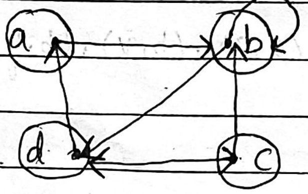
Step 2: Draw an arrow from vertex A to vertex B iff aRb called an edge.
The resulting graphical representation of R is called a digraph of R (or) directed graphs.

<u>Degree of vertex:</u>

The number of edges incident on a vertex.

① Consider the set $A = \{a, b, c, d\}$ and relation
$R = \{(a,b), (b,b), (b,d), (c,b), (c,d), (d,a), (d,c)\}$
defined on A. Represent this relation by digraph. Also find indegree and outdegree of each vertex.

$$M_R = \begin{array}{c|c} & a \ b \ c \ d \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



	a	b	c	d
ID (indegree)	1	3	1	2
OD (out degree)	1	2	2	2

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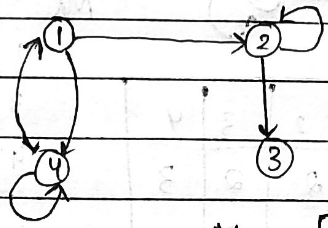
② Find the relation represented by the digraph represented below. and also find ID & OD of each vertex.

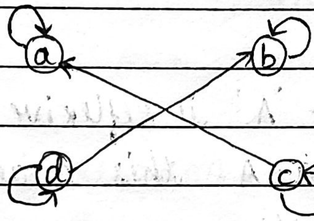
i)



$R = \{ (1,2), (2,2), (2,3), (1,4), (4,1), (4,4) \}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

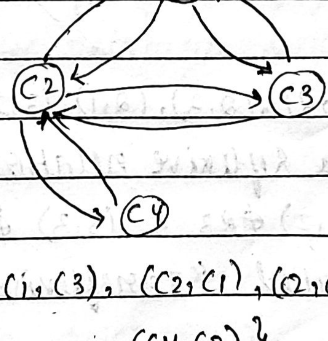
ii)



$R = \{ (a,a), (b,b), (c,c), (d,d), (c,a), (d,b) \}$

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

③ Find the relation represented by the digraph given below. write the matrix of the relation and also find indegrees and outdegrees of each vertex.



$R = \{ (C_1, C_2), (C_1, C_3), (C_2, C_1), (C_2, C_3), (C_2, C_4), (C_3, C_1), (C_3, C_2), (C_4, C_2) \}$

Vertices	C1	C2	C3	C4
Indegrees	2	3	2	1
outdegrees	2	3	2	1

$$M_R = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

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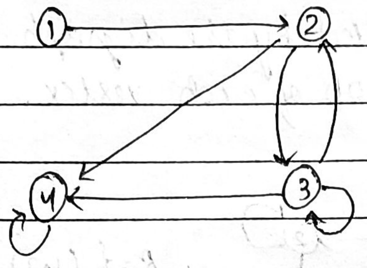
ii) 

$R = \{(1,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,4)\}$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 & 1 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$

Vertices	1	2	3	4
Indegrees	0	2	2	3
Out degrees	1	2	3	1

Types of relation: (Binary Relations)

① Reflexive Relation:

A relation 'R' on a set 'A' is Reflexive, if $aRa \ \forall a \in A$ i.e., if $(a,a) \in R \ \forall a \in A$, this means that each element of 'A' is related to itself.

Example: consider $A = \{1,2,3\}$. Let $R_1 = \{(1,1),(1,2),(2,2),(3,3),(2,1),(3,1)\}$ be a relation on 'A', then R_1 is a reflexive relation $\forall a \in A$, $(a,a) \in R_1$

a) Let $R_2 = \{(1,1),(1,2),(2,3),(3,3)\}$ be a relation on 'A', then R_2 is not a reflexive relation, since $2 \in A$, but $(2,2) \notin R_2$
$\therefore R_2$ is called "Non-reflexive relation".

b) Let $R_3 = \{(1,2),(2,3),(2,1),(3,2)\}$ be a relation on 'A', then R_3 is not a reflexive relation, since $1,2,3 \in A$, but $(1,1) \notin R_3$, $(2,2) \notin R_3$ & $(3,3) \notin R_3$.
$\therefore R_3$ is called "Irreflexive relation".

② Symmetric Relation:

A relation 'R' on a set 'A' is symmetric if, whenever $(a,b) \in R$ then $(b,a) \in R$ i.e., if aRb then bRa, this means that if one element is related to another element, then the second element is related to first.

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Example: The relation $R = \{(1,2),(2,1),(3,3),(2,2),(1,1)\}$ on $A = \{1,2,3\}$ is symmetric.

③ **Asymmetric relation:**

A relation 'R' on set 'A' is asymmetric if whenever $(a,b) \in R$ then $(b,a) \notin R$. i.e., if aRb then $b \not R a$, this means that the presence of $(a,b) \in R$ excludes the possibility of presence of $(b,a) \in R$.

Example: $R = \{(1,2),(2,2),(3,1),(3,3)\}$ on $A = \{1,2,3\}$ is assymmetric relation.

④ **Anti-symmetric Relation:**

A relation 'R' on set 'A' is anti-symmetric if $\forall$ $(a,b) \in A$ and $(a,b) \in R$ & aRb, bRa then $a = b$.

Example: ① $R = \{(x,y) / x \le y\}$ on set of $Z^+ \in A$ is an antisymmetric relation since, $x \le y$ and $y \le x$ then $x = y$.

② $R = \{(x,y) / x/y\}$ on $A \in Z^+$ is an antisymmetric relation since, x/y & $y/x \Rightarrow x = y$.

⑤ **Transitive Relation:**

A relation 'R' on set 'A' is transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$. i.e., if aRb & bRc then aRc

Example: ① $R = \{(x,y) / \text{"is parallel to"}\}$ on the set of lines in a plane is transitive because.

if $(x \parallel y)$ & $(y \parallel z)$ then $x \parallel z$

② R = "less than" & "greater than" are transitive on

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the set of real numbers, because
$x<y$ & $y<z \Rightarrow x<z$
(OR) $x>y$ & $y>z \Rightarrow x>z$

⑥ Equivalence Relation:

The relation 'R' on a set 'A' is called an equivalence relation if 'R' is reflexive, symmetric and transitive.
i.e., R is an equivalence relation on set A, if it has following properties.

i) $(a,a) \in R \;\forall a \in A$
ii) $(a,b) \in R \Rightarrow (b,a) \in R$. (not for every element of A).
iii) $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$.

Example: $R = \{(x,y) / \text{"="}\}$ on any set A is an equivalence relation because,
i) $a=a \quad \forall a \in A$
ii) $a=b \Rightarrow b=a$
iii) $a=b$ & $b=c \Rightarrow a=c$

* Let $R = \{(a,b) / a<b\}$ be a relation on set of all integers z. Discuss all relations on z.
i) Not reflexive because $a \nless a$.
ii) Not symmetric because $a<b \Rightarrow b \nless a$
iii) Transitive because $a<b$ & $b<c \Rightarrow a<c$

Given $R = \{(a,b) / a<b\}$ on z.
i) R is not reflexive on z since,
$1 \in Z$, $(1,1) \notin R$.

ii) R is not symmetric on z since, $(2,3) \in R$
but $(3,2) \notin R$.

iii) R is transitive on z since, $(a,b) \in R$ & $(b,c) \in R$
then $(a,c) \in R$.

* Let $R = \{(x,y) / x+y=5\}$, then be a relation defined on a set $A = \{1,2,3,4,5\}$, discuss all relations on A.

Given $R = \{(x,y) / x+y=5\}$
$R = \{(2,3),(1,4),(4,1),(3,2)\}$

i) R is not reflexive on A since, $1,2,3,4,5 \in A$ but $(1,1) \notin R$, $(2,2),(3,3),(4,4),(5,5) \notin R$.

ii) R is symmetric because $(1,4) \in R \Rightarrow (4,1) \in R$
& $(2,3) \in R \Rightarrow (3,2) \in R$

iii) R isn't transitive because $(1,4) \in R$ & $(4,1) \in R$
but $(1,1) \notin R$.

iv) R isn't antisymmetric on A.

* Let $A = \{1,2,3,4,5,6,7,8,9\}$ and * is the relation defined on AXA as
$R = \{(a,b) * (c,d)\} \in R \Rightarrow a+d = c+b$
P.T * is an equivalence relation on A.
$R(*) = \{(2,5)*(1,4),\ (1,4)*(2,5),\ (2,5)*(2,5),\ (1,4)*(1,4)\ \ldots\ldots\}$

Given, set $A = \{1,2,3,4,5,6,7,8,9\}$, * is the relation on A as $(a,b)*(c,d) \Rightarrow a+d = b+c$.

i) consider $(a,b)*(a,b) \in R \Rightarrow a+b = a+b$, so * is reflexive on A.

ii) consider $(a,b)*(c,d) \in R \Rightarrow a+d = c+b$.
then $(c,d)*(a,b) \in R \Rightarrow c+b = a+d$.
so, * is symmetric on A.

iii) consider $(a,b)*(c,d) \in R \Rightarrow a+d = c+b$ — ①
and $(c,d)*(e,f) \in R \Rightarrow c+f = d+e$ — ②

from ① & ②

$a + \cancel{d} = b + \cancel{d} + e - f$

$a + f = b + e \Rightarrow a + f = e + b$.

$\Rightarrow (a, b) * (e, f)$.

Hence * is transitive

$\therefore$ * is reflexive, symmetric & transitive. so it is an equivalence Relation.

* If R is a relation on set of integers Z defined as $R = \{(x, y) / x - y \text{ is divisible by } 6\}$ then prove that R is an equivalence relation.

Given,

$R = \{(x y) / x - y \text{ is divisible by } 6\}$

i) $(x, x) \in R \Rightarrow (x - x) = 0$ is divisible by 6 $\forall x \in Z$

so, R is reflexive on Z

ii) $(x, y) \in R \Rightarrow x - y = 6K$ — ①

$(y, x) \in R \Rightarrow y - x = 6(-K)$ from ①.

so, R is symmetric on Z

iii) $(x, y) \in R \Rightarrow x - y = 6K$ — ①

$(y, z) \in R \Rightarrow y - z = 6a$ — ②

then, ① + ②

$x - z = 6(K + a)$

$x - z = 6m$

$\therefore (x, z) \in R$.

so, R is transitive on Z.

$\therefore$ R is an equivalence relation.

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Equivalence class: [ ]

Let 'R' be an equivalence relation on set 'A' and let $a \in A$, the equivalence class of 'A' is denoted by [a] and is defined as $[a] = \{x \in A \mid xRa\}$.

It consists of all elements in A, that are linked to 'a' by the relation R. If $x \in [a]$, then x is called the representative of the class [a].

Example:

Relation $R = \{(a,a), (a,b), (b,a), (b,b), (c,c)\}$ on $A = \{a,b,c\}$ is an equivalence relation. Find the equivalence class of each element of A

$[a] = \{a,b\} \Rightarrow \{x \in A \mid xRa\}$

$[b] = \{a,b\} \Rightarrow \{x \in A \mid xRb\}$

$[c] = \{c\} \Rightarrow \{x \in A \mid xRc\}$

The representative of [a] is {a,b} & [b] is {a,b} & [c] is {c}

Partition sets:

Partition of set 'A' is a set of one or more non-empty subsets of 'A'. i.e., $A_1, A_2, A_3, \ldots$ such that every element of 'A' is in exactly one set i.e.,

① $A_1 \cup A_2 \cup A_3 \ldots \cup A_n = A$

② if $i \neq j$ then $A_i \cap A_j = \phi$

Example:

consider $A = \{1,2,3,4,5,6,7,8,9\}$

$A_1 = \{2,4,6,8\}$

$A_2 = \{1,3,5,7,9\}$

$A_1 \cup A_2 = A$

$A_1 \cap A_2 = \phi$

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* Show that the relation $(x,y)R(a,b) \Leftrightarrow x^2+y^2 = a^2+b^2$ is an equivalence relation on the plane & describe the equivalence class.

Given $(x,y)R(a,b) \Leftrightarrow x^2+y^2 = a^2+b^2$.

i) Reflexive: consider $(x,y)R(x,y) \Leftrightarrow x^2+y^2 = x^2+y^2$

Hence it is Reflexive.

ii) Symmetric: consider $(x,y)R(a,b) \Leftrightarrow x^2+y^2 = a^2+b^2$

$\Rightarrow a^2+b^2 = y^2+x^2$

$(a,b)R(x,y)$.

iii) Transitive: consider $(x,y)R(a,b)$ &

$(a,b)R(c,d)$

$\Rightarrow x^2+y^2 = a^2+b^2$ — ①

$a^2+b^2 = c^2+d^2$ — ②

from ① & ②

$x^2+y^2 = c^2+d^2$

$\Rightarrow (x,y)R(c,d)$

It is transitive.

So, R is an Equivalence relation.

Now for any point (x,y), the sum x^2+y^2 is the square of its distance from the origin.

$\therefore$ The equivalence classes are the set of points in the plane which have the same distance from the origin.

Thus, the equivalence classes are concentric circles centred at the origin.

A/c

* If R is the relation of set of integers Z defined as

$R = \{(x,y) / x,y \in Z, x-y \text{ is divisible by } 3\}$

Describe the equivalence class of R and the partition set of Z.

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i) $(x,x) \in R \Rightarrow 0$ divisible by 3
so Reflexive on Z

ii) $(x,y) \in R \Rightarrow (y,x) \Rightarrow$ $x-y = 3k$
$y-x = -3k \Rightarrow$ divisible
(y,x) is member of R
so, symmetric on Z

iii) $(x,y) \in R$ & $(y,z) \in R \Rightarrow (x,z) \in R$
$x-y = 3k$ —①
$y-z = 3a$ —②
$x-z = 3(m)$
so, $(x,z) \in R$.
so, Transitive on Z
$\therefore R$ is an equivalence relation.

For each integer a, $[a] = \{x \in Z / xRa\}$.
$= \{x \in Z / (x-a)$ divisible by $3\}$

$= \{x \in Z / (x-a) = 3k$ for some $k \in Z\}$

$[a] = \{x \in Z / x = 3k+a$ for some $k \in Z\}$

consider,
at $a=0 \Rightarrow [0] = \{x \in Z / x = 3k \ \forall\ k \in Z\}$

$[0] = \{-\infty, \ldots, -3, 0, 3, 6, \ldots \infty\}$

at $a=1 \Rightarrow$
$[1] = \{x \in Z / x = 3k+1$ for some $k \in Z\}$
$[1] = \{\ldots\ldots, -8, -5, -2, 1, 4, 7, \ldots\ldots\}$

at $a=2 \Rightarrow [2] = \{x \in Z / 3k+2$ for some $k \in Z\}$
$[2] = \{\ldots\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}$

Here $P = \{[0], [1], [2]\}$ is a partition set of Z.

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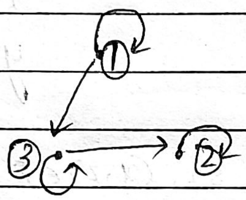
Types of Relations using Digraphs:

i) A relation R is said to be reflexive iff there is a loop at every vertex of directed graph.

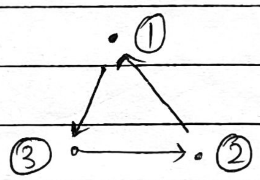
Example:

Let $A = \{1, 2, 3\}$

$R = \{(1,1), (1,3), (3,2), (2,2), (3,3)\}$



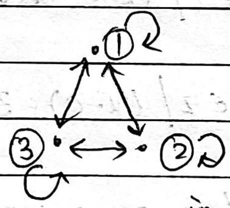
(Reflexive)



Irreflexive.

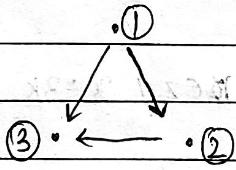
ii) A relation R is said to be irreflexive iff there is no loop at every vertex of the directed graph.

iii) (parallel edges)



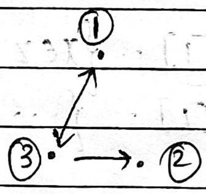
symmetric : A relation R is said to be symmetric iff $\forall$ every edge b/w distinct vertices in its digraph, there is an edge in the opposite direction. so, $(y,x) \in R \Rightarrow (x,y) \in R$

iv)



Transitive: A relation R is said to be transitive iff whenever $\exists$ an edge from vertex 'x' to 'y' & and from 'y' to 'z', there is an edge from 'x' to 'z'.

v) Assymettric:



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Operations on Relations:

① **complementary Relation:**

Given a relation of set A to set B, the complement of R is denoted by $\bar{R}$ and is defined as a relation from A to B with the property that $(a,b) \in \bar{R}$ iff $(a,b) \notin R$.

In other words $\bar{R} = (A \times B) - R$

② **Inverse of a Relation:**

The inverse of a Relation R is denoted by R^{-1} and is defined as $(b,a) \in R^{-1}$ iff $(a,b) \in R$

NOTE: * If M_R is the matrix relation of R, then $[M_R]^T = M_{R^{-1}}$

* $(R^{-1})^{-1} = R$

* consider the set, $A = \{a, b, c\}$ & $B = \{1, 2, 3\}$ and the two relations

$R_1 = \{(a,1), (b,2), (c,3)\}$

$R_2 = \{(a,1), (a,2), (b,1), (b,2)\}$ from A to B. Determine $\bar{R_1}$, $\bar{R_2}$, R_1^{-1}, R_2^{-1}, M_{R_1}, M_{R_2}, digraph of R_1 & R_2 also also $M_{R_1^{-1}}$, $M_{R_2^{-1}}$

$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\}$

i) $\bar{R_1} = \{(a,2), (a,3), (b,1), (b,2), (b,3), (c,1)\}$

ii) $\bar{R_2} = \{(a,3), (b,3), (c,1), (c,2), (c,3)\}$

$\bar{R_1}^{-1} = \{(2,a), (3,a), (1,b), (2,b), (3,b), (1,c)\}$

$\bar{R_2}^{-1} = \{(3,a), (3,b), (1,c), (2,c), (3,c)\}$

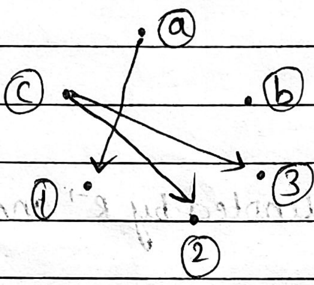
iii) $R_1^{-1} = \{(1,a),(2,c),(3,c)\}$

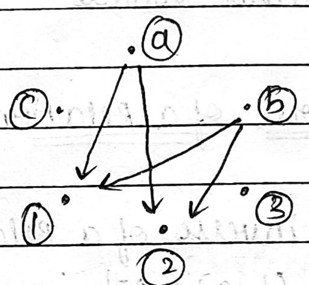
iv) $R_2^{-1} = \{(1,a),(2,a),(1,b),(2,b)\}$

v) $M_{R_1} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline a & & & \\ b & & & \\ c & & & \end{array}\;$ $\begin{array}{c} a \\ b \\ c \end{array}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

vi) $M_{R_2} = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

vii)





viii) $M_{R_1^{-1}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$

ix) $M_{R_2^{-1}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

* Let $A = \{1,2,3\}$ & $B = \{1,2,3,4\}$, the relations R & S from $A \to B$ are represented by the following matrices. Determine the relations $\overline{R}$, $\overline{S}$ and R^{-1}, S^{-1}. Also $M_{R^{-1}}$ and $M_{S^{-1}}$.

$A \times B = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4)\}$

$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$ $\quad M_S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$R = \{(1,1),(1,3),(2,4),(3,1),(3,2),(3,3)\}$

$S = \{(1,1),(1,2),(1,3),(1,4),(2,4),(3,2),(3,4)\}$

$\overline{R} = \{(1,2),(1,4),(2,1),(2,2),(2,3),(3,4)\}$

$\overline{S} = \{(2,1),(2,2),(2,3),(3,1),(3,3)\}$

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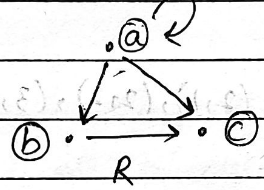
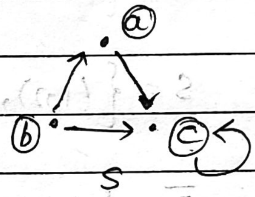
$R^{-1} = \{(1,1), (3,1), (4,2), (1,3), (2,3), (3,3)\}$

$S^{-1} = \{(1,1), (2,1), (3,1), (4,1), (4,2), (2,3), (4,3)\}$

$M_{R^{-1}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad M_{S^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

* The digraphs of two relations R and S on the set A, $A = \{a, b, c\}$ are given below: Draw the graph of $\bar{R}$ and R^{-1} and also $\bar{S}$ & S^{-1}. Also find $M_{\bar{R}}$, $M_{\bar{S}}$, $M_{R^{-1}}$, $M_{S^{-1}}$





$R = \{(a,a), (a,c), (a,b), (b,c)\}$

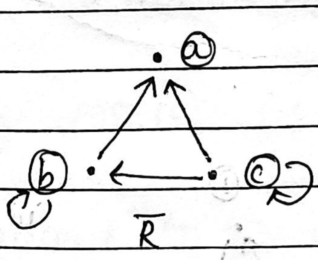
$S = \{(a,c), (b,a), (b,c), (c,c)\}$

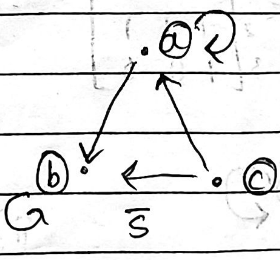
$\bar{R} = \{(a,a), (b,b), (c,a), (c,b), (c,c)\}$

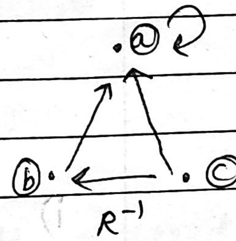
$\bar{S} = \{(a,a), (a,b), (b,b), (c,a), (c,b)\}$

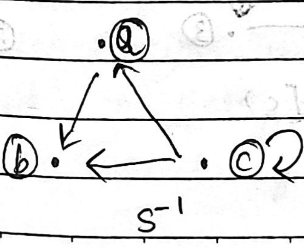
$R^{-1} = \{(a,a), (c,a), (b,a), (c,b)\}$

$S^{-1} = \{(c,a), (a,b), (c,b), (c,c)\}$









$$M_{\bar{R}} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

* Let $A = \{1,2,3\}$ and R is relation on A where matrices are given below. Find matrices of $\bar{R}, R^{-1}, \bar{S}, S^{-1}$. Also find the relation for $\bar{R}, R^{-1}, \bar{S}, S^{-1}$. Also draw the digraph for $R, S, R^{-1}, S^{-1}, \bar{R}, \bar{S}$

$$A \times A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}.$$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} ; \quad M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$R = \{(1,1),(1,3),(2,2),(2,3)\}$

$S = \{(1,2),(1,3),(2,1),(2,2),(3,2)\}$

$\bar{R} = \{(1,2),(2,1),(3,1),(3,2),(3,3)\}$

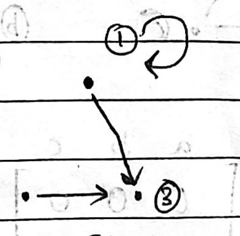
$\bar{S} = \{(1,1),(1,2),(2,3),(3,1),(3,3)\}$

$R^{-1} = \{(1,1),(3,1),(2,2),(3,2)\}$

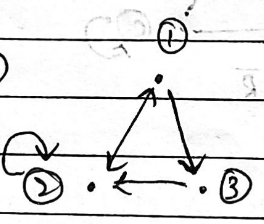
$S^{-1} = \{(2,1),(3,1),(1,2),(2,2),(2,3)\}$

$$M_{\bar{R}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \quad M_{\bar{S}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \quad M_{R^{-1}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

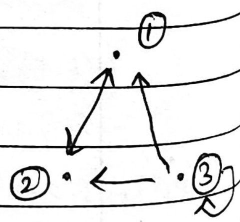
$$M_{S^{-1}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

i) 

G [R]

ii) 

[S]

iii) 

$[\bar{R}]$

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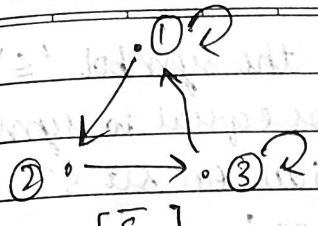
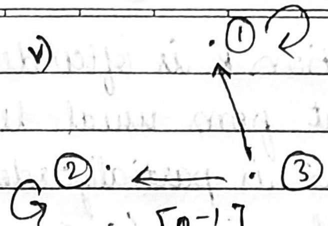
iv)



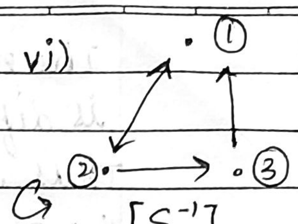
[$\bar{S}$]

v)



[R^{-1}]

vi)

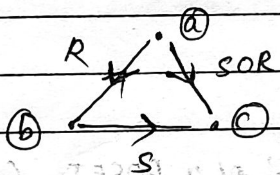


[S^{-1}]

composition of Relation:

Given a relation 'R' from set 'A' to set 'B'. And a relation 's' from set 'B' to set 'C'. we define a new relation from set 'A' to set 'C' called the composition of relation R & S. It is denoted by 'SOR' & is defined as follows:

If 'a' is in A, and 'c' is in 'C', then $(a,c) \in$ ~~SOR~~ ROS iff $\exists$ some 'b' in 'B' such that $(a,b) \in R$ & $(b,c) \in S$



ROS

* let $A = \{1,2,3\}$ and $B = \{a,b\}$ and $C = \{r,s\}$ and $R = \{(1,a),(2,a),(3,b)\}$ be a relation from A to B & $S = \{(a,s),(b,r)\}$ be a relation from B to C. then SOR from A to C is defined as follows:

since $(1,a) \in R$ & $(a,s) \in S \Rightarrow (1,s) \in SOR$
$(2,a) \in R$ & $(a,s) \in S \Rightarrow (2,s) \in SOR$
$(3,b) \in R$ & $(b,r) \in S \Rightarrow (3,r) \in SOR$

~~posets:~~ POSETS: (partially ordered sets)

A relation 'R' on a set 's' is called a partially ordered relation if it is reflexive, anti-symmetric and transitive

$\Rightarrow$ A set 's' together with a partially ordered relation R is called a partially ordered set or 'POSET'. It is denoted by (S, R).

The relation R is often denoted by the symbol '$\leq$' which is different from usual less than or equal to symbol. Thus if '$\leq$' is partially ordered relation on set 'S' then the ordered pair $(S, \leq)$ is a 'POSET'

Example: ① The relation '$\geq$' is a partial ordered relation on set of Integers.

② The relation '/' is a POSET on set of all +ve Integers

Preceeds & succeeds:

In a POSET, $(S, \leq)$ if any two elements 'a' and 'b', $a \leq b$ then we say that 'a' preceeds 'b' and 'b' succeeds 'a'.

Comparability:

The elements 'a' and 'b' of a POSET $(S, \leq)$ are comparable if either "$a \leq b$" or "$b \leq a$". i.e., a & b are related. When a & b are elements of 's' such that neither "$a < b$" nor "$b \leq a$" i.e., a & b are not related, then a & b are incomparable.

Example: In the POSET $(Z^+, /)$, the integers 2 & 4 are comparable since $2/4$ (2 divides 4) but the integers 3 & 5 are incomparable because 3 doesn't divide 5 or 5 doesn't divide 3.

Totally ordered sets: (OR) Linearly ordered sets.

If $(S, \leq)$ is a POSET and every two elements of 's' are comparable and the relation '$\leq$' is called a totally ordered relation.

This totally ordered set is also called linearly ordered set. (or) 'chain'

Examples: ① In the POSET $(Z, \leq)$, '$a \leq b$' or '$b \geq a$' for all integers 'a & b' with the relation $(Z, \leq)$ is a totally ordered set.

② The POSET $(Z^+, /)$, is not a totally ordered set, since it contains elements that are incomparable such as 3 & 5 i.e., $3 \nmid 5$ or $5 \nmid 3$.

NOTE: An ordered set may not be totally ordered but it is still possible for a subset 'A' of 'S' to be totally ordered.

$$A \subseteq S.$$

Example: consider the POSET $(Z^+, /)$ which is not totally ordered but $A = \{2, 6, 12, 36\} \subseteq Z^+$ is a totally ordered set since $2/6$, $6/12$, $12/36$, $2/12$, $2/36$, $6/36$.

Hasse Diagram :

A partially ordered relation '$\leq$' on a set 'X' can be represented by means of diagram known as 'Hasse diagram' of $(X, \leq)$. We represent the elements of 'X' by dots or small circles and if 'y' is an immediate successor of 'x', we take 'y' at a higher level than 'x' and join 'x' & 'y' by a straight line.

A diagram formed as above is known as Hasse diagram. Thus, there will not be any horizontal lines in the diagram of POSET as non-comparable elements are not joined.

Example:

Let $X = \{2, 3, 6, 12, 24, 36\}$

and the relation '|' (divides) such that $x | y$. Draw an

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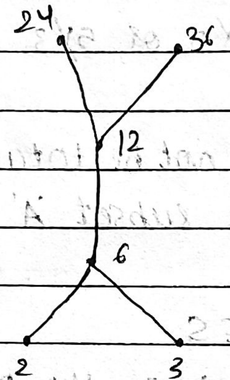
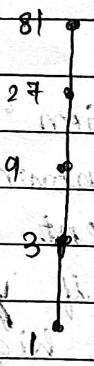
Hasse diagram of $(X, |)$

$2|6,\ 2|12,\ 2|24,\ 2|36,\ 3|6,\ 3|12,\ 3|24,\ 3|36,\ 6|12,\ 6|24,$
$6|36,\ 12|24,\ 12|36.$



* Let $A = \{1, 3, 9, 27, 81\}$. Draw Hasse Diagram of the POSET. $(A, |)$

$1|1,\ 3|3,\ 9|9,\ 27|27,\ 1|3,\ 1|9,\ 3|9,\ 9|27,\ 27|81,\ 81|81.$



* Construction of Hasse Diagram using for the given directed graph:

Procedure to obtain Hasse diagram of a directed graph of a partially ordered relation

Step ① Remove all the loops at all vertices.

Step ② Remove all edges whose existence is implied by transitive property.

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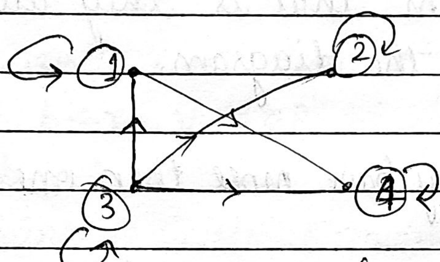
Step ③ Arrange all arrows pointing upwards, towards their terminal vertex. Remove all the arrows.

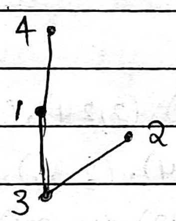
Example:

* Draw a Hasse diagram from the digraph 'G' for a partial ordered relation on a set $A = \{1, 2, 3, 4\}$



$R = \{(1,1), (2,2), (3,3), (4,4), (3,4)$
$(1,4), (3,1), (3,2)\}$

Hasse



Maximal element = 4, 2
Minimal element = 3.
least element = 3

because 4, 2 aren't connected, if they were connected then, 4 would be ⟵ greatest element = none.

First, we remove all the loops at all the vertices & edges b/w 3 & 4 which is transitively implied.

Arrange all the arrows pointing upwards.

Now remove all the arrows, we get the required Hasse diagram.

Special elements in POSETS

① Maximal element :

(A, R) $a \leq b$

An element 'a' in a POSET $(A, \leq)$ is called a maximal element if there is no element $b \in A$ such that $a < b$.

i.e., no other element of 'A' strictly succeeds 'a'

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② **Minimal element** -

An element 'c' in the POSET $(A, \leq)$ is called a minimal element if there is no other element $d \in A$, such that $d < c$ i.e., no element of 'A' strictly preceeds 'c'.

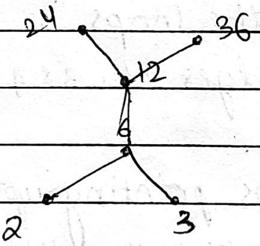
NOTE: * Maximal & Minimal elements are easy to spot in an Hasse diagram that is they are the top & bottom elements of the diagram.

* A POSET may have more than one maximal & minimal elements

Example: In the POSET $\{2, 3, 6, 12, 24, 36\}$ with divisibility relation.

$(A, |)$

$R = \{(2,6), (2,12), (2,24), (2,2), (2,36), (3,3), (3,6), (3,12), (3,24), (3,36), (6,6), (6,12), (6,24), (6,36), (12,12), (12,24), (12,36), (24,24), (36,36)\}$



2 & 3 are minimal elements

24, 36 are maximal elements

because 24, 36 aren't connected ← 36 ≠ greatest, 2 ≠ least (2, 3 aren't connected)

* A POSET may have a maximal element or a minimal element.

Example:

The POSET $(Z^-, \leq)$ has maximal element & no minimal element

whereas $(Z^+, \leq)$ has minimal element but no maximal element.

③ **Greatest element:**

Let $(P, \leq)$ be a POSET, an element $a \in P$ is the greatest element of 'P' if $x \leq a \ \forall \ x \in P$ i.e., every element in 'P' preceeds 'a'.

The greatest element if exists is unique.

④ **Least element:**

Let $(P, \leq)$ be a POSET, an element $b \in P$ is the least element of 'P' if $b \leq x \ \forall \ x \in P$ i.e., every element in 'P' succeeds 'b'.

The least element if exists is unique.

NOTE: * In Hasse diagram, a greatest element is connected to every other element by a path leading down & a least element is connected to every other element leading up.

* Maximal element may not be the greatest element & minimal element may not be the least element.

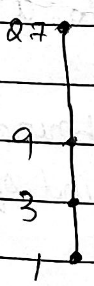
Example:

It has 2 maximal elements

36 ≠ greatest element

2 ≠ least element.

* Determine whether POSETS represented by Hasse diagrams have greatest element, least element and min & maximal element.

i) 

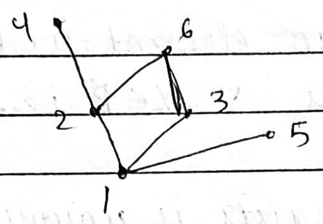
Max element = 27

Minimal element = 1

Greatest element = 27

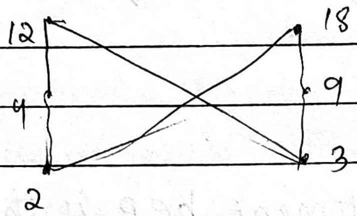
Least element = 1

ii)



Maximal = 4, 5, 6
Minimal = 1
greatest = none
least = 1

iii)



maximal = 12, 18
minimal = 2, 3
least = none
greatest = none.

Upperbound:

Let $(A, \leq)$ be a poset and $B \subset A$ and an element $u \in A$ is called an upperbound of 'B' if "u succeeds every element of B" i.e., $x \leq u \ \forall x \in B$.

Lowerbound:

Let $(A, \leq)$ be a poset and $B \subseteq A$, an element 'l' $\in A$ is called an lowerbound of 'B' if "l precedes every element of B" i.e., $x \geq l \ \forall x \in B$.

Least upperbound: (LUB) or supremum or Join or disjunction ($\vee$)

If $B \subset A$ and $a \in A$, then 'a' is called the least upperbound of 'B' if

i) 'a' is an upperbound of 'B'
ii) 'a' should be less than or equal to a' $(a \leq a')$ for every bound a' of B.

In other words, least upperbound of B = minimum of upperbound of B.

$$LUB(B) = \{\min UB(B)\}$$

Greatest lowerbound: (GLB) or Infimum or meet or conjuction ($\wedge$).

If $B \subseteq A$ and 'b' $\in A$, then 'b' is called greatest ~~upp~~ lowerbound of 'B' if

i) 'b' is a lowerbound of B.

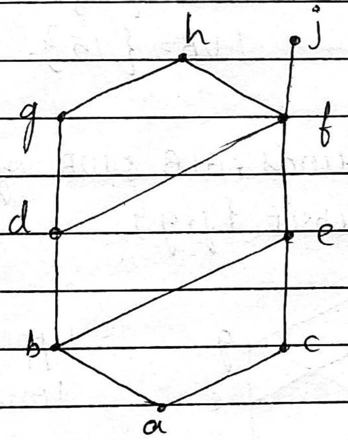
ii) $b' \leq b$ for every lowerbound b' of B.

In other words,

$$GLB(B) = \{\max LB(B)\}$$

Example:

① Find the lower & upperbounds of the subset $\{a, b, c\}$, $\{j, h\}$ and $\{a, c, d, f\}$ in the poset, with the Hasse diagram shown below, Also find GLB & LUB. for the above subsets.



i) for subset $\{a, b, c\}$

the upperbound $= \{e, f, j, h\}$

the lowerbound $= \{a\}$

$GLB = \{a\}$

$LUB = \{e\}$

ii) for subset $\{j, h\}$

$UB = \phi$

$LB = \{d, b, a, c, e, f\}$

$GLB = \{f\}$

$LUB = \phi$

iii) for subset $\{a, c, d, f\}$ $\Rightarrow$ $UB = \{j, h, f\}$ $\quad$ $GLB = \{a\}$

$LB = \{a\}$ $\quad$ $LUB = \{f\}$

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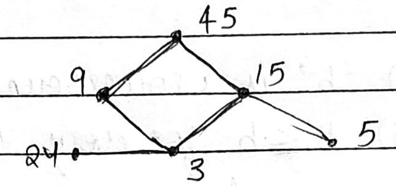
② Find all the UB & LB, GLB & LUB for the $A = \{3, 5\}$ and $B = \{15, 45\}$ for the following Hasse diagram



i) $A = \{3, 5\} \Rightarrow$ $UB = \{15, 45\}$

$LB = \phi$

$GLB = \phi$

$LUB = \{15\}$

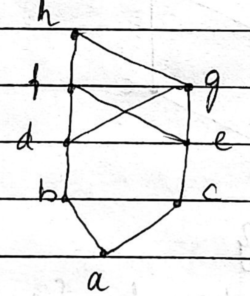
ii) $B = \{15, 45\} \Rightarrow$ $UB = \{45, 15\}$

$LB = \{5, 3\}$

$GLB = \{5\}$

$LUB = \{15\}$.

* Find lowerbounds, upperbounds, GLB & LUB of the following Hasse diagram. for the subset $\{f, g\}$



Upperbound : $\{h\}$

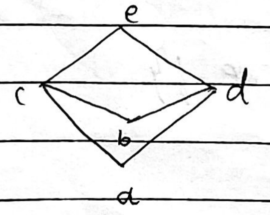
Lowerbound : $\{d, e, b, c, a\}$

$LUB = \{h\}$

$GLB = \{\phi\}$

because d & e are in the same position. and d & e aren't connected.

* $S = \{c, d\}$



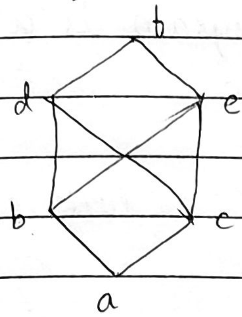
$UB = \{e\}$

$LB = \{b, a\}$

$GLB = \{\phi\}$ bcuz b & a aren't con-nected

$LUB = \{e\}$

* Find $S = \{d, e\}$ UB, LB, LUB, GLB for the given Hasse diagram.



LB = {b, c, a}
UB = {f}
LUB = {f}
GLB = {φ}

* **Lattices:**

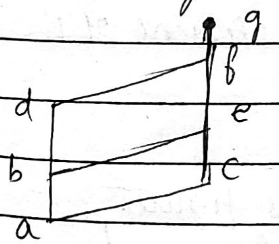
A poset $(P, \leq)$ is called a lattice if every two elements in subset of P has both LUB & GLB i.e., if LUB $\{x, y\}$ and GLB $\{x, y\}$ exist for every 'x & y' in 'P'.
In this case, we denote $x \vee y = LUB\{x, y\}$ [Join x & y] (+)
$x \wedge y = GLB\{x, y\}$ [meet x & y] (*)

Lattice is a mathematical structure equipped with a binary operations 'join' and 'meet'. other notations of join are ($\oplus$ or +). other notations of 'meet' are (• or *).

NOTE: Every chain is a lattice, since any 2 elements (x, y) of a chain are comparable.
We find $x \vee y = LUB\{x, y\} = y$
$x \wedge y = GLB\{x, y\} = x$

* Determine whether the posets represented by each of the Hasse diagram are lattices or not.

i)



every comparable elements have GLB & LUB.
non-comparable {d, g}

To check for a lattice, every pair must have GLB & LUB. In general, all comparable elements have GLB & LUB
∴ Better to check for incomparable elements.
In this Hasse diagram, incomparable pairs are {d, g} but $d \to f$ & $f \to g$ so, $d \to g$ (transitive)
UB = {φ} ; LB = {

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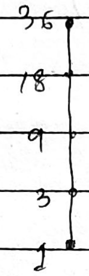
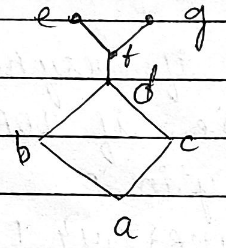
{a,c}, {a,d}, {a,f}, {a,g} .... all the pairs are comparable. ∴ Given Hasse diagram is a lattice.

ii)



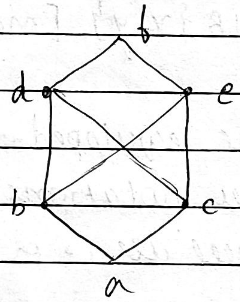
all chains are lattices.

iii)



for {e,g} no UB.
GLB {e,g} = {f}
LUB {e,g} = ϕ
∴ it is not a lattice.

iv)



LUB {d,e} = {f}
GLB {d,e} = ϕ

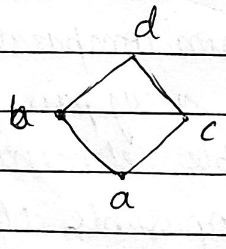
Greatest element of a lattice:

Let (L, ≤) be a lattice, an element g ∈ L is called the greatest element of 'L' if a ≤ g ∀ a ∈ 'L'

Least element of a lattice:

An element s ∈ 'L' is called the least element of 'L' if s ≤ a ∀ a ∈ L

Example:

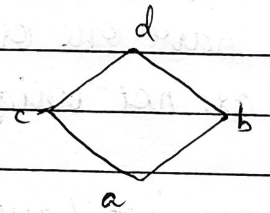


In this figure:
greatest element = d
least element = a

Bounded Lattice :

A lattice $(L, \leq)$ is called a bounded lattice if it has, both greatest element (1) and the least element (0).
In a lattice, they are also called as universal upperbound and universal lowerbound respectively.

Example:



least element = a
greatest element = d
so, it is a bounded lattice.

NOTE: Lattice $(Z^+, \leq)$ is not a bounded lattice as 1 is the least element and there is no greatest element.

Properties of bounded lattice :

(*) 0 → smallest element
1 → Greatest element.

① $0 \vee a = LUB(0, a) = a$
② $0 \wedge a = GLB(0, a) = 0$
③ $1 \wedge a = GLB(1, a) = a$
④ $1 \vee a = LUB(1, a) = 1 \quad \forall a \in L$

complemented lattice :

let $(L, \leq)$ be a bounded lattice for $a \in L$, $b \in L$, is called complement of 'a' if $LUB(a, b) = 1$ & $GLB(a, b) = 0$

or $a \vee b = 1$ or $a \wedge b = 0$

Where 1 & 0 are the greatest and least elements.

A bounded lattice $(L, \leq)$ is called a complemented lattice if every element in L, has a complement.

NOTE ①: If the set $(L, \leq)$ contains numbers, then $LUB(a,b) = a \vee b$
$= LCM(a,b) =$ greatest element (1)

$a \wedge b = GLB(a,b) = GCD(a,b) =$ least element (0)

② No two complements is symmetric.

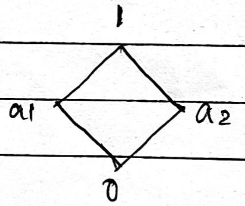
③ Any element $a \in L$ may have one or more complements in L. So, complements are not unique.

④ any element $a \in L$ may or may not have a complement

⑤ In any bounded lattice, the bounds 0 & 1 are unique complements of each other, because

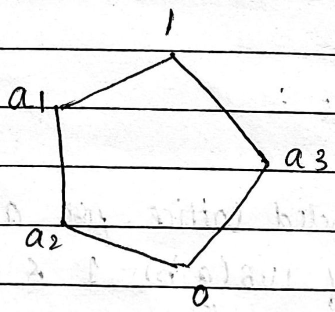
$0 \vee 1 = 1$
$0 \wedge 1 = 0$

Example: ①



$a_1 \wedge a_2 = GLB(a_1, a_2) = 0$
$a_1 \vee a_2 = LUB(a_1, a_2) = 1$

②



$a_2 \wedge a_3 = 0$
$a_2 \vee a_3 = 1$ $\Rightarrow a_2' = a_3$, $a_3' = a_2$

$a_1 \wedge a_3 = 0$
$a_1 \vee a_3 = 1$ $\Rightarrow a_1' = a_3$, ~~$a_3' = a_1$~~

'a_3' is the complement of 'a_1' & 'a_2'

$a_1 \wedge a_2 = 0$
$a_1 \vee a_2 = 1$ $\Rightarrow$ $a_1' = a_2$, $a_2' = a_1$

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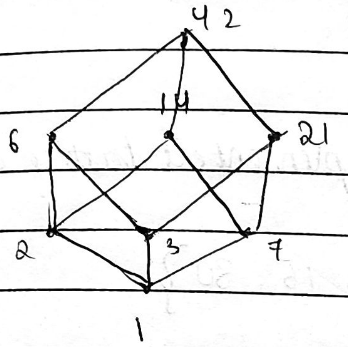
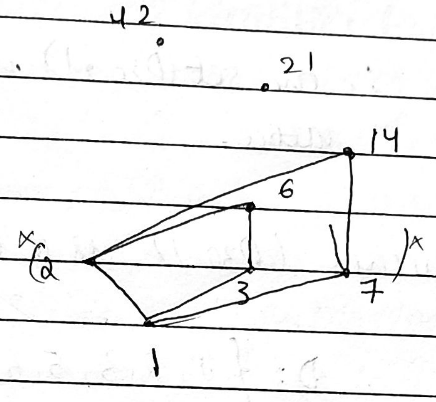
* let $(S_{42}, |)$ is a complemented lattice.

let $S_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$.





$1 \vee 42 = LUB(1, 42) = LCM(1, 42) = 42.$

$1 \wedge 42 = GLB(1, 42) = GCD(1, 42) = 1$

$\Rightarrow 1' = 42$

$42' = 1.$

$2 \vee 21 = 42 = LUB(2, 21) = LCM(2, 21) = 42.$

$2 \wedge 21 = 1 = GCD(2, 21).$

$\Rightarrow 2' = 21 \;\&\; 21' = 2$

$3 \wedge 14 = 1$

$3 \vee 14 = 42$

$3' = 42 \;\&\; 42' = 3$

$6 \wedge 7 = 1$

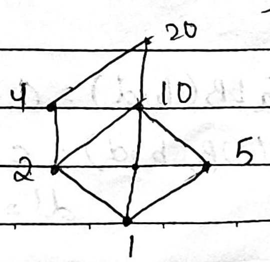
$6 \vee 7 = 42$

$6' = 7 \;\&\; 7' = 6.$

every element in S_{42} has complement, $(S_{42}, |)$ is a complemented lattice.

* consider $(D_{20}, |)$ check whether (D_{20}) is a complemented lattice or not.

$D = \{1, 2, 4, 5, 10, 20\}$



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$1 \vee 20 = 20$
$1 \wedge 20 = 1$
$1' = 20$

$2 \vee 10 = 10 \neq 20$
$2 \wedge 10 = 2 \neq 1$
2 has no complement.

$2 \vee 5 = 10 \neq 20$
$2 \wedge 5 = 1$

$2 \vee 4 = 4 \neq 20$
$2 \wedge 4 = 2 \neq 1$

$\therefore$ the set $(D_{20}, |)$ is not a complemented lattice.

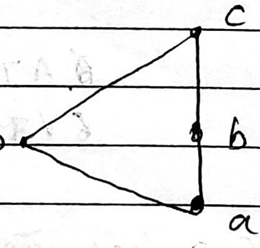
* check whether $(D_{30}, |)$ is a complemented lattice or not.

$D = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$1 \vee 30 = 30$ $2 \vee 15 = 30$ $3 \vee 10 = 30$ $5 \vee 6 = 30$
$1 \wedge 30 = 1$ $2 \wedge 15 = 1$ $3 \wedge 10 = 1$ $5 \wedge 6 = 1$
$1' = 30$ $2' = 15$ $3' = 10$ $5' = 6$

$6 \vee 5 = 30$ $10 \vee 3 = 30$ $15 \vee 2 = 30$
$6 \wedge 5 = 1$ $3 \wedge 10 = 1$ $15 \wedge 2 = 1$
$6' = 5$ $3 = 10'$ $15' = 2$

* Check if the following Hasse diagram is a complemented lattice or not.



least element = a
greatest element = c
$\therefore$ it is a bounded lattice

Then, check for complemented lattice.

$a \wedge c = GLB(a, c) = a = GLB(c, a)$
$a \vee c = LUB(a, c) = c = LUB(c, a)$
$a' = c$
$c' = a.$

$b \wedge d = GLB(b, d) = a = GLB(d, b)$
$b \vee d = LUB(b, d) = c = LUB(d, b)$
$d' = b, \; b' = d.$

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Distributive lattice:

A lattice $(L, \leq)$ is called a distributive lattice, if $\forall\, a, b, c \in L$. The following identities hold:

i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

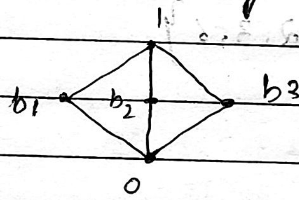
where $a \wedge b = GLB(a, b)$; $a \wedge c = GLB(a, c)$

$a \vee b = LUB(a, b)$; $a \vee c = LUB(a, c)$.

Here, both the equalities are equivalent to one another. Hence, to check the whether the lattice is distributive or not, it's sufficient to verify one of them.

Example: verify whether the following lattices are distributive or not.

i)



LHS $\Rightarrow$ $b_1 \wedge (b_2 \vee b_3) = b_1 \wedge (LUB(b_2, b_3))$

To verify ↗

$= b_1 \wedge (1)$

$= b_1$

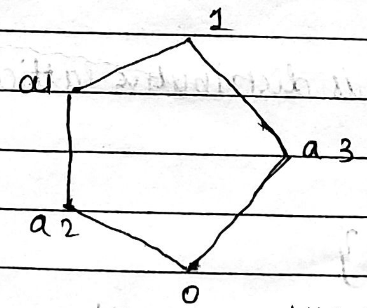
RHS $\Rightarrow$ $(b_1 \wedge b_2) \vee (b_1 \wedge b_3)$

$\Rightarrow 0 \vee 0 = 0$

LHS $\neq$ RHS

It is not a distributive lattice.

ii)



LHS $\Rightarrow$ $a_1 \wedge (a_2 \vee a_3)$

$= a_1 \wedge (LUB(a_2, a_3))$

$= a_1 \wedge 1$

$= a_1$

RHS $\Rightarrow$ $(a_1 \wedge a_2) \vee (a_1 \wedge a_3)$

$a_2 \vee 0$

$= a_2$

LHS $\neq$ RHS.

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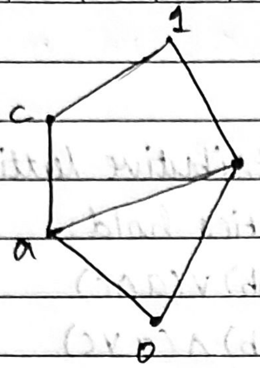
iii)



LHS ⇒ $a \wedge (b \vee c)$

$a \wedge (LUB(b, c))$

$a \wedge 1$

$= a$

RHS ⇒ $(a \wedge b) \vee (a \wedge c)$

$a \vee a$

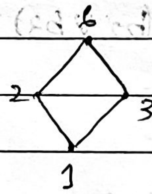
$= a$

LHS = RHS

so, it's a distributive lattice.

iv) check the set $(D_6, |)$ is a distributive lattice or not

$D = \{1, 2, 3, 6\}$



case 1: 1, 2, 3

LHS ⇒ $1 \wedge (2 \vee 3)$

$1 \wedge (LCM(2,3))$

$1 \wedge 6$

$GCD(1, 6) = 1$

RHS ⇒ $(1 \wedge 2) \vee (1 \wedge 3)$

$1 \vee 1$

1

LHS = RHS

so, it is distributive lattice.

case 2: 2, 3, 6

LHS ⇒ $2 \wedge (3 \vee 6)$

$2 \wedge (6)$

$= 2$

RHS ⇒ $(2 \wedge 3) \vee (2 \wedge 6)$

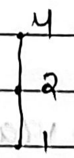
$1 \vee 2$

$= 2$

LHS = RHS

v) $(D_4, |)$

$D = \{1, 2, 4\}$



case 1: 1, 2, 4

LHS ⇒ $1 \wedge (2 \vee 4)$

$= 1 \wedge (4)$

$= 4$

RHS ⇒ $(1 \wedge 2) \vee (1 \wedge 4)$

$1 \vee 1 = 1$

LHS = RHS.

NOTE: In a distributive lattice, if an element has a complement then this complement is unique.

Boolean Algebra:

A complemented, distributive lattice is known as Boolean Algebra.
It is denoted by $(B, \vee, \wedge, ', 0, 1)$, where 'B' is a set on which two binary operations (and '$\wedge$' (*) & OR '$\vee$' (+)) and unary operation (') [complement] are defined. Here, 0 & 1 are two distinct elements of 'B'. Since $(B, \vee, \wedge)$ is a complemented distributive lattice, each element of 'B' has a unique complement.

NOTE:

① If 'n' is the product of distinct primes, then D_n is a boolean algebra i.e., if $n = p_1 \cdot p_2 \cdot p_3 \cdots p_n$ where p_i's are distinct, then D_n is a boolean algebra

Example: $D_6, D_{10}, D_{15}, D_{21}, D_{30}, D_{33}, \ldots$ etc are all boolean algebras since

$6 = 2 \times 3$ $\qquad 21 = 3 \times 7$
$10 = 2 \times 5$ $\qquad 30 = 3 \times 5 \times 2$
$15 = 3 \times 5$ $\qquad 33 = 3 \times 11$.

② If 'n' is divisible by squares of prime numbers, then D_n is not a boolean algebra i.e., if 'n' is the product of prime squares, then D_n is not a boolean algebra.

Example: $D_4, D_{12}, D_{18}, D_{16}, \ldots$ etc aren't boolean algebras because

$4 = 2 \times 2 = 2^2$
$12 = 2^2 \times 3$
$18 = 3^2 \times 2$
$16 = 2^2 \times 2^2$

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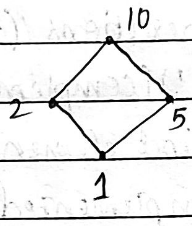
* Show that $(D_{10}, |)$ is a Boolean Algebra.

$D_{10} = \{1, 2, 5, 10\}$

Greatest element = 10
Least element = 1.
So, it is a bounded lattice.



To show that D_{10} is distributive lattice.

i) $1 \wedge (2 \vee 5) = (1 \wedge 2) \vee (1 \wedge 5)$

$1 \wedge 10 = 1 \vee 1$

$1 = 1$

LHS = RHS.

ii) $2 \wedge (5 \vee 10) = (2 \wedge 5) \vee (2 \wedge 10)$

$2 \wedge (10) = 1 \vee 2$

$2 = 2.$

LHS = RHS.

Hence $(D_{10}, |)$ is a distributive lattice.

check for complemented lattice:

$1 \wedge 10 = 1$ $\quad 2 \wedge 5 = 1$ $\quad 5 \wedge 2 = 1$ $\quad 10 \wedge 1 = 1$

$1 \vee 10 = 10$ $\quad 2 \vee 5 = 10$ $\quad 5 \vee 2 = 10$ $\quad 10 \vee 1 = 10$

So, $1' = 10$ $\quad$ so, $2' = 5$ $\quad 5' = 2$ $\quad 10' = 1$

So, every element of the lattice has a complement

$\therefore$ $(D_{10}, |)$ is a complemented lattice.

$\therefore$ $(D_{10}, |)$ is a boolean algebra

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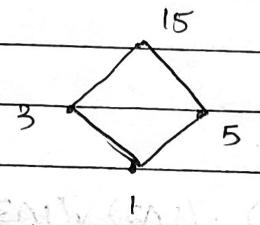
* show that $(D_{15}, |)$ is a boolean algebra.

$D_{15} = \{1, 3, 5, 15\}$



Distributive lattice:

i) $1 \wedge (3 \vee 5) = (1 \wedge 3) \vee (1 \wedge 5)$

$1 \wedge 15 = 1 \vee 1$

$1 = 1$ LHS = RHS.

ii) $3 \wedge (5 \vee 15) = (3 \wedge 5) \vee (3 \wedge 15)$

$3 \wedge (15) = 1 \vee 3$

$3 = 3$

LHS = RHS

Hence $(D_{15}, |)$ is a distributive lattice.

check for complemented lattice:

greatest element = 15

least element = 1

$1 \wedge 15 = 1$	$3 \wedge 5 = 1$	$5 \wedge 3 = 1$	$15 \wedge 1 = 1$
$1 \vee 15 = 15$	$3 \vee 5 = 15$	$5 \vee 3 = 15$	$15 \vee 1 = 15$
so, $1' = 15$	so, $3' = 05$	$5' = 3$	$15' = 1$.

so, every element of the lattice has a complement.

$\therefore (D_{15}, |)$ is a complemented lattice

$\therefore (D_{15}, |)$ is a boolean algebra.

* check whether $(D_{18}, |)$ is a boolean algebra or not.

$D_{18} = \{1, 2, 3, 6, 9, 18\}$

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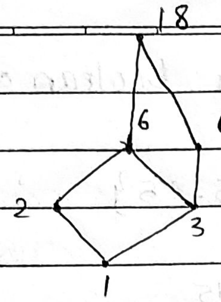


greatest element = 18
least element = 1

<u>Distributive:</u>

i) $1 \wedge (2 \vee 3) = (1 \wedge 2) \vee (1 \wedge 3)$

$1 \wedge (6) = 1 \vee 1$

$1 = 1$

LHS = RHS

ii) $3 \wedge (6 \vee 9) = (3 \wedge 6) \vee (3 \wedge 9)$

$3 \wedge (18) = 3 \vee 3$

$3 = 3$

iii) $6 \wedge (9 \vee 18) = (6 \wedge 9) \vee (6 \wedge 18)$

$6 \wedge (18) = 3 \vee 6$

$6 = 6$

LHS = RHS

It is a distributive lattice

<u>complemented lattice:</u>

$1 \wedge 18 = 1$	$2 \wedge 9 = 1$	$6 \wedge 9 = 3 \neq 1$
$1 \vee 18 = 18$	$2 \vee 9 = 18$	$6 \vee 9 = 18$
$1' = 18$	$2' = 9$	so, it doesn't have a complement.

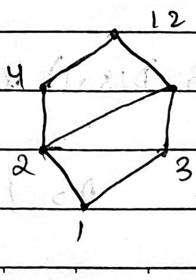
$\therefore$ It is not a complemented lattice

$\therefore$ $(D_{18}, 1)$ is not a boolean algebra.

* $(D_{12}, 1)$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

Do the same procedure above.



$12 = 3 \times 2^2$, so, it is not a boolean alg

Recurrence Relation:

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms, of the sequence namely $a_0, a_1, \ldots, a_{n-1}$. $\forall$ integers 'n' with $n \geq n_0$, where n_0 is a non-negative integer.

A sequence is called a solution of recurrence relation if it's terms satisfy the recurrence relation.

The initial conditions for a sequence specify the terms that preceed the first term when the recurrence relation takes effect.

Examples:

① Let $\{a_n\}$ be a sequence, that satisfies the recurrence relation $a_n = a_{n-1} + 3$, for $n = 1, 2, 3, \ldots$ and suppose that $a_0 = 2$.

② Let $\{a_n\}$ be the sequence, that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$, for $n = 2, 3, 4, \ldots$ and suppose that $a_0 = 3$ & $a_1 = 5$

③ Suppose $S = \{5, 8, 11, 14, \ldots\}$. Let $S_1 = 5$

$S_2 = 8 = 5 + 3 = S_1 + 3$

$S_3 = 11 = S_2 + 3$

$S_4 = 14 = S_3 + 3$

$\vdots$

$\boxed{S_n = S_{n-1} + 3}$

Explicit formula: (closed formula)

Finding a formula for the nth term for the sequence generated by a recurrence relation is called solving a recurrence relation. Such formula is called an explicit formula.

Important Methods to express a recurrence relation in an Explicit formula are:

① Back tracking method.

② Characteristic equation method.

* **Back tracking method:**

This method is suitable for linear non-homogeneous recurrence relation of the type

$$\boxed{a_n = r\,a_{n-1} + s}$$

where r & s are constants [s can be also a function of n]

① Find the explicit formula for the sequence defined by $a_n = a_{n-1} + 3$ for $n = 2, 3, \ldots\ldots$

where, $a_1 = 2$

replace $n \to n-1$

$a_{n-1} = a_{n-2} + 3$ — ①

replace $n-1 \to n-1$ again

$a_{n-2} = a_{n-3} + 3$ — ②

$a_{n-3} = a_{n-4} + 3$ — ③

$a_n = a_{n-1} + 3$

$= a_{n-2} + 3 + 3$

$= a_{n-3} + 3 + 3 + 3$

$= a_{n-4} + 3 + 3 + 3 + 3$

$\boxed{a_n = a_{n-k} + 3 \cdot k}$

$a_3 = a_2 + 3$

$a_2 = a_1 + 3$

$a_2 = 2 + 1 \times 3$

$a_2 = 2 + (2-1) \times 3$

$\Rightarrow a_n = a_{n-(n-1)} + 3(n-1)$

$a_n = a_1 + 3n - 3$

$a_n = 2 + 3n - 3$

$\boxed{a_n = 3n - 1}$

② solve $a_n = a_{n-1} + n$ where $a_1 = 4$.

$$a_n = a_{n-1} + n$$
$$a_{n-1} = a_{n-2} + (n-1)$$
$$a_{n-2} = a_{n-3} + (n-2)$$
$$a_{n-3} = a_{n-4} + (n-3)$$
$$\vdots$$
$$a_{n-(n-2)} = a_1 + (n-(n-2))$$
$$a_{n-(n-1)} = a_0 + [n-(n-1)]$$

$$a_n = a_{n-1} + n$$
$$= a_{n-2} + n + (n-1)$$
$$= a_{n-3} + n + (n-1) + (n-2)$$
$$= \vdots$$
$$= a_{n-(n-1)} + n + (n-1) + (n-2) + \cdots + [n-(n-2)]$$

$1+1$

$$a_n = a_1 + (n-2)n + n - \left(\frac{(n-2)(n-1)}{2}\right)$$

$$a_n = a_1 + n(n-1) - \frac{(n-1)(n-2)}{2}$$

$$= 4 + (n-1)\left[\frac{2n-n+2}{2}\right]$$

$$= 4 + \frac{(n-1)(n+2)}{2}$$

$$= 4 + \frac{n^2-n+2n-2}{2}$$

$$= \frac{8+n^2+n-2}{2} = \frac{n^2+n+6}{2}$$

③ $a_n = a_{n-1} + 2$, $a_1 = 2$.

$$a_{n-1} = a_{n-2} + 2$$
$$a_{n-2} = a_{n-3} + 2\cancel{+2}$$
$$a_{n-3} = a_{n-4} + 2\cancel{+2}\cancel{+2}$$
$$\vdots$$
$$a_{n-(n-2)} = a_{n-(n-1)} + 2\cancel{(n/2)}$$

$$a_n = a_{n-1} + 2$$
$$= a_{n-2} + 2 + 2$$
$$= a_{n-3} + 2 + 2 + 2$$
$$= \vdots$$
$$= a_1 + 2(n-1)$$
$$= \cancel{2} + 2n \cancel{-2}$$
$$= 2n$$

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* Find the explicit formula $a_n = 2a_{n-1} + 1$, $a_1 = 7$.

$a_n = 2a_{n-1} + 1$

$a_{n-1} = 2a_{n-2} + 1$

$a_{n-2} = 2a_{n-3} + 1$

$a_{n-3} = 2a_{n-4} + 1$

$\vdots$

$a_{n-(n-2)} = 2a_{n-(n-1)} + 1$

$a_n = 2a_{n-1} + 1$

$= 4a_{n-2} + 2 + 1$

$= 8a_{n-3} + 4 + 2 + 1$

$= 16a_{n-4} + 8 + 4 + 2 + 1.$

$\Rightarrow 2^k \cdot a_{n-k} + 2^{k-1} + 2^{k-2} + \dots + 2^0$

$\vdots$

$= 2^{n-1} \cdot a_{n-n+1} + 2^0 + 2^1 + 2^2 + \dots + 2^{n-2}$

$= 2^{n-1} \cdot a_1 + 1 + 2 + 4 + \dots + 2^{n-2}$

$= 7 \cdot 2^{n-1} + 1 \dfrac{(2^{n-1} - 1)}{2-1}$

$= 7 \cdot 2^{n-1} + 2^{n-1} - 1$

$= 2^n \left[\dfrac{7+1}{2}\right] - 1$

$= 2^n \dfrac{[8]}{2} - 1$

$= 2^{n-1} \cdot 8 - 1$

$k \to n-1$

$2^{n-1} a_1 + 2^{n-2} + 2^{n-1} + \dots + 2^0$

$= 7 \cdot 2^{n-1} + 1(2^{n-1} - 1)$

$= 2^{n-1}(7+1) - 1$

$= 2^{n-1}(8) - 1$

* $a_n = a_{n-1} + 3^n$, $a_1 = 4$

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* Find the explicit formula for $a_n = a_{n-1} + n^2$, $a_1 = 8$

$$a_n = a_{n-1} + n^2$$
$$a_{n-1} = a_{n-2} + (n-1)^2$$
$$a_{n-2} = a_{n-3} + (n-2)^2$$
$$\vdots$$
$$a_{n-(n-2)} = a_{n-(n-2)-1} + (n-(n-2))^2$$
$$= a_1 + (2)^2$$

$$a_n = a_{n-2} + n^2 + (n-1)^2$$
$$a_n = a_{n-3} + n^2 + (n-1)^2 + (n-2)^2$$
$$a_n = a_{n-4} + n^2 + (n-1)^2 + (n-2)^2 + (n-3)^2$$
$$a_n = \vdots$$
$$a_n = a_1 + n^2 + (n-1)^2 + (n-2)^2 + (n-3)^2 + \cdots + (n-(n-2))^2$$
$$a_n = 8 + n^2(n-2+1) + 1^2 + 2^2 + 3^2 + \cdots + (n-2)^2 - \cancel{2}n\frac{(n-2)(n-1)}{\cancel{2}}$$
$$a_n = 8 + n^2(n-1) + \frac{(n-2)(n-1)(2n-4+1)}{6} - (n-1)(n-2)n$$
$$a_n = 8 + (n-1)\left[n^2 + \frac{(n-2)(2n-3)}{6} - (n^2-2n)\right]$$
$$= 8 + (n-1)\left[\frac{(n-2)(2n-3)}{6} + n\right]$$
$$= 8 + (n-1)\left[\frac{2n^2 - 7n + 6 + 12n}{6}\right]$$
$$= 8 + (n-1)\left[\frac{2n^2 + 5n + 6}{6}\right]$$
$$= 8 + (n-1)\left[\frac{2n^2 + 5n + 6}{6}\right]$$
$$= 7 + \frac{6 + 2n^3 + 5n^2 + 6n - 2n^2 - 5n - 6}{6}$$
$$= 7 + \frac{2n^3 + 3n^2 + n}{6}$$
$$= 7 + \frac{n(2n^2 + 3n + 1)}{6} = 7 + \frac{n(2n+1)(n+1)}{6}$$

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* ① $a_n = n \cdot a_{n-1}$, $a_1 = 5$

$a_{n-1} = (n-1)\, a_{n-2}$

$a_{n-2} = (n-2)\, a_{n-3}$

$a_{n-3} = (n-3)\, a_{n-4}$

① $\Rightarrow n \cdot a_{n-1}$

$= n \cdot (a_{n-2}) \cdot (n-1)$

$= n \cdot (n-1)(n-2) \cdot a_{n-3}$

$a_n = n \cdot (n-1)(n-2)(n-3) \cdot a_{n-4}$.

$a_n = n \cdot (n-1)(n-2) \cdots (n-(k-1)) \cdot a_{n-k}$

replace $k \to n-1$

$a_n = n \cdot (n-1)(n-2) \cdots (2) \cdot a_1$

$= n! \cdot 5$.

Linear Homogeneous Recurrence Relation (characteristic equation):

A linear homogeneous Recurrence relation of degree k of constant coefficients is a recurrence relation of the form

$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$ — ①

where $r_1, r_2, r_3, \ldots, r_k$ are real numbers and $r_k \neq 0$

The solution to the above recurrence relation of the form $\boxed{a_n = x^n}$, where 'x' is a constant.

① is the characteristic equation.

$x^k - r_1 x^{k-1} - r_2 x^{k-2} \cdots - r_k = 0$ — ②

solving ②, we get 'k' roots,

then

$a_n = p_1 x_1^n + p_2 x_2^n + \cdots + p_k x_k^n$

where $p_1, p_2, \ldots, p_k$ depends on initial solution, is explicit formulas for sequence.

NOTE: Method / procedure.

* consider the recurrence relation, $a_n = r_1 a_{n-1} + r_2 a_{n-2}$

Let $a_1 = a$, $a_2 = b$ be the initial conditions.
The characteristic /auxiliary equation is

$$x^2 = r_1 x + r_2$$, where $a_n = x^2$, $a_{n-1} = x$, $a_{n-2} = \text{constant}$.

Solving the above equation,

$$x^2 - r_1 x - r_2 = 0$$

we get two roots, say α & β.

Case 1:

If α, β are real & distinct roots, then $\boxed{a_n = a\alpha^n + b\beta^n}$ satisfies the recurrence relation

Case 2:

If α, β are real & equal i.e., $[\alpha = \beta]$, then

$$\boxed{a_n = (a + nb)\alpha^n}$$

(*) Solve $a_n = 4a_{n-1} + 5a_{n-2}$; $a_1 = 2$, $a_2 = 6$

let $a_n = x^2$, $a_{n-1} = x$, $a_{n-2} = \text{constant}$

$$x^2 = 4x + 5$$

$$x^2 - 4x - 5 = 0$$

$$(x-5)(x+1) = 0$$

$$x = 5, -1$$

α, β are real & distinct.

So, $a_n = a \cdot 5^n + b \cdot (-1)^n$.

$a_1 = 2$. $\quad a_2 = 6$

$2 = 5a - b \quad 6 = 25a + b$

$30a = 8$

$a = \frac{4}{15}$, $b = \frac{-2}{3}$

$$a_n = \frac{4}{15} \cdot 5^n + (-1)^n \left(\frac{-2}{3}\right)$$

* solve $a_n = a_{n-1} + 2a_{n-2}$; $a_1 = 2$, $a_2 = 7$

$x^K = a_n$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

α, β are real & distinct

$a_n = 2^n \cdot a + b \cdot (-1)^n$.

$a_1 = 2$ $\qquad$ $a_2 = 7$

$2 = 2a - b$ $\qquad$ $7 = 4a + b$

$6a = 9$

$a = \frac{3}{2}$, $b = 1$

$a_n = \frac{3}{2} \cdot 2^n + (-1)^n (1)$.

$a_n = 3 \cdot 2^{n-1} + (-1)^n$

* solve $a_n = 4a_{n-1} + 5a_{n-2}$; $a_1 = 2$, $a_2 = 6$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$x = 5, -1$

α, β are real & distinct

$a_n = a \cdot 5^n + b(-1)^n$

* solve $a_n = 2a_{n-2}$, $a_1 = \sqrt{2}$, $a_2 = 6$

$x^2 - 2(1) = 0$

$(x(x) - 2) = 0$

$x = -\sqrt{2}$, $x = +\sqrt{2}$

α, β are real & distinct

$a_n = a \cdot (\sqrt{2})^n + b \cdot (-\sqrt{2})^n$

$a_n = b \cdot (-\sqrt{2})^n + (\sqrt{2})^n \cdot a$

$a_1 = \sqrt{2}$ $\qquad$ $a_2 = 6$

$\sqrt{2} = -\sqrt{2}b + \sqrt{2}a$. $\qquad$ $6 = (-\sqrt{2})^2 b + a(\sqrt{2})^2$

$\boxed{b = 1}$ $\quad$ $a - b = 1$ $\qquad$ $a + b = 3$

$\boxed{a = 2}$ $\quad$ $\boxed{b = 1}$