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UNIT-3

Stochastic process:

Let 'S' be a sample space of a random experiment and 'R' be a set of real numbers.

A random variable 'X' is a function 'f' from $S \to R$ such that

$X = f(s)$ and $s \in S$

we define an index set $T \subset R$.

Let us suppose that a random variable defined on 'S' depends on both 's' and 't', where

$s \in S$ and $t \in T$

* The stochastic process is a set of random variables defined on 'S' with parameter 't'

$\{X(s,t)\} \to$ stochastic process.

here, $X_0 = X(0)$ is the initial state of the system.

* The values assumed by the random variables $\{X(s,t)\}$ are called 'states'

* The set of all possible these values forms a state space 'S' of the process.

If the state space is discrete, then the stochastic process is called discrete state process (also called a chain).

probability vector:

Let $V = \{v_1, v_2, \cdots\cdots, v_n\}$, these are saide to be a probability vector if each v_i is $\geq 0 \; \forall i$

i) $\boxed{v_i \geq 0 \; \forall i}$

ii) $\sum_i v_i = 1$

Ex: $\{1,0\}$, $\{0, 1/2, 1/4, 1/4\}$ are probability vectors

but $\{1, 0, -1/2, -1/2, 0\}$ is not

* The stochastic matrix:

A square matrix 'P' is said to be a stochastic matrix if each row in 'P' is a probability vector.

EX: $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 2/3 & 0 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$

Regular stochastic matrix:

A stochastic matrix is said to be regular if the powers of 'P', $(P, P^2, P^3, \ldots)$ contains all non-zero elements

properties of Regular stochastic Matrix:

① Let 'P' be a stochastic matrix, 'P' has a unique fixed probability vector $V = \{v_1, v_2, v_3, \ldots, v_n\}$ such that $\boxed{VP = V}$

② If $P^2, P^3, P^4, \ldots$ approaches matrix 'V' where each row is a "fixed probabity vector" (unique)

* Find the fixed probability vector of matrix $P = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$
$V = \{v_1, v_2\}$.

NOTE: A stochastic matrix is not regular if '1s' are on the principal diagonal of 'P'.

$VP = V$

$[v_1 \ v_2] \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 + v_2/2 \\ v_2/2 \end{bmatrix}^T = [v_1 \ v_2]$

$v_1 + \frac{v_2}{2} = v_1$

$\frac{v_2}{2} = v_2$ $\Rightarrow$ Not a regular stochastic Matrix since '1' is on principal diagonal

* Find the unique fixed probability vector for the matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$, let $V = [v_1 \ v_2 \ v_3]$ such that

$VP = V$

and $\boxed{v_1 + v_2 + v_3 = 1}$ — (1)

$VP = V$

$$[v_1 \ v_2 \ v_3] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{v_3}{2} , & \frac{v_3}{2} + v_1 & v_2 \end{bmatrix}$$

$\frac{v_3}{2} = v_1$, $\frac{v_3}{2} + v_1 = v_2$, $v_3 = v_2$

$\boxed{v_3 = 2v_1}$

$\frac{2v_1}{2} + v_1 = v_2$

$v_1 + \frac{2v_1}{2} = v_2$

$\boxed{v_2 = 2v_1}$

$v_1 + v_2 + v_3 = 1$

$v_1 + 2v_1 + 2v_1 = 1$

$5v_1 = 1$

$\boxed{v_1 = 1/5}$

$\boxed{v_2 = \frac{2}{5}}$

$\boxed{v_3 = \frac{2}{5}}$

$V = \left\{ \frac{1}{5}, \frac{2}{5}, \frac{2}{5} \right\}$

Markov chain:

A stochastic process, which is such that, the generation of the probability distribution depends on the present state is called the "Markov process".

If this state space is discrete, we call this as "Markov chain"

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Let the outcomes $x_1, x_2, x_3, \ldots$ of sequence of trials satisfies the following properties:

i) Each outcome belongs to the finite set i.e., the state space of the outcome $\{a_1, a_2, \ldots, a_n\}$

ii) The outcome of any trial depends atmost upon the outcome of immediate previous trial.

Let P_{ij} is associated with every pair of outcomes $\{a_i, a_j\}$ then P_{ij} indicates probability of the occurence of a_j after a_i occurs. Such a stochastic process is called a "Markov chain"

These probabilities P_{ij} which are non-zero real numbers are called the transition probabilities and they form a square matrix called "transition probability matrix".

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & & & \vdots \\ P_{n1} & & & P_{nn} \end{bmatrix}$$

with each state a_i, there corresponds the ith row of transition probabilities.

Each row of P is a probability vector.

$\therefore$ The transition probability matrix is a stochastic matrix.

Higher Transition probabilities:

The entry P_{ij} in the matrix 'P' is the probability that the system changes from the state a_i to a_j in exactly n steps is denoted by $P_{ij}(n)$.
The matrix formed by the probabilities matrix $P_{ij}(n)$

$P^{(0)} = \{P_1^{(0)}, P_2^{(0)}, \ldots\ldots P_m^{(0)}\}$ is called the initial probability distribution at the start of the process.

$P^{(n)} = \{P_1^{(n)}, P_2^{(n)}, \ldots\ldots P_m^{(n)}\}$ denoted by nth step probability distribution of the end of 'n' steps.

Stationary distribution of regular Markov chain:

A markov chain is said to be regular if the associated transition probability matrix 'P' is regular if the n th step transition matrices $P^2, P^3, \ldots\ldots P^n$ approaches the matrix V where each row in V is a unique fixed probability vector.

* Ex: A person's playing habits are as follows:
If he plays one day he is 70% sure not to play the next day.
On the other hand if he does not play one day he is 60% sure not to play the next day as well.

In the long run, how often does he play?

State S = {play, doesn't play}

$$P = \begin{array}{c} \\ \text{play} \\ \text{don't play} \end{array} \begin{array}{c} \begin{array}{cc} \text{play} & \text{don't play} \end{array} \\ \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

To find the probability for how long he plays in long run, we find the unique fixed probability vector.

$$VP = V$$

$$[V_1 \; V_2] \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [V_1 \; V_2]$$

$$0.3V_1 + 0.4V_2 = V_1 \quad -① $$

$$0.7V_1 + 0.6V_2 = V_2 \quad -②$$

$$0.4V_2 = 0.7V_1$$

$$\boxed{V_1 = \frac{4}{7}V_2}$$

$$0.7V_1 = 0.4V_2$$

$$V_1 + V_2 = 1$$

$$V_1 + \frac{7V_1}{4} = 1$$

$$\frac{11V_1}{4} = 1$$

$$\boxed{V_1 = \frac{4}{11}}, \quad \boxed{V_2 = \frac{7}{11}}$$

probability that he plays in the long run $= \frac{4}{11}$

* A man's smoking habits are as follows:
If he smokes filter cigarettes one week, he switches to no filter cigarettes next week with probability 0.2. On the other hand if he smokes no filter cigarettes one week there is a probability of 0.7 that he will smoke no filter cigarettes next week as well. In the long run how often does he smoke filter cigarettes.

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state $S = \{ \text{filter}, \text{no filter} \}$

$$P = \begin{array}{c c} & \begin{array}{cc} f & nf \end{array} \\ \begin{array}{c} f \\ nf \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$$

$$VP = V$$

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

$0.8V_1 + 0.3V_2 = V_1$ $\qquad$ $0.2V_1 + 0.7V_2 = 0.1V_2$

$0.3V_2 = 0.2V_1$ $\qquad$ $0.3V_2 = 0.2V_1$

$$\boxed{V_2 = \frac{2}{3}V_1}$$

$$V_1 + V_2 = 1$$

$$V_1 + \frac{2}{3}V_1 = 1$$

$$\frac{5V_1}{3} = 1$$

$$\boxed{V_1 = \frac{3}{5}}, \quad V_2 = \frac{2}{\cancel{3}} \times \frac{\cancel{3}}{5} = \frac{2}{5}$$

$$\boxed{V_2 = \frac{2}{5}}$$

In the long $P = \frac{3}{5} = 0.6.$

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* Three boys A, B & C are throwing ball to each other. 'A' always throws the ball to 'B' & 'B' always throws the ball to 'C' but 'C' is just likely to throw the ball to 'B' as to 'A'. If 'C' was the first person to throw the ball, find the probabilities that i) A has the ball, ii) B has the ball and iii) C has the ball, for the fourth throw.

State space = {A, B, C}

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

initially ⇒ ball is with 'C' ⇒ $P^{(0)} = \{0, 0, 1\}$

$P^{(1)} = P^{(0)}.P$

$P^{(2)} = P^{(1)}.P$

$P^{(3)} = P^{(2)}.P$

$P^{(4)} = P^{(3)}.P$

$$P^{(1)} = [0 \;\; 0 \;\; 1] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [1/2 \;\; 1/2 \;\; 0]$$

$$P^{(2)} = [1/2 \;\; 1/2 \;\; 0] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [0 \;\; 1/2 \;\; 1/2]$$

$$P^{(3)} = [0 \;\; 1/2 \;\; 1/2] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [1/4 \;\; 1/4 \;\; 1/2]$$

$$P^{(4)} = [1/4 \;\; 1/4 \;\; 1/2] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [1/4 \;\; 1/2 \;\; 1/4]$$

After 4^{th} throw, 'A' has the ball with probability $1/4$,
'B' has the ball with probability $1/2$,
'C' has the ball with probability $1/4$.

* There are 2 white marbles in bag 'A' and 3 red marbles in bag B. At each step of the process a marble is selected at random from each bag and the two marbles selected are independent/interchanged then find,
  i) transition probability matrix.
  ii) what is the probability that there are 2 red marbles in A after 3 steps.
  iii) In the long run what is the probability that there are 2 red marbles in A?

<u>state 1:</u> [2W | 3R]
A B

] $S_1 \rightarrow S_2$

<u>state 2:</u> [1W 1R | 2R 1W]
A B

<u>state 3:</u> [2R | 1R 1W]
A B

after state 3, we can't achieve state 1, immediately.

so, state 3 $\rightarrow$ state 1 is an impossible event

state space - $\{S_1, S_2, S_3\}$

$$P = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array}\begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1/2 \times 1/3 & 1/2 \times 1/3 + 1/2 \times 2/3 & 1/2 \times 2/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} \end{array} = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

Row-wise Explanation:

Row 1:

The system has to move from S_1 to S_2 with a probability 1.

Row 2:

The system changing $S_2 \to S_1$:

$S_1 \Rightarrow$ | 2W | 3R |, A B

$S_2 \Rightarrow$ | 1W 1R | 2R 1W |, A B

to make S_2 to S_1, we should chose 1R from A in S_2 & 1 W from B in S_2,

so, $P(1R \text{ from } A) = \frac{1}{2}$

$P(1W \text{ from } B) = \frac{1}{3}$

so, $P(S_2 \to S_1) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

The system remains in S_2 state:

Interchanging $\Rightarrow$ 1W from A & B + 1R from A & B

S_2: | 1W 1R | 2R 1W |, A B

$\Rightarrow$ from A $\Rightarrow P(1W) = \frac{1}{2}$
from B $\Rightarrow P(1W) = \frac{1}{3}$
$\Big\} = P = \frac{1}{6}$

+

from A $\Rightarrow P(1R) = \frac{1}{2}$
from B $\Rightarrow P(1R) = \frac{2}{3}$
$\Big] \Rightarrow P = \frac{1}{3}$

$\Rightarrow P(S_2 \to S_2) = \frac{1}{6} + \frac{1}{3} = \frac{3}{6} = \frac{1}{2}$

The system changing $S2 \rightarrow S3$

S2: | 1W 1R | 2R 1W | $\longrightarrow$ S3: | 2R | 2W 1R |

(A, B)

$\Rightarrow$ P(1W from A) in S2

& P(1R) from B in S2

$\Rightarrow \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$

Row 3:

state never changes from $S3 \rightarrow S1 \Rightarrow P = 0$

state change from $S3 \rightarrow S2$

S3: | 2R | 2W 1R | $\longrightarrow$ S2: | 1W 1R | 2R 1W |

(A, B)

P(1R from A) in S3

& P(1W from B) in S3 $\Rightarrow \frac{2}{2} \times \frac{2}{3} = \frac{2}{3}$

state remains in S3 $\Rightarrow$

S3: | 2R | 2W 1R | $\Rightarrow$ 1R from (A & B)

(A, B)

$= \frac{2}{2} \times \frac{1}{3} = \frac{1}{3}$.

ii) $p^{(0)} = [1\ 0\ 0]$ initially in state 1.

$$p^{(1)} = p^{(0)} \cdot P = [1\ 0\ 0] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [0\ 1\ 0]$$

$$p^{(2)} = p^{(1)} \cdot P = [0\ 1\ 0] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [1/6\ \ 1/2\ \ 1/3]$$

$$p^{(3)} = p^{(2)} \cdot P \quad [1/6\ \ 1/2\ \ 1/3] \cdot P = [1/2\ \ 23/36\ \ 5/18]$$

P (having 2 red in A after 3 steps) = only at state 3 it's possible, so

$$P = \frac{5}{18}$$

iii) $V = \{V_1, V_2, V_3\}$

$V \cdot P = V$

$V_1 + V_2 + V_3 = 1$

$$[V_1 \quad V_2 \quad V_3] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [V_1 \quad V_2 \quad V_3]$$

$\frac{V_2}{6} = V_1$; $V_1 + \frac{V_2}{2} + \frac{2V_3}{3} = V_2$; $\frac{V_2}{3} + \frac{V_3}{3} = V_3$

$\boxed{V_2 = 6V_1}$ — ①

$\frac{6V_1}{3} + \frac{V_3}{3} = V_3$

$6V_1 = 2V_3$

$\boxed{V_3 = 3V_1}$

$V_1 + V_2 + V_3 = 1$

$V_1 + 3V_1 + 6V_1 = 1$

$\boxed{V_1 = \frac{1}{10}}$, $\boxed{V_2 = \frac{6}{10}}$, $\boxed{V_3 = \frac{3}{10}}$

In the long run, A has 2 red marbles in state 3 so,

$V_3 \Rightarrow P = \frac{3}{10}$

* There are two boys b_1 & b_2 and 2 girls g_1 & g_2 are throwing a ball from one to the other. Each boy throws the ball to the other with probability 0.5 & to each girl with probability $\frac{1}{4}$. On the other hand, each girl throws the ball to each boy with probability $\frac{1}{2}$ and never to the girl. In the long run, how often does each recieve the ball?

$S = \{b_1, b_2, g_1, g_2\}$

$$P = \begin{array}{c|cccc} & b_1 & b_2 & g_1 & g_2 \\ \hline b_1 & 0 & 1/2 & 1/4 & 1/4 \\ b_2 & 1/2 & 0 & 1/4 & 1/4 \\ g_1 & 1/2 & 1/2 & 0 & 0 \\ g_2 & 1/2 & 1/2 & 0 & 0 \end{array}$$

To find in the long run, how often does each recieve the ball, we find the unique fixed probability vector

$$V = [V_1 \ V_2 \ V_3 \ V_4].$$

$$VP = V$$

$$V_1 + V_2 + V_3 + V_4 = 1$$

$$[V_1 \ V_2 \ V_3 \ V_4] \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = [V_1 \ V_2 \ V_3 \ V_4]$$

$$\frac{V_2}{2} + \frac{V_3}{2} + \frac{V_4}{2} = V_1 \quad \text{— (1)}$$

$$\frac{V_1}{2} + \frac{V_3}{2} + \frac{V_4}{2} = V_2 \quad \text{— (2)}$$

$$\frac{V_1}{4} + \frac{V_2}{4} = V_3 \quad \text{— (3)}$$

$$\frac{V_1}{4} + \frac{V_2}{4} = V_4 \quad \text{— (4)}$$

from (3) & (4)

$$V_3 = V_4.$$

$$\frac{V_2}{2} + \frac{2V_4}{2} = V_1$$

$$\boxed{V_2 = 2(V_1 - V_4)}$$

$$\frac{V_1}{2} + \frac{2V_4}{2} = V_2$$

$$\boxed{V_2 = V_4 + \frac{V_1}{2}}$$

$$\boxed{V_2 = V_1}$$

$$\boxed{V_4 = \frac{V_1}{2}}, \ \boxed{V_3 = \frac{V_1}{2}}$$

$$v_1 + v_1 + \frac{v_1}{2} + \frac{v_1}{2} = 1$$

$$3v_1 = 1$$

$$\boxed{v_1 = \frac{1}{3}}, \boxed{v_2 = \frac{1}{3}}, \boxed{v_3 = \frac{1}{6}}, \boxed{v_3 = \frac{1}{6}}$$

In the long run $= [v_1 \ v_2 \ v_3 \ v_4]$
$= [1/3 \ 1/3 \ 1/6 \ 1/6]$

* A player has Rs 300. At each play of a game, he loses Rs. 100 with probability 3/4 but wins with Rs. 200 with probability 1/4. He stops playing if he has lost his Rs. 300 or he has won at least Rs. 300.

a) Determine the transition probability matrix of the markov chain

b) Find the probability that there are atleast 4 plays to the game.

state space $S = \{0, 100, 200, 300, 400, 500, 600\}$

$$P = \begin{array}{c|ccccccc} & 0 & 100 & 200 & 300 & 400 & 500 & 600 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 100 & 3/4 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 200 & 0 & 3/4 & 0 & 0 & 1/4 & 0 & 0 \\ 300 & 0 & 0 & 3/4 & 0 & 0 & 1/4 & 0 \\ 400 & 0 & 0 & 0 & 3/4 & 0 & 0 & 1/4 \\ 500 & 0 & 0 & 0 & 0 & 3/4 & 0 & 1/4 \\ 600 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$p^{(0)} = [0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0]$$

$$p^{(1)} = p^{(0)} \cdot P = [0 \ 0 \ 3/4 \ 0 \ 0 \ 1/4 \ 0]$$

$$p^{(2)} = p^{(1)} \cdot P = [0 \ 9/16 \ 0 \ 0 \ 6/16 \ 0 \ 1/16]$$

$$p^{(3)} = p^{(2)} \cdot P = [27/64 \ 0 \ 0 \ 27/64 \ 0 \ 0 \ 10/64]$$

$$p^{(4)} = p^{(3)} \cdot P = [\tfrac{27}{64} \ 0 \ \tfrac{81}{256} \ 0 \ 0 \ \tfrac{27}{256} \ \tfrac{10}{64}]$$

probability that atleast thee all 4 Plays = $\frac{81}{256} + 0 + 0 + \frac{27}{256}$

$= \frac{108}{256}$

Queuing Theory

Queuing Models:

The queuing models are very helpful for determining how to operate a queuing system in the most effective way i.e., if too much service capacity to operate the system it invokes excessive cost.
These models enable us to find an appropriate balance between the cost of service and the amount of waiting time.

Basic characteristics of Queuing system:

1. Input source
2. Queue discipline
3. service Mechanism

Input source:

The input source is the size in the total no. of units that might require service from time to time, it may be assumed to be finite or infinite.
* Assumption is that, the queue generates according to poisson distribution at an average rate.
* 1. The equivalent assumption is that they generate accord-ing to exponential distribution b/w consecutive intervals.

Queue discipline:

This refers to the order in which the units or customers are

selected for services.
The queue disciplines are:

i) FIFO - first in first out.
ii) LIFO - last in first out.
iii) SIRO - service in Random order.

Service Mechanism

This consists of one or more service facility, each of which contains one or more parallel service channels.

If there is more than one service facility, the arrival unit may recieve the service from a sequence of service channels

at a given time. The arrival enters the service facility & is completely served by the ~~counter~~ server.

From the commencement of the service to its complet-ion for an unit at the service facility is known as service time. and it follows exponential distribution

Classification of Queuing Models:

* Any queuing model follows Kendal & Lee notation.

* Any queuing model may be completely specified in the following symbolic form:

a/b/c : d/e

a : Type of distribution of the inter-arrival time of the unit

b: Type of distribution of inter-service time of a unit
c: The no. of servers
d: capacity of the system.
e: Queue discipline

In all the cases we take
M: The arrival follows the poisson distribution & service follows exponential distribution.

There are four important Queuing models:

1. M/M/1 : ∞/FIFO → single server with infinite capacity
2. M/M/1 : K/FIFO → single server with finite capacity
3. M/M/s : ∞/FIFO → Multi server with ∞ capacity
4. M/M/s : K/FIFO → Multi server with finite capacity.
5. M/G/1 — G: General distribution (usually normal)

M/M/1 : ∞/FIFO model

If 'λ' is mean arrival rate, 'μ' is mean service rate and $\mu > \lambda$

1. Traffic ~~in the~~ intensity (or) utilisation factor (or) busy period is denoted by ρ

$$\rho = \frac{\lambda}{\mu}$$

2. Probability that there are 'n' customers in the queue including the one being served.

$$P(n) = (1-\rho)\rho^n$$

(3) probability that there are exactly zero units in the system.

$$P_0 = P(0) = 1-\rho \quad \text{probability}$$

(4) probability that the system is idle = probability that the arrival will not have to wait in the queue first service.

$$P(0) = 1-\rho.$$

(5) probability of n or more units in the system is given by

$$\rho^n$$

(6) Expected no. of customers in the queue / queue length

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

(7) Expected no. of customers in the system is given by:

$$L_s = L_q + \rho$$

(8) Expected waiting time in the queue i.e.,

$$W_q = \frac{L_q}{\lambda}$$

(9) Expected waiting time in the system is: $W_s = W_q + \frac{1}{\mu}$

(10) Non-empty queue size $D = \frac{\mu}{\mu-\lambda}$

(11) probability that an arrival will have to wait in queue more than 'W' in queue $= \left(\frac{\lambda}{\mu}\right) e^{(\lambda-\mu)W}$

(12) probability that an arrival will have to wait more than 'V' in the system $= e^{(\lambda-\mu)V}$

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* what is the probability that a customer has to wait more than 15 min to get his service completed in M/M/1 : ∞/FIFO system if $\lambda = 6$, $\mu = 10$ per hour.

↓

formula ⑪ 15 mins → 15/60 hrs

$$P = \frac{6}{10} e^{(-4)(15)/60}$$

$$P = \frac{3}{5} e^{-60/60}$$

$$P = 0.2207$$

* Find the probability that there are atleast 10 customers in the system in M/M/1 : ∞/FIFO model. $\mu = 6$, $\lambda = 8$

↓

formula ⑤

$$\rho^n = (\rho)^{10} = \left(\frac{6}{8}\right)^{10} = 0.0563$$

* In the usual notation of an M/M/1 : ∞/FIFO model, $\rho = 0.6$. what is the probability that queue contains 5 or more customers.

$$\rho^n = (0.6)^5 = 0.07776$$

* Suppose that customers arrive at a poisson rate of one per every 12 minutes that the service time is exponential at a rate of one service per 8 minutes. what is

a) The average no. of customers in the system

b) The average time a customer spends in the system

$$\lambda = \frac{60}{12} = 5/hr$$

since service rate follows exponential distribution,

so, mean $\Rightarrow \frac{1}{\mu} \Rightarrow \frac{1}{\mu} = 8$ $\boxed{\mu = \frac{1}{8}}$ $\Rightarrow \mu = 60 \times \frac{1}{8} = 7.5/hr$

i) $Ls = Lq + \rho$

$$Lq = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{5^2}{7.5(7.5-5)} = \frac{25}{7.5 \times 2.5} = 1.333$$

$$Ls = Lq + \rho$$

$$Ls = \frac{4}{3} + \frac{5}{7.5} = 2$$

ii) $Wq = \frac{Lq}{\lambda} = \frac{4}{3} \times \frac{1}{5} = \frac{4}{15}$

$$Ws = Wq + \frac{1}{\mu} = \frac{4}{15} + \frac{1}{7.5} \quad \frac{20+10}{75} = \frac{30}{75} = \frac{2}{5}$$

* customers arrive at a sales counter manned by a single person according to a Poisson process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of 100 seconds. Find the average time of a customer.

$$\mu = \frac{1}{100} \text{ sec (poisson)} \Rightarrow \mu = \frac{60 \times 60}{100} = 36/hr$$

$$\lambda = 20/hr$$

$$Lq = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(20)^2}{36(36-20)} = 0.6944$$

$$Wq = \frac{Lq}{\lambda} = \frac{0.6944}{20} = 0.0347$$

* In a city airport flights arrive at a rate of 24 flights per day. It is known that the inter-arrival time follows an exponential distribution and the service time is also exponential with an average of 30 mins.

i) probability that the system will be idle

ii) The mean queue size
iii) The avg. no of flights in the queue
iv) The probability that the size exceeds 7.

$$\frac{1}{\lambda} = 24$$

$\lambda = 24$ per day $\Rightarrow \lambda = 1$ per hour

$$\frac{1}{\mu} = 30 \text{ minutes}$$

$\mu = \frac{1}{30}$ /min $\Rightarrow \mu = \frac{60}{30} = 2/hr$

i) $P_0 = 1 - \rho = 1 - \frac{\lambda}{\mu} = 1 - \frac{1}{2} = 0.5$

ii) $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{1}{2(2-1)} = \frac{1}{2} = 0.5$

iii) $L_q = 0.5$

iv) exceeds 7 $\Rightarrow$ atleast 8

$= \rho^n = \left(\frac{\lambda}{\mu}\right)^8 = (0.5)^8 = 0.0039$

49. A supermarket has a single cashier. During peak hours, customers arrive at a rate of 20/hr. the avg no. of customers that can be processed by the cashier is 24/hr.
i) the probability that the cashier is idle.
ii) the avg no. of customers in the queuing system.
iii) The avg time customer spends in the system.
iv) the avg no. of customers in the queue.
v) the avg time a customer spends in the queue waiting for service.

$\lambda = 20/hr$

$\mu = 24/hr$

i) $P_0 = 1 - \rho = 1 - \frac{20}{24} = \frac{4}{24} = \frac{1}{6}$

ii) $L_s = L_q + \rho = \frac{\lambda^2}{\mu(\mu-\lambda)} + \frac{1}{6} = \frac{\cancel{400}^{100}}{24(\cancel{4})} + \frac{1}{6} = \frac{25}{6} + \frac{1}{6} = \frac{26}{6}$

$= 4.333$

iii) $W_s = W_q + \frac{1}{\mu} = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{\cancel{25}^{5}}{6} \times \frac{1}{\cancel{20}_{4}} + \frac{1}{24}$

$= \frac{5+1}{24} = \frac{1}{4} = 0.25$

iv) $L_q = \frac{25}{6}$

v) $W_q = \frac{\cancel{25}^{5}}{6} \times \frac{1}{\cancel{20}_{4}} = \frac{5}{24}$

(M/M/1): (K/FIFO) single server with Finite capacity.

If λ is the arrival rate, μ is the mean service rate then,
K = capacity (consider the one taking service too)

i) Traffic intensity or Busy period $= \rho = \frac{\lambda}{\mu}$

ii) Probability that the system is idle or there are exactly zero units $= P_0 = \frac{1-\rho}{1-\rho^{K+1}}$ for $\rho \neq 1$

iii) Effective arrival rate $= \lambda_e = \mu(1-P_0)$

iv) Average number of customers in the system is

$$L_s = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} \quad \text{when } \rho \neq 1$$

v) Waiting time of customers in the system is

$$W_s = \frac{L_s}{\lambda_e}$$

vi) No. of customers (Average) in the queue is

$$L_q = L_s - \frac{\lambda_e}{\mu}$$

vii) Average waiting time of customers in the queue is

$$W_q = \frac{L_q}{\lambda_e}$$

NOTE: When $\rho = 1$, $L_s = \frac{K}{2}$, $P_0 = \frac{1}{K+1}$

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* Find the expected no. of customers in the system for $(M/M/1):(K/FIFO)$ model if $\lambda = 4$ & $\mu = 4$ & maximum capacity is $= 7$

$$K = 7,\ \lambda = 4,\ \mu = 4$$

$K' = K+1 = 7+1 = 8$ (we should also consider the one being served too)

$$L_s = \frac{K'}{2} = \frac{8}{2} = 4$$

* Patients arrive at a doctor's clinic according to poisson distribution at a rate of 30 patients per hour. The waiting room doesn't accomodate more than 9 patients. Exam time per patient is poisson ~~exponential~~ with mean rate of 20/hr. Find the

i) probability that an arriving patient will not wait.

ii) Effective arrival rate

iii) Avg no. of patients in the clinic

iv) Expected time a patient spends in the clinic.

v) Avg no. of patients in the queue.

vi) Expected waiting time of a patient in the queue.

$\lambda = 30$

$\mu = 20$

$$\rho = \frac{\lambda}{\mu} = \frac{30}{20} = 1.5$$

i) $P_0 = \dfrac{1-\rho}{1-\rho^{K+1}} = \dfrac{1-1.5}{1-(1.5)^{11}} = 0.0058$

$\boxed{K = 9+1 = 10}$ {9 in the queue + 1 getting service}

ii) $\lambda_e = \mu(1-P_0) = 20(1-0.0058)$
$= 19.88$

iii) $L_s = \dfrac{\rho}{1-\rho} - \dfrac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} = \dfrac{1.5}{-0.5} - \dfrac{11(1.5)^{11}}{1-(1.5)^{11}} = 8.128$

iv) $W_s = \frac{L_s}{\lambda_e} = \frac{8.128}{19.88} = 0.4084$

v) $L_q = L_s - \frac{\lambda_e}{\mu} = 8.128 - \frac{19.88}{20} = 7.134$

vi) $W_q = \frac{L_q}{\lambda_e} = \frac{7.134}{19.88} = 0.3588$

* A city has 1 person barber shop which can accomodate a maximum of 5 people at a time (4 waiting & 1 getting haircut). On average, customers arrive at the rate of 8 per hour and the barber takes 6 mins for serving each customer. It is estimated that the arrival process is poisson & the service time is exponential. Find:

a) % of time the barber is idle

b) Fraction of potential customers who will be turned away

c) Effective arrival rate of customers for the shop.

d) Expected no. of customers in the barber shop.

e) Expected time a person spends in barber shop.

f) Expected no. of customers waiting for a hair-cut

g) Expected waiting time of a customer in the queue.

$K = 5$

$\mu = \frac{1}{6} \Rightarrow 60 \times \frac{1}{6} = 10/hr$

$\lambda = 8/hr$

a) $P_0 = \frac{1-\rho}{1-\rho^{K+1}} = \frac{1-\frac{\lambda}{\mu}}{1-\rho^6} = \frac{1-\frac{8}{10}}{1-(0.8)^6} = \frac{0.2}{0.737} = 0.271$

$= 27.1\%$

b) $P(\text{customers turned away}) = P(K>5)$

$= P_0 \cdot \rho^{K+1}$

$= 0.271 \times (0.8)^6 = 0.0710 4 = 0.7104$

c) $\lambda e = \mu(1-P_0) = 10(1-0.271) = 7.29$

d) $L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} = \frac{0.8}{1-0.8} - \frac{(6)\rho^6}{1-\rho^6}$

$= \frac{8}{2} - \frac{6(0.8)^6}{1-(0.8)^6} = 1.8683$

NOTE: Probability that there are 'n' customers in the system

$\Rightarrow P_n = P(\text{n customers in system})$

$= \frac{(1-\rho)\rho^n}{1-\rho^{k+1}}, \quad \rho \neq 1$

e) $W_s = \frac{L_s}{\lambda e} = \frac{1.8683}{7.29} = 0.2562$

f) $W_q = \frac{L_q}{\lambda e} = \frac{L_s - \frac{\lambda e}{\mu}}{\lambda e} = \frac{1.8683 - \frac{7.29}{10}}{7.29} = 0.1562$

g) b) $L_q = 1.1393$

(M/M/S) : (∞/FIFO) model $\Rightarrow$ Multiserver with infinite capacity.

The service rate and arrival rate follow exponential & poisson.
The no. of servers is finite = s
capacity is infinite.

λ = mean arrival rate
μ = mean service rate

i) $\rho = \frac{\lambda}{\mu}$ $\Rightarrow$ Traffic intensity or Busy period.

ii) P_0 = probability that the system is idle

$\Rightarrow P_0 = \frac{1}{\left[\sum_{n=0}^{s-1} \frac{1}{n!}\rho^n + \frac{\rho^s}{s!(1-\frac{\rho}{s})}\right]}$

iii) $P_n = \frac{P_0 \cdot \rho^n}{s!\, s^{n-s}}$ probability that there are 'n' customers in the queue.

iv) $L_q = \frac{\rho^{s+1} \cdot P_0}{s \cdot s! \left(1-\frac{\rho}{s}\right)^2}$ Length of the queue

v) $L_s = L_q + \frac{\lambda}{\mu}$ no. of customers in the system

vi) $W_s = \frac{L_s}{\lambda}$

vii) $W_q = \frac{L_q}{\lambda}$

viii) probability that an arrival has to wait for service is
$P(N \geq s) = \frac{\rho^s \cdot P_0}{s! \left(1-\frac{\rho}{s}\right)}$

ix) probability that an arrival enters a service without waiting $= 1 - \frac{\rho^s \cdot P_0}{s! \left(1-\frac{\rho}{s}\right)}$

* A travel centre has 3 services counters to recieve people who visit to book air tickets. The customers arrive in a poisson distribution with the average arrival of 100 persons in a 10 hour service day. It has been estimated that the service time follows an exponential distribution. The avg service time is 15 mins. Find

i) Expected no. of customers in the system
ii) Expected no. of customers in the queue
iii) Expected time customer spends in the system.
iv) Expected waiting time for a customer in the queue.
v) probability that a customer must wait before he gets active

$$\lambda = \frac{100}{10} = 10/hr$$

$$\mu = \frac{1}{15} \times 60 = 4/hr$$

$$\rho = \frac{\lambda}{\mu} = \frac{10}{4} = 2.5$$

$$P_0 = \frac{1}{\left[\sum_{n=0}^{2} \frac{1}{n!}(2.5)^n + \frac{(2.5)^3}{3!\,(1-\frac{2.5}{3})}\right]} = \frac{1}{1+2.5+\frac{(2.5)^2}{2!}+\frac{(2.5)^3}{3!\,(1-\frac{2.5}{3})}}$$

$$P_0 = 0.045$$

i) $$L_q = \frac{(2.5)^4 \cdot 0.045}{3 \cdot 3!\,(1-\frac{2.5}{3})^2} = 3.5$$

ii) $$L_s = L_q + \frac{\lambda}{\mu} = 3.5 + 2.5 = 6$$

iii) $$W_s = \frac{L_s}{\lambda} = \frac{6}{10} = 0.6$$

iv) $$W_q = \frac{L_q}{\lambda} = \frac{3.5}{10} = 0.35$$

v) $$P(N \geq 3) = \frac{(2.5)^3 \cdot 0.045}{3!\,(1-\frac{2.5}{3})} = 0.702$$

* A local hospital has 3 doctors for treating patients. The patients arrive at the hospital according to poissons distribution with an average of 8 per hour. On an average, a doctor takes about 15 minutes to treat each patient & the actual time is known to vary approximately exponentially around this average. Find
  - a) Traffic intensity of the system.
  - b) probability that a customer has to wait for a service
  - c) Avg. no. of customers waiting in the queue.
  - d) Avg no. of customers in the system.

e) Avg. waiting time a customer spends in the queue
f) Expected time a customer spends in the system.

a) $\rho = \frac{\lambda}{\mu}$ $\lambda = 8/\text{hour}$

$\mu = \frac{1}{15} \times 60 = 4/\text{hr}$

$\rho = \frac{\lambda}{\mu} = \frac{8}{4} = 2.$

b) $P(N \geq 3) = \frac{\rho^s \cdot P_0}{s!\,(1 - \frac{\rho}{s})}$

$P_0 = \frac{1}{\sum_{n=0}^{2} \frac{1}{n!}(\rho)^n + \frac{\rho^s}{s!(1-\frac{\rho}{s})}} = \frac{1}{\sum_{n=0}^{2} \frac{1}{n!}(2)^n + \frac{(2)^3}{3!(1-\frac{2}{3})}}$

$P_0 = \frac{1}{1+2+\frac{4}{2!} + \frac{8}{3!(1-\frac{2}{3})}} = \frac{1}{9} = 0.1111$

$P(N \geq 3) = \frac{(2)^3 \cdot \frac{1}{9}}{3!(1-\frac{2}{3})} = \frac{8 \times 3}{9 \times 6 \times 1} = \frac{4}{9}$

c) $L_q = \frac{(2)^4 \cdot 1}{3 \cdot 3! \cdot 9\,(1-\frac{2}{3})^2} = \frac{16 \times 9}{3 \times 6 \times 9 \times 1} = \frac{8}{9}$

d) $L_s = L_q + \frac{\lambda}{\mu} = \frac{8}{9} + 2 = \frac{26}{9}$

e) $W_q = \frac{L_q}{\lambda} = \frac{8}{9 \times 8} = \frac{1}{9}$

f) $W_s = \frac{L_s}{\lambda} = \frac{26}{9 \times 8} = \frac{13}{36}$

* A petrol pumb has 4 pumps. The service time follows an exponential distribution with a mean of 6 minutes and cars arrive for service in a Poisson process at the rate of

30 cars per hour. Find the average waiting time in the queue, average time spent in the system and the average no. of cars in the system.

$$\mu = \frac{1}{6} \times 60 = 10/hr$$

$$\lambda = 30/hr$$

$$\rho = \frac{\lambda}{\mu} = \frac{30}{10} = 3$$

$$P_0 = \frac{1}{\sum_{n=0}^{3} \frac{1}{n!}(3)^n + \frac{(3)^4}{4!(1-\frac{3}{4})}} = \frac{1}{1+3+\frac{9}{2}+\frac{27}{6}+\frac{81\times 4}{24(1)}} = \frac{1}{4+\frac{9}{2}+\frac{27}{2}}$$

$$= \frac{1}{22+\frac{27}{6}} = \frac{1}{22+\frac{9}{2}} = \frac{1}{26.5}$$

$$= 0.03$$

$$L_q = \frac{(3)^5 \cdot 0.03}{4 \cdot 4!(1-\frac{3}{4})^2} = \frac{3^5 \times 16 \times 0.03}{4 \times 24 \times 1} = 1.215$$

$$L_s = L_q + \frac{\lambda}{\mu} = 1.215 + 3 = 4.215$$

$$W_q = \frac{L_q}{\lambda} = \frac{1.215}{30} = 0.0405$$

$$W_s = \frac{L_s}{\lambda} = 0.1405$$

M/G/1 Queuing system

$$M/G/1 : \infty/GD$$

In this model, the arrival rate follows Poisson distribution, the service rate follows general distribution & is single server with infinite capacity.

GD - General service discipline

such as i) FIFO
ii) LIFO
iii) SIRO

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① Average no. of customers in the system $= L_s = \dfrac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}+\rho$, $\rho \neq 1$

② No. of customers in the queue $= L_q = \dfrac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}$, $\rho \neq 1$

③ Avg waiting time that a customer spends in the system

$$= W_s = \frac{\lambda^2\sigma^2+\rho^2}{2\lambda(1-\rho)}+\frac{1}{\mu} = \frac{L_s}{\lambda}+\frac{1}{\mu}$$

④ Avg waiting time of a customer in the queue =

$$W_q = \frac{\lambda^2\sigma^2+\rho^2}{2\lambda(1-\rho)} = \frac{L_q}{\lambda}$$

* 'σ' is the standard deviation of the service time

* Automatic car wash facility operates with only one bay. Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking lot if the bay is busy. If the service time for all cars is constant and equal to 10 minutes, then determine L_s, L_q, W_s & W_q.

$$\lambda = 4/hr$$

$$\mu = \frac{1\times 60}{10} = 6/hr$$

$$\rho = \frac{\lambda}{\mu} = \frac{4}{6} = 0.666$$

Since service time is constant for all cars. so, $\sigma = 0$

$$L_s = \frac{(4)^2(0)^2+\left(\frac{2}{3}\right)^2}{2\left(1-\frac{2}{3}\right)}+\frac{2}{3}$$

$$L_s = \frac{\cancel{4}^{2\times 3}}{9\times \cancel{2}}+\frac{2}{3} = \frac{12}{9}$$

$$L_q = L_s-\rho = \frac{12}{9}-\frac{2}{3} = \frac{6}{9} = \frac{2}{3}$$

$$Ws = \frac{16(0) + \left(\frac{2}{3}\right)^2 \times 3}{2 \times 4 \times 1} + \frac{1}{6}$$

$$= \frac{4 \times 8}{\cancel{9}^{3} \times 2 \times 4} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$$Wq = \frac{2}{3 \times 6} = \frac{1}{9}$$

* Customers arrive at a one-man barber shop in a remote village according to a Poisson distribution with the avg arrival rate of 8/hr. It is estimated that the service time follows a random distribution with the mean service time of 6 minutes & standard deviation equal to 15 mins.

i) Lq
ii) Ls
iii) Wq
iv) Ws

$\lambda = 8/hr$ $\qquad \rho = \frac{\lambda}{\mu} = 0.8$

$\mu = \frac{1 \times 60}{6} = 10/hr$

$\cancel{\sigma = 15 \times \frac{1}{60} = \frac{1}{4} hr}$ $\quad \sigma = 1/4$

$$Lq = \frac{(8)^2\left(\frac{1}{4}\right)^2 + \left(\frac{4}{5}\right)^2}{2\left(1 - \frac{4}{5}\right)} = \frac{\left(64 \times \frac{1}{16} + \frac{16}{25}\right) \times 5}{2}$$

$\cancel{Lq = \frac{16 \times 5}{2}\left(64 + \frac{1}{25}\right) = 40(64 + 0.04)}$

$\cancel{= 2561.6}$

$Lq = 11.6$

$Ls = Lq + \rho = \cancel{2561}\ 11.6 + 0.8$

$= \cancel{2562.4}\ 12.4$

$$Wq = \frac{\left(64 \times \frac{1}{16} + \left(\frac{4}{5}\right)^2\right) \times 5}{2 \times 8(1)} = \frac{\left(4 + \frac{16}{25}\right)5}{16} = 1.45$$

* In a heavy machine shop, the overhead crane is 75% utilized. Time study observations gave the avg service time as 10.5 minutes with standard deviation of 8.8 mins. what is the avg calling rate for the service of the crane and what is the delay in getting service if the average service time is cut to 8 minutes with $\sigma = 6$ mins how much reduction will occur on average in the delay of getting saved.

$\rho = 0.75$

$\mu = 10.5$ mins $\Rightarrow \frac{1}{10.5} \times 60 = 5.714$

$\sigma = \frac{8.8}{60} = \frac{0.88}{6} = 0.1467$

$\frac{\lambda}{\mu} = \rho$

$\frac{\lambda}{5.714} = 0.75$

$\lambda = 4.2855.$

Avg calling rate = avg waiting time of customer in the queue.

$$W_q = \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho) \times \lambda}$$

$$= \frac{(4.285)^2 (0.1467)^2 + (0.75)^2}{2(1-0.75) \times 4.285}$$

$$= \frac{1.9152}{4.285} = 0.44/\text{hour}$$

$= 26/\text{min}.$

case 2: $\mu = 8$ mins $= \frac{1}{8} \times 60 = \frac{30}{4} = \frac{15}{2} = 7.5/\text{hr}$

$\sigma = 6$ mins $= \frac{6}{60} = \frac{1}{10} = 0.1$

$\frac{\lambda}{\mu} = 0.75$

$\lambda = 0.75 \times 7.5 = 5.625$

$$W_q = \frac{(5.625)^2(0.1)^2 + (0.75)^2}{2(1-0.75)\times 5.625} = 0.3125/\text{hour}$$

$$= 18.75/\text{min.}$$

reduction in delay = $26 - 18.75 = 7.25$