Binomial distribution

Consider an experiment with only two outcomes – SUCCESS with probability p and FAILURE with probability q (q=1-p). Independent repeated trials of such an experiment are called Bernoulli trials (where the probability of success is same for each trial).

A binomial experiment consists of a fixed number of Bernoulli trials and it is denoted by B(n,p).

Bernoulli's theorem

The probability of *x* success in *n* trials is equal to $\boxed{n_{C_x}\, p^x q^{n-x}}$

Example:

The probability of getting 15 heads when a coin is tossed 25 times is

$$n_{C_x}\, p^x q^{n-x} = 25_{C_{15}} \left(\frac{1}{2}\right)^{15} \left(\frac{1}{2}\right)^{25-15}$$

Binomial Distribution

Consider a binomial experiment B(n, p) which consists of n independent repeated trials with two outcomes SUCCESS with probability p and FAILURE with probability q. (where $q = 1 - p$).

The number X of x successes is a random variable with the following distribution.

X	0	1	2	…	n
P(X)	q^n	$n_{C_1} p q^{n-1}$	$n_{C_2} p^2 q^{n-2}$	…	p^n

This distribution is called the binomial distribution since it corresponds to the successive terms of the binomial expansion

$$\boxed{(q + p)^n = n_{C_0} q^n + n_{C_1} p q^{n-1} + n_{C_2} p^2 q^{n-2} + \cdots \ldots + n_{C_n} p^n}$$

Clearly $\sum P(x_i) = (q + p)^n = 1^n = 1$

Mean & Variance of Binomial Distribution

Mean $= \mu = E(X) = \sum_{x=0}^{n} xP(x)$

$$\mu = \sum_{x=0}^{n} x\, n_{C_x}\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!\ (n-x)!}\, p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n(n-1)!}{x(x-1)!\ (n-x)!}\, p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!\ (n-x)!}\, pp^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\ (n-x)!}\ p^{x-1} q^{n-x}$$

$$\mu = np \sum_{x=1}^{n} (n-1)_{C_{(x-1)}}\ p^{x-1} q^{(n-1)-(x-1)}$$

$$\boxed{\mu = np\ (q+p)^{n-1} = np}$$

Variance

$$\sigma^2 = E(X^2) - \big(E(X)\big)^2$$

Now,

$$E(X^2) = \sum_{x=0}^{n} x^2 P(x)$$

$$= \sum_{x=0}^{n} [x(x-1) + x] P(x)$$

$$E(X^2) = \sum_{x=0}^{n} [x(x-1)P(x)] + \sum_{x=0}^{n} xP(x)$$

$$= \sum_{x=0}^{n} \left[x(x-1)n_{C_x}\, p^x q^{n-x}\right] + \mu$$

$$= \sum_{x=0}^{n} \left[x(x-1)\frac{n!}{x!\ (n-x)!}\, p^x q^{n-x}\right] + np$$

$$= \sum_{x=2}^{n} \left[x(x-1)\frac{n(n-1)(n-2)!}{x(x-1)(x-2)!\ (n-x)!}\, p^x q^{n-x}\right] + np$$

$$= n(n-1)\sum_{x=2}^{n} \left[\frac{(n-2)!}{(x-2)!\ (n-x)!}\, p^x q^{n-x}\right] + np$$

$$= n(n-1)\sum_{x=2}^{n} \left[\frac{(n-2)!}{(x-2)!\ \left((n-2)-(x-2)\right)!}\, p^2\, p^{x-2} q^{n-x}\right] + np$$

$$= n(n-1)\, p^2 \sum_{x=2}^{n} \left[(n-2)_{C_{(x-2)}}\, p^{x-2} q^{(n-2)-(x-2)}\right] + np$$

$$E\left(X^2\right) = n(n-1)\,p^2(q+p)^{n-2} + np$$

$$\boxed{E\left(X^2\right) = n(n-1)\,p^2 + np}$$

Now,

$$\therefore \sigma^2 = E\left(X^2\right) - \left(E(X)\right)^2$$

$$\Rightarrow \sigma^2 = n(n-1)\,p^2 + np - (np)^2$$

$$\Rightarrow \sigma^2 = n^2p^2 - np^2 + np - n^2p^2$$

$$\Rightarrow \sigma^2 = np(1-p)$$

$$\Rightarrow \boxed{\sigma^2 = npq}$$

Conclusion:

Mean of Binomial distribution is $\mu = np$ & variance is $\sigma^2 = npq$

1) Let x be a binomially distributed random variable with mean 2 and standard deviation $2/\sqrt{3}$. Find the corresponding probability function.

- Given μ=2, $\sigma = 2/\sqrt{3}$

Wkt μ= np and $\sigma = \sqrt{npq}$

∴np =2 and npq = $4/3$

Simplifying, we get

q=2/3 , p =1-q =1/3, n=6

Therefore the probability function for the distribution is

P(x) = b (n,p,x) = b(6,1/3,x)

$$P(x) = {}^{6}C_{x}\left(\frac{1}{3}\right)^{x}\left(\frac{2}{3}\right)^{6-x}$$

2) When a coin is tossed 4 times, find the probability of getting (i) exactly one head, (ii) at most 3 heads, and (iii) at least two heads.

- Given n=4

The probability of getting head in each trial is p=1/2.

∴ Probability of getting x heads in 4 trials is

$$P(x) = b(4,1/2,x) = {}^4C_x\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{4-x}$$

$$= {}^4C_x\left(\frac{1}{2}\right)^{4} = \left(\frac{1}{16}\right){}^4C_x$$

(i)Probability of getting exactly one head is

$$P(x=1) = \left(\frac{1}{16}\right){}^4C_1 = 1/4$$

(ii)Probability of getting at most 3 heads

$$P(x\leq 3) = P(0) + P(1) + P(2) + P(3)$$

$$=\frac{1}{16}\{ {}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 \}= \frac{15}{16}$$

(iii) Probability of getting at least two heads

$$P(x \geq 2) = 1 - P(x < 2)$$

$$= 1 - \{ P(0) + P(1) \}$$

$$= 1 - \frac{1}{16} \{ {}^4C_0 + {}^4C_1 \} = \frac{11}{16}$$

3) The probability that a pen manufactured by a company will be defective is 0.1.
If 12 such pens are selected at random, find the probability that

(i) Exactly two pens will be defective
(ii) At most two pens will be defective.
(iii) At least two pens will be defective
(iv) none will be defective.

- Hint: n=12

Let p be the probability that a pen manufactured is defective.

Then $p=0.1$, $q = 1 - p = 0.9$

∴ Probability that x pens are defective out of 12 is

$P(x) = b(12,0.1,x) = {}^{12}C_x(0.1)^x(0.9)^{12-x}$

(i) Probability that exactly two pens will be defective is $P(x=2) =$

(ii) Probabilitythat atmost 2 pens will be defective is $P(x\leq 2)$

(iii) Probabilitythat atleast two pens will be defective is $P(x\geq 2)$

(iv) Probability that none of the pens will be defective is $P(x=0)$

4) The number of telephone lines busy at an instant of time is a binomial variate with probability 0.2. If at an instant 10 lines are chosen at random, what is the probability that (i) 5 lines are busy? (ii) at most 2 lines are busy?

- Hint: n=10

Let p be the probability that a telephone line is busy.

Then p=0.2 , q = 1- p = 0.8

∴Probabilitythat x lines arebusy out of 10 is

$$P(x) = b(10,0.2,x) = {}^{10}C_x(\mathbf{0.2})^{x}(\mathbf{0.8})^{\mathbf{10}-x}$$

(i) Probability that 5 lines are busy is

$P(x=5)=$

(ii) Probability that atmost 2 lines are busy is

$P(x\leq2)=$

5) The probability that a man aged 60 will live up to 70 is 0.65. What is the probability that out of 10 men, now aged 60, at least 7 will live up to 70.

- Hint: $n=10$

  Let p be the probability that the man aged 60 will live to be 70.

  Then $p = 0.65$, $q = 0.35$

  Required probability is $P(x \geq 7)=$

6) In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such items, how many would be expected to contain at least 3 defective parts.

- Let p be the probability that a sample of items contain defective part.

Given $\mu=2$ for n=20.

$\therefore$ μ= np $\Rightarrow$ p = 2/20 = 0.1

$\therefore$ q = 0.9

The probability function for the distribution is

$$P(x) = {}^{20}C_x(0.1)^x(0.9)^{20-x}$$

Therefore the probability that there are atleast 3 defective parts in an item is

P(x $\geq$ 3) = 1- P(x< 3) = 0.323

Hence, the expected number of items that contain at least three defective parts in a sample of 1000 items is

0.323 x 1000 = 323

7) In a bombing action there is a 50% chance that any bomb will strike the target. Two direct hits are needed to destroy the target completely. How many bombs are required to be dropped to give a 99% chance or better chance of completely destroying the target?

- $p = 50\% = ½ , q = ½$

Let n be the number of bombs to be dropped, then the probability that x bombs will strike the target is given by the probability function

$$P(x) = {}^nC_x\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x} = {}^nC_x\left(\frac{1}{2}\right)^n$$

Since at least two bombs are required to destroy the target completely, the probability that at most n bombs would destroy the target is $P(2 \leq x \leq n)$.

$$P(2 \leq x \leq n) = 1 - P(x < 2)$$

$$= 1 - \{P(0) + P(1)\}$$

$$= 1 - P(0) - P(1)$$

$$= 1 - {}^{n}C_0\left(\frac{1}{2}\right)^n - {}^{n}C_1\left(\frac{1}{2}\right)^n$$

$$= 1 - \left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)^n$$

$$P(2 \leq x \leq n) = 1 - \frac{n+1}{2^n}$$

This probability is greater than or equal to 99% if

$$P(2 \leq x \leq n) \geq \frac{99}{100}$$

$$\Rightarrow 1 - \frac{n+1}{2^n} \geq \frac{99}{100}$$

$$\Rightarrow \quad -\frac{n+1}{2^n} \geq \frac{99}{100} - 1$$

$$\Rightarrow \quad -\frac{n+1}{2^n} \geq -\frac{1}{100}$$

$$\Rightarrow \quad \frac{n+1}{2^n} \leq \frac{1}{100}$$

The above inequality holds good if $n \geq 11$.

Thus a minimum of 11 bombs are to be dropped to get a 99% or better chance of completely destroying the target.

8) An airline knows that 5 percent of people making reservations on a certain flight will not turn up. Consequently, their policy is to sell 52 tickets for a flight that can only hold 50 passengers. What is the probability that there will be a seat available for every passenger who turns up?

{Hint: p =P[passenger will not turn up}

$$p = 0.05$$

n=52, required probability is $P[x \geq 2]$

Ans: 0.7405 }