Example 1. *A random variable x is uniformly distributed for $-2 < x < 2$. Find the mean and standard deviation. Also, evaluate $P(x < 1)$, $P(|x| > 1)$ and $P(|x - 1| \geq 1/2)$.*

» Here, the density function is (by virtue of expression (1))

$$u(x) = u(-2, 2, x) = \begin{cases} \frac{1}{4} & \text{for } -2 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Therefore, the mean is (by expression (2)) $\mu = \frac{1}{2}(-2+2) = 0$ and the standard deviation is (by expression (4))

$$\sigma = \frac{1}{2\sqrt{3}}(2 + 2) = \frac{2}{\sqrt{3}}.$$

Next, we find that

(i) $P(x < 1) = \int_{-2}^{1} u(x)\,dx = \int_{-2}^{1} \frac{1}{4}\,dx = \frac{3}{4}$,

(ii) $P(|x| > 1) = 1 - P(|x| \leq 1) = 1 - P(-1 \leq x \leq 1)$

$$= 1 - \int_{-1}^{1} u(x)\,dx = 1 - \int_{-1}^{1} \frac{1}{4}\,dx = \frac{1}{2},$$

(iii) $P\left(|x - 1| \geq \frac{1}{2}\right) = 1 - P\left(-\frac{1}{2} < (x-1) < \frac{1}{2}\right)$

$$= 1 - P\left(\frac{1}{2} < x < \frac{3}{2}\right) = 1 - \int_{1/2}^{3/2} u(x)\,dx$$

$$= 1 - \int_{1/2}^{3/2} \frac{1}{4}\,dx = 1 - \frac{1}{4}\left(\frac{3}{2} - \frac{1}{2}\right) = \frac{3}{4}$$ ■

Example 2. *On a certain city transport route, buses ply every 30 minutes between 6 a.m. and 10 p.m. If a person reaches a bus stop on this route at a random time during this period, what is the probability that he will have to wait for at least twenty minutes ?*

» Let x denote the waiting time (in minutes) for the next bus. Then x is distributed uniformly over the interval (0, 30) with probability density function

$$u(0, 30, x) = \begin{cases} \frac{1}{30}, & 0 < x < 30 \\ 0, & \text{elsewhere} \end{cases}$$

Therefore, the probability that a person has to wait for atleast 20 minutes is

$$P(x \geq 20) = \int_{20}^{30} u(0, 30, x)\,dx = \frac{1}{30}\int_{20}^{30} dx = \frac{1}{3}.$$ ■

Example 3. *For the uniform distribution over the interval* (a, b), *find the moment generating function about 0.*

» The m.g.f. about 0 is given by

$$E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} p(x)\, dx$$

$$= \int_{a}^{b} e^{tx} \frac{1}{b-a}\, dx, \quad \text{for the uniform distribution.}$$

$$= \frac{1}{b-a} \cdot \frac{(e^{tb} - e^{ta})}{t}$$ ∎

Example 4. *Find the cumulative distribution function* (C.D.F) *for a uniform distribution in the interval* (a, b).

» The $P.D.F.$ for a uniform distribution in (a, b) is

$$P(x) = u(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a < x < b \\ 0 & \text{elsewhere} \end{cases} \quad \ldots \text{(i)}$$

Therefore the $C.D.F.$ is given by*

$$F(t) = \int_{-\infty}^{t} p(x)\, dx = \int_{-\infty}^{t} u(x)\, dx \quad \ldots \text{(ii)}$$

If $t \leq a$, (ii) yields $F(t) = 0$, because $u(t) = 0$ for $t \leq a$.

If $a < t < b$, then (i) and (ii) yield

$$F(t) = \int_{-\infty}^{a} 0 \cdot dx + \int_{a}^{t} \frac{1}{b-a}\, dx = \frac{(t-a)}{b-a}$$

If $t \geq b$, then (i) and (ii) yield

$$F(t) = \int_{-\infty}^{a} 0 \cdot dx + \int_{a}^{b} \frac{1}{b-a}\, dx + \int_{b}^{t} 0 \cdot dx$$

$$= \frac{1}{b-a} \cdot (b-a) = 1$$

Thus, the required $C.D.F.$ is

$$F(t) = \begin{cases} 0 & \text{for } t \leq a \\ \dfrac{t-a}{b-a} & \text{for } a < t < b. \\ 1 & \text{for } t \geq b \end{cases}$$

■