Exponential distribution

A continuous random variable X is said to follow Exponential distribution if its probability function is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, 0 < x < \infty \\ 0, \quad\quad otherwise \end{cases}$$

where α is an arbitrary positive real constant.

Clearly,

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} \alpha e^{-\alpha x}\, dx = \alpha \left[\frac{e^{-\alpha x}}{-\alpha}\right]_0^{\infty}$

$$= -\left[e^{-\infty} - e^{0}\right] = -[0 - 1] = 1$$

Mean & Variance:

Mean$= \mu = E(X) = \int_{-\infty}^{\infty} x.f(x)dx$

$$\mu = \int_{0}^{\infty} x.\alpha e^{-\alpha x}\, dx$$

$$\mu = \alpha\left\{x\left(\frac{e^{-\alpha x}}{-\alpha}\right) - (1)\left(\frac{e^{-\alpha x}}{\alpha^2}\right)\right\}_0^{\infty}$$

$$\mu = \alpha\left\{[0-0] - \left(\frac{1}{\alpha^2}\right)\left(e^{-\infty} - e^0\right)\right\}$$

$$\mu = \alpha\left\{0 - \left(\frac{1}{\alpha^2}\right)(0-1)\right\} = \frac{1}{\alpha}$$

$$\boxed{\mu = \frac{1}{\alpha}}$$

Variance

$$\sigma^2 = E(X^2) - \left(E(X)\right)^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2.f(x)dx$$

$$E(X^2) = \int_0^{\infty} x^2\,\alpha e^{-\alpha x}\,dx$$

$$E(X^2) = \alpha\left\{x^2\left(\frac{e^{-\alpha x}}{-\alpha}\right) - (2x)\left(\frac{e^{-\alpha x}}{\alpha^2}\right) + (2)\left(\frac{e^{-\alpha x}}{-\alpha^3}\right)\right\}_0^{\infty}$$

$$E(X^2) = \alpha\left\{(0-0) - (0-0) + \left(\frac{2}{-\alpha^3}\right)\left(e^{-\infty} - e^0\right)\right\}$$

$$E(X^2) = \alpha\left\{\left(\frac{2}{-\alpha^3}\right)(0-1)\right\} = \frac{2}{\alpha^2}$$

$$\therefore \sigma^2 = E(X^2) - \left(E(X)\right)^2 = \frac{2}{\alpha^2} - \left(\frac{1}{\alpha}\right)^2$$

$$\boxed{\therefore \sigma^2 = \frac{1}{\alpha^2}}$$

Cumulative distribution function of the exponential distribution:

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le \mathrm{x}) = \int_{-\infty}^{x} f(x)dx$$

Case(1): When $\mathrm{X} < 0$

$f(x) = 0$, $\quad \therefore \mathrm{F(x)} = 0$

Case(2): When $\mathrm{X} > 0$

$$\mathrm{F(x)} = \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx$$

$$\mathrm{F(x)} = \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} \alpha e^{-\alpha x}\,dx$$

$$\Rightarrow \mathrm{F(x)} = \alpha \left\{\frac{e^{-\alpha x}}{-\alpha}\right\}_0^x$$

$$\Rightarrow \mathrm{F(x)} = -\left\{e^{-\alpha x} - e^{0}\right\} = -e^{-\alpha x}$$

Therefore, cumulative distribution function of exponential distribution is

$$\boxed{\mathrm{F(x)} = \begin{cases} -e^{-\alpha x}, & 0 < x < \infty \\ 0, & otherwise \end{cases}}$$

Clearly,

$$\boxed{f(x) = \frac{d}{dx}\left(\mathrm{F(x)}\right)}$$

Problems:

1. If x is an exponential variate with mean 3 find (i) P (x > 1) (ii) P (x < 3)

Soln: Given that

$$\mu = 3 = \frac{1}{\alpha} \qquad \therefore \boxed{\alpha = \frac{1}{3}}$$

We know that PDF of Exponential distribution is

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0 < x < \infty \\ 0, & otherwise \end{cases}$$

$$\therefore f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & 0 < x < \infty \\ 0, & otherwise \end{cases}$$

(i) $P(X > 1) = \int_1^\infty f(x)\, dx = \int_1^\infty \frac{1}{3} e^{-x/3}\, dx$

$$P(X > 1) = \frac{1}{3}\left(\frac{e^{-x/3}}{-1/3}\right)_1^\infty$$

$$P(X > 1) = -\left(e^{-\infty} - e^{-1/3}\right) = 0.7165$$

(ii) $P(X < 3) = 1 - P(X \geq 3)$

$$= 1 - \int_{3}^{\infty} \frac{1}{3} e^{-x/3}\, dx$$

$$= 1 - \frac{1}{3}\left(\frac{e^{-x/3}}{-1/3}\right)_{3}^{\infty}$$

$$= 1 + \left(e^{-\infty} - e^{-1}\right) = 1 + (0 - 0.36788)$$

$$P(X < 3) = 0.6321$$

2. If X is an exponential variate with mean 5, evaluate
   (i) $P(0 < X < 1)$ (ii) $P(-\infty < X < 10)$
   (iii) $P(X \le 0 \text{ or } X \ge 1)$

3. The length of telephone conversation in a booth has been an exponential distribution and found on average to be 5 minutes. Find the probability that a random call made from this booth
   (i) ends less than 5 minutes
   (ii) between 5 and 10 minutes.

4. The mileage which car owner get with a certain kind of radial tire is a random variable having an exponential distribution with mean 40,000km. Find the probabilities that one of these tires will last
   (i) At least 20,000km
   (ii) At most 30,000km.

5. The increase in sales per day in a shop is exponentially distributed with mean Rs.4000. The sales tax is to be levied at the rate of 18%. What is the probability that the sales tax will exceed Rs.810 per day?

Soln:

Let us define two random variables

X - Sales

Y – Sales tax

Given $\mu = 4000 = \frac{1}{\alpha}$ $\quad \therefore \boxed{\alpha = \frac{1}{4000}}$

This is related to X

We know that PDF of Exponential distribution is

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, 0 < x < \infty \\ 0, \quad\quad otherwise \end{cases}$$

Sales tax Y	Sales X
18	100
810	A=?

$$A = \frac{100 \times 810}{18} = 4500$$

$$P\binom{\text{Sales tax}}{\text{exceeds Rs. 810}} = P(Y \geq 810)$$

$$= P(X \geq 4\ 500)$$

$$= \int_{4\ 500}^{\infty} f(x)\, dx$$

$$= \int_{4\ 500}^{\infty} \alpha e^{-\alpha x}\, dx$$

$$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha}\right]_{4\ 500}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-1.125}\right]$$

$$P\binom{\text{Sales tax}}{\text{exceeds Rs. 810}} = 0.324\ 65$$

6. The sales per day in a shop are exponentially distributed with average sale amounting to Rs.100 and net profit is 8%. Find the probability that the net profit exceeds Rs.30 on two consecutive days.