Probability

1. Probability

The aim of probability theory is to provide mathematical solution to the future event of which the outcome is uncertain and about which we want to make a prediction.

2. Random Experiment

An experiment is called a random experiment if, when repeated under the same conditions, it is such that the outcome can not be predicted with certainty but all possible outcomes can be determined prior to the performance of the experiment.

Each performance of the random experiment is called a trial.

3. Sample space

The collection of all possible outcomes of a random experiment is called the sample space and is denoted by S.

Example(1):

The sample space for the random experiment of tossing two coins is

S={HH, HT, TH, TT}   O(S) = 4

Example(2):

When a die is thrown, sample space is

S={1,2,3,4,5,6} O(S)=6

4. Event

Any subset A of a sample space S is called an event.

In example(1), A={HT, TH} is the event of getting exactly one head.

In example(2), B={1,3,5} is the event of getting an odd number.

5. Exhaustive Events

A set of events is said to be Exhaustive, if it includes all the possible outcomes of a trial, out of which one or more events is to occur.

6. Equally likely events

If in a trial, all the events have same chances of happening, then events are said to be equally likely.

7. Mutually exclusive events

If in a trial, happening of an event prevents the happening of all other events then such events are said to be mutually exclusive.

Example: When a coin is tossed, either head or tail can appear but not both. Thus appearances of head or tail are mutually exclusive.

8. Independent events

If happening of an event does not affect the chance of happening of other events, then such events are said to be independent events.

Example: When a pair of coins is tossed, appearance of head or tail on one coin does not affect the chance of appearance of head or tail on the other coin.

9. Classical Definition of Probability of an event

If there are n exhaustive , mutually exclusive and equally likely cases of which m are favorable to an event A, then the probability of happening of an event A denoted by P(A) is defined as

$$\boxed{P(A) = \frac{\left(\begin{array}{c}\text{Number of cases}\\ \text{favorable to happening}\\ \text{of event A}\end{array}\right)}{\text{Total number of ways}}}$$

$$\boxed{P(A) = \frac{m}{n} = \frac{O(A)}{O(S)}}$$

NOTE:

If m out of n cases are favorable to event A then there are (n-m) cases unfavorable to the event A. This set of unfavorable events is known as complement of A denoted by $\bar{A}$.

Clearly,

$$P(A) + P(\bar{A}) = \frac{m}{n} + \frac{n - m}{n} = \frac{n}{n} = 1$$

Thus,

$$\boxed{P(\bar{A}) = 1 - P(A) \ \text{ or } \ P(A) + P(\bar{A}) = 1}$$

10. <u>Sure Event</u>

If $\mathbf{P(A) = 1}$, then $\mathbf{A}$ is said to be sure event.

11. <u>Impossible event</u>

If $\mathbf{P(A) = 0}$, then $\mathbf{A}$ is said to be impossible event.

*** <u>Elementary Problems</u> ***

12. Find the probability of getting a number greater than 3 when an unbiased dice is rolled.

Soln: S={1,2,3,4,5,6} A={4,5,6}

$$\mathbf{P(A) = \frac{m}{n} = \frac{O(A)}{O(S)} = \frac{3}{6} = 0.5}$$

13. In a single throw of two unbiased dice, find the probability of obtaining a total of 8.

Soln:

S= {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

A={(2,6), (3,5), (4,4), (5,3), (6,2)}

$$\mathbf{P(A) = \frac{O(A)}{O(S)} = \frac{5}{36}}$$

14 A bag contains 8 white and 6 black balls, out of which 2 balls are drawn at random. Find the probability that both balls are black.

Soln:

$$m = \begin{pmatrix} \text{number of ways} \\ \text{of selecting 2 black} \\ \text{balls out of 6} \end{pmatrix} = 6_{C_2} = 15$$

$$n = \begin{pmatrix} \text{Total number of} \\ \text{ways of selecting} \\ \text{2 balls out of 14} \end{pmatrix} = 14_{C_2} = 91$$

$$\mathrm{P(A)} = \frac{\mathrm{m}}{\mathrm{n}} = \frac{15}{91}$$

15 A leap year is selected at random, find the probability that it contains 53 Sundays.

Soln:

Leap year: 366 days

52 Sundays + 2 days

These two days can be

i) SUN and MON
ii) MON and TUE
iii) TUE and WED
iv) WED and THU
v) THU and FRI
vi) FRI and SAT
vii) SAT and SUN

Total number of ways in which two days can be selected is n = 7.

Only those cases are favorable which contain a Sunday. Therefore m = 2.

$$P(A) = \frac{m}{n} = \frac{2}{7}$$

15. Axiomatic definition of probability

Let S be a sample space, then

[P1] : for any event A, $\mathbf{P(A) \geq 0}$

[P2]: $\mathbf{P(S) = 1}$

[P3]: for any two disjoint events A and B,

$$\mathbf{P(A \cup B) = P(A) + P(B)}$$

In general, for any infinite sequence of mutually exclusive events $\mathbf{A_1, A_2, \ldots \ldots ..}$

$$\mathbf{P(A_1 \cup A_2 \cup A_3 \cup \ldots \ldots)}$$
$$\mathbf{= P(A_1) + P(A_2) + P(A_3) + \cdots}$$

16. For any event $\mathbf{A}$, $\mathbf{0 \leq P(A) \leq 1}$

From [P1], $\mathbf{P(A) \geq 0}$.
Hence we have to only show that
$\mathbf{P(A) \leq 1}$.

Since $\mathbf{S = A \cup \bar{A}}$ where $\mathbf{A}$ and $\mathbf{\bar{A}}$ are mutually exclusive, we have

$$P(S) = P(A \cup \bar{A})$$
$$1 = P(A) + P(\bar{A}) \quad \{\ [P2]\ \&\ [P3]\ \}$$

$$\boxed{P(A) = 1 - P(\bar{A})}$$

As $P(\bar{A}) \geq 0$, it follows that $P(A) \leq 1$.

Thus, $\boxed{0 \leq P(A) \leq 1}$

17. NOTE:

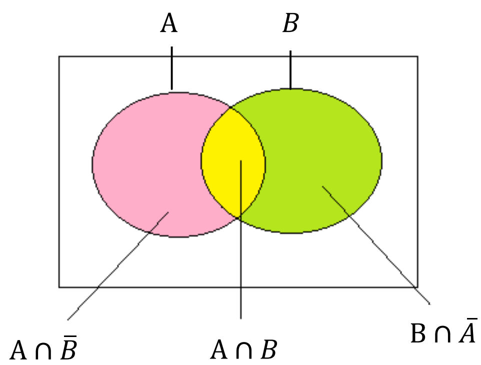

Fig.1

a) $\mathbf{A} = (\mathbf{A} \cap \overline{\mathbf{B}}) \cup (\mathbf{A} \cap \mathbf{B})$ union of disjoint sets. Then by property [P3]

$$\boxed{\mathbf{P}(\mathbf{A}) = \mathbf{P}(\mathbf{A} \cap \overline{\mathbf{B}}) + \mathbf{P}(\mathbf{A} \cap \mathbf{B})}$$

b) $\mathbf{B} = (\mathbf{B} \cap \overline{\mathbf{A}}) \cup (\mathbf{A} \cap \mathbf{B})$ union of disjoint sets. Then by property [P3]

$$\boxed{\mathbf{P}(\mathbf{B}) = \mathbf{P}(\mathbf{B} \cap \overline{\mathbf{A}}) + \mathbf{P}(\mathbf{A} \cap \mathbf{B})}$$

c) $\boxed{\mathbf{A} \cup \mathbf{B} = (\mathbf{A} \cap \overline{\mathbf{B}}) \cup \mathbf{B}}$

d) $\boxed{\mathbf{A} \cup \mathbf{B} = \boldsymbol{A} \cup (\mathbf{B} \cap \overline{\mathbf{A}})}$

18. Addition law of probability

If $\mathbf{A}$ and $\mathbf{B}$ are any two events which are not mutually exclusive then

$$\boxed{\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}}$$

Proof:

Wkt $\mathbf{A \cup B = (A \cap \overline{B}) \cup B}$--------(1)

{Refer Fig.1 or Result 17(c)}

Applying property [P3] to Eqn.(1) which is union of disjoint sets, we get

$$\mathbf{P(A \cup B) = \mathit{P}(A \cap \overline{B}) + P(B)} \text{ -------(2)}$$

From result 17(a), we have

$$\mathbf{\mathit{P}(A \cap \overline{B}) = P(A) - P(A \cap B)} \text{ ---------(3)}$$

Using (2) in (1), we get

$$\mathbf{P(A \cup B) = [P(A) - P(A \cap B)] + P(B)}$$

$$\therefore \boxed{\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}}$$

19. $P(A \cup B \cup C) = P(A) + P(B) + P(C) + P(A \cap B \cap C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$

Proof:

Let $\mathbf{D = B \cup C}$

Then,

$$\begin{aligned}
\mathbf{P(A \cup B \cup C)} &= \mathbf{P(A \cup D)} \\
&= \mathbf{P(A) + P(D) - P(A \cap D)} \\
&= \mathbf{P(A) + P(B \cup C) - P\big(A \cap (B \cup C)\big)} \\
&= \mathbf{P(A) + P(B) + P(C) - P(B \cap C)} \\
&\quad \mathbf{- P\big((A \cap B) \cup (A \cap C)\big)}
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{P(A) + P(B) + P(C) - P(B \cap C)} \\
&\quad \mathbf{- \{P(A \cap B) + P(A \cap C)} \\
&\quad \mathbf{- P(A \cap B \cap C)\}}
\end{aligned}$$

$$\begin{aligned}
\therefore\ \mathbf{P(A \cup B \cup C)} \\
&= \mathbf{P(A) + P(B) + P(C) - P(B \cap C)} \\
&\quad \mathbf{- P(A \cap B) - P(A \cap C)} \\
&\quad \mathbf{+ P(A \cap B \cap C)}
\end{aligned}$$

20. Multiplication Rule for independent events

If A and B are independent events then

$$\mathbf{P(A \cap B) = P(A).P(B)}$$

Proof:

Let the total number of ways for happening of A and B be N_1, N_2 respectively, out of which favourable number of ways for happening of A and B be $m_1 \& m_2$ respectively.

According to fundamental theorem of multiplication, *if an event can happen in m ways and after it has happened in one of the ways, the other independent event can happen in n ways, then the two events together can happen in mn ways.*

Now, the total number of ways is $N_1.N_2$ out of which $m_1.m_2$ are favourable to the happening of both the events.

$$\therefore \mathbf{P(A \cap B)} = \frac{m_1.m_2}{N_1.N_2} = \frac{m_1}{N_1}\frac{m_2}{N_2}$$

$$\boxed{\mathbf{P(A \cap B) = P(A).P(B)}}$$

21. Probability of happening of at least one event

If $P(A_1), P(A_2), \ldots\ldots\ldots.. P(A_n)$ are probabilities of happening of n independent events, then the probability p of happening of at least one of these events is given by

$$\boxed{p = 1 - P(\overline{A_1}).P(\overline{A_2}).P(\overline{A_3})\ldots P(\overline{A_n})}$$

22. Note:

Events	Set theoretical expression
Only A happens	$A \cap \bar{B} \cap \bar{C}$
All the 3 events A,B,C happens	$A \cap B \cap C$
None of the three events happens	$\bar{A} \cap \bar{B} \cap \bar{C}$ or $\overline{A \cup B \cup C}$
A and B occurs but not C	$A \cap B \cap \bar{C}$

At least one of the three events happens	$A \cup B \cup C$
Exactly one event happens $(A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$	

23. The probability that 3 students A, B, C solve a problem are 1/2, 1/3, 1/4 respectively. If the problem is simultaneously assigned to all of them what is the probability that the problem is solved?

Soln:

Given $P(A) = \frac{1}{2}$; $P(B) = \frac{1}{3}$; $P(C) = \frac{1}{4}$

$$\therefore P(\bar{A}) = \frac{1}{2} ; P(\bar{B}) = \frac{2}{3}; P(\bar{C}) = \frac{3}{4}$$

The probability that the problem cannot be solved by any one of them is

$$P(\bar{A}).P(\bar{B}).P(\bar{C}) = \frac{1}{2}\frac{2}{3}\frac{3}{4} = \frac{1}{4}$$

Hence, the probability that the problem is solved by at least one of them is

$$p = 1 - P(\bar{A}).P(\bar{B}).P(\bar{C}) = 1 - \frac{1}{4} = \frac{3}{4}$$

24. A book shelf contains 20 books of which 12 are on Numerical analysis and 8 are on Differential equations. If three books are taken out at random, find the probability that all the three are on the same subject.

Soln:

Out of 20 books, 3 can be drawn in 20_{C_3} ways.

Out of 12 numerical analysis books, 3 can be drawn in 12_{C_3} ways.

Let $\mathrm{P(A)}$ be the probability of drawing 3 Numerical analysis books, then

$$\mathrm{P(A)} = \frac{12_{C_3}}{20_{C_3}} = \frac{11}{57}$$

Similarly,
if $\mathbf{P(B)}$ is probability of drawing 3 Differential equation books, then

$$\mathbf{P(B) = \frac{8_{C_3}}{20_{C_3}} = \frac{14}{285}}$$

Hence, probability of drawing three books on the same subject is $\mathbf{P(A \cup B)}$.

Now, using addition rule for mutually exclusive events, we get

$$\mathbf{P(A \cup B) = P(A) + P(B) = \frac{11}{57} + \frac{14}{285}}$$
$$\mathbf{= 0.24}$$

25. The probability that India wins a cricket test match against West Indies is known to be 2/5. If India and West Indies play 3 test matches what is the probability that
(i) India will lose all the three matches.
(ii) India will win at least one test match.
(iii) India will win all the tests.
(iv) India will win at most one match.

Soln: Let A, B and C denotes the events that India wins the first, second, third test match against West Indies respectively. Then,

$$\mathbf{P(A) = P(B) = P(C) = \frac{2}{5}}$$

$$\therefore \mathbf{P(\bar{A}) = P(\bar{B}) = P(\bar{C}) = \frac{3}{5}}$$

We note that the three events are independent.

Let p_1 be the probability of India loosing all the three matches, then

$$\boldsymbol{p_1} = \mathbf{P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\bar{A}).P(\bar{B}).P(\bar{C})}$$

$$\therefore \boxed{\boldsymbol{p_1} = \mathbf{\frac{3}{5}.\frac{3}{5}.\frac{3}{5} = \frac{27}{125}}}$$

Let p_2 be the probability of India winning at least one match , then

$$p_2 = 1 - \mathrm{P}(\text{loosing all the 3 matches})$$

$$\boxed{p_2 = 1 - \mathrm{P}(\bar{\mathrm{A}} \cap \bar{\mathrm{B}} \cap \bar{\mathrm{C}}) = 1 - \frac{27}{125} = \frac{98}{125}}$$

Let p_3 be the probability of India winning all the three matches, then

$$p_3 = \mathrm{P}(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C}) = \mathrm{P}(\mathrm{A}).\mathrm{P}(\mathrm{B}).\mathrm{P}(\mathrm{C})$$

$$\therefore \boxed{p_3 = \frac{2}{5}.\frac{2}{5}.\frac{2}{5} = \frac{8}{125}}$$

Let p_4 be the probability of India winning at most one matches, then

$$p_4 = \mathrm{P}\binom{\text{losing}}{\text{all}} + \mathrm{P}\binom{\text{winning exactly}}{\text{one match}}$$

$$\therefore p_4 = \mathrm{P}(\bar{\mathrm{A}} \cap \bar{\mathrm{B}} \cap \bar{\mathrm{C}}) + \mathrm{P}(A \cap \bar{B} \cap \bar{C})$$
$$+\mathrm{P}(\bar{A} \cap B \cap \bar{C}) + \mathrm{P}(\bar{A} \cap \bar{B} \cap C)$$

$$\therefore \boxed{p_4 = \frac{81}{125}}$$

26. A husband and wife appear in an interview for 2 vacancies in the same post. The probability of husband's selection is 1/7 and that of wife's selection is 1/5. What is the probability that
(i) Both of these will be selected
(ii) Only one of them will be selected
(iii) none of them will be selected?

Soln:
Let $\mathbf{P(A)}$ be the probability of husband's selection, then

$$\boxed{\mathbf{P(A) = \frac{1}{7}}} \;\&\; \boxed{\mathbf{P(\bar{A}) = 1 - P(A) = \frac{6}{7}}}$$

Let $\mathbf{P(B)}$ be the probability that wife is selected, then

$$\boxed{\mathbf{P(B) = \frac{1}{5}}} \;\&\; \boxed{\mathbf{P(\bar{B}) = 1 - P(B) = \frac{4}{5}}}$$

i) Let p_1 be the probability for the selection of both husband and wife, then

$$p_1 = \mathbf{P(A \cap B) = P(A).P(B)}$$

{ ∵ A & B are independent events }

$$\boxed{\therefore\ p_1 = \frac{1}{7}.\frac{1}{5} = 0.02857}$$

ii) Let p_2 be the probability that only one of them is selected, then

$$p_2 = \mathbf{P(A)P(\bar{B}) + P(B)P(\bar{A})}$$

$$\boxed{\therefore\ p_2 = \frac{1}{7}.\frac{4}{5} + \frac{1}{5}.\frac{6}{7} = \frac{10}{35} = 0.2857}$$

iii) Let p_3 be the probability that none of them is selected, then

$$p_3 = \mathbf{P(\bar{A})P(\bar{B})} = \frac{6}{7}.\frac{4}{5} = 0.6857$$

27. A and B throw alternatively a pair of dice. The one who throws 9 first wins. If A begins then show that the chances of their winning are 9:8.

Soln: The total number of exhaustive ways in which two dice can fall=6x6=36.

The number of ways of getting a sum total of 9 by tossing the two dice are (4+5), (5+4), (3+6), (6+3), i.e, 4.

Probability of getting 9 $=\frac{4}{36}=\frac{1}{9}$

Probability of not getting 9 $= 1-\frac{1}{9}=\frac{8}{9}$

If A begins, then he can win in 1st or 3rd or 5th ....throws. Let A_i be the event of A winning in ith throw.

Then, $P(A_1)=\frac{1}{9}$

$$\mathrm{P}(\mathrm{A}_3)=\mathrm{P}(\bar{\mathrm{A}})\mathrm{P}(\bar{\mathrm{B}})\mathrm{P}(\mathrm{A})=\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{1}{9}\ \text{ and so on}$$

Let p_1 be the probability that A wins, then

$$p_1=\mathrm{P}(\mathrm{A}_1)+\mathrm{P}(\mathrm{A}_3)+\mathrm{P}(\mathrm{A}_5)+\cdots\ldots\ldots$$

$$p_1=\frac{1}{9}+\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{1}{9}+\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{1}{9}+\cdots\ldots\ldots$$

$$p_1=\frac{1}{9}\left[1+\left(\frac{8}{9}\right)^2+\left(\frac{8}{9}\right)^4+\cdots\ldots\ldots.\right]$$

$$p_1=\frac{1}{9}\left\{\begin{matrix}\text{Sum upto infinity of Geometric}\\ \text{series with a}=1\text{ and r}=\left(\frac{8}{9}\right)^2\end{matrix}\right\}$$

$$\therefore p_1=\frac{1}{9}\left(\frac{\mathrm{a}}{1-\mathrm{r}}\right)$$

$$\boxed{\therefore p_1=\frac{1}{9}\left(\frac{1}{1-\left(\frac{8}{9}\right)^2}\right)=\frac{9}{17}}$$

$\therefore$ The chance of B winning is

$$\boxed{p_2 = 1 - \frac{9}{17} = \frac{8}{17}}$$

Hence, the ratio of the chances of winning of A and B are

$$\boxed{p_1 : p_2 = \frac{9}{17} : \frac{8}{17} = 9 : 8}$$