Unit - 2

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EXPERIMENT

- DETERMINISTIC → know the outcome beforehand
- STOCHASTIC

A random variable (RV) is the one whose value is determined by the outcome of a random experiment

Also, a random function on a sample space S is a function $f : S \to R$ which assigns a real number $f(s)$ to each sample point s of S.

$R \to$ Real Number

2 types of R.V.'s

DISCRETE	CONTINUOUS
a variable whose value is obtained by counting	a variable whose value is obtained by measuring

If in a random experiment, a variable takes a finite set of values, it is called a discrete variable.

If it takes infinite no. of uncountable values, it is called a continuous variable.

Discrete Probability Distribution

Suppose a discrete random variable X is the outcome of an experiment. If $P(X = x_i) = p_i$, $i = 1, 2, 3, \ldots$

where (i) $p_i \geq 0 \ \forall i$

(ii) $\sum_i p_i = 1$

then $\langle x_i, p_i \rangle$ constitutes a probability distribution called DPD

Eg Toss 2 coins

HH HT TH TT

Define R.V. X to be the no. of heads

$X = x_i$	0	1	2
p_i	1/4	2/4	1/4

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Eg Roll 2 dice

X = sum of the 2 nos.

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
p_i	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

Eg The prob. dist^n of a R.V. X is given below

x_i	0	1	2	3	4	5	6	7
$P(x_i)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find k, evaluate $P(X<6)$, $P(3<X\leq 5)$

$\sum P(x_i) = 1$

$\Rightarrow 0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$

$\Rightarrow 10k^2+9k = 1$

$\Rightarrow k = 1/10, -1$

As $p_i \geq 0$

$k = 1/10$

$P(X<6) = P(0)+P(1)+P(2)+P(3)+P(4)+P(5)$

$= 0+0.1+0.2+0.2+0.3+0.01 = 0.81$

$P(3<X\leq 5) = P(4)+P(5)$

$= 0.3+0.01$

$= 0.31$

Distribution function

The distribution function $F(x)$ of the DRV X is defined by

$$F(x) = P(X\leq x) = \sum_{i=1}^{x} P(x_i)$$

where x is any integer between 1 & n

Continuous Random Variable (C.R.V.)

A C.R.V. can take infinite no. of possible values

Eg height, weight, time required, etc.

A C.R.V. is defined over an interval of values.

The probability of obtaining a single value is zero.

If $f(x)$ is a function such that,

(i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$,

then $f(x)$ defines a probability density function (pdf)

Also, in this case, the probability that X lies b/w a & b is given as

$$P(a<X<b) = \int_{x=a}^{b} f(x)\,dx = P(a\leq X\leq b)$$

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Distribution Function for a CRV

$$F(x) = P(X \le x) = P(-\infty \le X \le x)$$

$$= \int_{-\infty}^{x} f(x)\,dx$$

Eg Find the distribution function for the RV given by

$$f(x) = \begin{cases} cx^2, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

To find c

As $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_{0}^{3} f(x)\,dx = 1$

$\Rightarrow c\int_{0}^{3} x^2\,dx = 1$

$\Rightarrow \left. \frac{cx^3}{3} \right]_0^3 = 1$

$\Rightarrow 9c = 1$

$\Rightarrow c = 1/9$

If $x \le 0$

$F(x) = 0$

If $0 < x < 3$

$$F(x) = \int_0^x \frac{x^2}{9}\,dx$$

$$= \frac{1}{9}\left[\frac{x^3}{3}\right]_0^x$$

$$= \frac{x^3}{27}$$

If $x \geq 3$

$F(x) = 1$

$$F(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x^3/27, & \text{if } 0 < x < 3 \\ 1, & \text{if } x \geq 3 \end{cases}$$

HW. A box contains 12 balls, 3 white & 9 red. A sample of 3 balls is selected. Let X denote the no. of white balls in the sample. Find the distribution of X and its distribution function.

X.	0	1	2	3
P(x)	$\frac{21}{55}$	$\frac{27}{55}$	$\frac{27}{220}$	$\frac{1}{220}$
F(x)	$\frac{21}{55}$	$\frac{48}{55}$	$\frac{219}{220}$	1

$P(x=0) = P(0W, 3R)$

$= \frac{{}^9C_3}{{}^{12}C_3}$

$P(X=1) = P(1W, 2R) = \frac{{}^3C_1 \times {}^9C_2}{{}^{12}C_3}$

$P(x=2) = P(2W, 1R) = \frac{{}^3C_2 \times {}^9C_1}{{}^{12}C_3}$

$P(x=0) = P(3W, 0R) = \frac{{}^3C_3}{{}^{12}C_3}$

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MATHEMATICAL EXPECTATION

D.R.V. Consider a D.R.V. X and a function $\phi(X)$ of X.

Let $\{P(x_i)\}$ be a probability distribution of X

Then $E\{\phi(X)\} = \sum_i \phi(x_i)P(x_i)$ — (*)

is called the mathematical expectation of $\phi(x)$ in the distribution

Mean, Moments & Standard Derivation

Case I $\phi(X) = X$

(*) $\Rightarrow E\{X\} = \sum_i x_i P(x_i) = \mu$ — ①

$E\{X\}$ is called the mean of the distribution & is denoted by μ

Case II $\phi(X) = (X-a)^r$ where $a \in \mathbb{R}$ & $r \in \mathbb{Q}^+$

$E\{(X-a)^r\} = \sum_i (x_i - a)^r P(x_i)$ — ②

This is called the r^{th} moment about 'a' of the distribution & is denoted by $\mu_r(a)$

Special cases: $\mu_0(a) = \sum_i (x_i - a)^0 P(x_i)$

$= 1$

$\mu_1(a) = \sum_i (x_i - a) P(x_i)$

$= \sum x_i P(x_i) - a \sum P(x_i)$

$= \mu - a$

$\mu_1(0) = \mu - 0 = \mu$

Case (III) If $\phi(x) = (X-\mu)^r$.

then (*) $\Rightarrow E\{(X-\mu)^r\} = \sum_i (x_i - \mu)^r P(x_i)$ — ③

This is called the r^{th} central moment of the distribution and is denoted by μ_r

Also,

$\mu_0 = \sum (x_i-\mu)^0 P(x_i) = \sum P(x_i) = 1$

$\mu_1 = \sum (x_i - \mu) P(x_i) = \sum x_i P(x_i) - \mu \sum P(x_i)$

$= \mu - \mu = 0$

Variance & Standard Deviation

In ③, put $r = 2$

$\Rightarrow \mu_2 = \sum (x_i - \mu)^2 P(x_i) \rightarrow$ This is the 2^{nd} central moment of the distribution and is called the VARIANCE of the distribution.

VARIANCE is denoted by σ^2, V or $var(X)$.

A positive square root of the variance is called the Standard ~~Derivation~~ Deviation (σ)

Relation between Mean & Standard Deviation

$\sigma^2 = \sum (x_i - \mu)^2 P(x_i)$

$= \sum (x_i^2 - 2\mu x_i + \mu^2) P(x_i)$

$= \sum x_i^2 P x_i - 2\mu \sum x_i P(x_i) + \mu^2 \sum P(x_i)$

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$$= E\{x^2\} - 2\mu^2 + \mu^2 = E\{x^2\} - \mu^2$$

$$\sigma^2 = E\{x^2\} - [E\{x\}]^2$$

$$\therefore \sigma = \sqrt{E\{x^2\} - [E\{x\}]^2}$$

C.R.V Consider a C.R.V. X and a function $\phi(x)$ of X. Let $\{P(x_i)\}$ be a prob. dist. of X. Then

$$E\{\phi(x)\} = \int_{x=-\infty}^{\infty} \phi(x) P(x)\, dx \quad - \circledast$$

is called the mathematical expectation of $\phi(x)$ in the distribution.

Mean, Moments & Standard Deviation

Case I $\phi(x) = x$, $\circledast \Rightarrow E\{x\} = \int_{-\infty}^{\infty} x P(x)\, dx = \mu$ — ①

$E\{x\}$ is called the mean of the dist. & is denoted by μ

Case II $\phi(x) = (x-a)^r$, where $a \in \mathbb{R}$ & $r \in \mathbb{Q}^+$

then $\circledast \Rightarrow E\{(x-a)^r\} = \int_{-\infty}^{\infty} (x-a)^r P(x)\, dx = \mu_r(a)$ — ②

This is called the r^{th} moment about 'a' of the distribution & is denoted by $\mu_r(a)$.

Special cases: $\mu_0(a) = \int_{-\infty}^{\infty} (x-a)^0 P(x)\, dx$

$$= \int_{-\infty}^{\infty} P(x)\, dx = 1$$

$$\mu_1(a) = \int_{-\infty}^{\infty} (x-a)^1 P(x)\, dx$$

$$= \int_{-\infty}^{\infty} x P(x)\, dx - a\int_{-\infty}^{\infty} P(x)\, dx$$

$$= \mu - a$$

$$\mu_1(0) = \mu - 0 = \mu$$

Case (III): If $\phi(x) = (x-\mu)^r$

then $\circledast \Rightarrow E\{(x-\mu)^r\} = \int_{-\infty}^{\infty} (x-\mu)^r P(x)$ — ③

This is called the r^{th} central moment of the distribution & is denoted by μ_r.

Also,

$$\mu_0 = \int_{-\infty}^{\infty} (x-\mu)^0 P(x)\, dx = \int_{-\infty}^{\infty} P(x)\, dx = 1$$

$$\mu_1 = \int_{-\infty}^{\infty} (x-\mu) P(x)\, dx = \int_{-\infty}^{\infty} x P(x)\, dx - \mu\int_{-\infty}^{\infty} P(x)\, dx$$

$$= \mu - \mu = 0$$

Variance & Standard Deviation

In ③, put $r = 2$

$\Rightarrow \mu_2 = \int_{-\infty}^{\infty} (x-\mu)^2 P(x)$ → This is the 2^{nd} central moment of the distribution &

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is called the VARIANCE of the distribution.

VARIANCE is denoted by σ^2, V or $var(x)$.

A positive square root of the variance is called the standard deviation (S.D.) or (σ)

Relation b/w Mean & Standard Deviation

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 P(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 P(x)\, dx - 2\mu \int_{-\infty}^{\infty} x P(x)\, dx + \mu^2 \int_{-\infty}^{\infty} P(x)\, dx$$

$$= E\{x^2\} - 2\mu^2 + \mu^2$$

$$= E\{x^2\} - \mu^2$$

$$\sigma^2 = E\{x^2\} - \{E\{x\}\}^2 \Rightarrow \sigma = \sqrt{E\{x^2\} - [E\{x\}]^2}$$

Eg Find the mean & SD of the dist. from the table

x_i	-5	-4	1	2
$P(x_i)$	1/4	1/8	1/2	1/8

$$\text{Mean}(\mu) = \sum x_i P(x_i)$$

$$= -5(1/4) - 4(1/8) + 1/2 + 2(1/8)$$

$$= -1$$

$$E\{x^2\} = \sum x_i^2 P(x_i)$$

$$= \frac{(-5)^2}{4} + \frac{(-4)^2}{8} + \frac{1}{2} + \frac{(2)^2}{8}$$

$$= \frac{37}{4}$$

$$\therefore \sigma^2 = E\{x^2\} - [E\{x\}]^2$$

$$= \frac{37}{4} - (-1)^2$$

$$= \frac{37}{4} - 1 = \frac{33}{4}$$

$$\sigma = \frac{\sqrt{33}}{2}$$

Eg A R.V. X has the pdf given by $f(x) = \begin{cases} ke^{-3x}, & x > 0 \\ 0, & x \le 0 \end{cases}$

Find i) k ii) $P(1 \le x \le 3)$ iii) $P(x \ge 0.5)$

iv) Mean of the dist. v) S.D. of the dist.

Ans As $\int_{-\infty}^{\infty} P(x_i)\, dx = 1$

$$\Rightarrow \int_0^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \int_0^{\infty} ke^{-3x}\, dx = 1$$

$$\Rightarrow k\int_0^{\infty} e^{-3x}\, dx = 1$$

$$\Rightarrow k\left[\frac{e^{-3x}}{-3}\right]_0^{\infty} = 1$$

$$\Rightarrow k\left[0 + \frac{1}{3}\right] = 1$$

$$\Rightarrow \frac{k}{3} = 1$$

$$\Rightarrow k = 3$$

ii) $P(1 \le x \le 3)$

$$= \int_1^3 3e^{-3x}\, dx$$

$$= 3\int_1^3 e^{-3x}\, dx$$

$$= 3\left[\frac{e^{-3x}}{-3}\right]_1^3$$

$$= 3\left[\frac{e^{-9}}{-3} - \frac{e^{-3}}{-3}\right]$$

$$= -1(e^{-9} - e^{-3})$$

$$= e^{-3} - e^{-9}$$

$$= 0.04966$$

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iii) $P(x \geq 0.5)$

$$= \int_{0.5}^{\infty} 3e^{-3x}\,dx$$

$$= 3\left[\frac{e^{-3x}}{-3}\right]_{0.5}^{\infty}$$

$$= \frac{3}{-3}\left[0 - e^{-1.5}\right]$$

$$= e^{-1.5}$$

iv) Mean of the dist

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{0}^{\infty} x(3e^{-3x})\,dx$$

$$= 3\int_{0}^{\infty} xe^{-3x}\,dx$$

$$= 3\left[x\frac{e^{-3x}}{-3} - \frac{e^{-3x}}{(-3)^2}\right]_0^{\infty}$$

$$= 3\left[0 - 0 - \left(\frac{0}{-3} - \frac{1}{9}\right)\right]$$

$$= 3\left[\frac{1}{9} - \frac{0}{-3}\right]$$

$$= \frac{1}{3} + 0 = \frac{1}{3}$$

v) $\sigma^2 = E\{x^2\} - \left[E\{x\}\right]^2$

$$E\{x^2\} = 3\int_0^{\infty} x^2 e^{-3x}\,dx$$

$$= 3\left[x^2\frac{e^{-3x}}{-3} - 2x\frac{e^{-3x}}{(-3)^2} + 2\frac{e^{-3x}}{(-3)^3}\right]_0^{\infty}$$

$$= 3\left[0 - 0 + 0 - \left(0 - 0 + \frac{2}{-27}\right)\right]$$

$$= \frac{+2}{9}$$

$$\sigma^2 = \frac{+2}{9} - \left[\frac{1}{3}\right]^2$$

$$= \frac{+2}{9} - \frac{1}{9}$$

$$= \frac{1}{9}$$

$$\sigma = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

$$\left[\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty}\frac{x}{e^x} \overset{\text{By L'H rule}}{=} \lim_{x\to\infty}\frac{1}{e^x} = 0\right]$$

Eg A R.V. X has the pdf given by $f(x) = \frac{k}{1+x^2}$, $-\infty < x < \infty$. Find (i) k (ii) $P(x \geq 0)$ iii) $P(0 < x < 1)$

iv) Mean

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As $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_{-\infty}^{\infty} \frac{k}{1+x^2}\,dx = 1$

$\Rightarrow k\left[\tan^{-1}x\right]_{-\infty}^{\infty} = 1$

$\Rightarrow k\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right] = 1$

$\Rightarrow k\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$

$\Rightarrow k\pi = 1$

$\Rightarrow k = 1/\pi$

ii) $P(x \geq 0)$

$= \int_0^{\infty} \frac{1}{\pi(1+x^2)}\,dx$

$= \frac{1}{\pi}\left[\tan^{-1}x\right]_0^{\infty}$

$= \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right]$

$= \frac{1}{2}$

iii) $P(0 < x < 1)$

$= \frac{1}{\pi}\int_0^1 \frac{1}{1+x^2}\,dx$

$= \frac{1}{\pi}\left[\tan^{-1}x\right]_0^1$

$= \frac{1}{\pi}\left[\frac{\pi}{4} - 0\right]$

$= \frac{1}{4}$

iv) Mean

$\mu = \int_{-\infty}^{\infty} x\,\frac{1}{\pi(1+x^2)}\,dx$

$= \frac{1}{\pi}\int_0^{\infty} \frac{x}{1+x^2}\,dx$

$= \frac{1}{2\pi}\int_1^{\infty} \frac{dt}{t} = \frac{1}{2\pi}\left[\log t\right]_1^{\infty} =$

Q. A R.V X has the pdf given by $f(x) = \begin{cases} cxe^{-x/2}, & 0<x<\infty \\ 0, & \text{elsewhere} \end{cases}$

Find (i) c (ii) $P(2 \leq x \leq 4)$ (iii) $P(x \geq 0.5)$

(iv) Mean of the dist. (v) S.D. of the dist

As $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow c\int_0^{\infty} xe^{-x/2}\,dx = 1$

$\Rightarrow c\left[x\frac{e^{-x/2}}{(-1/2)} - \frac{e^{-x/2}}{(-1/2)^2}\right]_0^{\infty} = 1$

$\Rightarrow c[4] = 1 \Rightarrow c = 1/4$

iv) Mean $= \frac{1}{4}\int_0^{\infty} x^2 e^{-x/2}$

$= \frac{1}{4}\left[x^2\frac{e^{-x/2}}{-1/2} - 2x\frac{e^{-x/2}}{(-1/2)^2} + 2\frac{e^{-x/2}}{(-1/2)^3}\right]_0^{\infty}$

$= \frac{1}{4}[16] = 4$

ii) $P(2 \leq x \leq 4)$

$= \int_2^4 f(x)\,dx$

$= \frac{1}{4}\int_2^4 xe^{-x/2}\,dx = \frac{1}{4}\left[x\frac{e^{-x/2}}{-1/2} - \frac{e^{-x/2}}{(-1/2)^2}\right]_2^4$

$= \frac{1}{4}\left[-8e^{-2} - 4e^{-2} - (-4e^{-1} - 4e^{-1})\right]$

$= \frac{1}{4}\left[-12e^{-2} + 8e^{-1}\right] = 0.32975$

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iii)

$P(x \geq 0.5)$

$$= \int_{0.5}^{\infty} f(x)\,dx$$

$$= \frac{1}{4}\int_{0.5}^{\infty} x e^{-x/2}\,dx$$

$$= \frac{1}{4}\left[x\frac{e^{-x/2}}{-1/2} - \frac{e^{-x/2}}{(-1/2)^2}\right]_{0.5}^{\infty}$$

$$= \frac{1}{4}\left[0 - 0 - \left(-1e^{-1/4} - 4e^{-1/4}\right)\right]$$

$$= \frac{5}{4}e^{-1/4} = 0.9735$$

v) $SD^2 = E\{X^2\} - \{E\{X\}\}^2$

$$= \frac{1}{4}\int_{0}^{\infty} x^3 e^{-x/2}\,dx - 4^2$$

$$= \frac{1}{4}\left[x^3\frac{e^{-x/2}}{-1/2} - 3x^2\frac{e^{-x/2}}{(-1/2)^2} + 6x\frac{e^{-x/2}}{(-1/2)^3} - 6\frac{e^{-x/2}}{(-1/2)^4}\right]_0^{\infty} - 16$$

$$= \frac{1}{4}\left[0 - (-96)\right] - 16$$

$$= 24 - 16$$

$$= 8$$

$\therefore SD = \sqrt{8} = 2\sqrt{2}$

Eg. The diameter of an electric cable is a CRV with $f(x) = 6x(1-x)$, $0 \leq x \leq 1$ & 0 elsewhere

Verify that it defines a pdf.

Find its μ & σ.

Eg. If $f(x) = \frac{c}{2}\begin{cases} \sin x, & 0 \leq x \leq \pi \\ 0, & \text{elsewhere} \end{cases}$

Find (i) c (ii) μ & σ (iii) $P(0 \leq x \leq \pi/2)$

Answers

1. $f(x) = 6x(1-x)$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 6x(1-x)\,dx$$

$$= 6\int_0^1 (x - x^2)\,dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 6\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 6 \times \frac{1}{6} = 1$$

Now $f(x) \geq 0$ for every value of x and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$\therefore$ It defines a p.d.f.

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= 6\int x^2(1-x)\, dx$$

$$= 6\int_0^1 (x^2 - x^3)\, dx$$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= 6\left[\frac{1}{3} - \frac{1}{4}\right]$$

$$= 6 \times \frac{1}{12} = \frac{1}{2}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= 6\int_0^1 (x^3 - x^4)\, dx$$

$$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$$

$$= 6\left[\frac{1}{4} - \frac{1}{5}\right]$$

$$= \frac{6}{20} = \frac{3}{10}$$

$$\sigma = \sqrt{\frac{3}{10} - \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{3}{10} - \frac{1}{4}} = \frac{\sqrt{5}}{10} = \frac{1}{2\sqrt{5}}$$

ii) $f(x) = \begin{cases} c \sin x, & 0 \le x \le \pi \\ 0, & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \int_0^{\pi} c \sin x\, dx = 1$$

$$\Rightarrow c\int_0^{\pi} \sin x\, dx = 1$$

$$\Rightarrow c\left[-\cos x\right]_0^{\pi} = 1$$

$$\Rightarrow c\left[1 + 1\right] = 1$$

$$\Rightarrow c = \frac{1}{2}$$

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \frac{1}{2}\int_0^{\pi} x \sin x\, dx$$

$$= \cancel{\frac{1}{2}\int x \sin x} = \frac{1}{2}\left[x(-\cos x) - (-\sin x)\right]_0^{\pi}$$

$$= \frac{1}{2}\left[\pi - 0 - (0 - 0)\right]$$

$$= \pi/2$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \frac{1}{2}\int_0^{\pi} x^2 \sin x\, dx$$

$$= \frac{1}{2}\left[x^2(-\cos x) - 2x(-\sin x) + 2\cos x\right]_0^{\pi}$$

$$= \frac{1}{2}\left[\pi^2 - 0 - 2 - 2\right]$$

$$= \frac{\pi^2 - 4}{2} = 2.9348$$

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$$\sigma = \sqrt{\frac{\pi^2-4}{2} - \frac{\pi^2}{4}}$$

$$= \sqrt{\frac{2\pi^2-8-\pi^2}{4}}$$

$$= \sqrt{\frac{\pi^2-8}{4}} = \frac{\sqrt{\pi^2-8}}{2} = 0.6837$$

iii) $P(0 \le x \le \pi/2)$

$$= \frac{1}{2}\int_0^{\pi/2} \sin x \, dx$$

$$= \frac{1}{2}\left[-\cos x\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[0+1\right]$$

$$= \frac{1}{2}$$

Eg The p.d.f of x given by $P(x) = y_0 e^{-|x|}$, $-\infty < x < \infty$

Find y_0, μ & σ of the dist.

$$P(x) = y_0 e^{-|x|}$$

$$|x| = \begin{cases} x, & 0 < x \\ -x, & x < 0 \end{cases}$$

As $\int_{-\infty}^{\infty} P(x)\,dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} y_0 e^{-|x|} dx = 1$$

$$\Rightarrow y_0 \int_{-\infty}^{0} \cancel{y_0} e^{x} dx + y_0 \int_0^{\infty} e^{-x} dx = 1$$

$$\Rightarrow y_0 \left[e^{x}\right]_{-\infty}^{0} + y_0 \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$\Rightarrow y_0 \left[1 - 0 + 0 + 1\right] = 1$$

$$\Rightarrow y_0 = 1/2$$

Mean, $\mu = \int_{-\infty}^{\infty} x P(x)\,dx$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x e^{-|x|} dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} x e^{x} dx + \int_0^{\infty} x e^{-x} dx\right]$$

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$$= \frac{1}{2}\left[\left[xe^{x}-e^{x}\right]_{-\infty}^{0}+\left[x\frac{e^{-x}}{-1}-\frac{e^{-x}}{(-1)^2}\right]_{0}^{\infty}\right]$$

$$= \frac{1}{2}\left[\left[0-1-(0-0)\right]+\left[0-0-(0-1)\right]\right]$$

$$= \frac{1}{2}(+0) = 0$$

$$E\{x^2\} = \int_{-\infty}^{\infty} x^2 p(x)\,dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} x^2 e^{x}\,dx + \int_{0}^{\infty} x^2 e^{-x}\,dx\right]$$

$$= \frac{1}{2}\left[\left[x^2e^{x}-2xe^{x}+2e^{x}\right]_{-\infty}^{0} + \left[x^2\frac{e^{-x}}{-1}-2x\frac{e^{-x}}{(-1)^2}+2\frac{e^{-x}}{(-1)^3}\right]_{0}^{\infty}\right]$$

$$= \frac{1}{2}\left[\left[2-0\right]+\left[0-(-2)\right]\right]$$

$$= \frac{4}{2} = 2$$

$$\sigma^2 = E\{x^2\} - \left[E\{x\}\right]^2$$

$$= 2 - 0^2$$

$$= 2$$

$$\sigma = \sqrt{2}$$

Moment Generating Function (M.G.F)

For a D.R.V.

The m.g.f. of a DRV X about the point $x=a$ is defined as the expected value of $e^{t(x-a)}$ & is denoted by $M_a(t)$

$$\Rightarrow M_a(t) = E\{e^{t(x-a)}\}$$

$$= \sum_i e^{t(x_i-a)}P(x_i), \quad t - \text{parameter} \quad — (1)$$

$$= \sum p_i \left\{1 + t(x_i-a) + \frac{t^2(x_i-a)^2}{2!} + \dots\right\}$$

$$= \sum p_i + t\sum p_i(x_i-a) + \frac{t^2}{2!}\sum p_i(x_i-a)^2 + \dots$$

$$= 1 + t\mu_1(a) + \frac{t^2}{2!}\mu_2(a) + \dots + \frac{t^r}{r!}\mu_r(a) + \dots \quad — (2)$$

$M_a(t)$ generates moments $\{\mu_r(a)\}$ & that is why it is called M.G.F.

(2) $\Rightarrow \mu_r(a) =$ coefficient of $\frac{t^r}{r!}$ in the expression of $M_a(t)$

Also, ② ⟹ $\mu_n(a) = \left[\frac{d^n}{dt^n} M_a(t)\right]_{t=0}$ — ③

∴ The moment about any point $x = a$ can be found using ③

Also ① ⟹ $M_a(t) = e^{-at} \sum_i e^{tx_i} p_i$

$= e^{-at} M_0(t)$

⟹ m.g.f. about a point $x = a$ = $e^{-at} \times$ mgf. about the origin

For Continuous R.V.'s

If $f(x)$ is the density function of a CRV X, then the m.g.f. of X about $x = a$ is given by

$$M_a(t) = \int_{-\infty}^{\infty} e^{t(x-a)} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-at} e^{tx} f(x)\, dx$$

$$= e^{-at} \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

Eg. Find the M.G.F. about the origin of the exponential distribution given by $f(x) = \frac{1}{c} e^{-x/c}$, $0 \le x < \infty$, $c > 0$. Hence find its mean & SD

$$M_0(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \frac{1}{c} \int_0^{\infty} e^{tx} e^{-x/c}\, dx$$

$$= \frac{1}{c} \int_0^{\infty} e^{-(\frac{1}{c} - t)x}\, dx$$

$$= \frac{1}{c} \left[\frac{e^{-(\frac{1}{c}-t)x}}{-(\frac{1}{c}-t)}\right]_0^{\infty}$$

$$= \frac{1}{c}\left[0 - \frac{1}{-(\frac{1}{c}-t)}\right]$$

$$= \frac{1}{c} \times \frac{c}{1-tc} = \frac{1}{1-ct}$$

$$= (1-ct)^{-1}$$

$$= 1 + ct + c^2t^2 + \dots + c^n t^n + \dots$$

$$\mu_1'(0) = \left[\frac{d}{dt}(M_0(t))\right]_{t=0}$$

$$= \left[0 + c + 2c^2 t + \dots\right]_{t=0} = c$$

$$\mu_2(0) = \left[\frac{d^2}{dt^2}(M_0(t))\right]_{t=0}$$

$$= \left[2c^2 + 6c^3t + \dots\right]_{t=0}$$

$$= 2c^2$$

$$\sigma = \sqrt{E\{x^2\} - [E\{x\}]^2}$$

$$= \sqrt{\mu_2(0) - (\mu_1(0))^2}$$

$$= \sqrt{2c^2 - c^2} = \sqrt{c^2} = c$$

Probability Distributions

A probability distribution gives the probabilities of events in an experiment.

BINOMIAL DISTRIBUTION

A B.D. gives the discrete probability distribution of obtaining exactly x successes out of n trials.

Let n be a +ve integer & $p \in R$, $0 \le p < 1$

Then $B(n, p, x) = {}^nC_x \, p^x q^{n-x}$

gives the probability of obtaining x successes in n trials $\left(\begin{matrix} p+q=1 \\ q = 1-p \end{matrix}\right)$ and is known as Binomial distribution.

It can be verified that

(i) $B(n, p, x) \geq 0$

(ii) $\sum_{x=0}^{n} B(n, p, x) = 1$

For (ii) $LHS = \sum_{x=0}^{n} B(n, p, x)$

$$= {}^nC_0 \, p^0 q^n + {}^nC_1 \, p q^{n-1} + {}^nC_2 \, p^2 q^{n-2}$$

$$+ \dots + {}^nC_{n-1} \, p^{n-1} q + {}^nC_n \, p^n$$

$$= (p+q)^n$$

$$= 1^n = 1$$

Eg ~~Out of~~ In 10 tosses of a fair coin, what is the probability of getting (i) 5H, 5T (ii) 7H, 3T (iii) atmost 2 H (iv) atleast 3 heads (v) 6 or more heads.

Ans Given: $n = 10$, ~~p~~

Let X = no. of heads

$p = \frac{1}{2}$, $q = 1 - \frac{1}{2} = \frac{1}{2}$

(i) $P(5H) = B(10, \frac{1}{2}, 5)$

$$= {}^{10}C_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^5$$

$$= 0.2461$$

ii) $P(7H) = {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 = 0.1172$

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iii) $P(X \le 2) = P(0H) + P(1H) + P(2H)$

$= {}^{10}C_0 \left(\frac{1}{2}\right)^{10} + {}^{10}C_1 \left(\frac{1}{2}\right)^{10} + {}^{10}C_2 \left(\frac{1}{2}\right)^{10}$

$= 0.0547$

iv) $P(X \ge 3) = 1 - P(X \le 2)$

$= 1 - 0.0547$

$= 0.9453$

v) $P(X \ge 6) = P(6H) + P(7H) + P(8H) + P(9H) + P(10H)$

$= \left({}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right) \times \frac{1}{2}^{10}$

$= 0.3769$

Now, for a biased coin where a heads appears 3 times as much as tails.

$\left[\begin{array}{l} p = 3x,\ q = x \\ p+q = 1 \\ 4x = 1 \\ x = 1/4 \end{array}\right]$

then $p = \frac{3}{4}$, $q = 1 - \frac{3}{4} = \frac{1}{4}$

i) $P(5H) = {}^{10}C_5 \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^5$

$= 0.0584$

ii) $P(7H) = {}^{10}C_7 \left(\frac{3}{4}\right)^7 \left(\frac{1}{4}\right)^3$

$= 0.2503$

iii) $P(X \le 2) = 0.0004158$

iv) $P(X \ge 3) = 1 - 0.0004158 = 0.9995842$

v) $P(X \ge 6) = 0.92187$

Mean & Variance of Binomial Distribution

$\mu = E\{x\} = \sum_{x=0}^{n} x\, B(n,p,x)$

$= \sum x\, {}^nC_x\, p^x q^{n-x}$

$= {}^nC_1\, p q^{n-1} + 2\, {}^nC_2\, p^2 q^{n-2} + 3\, {}^nC_3\, p^3 q^{n-3} + \dots + n\, {}^nC_n\, p^n$

$= npq^{n-1} + 2\frac{n(n-1)}{2!} p^2 q^{n-2} + 3\frac{n(n-1)(n-2)}{3!} p^3 q^{n-3} + \dots + np^n$

$= np\left[q^{n-1} + (n-1)pq^{n-2} + \frac{(n-1)(n-2)}{2!} p^2 q^{n-3} + \dots + p^{n-1}\right]$

$= np(p+q)^{n-1}$

$= np \qquad [\because p+q = 1]$

Variance of Binomial Dist.

$\sigma^2 = E\{x^2\} - [E\{x\}]^2$

Now $E\{x^2\} = \sum x^2\, B(n,p,x)$

$= \sum \{x(x-1) + x\}\, {}^nC_x\, p^x q^{n-x}$

$= \sum x(x-1)\, {}^nC_x\, p^x q^{n-x} + \sum x\, {}^nC_x\, p^x q^{n-x}$

$= 2\, {}^nC_2 p^2 q^{n-2} + 3(2)\, {}^nC_3 p^3 q^{n-3}$

$+ \dots + n(n-1)\, {}^nC_n p^n + \mu$

$= 2 \frac{n(n-1)}{2!} p^2 q^{n-2} + 3(2) \frac{n(n-1)(n-2)}{3!} p^3 q^{n-3}$

$+ \dots + n(n-1)p^n + \mu$

$= \frac{n(n-1)}{0!} p^2 q^{n-2} + \frac{n(n-1)(n-2)p^3 q^{n-3}}{1!}$

$+ \dots + n(n-1)p^n + \mu$

$= \cancel{np^2} \; n(n-1)p^2 \left[q^{n-2} + (n-2)pq^{n-3} + \dots + p^{n-2} \right]$

$+ \mu$

$= n(n-1)p^2 (p+q)^{n-2} + \mu$

$= n(n-1)p^2 + np$

$\cancel{np}$

$\therefore \sigma^2 = n(n-1)p^2 + np - n^2p^2$

$= n^2p^2 - np^2 + np - n^2p^2$

$= np - np^2$

$= np(1-p)$

$= npq \qquad \Rightarrow \sigma = \sqrt{npq}$

In sampling a large no of parts produced by a machine, the mean no of defectives in sample of 20 is 2. Out of 1000 such samples, how many could be expected to contain i) no defective part ii) at least 3 defective parts. iii) At most 1 defective part

Given, $\mu = 2$, $n = 20$

$p = \frac{\mu}{n} = \frac{2}{20} = \frac{1}{10}$.

$q = 1 - \frac{1}{10} = \frac{9}{10}$

i) $P(x=0) = {}^{20}C_0 \left(\frac{9}{10}\right)^{20}$

$= 0.1216$

Out of 1000 samples, 122 samples will have no defectives.

ii) $P(x \geq 3) = 1 - P(x=0) - P(x=1) - P(x=2)$

$= 1 - {}^{20}C_1 \left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{19} - {}^{20}C_2 \left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{18} - {}^{20}C_0 \left(\frac{9}{10}\right)^{20}$

$= 0.3231$

Out of 1000 samples, 323 samples will have at least 3 defectives.

iii) $P(x \leq 1) = P(x=0) + P(x=1)$

$= {}^{20}C_0 \left(\frac{9}{10}\right)^{20} + {}^{20}C_1 \left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{19}$

$= 0.3917$ Out of 1000 samples, 392 samples will have at most 1 defective

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Eg The probability that an entering student will graduate in 4 years is 0.4. Determine the probability that out of 5 students

i) none $= {}^5C_0 (0.6)^5 = \cancel{0.0060466}\ 0.07776$

ii) one $= {}^5C_1 (0.4)(0.6)^4 = \cancel{0.020155}\ 0.2592$

iii) at least 1 will graduate. $= 1 - {}^5C_0 (0.6)^5 = \cancel{0.99395}\ 0.92224$

Eg In a consignment of items, 5% are defective. A random sample of 8 items is taken for inspection. What is the probability that it has 1 or more defective?

$p = 5\% = \frac{5}{100}$ $\quad n = 8$

$q = 1 - \frac{5}{100} = \frac{95}{100}$

$P(X \geq 1) = 1 - P(X=0) \cancel{- P(X=1)}$

$= 1 - {}^8C_0 \left(\frac{95}{100}\right)^8 \cancel{- {}^8C_1 \left(\frac{5}{100}\right)\left(\frac{95}{100}\right)^7}$

$= \cancel{0.05724}\ 0.3366$

Eg If the probability of success is 0.01, then how many trials are necessary in order that probability of atleast one success is 0.5 or more?

Ans $p = 0.01,\ q = 0.99$

$n = ?\quad P(X \geq 1) \geq 0.5$

$\Rightarrow 1 - q^n \geq 0.5$

$\Rightarrow 1 - (0.99)^n \geq 0.5$

$\Rightarrow 1 - 0.5 \geq 0.99^n$

$\Rightarrow 0.5 \geq 0.99^n$

$\Rightarrow n \leq \log_{0.99} 0.5$

$\Rightarrow n \leq 69$

At least 69 trials are necessary

Eg In a bombing action, there is a 50% chance that any bomb will strike the target. Two direct hits are needed to destroy the target completely. How many bombs are required to be dropped to have 99% chance or better to of completely destroying the target?

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Ans. $p = 0.5$, $q = 0.5$, $n = ?$

$$P(x \geq 2) \geq 0.99$$

$$1 - P(x=0) - P(x=1) \geq 0.99$$

$$1 - 0.5^n - n(0.5)^n \geq 0.99$$

$$1 - 0.99 \geq 0.5^n (n+1)$$

$$0.01 \geq 0.5^n (n+1)$$

On solving, we get $n \approx 11$

Q. In a B.P., if mean is 6 and s.d. is 2, find the total no of trials & the prob. of success in each trial

Ans $\mu = 6$, $\sigma = 2$

$np = 6$, $npq = \sigma^2 = 4$

$$q = \frac{npq}{np} = \frac{4}{6} = \frac{2}{3}$$

$$p = 1 - \frac{2}{3} = \frac{1}{3}$$

$$n \times \frac{1}{3} = 6$$

$\Rightarrow n = 18$

Q. In a biased coin in which heads appear 4 times as much as tails, find the prob of obtaining the foll in 10 tosses i) 5H, 5T, ii) 8H, 2T iii) At least 3H

Ans $p + q = 1$

$4x + x = 1$

$5x = 1$

$x = 1/5$

$p = 4/5$, $q = 1/5$

i) $P(5H, 5T) = {}^{10}C_5 \left(\frac{4}{5}\right)^5 \left(\frac{1}{5}\right)^5$

$= 0.02642$

ii) $P(8H, 2T) = {}^{10}C_8 \left(\frac{4}{5}\right)^8 \left(\frac{1}{5}\right)^2$

$= 0.30199$

iii) $P(\text{at least } 3H) = 1 - P(0H) - P(1H) - P(2H)$

$= 1 - {}^{10}C_0 \left(\frac{1}{5}\right)^{10} - {}^{10}C_1 \frac{4}{5}\left(\frac{1}{5}\right)^9 - {}^{10}C_2 \left(\frac{4}{5}\right)^2 \left(\frac{1}{5}\right)^8$

$= 0.99992$

Eg

Q. The ~~probability~~ cases of twin births is are 1 out of 60. There are 20 births in a hospital in a single day. Find the probability that there are 2 or more twin births on a given day.

~~The~~ Ans $p = \frac{1}{60}$, $n = 20$

$P(x \geq 2) = 1 - P(x=0) - P(x=1)$

$= 1 - {}^{20}C_0 \left(\frac{59}{60}\right)^{20} - {}^{20}C_1 \left(\frac{1}{60}\right)\left(\frac{59}{60}\right)^{19}$

$= 0.04327$

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Poisson Distribution

In Binomial dist, $B(n,p,x) = {}^nC_x\, p^x (1-p)^{n}$

with $\mu = np$

If n becomes very large & p is very small such that μ is finite, then

$$B(n,p,x) \approx \frac{e^{-\mu}\mu^x}{x!} \quad \text{—} \circledast$$

$\circledast$ is gives the expression for Poisson Dist.

$\circledast$ defines a prob dist:

(i) $\frac{e^{-\mu}\mu^x}{x!} \geq 0$

(ii) $\sum_x P(\mu,x) = \sum_x \frac{e^{-\mu}\mu^x}{x!}$

$$= e^{-\mu}\sum \frac{\mu^x}{x!}$$

$$= e^{-\mu}\left[1 + \frac{\mu}{1!} + \frac{\mu^2}{2!} + \cdots\right]$$

$$= e^{-\mu}\left[e^{\mu}\right]$$

$$= 1$$

Mean & Variance of P.D.

Mean:

$$\mu = E[x] = \sum_x x\, P(\mu,x)$$

$$= \sum x \frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum \frac{e^{-\mu}\mu^x}{(x-1)!}$$

$$= e^{-\mu}\sum \frac{\mu^x}{(x-1)!}$$

$$= e^{-\mu}\mu \sum \frac{\mu^{x-1}}{(x-1)!}$$

$$= \mu e^{-\mu}\left[1 + \frac{\mu}{1!} + \frac{\mu^2}{2!} + \cdots\right]$$

$$= \mu e^{-\mu} e^{\mu}$$

$$= \mu$$

Variance $\sigma^2 = E\{x^2\} - \{E[x]\}^2$

$$E\{x^2\} = \sum_{x=0}^{\infty} x^2 \frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum_{x=0}^{\infty} [x(x-1)+x]\frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\mu}\mu^x}{x!} + \sum_{x=0}^{\infty} x\frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\mu}\mu^x}{(x-2)!} + \sum_{x=1}^{\infty} \frac{e^{-\mu}\mu^x}{(x-1)!}$$

$$= \mu^2 \sum_{x=2}^{\infty} \frac{e^{-\mu}\mu^{x-2}}{(x-2)!} + e^{-\mu}\mu \sum_{x=1}^{\infty} \frac{\mu^{x-1}}{(x-1)!}$$

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$= \mu^2 [e^{-\mu}[1+\mu+\frac{\mu^2}{2!}+\cdots]] + \mu e^{-\mu} e^{\mu}$

$= \mu^2 e^{-\mu} e^{\mu} + \mu$

$= \mu^2 + \mu$

$\sigma^2 = \mu^2 + \mu - \mu^2 = \mu$

$\sigma = \sqrt{\mu}$

Eg A Poisson variate is such that $P(x=3) = P(x=4)$. Find the prob. $P(x=2)$

As $P(\mu, x) = \dfrac{e^{-\mu}\mu^x}{x!}$

Given $P(x=3) = P(x=4)$

$\Rightarrow \dfrac{e^{-\mu}\mu^3}{3!} = \dfrac{e^{-\mu}\mu^4}{4!}$

$\Rightarrow 1 = \dfrac{\mu}{4} \Rightarrow \mu = 4$

$P(x=2) = \dfrac{e^{-\mu}\mu^2}{2!} = \dfrac{e^{-4}(4)^2}{2!}$

$= 0.1465$

Eg A Poisson variate is such that $P(x=2) = 9P(x=4) + 90P(x=6)$.

Find the mean & variance of the dist.

As $P(\mu, x) = \dfrac{e^{-\mu}\mu^x}{x!}$

Given $P(x=2) = 9P(x=4) + 90P(x=6)$

$\dfrac{e^{-\mu}\mu^2}{2!} = \dfrac{9e^{-\mu}\mu^4}{4!} + \dfrac{90e^{-\mu}\mu^6}{6!}$

$1 = \dfrac{9\mu^2}{4\times 3} + \dfrac{90\mu^4}{6\times5\times4\times3}$

$\Rightarrow \dfrac{270\mu^2 + 90\mu^4}{6\times5\times4\times3} = 1$

$\Rightarrow 270\mu^2 + 90\mu^4 = 6\times5\times4\times3$

$\Rightarrow 3\mu^2 + \mu^4 = 4$

$\Rightarrow \mu = 1$

Eg In a factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. Use Poisson dist. to calculate the approx. no. of packets containing (i) no defective (ii) one defective blades in a consignment of 10000 packets.

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Ans. $\mu = np = 10 \times 0.002$

$= 0.02$

$$P(x=0) = \frac{e^{-0.02}(0.02)^0}{0!}$$

$= 0.9802$

Out of 10000 packets, approx. 9802 packets will have no defective ~~part~~ blades.

$$P(x=1) = \frac{e^{-0.02}(0.02)^1}{1!}$$

$= 0.0196$

Out of 10000 packets, approx. 196 packets will have one defective blade.

Eg A car hire firm has 2 cars which it rents out day by day. The demand for a car on each day follows Poisson distribution with mean 1.5. Calculate the proportion of days on which

i) there is no demand.

ii) demand is refused

$\mu = 1.5$

i) $P(X=0) = \frac{e^{-1.5}(1.5)^0}{0!} = 0.2231$

ii) $P(x>2) = 1 - P(x=0) - P(x=1) - P(x=2)$

$= 1 - e^{-1.5} - e^{-1.5} \times 1.5 - \frac{e^{-1.5} \times 1.5^2}{2}$

$= 0.19115$

$\simeq 0.1912$

Eg If the probability of a bad reaction from a certain vaccine is 0.001, determine the chance that out of 2000 individuals, more than 2 will have a bad reaction.

$n = 2000, \; p = 0.001$

$\mu = 2000 \times 0.001$

$= 2$

~~$P(x \geq 2) = \frac{e^{-\mu}\mu^2}{2!} = \frac{e^{-2}(2)^2}{2!}$~~

$P(x>2) = 1 - P(x=0) - P(x=1) - P(x=2)$

$= 1 - e^{-2} - e^{-2} \times 2 - \frac{e^{-2} \times 2^2}{2!}$

$= 0.3233$

Eg The no. of monthly breakdowns of a computer is a R.V. having Poisson distribution with mean equal to 1.8. Find the probability that this computer will work for a month

(i) without breakdown (ii) with only one breakdown

(iii) with at least one breakdown (iv) with atmost 1 breakdown

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Eg In sampling a large no of parts made by a Machine, the mean no. of defectives in a sample of 20 is 2. Out of 2000 such samples, how many would be expected to contain at least 3 defective parts ?

Answers

1. $\mu = 1.8$

i) $P(x=0) = \frac{e^{-1.8}(1.8)^0}{0!}$

$= 0.1653$

ii) $P(x=1) = e^{-1.8} \times 1.8$

$= 0.2975$

iii) $P(x \geq 1) = 1 - P(x=0)$

$= 1 - e^{-1.8}$

$= 0.8347$

iv) $P(x \leq 1) = P(x=0) + P(x=1)$

$= e^{-1.8} + e^{-1.8} \times 1.8$

$= 0.4628$

2. $\mu = 2, \quad n = 20$

$P(X \geq 3) = 1 - P(x=0) - P(x=1) - P(x=2)$

$= 1 - e^{-2} - e^{-2}(2) - \frac{e^{-2}(2)^2}{2!}$

$= 0.3233$

Prove that Poisson distribution is a limiting case of Binomial Distribution

In Binomial dist.,

$B(n, p, x) = {}^nC_x p^x q^{n-x}$, $\mu = np \Rightarrow p = \frac{\mu}{n}$

If $n \to \infty$ & $p \to 0$, s.t. μ is finite

$$\lim_{n\to\infty} B(n,p,x) = \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1)\dots(n-x+1)}{x!} \frac{\mu^x}{n^x} \left(1-\frac{\mu}{n}\right)^{-x} \left(1-\frac{\mu}{n}\right)^n$$

$$= \frac{\mu^x}{x!} \lim_{n\to\infty} \frac{n(n-1)\dots(n-x+1)}{n^x} \left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x}$$

$$= \frac{\mu^x}{x!} \left[\lim_{n\to\infty} \frac{n(n-1)\dots(n-x+1)}{n \cdot n \cdots n}\right] \left[\lim_{n\to\infty} \left(1-\frac{\mu}{n}\right)^n\right] \left[\lim_{n\to\infty} \left(1-\frac{\mu}{n}\right)^{-x}\right]$$

$$= \frac{\mu^x}{x!} \cdot (1) \cdot e^{-\mu} \cdot (1)$$

$$= \frac{e^{-\mu}\mu^x}{x!}$$

$$= P(\mu, x)$$

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EXPONENTIAL DISTRIBUTION

(This is a continuous prob. dist.)

$$P(\alpha, x) = \begin{cases} \alpha e^{-\alpha x}, & 0 \le x < \infty, \ \alpha \in \mathbb{R}^+ \\ 0, & \text{elsewhere} \end{cases}$$

is a pdf called the Exponential Distribution.

α is the parameter of the distribution.

To verify that ① defines a pdf,

(i) $P(\alpha, x) \ge 0$

(ii) $\int_{-\infty}^{\infty} P(\alpha, x)\, dx = 1$

$$LHS = \int_0^{\infty} \alpha e^{-\alpha x} dx$$

$$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha}\right]_0^{\infty} = -1\left[e^{-\infty} - e^{-0}\right]$$

$$= -1[0-1] = 1$$

Mean of exp. dist.

$$\text{Mean} : \int_{-\infty}^{\infty} x\, P(\alpha, x)\, dx$$

$$= \int_0^{\infty} x \alpha e^{-\alpha x} dx$$

$$= \alpha \left[x \frac{e^{-\alpha x}}{-\alpha} - \frac{e^{-\alpha x}}{(-\alpha)^2}\right]_0^{\infty}$$

$$= \alpha\left[0 - 0 - \left(0 - \frac{1}{\alpha^2}\right)\right] = \frac{1}{\alpha}$$

Variance of Exp dist.

$$\sigma^2 = E[x^2] - \{E[x]\}^2$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 P(\alpha, x)\, dx = \int_0^{\infty} x^2 \alpha e^{-\alpha x} dx$$

$$= \alpha \int_0^{\infty} x^2 e^{-\alpha x} dx$$

$$= \alpha\left[x^2 \frac{e^{-\alpha x}}{-\alpha} - (2x)\frac{e^{-\alpha x}}{(-\alpha)^2} + 2\frac{e^{-\alpha x}}{(-\alpha)^3}\right]_0^{\infty}$$

$$= \alpha\left[0 - \left\{0 - 0 - \frac{2}{\alpha^3}\right\}\right] = \frac{2}{\alpha^2}$$

$$\sigma^2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2} \Rightarrow \sigma = \frac{1}{\alpha}$$

Evaluation of Probability

Since $P(\alpha, x)$ is a cont. prob. dist,

$$\Rightarrow P(0 \le x \le a) = \int_0^a P(\alpha, x)\, dx$$

$$P(x \ge a) = \int_a^{\infty} P(\alpha, x)\, dx$$

$$= 1 - \int_0^a P(\alpha, x)\, dx$$

Eg of Exp. Dist.

1. Time b/w successive arrivals
2. Duration of phone calls / rain
3. Service time at hotel / restaurant
4. Time req. to repair a part

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Eg The duration of a telephone conversation follows exponential dist. with mean 3 minutes. Find the prob. that the conversation may last

(i) more than 1 minute

(ii) less than 3 minutes

(iii) between 2 & 4 minutes

Ans.

$\mu = 3, \quad a = \frac{1}{3}. \quad P(a, x) = \begin{cases} \frac{1}{3} e^{-\frac{1}{3}x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

i) $P(x > 1) = \int_1^\infty \frac{1}{3} e^{-\frac{x}{3}} dx$

$= \cancel{1 - P(x=0)} = \frac{1}{3} \left[\frac{e^{-x/3}}{(-1/3)} \right]_1^\infty$

$= \cancel{1 - f}$ $= (-1) \left[e^{-\infty} - e^{-1/3} \right]$

$= e^{-1/3} = \frac{1}{e^{1/3}} = 0.71653$

ii) $P(x < 3) = \int_0^3 \frac{1}{3} e^{-\frac{x}{3}} dx$

$= \frac{1}{3} \left[\frac{e^{-x/3}}{(-1/3)} \right]_0^3$

$= \frac{\cancel{3}}{(-1)} \left[e^{-1} - 1 \right] = 1 - e^{-1}$

$= 0.63212$

iii) $P(2 < x < 4) = \int_2^4 \frac{1}{3} e^{-\frac{x}{3}} dx$

$= \frac{1}{3} \left[\frac{e^{-x/3}}{-1/3} \right]_2^4$

$= \cancel{\frac{3}{3}} (-1) \left[e^{-4/3} - e^{-2/3} \right]$

$= 0.24982$

Eg The mileage that a car owner gets with a certain kind of radial tyre is a random variable that follows exp. dist. with mean 40,000 km. Find the prob. that these tyres will last

(i) at least 20,000 km (ii) at most 30,000 km

Ans $\mu = 40{,}000\,km \quad a = \frac{1}{40000}$

i) $P(x \geq 20000) = \int_{20000}^\infty \frac{1}{40000} e^{-\frac{x}{40000}} dx$

$= \frac{1}{40000} \left[\frac{e^{-x/40000}}{(-1/40000)} \right]_{20000}^\infty$

$= -1 \left[e^{-\infty} - e^{-1/2} \right]$

$= 0.60653$

ii) $P(x \leq 30000) = \int_0^{30000} \frac{1}{40000} e^{-\frac{x}{40000}} dx$

$= \frac{1}{40000} \left[\frac{e^{-x/40000}}{(-1/40000)} \right]_0^{30000}$

$= -1 \left[e^{-3/4} - e^{-0} \right] = 0.5276$

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Eg The avg daily turnout in a store is ₹ 10,000 and the avg profit is 8%. If the turnout has an expo dist., find the prob that the net profit will exceed ₹ 3000 on

i) two consecutive days

ii) 3 out of 6 days

Ans X : R.V. associated with profit

$\mu = \frac{8}{100} \times 10000 = 800$

$\alpha = \frac{1}{800}$

$P(x > 3000) = \int_{3000}^{\infty} \frac{1}{800} e^{-\frac{x}{800}} dx$

$= \frac{1}{800} \left[\frac{e^{-x/800}}{-1/800} \right]_{3000}^{\infty}$

$= -1 \left[e^{-\infty} - e^{-15/4} \right]$

$= 0.02352$

i) $P(x > 3000$ on two consecutive days$)$

$= e^{-\frac{15}{4}} \times e^{-\frac{15}{4}}$

$= 0.00055311$

ii) $P(x > 3000$ on 3 out of 6 days$)$

$= {}^6C_3 \left(e^{-\frac{15}{4}}\right)^3 \left(1 - e^{-\frac{15}{4}}\right)^3$

$= 0.0002422$

Eg The life of a bulb is advertised to have a mean of 200 hrs. Assuming that it follows exp dist, find the prob that the life of a bulb is

i) less than 200 hrs

ii) b/w 100 & 300 hrs

Ans $\mu = 200$, $\alpha = \frac{1}{200}$

i) $P(x < 200) = \int_0^{200} \frac{1}{200} e^{-\frac{x}{200}} dx$

$= \frac{1}{200} \left[\frac{e^{-x/200}}{-1/200} \right]_0^{200}$

$= -1 \left[e^{-1} - e^{0} \right] = 0.63212$

ii) $P(100 < x < 300) = \int_{100}^{300} \frac{1}{200} e^{-\frac{x}{200}} dx$

$= \frac{1}{200} \left[\frac{e^{-x/200}}{-1/200} \right]_{100}^{300}$

$= -1 \left[e^{-3/2} - e^{-1/2} \right] = 0.3834$

Eg The time reqd. (in hrs) to repair a machine is exp. distributed with parameter $\alpha = 1/2$

i) What is the prob. that the repair time exceeds 2 hrs.

ii) What is the conditional prob. that a repair takes at least 10 hrs given that its duration exceeds 9 hrs?

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Ans $\alpha = \frac{1}{2}$; $P(\alpha, x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

ii) $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$P(x \geq 10 / x > 9) = \frac{P(x \geq 10)}{P(x > 9)}$

$P(x > 10) = \int_{10}^{\infty} \frac{1}{2} e^{-x/2} dx$

$= \frac{1}{2} \left[\frac{e^{-x/2}}{-1/2} \right]_{10}^{\infty}$

$= -1 \left[e^{-\infty} - e^{-5} \right]$

$= 0.006738$

$P(x > 9) = \int_{9}^{\infty} \frac{1}{2} e^{-x/2} dx$

$= -1 \left[e^{-\infty} - e^{-4.5} \right]$

$= 0.01111$

$P(x \geq 10 / x > 9) = \frac{e^{-5}}{e^{-4.5}}$

$= 0.60653$

i) $P(x > 2) = \int_{2}^{\infty} \frac{1}{2} e^{-x/2} dx = \frac{1}{2} \left[\frac{e^{-x/2}}{-1/2} \right]_{2}^{\infty}$

$= \cancel{\frac{2}{2}} -1 \left[e^{-\infty} - e^{-1} \right] = 0.3679$

GAMMA DISTRIBUTION

$\Gamma n = \int_0^{\infty} x^{n-1} e^{-x} dx$ for $n > 0$

$\Gamma(n+1) = n \Gamma n$

If n is a +ve integer

$\Gamma(n+1) = n!$,

$\Gamma\frac{1}{2} = \sqrt{\pi}$

The continuous random variable x follows Gamma distribution with parameters α & β if its density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta^{\alpha} \Gamma \alpha} x^{\alpha - 1} e^{-\frac{x}{\beta}}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \text{where } \alpha > 0, \beta > 0$$

If $\alpha = 1$

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}}, & x \geq 0 \\ 0, & x < 0 \end{cases} \equiv \text{Exponential dist.}$$

Mean of Gamma distribution

$\mu = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x \frac{1}{\beta^{\alpha} \Gamma \alpha} x^{\alpha - 1} e^{-\frac{x}{\beta}} dx$

$= \int_0^{\infty} \frac{1}{\beta^{\alpha} \Gamma \alpha} x^{\alpha} e^{-\frac{x}{\beta}} dx$

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Let $\frac{x}{\beta} = t$ $x = 0, t = 0$; $x = \infty, t = \infty$

$\Rightarrow dx = \beta\, dt$

$$\Rightarrow \mu = \int_0^\infty \frac{t^{\alpha} e^{-t}}{\Gamma\alpha} \beta\, dt$$

$$= \frac{\beta}{\Gamma\alpha} \int_0^\infty t^{\alpha} e^{-t}\, dt$$

$$= \frac{\beta}{\Gamma\alpha} \Gamma(\alpha+1)$$

$$= \frac{\beta \times \alpha \cancel{(\alpha+1)}\, \Gamma\alpha}{\Gamma\alpha}$$

$$= \alpha\beta$$

Variance of Gamma Distribution

$$\sigma^2 = E[x^2] - \{E[x]\}^2$$

$$E[x^2] = \int_0^\infty x^2 \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha}\Gamma\alpha} dx$$

$$= \int_0^\infty \frac{x^{\alpha+1} e^{-x/\beta}}{\beta^{\alpha}\Gamma\alpha} dx$$

Put $\frac{x}{\beta} = t$ $x = 0, t = 0$

$\Rightarrow dx = \beta\, dt$ $x = \infty, t = \infty$

$$E[x^2] = \int_0^\infty \beta t \frac{t^{\alpha} e^{-x}}{\Gamma\alpha} \beta\, dt$$

$$= \frac{\beta^2}{\Gamma\alpha} \int_0^\infty t^{\alpha+1} e^{-t}\, dt$$

$$= \frac{\beta^2}{\Gamma\alpha} \Gamma(\alpha+2)$$

$$= \frac{\beta^2 \times (\alpha+1)\Gamma(\alpha+1)}{\Gamma\alpha} = \frac{\beta^2 \times (\alpha+1) \times \alpha \times \Gamma\alpha}{\Gamma\alpha}$$

$$= \alpha(\alpha+1)\beta^2$$

$$\sigma^2 = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2$$

$$= \alpha\beta^2$$

$$\sigma = \beta\sqrt{\alpha}$$

Eg A random variable x is Gamma distributed with $\alpha = 3$, $\beta = 2$

Find (i) $P(x \le 1)$ (ii) $P(1 \le x \le 2)$

i) $$P(x \le 1) = \int_0^1 \frac{1}{2^3 \Gamma 3} x^2 e^{-\frac{x}{2}} dx$$

$$= \frac{1}{8 \times 2} \int_0^1 x^2 e^{-\frac{x}{2}} dx$$

$$= \frac{1}{16}\left[x^2 \frac{e^{-x/2}}{-1/2} - 2x \frac{e^{-x/2}}{(-1/2)^2} + 2\frac{e^{-x/2}}{(-1/2)^3}\right]_0^1$$

$$= \frac{1}{16}\left[-2e^{-1/2} - 8e^{-1/2} - 16e^{-1/2} + 16\right]$$

$$= 0.0143877$$

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ii) $P(1 \le x \le 2) = \int_1^2 \frac{1}{16} x^2 e^{-x/2} dx$

$= \frac{1}{16}\left[x^2 \frac{e^{-x/2}}{-1/2} - 2x\frac{e^{-x/2}}{(-1/2)^2} + \frac{2e^{-x/2}}{(-1/2)^3}\right]_1^2$

$= 0.065914$

Eg The length of time reqd. to repair an article at a repair shop follows Gamma dist. By observing a large no. of cases, the mean & variance are found to be 2 hrs & 1 hr² resp. Estimate the prob. that the next article to arrive will req. less than 1 hr. to be repaired.

Ans $\mu = 2$

$\sigma^2 = 1$

$\Rightarrow \alpha\beta = 2$

& $\alpha\beta^2 = 1$

$\beta = \frac{\sigma^2}{\mu} = \frac{1}{2}$

$\alpha\beta = 2 \Rightarrow \frac{\alpha}{2} = 2 \Rightarrow \alpha = 4$

$f(x) = \frac{1}{(\frac{1}{2})^4 \Gamma 4} x^3 e^{-2x}$

$= \frac{8}{3} x^3 e^{-2x}$

$P(x<1) = \int_0^1 \frac{8}{3} x^3 e^{-2x} dx$

$= \frac{8}{3}\left[x^3 \frac{e^{-2x}}{-2} - 3x^2\frac{e^{-2x}}{(-2)^2} + 6x\frac{e^{-2x}}{(-2)^3} - 6\frac{e^{-2x}}{(-2)^4}\right]_0^1$

$= \frac{8}{3}\left[\frac{e^{-2}}{-2} - \frac{3}{4}e^{-2} - \frac{6}{8}e^{-2} - \frac{6}{16}e^{-2} + \frac{6}{16}\right]$

$= 0.1428765$

Eg The daily consumption of milk in a city, in excess of 20,000 litres is distributed as a gamma variate with parameters $\alpha = 2$, $\beta = 10,000$. The city has a daily stock of 30000 litres. What is the prob. that the stock is insufficient on a given day?

If the R.V. X denotes the daily consumption of milk in the city,

then the RV, $Y = X - 20000$ follows Gamma dist.

$f(y) = \frac{y^{2-1} e^{-\frac{y}{10000}}}{(10000)^2 \Gamma 2}$

$= \frac{1}{10000^2} y e^{\frac{-y}{10000}}$

$P(X > 30000) = P(Y > 10000)$

$= \int_{10000}^{\infty} \frac{1}{10000^2} \cdot y e^{\frac{-y}{10000}} dy$

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$$= \frac{1}{10000^2}\left[y\frac{e^{-y/10000}}{-1/10000} - \frac{e^{-y/10000}}{(-1/10000)^2} \right]_{10000}^{\infty}$$

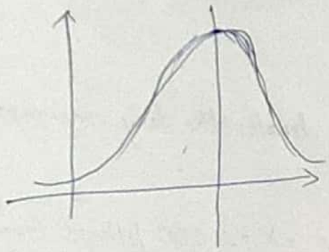
$$= \frac{1}{10000^2}\left[0 - \{-(10000)^2 e^{-1} - (10000)^2 e^{-1}\} \right]$$

$$= e^{-1} + e^{-1} = \frac{2}{e} = 0.73576$$

NORMAL DISTRIBUTION

(For C.R.V)

Any quantity whose variation depends on random causes follows N.D



The N.D. is defined as

$$N(\mu, \sigma, x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\}$$

μ, σ — parameters of the dist.

Remark: Mean, mode & median coincide for a N.D

The graph of N.D is a Bell shaped curve

Note that (i) $N(\mu, \sigma, x) \geq 0$

(ii) $\int_{-\infty}^{\infty} N(\mu, \sigma, x)\, dx = 1$

$$LHS = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} dx$$

Put $\frac{x-\mu}{\sigma} = z$ When $x = \infty$, $z = \infty$; $x = -\infty$, $z = -\infty$

$\Rightarrow dx = \sigma dz$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}z^2 \right\} \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\right) dz$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \exp\left(-\frac{z^2}{2}\right) dz$$

Put $\frac{z^2}{2} = t$ when $z = 0$, $t = 0$; $z = \infty$, $t = \infty$

$z\,dz = dt$

$dz = \frac{dt}{\sqrt{2t}}$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(-\tfrac{1}{2}+1\right) = \frac{1}{\sqrt{\pi}} \Gamma\tfrac{1}{2} = \frac{1}{\sqrt{\pi}} \times \sqrt{\pi}$$

$$= 1$$

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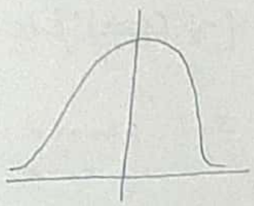
Standard Normal Distribution

The ND for which $\mu = 0$ & $\sigma = 1$ is called SND

The pdf for S.N.D. is $\phi(z) = N(0,1,x) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$



Area under the curve is unity $= 1$

Evaluation of $P(a \le x \le b)$

Let $z = \frac{x-\mu}{\sigma}$. Find z corresponding to $x = a$ & $x = b$ (say z_1 & z_2)

$$P(a \le x \le b) = P(z_1 \le z \le z_2)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}} dz$$

Eg If X is normally distributed with mean 5 & variance 4, find $P(3 < x < 7)$.

Ans $\mu = 5$, $\sigma^2 = 4$

$\Rightarrow \sigma = \sqrt{4} = 2$

As $z = \frac{x-\mu}{\sigma} \Rightarrow z = \frac{x-5}{2}$

If $x = 3$, $z = \frac{3-5}{2} = -1$

If $x = 7$, $z = \frac{7-5}{2} = 1$

$$P(3 < x < 7) = P(-1 < z < 1)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\frac{z^2}{2}} dz$$
$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} e^{-z^2/2} dz$$
$$= 2(0.3413) = 0.6826$$

NOTE : $P(z > z_1) = 0.5 - P(0 < z < z_1)$

Eg If X is a R.V. following ND with $\mu = 50$ & $\sigma = 10$, find the prob. that X lies b/w 45 & 62.

$\mu = 50$, $\sigma = 10$

As $z = \frac{x-\mu}{\sigma}$. $z = \frac{x-50}{10}$

If $x_1 = 45 \Rightarrow z_1 = \frac{45-50}{10} = -0.5$

$x_2 = 62 \Rightarrow z_2 = \frac{62-50}{10} = 1.2$

$$P(45 < x < 62) = P(-0.5 < z < 1.2)$$
$$= P(-0.5 < z < 0) + P(0 < z < 1.2)$$
$$= P(0 < z < 0.5) + P(0 < z < 1.2)$$
$$= 0.1915 + 0.3849$$
$$= \cancel{0.5674}\ 0.5764$$

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Eg The avg. ~~of~~ of lengths of metal bars produced by a company is 68.22 cm with variance 10.8 cm². How many bars in a consignment of 1000 are expected to be over 72 cms?

$\mu = 68.22$ cm, $\sigma^2 = 10.8$ cm².

$\sigma = \sqrt{10.8}$

As $z = \frac{x-\mu}{\sigma} \Rightarrow z = \frac{x - 68.22}{\sqrt{10.8}}$

As $x = 72$, $z = \frac{72 - 68.22}{3.2863} = 1.15$

$P(72 < x < \infty) = P(1.15 < z < \infty)$

$= 0.5 - P(0 < z < 1.15)$

$= 0.5 - 0.3749$

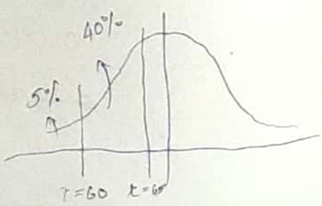
$= 0.1251$

In a consignment of 1000 bars, approx. 125 bars can be expected to be over 72 cms.

Eg In a N.D., 5% of the items are under 60 & 40% are b/w 60 & 65. Find μ & σ of the dist.

$P(x < 60) = 5\%$

$P(60 < x < 65) = 40\%$



For $x_1 = 60$

$A(z_1) = 0.5 - 0.05$
$= 0.45$

For $A(z_1) = 0.45$,
$z_1 = -1.65$

As $z = \frac{x-\mu}{\sigma}$,

$\Rightarrow -1.65 = \frac{60-\mu}{\sigma}$

$\Rightarrow 60 - \mu = -1.65\sigma$

$\Rightarrow \cancel{\mu - 1.6} \; \mu - 1.65\sigma = 60$ — (1)

For $x_2 = 65$,

$A(z_2) = 0.5 - 0.45$
$= 0.05$

For $A(z_2) = 0.05$
$z_2 = -0.13$

As $z = \frac{x-\mu}{\sigma}$

$\Rightarrow -0.13 = \frac{65-\mu}{\sigma}$

$\Rightarrow 65 - \mu = -0.13\sigma$

$\Rightarrow \mu - 0.13\sigma = 65$ — (2)

From (1) & (2),

$\mu = 65.427$

$\sigma = 3.289$

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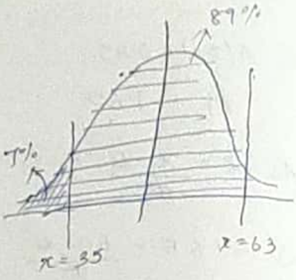
Eg Find μ & σ of a ND in which 7% of the items are under 35 & 89% are under 63.



For $x_1 = 35$,

$A(z_1) = 0.5 - 0.07$

$= 0.43$

$z_1 = -1.48$

As $z = \frac{x-\mu}{\sigma}$,

$-1.48 = \frac{35-\mu}{\sigma}$

$\Rightarrow \mu - 1.48\sigma = 35$ — ①

For $x_2 = 63$,

$A(z_2) = 0.89 - 0.5$

$= 0.39$

$z_2 = 1.23$

$1.23 = \frac{63-\mu}{\sigma}$

$\Rightarrow \mu + 1.23\sigma = 63$ — ②

From ① & ②,

$\mu = 50.29$,

$\sigma = 10.33$

Eg If in a N.D., 31% of items are under 45 & 8% are over 64, find μ & σ

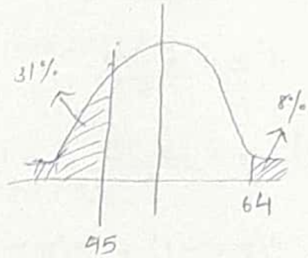
Eg In a N.D., if 60% are under 25 & 80% are under 50, find μ & σ.

Answers

1. For $x_1 = 45$,

$A(z_1) = 0.5 - 0.31$

$= 0.19$



$z_1 = -0.50$

As $z = \frac{x-\mu}{\sigma}$

$\Rightarrow -0.5 = \frac{45-\mu}{\sigma}$

$\Rightarrow \mu - 0.5\sigma = 45$ — ①

From ① & ②

$\mu = 49.9738$

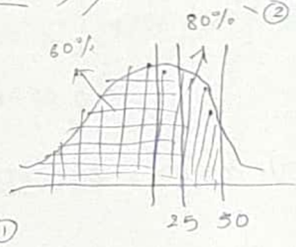
$\sigma = 9.9476$

For $x_2 = 64$

$A(z_2) = 0.5 - 0.08$

$= 0.42$

$z_2 = 1.41$ $\Rightarrow 1.41 = \frac{64-\mu}{\sigma}$ $\Rightarrow \mu + 1.41\sigma = 64$ — ②



2. For $x_1 = 25$

$A(z_1) = 0.6 - 0.5$

$= 0.1$

$z_1 = 0.25$ $\Rightarrow \mu + 0.25\sigma = 25$ — ①

For $x_2 = 50$

$A(z_2) = 0.8 - 0.5 = 0.3$

$z_2 = 0.84$ $\Rightarrow \mu + 0.84\sigma = 50$ — ②

From ① & ②, $\mu = 14.40678$

$\sigma = 42.37288$

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Eg The marks of 1000 students are N.D. with mean 70 & s.d 5. Find the no. of students whose marks are

(i) less than 65 (ii) more than 78 (iii) b/w 60 & 80

Ans $\mu = 70$, $\sigma = 5$

$Z = \frac{x-\mu}{\sigma}$

$\Rightarrow Z = \frac{x-70}{5}$

i) $Z = \frac{65-70}{5} = -1$

$P(Z<-1) = P(Z>1)$

$= 0.5 - P(0<Z<1)$

$= 0.5 - 0.3413$

$= 0.1587$

∴ Out of 1000 students, approx. 159 students have marks less than 65

(ii) $Z = \frac{78-70}{5} = 1.6$

$P(Z>1.6) = 0.5 - P(0<Z<1.6)$

$= 0.5 - 0.4452$

$= 0.0548$

∴ Out of 1000 students, approx. 55 students have more than 78 marks.

iii) $x = 60$, $Z = \frac{60-70}{5} = -2$

$x = 80$, $Z = \frac{80-70}{5} = 2$

$P(-2<Z<2) = 2 \times P(0<Z<2)$

$= 2 \times 0.4772$

$= 0.9544$

Out of 1000 students, approx. 954 students have marks b/w 60 & 80.

Eg In a test on 2000 electric bulbs, the life of a particular make follows N.D. with an average life of 2040 hrs & sd of 60 hrs. Estimate the no. of bulbs likely to burn for

i) more than 2130 hrs

ii) less than 1950 hrs

iii) more than 1920 hrs & less than 2080 hrs.

Ans i) $Z = \frac{2130-2040}{60} = 1.5$

$P(Z>1.5) = 0.5 - P(0<Z<1.5)$

$= 0.5 - 0.4332$

$= 0.0668$

Out of 2000 electric bulbs, approx. 134 bulbs will burn for more than 2130 hrs

ii) $Z = \frac{1950-2040}{60} = -1.5$

$P(Z<-1.5) = P(Z>1.5)$

$= 0.0668$

Out of 2000 electric bulbs, approx. 134 bulbs will burn for less than 1950 hrs.

iii) $x = 1920 \Rightarrow Z = \frac{1920-2040}{60} = -2$

$x = 2080 \Rightarrow Z = \frac{2080-2040}{60} = 0.67$

$P(-2<Z<0.67) = P(Z>-2) + P(Z<0.67)$

$= P(0<Z<2) + P(0<Z<0.67)$

$= 0.4772 + 0.2486$

$= 0.7258$

Out of 2000 electric bulbs, approx. 1452 bulbs will burn for more than 1920 hrs & less than 2080 hrs.

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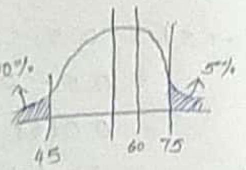
Eg. In a certain exam, a student is considered to have failed, secured second class, first class or distinction according as ~~the~~ he scores $< 45\%$, between 45 & 60%, b/w 60 & 75% & above 75% marks. In a particular year, 10% of student failed & 5% got distinction. Find the % of students who got I class & II class.



Ans For $x = 45$

$A(z_1) = 0.5 - 0.1$

$= 0.4$

$\therefore z_1 = -1.29$ ~~-1.27~~

As $z = \frac{x-\mu}{\sigma}$

$\Rightarrow -1.29 = \frac{45-\mu}{\sigma}$

$\Rightarrow \mu - 1.29\sigma = 45$ — (1)

For $x = 75$

$A(z_2) = 0.5 - 0.05$

$= 0.45$

$z_2 = 1.65$

$\Rightarrow 1.65 = \frac{75-\mu}{\sigma}$ $\Rightarrow \mu + 1.65\sigma = 75$ — (2)

From (1) & (2), $\mu = 58.1633$

$\sigma = 10.2041$

% of students who got I Class

$P(60 < x < 75) = P(0.18 < z < 1.65)$

$= \cancel{P(0.18 < z < 0) + P(0 < z < 1.65)}$

$= P(0 < z < 1.65) - P(0 < z < 0.18)$

$= 0.4505 - 0.0714$

$= 0.3791$

$= 37.91\% \simeq 38\%$

% of students who got II Class

$P(45 < x < 60) = P(-1.29 < z < 0.18)$

$= P(-1.29 < z < 0) + P(0 < z < 0.18)$

$= P(0 < z < 1.29) + P(0 < z < 0.18)$

$= 0.4015 + 0.0714$

$= 0.4729$

$= 47.29\% \simeq 47\%$

Eg If the time that a bus passes a certain stop is N.D. with mean 25 mins. & S.D. 3 mins., what is the least time that one should arrive at a stop & still have a probability of 99% of catching the bus?

$\mu = 25, \ \sigma = 3$

$A(z) = 0.99$

$\left(\begin{array}{l} 0.99 - 0.5 \\ = 0.49 \end{array}\right)$

$z = 2.33$

$$2.33 = \frac{x-25}{3}$$

$\Rightarrow x = 31.99$

UNIFORM DISTRIBUTION

A uniform distribution for a CRV X is defined as

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}$$

If $a \le x \le b$

H.W. Find the prob. dist. f^n $F(x)$ for uniform dist.

Ans $F(x) = \int_{-\infty}^{x} f(u)\, du$

$= \int_{-\infty}^{x} \frac{1}{b-a}\, du$

$= \frac{1}{b-a} \int_{-\infty}^{x} du$

$= \frac{1}{b-a} \left[u\right]_{-\infty}^{x}$

$= \frac{1}{b-a}\left[x+\infty\right]$

Mean of uniform dist.

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_a^b x \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) = \frac{a+b}{2}$$

Variance of uniform dist.

$$\sigma^2 = E[X^2] - \{E[X]\}^2$$

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right)$$

$$\sigma^2 = \frac{b^2+a^2+ab}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^2+a^2+ab}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4b^2+4a^2+4ab-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12} = \frac{(b-a)^2}{12}$$

Eg If X is uniformly distributed with mean 1 & variance 4/3, find $P(X<0)$

$$\frac{a+b}{2} = 1, \quad \frac{(b-a)^2}{12} = \frac{4}{3}$$

$\Rightarrow a+b = 2, \quad b-a = 4$

$\Rightarrow a = -1, \; b = 3$

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$$P(x<0) = \int_{-\infty}^{0} f(x)\,dx$$

$$= \int_{-1}^{0} \frac{1}{3-(-1)}\,dx$$

$$= \frac{1}{4}\left[x\right]_{-1}^{0}$$

$$= \frac{1}{4}\left[0+1\right] = \frac{1}{4}$$

Eg The buses arrive at a stop in 15 min intervals, starting at 7 a.m. If a passengers arrives at the stop at a random time that is uniformly distributed b/w 7 & 7.30 a.m., find the prob. that he waits for

i) less than 5 mins

ii) at least 12 minutes.

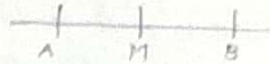
$$f(x) = \begin{cases} \frac{1}{30}, & 7 \le x \le 7.30 \\ 0, & \text{elsewhere} \end{cases}$$

i) $P(x<5) = \int_{0}^{5} \frac{1}{30}\,dx = \frac{1}{30}\left[x\right]_{0}^{5} = \frac{5}{30} = \frac{1}{6}$

ii) $P(x \ge 12) = \int_{12}^{30} \frac{1}{30}\,dx = \frac{1}{30}\left[x\right]_{12}^{30} = \frac{18}{30} = \frac{3}{5}$

Eg A bus travels b/w two cities A & B 100 miles apart. The distance X of the point of breakdown from city A is a uniform variate. There are service garages in the city A, city B & midway b/w the cities b/w A & B. If a breakdown occurs, a tow truck is sent from the garages closest to the point of breakdown. Estimate the probability that the tow truck has to travel more than 10 miles to reach the bus.

$$f(x) = \begin{cases} \frac{1}{50}, & \\ 0, & \text{otherwise} \end{cases}$$



$$P(X>10) = \int_{10}^{25} \frac{1}{50}\,dx$$

$$= \frac{1}{50}\left[x\right]_{10}^{25}$$

$$= \frac{15}{50} = \frac{3}{10}$$

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