UNIT II :

CONTINUOUS PROBABILITY DISTRIBUTION

Exponential Distribution :

$P(\alpha, x) = \begin{cases} \alpha e^{-\alpha x} & , 0 \le x \le \infty, \alpha \in R^+ \\ 0 & \text{elsewhere} \end{cases}$

α is the parameter of the distribution.

$\mu = \frac{1}{\alpha}$; SD, $\sigma = \frac{1}{\alpha}$

$P(a<x<b) = \int_a^b P(\alpha, x)\, dx$

Q. The duration of a telephone conversation follows ED with mean 3 mins. Find prob. that convo may last i) more than 1 min, ii) less than 3 min, iii) btw 2 & 4 min

$\mu = 3 \Rightarrow \frac{1}{\alpha} = 3 \Rightarrow \alpha = \frac{1}{3}$

$P(\alpha, x) = \begin{cases} \frac{1}{3} e^{-\frac{1}{3}x} & x \ge 0 \\ 0 & \text{elsewhere} \end{cases}$

i) $P(\alpha, x>1) = \int_1^\infty \frac{1}{3} e^{-\frac{1}{3}x} dx = \frac{1}{3}\left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right]_1^\infty = -1(e^{-\infty} - e^{-1/3}) = \frac{1}{e^{1/3}} = \cancel{1.3956}\ 0.7165$

ii) $P(\alpha, x<3) = \int_0^3 \frac{1}{3} e^{-\frac{1}{3}x} dx = \frac{1}{3}\left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right]_0^3 = -1(e^{-1} - e^{-0}) = 0.6321$

iii) $P(\alpha, 2<x<4) = \int_2^4 \frac{1}{3} e^{-\frac{1}{3}x} dx = \frac{1}{3}\left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right]_2^4 = -1(e^{-\frac{4}{3}} - e^{-\frac{2}{3}}) = 0.2498$

Q. The mileage that a car owner gets with a certain tyre is an exponential variate with mean 40,000 km. Find prob. that these tyres will last i) at least 20,000 km, ii) at most 30,000 km

$\mu = 40000 \Rightarrow \alpha = \frac{1}{\mu} = \frac{1}{40,000} = 2.5 \times 10^{-5}$

i) $P(\alpha, x \ge 20,000) = \int_{20,000}^\infty \frac{1}{40,000} e^{-\frac{1}{40,000}x} dx = \frac{1}{40,000}\left[\frac{e^{-\frac{x}{40,000}}}{-1/40,000}\right]_{20,000}^\infty = -1(e^{-\infty} - e^{-\frac{1}{2}})$

$= 0.6065$

ii) $P(\alpha, x \le 30,000) = \int_0^{30,000} \frac{1}{40,000} e^{-\frac{1}{40,000}x} dx = \frac{1}{40,000}\left[\frac{e^{-\frac{x}{40,000}}}{-1/40,000}\right]_0^{30,000} = -1(e^{-3/4} - e^{-0})$

$= 0.5276$

Q. The avg. daily turnout of a store is ₹10,000 & the avg. profit is 8%. If turn out is exponential

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variate, find the prob. that the net profit will exceed ₹1000 on 2 consecutive days.

$\mu = 800 \Rightarrow \alpha = \frac{1}{\mu} = \frac{1}{800}$

$P(\alpha, x) = \int_{1000}^{\infty} e^{-\alpha x} dx = \alpha \frac{1}{800}\left[\frac{e^{-\frac{x}{800}}}{-1/800}\right]_{1000}^{\infty} = -1(e^{-\infty} - e^{-\frac{10}{8}}) = 0.2865$

$P(x > 1000 \text{ on 2 cons. days}) = (0.2865)^2 = 0.0821$ [Multiplication law of Probability]

Q. The life of a bulb is advertised to have a mean of 200 hrs. If it has ED, find the prob. that the bulb will last for i) less than 200 hrs, ii) between 100 & 300 hrs, iii) ~~between~~ more than 250 hrs

$\mu = 200 \Rightarrow \alpha = \frac{1}{\mu} = \frac{1}{200}$

i) $P(\alpha, x < 200) = \int_{0}^{200} e^{-\frac{1}{200}x} dx = \frac{1}{200}\left[\frac{e^{-\frac{x}{200}}}{-1/200}\right]_0^{200} = -1(e^{-1} - e^{-0}) = 1$

ii) $P(\alpha, 100 < x < 300) = \frac{1}{200}\int_{100}^{300} e^{-\frac{1}{200}x} dx = \frac{1}{200}\left[\frac{e^{-\frac{x}{200}}}{-1/200}\right]_{100}^{300} = -1(e^{-3/2} - e^{-1/2})$
$= 0.03834$

iii) $P(\alpha, x > 250) = \frac{1}{200}\int_{250}^{\infty} e^{-\frac{1}{200}x} du = \frac{1}{200}\left[\frac{e^{-\frac{x}{200}}}{-1/200}\right]_{250}^{\infty} = -1(e^{-\infty} - e^{-5/4})$
$= 0.2865$

Q. The length of time for a person to be served at a cafe is an exp variate with $\mu = 4$ minutes. Find the prob. that a person is served in less than 3 mins on at least 4 of the next 6 days.

$\mu = 4$ mins, $\alpha = \frac{1}{\mu} = \frac{1}{4}$

$P(x=4) + P(x=5) + P(x=6)$

Table
row : 1st decimal place, column : 2nd decimal place

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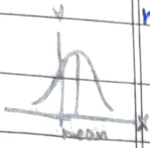
NORMAL DISTRIBUTION

Any quantity whose variation depends on random causes is distributed according to normal law. It is defined by as



$$N(\mu, \sigma, x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

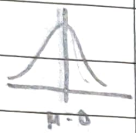
This is also called Gaussian distribution

NOTE: i) $N(\mu, \sigma, x) \geq 0$

ii) $\int_{-\infty}^{\infty} N(\mu, \sigma, x)\, dx = 1$

STANDARD NORMAL DISTRIBUTION (draw figure)

The normal distribution for which $\mu = 0$ & $\sigma = 1$ is called the SND.



In this case, $z = \frac{x-\mu}{\sigma}$ (standard normal variant)

$P(a \leq x \leq b) = P(z_1 \leq z \leq z_2) = \cancel{P(z \geq z_1) + P(z \leq z_2)} = P(0 \leq z \leq z_1) + P(0 \leq z \leq z_2)$

Q. If X is normally distributed with mean 5 and variance 4, find the prob. that X lies between 2 and 6.

$\mu = 5, \sigma^2 = 4, \sigma = 2$

$P(2 \leq x \leq 6)$: If $x = 2$, $z = \frac{2-5}{2} = \frac{-3}{2} = -1.5$

If $x = 6$, $z = \frac{6-5}{2} = \frac{1}{2} = 0.5$

$P(2 \leq x \leq 6) = P(-1.5 \leq z \leq 0.5) = P(0 \leq z \leq 1.5) + P(0 \leq z \leq 0.5)$

$= 0.4332 + 0.1915 = 0.6247$

Q. In a city, the no. of power breakdowns in a week is a normal variate with $\mu = 11.6$ & $\sigma = 3.3$. Find the probability that there will be atleast 8 breakdowns in a week

$\mu = 11.6, \sigma = 3.3$

If $x = 8$, $\frac{8-11.6}{3.3} = \frac{-3.6}{3.3} = -1.09$

$P(x \geq 8) = P(z > -1.09) = P(0 < z < 1.09) + P(0 < z < \infty)$

$= 0.3621 + 0.5 = 0.8621$

Q. The avg. length of metal bars produced by a company is 68.22 cm with variance 10.8 cm. How many bars in a consignment of 1000 are expected to be over 72 cm?

$\mu = 68.22 cm$, $\sigma^2 = 10.8 cm$, $\sigma = 3.2863$ $\Rightarrow z = \frac{72-68.22}{3.2863} = 1.1502$

$P(x>72) = P(0<z<\infty) - P(0<z<1.1502)$

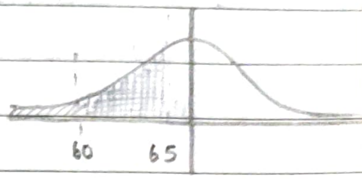
$= 0.5 - 0.3749 \doteq 0.1251$ $\Rightarrow$ For 1000, $1000 \times 0.1251 = 125$ bars over 72 cm

Q. In a normal distribution, 5% of the items are under 60 and 40% are between 60 & 65. Find the mean & s.d of the distribution

When $x = 60$, $A(z) = 0.05 \Rightarrow A(z) = 0.5 - 0.05 = 0.45$

$\Rightarrow z_1 = \cancel{0.} -1.64 = \frac{60-\mu}{\sigma} \Rightarrow 60 - \mu = -1.64\sigma$

$\Rightarrow \mu - 1.64\sigma = 60$ — ①



When $x = 65$, $A(z) = 0.05$

$\Rightarrow z = -0.12 \Rightarrow \cancel{\mu - 1.64\sigma} \; \mu - 0.12\sigma = 65$ ②

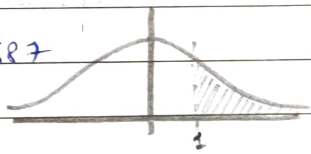
Solving ① & ②, $\mu = 65.3947$, $\sigma = 3.2894$

Q. A sample of 500 items have a mean of 12 and s.d of 3. If it follows normal distribution, how many items are expected to have ~~life~~ i) more than 15, ii) less than 6, iii) between 10 to 14.

$\mu = 12$, $\sigma = 3$

i) $x = 15$, $z = \frac{15-12}{3} = \frac{3}{3} = 1$

$P(x>15) = P(z>1)$ $\cancel{P(0<z<\infty)}$ $P(z>1) = 0.5 - 0.3413 = 0.1587$

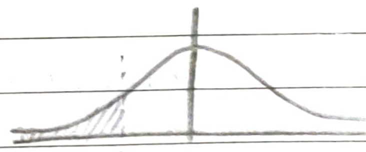


ii) $x = 6$, $z = \frac{6-12}{3} = \frac{-6}{3} = -2$

$P(x<6) = P(z<-2) = P(0<z<2) - P(0<z<\infty)$

$= \cancel{0.4772 + 0.5 = 0.9772}$

$= 0.5 - 0.4772 = 0.0228$



iii) $10<x<14$, If $x = 10$, $z = \frac{10-12}{3} = \frac{-2}{3} = 0.67$

If $x = 14$, $z = \frac{14-12}{3} = \frac{2}{3} = 0.67$

$P(10<x<14) = P(-0.67<z<0.67) = P(0<z<0.67) + P(0<z<0.67)$

$= 0.2486 + 0.2486 = 0.4972$

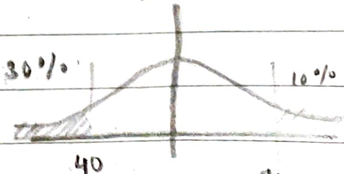
Out of 500 items, i) ~79, ii) ~11, iii) ~248

Q. In a ND, 30% of the items are under 40 & 10% are over 70. Find the μ & sd of distribution

When $x = 40$, $A(z_1) = 0.5 - 0.3 = 0.2$

$\Rightarrow z_1 = -0.525$ (btw 0.52 & 0.53)

$\Rightarrow -0.525 = \frac{40-\mu}{\sigma} \Rightarrow 40 - \mu = -0.525\sigma \Rightarrow \mu - 0.525\sigma = 40$ -①



Find z using the value of where the probability is in the table

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When $x = 70$, $A(z) = 0.5 - 0.1 = 0.4$

$\Rightarrow z = 1.28$

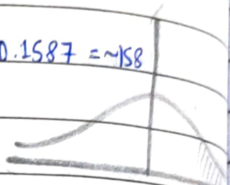
$\Rightarrow 1.28 = \frac{70-\mu}{\sigma} \Rightarrow 1.28\sigma = 70 - \mu \Rightarrow \mu + 1.28\sigma = 70$ — (2)

Solving (1) & (2), $\mu = 48.7259$, $\sigma = 16.6205$

Q. The marks of 1000 students in an exam follows ND with mean 70 and sd 5. Find the number of students whose marks will be i) less than 65, ii) more than 75, iii) between 64 & 73

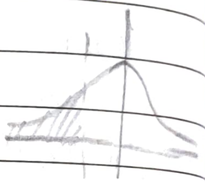
ii) $x = 75$, $z = \frac{75-70}{5} = \frac{5}{5} = 1$

$P(x>75) = P(z>1) = P(0<z<\infty) - P(0<z<1) = 0.5 - 0.3413 = 0.1587 = \sim 158$



i) $x = 65$, $z = \frac{65-70}{5} = \frac{-5}{5} = -1$

$P(x<65) = P(z<-1) = P(0<z<\infty) - P(0<z<-1) = 0.5 - 0.3413 = 0.1587 = \sim 158$

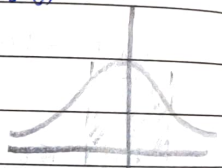


iii) $64<x<73$, if $x = 64$, $\frac{64-70}{5} = \frac{-6}{5} = -1.2$

if $x = 73$, $\frac{73-70}{5} = \frac{3}{5} = 0.6$

$P(64<x<73) = P(-1.2<z<0.6) = P(0<z<-1.2) + P(0<z<0.6)$

$= 0.3849 + 0.2257 = 0.6106 = \sim 610$



Q. If 4 students randomly selected, atleast 1 is more than 75.

$p = 0.1587$, $q = 1 - 0.1587 = 0.8413$

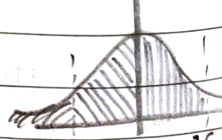
$P(\text{atleast 1 std}) = 1 - P(x=0) = 1 - {}^4C_0\, p^0 q^4 = 1 - 0.50096 = 0.49904$

Q. For a ND with mean μ and sd σ, evaluate i) $P(|x-\mu| \le \sigma)$, ii) $P(|x-\mu| \le 2\sigma$, iii) $P(|x-\mu| \le 3\sigma$

Q. In a ND, 10% of the items are under 25 and 80% are under 75.

When $x = 25$, $A(z) = 0.5 - 0.1 = 0.4$

$z_1 = -1.28 \Rightarrow -1.28 = \frac{25-\mu}{\sigma} \Rightarrow \mu - 1.28\sigma = 25$ — (1)



When $x = 75$, $A(z) = 0.5 - 0.2 = 0.3$

$z = 0.525 \Rightarrow 0.525 = \frac{75-\mu}{\sigma} \Rightarrow \mu + \cancel{0.25\sigma}\ 0.525\sigma = 75$ — (2)

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Solving ① & ②, $\mu = 60.4571$, $\sigma = 27.7008$

1. i) $z = \frac{x-\mu}{\sigma}$

$P(|x-\mu| \leq \sigma) = P(-\sigma \leq x-\mu \leq \sigma) = P(-1 \leq z \leq 1) = 2P(0 \leq z \leq 1) = 2 \times 0.3413 = 0.6826$

ii) $P(|x-\mu| \leq \sigma) = P(-2\sigma \leq x-\mu \leq 2\sigma) = P(\cancel{-0.9544}\,-2 \leq z \leq \cancel{0.9544}\,2) = 2P(0 \leq z \leq \cancel{0.9544}\,2) =$

$2 \times \cancel{0.3289}\,0.4773 = 0.9546$

iii) $P(|x-\mu| \leq \sigma) = P(-3\sigma \leq x+\mu \leq 3\sigma) = 2P(0 \leq z \leq 3) = 0.9974$

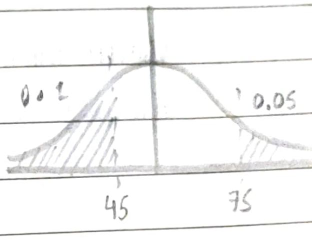
3. <45% is 10% of students, >75% is 5% students, find btw 45%–50% & 60%–75%

$-1.28 = \frac{45-\mu}{\sigma}$

$\mu - 1.28\sigma = 45$

$1.645 = \frac{75-\mu}{\sigma}$

$\mu + 1.6$



When $x = 45$, $A(z_1) = 0.5 - 0.1 = 0.4$

$z_1 = -1.28 \Rightarrow \mu - 1.28\sigma = 45$

When $x = 75$, $A(z) = 0.5 - 0.05 = 0.45$

$z = 1.645 \Rightarrow \mu + 1.645\sigma = 75$

Solving the above eqn's, $\mu = 58.1282$, $\sigma = 10.2564$

i) $45 < x < 60$

When $x = 45$, $z = \frac{45 - 58.1282}{10.2564} = -1.28$

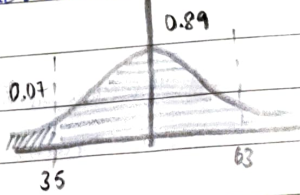
When $x = 60$, $z = \frac{60 - 58.1282}{10.2564} = 0.1825$

$P(45 < x < 60) = P(-1.28 < z < 0.1825) = P(0 \leq z \leq 1.28) + P(0 \leq z \leq 0.1825)$

$= 0.3997 + 0.0714 = 0.4711$

Q. Find μ & σ of a ND if 7% of the items are under 35 and 89% are under 63.

$-1.475 = \frac{35-\mu}{\sigma}$



When $x = 35$, $A(z_1) = 0.5 - 0.07 = 0.43$

$z_1 = -1.48$ — ①

$\Rightarrow \mu - 1.48\sigma = 35$

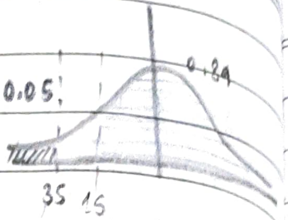
When $x = 63$, $A(z) = 0.5 - 0.89 = -0.39$ | $0.89 - 0.5 = 0.39$

$z = -(-1.23)$ | 1.23 — ②

$\Rightarrow \mu + 1.23\sigma = 63$

Solving ① & ②, $\mu = 50.2915$, $\sigma = 10.3321$

Q. Find μ & σ if 5% of the items are under 35 and 89% are over 15.



UNIFORM DISTRIBUTION

A random variable X is said to be uniformly distributed if $a \le x \le b$ if

$$f(x) = \begin{cases} \frac{1}{b-a} & , a \le x \le b \\ 0 & , \text{otherwise} \end{cases}$$

$$\int_a^b f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}[x]_a^b = \frac{1}{b-a}(b-a) = 1$$

Mean of UD : $\mu = \frac{1}{2}(a+b)$

Variance of UD : $\sigma^2 = \frac{1}{12}(b-a)^2$

CUMULATIVE PROBABILITY FUNCTION

$$F(x) = \begin{cases} 0 & , -\infty \le x \le a \\ \frac{x-a}{b-a} & , a \le x \le b \\ 1 & , x > b \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_a^x \frac{1}{b-a}\,dx = \frac{1}{b-a}[x]_a^x = \frac{1}{b-a}(x-a) = \frac{x-a}{b-a}$$

Q. If X is uniformly distributed with mean 1 and variance $\frac{4}{3}$, find $P(x<0)$

$\mu = \frac{1}{2}(a+b) \Rightarrow 1 = \frac{1}{2}(a+b) \Rightarrow a+b = 2$

$\sigma^2 = \frac{1}{12}(b-a)^2 \Rightarrow \frac{4}{3} = \frac{1}{12}(b-a)^2 \Rightarrow (b-a)^2 = 16 \Rightarrow b-a = \pm 4$

When $b-a = 4$, $a = -1$, $b = 3$

When $b-a = -4$, $a = 3$, $b = -1$

$P(x<0) = \int_{-1}^{0} f(x)\,dx = \int_{-1}^{0} \frac{1}{4}\,dx = \frac{1}{4}[x]_{-1}^{0} = \frac{1}{4}(0-(-1)) = \frac{1}{4}$

Q. Buses arrive at a stop at 15 min intervals starting at 7 a.m. If a passenger arrives at the stop at

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random time that is uniformly distributed b/w 7 & 7:30 a.m, find the prob. that he waits for i) < 5 min, ii) at least 12 min for the bus.

$f(x) = \begin{cases} \frac{1}{30}, & 7 \le x \le 7{:}30 \\ 0, & \text{otherwise} \end{cases}$

i) $P(x<5) = \int_{7:10}^{7:15} \frac{1}{30} dx + \int_{7:25}^{7:30} \frac{1}{30} dx = \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$

ii) $P(\text{atleast } 12, x \ge 12) = \int_{7:00}^{7:03} \frac{1}{30} dx + \int_{7:15}^{7:18} \frac{1}{30} dx = \frac{3}{30} + \frac{3}{30} = \frac{1}{5}$

Q. In a certain city transport route, buses ply every 30 mins between 6 to 10 a.m. If a person reaches a bus stop at a random time, what is the prob. that he has to wait for atleast 20 mins

$f(x) = \int_6^{10} = \frac{1}{60 \times 4} = \frac{1}{240}$

$P(x \ge 20) = 8 \times \int_{6:00}^{6:10} \frac{1}{240} dx = \cancel{8} \times \frac{1}{\cancel{240}_{30}} \times [x]_0^{10} = \frac{10}{30} = \frac{1}{3}$

Q. A & B are 10 miles apart. The distance X of the point of breakdown from A is a uniform variant. There are service garages in city A, city B and mid way. If a breakdown occurs a tow truck is sent from the garage closest to the breakdown point. Find the prob. that the truck has to travel i) > 10 miles to reach

GAMMA FUNCTION

$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx, \; n > 0$

$\Gamma n+1 = n \Gamma n$

If n is an integer, $\Gamma(n+1) = n!$

GAMMA DISTRIBUTION

The continuous random variable X has a gamma distribution with parameter α & β if its density function is given by

$f(x) = \begin{cases} \frac{1}{\beta^\alpha \sqrt{\alpha}} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

If $\alpha = 1$, $f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$

For $\alpha = 1$, gamma distribution reduces to exponential.

Mean, $\mu = \alpha\beta$ and variance, $\sigma^2 = \alpha\beta^2$

Q. A random variable X is gamma distributed with $\alpha = 3, \beta = 2$, find i) $P(X \le 1)$, ii) $P(1 \le X \le 2)$

i) $P(X \le 1) = \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{2^3\sqrt{3}} \cdot x^{3-1} \cdot e^{-x/2}\,dx$

$= \frac{1}{8.2!} \int_0^1 x^2 e^{-x/2}\,dx = \frac{1}{8.2!}\left[x^2 \frac{e^{-x/2}}{-1/2}\right]_0^1 - \int_0^1 2x \frac{e^{-x/2}}{-1/2}$

$= [-x^2 e^{-x/2}(2)]_0^1 + 4\int_0^1 x e^{-x/2}\,dx = +[2x^2 e^{-x/2}]_1^0 + 4\left[\left(\frac{xe^{-x/2}}{-1/2}\right)_0^1 - \int e^{-x/2}\,dx\right]$

$= ((-2)(1)^{-1/2} e^{-1/2}) + 4((-2(1)e^{-1/2} - 0) + 2\left(\frac{e^{-x/2}}{-1/2}\right)_0^1$

$= -2e^{-1/2} + 4[(-2e^{-1/2} + 2)(-2e^{-1/2} + 2))]$

$= -2e^{-1/2} - 8e^{-1/2} - 16e^{-1/2} + 16 = -26e^{-1/2} + 16$

$= 1 - \frac{26}{16} e^{-1/2} = 0.01438$

ii) $\int_1^2 f(x)\,dx = \int_1^2 \frac{1}{2^3\sqrt{3}} \cdot x^{3-1} \cdot e^{-x/2}\,dx = \frac{1}{16}\int_1^2 x^2 e^{-x/2}$

$= \frac{1}{16}\left[x^2 \frac{e^{-x/2}}{(-1/2)} - 2x \frac{e^{-x/2}}{(-1/2)^2} + 2(1)\frac{e^{-x/2}}{(-1/2)^3}\right]_1^2$

$= \frac{1}{16}\left[(4(-2)e^{-1} - 4(4)e^{-1} - 16e^{-1}) - (-2e^{-1/2} - 8e^{-1/2} - 16e^{-1/2})\right]$

$= \frac{1}{16}[-40e^{-1} - (-26e^{-1/2})] = \frac{1}{16}[-40e^{-1} + 26e^{-1/2}]$

$= 0.06591$

Q. The demand for certain items follows gamma distribution with $\mu = 8$ & $\sigma^2 = 32$, find prob. i) atleast 10 items, $P(x \ge 10)$

Given $\mu = 8 \Rightarrow \alpha\beta = 8$ and $\sigma^2 = 32 \Rightarrow \alpha\beta^2 = 32$

$\therefore (\alpha\beta)\beta = 32 \Rightarrow \beta = \frac{32}{8} = 4 \Rightarrow \alpha\beta = 8 \Rightarrow \alpha = \frac{8}{4} = 2$

$\therefore \alpha = 2, \beta = 4$

i) $\int_{10}^{\infty} f(x)\,dx = \int_{10}^{\infty} \frac{1}{4^2\sqrt{2}} \cdot x^{2-1} \cdot e^{-x/4}\,dx = \frac{1}{16}\int_{10}^{\infty} x e^{-x/4}\,dx = \frac{1}{16}\int$

$= \frac{1}{16}\left[\left(\frac{xe^{-x/4}}{-1/4}\right)_{10}^{\infty} - \int \frac{e^{-x/4}}{-1/4}\,dx\right.$

$= \frac{1}{16}\left[\left(4xe^{-x/4}\right)_{\infty}^{10} - \left(4e^{-x/4}(4)\right)_{10}^{\infty}\right] = \frac{1}{16}\left[40e^{-10/4} + 16e^{-10/4}\right]$

$= \frac{56}{16}e^{-10/4} = 0.2872$

Q. The daily consumption of a mill in a city is excess of 20,000 items is distributed as a gamma variate, $\alpha = 2$, $\beta = 10{,}000$. The city has a daily stock of 30,000 items, find prob. that the stock is insufficient on a given day.

If X is daily consumption $\Rightarrow Y = X - 20{,}000$ (gamma variant)

$P(X > 30000) = P(Y > 10000) = \int_{10000}^{\infty} \frac{1}{(10000)^2 \sqrt{2}} \cdot x^{2-1} \cdot e^{-x/\beta}$

$= \frac{1}{(10000)^2} \int_{10000}^{\infty} y e^{-y/\beta} dy = \frac{1}{(10000)^2}\left[\frac{ye^{-y/\beta}}{-1/\beta} - \frac{(1)\cdot e^{-y/\beta}}{(-1/\beta)^2}\right]_{10000}^{\infty}$

$= \frac{1}{10^8}\left[10^4/10^4\right) e^{-\frac{10^4}{10^4}} + (10^4)^2 e^{-10^4/10^4}\right]$

$= \frac{1}{10^8}\left[10^8 e^{-1}\right] \cdot 2 = 2e^{-1} = 0.7357$

Q. The time required for an item at a repair shop follows gamma distribution with $\mu = 2$ & $\sigma^2 = 1$, find prob. to service in less than 1 hr.

$P(X < 1) \Rightarrow \alpha\beta = 2$, $\alpha\beta^2 = 1 \Rightarrow (\alpha\beta)\beta = 1 \Rightarrow \beta = 1/2$

$\alpha = \frac{2}{\beta} = 4$

$\alpha = 4$, $\beta = \frac{1}{2}$

$= \int_0^1 \frac{1}{\left(\frac{1}{2}\right)^4 \sqrt{4}} x^{4-1} \cdot e^{-x(2)} dx = \frac{16}{3!}\int_0^1 x^3 e^{-2x} dx = \cancel{\frac{16}{3!}\left[\frac{x^3 e^{-2x}}{-2}\right.}$

$= \frac{16}{3!}\left[\frac{x^3 e^{-2x}}{-2} - \frac{3x^2 e^{-2x}}{(-2)^2} + \frac{6x e^{-2x}}{(-2)^3} - \frac{6e^{-2x}}{(-2)^4}\right]_0^2$

$= \frac{16}{3!}\left[\left[\frac{e^{-2}}{-2} - \frac{3e^{-2}}{4} + \frac{6e^{-2}}{-8} - \frac{6e^{-2}}{16}\right] - \left[\frac{-6}{16}\right]\right]$

$= \frac{16}{3!}\left[e^{-2}\left[-\frac{1}{2} - \frac{3}{4} + \frac{6}{8} - \frac{6}{16}\right] + \frac{6}{16}\right] = \frac{16}{3!}\left[-\frac{19}{8}e^{-2} + \frac{6}{16}\right]$

$= 0.14287$

JOINT PROBABILITY DISTRIBUTION

A JPD is used to determine the likelihood of certain events happening, given certain other events happening.

If X and Y are 2 discrete random variables then the joint prob. function of X & Y can

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be defined as

$P(X=x, Y=y) = h(x,y)$ = Prob. that x & y are together.

$h(x,y)$ satisfies 2 conditions

i) $h(x,y) \geq 0$, ii) $\sum_y \sum_x h(x,y) = 1$

Let $X = \{x_1, x_2, \dots, x_n\}$ & $Y = \{y_1, y_2, \dots, y_n\}$

Then $P(X=x_i, Y=y_j) = h(x_i, y_j) = J_{ij}$ $\quad 1 \leq i \leq x$, $1 \leq j \leq y$

J_{ij} is called the joint prob. distribution of X & Y.

This is written in the form of a table.

X \ Y	y_1	y_2	... y_m
x_1	J_{11}	J_{12}	... J_{1n}
x_2	J_{21}	J_{22}	... J_{2n}
⋮			
x_n	J_{n1}	J_{n2}	... J_{nm}

Q. A fair coin is tossed twice. Let X denote 0 or 1 according to head or tail appearing first. Let Y denote the no. of heads obtained. Write the JPD of X & Y.

	HH	HT	TH	TT
X :	0	0	1	1
Y :	2	1	1	0

X \ Y	0	1	2
0	0	1/4	1/4
1	1/4	1/4	0

MARGINAL PROBABILITY DISTRIBUTION

If we add up the rows and columns of a joint prob. dist., we get

X \ Y	y_1	y_2 ...	y_n	$f(x)$
x_1	J_{11}	J_{12}	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	J_{2n}	$f(x_2)$
⋮				⋮
x_m	J_{m1}	J_{m2}	J_{mn}	$f(x_m)$
$g(y)$	$g(y_1)$	$g(y_2)$	$g(y_m)$	1

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$f(x_i) = J_{i1} + J_{i2} + \dots + J_{in}$

$g(y_j) = J_{1j} + J_{2j} \dots + J_{nj}$

$\Rightarrow \sum_i f(x_i) = \sum_j g(y_j) = \sum_j \sum_i J_{ij} = 1$

$f(x_i)$ & $g(y_j)$ are called the marginal prob. dist. of X & Y.

INDEPENDENT RANDOM VARIABLES:

X & Y are independent iff $P(X=x, Y=y) = P(X=x)P(Y=y)$

$\Rightarrow P(X=x_i, Y=y_j) = P(X=x_i)\,P(Y=y_j)$

$\Rightarrow J_{ij} = f(x_i)\cdot g(y_j) \quad \forall i,j$

Expectation, Variance, Covariance, & Correlation

Expectation: $\mu_X = E(X) = \sum_{i=1}^{m} x_i f(x_i)$, $\mu_Y = \sum_{j=1}^{n} y_j g(y_j)$

$E(XY) = \sum_j \sum_i x_i y_j J_{ij}$

Variance: $\sigma_X^2 = \sum_{i=1}^{m} (x_i - \mu_X)^2 f(x_i)$

$\sigma_Y^2 = \sum_{j=1}^{n} (y_j - \mu_Y)^2 g(y_j)$

Covariance: $cov(X,Y) = E(XY) - E(X)E(Y)$

Correlation: $\rho(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$

NOTE: If X and Y are independent, then $E(XY) = E(X).E(Y)$

$\Rightarrow cov(X,Y) = 0$, $\rho(X,Y) = 0$

Q. The JPD of 2 random variables X & Y is given below:

X \ Y	-2	-1	4	5	$f(x_i)$
1	0.1	0.2	0	0.3	0.6
2	0.2	0.1	0.1	0	0.4
$g(y_j)$	0.3	0.3	0.1	0.3	

Find i) MPD of X & Y, ii) Expectation of X, Y and XY, iii) SD's of X & Y, iv) Cov(X, Y), v) Cor(X, Y), vi) Are X & Y independent.

i)

ii) $E(X) = \mu_X = \sum_{i=1}^{2} x_i f(x_i) = 1(0.6) + 2(0.4) = 1.4$

$E(Y) = \mu_Y = \sum_{j=-2} y_j g(y_j) = -2(0.3) + (-1)(0.3) + 4(0.1) + 5(0.3) = -0.6 - 0.3 + 0.4 + 1.5 = 1$

$$E(XY) = \sum_j \sum_i x_i y_j J_{ij} = 1[(-2)(0.1)+(-1)(0.2)+(4)(0)+5(0.3)] + 2[(-2)(0.2)(-1)(0.1) + 4(0.1)+5(0)] = 0.9$$

iii) $$\sigma_X^2 = E(X^2) - \{E(X)\}^2 = \sum x_i^2 f(x_i) - (1.4)^2 = (1)^2(0.6) + (2)^2(0.4) - (1.4)^2 = 0.24$$

$$\sigma_X = \sqrt{0.24} = 0.4899$$

$$\sigma_Y^2 = E(Y^2) - \{E(Y)\}^2 = \sum y_j^2 g(y_j) - (1)^2 = (-2)^2(0.3) + 1^2(0.3) + 4^2(0.1) + 5^2(0.3) - 1^2 = 9.6$$

$$\sigma_Y = \sqrt{9.6} = 3.0984$$

iv) $Cov(X,Y) = E(XY) - E(X)E(Y) = 0.9 - [1.4 \times 1] = -0.5$

v) $$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-0.5}{0.4899 \times 3.0984} = -0.3294$$

vi) X & Y are not independent variables.

Q. From the given table, find the correlation between X and Y.

Y \ X	-1	0	1	$g(y_j)$
0	1/15	2/15	1/15	4/15
1	3/15	2/15	1/15	6/15
2	2/15	1/15	2/15	5/15
$f(x_i)$	6/15	5/15	4/15	

$$E(X) = \mu_X = \sum_{i=-1}^{1} x_i f(x_i) = -1(6/15) + 0(5/15) + 1(4/15) = -2/15 = -0.1334$$

$$E(Y) = \mu_Y = \sum_{j=0}^{2} y_j f(y_j) = 0(4/15) + 1(6/15) + 2(5/15) = 16/15 = 1.0667$$

$$E(XY) = \sum_j \sum_i x_i y_j J_{ij} = -1[0(1/15) + 1(3/15) + 2(2/15)] + 0[0(2/15) + 1(2/15) + 2(1/15)] + 1[0(1/15) + 1(1/15) + 2(2/15)]$$
$$= -7/15 + 5/15 = -2/15$$

$$\sigma_X^2 = \sum_{i=1}^{m} (x_i - \mu_x)^2 f(x_i) = E(X^2) - \{E(X)\}^2 = \sum x_i^2 f(x_i) - (-2/15)^2$$
$$= [(-1)^2(6/15) + 0^2(5/15) + 1^2(4/15)] - (-2/15)^2 = \frac{10}{15} - \frac{4}{225} = \frac{146}{225}$$

$$\sigma_X = \sqrt{\frac{146}{225}} = \frac{\sqrt{146}}{15}$$

$$\sigma_Y^2 = E(Y^2) - \{E(Y)\}^2 = \sum y_j^2 f(y_j) - \left(\frac{16}{15}\right)^2 = \left[0\left(\frac{4}{15}\right) + 1^2\left(\frac{6}{15}\right) + 2^2\left(\frac{5}{15}\right)\right] - \left(\frac{16}{15}\right)^2$$
$$= \frac{134}{225}$$

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$\sigma_Y = \sqrt{\frac{134}{225}} = \frac{\sqrt{134}}{15}$

$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{E(XY) - [E(X).E(Y)]}{\sigma_X \sigma_Y} = \frac{2}{225}$ $= 0.01429$

Q. A fair coin is tossed 3 times and the sample space S is {HHH, HHT, HTH, HTT, TTH, THT, THH, TTT}. Let X be 0 or 1 according as a head or tail occurs in the first toss & Y be the total no. of heads that occur.

i) Find the JPD of X & Y, ii) Find cov (X,Y), iii) Find if X & Y are independent, iv) Cor (X,Y)

	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X:	0	0	0	0	1	1	1	1
Y:	3	2	2	1	2	1	1	0

X \ Y	0	1	2	3	$f(x_i)$
0	0	1/8	2/8	1/8	4/8
1	1/8	2/8	1/8	0	4/8
$f(y_j)$	1/8	3/8	3/8	1/8	1

$\frac{3}{8} + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$

$\frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8}$

$E(X) = \mu_X = \sum_{i=0}^{1} x_i f(x_i) = \cancel{0(4/8)} + 1(4/8) = 4/8$

$E(Y) = \mu_Y = \sum_{j=0}^{3} y_j f(y_j) = \cancel{0(1/8)} + 1(3/8) + 2(3/8) + 3(1/8) = 12/8$

$E(XY) = \sum_j \sum_i x_i y_j J_{ij} = 0[\cancel{0(0) + 1(1/8) + 2(2/8) + 3(1/8)}] + 1[\cancel{0(1/8)} + 1(2/8) + 2(1/8) + 3(0)]$
$= \frac{4}{8}$

$\sigma_X^2 = E(X^2) - \{E(X)\}^2 = \sum x_i^2 f(x_i) - (4/8)^2 = 0^2(4/8) + 1^2(4/8) - (4/8)^2 = 2/8$

$\sigma_X = 1/2$

$\sigma_Y^2 = E(Y^2) - \{E(Y)\}^2 = \sum y_j^2 g(y_j) - (12/8)^2 = \cancel{0^2(1/8)} + 1^2(3/8) + 2^2(3/8) + 3^2(1/8) - (12/8)^2$
$= 3/4$

$\sigma_Y = \frac{\sqrt{3}}{2}$

$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{4}{8} - \left(\frac{4}{8} \times \frac{12}{8}\right) = -1/4$

$Cor(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-1/4}{1/2 \times \sqrt{3}/2} = -\frac{1}{\sqrt{3}}$

∴ X & Y are not independent variables.

Q. 2 balls are selected at random from a box containing 3 blue, 2 red, 3 green balls.

If X is the no. of blue balls & Y is the no. of red balls selected, find
i) JPD of X & Y, ii) P[(X,Y) ∈ A] where A is the region $\{(x,y) \mid x+y \le 1\}$, iii) Cov(X,Y)

i)

X \ Y	0	1	2	$f(x_i)$
0	3/28	6/28	1/28	10/28
1	9/28	6/28	0	15/28
2	3/28	0	0	3/28
$g(y_j)$	18/28	12/28	1/28	1

$E(X) = \mu_X = \sum_{i=0}^{2} x_i f(x_i) = 0(10/28) + 1(15/28) + 2(3/28) = 21/28 = 3/4$

$E(Y) = \mu_Y = \sum_{j=0}^{2} y_j f(y_j) = 0(18/28) + 1(12/28) + 2(1/28) = 14/28 = 2/4 = 1/2$

$E(XY) = \sum_j \sum_i x_i y_j f_{ij} = 0[(0)(3/28) + 1(6/28) + 2(1/28)] + 1[0(9/28) + 1(6/28) + 2(0)] + 2[0(3/28) + 1(0) + 2(0)] = 6/28$

iii) $Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{6}{28} - \left[\frac{21}{28} \times \frac{14}{28}\right] = \frac{-9}{56} = -0.1607$

ii) $P[(X,Y) \in A]\ \{x,y \mid x+y \le 1\} = \frac{3}{28} + \frac{6}{28} + \frac{9}{28} = \frac{18}{28}$

Q. The JPD of 2 DRV is X and Y is given by $f(x,y) = C(2x+y)$ where X & Y ~~are~~ can assume all integer values for $0 \le x \le 2$, $0 \le y \le 3$ and $f(x,y) = 0$ otherwise. Find i) C, ii) $P(X \ge 1, Y \le 2)$, iii) Marginal dist. of X & Y.

X \ Y	0	1	2	3	$f(x_i)$
0	0	C	2C	3C	6C
1	2C	3C	4C	5C	14C
2	4C	5C	6C	7C	22C
$g(y_j)$	6C	9C	12C	15C	42C

i) $42C = 1 \Rightarrow C = 1/42$

ii) $P(X \ge 1, Y \le 2) = 2C + 3C + 4C + 4C + 5C + 6C = 24C = \frac{24}{42} = \frac{4}{7}$

iii) From table

Q. Conditional Probability Distribution

$h(y|x) = \frac{h(x,y)}{f(x)}$ & $h(x|y) = \frac{h(x,y)}{g(y)}$

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Q. If X & y are independent random variables with the distributions given by

X	1	2
f(x)	0.7	0.3

Y	-2	6	8
g(y)	0.3	0.5	0.2

Find JPD of X & Y. Also find $P(Y=8|X=1)$.

X \ Y	-2	5	8	$f(x_i)$
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
$g(y_j)$	0.3	0.5	0.2	1

$$P(Y=8|X=1) = \frac{0.14}{0.7} = 0.02$$

Q. Two cards are selected at random from a box which contains 5 cards numbered 1, 1, 2, 2, 3. If X denotes the sum and Y the maximum of the 2 numbers, find the JPD of X & Y. Also find cov(X,Y) & cor(X,Y).

X \ Y	1	2	3	$f(x_i)$
2	1/10	0	0	1/10
3	0	4/10	0	4/10
4	0	1/10	2/10	3/10
5	0	0	2/10	2/10
$g(y_j)$	1/10	5/10	4/10	1

$$\frac{2}{10} + \frac{12}{10} + \frac{12}{10} + \frac{10}{10} = \frac{36}{10}$$

$$E(X) = \mu_X = \sum_{i=2}^{5} x_i f(x_i) = 2(1/10) + 3(4/10) + 4(3/10) + 5(2/10) = 36/10$$

$$\frac{1}{10} + \frac{10}{10} + \frac{12}{10}$$

$$E(Y) = \mu_Y = \sum_{j=1}^{3} y_j g(y_j) = 1(1/10) + 2(5/10) + 3(4/10) = 23/10$$

$$\frac{2}{10} + \frac{24}{10} + \frac{32}{10} + \frac{30}{10} = \frac{88}{10} \qquad \frac{2}{10} + \frac{6}{10} = \frac{8}{10}$$

$$E(XY) = \sum_j \sum_i x_i y_j J_{ij} = 2[1(1/10) + \cancel{2(0)} + \cancel{3(0)}] + 3[\cancel{1(0)} + 2(4/10) + \cancel{3(0)}] + 4[\cancel{1(0)} + 2(1/10) + 3(2/10)] + 5[\cancel{1(0)} + \cancel{2(0)} + 3(2/10)] = \frac{88}{10}$$

$$\sigma_X^2 = E(X^2) - \{E(X)\}^2 = \sum x_i^2 f(x_i) - (36/10)^2 = 2^2(1/10) + 3^2(4/10) + 4^2(3/10) + 5^2(2/10) - (36/10)^2 = \frac{21}{25}$$

$$\sigma_X = \frac{\sqrt{21}}{5}$$

$$\sigma_Y = E(Y^2) - \{E(Y)\}^2 = \sum y_j^2 g(y_j) - (23/10)^2 = 1^2(1/10) + 2^2(5/10) + 3^2(4/10) - (23/10)^2$$

$$= \frac{41}{100} \Rightarrow \sigma_Y = \frac{\sqrt{41}}{10}$$

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$Cov(X,Y) = E(XY) - E(X).E(Y) = \frac{88}{10} - \left(\frac{36}{10} \times \frac{23}{10}\right) = \frac{13}{25} = 0.52$

$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{13/25}{\sqrt{21}/5 \cdot \sqrt{41}/10} = \frac{0.52}{0.9165 \times 0.6403} = 0.8861$

JPD (CRV)

If X & Y are 2 ~~continut~~ continuous random variables & $f(x,y)$ is a real-valued function s.t. $f(x,y) \geq 0$ & $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$, then $f(x,y)$ is called the ~~just~~ joint prob. func. of X & Y.

$P(a \leq x \leq b, c \leq y \leq d) = \int_c^d \int_a^b f(x,y)\,dx\,dy$

MD

$P(X \leq x) = F_1(X) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx$ is the marginal dist. func. of X.

$P(Y \leq y) = F_2(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$ "

Expectation, Variance, Covariance, & Correlation.

$\mu_X = E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,f(x,y)\,dx\,dy$

$\sigma_X^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)^2 f(x,y)\,dx\,dy = E[(x-\mu_X)^2]$

$\mu_Y = E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\,f(x,y)\,dx\,dy$

$\sigma_Y^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-\mu_Y)^2 f(x,y)\,dx\,dy = E[(y-\mu_Y)^2]$

$Cov(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x,y)\,dx\,dy = E(XY) - E(X)E(Y)$

If X & Y are independent, $cov(X,Y) = 0$.

Q. The JPD func. of X & Y is $f(x,y) = \begin{cases} Cxy, & 0 < x < 4,\ 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$

Find i) C, ii) $P(1 < x < 2, 2 < y < 3)$ iii) MD func. of X & Y

i) As $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_1^5\int_0^4 Cxy\,dx\,dy = 1$

$\Rightarrow C\left[\frac{x^2}{2}\right]_0^4 \left[\frac{y^2}{2}\right]_1^5 = 1 \Rightarrow C\left[\frac{16}{2} - 0\right]\left[\frac{25}{2} - \frac{1}{2}\right] = 1 \Rightarrow C\left[\frac{16}{2} \times \frac{24}{2}\right] = 1$

$\Rightarrow 96C = 1 \Rightarrow C = \frac{1}{96}$

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ii) $P(1<x<2, 2<y<3) = \int_2^3\int_1^2 \frac{1}{96} xy \, dx \, dy = \frac{1}{96}\left[\frac{x^2}{2}\right]_1^2 \left[\frac{y^2}{2}\right]_2^3 = \frac{1}{96} \times \left(\frac{4}{2}-\frac{1}{2}\right) \times \left(\frac{9}{2}-\frac{4}{2}\right)$

$= \frac{16}{384}$

iii) $F_1(x) = \int_{-\infty}^{x}\int_1^5 \frac{1}{96} xy \, dy \, dx = \frac{1}{96}\int_0^x x\left[\frac{y^2}{2}\right]_1^5 dx = \frac{24}{192}\int_0^x x \, dx = \frac{1}{8}\left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{16}$

$$F_1(x) = \begin{cases} 0, & \text{if } x<0 \\ \frac{x^2}{16} & \text{if } 0<x<4 \\ 1 & \text{if } x>4 \end{cases}$$

$F_2(y) = \int_{-\infty}^{y}\int_0^4 \frac{1}{96} xy \, dx \, dy = \frac{1}{96}\int_1^y y\left[\frac{x^2}{2}\right]_0^4 dy = \frac{16}{192}\int_1^y y \, dy = \frac{1}{12}\left[\frac{y^2}{2}\right]_1^y = \frac{1}{24}(y^2-1)$

$$F_2(y) = \begin{cases} 0 & \text{, if } y<1 \\ \frac{y^2-1}{24} & \text{, if } 1<y<5 \\ 1 & \text{, if } y>5 \end{cases}$$

Q. If X & Y are CRV's having joint density func. $f(x,y) = \begin{cases} C(x^2+y^2) & , 0\le x,y\le 1 \\ 0 & , \text{otherwise} \end{cases}$

Find i) C, ii) $P(X<\frac{1}{2}, Y>\frac{1}{2})$, iii) $P(\frac{1}{4}<X<\frac{3}{4})$, iv) $P(Y<\frac{1}{2})$

i) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_0^1\int_0^1 C(x^2+y^2)\,dx\,dy = 1 \Rightarrow C\int_0^1 \left(\frac{x^3}{3}+xy^2\right)_0^1 dy = 1$

$\Rightarrow C\int_0^1 \left(\frac{1}{3}+y^2\right) dy = 1 \Rightarrow C\left[\frac{1}{3}y + \frac{y^3}{3}\right]_0^1 = 1 \Rightarrow \frac{C}{3}[1+1] = 1 \Rightarrow C = \frac{3}{2}$

ii) $P(X<\frac{1}{2}, Y>\frac{1}{2}) = \frac{3}{2}\int_{1/2}^{1}\int_0^{1/2} (x^2+y^2)\,dx\,dy = \frac{3}{2}\int_{1/2}^{1}\left(\frac{x^3}{3}+xy^2\right)_0^{1/2} dy = \frac{3}{2}\int_{1/2}^{1}\left(\frac{1}{24}+\frac{1}{2}y^2\right)dy$

$= \frac{3}{2}\left[\frac{1}{24}y + \frac{1}{2}\frac{y^3}{3}\right]_{1/2}^{1} = \frac{3}{2}\left[\frac{1}{24}\left(1-\frac{1}{2}\right)+\frac{1}{6}\left(1-\frac{1}{8}\right)\right] = \frac{3}{2}\left(\frac{1}{48}+\frac{7}{48}\right) = \frac{3}{2}\left(\frac{8}{48}\right) = \frac{1}{4}$

iii) $P(\frac{1}{4}<X<\frac{3}{4}) = \frac{3}{2}\int_0^1\int_{1/4}^{3/4}(x^2+y^2)\,dx\,dy = \frac{3}{2}\int_0^1\left(\frac{x^3}{3}+xy^2\right)_{1/4}^{3/4} dy = \frac{3}{2}\int_0^1\left(\frac{9}{64}+\frac{3}{4}y^2 - \frac{1}{192}+\frac{1}{4}y\right)$

$dy = \frac{3}{2}\left[\frac{13}{96}y + \frac{1}{2}\frac{y^3}{3}\right]_0^1 = \frac{3}{2}\left[\frac{13}{96}(1)+\frac{1}{2}\left(\frac{1}{3}\right)\right] = \frac{29}{64}$

iv) $P(Y<\frac{1}{2}) = \frac{3}{2}\int_0^{1/2}\int_0^1 (x^2+y^2)\,dy\,dx = \frac{3}{2}\int_0^{1/2}\left(\frac{x^3}{3}+xy^2\right)_0^1 dy = \frac{3}{2}\int_0^{1/2}\left(\frac{1}{3}+y^2\right)dy$

$= \frac{3}{2}\left[\frac{1}{3}y+\frac{y^3}{3}\right]_0^{1/2} = \frac{3}{2}\left[\frac{1}{6}+\frac{1}{24}\right] = \frac{5}{16}$

$C\int_2^4 \left[6x - \frac{x^2}{2} - xy\right]_0^2 dy = 1 \Rightarrow C\int_2^4 (6(2) - 2 - 2y)\,dy = 1$

$\Rightarrow C\left[10y - \frac{2y^2}{2}\right]_2^4 = 1 \Rightarrow C\left[10(4-2) - 2\left(\frac{16}{2} - \frac{4}{2}\right)\right] = 8C = 1 \Rightarrow C = 1/8$

Q. $f(x,y) = \begin{cases} C(6-x-y), & 0<x<2,\ 2<y<4 \\ 0 & \text{otherwise} \end{cases}$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_2^4\int_0^2 C(6-x-y)\,dx\,dy = 1 \Rightarrow 6C\int_2^4 \left(-\frac{x^2}{2}\right)_0^2 dy = 1$

$\Rightarrow 6C \cdot -\frac{4}{2} \cdot -\frac{16}{2} + \frac{4}{2} = 1 \Rightarrow 6C \cdot -\frac{4}{2} \cdot -\frac{12}{2} = 1 \Rightarrow 8C = 1 \Rightarrow C = \frac{1}{8}$

i) $P(X<1, Y<3) = \frac{1}{8}\int_0^3\int_0^1 (6-x-y)\,dx\,dy$

ii) $P(X+Y<3)$

$= \frac{1}{8}\int_0^1\int_2^{3-x} (6-x-y)\,dy\,dx$

iii) $P(X<1 \mid Y>3)$