Example 12.5. Suppose X, Y are random variables whose joint PDF is given by

$$f(x,y)=\begin{cases}\frac{1}{y} & 0<y<1, 0<x<y\\ 0 & \text{otherwise}\end{cases}.$$

(a) Find the covariance of X and Y.
(b) Find $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.
(c) Find $\rho(X,Y)$.

Solution:

(a) Recall that $\mathrm{Cov}(X,Y)=\mathbb{E}XY-\mathbb{E}X\mathbb{E}Y$. So

$$\begin{aligned}\mathbb{E}XY &= \int_0^1\int_0^y xy\frac{1}{y}dxdy=\int_0^1\frac{y^2}{2}dy=\frac{1}{6},\\ \mathbb{E}X &= \int_0^1\int_0^y x\frac{1}{y}dxdy=\int_0^1\frac{y}{2}dy=\frac{1}{4},\\ \mathbb{E}Y &= \int_0^1\int_0^y y\frac{1}{y}dxdy=\int_0^1 ydy=\frac{1}{2}.\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}XY-\mathbb{E}X\mathbb{E}Y\\ &= \frac{1}{6}-\frac{1}{4}\frac{1}{2}\\ &= \frac{1}{24}.\end{aligned}$$

(b) We have that

$$\begin{aligned}\mathbb{E}X^2 &= \int_0^1\int_0^y x^2\frac{1}{y}dxdy=\int_0^1\frac{y^2}{3}dy=\frac{1}{9},\\ \mathbb{E}Y^2 &= \int_0^1\int_0^y y^2\frac{1}{y}dxdy=\int_0^1 y^2dy=\frac{1}{3}.\end{aligned}$$

Thus recall that

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}X^2-(\mathbb{E}X)^2\\ &= \frac{1}{9}-\left(\frac{1}{4}\right)^2=\frac{7}{144}.\end{aligned}$$

Also

$$\begin{aligned}\mathrm{Var}(Y) &= \mathbb{E}Y^2-(\mathbb{E}Y)^2\\ &= \frac{1}{3}-\left(\frac{1}{2}\right)^2=\frac{1}{12}.\end{aligned}$$

(c)

$$\rho(X,Y)=\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}=\frac{\frac{1}{24}}{\sqrt{\left(\frac{7}{144}\right)\left(\frac{1}{12}\right)}}\approx 0.6547.$$