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Unit - III

Probability Vectors and Stochastic Matrices.

A vector $q = [q_1\ q_2 \cdots q_n]$ is called a probability vector if
i) $q_i \geq 0 \ \forall i$ ii) $\sum_{i=1}^{n} q_i = 1$

A square matrix $P = [p_{ij}]$ is called a Stochastic Matrix if each row of P is a probability vector.

If A is an $n \times n$ square matrix, then we can define $A^2, A^3 \ldots$ and all are $n \times n$ square matrices.

Also, if u is a vector with n components, then the product uA can be defined

$$[u]_{1\times n}\ [A]_{n\times n} = [uA]_{1\times n}.$$

If $u \neq 0$, then u is said to be a fixed point (vector) of A if
$$uA = u.$$

If k is a scalar s.t $k \neq 0$, then
$$(ku)A = k(uA) = ku$$

Result - 1

If u is a fixed point vector of a matrix A, then every multiple of ku of u is also a fixed pt. vector of A.

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Result - 2

If A and B are stochastic matrices, then their product AB is also stochastic.

In particular, all powers of A i.e. $A^2, A^3, \ldots A^n$ are also stochastic.

Defn :-
A stochastic matrix P is said to be REGULAR if all the entries of some powers of P (say P^m) are +ve.

Eg:- Consider,

$$A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \qquad B^2 \begin{bmatrix} 1 & 0 \\ 3/4 & 1/4 \end{bmatrix}$$

$\frac{3}{8} + \frac{1}{8} = \frac{1}{2}$

$$B^3 = B^2 B = \begin{bmatrix} 1 & 0 \\ 3/4 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 7/8 & 1/8 \end{bmatrix}.$$

A → Regular (at all power, entries are non zero).
B → Not Regular.

→ Questions.

1) Find fixed prob vector of $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

If u is the fixed prob vector of u - u it is 1×3

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Let $u = [x \quad y \quad z]$ & $x+y+z=1$.

$z = 1-x-y$

$$[x \quad y \quad 1-x-y] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x \quad y \quad 1-x-y]$$

$\frac{1}{2}(1-x-y) = x$

$x + \frac{1}{2}(1-x-y) = y$

$y = 1-x-y \Rightarrow 2y = 1-x$

$y = \frac{1-x}{2} = \frac{1}{2} - \frac{x}{2}$

$\frac{1}{2} - \frac{x}{2} - \frac{y}{2} = x$

$\frac{1}{2} - y = \frac{3}{2}x$

$x = 3 - \frac{2}{3}y$

$\frac{2}{3}\left(\frac{1}{2} - \frac{x}{2}\right)$

$\Rightarrow x = \frac{1}{5}, \; y = \frac{2}{5}$

$\frac{1}{3} - \frac{x}{3}$

$z = \frac{2}{5}$

$$\begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix} P = \begin{bmatrix} 1/5 & 2/5 & 2/5 \end{bmatrix}$$

Result - 3

If P is a regular stochastic matrix then,

i) P has a unique fixed pt. vector u and all components of u are +ve.

ii) The sequence $P, P^2, \ldots P^n$ of powers of P approaches the matrix V each of whose rows is u

2) Find the fixed prob. vector of $\begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$

$u = [x \quad y \quad z]$ $\quad x+y+z=1$

$z = 1-x-y$

$$[x \quad y \quad 1-x-y] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x \quad y \quad 1-x-y]$$

$\frac{y}{6} = x$

$y = 6x$

$\boxed{y = \frac{6}{10}}$

$\frac{y}{3} + \frac{1}{3} - \frac{x}{3} - \frac{y}{3} = 1-x-y$

$y = 1 - x + \frac{x}{3} - \frac{1}{3}$

$y = \frac{2}{3} - \frac{2x}{3}$

$6x + \frac{2x}{3} = \frac{2}{3}$

$z = 1 - \frac{1}{10} - \frac{6}{10}$

$x\left(\frac{6+2}{3}\right) = \frac{2}{3}$

$\boxed{z = \frac{3}{10}}$

$x\left(\frac{20}{3}\right) = \frac{2}{3}$

$x = \frac{1}{10}$

$$\begin{bmatrix} 1/10 & 6/10 & 3/10 \end{bmatrix} P = \begin{bmatrix} 1/10 & 6/10 & 3/10 \end{bmatrix}$$

$P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$ $\quad u = [x \quad y] \quad x+y=1$

$y = 1-x.$

~~$[x \quad y]$~~

$$[x \quad 1-x] \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = [x \quad 1-x]$$

$$\frac{1}{2} - \frac{x}{2} = x$$

$$\frac{1}{2} = x\left(\frac{3}{2}\right)$$

$$x = 1/3.$$

$$y = 1-x = \frac{2}{3}$$

→ Markov Process.

A markov process consists of a seq. of repeated trials of an experiment whose outcomes have the foll^n properties.

1. Each outcome belongs to a finite set $\{a_1, a_2, \ldots a_n\}$ called the STATE SPACE of the system.
2. The outcome of any trial depends, at most, on the outcome of the preceding trial and not on any earlier outcome.

Accordingly, with each pair of states (a_i, a_j), there is given a prob p_{ij} that a_j occurs right after a_i

$$M = \begin{array}{c|cccc} & a_1 & a_2 & \cdots & a_n \\ \hline a_1 & P_{11} & P_{12} & \cdots & P_{1n} \\ a_2 & P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & & & \\ a_n & P_{n1} & P_{n2} & \cdots & P_{nn} \end{array}$$

The matrix M is called the transi^n matrix of the Markov Process.

Eg:- i) Brownanian Motion.
ii) Random walk.

→ Result:-
The transi^n matrix of a Markov process is a stochastic Matrix

→ Questions:-

1) A man goes to work either by bus or drives his car. Suppose he never takes the bus two days in a row, if he drives to work, then the next day, he is twice as likely to take the bus as to drive. Write the trans^n matrix & find the fixed pt prob.

→ $M = \begin{bmatrix} 0 & 1 \\ 2/3 & 1/3 \end{bmatrix}$

If u is the fixed pt. prob prob vector of M, then $u = [x \quad y]$

$u = Mu$

$$[x \quad 1-x] \begin{bmatrix} 0 & 1 \\ 2/3 & 1/3 \end{bmatrix} = [x \quad 1-x]$$

$$\frac{2}{3}(1-x) = x \cdot \cancel{\frac{1}{3}(1-x)}$$

$$\cancel{\frac{1}{3}(1-x) = x}$$

$x = \frac{2}{5}$

$$\cancel{\frac{1}{3} - \frac{1}{3}x = x}$$

$1 - x = \frac{3}{5}$

$$\cancel{x\left(1+\frac{1}{3}\right) = \frac{1}{3}}$$

$$\cancel{x\left(\frac{4}{3}\right) = \frac{1}{3}} \qquad \cancel{x = \frac{1}{4}}$$

$$\begin{bmatrix} 2/3 & 3/5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 & 3/5 \end{bmatrix}$$

2. A, B & C are throwing ball at each other. A always throws it to B and B always throws it to C. But C is as likely to throw the ball to B as to A. Write the transⁿ matrix.

$$M = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

→ State Distributⁿ :-

Consider a Markov process with transitⁿ matrix M. The k^{th} state distributⁿ of the M-process in the prob vector

$$q_k = [q_{k_1} \quad q_{k_2} \quad \cdots \quad q_{k_n}]$$

where q_{k_i} is the prob that the state a_i occurs at the k^{th} trial of the Markov chain.

If the initial data distributⁿ q_0 (at $t=0$) is given, then we can find $q_1, q_2 \ldots q_n$ as follows :-

$$q_1 = q_0 M$$

$$q_2 = q_1 M = q_0 M^2$$

$$q_x = q_0 M^x$$

→ Questions :-

1) If B is the first person to throw the ball, what is the prob of each of them having ball after 3 throws?

$$q_0 = [0 \quad 1 \quad 0]$$

$$q_1 = q_0 M = [0 \quad 1 \quad 0] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [0 \quad 0 \quad 1]$$

$$q_2 = q_1 M = [0 \quad 0 \quad 1] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [1/2 \quad 1/2 \quad 0]$$

$$q_3 = q_2 M = [1/2 \quad 1/2 \quad 0] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [0 \quad 1/2 \quad 1/2]$$

→ In a long run, what is the prob of each of them having a ball

$uM = u \quad , \quad u = [x \quad y \quad z]$

$$[x \quad y \quad 1-y-x]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x \quad y \quad 1-y-x]$$

$$\frac{1}{2}(1-x-y) = x$$

$$3x + y = 1 \quad \text{—①} \times 3.$$

$$x + \frac{1}{2}(1-x-y) = y$$

$$2x + 1 - x - y = 2y$$

$$x - 3y = -1 \quad \text{—②}$$

$$9x + 3y = 3$$
$$x - 3y = -1$$
$$10x = 2$$
$$x = \frac{1}{5}$$

$$\frac{3}{5} + y = 1$$
$$y = 1 - \frac{3}{5} = \frac{2}{5} \qquad z = \frac{2}{5}$$

$$\therefore \quad u = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1/5 & 2/5 & 2/5 \end{bmatrix} \end{array} \quad \text{(longer run).}$$

→ Questions :

1) Write the transition matrix of a simple weather model in which a rainy day is 50% likely to be followed by another rainy day and a sunny day is 70% likely to be followed by a sunny day.

→ If it is raining today, what is the proba of a sunny day on the 3rd day? In the long run, what % of days are rainy?

$$M = \begin{array}{c|cc} & R & S \\ R & 1/2 & 1/2 \\ S & 1/10 & 9/10 \end{array} \qquad q_0 = \begin{array}{c} \begin{array}{cc} R & S \end{array} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array}$$

$$q_1 = q_0 M = [1 \quad 0]\begin{bmatrix} 1/2 & 1/2 \\ 1/10 & 9/10 \end{bmatrix} = [1/2 \quad 1/2]$$

$$q_2 = q_1 M = [1/2 \quad 1/2]\begin{bmatrix} 1/2 & 1/2 \\ 1/10 & 9/10 \end{bmatrix} = [3/10 \quad 7/10]$$

$\frac{1}{4} + \frac{1}{20}$

$\frac{6}{20} = \frac{3}{10}$

$\frac{1}{4} + \frac{9}{20}$

$\frac{14}{20} = \frac{7}{10}$

$\frac{3}{20} + \frac{7}{100}$

$\frac{22}{100}$ $\frac{11}{50}$

$$q_3 = q_2 M = [3/10 \quad 7/10]\begin{bmatrix} 1/2 & 1/2 \\ 1/10 & 9/10 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} R & S \end{array} \\ \begin{bmatrix} 0.22 & 0.78 \end{bmatrix} \end{array}$$

0.78 prob of sunny day on 3rd day.

$$u = [x \quad y]$$

$$uM = u$$

$$[x \quad 1-x]\begin{bmatrix} 1/2 & 1/2 \\ 1/10 & 9/10 \end{bmatrix} = [x \quad 1-x]$$

$$\frac{x}{2} + \frac{1}{10}(1-x) = x$$

$$5x + 1 - x = 10x$$
$$6x = 1$$
$$x = \frac{1}{6} \qquad y = \frac{5}{6}$$

$$u = \begin{array}{c} \begin{bmatrix} 1/6 & 5/6 \end{bmatrix} \\ \begin{array}{cc} R & S \end{array} \end{array} \quad \rightarrow \frac{100 \times 1}{6} = \frac{50}{3}\% \text{ days are rainy.}$$

2) A students study habits are as follows.

i) If he studies one night, he is 70% sure not to study next night.

ii) If he does not study one night, he 60% not to study next night.

In long run, how often does he study.

$$M = \begin{array}{c|cc} & S & N \\ \hline S & 0.3 & 0.7 \\ N & 0.4 & 0.6 \end{array}$$

$$u = [x \quad y]$$

$$uM = u$$

$$[x \quad 1-x] \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x \quad 1-x]$$

$$0.3x + 0.4 - 0.4x = x$$

$$0.4 = 1.1x$$

$$x = \frac{4}{11}$$

$$y = \frac{7}{11}$$

$$u = \left[\frac{4}{11} \quad \frac{7}{11}\right]$$

$\frac{4 \times 100}{11} = 36.36\%$ he studies.

3) A salesman's territory consists of 3 cities A, B & C. He never sells in the same city on 2 consecutive days. If he sells in city A, next day in B, But if he sells in B or C, the next day, he is thrice as likely to sell in city A than in other city. In the long run, how often does he sell in each of the cities.

$$M = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0.75 & 0 & 0.25 \\ C & 0.75 & 0.25 & 0 \end{array}$$

$$u = [x \quad y \quad z] \qquad x+y+z=1$$

$$uM = u$$

$$[x \quad y \quad 1-x-y] \begin{bmatrix} 0 & 1 & 0 \\ 0.75 & 0 & 0.25 \\ 0.75 & 0.25 & 0 \end{bmatrix} = [x \quad y \quad 1-x-y]$$

$$\cancel{0.75y} + 0.75 - 0.75x - \cancel{0.75y} = x$$

$$1.75x = 0.75$$

$$x = \frac{0.75}{1.75} = \frac{3}{7}$$

$$x + 0.25 - 0.25x - 0.25y = y$$

$$\cancel{0.75x + 1.25y = 0.25}$$

$$1.25y = 0.75x + 0.25$$

$$1.25y = 0.75 \times \frac{3}{7} + 0.25$$

$$y = \frac{0.3214 + 0.25}{1.25} = \frac{0.5714}{1.25}$$

$y\,0.25 = 1-x-y$

$1.25y = 1 - \frac{3}{7}$

$1.25y = \frac{4}{7}$

$y = \frac{4}{1.25 \times 7}$

$\boxed{y = \frac{3.2}{7}}$

$$\boxed{\overset{A}{x = \frac{3}{7}}, \quad \overset{B}{y = \frac{3.2}{7}}, \quad \overset{C}{z = \frac{0.8}{7}}}$$

probability of selling in cities A, B & C

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4) In a survey, the folⁿ for moving from one brand to another is given:-

From brand \ To	1	2	3
1	0.8	0.1	0.1
2	0.03	0.95	0.02
3	0.2	0.05	0.75

If the current market share are 45%, 25%, 30%

i) What is the expected market share after 2 years
ii) What is the longterm predⁿ of market share

i)
$$M = \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 0.8 & 0.1 & 0.1 \\ 2 & 0.03 & 0.95 & 0.02 \\ 3 & 0.2 & 0.05 & 0.75 \end{array}$$

$$q_0 = \begin{bmatrix} 0.45 & 0.25 & 0.3 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0.45 & 0.25 & 0.3 \end{bmatrix} \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.03 & 0.95 & 0.02 \\ 0.2 & 0.05 & 0.75 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0.4275 & 0.2976 & 0.275 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 0.4275 & 0.2976 & 0.275 \end{bmatrix} \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.03 & 0.95 & 0.02 \\ 0.2 & 0.05 & 0.75 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 0.4059 & 0.3391 & 0.2549 \end{bmatrix}$$

∴ After 2 years, share market = 40.59%, 33.91% & 25.49%

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ii) $u = Mu$ for longer run.

$$u = \begin{bmatrix} x & y & z \end{bmatrix}$$

$$\begin{bmatrix} x & y & 1-x-y \end{bmatrix} \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.03 & 0.95 & 0.02 \\ 0.2 & 0.05 & 0.75 \end{bmatrix} = \begin{bmatrix} x & y & 1-x-y \end{bmatrix}$$

$x = 0.2371$, $y = 0.6185$, $z = 0.1444$.

∴ In longer run, market share will be 23.71%, 61.85% & 14.44%.

5) A man is at an integer point b/w 0 & 5 on X-axis. He takes one step to right with prob of 3/4 or to the left with prob 1/4, unless he is at 0 or 5. At 0, he always takes a step to the right to reach point 1 and at 5, he takes a step to the left to reach point 4. Find

i) The transⁿ matrix
ii) If he stands at 2, what is the prob that he is at 1 after 3 steps.
iii) In the long run, how often is he at 1.

5

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1/4 & 0 & 3/4 & 0 & 0 & 0 \\ 2 & 0 & 1/4 & 0 & 3/4 & 0 & 0 \\ 3 & 0 & 0 & 1/4 & 0 & 3/4 & 0 \\ 4 & 0 & 0 & 0 & 1/4 & 0 & 3/4 \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

$q_0 = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]$

$q_1 = q_0 M = [0 \quad 1/4 \quad 0 \quad 3/4 \quad 0 \quad 0]$

$q_2 = q_1 M = [1/16 \quad 0 \quad 6/16 \quad 0 \quad 9/16 \quad 0]$

$q_3 = q_2 M = [0 \quad 10/64 \quad 0 \quad 27/64 \quad 0 \quad 27/64]$

$\frac{1}{16} + \frac{6}{16 \times 4}$

$\frac{10}{64}$

prob that he at 1 after 3 steps $= \frac{10}{64}$

$18 + 9$

In long sum, $uM = u$.

$u = [x \quad y \quad z \quad p \quad q \quad r]$

$[x \quad y \quad z \quad p \quad q \quad 1-x-z-p-q] \, [M] = u$

$\frac{y}{4} = x \qquad \boxed{y = 4x}$

$x + \frac{z}{4} = y$

$\frac{z}{4} = 3x \qquad \boxed{z = 12x}$

$y \times \frac{3}{4} + \frac{p}{4} = z$

$3x + \frac{p}{4} = 12x$

$\frac{p}{4} = 9x \Rightarrow \boxed{p = 36x}$

$z \times \frac{3}{4} + \frac{q}{4} = p$

$9x + \frac{q}{4} = 36x$

$\frac{q}{4} = 27x \qquad \boxed{q = 108x}$

$p \times \frac{3}{4} + 1 - x - z - p - q = q$

$27x + 1 - x - 12x - 36x - 108x = 108x$

$1 - 130x = 108x$

$1 = 238x$

$x = \frac{1}{238}$

$u = [1/238 \quad 1/59.5 \quad 1/19.83 \quad 1/6.61 \quad 1/2.2 \quad 1-(all)]$

In long sum

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6) A player has Rs. 300. At each play of a game. He loses Rs.100 with prob. 3/4 but makes Rs. 200 with prob 1/4. He stops playing if he has lost Rs. 300 or won at least 300. Find the prob that there are at least 4 plays to game.

$$\begin{array}{c|ccccccc} & 0 & 100 & 200 & 300 & 400 & 500 & 600 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 100 & 3/4 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 200 & 0 & 3/4 & 0 & 0 & 1/4 & 0 & 0 \\ 300 & 0 & 0 & 3/4 & 0 & 0 & 1/4 & 0 \\ 400 & 0 & 0 & 0 & 3/4 & 0 & 0 & 1/4 \\ 500 & 0 & 0 & 0 & 0 & 3/4 & 0 & 1/4 \\ 600 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$q_0 = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]$

$q_1 = q_0 M = [0 \quad 0 \quad 3/4 \quad 0 \quad 0 \quad 1/4 \quad 0]$

$q_2 = q_1 M = [0 \quad 9/16 \quad 0 \quad 0 \quad 6/16 \quad 0 \quad 1/16]$

$q_3 = q_2 M = [27/64 \quad 0 \quad 0 \quad 27/64 \quad 0 \quad 0 \quad 10/64]$

$q_4 = q_3 M = [108/256 \quad 0 \quad 81/256 \quad 0 \quad 0 \quad 27/256 \quad 40/256]$

prob that game has atleast 4 play

$= \frac{81}{256} + \frac{27}{256}$

$= \frac{108}{256} =$

$= \frac{27}{64}$

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Markov Process Defns.

1. Absorbing State :

A state a_i of the Markov chain is said to be an absorbing state if the system remains in the state a_i once it enters there.

A state a_i is absorbing if $p_{ii} = 1$ i.e. there is a '1' on the principle diagonal.

Note :

A stothastic matrix is not regular if it has '1' on the principal diagonal.

2. Transient state :

A state is said to be transient if it is possible to leave the state entering it once.

3. Periodic state :

A state is said to be periodic if it can be returned to in multiples of some positive integer greater than 1. The integer is known as the period of the state.

4) Recurring state :

A state is said to be recurring if the process returns to the state after a finite no. of steps.

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7) Suppose an urn A contains 2 white marbles and urn B contains 4 red marbles. At each step of the process, a marble is selected at random from each urn and the 2 marbles are interchanged.

i) Find the trans^n matrix M

ii) Find the prob. that there are 2 red in A after 3 steps.

iii) In the long run, what is the prob that A has 2 red marbles.

States :-

a_0		a_1		a_2	
2W	4R	1R 1W	1W 3R	2R	2W 2R
A	B				

$$M = \begin{array}{c|ccc} & a_0 & a_1 & a_2 \\ \hline a_0 & 0 & 1 & 0 \\ a_1 & 1/8 & 4/8 & 3/8 \\ a_2 & 0 & 1/2 & 1/2 \end{array}$$

$a_1 \to a_0$.

$M_{21} = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}$

$M_{22} = \frac{1}{2}\cdot\frac{1}{8} + \frac{1}{2}\cdot\frac{3}{8} = \frac{4}{8}$

$q_0 = [1 \quad 0 \quad 0]$

$$q_1 = q_0 M = [1 \quad 0 \quad 0]\begin{bmatrix} 0 & 1 & 0 \\ 1/8 & 4/8 & 3/8 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$q_1 = [0 \quad 1 \quad 0]$

$$q_2 = q_1 M = [0 \quad 1 \quad 0]\begin{bmatrix} 0 & 1 & 0 \\ 1/8 & 4/8 & 3/8 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$q_2 = [1/8 \quad 4/8 \quad 3/8]$.

$\frac{8}{8\times8} + \frac{16}{64} + \frac{3\times4}{16\times4} = \frac{12+8+16}{64} = \frac{31}{64}$

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$$q_3 = q_2 M = [1/8 \quad 4/8 \quad 3/8]\begin{bmatrix} 0 & 1 & 0 \\ 1/8 & 4/8 & 3/8 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$q_3 = [4/64 \quad 36/64 \quad 24/64]$

$q_2 = \frac{24}{64} = \frac{3}{8} \Rightarrow$ prob of A containing two red.

In long run, $uM = u$.

$u = [x \quad y \quad z]$

$$[x \quad y \quad 1-x-y]\begin{bmatrix} 0 & 1 & 0 \\ 1/8 & 4/8 & 3/8 \\ 0 & 1/2 & 1/2 \end{bmatrix} = [x \quad y \quad 1-x-y]$$

$\frac{y}{8} = x$

$y = 8x$

$x + \cancel{\frac{4y}{8}}_{2} + \frac{1}{2} - \frac{x}{2} - \cancel{\frac{y}{2}} = y$

$\frac{x}{2} + \frac{1}{2} = 8x$

$x + 1 = 16x$

$15x = 1$

$x = \frac{1}{15}$

$y = \frac{8}{15}$

$z = \frac{6}{15}$

$u = [1/15 \quad 8/15 \quad 6/15]$

prob that A has 2 red.

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→ Queuing Theory

Mathematical study of waiting lines or queues. ERLANG in 1909.

- Basic Characteristics of Queueing Theory

1) Input or Arrival Pattern:-
The arrival rate (interval between 2 successive arrivals) in a random variable. It is expressed by the probability distribution of the number of arrivals per units of time. This is usually considered to be a Poisson variate. In this case, ~~the~~ time b/w two consecutive intervals follows exponential distributⁿ.

2) Service Mechanism.
This represents the no. of people served in one time period. The service pattern should indicate the no. of servers and whether they are arranged in series or in parallel.

3) Queue Discipline.

*) FIFO (FCFS) : First In First Out

*) FILO (FCLS) : First In Last Out.

*) SIRO : Selectⁿ in random order.

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*) PIR : Priority in Selection.

4) System Capacity :
The max. no. of customers in the queueing System. It can be finite or Infinite.

5) Transient and Steady states.
T.S → depends on time
S.S → doesn't depend

6) KENDALL'S Notation
The form is (a/b/c):(d/e), where
a → Inter arrival distributⁿ
b → service time " (usually exponential)
c → No. of channels or servers.
d → system capacity
e → Queue displine.

a & b are usually considered Markovian and denoted by M.

→ Four imp. queueing systems are

i) (M/M/1) : (∞/FIFO)
ii) (M/M/s) : (∞/FIFO)
iii) (M/M/1) : (K/FIFO)
iv) (M/M/s) : (K/FIFO)

* Line length : No. of customers in the queueing System.

(*) $P(N \geq n) = \rho^n$

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* Queue length : Line length – No. of customers being served.

I) Model :- $(M/M/1) : (\infty/FIFO)$

Let $N(t)$ be the no. of customers in the system at time t. and $P_n(t)$ be the probability that there are n customers in the system at time t where $n \geq 1$

i.e. $P_n(t) = P[N(t) = n]$

- For Ist Model, the foll^n assumptions are made :-

1) The mean arrival rate λ is constant.
2) The mean service rate μ is constand
3) $\lambda < \mu$ i.e. $\rho = \frac{\lambda}{\mu} < 1$ (traffic intensity)

- Characteristics of Model I.

i) $P_n = P(N=n) = \rho^n(1-\rho)$ for $n = 0, 1, 2 \dots$

ii) Avg ~~or~~ Expected no. of customers in the system (L_s) :

$$L_s = E(N) = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}$$

iii) Avg or Exp no. of customers in the queue (L_q):

$$L_q = E(N-1) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$

(*) $P_0 = 1-\rho$ (prob that there are zero customers).

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iv) Avg or Exp waiting time of the customer in the (W_s) :

$$W_s = \frac{1}{\mu-\lambda}$$

v) Average or expected waiting time of a customer in the queue (W_q):

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$$

→ Little's Formula:

i) $L_s = \lambda W_s$ ii) $L_q = \lambda W_q$

iii) $W_s = W_q + \frac{1}{\mu}$ iv) $L_s = L_q + \frac{\lambda}{\mu}$

v) Prob that there are atleast n customers in the system

$$P(N \geq n) = \rho^n = \left(\frac{\lambda}{\mu}\right)^n$$

→ Questions :-

1) If $\lambda = 6$ & $\mu = 8$ in $M/M/1$ Q- system, find
i) the prob of at least 10 customers in the sy.
ii) " " " " " " 5 " " "
iii) avg no. of customers in the system.

→ iii) $L_s = \frac{\lambda}{\mu-\lambda} = \frac{6}{8-6} = 3$

ii) $P(N \geq 10) = \left(\frac{\lambda}{\mu}\right)^{10} = \left(\frac{6}{8}\right)^{10} = 0.0563.$

iii) $P(N \geq 5) = \left(\frac{6}{8}\right)^{5} = 0.2373.$

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(*) $P(T_s > t)$ = Prob that the waiting time of a cust. exceeds t.
$= e^{-(\mu-\lambda)t}$

2) For MM1 model, what is the prob. that a customer has to wait for more than 15 mins. in a queue system if $\lambda = 6$ per hour and $\mu = 10$ per hour.

$\mu = 10 \quad \lambda = 6 \quad t = 0.25$ hours.

$P(T_s > 0.25) = e^{-(10-6)0.25}$

$= 0.3678$

3) Suppose that customers arrive at a Poisson rate of one every 12 minutes and the service time is exponential at a rate of one service every 8 minutes. What is
i) the average number of customers in the system.
ii) " " time " " spends in the system.

$\mu = 1/8 \quad \lambda = 1/12$

i) $L_s = \frac{\lambda}{\mu-\lambda} = \frac{1/12}{\frac{1}{8} - 1/12} = \frac{0.0833}{0.0417}$

$L_s \approx 2$ customers.

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ii) $W_s = \frac{1}{\mu-\lambda} = \frac{1}{0.0417}$

$W_s = 23.9808$ mins.

4). A supermarket has a single cashier. During peak hours, customers arrive at a rate of 20 per hour and get serviced at a rate of 24 per hour. Calculate
i) the prob. that the cashier is idle
ii) the average no. of customers in the system.
iii) " " " " in the queue.
iv) " " time that a cust. spends in system
v) " " " " " " queue.

$\mu = 24 \quad \lambda = 20$

ii) $L_s = \frac{\lambda}{\mu-\lambda} = \frac{1/20}{\frac{1}{24} - 1/20} = \frac{20}{24-20} = 5$ customers

iii) $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{20 \times 20}{24(4)} = \frac{25}{6} \approx 4$ customers

iii) $W_s = \frac{1}{\mu-\lambda} = \frac{1}{24-20} = 0.25$ hours. (15 mins)

iv) $W_q = \frac{L_q}{\lambda} = \frac{4}{20} = 0.2$ hours. (12 mins)

i) $P_0 = 1-\rho = 1-\frac{\lambda}{\mu} = 1-\frac{20}{24}$

$P_0 = 0.167$

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5) Customers arrive at a sales counter manned by a single person, with a mean rate of 20 per hour. The time required to serve a cust. has exp dist. with a mean of 100 secs. Find the average waiting time of
i) In the system ii) In the queue

$\lambda = 20$ $\mu = \frac{1}{100} \times 60 \times 60 = 36$

$W_s = \frac{1}{\mu - \lambda} = \frac{1}{36-20} = \frac{1}{16} = 0.0625$ hrs

$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{20}{36(16)} = 0.03472$ hrs.

6) Customers arrive at a repair shop foll^n Poisson dist at a rate of one per 10 mins and the service rate is exponential with mean 8 minutes (i) Find the average no of customers in the shop. (L_s)
ii) ~~Avg~~ waiting time (W_s)
iii) avg time a customer spends in the queue (W_q)

$\lambda = 1$ per 10 min = $\frac{1}{10}$ per min.
$\mu = \frac{1}{8}$

$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1/10}{1/8 - 1/10}$

$L_s = \frac{1/10}{\frac{2}{80_{40}}} = 4$ cust.

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$W_s = \frac{1}{\mu-\lambda} = \frac{1}{\frac{1}{8}-\frac{1}{10}}$; $1 - P_0$

$= \frac{1}{2/80}$

$= 40$ mins

$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1/10}{\frac{1}{8}\left(\frac{1}{8}-\frac{1}{10}\right)}$

$= \frac{\frac{1}{10}}{\frac{1}{8_4}\left(\frac{2}{80}\right)}$

$= 32$ mins.

7. Patients arrive at a hospital at the rate of one per hour. At present, only one emergency can be handled at a time. Patients spend an avg of 20 mins receiving emergency care. Find
i) avg. time a patient spends in the queue W_s
ii) " length of ~~patient~~ queue formed. L_q
iii) " time a patient spends in the sys.
iv) prob. that a patient arriving at the hospital has to wait.

$\frac{60}{20}$

$\lambda = 1$ $\mu = 3 \left(\frac{60}{20}\right)$

$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1}{3(3-1)} = \frac{1}{6}$ hours.

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$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{1}{3(3-1)} = \frac{1}{6}$$

$$W_s = \frac{1}{\mu-\lambda} = 0.5 \text{ hours.}$$

iv) $1 - P_0 = \rho = \frac{\lambda}{\mu} = \frac{1}{3}$

v) $P(N \geq 5) = \rho^5$

$$= \left(\frac{1}{3}\right)^5 = \frac{1}{243}$$

8. Patients arrive at doctors clinic at mean interval of 20 mins and spend a mean of 15 mins in consultation. The doctor wishes to have enough seats in the waiting room so that not more than 2% of patients will have to stand.

$\lambda = 1/20$ $\quad \mu = 1/15$

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{1/400}{1/15\left(\frac{1}{15}-\frac{1}{20}\right)}$$

$$= \frac{1/400}{\frac{1}{15} \times \frac{5}{300}}$$

$$= \frac{900}{400} = \frac{9}{4}$$

$$= 2.25$$

$\frac{4.5}{100} = 0.045$

2.295

$\frac{2.25}{100} \times 2$

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Prob that there are n people wait $= \rho^n$.

$\rho^n \leq 0.02$ $\quad n = ?$

$(0.75)^n \leq 0.02$

$10^n = 2$

$x = \log_{10} 2$

$$n = \frac{\log 0.02}{\log 0.75}$$

$n = 13.59 \approx 14$

9. An organization has one reservation clerk on duty at a given time. Assuming customers arrive at a rate of 8 per hour and the clerk can serve a customer in 6 mins, then find.
i) prob of system is busy (ρ)
ii) avg time a cust spends in the system
iii) length of queue
iv) avg. no of cust in the systems.

$\lambda = 8$ $\quad \mu = \frac{60}{6} = 10.$

i) $\rho = \frac{\lambda}{\mu} = \frac{8}{10} = 0.8$

ii) $W_s = \frac{1}{\mu-\lambda} = \frac{1}{10-8} = 0.5 \text{ hr}$

iii) $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{8^2}{10(10-8)} = 3.2$

iv) $L_s = \lambda W_s = 8 \times \frac{1}{2} = 4$

(4) $\mu s > \lambda$.

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→ Model II : $(M/M/s):(\infty/FIFO)$.
Multiple servers with infinite capacity.

→ Characteristics of the Model.

1. $$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!\left(1-\frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right]^{-1}$$

2. $$L_q = \frac{1}{s \cdot s!}\frac{(\lambda/\mu)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0$$

3. $$L_s = L_q + \frac{\lambda}{\mu}$$

4. $$W_q = \frac{1}{\lambda} L_q$$

5. $$W_s = \frac{1}{\lambda} L_s$$

6. Prob. that a customer has to wait.

$$P(N \geq s) = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{1}{\left(1-\frac{\lambda}{s\mu}\right)} P_0$$

→ Questions :-

1) A travel centre has 3 service counters to book tickets. The customers arrive at a Poisson dist. with an average arrival of 100 ppl in a 10-hr service day. The average service time is 15 ~~Cans~~ mins. Find
i) Expected no. of custs in the queue
ii) " " " " " " system.

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iii) Expected time a cust spends in queue
iv) " " " " " " system
v) Prob that a cust must wait.

$\lambda = 10$

$\mu = \frac{60}{15} = 4$ $\quad s = 3$.

i) $$P_0 = \left[\sum_{0}^{2}\left\{\frac{1}{n!}\left(\frac{10}{4}\right)^n + \frac{1}{3!\left(1-\frac{10}{12}\right)}\left(\frac{10}{4}\right)^3\right\}\right]^{-1}$$

$$= \left[1 + \frac{5}{2} + \frac{1}{2}\left(\frac{5}{2}\right)^2 + \frac{12}{6(2)}\left(\frac{1000}{64}\right)\right]^{-1}$$

$$= \left[1 + \frac{2}{5} + \frac{2}{25} + \frac{1000}{64}\right]^{-1}$$

$= 0.045$

ii) $$L_q = \frac{1}{3\times 6}\frac{(10/4)^4}{\left(1-\left(\frac{10}{12}\right)\right)^2} P_0$$

$$= \frac{1}{18}\frac{39.0625}{(1/6)^2} \times 0.045$$

$$= \frac{1}{18}\frac{39.0625}{0.0277} \times 0.045$$

$= 3.5255$

ii) $L_s = L_q + \frac{\lambda}{\mu}$

$= 3.5255 + \frac{10/4}{}$ $\left(\frac{2.5}{}\right)$

$= 6.025$

iii) $W_s = \frac{1}{\lambda} L_s$

$= \frac{1}{10} 6.025$

$= 0.6025$ hrs

iv) $W_q = \frac{1}{\lambda} L_q$

$= \frac{1}{10} \times 3.5255$

$= 0.3525$ hrs.

vi) $P(N \geq 3) = \frac{1}{3!} \left(\frac{10}{4}\right)^3 \left[\frac{1}{1 - \frac{10}{12}}\right] (0.045)$

$= 0.7016$

2). $S = 2$ $\lambda = 12$ $\mu = \frac{60}{6} = 10$

$P_0 = \left\{ \sum_{0}^{1} \frac{1}{n!} \left(\frac{12}{10}\right)^n + \frac{1}{2\left(1 - \frac{12}{20}\right)} \left(\frac{12}{10}\right)^2 \right\}^{-1}$

$= \left[\underset{n=0}{1} + \underset{n=1}{1.2} + \frac{20}{16}(1.44) \right]^{-1}$

$P_0 = 0.25$

$L_q = \frac{1}{2 \times 2} \cdot \frac{\left(12/10\right)^3}{\left(1 - \frac{12}{20}\right)^2} \times 0.25$

$= \frac{1}{4} \times \frac{1.728}{0.16} \times 0.25$

$= 0.675$

$L_s = L_q + \frac{\lambda}{\mu}$

$= 0.675 + \frac{12}{10}$

$= 1.875$

$W_s = \frac{1}{12} \times 1.875$

$= 0.1562$ hrs

$W_q = \frac{1}{12} \times 0.675$

$= 0.0562$ hrs.

$P(N \geq 2) = \frac{1}{2} \left(\frac{12}{10}\right)^2 \frac{1}{\left(1 - \frac{12}{20}\right)} \times 0.25$

$= \frac{1}{2} \times 1.44 \times \frac{20}{8} \times 0.25$

$= 0.45$

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3) $s = 3$, $\lambda = 12$ $\mu = \frac{60}{10} = 6$

$$P_0 = \left[\sum_{0}^{2} \frac{1}{n!}\left(\frac{12}{6}\right)^n + \frac{1}{6\left(1-\frac{12}{18}\right)}\left(\frac{12}{6}\right)^3\right]^{-1}$$

$$= \left[1 + 2 + \frac{4}{2} + \left(\frac{18}{6\times 6}\right) 8\right]^{-1}$$

$$= [5 + 4]^{-1}$$

$$= 0.111$$

→ Model-III : $(M/M/1):(K/FIFO)$

This is the case when the system can accomodate only a finite no. of arrivals. If a customer arrives and the queue is full, then the cust has to leave.

→ Characteristics :-

$\rho = \lambda/\mu$

1. $P_0 = \frac{1-\rho}{1-\rho^{k+1}}$ if $\rho \neq 1$ / $\frac{1}{k+1}$ if $\rho = 1$

2. $P_n = \frac{(1-\rho)\rho^n}{1-\rho^{k+1}}$ if $\rho \neq 1$ / $\frac{1}{k+1}$ if $\rho = 1$

(*) Here we need not have $\lambda < \mu$
The steady state probability P_n exist even if $\lambda \geq \mu$.

3. $L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}$ if $\rho \neq 1$ / $\frac{k}{2}$ if $\rho = 1$

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4. Effective arrival rate $(\lambda') = \mu(1-P_0)$

5) $L_q = L_s - \frac{\lambda'}{\mu}$ (6) $W_q = \frac{1}{\lambda'} L_q$ (7) $W_s = \frac{1}{\lambda'} L_s$.

→ Questions

1) Patients arrive at a clinic in Poisson dist at a rate of 30 per hour. The waiting room can accomodate at most 9 patients. Examint time is 20 per hour. Find the
i) Prob that a patient doesn't have to wait.
ii) Effective arrival rate
iii) Avg no. of patients in the queue
iv) W_q.

→ i) $P_0 = \frac{1-\rho}{1-\rho^{k+1}}$ $\rho = \frac{\lambda}{\mu} = \frac{30}{20} = 1.5$

$\frac{1-1.5}{1-(1.5)^{10}} = \frac{0.5}{56.665}$ $k = 9+1 = 10$

$= 0.00584$

ii) $\lambda' = \mu(1-P_0) = 20(1-0.00584)$
$\lambda' = 19.8832$

iii) $L_s = \frac{1.5}{1-1.5} - \frac{11(1.5)^{11}}{1-(1.5)^{11}}$

$= -3 - \frac{951.473}{-85.495} = -3 + 11.1286$

$= 8.1286$

iii) $L_q = L_s - \frac{\lambda'}{\mu}$

$= 8.1286 - \frac{19.8832}{20} = 7.1349$

iv) $W_q = \frac{1}{\lambda'} L_q$

$= \frac{7.1344}{19.8834} = 0.3588$ hrs.

2). $\lambda = 3$ /hr

$\mu = 2$ /hr

$k = 5$

i) $\rho = \frac{\lambda}{\mu} = \frac{3}{2} = 1.5$

$P_0 = \frac{1-\rho}{1-\rho^{k+1}} = \frac{1-1.5}{1-(1.5)^6} = \frac{0.5}{10.3906}$

$= 0.0481$

ii) $L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}$

$= \frac{1.5}{1-(1.5)} - \frac{6(1.5)^6}{1-(1.5)^6}$

$= -3 + \frac{68.3437}{10.3906}$

$= 3.5774$

3) max 5 ppl (4 waiting & 1 getting haircut),

$\lambda = 8$

$\mu = \frac{60}{6} = 10$

$k = 5$

$\rho = \frac{8}{10} = 0.8$

$P_0 = \frac{1-\rho}{1-\rho^{k+1}} = \frac{1-0.8}{1-(0.8)^6} = \frac{0.2}{0.7379}$

$P_0 = 0.2710$

% of time barber is idle = 27.1%

ii) Effective arrival rate $\lambda' = \mu(1-P_0)$

$= 10(1-0.271)$

$= 7.29$

iii) $L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}$

$= \frac{0.8}{1-0.8} - \frac{6(0.8)^6}{1-(0.8)^6}$

$= 4 - \frac{1.5728}{0.7399}$

$= 4 - 2.1257$

$= 1.8743$

$L_q = L_s - \frac{\lambda'}{\mu} = 1.8743 - 0.729$

$L_q = 1.1453$.

iv) $W_q = \frac{1}{\lambda'} L_q = \frac{1}{7.29} \times 1.1453$

$= 0.157$ hrs

v) ~~Prob~~ Fraction of potential outcome who are turned away.

$\Rightarrow$ ~~$P(n \geq 6) = 1 - \{P_6 + P_5 + P_4 + P_3 + P_2 + P_1 + P_0\}$~~

~~$= 1 - \frac{1-0.8}{1-(0.8)^{k+1}} (0.8^6 + 0.8^5 + 0.8^4 + \dots)$~~

$P(n \geq 6) = P_6 + P_7 + P_8 + \dots$

$= \frac{1-\rho}{1-\rho^6} (\rho^6 + \rho^7 + \rho^8 + \dots)$

$= \frac{1-\rho}{1-\rho^6} \times \rho^6 (1 + \rho + \rho^2 + \dots)$

$= \frac{\cancel{1-\rho}}{1-\rho^6} \times \rho^6 \left(\frac{1}{\cancel{1-\rho}}\right)$ ← sum of g.p.

$= \frac{\rho^6}{1-\rho^6} = \frac{(0.8)^6}{1-(0.8)^6} = \frac{0.2621}{0.7379}$

$= 0.35519$