Uniform distribution

A continuous random variable X is said to be uniformly distributed over the interval $-\infty < a < b < \infty$, if its probability function is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & otherwise \end{cases}$$

Clearly,

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)dx = \int_a^b \frac{1}{b-a}\, dx = \frac{1}{b-a}[x]_a^b$

$$= \frac{1}{b-a}[b-a] = 1$$

Mean & Variance:

Mean $= \mu = E(X) = \int_{-\infty}^{\infty} x.f(x)dx$

$$\mu = \int_a^b x.\frac{1}{b-a}\, dx$$

$$\mu = \frac{1}{b-a}\left\{\frac{x^2}{2}\right\}_a^b = \frac{1}{2(b-a)}\left[b^2 - a^2\right]$$

$$\boxed{\mu = \frac{b+a}{2}}$$

Variance

$$\sigma^2 = E\left(X^2\right) - \left(E(X)\right)^2$$

$$E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 . f(x) dx$$

$$E\left(X^2\right) = \int_a^b x^2 \frac{1}{b-a} dx$$

$$E\left(X^2\right) = \frac{1}{b-a}\left\{\frac{x^3}{3}\right\}_a^b$$

$$E\left(X^2\right) = \frac{1}{3(b-a)}\left[b^3 - a^3\right]$$

$$E\left(X^2\right) = \frac{1}{3(b-a)}(b-a)\left[b^2 + ba + a^2\right]$$

$$\therefore \boxed{E(X^2) = \frac{a^2 + ab + b^2}{3}}$$

$$\therefore \sigma^2 = E(X^2) - \left(E(X)\right)^2$$

$$\therefore \sigma^2 = \frac{a^2 + ab + b^2}{3} - \left(\frac{b + a}{2}\right)^2$$

$$\therefore \sigma^2 = \frac{4(a^2 + ab + b^2) - 3(a + b)^2}{12}$$

$$\therefore \sigma^2 = \frac{(4a^2 + 4ab + 4b^2 - 3a^2 - 3b^2 - 6ab)}{12}$$

$$\therefore \sigma^2 = \frac{(a^2 - 2ab + b^2)}{12}$$

$$\boxed{\therefore \sigma^2 = \frac{(b - a)^2}{12}}$$

Cumulative distribution function of the exponential distribution:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$$

Case(1): When $X < a$

$f(x) = 0\,, \quad \therefore F(x) = 0$

Case(2): When $a < X < b$

$$F(x) = \int_{-\infty}^{a} f(x)dx + \int_{a}^{x} f(x)dx$$

$$F(x) = \int_{-\infty}^{a} 0\, dx + \int_{a}^{x} \frac{1}{b-a}\, dx$$

$$\Rightarrow F(x) = \frac{1}{b-a}\{x\}_{a}^{x}$$

$$\Rightarrow F(x) = \frac{x-a}{b-a}$$

Case(3): When $X > b$, $F(x) = 1$

Therefore, cumulative distribution function of uniform distribution is

$$\boxed{\mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{0}\ , & \boldsymbol{X} < a \\ \dfrac{\boldsymbol{x} - \mathbf{a}}{\boldsymbol{b} - \boldsymbol{a}}\ , & \boldsymbol{a} < X < b \\ \mathbf{1}\ , & \boldsymbol{X} > b \end{cases}}$$

Clearly,

$$\boxed{f(x) = \frac{d}{dx}\left(\mathbf{F}(\mathbf{x})\right)}$$

Problem:

1. A random variable X has a uniform distribution over $(-3, 3)$. Find k for which $P(X > k) = \frac{1}{3}$. Also evaluate $P(X < 2)$ and $P(|X - 2| < 2)$

Soln:

We know that PDF of Uniform distribution is

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & otherwise \end{cases}$$

Here, $(a, b) = (-3, 3)$

$$\therefore f(x) = \begin{cases} \dfrac{1}{6}, & -3 < x < 3 \\ 0, & otherwise \end{cases}$$

(i) Given that $P(X > k) = \frac{1}{3}$

$$1 - P(X \leq k) = \frac{1}{3}$$

$$1-\int_{-\infty}^{k} f(x)\,dx=\frac{1}{3}$$

$$1-\int_{-3}^{k} \frac{1}{6}dx=\frac{1}{3}$$

$$1-\frac{(k+3)}{6}=\frac{1}{3}$$

$$\Rightarrow \boxed{k=1}$$

(ii) $P(X<2)=\int_{-3}^{2}\frac{1}{6}dx=\frac{5}{6}$

(iii) $P(|X-2|<2)=P(-2<X-2<2)$

$$=P(-2+2<X-2+2<2+2)$$

$$P(|X-2|<2)=P(0<X<4)$$

$$=\int_{0}^{3}\frac{1}{6}dx=\frac{1}{2}$$