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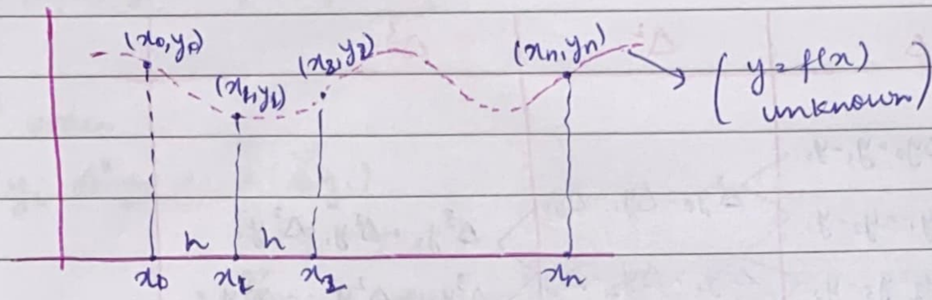
UNIT - I

FINITE DIFFERENCES, interpolation, numerical differentiation & intersection

FINITE DIFFERENCES :-



consider a set of points $(x_0,y_0), (x_1,y_1), \ldots (x_n,y_n)$ of an unknown function $y=f(x)$. Let x_i's be equispaced with spacing h, then

$$\left.\begin{aligned} x_1 - x_0 &= h \\ x_2 - x_1 &= h \\ &\vdots \\ x_n - x_{n-1} &= h \end{aligned}\right\}$$

$x_2 = x_1 + h = (x_0+h)+h = x_0 + 2h$

$x_3 = x_0 + 3h$

$\boxed{x_n = x_0 + nh}$

- forward difference

→ 1st order forward difference

$\boxed{\Delta y_i = y_{i+1} - y_i}$

eg $\Delta y_4 = y_5 - y_4$

→ 2nd order forward difference

$$\begin{aligned}\Delta^2 y_i &= \Delta(\Delta y_i) \\ &= \Delta(y_{i+1} - y_i) \\ &= \Delta y_{i+1} - \Delta y_i \\ &= (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i) \\ &= y_{i+2} - 2y_{i+1} + y_i\end{aligned}$$

→ 3rd order forward difference

$$\begin{aligned}\Delta^3 y_i &= \Delta(\Delta^2 y_i) \\ &= \Delta(y_{i+2} - 2y_{i+1} + y_i) \\ &= \cancel{y_i}\ y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i\end{aligned}$$

Pascal's Pattern

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

4^{th} order forward difference

$$\Delta^4 y_i = y_{i+4} - 4y_{i+3} + 6y_{i+2} - 4y_{i+1} + y_i$$

- **Forward difference table**

	x	y	Δ	Δ^2	Δ^3
arguments	x_0	y_0			
			$\Delta y_0 = y_1 - y_0$		
	x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
			$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
	x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
			$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$
	x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$	
			$\Delta y_3 = y_4 - y_3$		
	x_4	y_4			
		entities			

Here,
y_0 is known as leading entry
$\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$ — leading differences

Q construct the forward difference for

$f(x) = x^3 + 4x$ for $x = 0\ (1)\ 5$

$x_0 \quad h \quad x_n$

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	0					
		$\Delta y_0 = 5$				
1	5		6			
		$\Delta y_1 = 11$		6		
2	16		12		0	
		$\Delta y_2 = 23$		6		0
3	39		18		0	
		$\Delta y_3 = 41$		6		
4	80		24			
		$\Delta y_4 = 65$				
5	145					

What is $\Delta^2 y_2$?
Ans (18)

NOTE: If $f(x)$ is a polynomial of degree n then n^{th} order forward difference is ~~for~~ all constant & differences of order $> n$ are all zero.

- Backward difference

→ 1^{st} order backward difference ∇ (∇ - nabla or del)

$$\nabla \leftarrow \boxed{\nabla y_i = y_i - y_{i-1}}$$

→ 2^{nd} order

$$\begin{aligned}
\nabla^2 y_i \cancel{\Delta^2 y_i} &= \nabla(\nabla y_i) \\
&= \nabla(y_i - y_{i-1}) \\
&= \nabla y_i - (\nabla y_{i-1} - y_{i-2}) \\
&= \nabla y_i - y_{i-1} + y_{i-2} \\
&= y_i - 2y_{i-1} + y_{i-2}
\end{aligned}$$

→ 3^{rd} order

$$\begin{aligned}
\nabla^3 y_i \cancel{\Delta^3 y_i} &= \nabla(\nabla^2 y_i) \\
&= \cancel{\Delta(y_i - 2y_{i-1} + y_{i-2})} \\
&= \cancel{\Delta y_i - \Delta 2y_{i-1} + \Delta y_{i-2}}
\end{aligned}$$

NOTE: (i) Backward difference table is same as forward difference table but only naming is ~~diff.~~ reversed

x	y	∇	∇^2	∇^3
x_0	y_0			
		$y_1 - y_0 = \nabla y_1$		
x_1	y_1		$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$	
		$y_2 - y_1 = \nabla y_2$		$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$
x_2	y_2		$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$	
		$y_3 - y_2 = \nabla y_3$		
x_3	y_3			

NOTE: (ii) If $f(x)$ is a polynomial of degree n then n^{th} order backward difference is all constant and the differences of order greater than n are all zero

(iii) Relation b/w forward & backward difference operator.

PTO.

$(a+b)^3 = {}^3C_0 a^3 + {}^3C_1 a^2 b + {}^3C_2 ab^2 + {}^3C_3 b^3$

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Wkt,

$$\Delta y_i = \underbrace{y_{i+1} - y_i}_{\nabla y_{i+1}}$$

$$\boxed{\Delta y_i = \nabla y_{i+1}}$$

(iv) SHIFT OPERATOR (E):

$$E(y_i) = y_{i+1}$$
$$E^2(y_i) = y_{i+2}$$
$$E^{-1}(y_i) = y_{i-1}$$

$$\boxed{E^n[f(x)] = f(x+nh)}$$

(Questions on finite differences are to be done later)

(v) Relation b/w E, Δ & ∇

$$\Delta(y_i) = y_{i+1} - y_i = E(y_i) - y_i = y_i(E-1)$$

$$\boxed{\Delta = E - 1} \text{ — (1)}$$

Now,

$$\nabla y_i = y_i - y_{i-1}$$
$$= y_i - E^{-1}(y_i)$$
$$= y_i\left(1 - \frac{1}{E}\right)$$

$$\boxed{\nabla = 1 - E^{-1}} \text{ — (2)}$$

(vi) Binomial Expansion

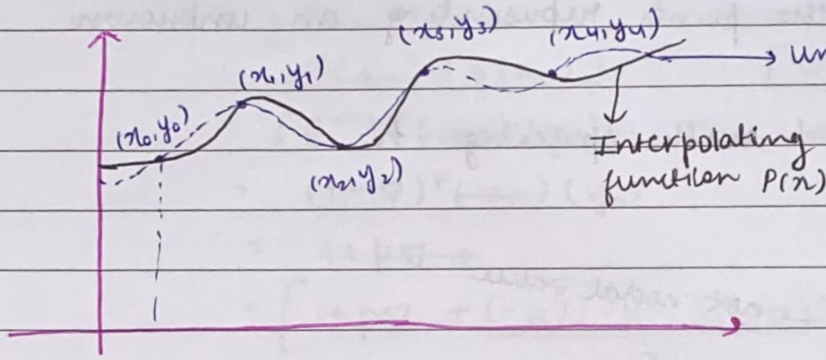
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \text{ ------}$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \text{ ----}$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^2 + \text{ ----}$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \text{ ----}$$

* INTERPOLATION *



$$\boxed{P(x_i) = f(x_i)} \quad \forall\ i = 0 \text{ to } n$$

Consider $(n+1)$ points $\overbrace{(x_i, y_i)}^{\text{nodal values}}$, $i=0$ to n of an unknown function $y=f(x)$. we try to determine a function $P(x)$ passing through all the given points exactly and may differ from $f(x)$ but ~~known~~ non-nodal values.

i.e $\quad P(x) = f(x)\ \forall\ i=0$ to n

where $P(x)$ is known as interpolating function and the technique of determining $P(x)$ is known as Interpolation.

NOTE: (i) Interpolation is the process of estimating the value of a dependant variable b/w the intermidiate values of independant variables.

(ii) Interpolation can be used to determine the y value outside the given range also but only in the small neighbourhood otherwise the error will be large (This process is actually called extra-

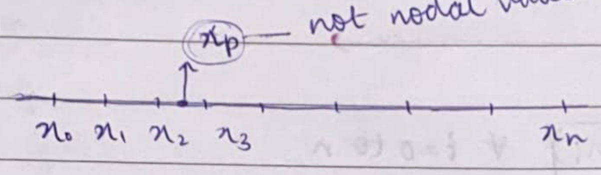
NEWTON GREGORY FORWARD INTERPOLATION FORMULA

Let (x_i, y_i) $i = 0$ to n be the points representing an unknown function, $y = f(x)$

Let the data be equispaced with spacing h

i.e. $x_n = x_0 + nh$

If y is required at some x_p — not nodal value



$x_p = x_0 + ph$

then,

$$y = y(x_p)$$
$$= y(x_0 + ph)$$
$$= E^p[y(x_0)]$$
$$= (\Delta + 1)^p y(x_0)$$
$$= \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)\Delta^3}{3!} + \cdots\right] y_0$$

$$\boxed{y = y_0 + p\Delta y_0 + \frac{p(p-1)\Delta^2 y_0}{2!} + \frac{p(p-1)(p-2)\Delta^3 y_0}{3!} + \cdots} \quad (1)$$

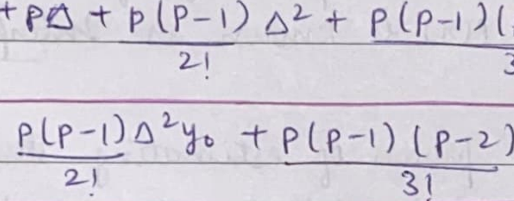
$$\boxed{p = \frac{x_p - x_0}{h}}$$

eq^n (1) is NGFIF which is used to interpolate value of y in the neighbourhood of x_0.

NEWTON GREGORY BACKWORD INTERPOLATION FORMULA

Let (x_i, y_i) $i = 0$ to n be the points representing an unknown function, $y = f(x)$

Let



$x_p = x_n + ph$ (p is $-$ve here)

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then, $y = y(x_p)$

$= y(x_n + ph)$

$= E^p \cancel{[x_n]} [y(x_n)]$

$= (E^{-1})^p \cancel{(x_n)} (y_n)$

$= (1-\nabla)^p \cancel{(x_n)} (y_n)$

$= \cancel{1+p\nabla+}$

$= \left[1 + p\nabla + \frac{(-p)(-p-1)(-\nabla)^2}{2!} - \frac{(-p)(-p-1)(-p-2)(-\nabla)^3}{3!} + \ldots\right](y_n)$

WKt,
$\boxed{\nabla = 1 - E^{-1}}$

Here,
$y_n = y(x_n)$

$$\boxed{y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n - - -} \quad —(2)$$

$$\boxed{p = \frac{x_p - x_n}{h}}$$

eq^n (2) is known as newton gregory backword interpolation formula which is used to estimate the value of y in the neighbourhood of x_n.

NOTE: Both the Newton gregory backward & forward interpolation techniques are used only if the data is equispaced.

Q1. Find y(1.5) from the following data

x	1 (x_0)	2	3	4	5 (x_n)
$y = x^3$ (unknown)	1	8	27	64	125

sol^n:- data is equispaced with spacing $h = 1$

$x_p = 1.5$ is near ~~to~~ x_0 & hence we can use NGFIF

$$P = \frac{x_p - x_0}{h} = \frac{1.5-1}{1} = 0.5$$

WKt,

$$y = y(x_p) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \text{-------} \quad —(1)$$

Let us construct the forward difference table

x	y	Δ	Δ^2	Δ^3	Δ^4
1	$1 = y_0$				
		$7 = \Delta y_0$			
2	8		$12 = \Delta^2 y_0$		
		19		$6 = \Delta^3 y_0$	
3	27		18		$0 = \Delta^4 y_0$
		37		6	
4	64		24		
		61			
5	125				

→ Leading entries

$\therefore$ From eq^n (1)

$$y(x_p) = (1) + (0.5)(1) + \frac{(0.5)(0.5-1)}{2!}(12) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(6) + 0 + 0 + \ldots$$

Remember: we are not truncating the series higher differences are all zero.

$y(1.5) = 1 + 3.5 - 1.5 + 0.375 = 3.375$

What happens if we use backward difference formula to compute $y(1.5)$?

Let us construct backward difference table

x	y	∇	∇^2	∇^3	∇^4
1	1				
		7			
2	8		12		
		19		6	
3	27		18		$0 = \nabla^4 y_n$
		37		$6 = \nabla^3 y_n$	
4	64		$24 = \nabla^2 y_n$		
		$61 = \nabla y_n$			
5	$125 = y_n$				

→ Leading entries

$$p = \frac{x_p - x_n}{h} = \frac{1.5 - 5}{1} = -3.5$$

By using NGBIF,

$$y(x_p) = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n \dots$$

$$y(1.5) = 125 + (-3.5)(61) + \frac{(-3.5)(-3.5+1)(24)}{2} + \frac{(-3.5)(-3.5+1)(-3.5+2)(6)}{6} + 0$$

$\therefore\ y(1.5) = 3.375$ ← same & exact answer

NOTE : When we are getting same answer in both forward & backward interpolation techniques, then why do we need two difference formulae ??

Q2 Determine the interpolating function satisfying the following data using NGFIF.

x	1	2	3	4	5
$y = x^3$	1	8	27	64	125

$P = \frac{x_p - x_0}{h} = \frac{x-1}{1} = x-1$ { $x_p = x$, as we need the interpolating funⁿ

By using NGFIF,

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)\Delta^2 y_0}{2!} + \frac{P(P-1)(P-2)\Delta^3 y_0}{3!} + \dots$$

(By using previous ques table)

$$y = 1 + (x-1)(7) + \frac{(x-1)(x-2)(12)}{2} + \frac{(x-1)(x-2)(x-3)(6)}{6} + 0 + 0 \dots$$

$$y = 1 + (7x-7) + (6x^2 -$$

$\therefore\ y = x^3$

(If we use NGBIF, we'll get the same answer)

Q Find the interpolating function satisfying the following data

x	1	2	3
$y = x^3$	1	8	27

$p = \frac{x_p - x_0}{h} = \frac{x-1}{1} = x-1$

Forward diff table,

x	y	Δ	Δ^2
1	1		
		7	
2	8		12
		19	
3	27		

NGFIF,

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 \text{ ------}$$

As $\Delta^3 y_0$, $\Delta^4 y_0$ & so on are not available, we have to truncate the series

$$\Rightarrow \quad y = 1 + (x-1)(7) + \frac{(x-1)(x-2)}{2!}(12) + \text{ ---}$$

$$y = 1 + 7x - 7 + (x^2 - 3x + 2)6 \leftarrow \text{Truncated series}$$

$$\boxed{y = 6x^2 - 11x + 6} \rightarrow \text{Interpolating fun}^n \; P(x)$$

Actual fun^n - $f(x) = x^3$

Note: $P(x_i) = f(x_i)$, $i = 0, 1, 2$

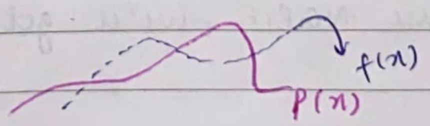
x	$P(x)$ $(6x^2 - 11x + 6)$	$f(x)$ x^3
1	1	1
2	8	8
3	27	27

So, $P(x)$ is satisfying the conditions given.

But,

$P(1.5) \neq f(1.5)$

$\downarrow$ 3 $\downarrow$ 3.375



NOTE: If the data is given at (n+1) points then the degree of interpolating polynomial will be less than or equal to n.

Q Determine the interpolating funⁿ satisfying the following data

x	1	2	3	4	5 → radian
$y = \sin x$	0.8414	0.9093	~~0.9568~~ 0.1411	-0.9568	-0.9589

forward diff table,

x	y	Δ	Δ^2	Δ^3	Δ^4
1	0.8414				
		0.0678			
2	0.9093		-0.8359		
		-0.7681		0.7061	
3	0.1411		-0.1298		0.1195
		-0.8979		0.8256	
4	-0.9568		0.6958		
		-0.2021			
5	-0.9589				

$$P = \frac{x_p - x_0}{h} = \frac{x-1}{1} = x-1$$

NGFIF,

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0 + \dots$$

$$y = 0.8414 + (x-1)(0.0678) + \frac{(x-1)(x-2)(-0.8359)}{2} + \frac{(x-1)(x-2)(x-3)(0.7061)}{6} + \frac{(x-1)(x-2)(x-3)(x-4)(0.1195)}{24}$$

$$= 0.8414 + 0.0678x - 0.0678 + \frac{(x^2-3x+2)(-0.8359)}{2} + \frac{(x^3-6x^2+11x-6)(0.7061)}{6} + \frac{(x^4-10x^3+35x^2-50x+24)(0.1195)}{24}$$

$$= 0.7734 + 0.0678x + (x^2-3x+2)(0.4179) + (x^3-6x^2+11x-6)(0.1176) + (x^4-10x^3+35x^2-50x+24)(4.979\times10^{-3})$$

$=$

H.W.

Q Find $y(0.3)$ from the following table

x	0	0.2	0.4	0.6	0.8
$y = \cancel{\sin(x+4)}$ $\sin x + 4x$	0	0.9986	1.9894	2.9646	3.9173

$$P = \frac{x_p - x_0}{h} = \frac{0.3-0}{0.2}$$

$$= 1.5$$

$$y = y_0 + P\Delta y_0 + P(P-1)\Delta y_0^2 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0 \ldots$$

x	y	Δ	Δ^2	Δ^3	
0	0				
		0.9986			
0.2	0.9986		-0.0078		
0.4	1.9894	0.9908		-0.0078	
			-0.0156		0.0009
0.6	2.9646	0.9752		-.0069	
			-0.0225		
0.8	3.9173	0.9527			

$$y = 1.5 \times 0.9986 + \frac{(1.5)(0.5)(-0.0078)}{2} + \frac{(1.5)(0.5)(0.5)(0.0078)}{6}$$

$$+ \frac{(1.5)(0.5)(-0.5)(-1.5)(0.0009)}{24}$$

$$= 1.49548$$

Q Find the approximate no. of students who scored marks between 90 & 95. from the following data

marks	no. of students
0-20	2
20-40	4
40-60	10
60-80	25
80-100	12

First let us construct a cumulative frequency table

x (marks less than)	20	40	60	80	100
y (no of students)	2	2+4 = 6	2+4+10 = 16	2+4+10+25 = 41	2+4+10+25+12 = 53

$\left(\text{No of students b/w } 90 \,\&\, 95 \text{ marks}\right) = y(95) - y(90)$ —— ①

Now,

Let us construct the backward difference table

x	y	∇	∇^2	∇^3	∇^4
20	2				
		4			
40	6		6		
		10		9	
60	10		5		-37
		25		-28	
80	41		-13		
		12			
100	53				

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n \text{ ------}$$

For $y(95)$,

$$p = \frac{95-100}{20} = -0.25$$

$$y(95) = 53 - 0.25\times 12 + \frac{0.25(0.75)}{2}\times 13 + 0.25(0.75)(1.75)\cancel{(2.75)}\times 28/6 + \frac{0.25(0.75)(1.75)(2.75)}{24}\times 37$$

$$= 53 - 3 + 1.21875 + 1.53125 + 1.39113$$

$$= 54.1411 \simeq 54$$

For $y(90)$,

$$p = \frac{90-100}{20} = -0.5$$

$y(90) = 53 - 0.5\times12 + \frac{0.5\times0.5\times13}{2} + \frac{0.5\times0.5\times1.5\times28}{6} + \frac{0.5\times0.5\times1.5\times2.5\times37}{24}$

$= 47 + 1.625 + 1.75 + 1.4453125$

$= 54.8203 - 3$

$= 51.8203 \simeq 52$

$\therefore$ No. of students b/w 90 & 95 $= 54 - 52$

$= \underline{\underline{2}}$

INTERPOLATION FOR UNEQUISPACED DATA

* Lagrange's interpolation

consider the following data which need not be equispaced (can or cannot be)

x	x_0	x_1	x_2	x_3
$f(x)=y$	y_0	$\cancel{x_2}y_2$	y_2	y_3

If we define,

$$P(x) = y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3 \quad \text{— (1)}$$

$P(x) \downarrow$ Interpolating function

clearly $P(x)$ passes through the given points

i.e., $P(x_0) = (1)y_0 + 0 + 0 + 0 = y_0 = f(x_0)$

similarly, $P(x_i) = y_i = f(x_i) \; \forall i = 1 \text{ to } 3$

$\therefore$ eq^n (1) is the interpolating function representing the given data and it is known as Lagrange's interpolating polynomial

we rewrite eq^n (1) as

$$\boxed{y = l_0(x)\times y_0 + l_1(x)\times y_1 + l_2(x)\times y_2 + l_3(x)\times y_3}$$

$$\begin{Bmatrix} \ln = \log_e x \\ \log x = \log_{10} x \end{Bmatrix}$$

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In general,

for (x_i, y_i) $i = 0$ to n

$$\boxed{y = l_0(x) \times y_0 + l_1(x) \times y_1 + l_2(x) \times y_2 + \text{------} \; l_n(x) \times y_n}$$

where, $l_i(x) = \dfrac{(x-x_0)(x-x_1) \text{---} (x-x_{i-1})(x-x_{i+1})(x_i - x_n)}{(x_i - x_0)(x_i - x_1) \text{---} (x_i - x_{i-1})(x_i - x_{i+1})(x_i - x_n)}$

Here, $l_i(x)$ is known as Lagrange's fundamental polynomial

clearly, $l_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

eg $l_0(x) = \dfrac{(x-x_1)(x-x_2)}{(x_0 - x_1)(x_0 - x_2)}$

$l_0(x_0) = 1$

$l_0(x_1) = 0$

$l_0(x_2) = 0$

Q Find $y(0.5)$ from the following data using lagrange's interpolation technique.

	x_0	x_1	x_2	x_3
x	0.2	1	1.4	2.2
$y = \log_{10} x$	-0.6989	0	0.1461	0.3424
	y_0	y_1	y_2	y_3

$$y(x) = l_0(x) \times y_0 + l_1(x) \times y_1 + l_2(x) \times y_2 + l_3(x) \times y_3 \quad \text{—} \; (1)$$

where,

$\Rightarrow$ $l_0(x) = \dfrac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \dfrac{(0.5-1)(0.5-1.4)(0.5-2.2)}{(0.2-1)(0.2-1.4)(0.2-2.2)}$

$= \dfrac{(-0.5) \times (-0.9) \times (-1.7)}{(-0.8) \times (-1.2) \times (-2)} = 0.3984$

$\Rightarrow$ $l_1(x) = \dfrac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \dfrac{(0.5-0.2)(0.5-1.4)(0.5-2.2)}{(1-0.2)(1-1.4)(1-2.2)}$

$= \dfrac{0.3 \times (-0.9)(-1.7)}{(0.8)(-0.4)(-1.2)} = 1.1953$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(0.5-0.2)(0.5-1)(0.5-2.2)}{(1.4-0.2)(1.4-1)(1.4-2.2)} = -0.6640$$

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(0.5-0.2)(0.5-1)(0.5-1.4)}{(2.2-0.2)(2.2-1)(2.2-1.4)}$$

$$= \frac{0.3\times(-0.5)(-0.9)}{2\times 1.2\times 0.8} = 0.07031$$

Finally,

$$y(0.5) = (0.3984)(-0.6989) + 0 + (-0.6640)(0.1961) + (0.07031)(0.3424)$$

$$= -0.2784 - 0.1302 + 0.02407$$

$$= \cancel{-0.3845} \quad -0.3514$$ {exact ans is -0.3010}

Q Find lagrange's interpolating polynomial from the following data

x	1	2	3
$y = x^4$	1	16	81

$$y(x) = l_0(x)y_0 + l_1(x)y_1 + l_2(x)y_2$$

$$\Rightarrow l_0(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{x^2-5x+6}{2}$$

$$\Rightarrow l_1(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = \frac{x^2-3x-x+3}{-1} = -x^2+4x-3$$

$$\Rightarrow l_2(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{x^2-x-2x+2}{2} = \frac{x^2-3x+2}{2}$$

$$y(x) = \frac{x^2-5x+6}{2}\times 1 + (-x^2+4x-3)\times 16 + \frac{x^2-3x+2}{2}\times 81$$

$$= (x^2-5x+6 - 32x^2 + \cancel{64x}^{128x} - \cancel{48}^{96} + 81x^2 - 243x + \cancel{162}^{162})/2$$

$$= (50x^2 - 120x + 72)/2 = 25x^2 - 60x + 36$$

$$\approx x^4 \text{ in } (1,3)$$

check

nodal values

x	y	y_{exact}
1	1	1
2	16	16

non nodal values

x	y	y_{exact}
1.5	2.25	5.0625

large deviation

NOTE: The reason for large deviation in numerical value of y & exact value is the large gap between the x_i values and also the limited data.

INVERSE INTERPOLATION using Lagrange's technique

Inverse interpolation is the technique of determining the value of independent variable (x) for a given value of dependent variable (y).

Consider following data

x	x_0	x_1	x_2	x_3
y	y_0	y_1	y_2	y_3

For eg, if roots of algebraic or trancendental eq^n (say $f(x)=0$) is required then we have to determine the value of x at which $y=0$.

→ Lagrange's formula can be suitably adjusted for inverse interpolation.

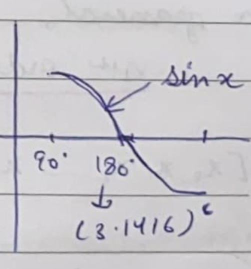
Lagrange's inverse interpolation formula is

$$x(y) = l_0(y)x_0 + l_1(y)x_1 + l_2(y)x_2 + l_3(y)x_3$$

where, $l_0(y) = \dfrac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}$, $|||^{ly}$ $l_i(y)$ are defined for $i=1$ to 3

Q Determine an approximation root of $f(x)=0$ from ~~which~~ the following table using Lagrange method.

	x_0	x_1	x_2	x_3
x	3	3.1	3.3	3.6
$y=\sin x$	0.1411	0.0415	-0.1577	-0.4425
	y_0	y_1	y_2	y_3



we need x value or which $y=0$

Lagrange's inverse interpolation formula is

$$x(y) = l_0(y)x_0 + l_1(y)x_1 + l_2(y)x_2 + l_3(y)x_3$$

$$x(0) = l_0(0)(3) + l_1(0)(3.1) + l_2(0)(3.3) + l_3(0)(3.6) \quad \text{— (1)}$$

$$l_0(y) = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}$$

$$\Rightarrow l_0(0) = \frac{(0-0.0415)(0+0.1577)(0+0.4425)}{(0.1411-0.0415)(0.1411+0.1577)(0.1411+0.4425)}$$

$$= 0.1668$$

$\Rightarrow l_1(0) = 1.0254$

$\Rightarrow l_2(0) = 0.1528$

$\Rightarrow l_3(0) =$

$\therefore x(0) = 0.1668 \times 3 + 1.0254 \times 3.1 + 0.1528 \times 3.3 + \quad \times 3.6$

$= 3.1418$

DIVIDED DIFFERENCES

First order divided difference (D.D) of x_0 & x_1 denoted by $[x_0, x_1]$ is defined as $[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$

|||ly 2^{nd} order D.D of x_0, x_1, x_2 is definitely,

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

In general,

n^{th} order D.D is,

$$[x_0, x_1, x_2 \ldots x_n] = \frac{[x_1, x_2 \ldots x_n] - [x_0, x_1 \ldots x_{n-1}]}{x_n - x_0}$$

Divided difference table

consider the following data,

x	1	2	3	5	6
$y = x^3$	1	8	27	125	512

	x	y	1st D.D	2nd D.D	3rd D.D	4th D.D
x_0	1	1	$[x_0,x_1] = \frac{8-1}{2-1} = 7$	$[x_0,x_1,x_2] = 6$	$[x_0,x_1,x_2,x_3]$ 1	0
x_1	2	8	$[x_1,x_2] = \frac{27-8}{3-2} = 19$	$[x_1,x_2,x_3] = 10$	$[x_1,x_2,x_3,x_4]$ 1	
x_2	3	27	$[x_2,x_3] = \frac{125-27}{5-3} = 49$	$[x_2,x_3,x_4] = 16$		
x_3	5 ~~4~~	125	$[x_3,x_4] = \frac{512-125}{6-5} = 387$			
x_4	6	512				

NEWTON'S D.D INTERPOLATION FORMULA

$$y(x) = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3] + \text{------}$$

Q Find NDD interpolating polynomial from the data. (Refer above table)

$$y(x) = 1 + (x-1)(7) + (x-1)(x-2)(6) + (x-1)(x-2)(x-3)(1)$$
$$= 1 + 7x - 7 + (x^2-3x+2)6 + (x^3-6x^2+11x-6)$$
$$= \cancel{x^3+x^2-5} \quad x^3$$

Q Find NDD interpolating polynomial from the following data.

x	0	1	3	4
$y = x^6+1$	1	2	730	4097

x	y	1st D.D	2nd D.D	3rd D.D
0	1	1	$\frac{\cancel{728}^{364}-1}{3-0} = \cancel{242.67}^{121}$	$\frac{879.67-242.67}{4-0} = \cancel{159.25}^{220}$
1	2	$\frac{730-2}{3-1} = \cancel{728}^{364}$	$\frac{3367-\cancel{728}^{364}}{4-1} = \cancel{879.67}^{1001}$	
3	730	$\frac{4097-730}{4-3} = 3367$		
4	4097			

$$y(x) = 1 + (x-0)(1) + (x-0)(x-1)(242.67) + (x-0)(x-1)(x-3)(159.25)$$
$$= 1 + x + 242.67x^2 - 242.67x + 159.25(x^3 - 4x^2 + 3x)$$
$$= 159.25x^3 - 394.33x^2 \quad \times$$

$$y(x) =$$

H.W.

Q Find $y(1.2)$ from the following data using NDDIF

x	1	1.5	2.5	4
$y = \cos x + x^2$				

also determine the absolute error.

SUMMARY

1.) If the data is equispaced then we can use all the four interpolation techniques.

2.) If the data is unequispaced then use Lagrange's interpolation or NDDIF.

3.) If a missing term is required from the given data then use interpolation techniques with unequispaced data or assuming it as a polynomial of specific degree, extract the unknowns from forward or backward difference table only.

eg

x	1	2	3	4	5
y	8	a	66	b	228

a & b

$x^3 + 4x^2 + 3$

Method 1: Using forward diff. table as the data is equispaced

As we know 3 data points, we can assume that y is a polynomial of degree 2 (here, y is the interpolating function not the actual polynomial)

$\therefore$ All 2^{nd} order differences will be constant and differences of order greater than 2 are all zero.

Now,

constructing forward diff. table,

x	y	Δ	Δ^2	Δ^3	Δ^4
1	8				
		$a-8$			
2	a		$74-2a$		
		$66-a$		$b+3a-206$ $=0$	
3	66		$b+a-132$		not needed
		$b-66$		$426-a-3b$ $=0$	
4	b		$294-2b$		
		$228-b$			
5	228				

3^{rd} order differences are zero,

$b+3a = 206$ — (1)

$a+3b = 426$ — (2)

Simplify (1) & (2), we get

$a = 24$ & $b = 134$

Exact values,

$y = x^3+4x^2+3$, $y(2) = 27 = a$

$y(4) = 131 = b$

Method 2: Use Lagrange's interpolation formula for the data

x	1	3	5
y	8	66	228

(equispaced not necessary for all ques)

& then find $y(2)$ & $y(4)$

PART B

NUMERICAL DIFFERENTIATION

Numerical differentiation is the technique of estimating the value of derivate of function at a given point from the specified data points.
i.e. when y is not known explicitly in terms of x, only the table is given.

* Numerical differentiation using Newton's forward difference interpolating technique

WKT, $y(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0$ ----

where, $P = \frac{x-x_0}{h}$ — ① $\boxed{\frac{dP}{dx} = \frac{1}{h}(1-0) = \frac{1}{h}}$ — ②

$\therefore y(x) = y_0 + P\Delta y_0 + (P^2-P)\frac{\Delta^2 y_0}{2!} + (P^3-3P^2+2P)\frac{\Delta^3 y_0}{3!} + (P^4-6P^3+11P^2-6P)\frac{\Delta^4 y_0}{4!}$

$\frac{dy}{dP} = 0 + \Delta y_0 + (2P-1)\frac{\Delta^2 y_0}{2!} + (3P^2-6P+2)\frac{\Delta^3 y_0}{3!} + (4P^3-18P^2+22P-6)\frac{\Delta^4 y_0}{4!}$

From (2),

$$\boxed{\frac{dy}{dx} = \left(\frac{dy}{dP}\right)\left(\frac{dP}{dx}\right) = \frac{1}{h}\left[\Delta y_0 + (2P-1)\frac{\Delta^2 y_0}{2!} + (3P^2-6P+2)\frac{\Delta^3 y_0}{3!} + (4P^3-18P^2+22P-6)\frac{\Delta^4 y_0}{4!} --\right]}$$ — (3)

eqⁿ(3) is the general formula to determine $\frac{dy}{dx}$ at $x = x_p$, ~~i.e.~~ $\boxed{x = x_0 + Ph = x_p}$

Second order derivative

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dP}\left(\frac{dy}{dx}\right) \times \frac{dP}{dx}$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (P-1)\Delta^3 y_0 + (12P^2-3(P-22)\frac{\Delta^4 y_0}{4!} + ---\right]}$$ — (4)

eqⁿ(4) is the general formula to determine 2nd order derivative at $x = x_p = x_0 + Ph$

* Numerical differentiation using newton's backward difference technique

WKt,

$$y = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n - \cdots$$

where, $P = \frac{x_p - x_n}{h}$, $\boxed{\frac{dP}{dx} = \frac{1}{h}(1-0) = \frac{1}{h}}$ —①

$$\therefore \; y(x) = y_n + P\nabla y_n + \frac{(P^2+P)\nabla^2 y_n}{2!} + + \frac{(P^3+3P^2+2P)\nabla^3 y_n}{3!} + \cdots$$

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[0 + \nabla y_n + \frac{(2P+1)\nabla^2 y_n}{2!} + \frac{(3P^2+6P+2)\nabla^3 y_n}{3!} + \frac{(4P^3+18P^2+22P-6)\nabla^4 y_n}{4!} \cdots\right)} \quad —(4)$$

eq^n (4) is the general formula to determine $\frac{dy}{dx}$ at $x = x_p$ $\boxed{x = x_n + ph}$

second order derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dP}\left(\frac{dy}{dx}\right) \times \frac{dP}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (P+1)\nabla^3 y_n + \frac{(12p^2+36p+22)\nabla^4 y_n}{4!} + \cdots\right] \quad —(5)$$

eq^{ns} (4) & (5) are used to determine the first & 2nd order derivatives of in the neighbourhood of x_0 & x_n.

Special case:

1.) At $x = x_0$, $P = \frac{x - x_0}{h} = 0$

then,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \frac{\Delta^5 y_0}{5} \text{———}\right]$$

&

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \text{—} \text{—}\right]$$

2.) At $x = x_n$, $P = \frac{x - x_n}{h} = 0$

From (4) & (5),

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \text{———}\right]$$

&

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \text{———}\right]$$

Q Find y' of 1.3 & $y''(1.3)$ from the following data

x	1	1.2	1.4	1.6	1.8
$y = 2x\sin x$	1.6829	2.2638	2.7592	3.1986	3.5058

$P = \frac{1.3 - 1}{0.2} =$

Using general formula,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + (2P-1)\frac{\Delta^2 y_0}{2!} + (3P^2 - 6P + 2)\frac{\Delta^3 y_0}{3!} + (4P^3 - 18P^2 + 22P - 6)\frac{\Delta^4 y_0}{4!} \text{——}\right)$$

construct the forward difference table,

x	y	Δ	Δ^2	Δ^3	Δ^4
1	1.6829				
		0.5539 ~~0.5809~~			
1.2	2.2368		−0.0315		
		0.5224 ~~0.4954~~		−0.0515	
1.4	2.7592		−0.083		0.00235
		0.4394		−0.04915	
1.6	3.1986		−0.13215		
		0.30725			
1.8	3.5058				

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$P = \frac{1.3 - 1}{0.2} = 1.5$

$\frac{dy}{dx} = \frac{1}{\ } \left[0.5539 + \right.$

$\boxed{\frac{dy}{dx} = \cancel{2.614}\ 2.62}$

$\boxed{\frac{d^2y}{dx} = \cancel{-1.435}\ -1.4432}$

Exact Ans :- $y = 2x\sin x$

$y' = 2\sin x + 2x\cos x$

$y'' = 2\cos x - 2x\sin x + 2\cos x$

$\boxed{y'(1.3) = 2.6226}$ & $\boxed{y''(1.3) = -1.435}$

Q From previous problem, find $y'(1.8)$ & $y''(1.8)$ (backward diff) (use special case)

$\Rightarrow \frac{dy}{dx} = \frac{1}{0.2}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots \right]$

$= \frac{1}{0.2}\left[0.30725 + \frac{(-0.13215)}{2} + \frac{(-0.04915)}{3} + \frac{(0.00235)}{4} \right]$

$= \frac{1}{0.2}\left[0.30725 - 0.066075 - 0.01638 + 5.875 \times 10^{-4} \right]$

$= 1.1269$ X

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n - \right]$

$= \frac{1}{(0.2)^2}\left[-0.132215 + (-0.04915) + \frac{11}{2}(0.00235) \right]$

$= \frac{1}{0.04}\left[-0.1813 + 0.012925 \right]$

$= -4.209$ X

$\frac{dy}{dx} =$

$\frac{d^2y}{dx^2} =$

Q Find the radius of curvature of $y = f(x)$ when $x = 24$ given the following data.

x	19	20	21	22	23
y	91	100.25	110	120.25	131

Radius of curvature $\rho = \frac{(1+(y')^2)^{3/2}}{y''}$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
19	91			
		9.25		
20	100.25		0.5	
		9.75		0
21	110		0.5	
		10.25		0
22	120.25		0.5	
		10.75		
23	131			

To find y' & y'' at $x = 24$

$p = \frac{x - x_n}{h} = \frac{24-23}{1} = 1$

$\Rightarrow y' = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2!}(2p+1)\right]$

$\Rightarrow y' = \frac{1}{1}\left[10.75 + (2+1)\frac{0.5}{2}\right]$

$\Rightarrow y' = 10.75 + 0.75$

$\therefore y' = \underline{\underline{11.5}}$

$\Rightarrow y'' = \frac{1}{h^2}[\nabla^2 y]$

$\therefore y'' = 0.5$

now,

$\rho = \frac{(1+(y')^2)^{3/2}}{y''}$

$\rho = \frac{(1+(11.5)^2)^{3/2}}{0.5}$

$\therefore \rho = \underline{\underline{3076.3}}$

Q A slider in a machine moves along a fixed string. Its distance x (cm) along the rod is given bellow for the values of time t(sec). Find the velocity of the slides and its acceleration at t = 0.5 s.

t	0	0.1	0.2	0.3	0.4	0.5
x	30.13	31.62	32.87	33.64	33.95	33.81

Let $x \to y$ & $t \to x$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
0	30.13					
		1.49				
0.1	31.62		−0.24			
		1.25		−0.24		
0.2	32.87		−0.48		0.26	
		0.77		0.02		−0.27
0.3	33.64		−0.46		−0.01	
		0.31		0.01		
0.4	33.95		−0.45			
		−0.14				
0.5	33.81					

velocity at t = 0.5 s = ?

$$p = \frac{t - t_n}{h} = \frac{0.5 - 0.5}{0.1} = 0$$

at $p = 0$,
$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \frac{\nabla^5 y_n}{5}\right]$$

$$= \frac{1}{0.1}\left[-0.14 - \frac{0.45}{2} + \frac{0.01}{3} - \frac{0.01}{4} - \frac{0.27}{5}\right]$$

$$= \underline{-4.18}$$

so, velocity at t = 0.5 s is −0.836 cm/s

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{22}{4!}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n\right]$$

$$= \underline{-67.42}$$

Q The following table gives the temp θ in (°C) of a cooling body at diff instance of time t in sec, Find rate of cooling at t = 8 s.

t	1	3	5	7	9
θ	85.3	74.5	67	60.5	54.3

x(t)	y(θ)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	85.3				
		-10.8			
3	74.5		3.3		
		-7.5		-2.3	
5	67		1		1.6
		-6.5		-0.7	
7	60.5		0.3		
		-6.2			
9	54.3				

$\frac{d\theta}{dt}$ at $t=8 \Rightarrow \frac{1}{h}\left[\nabla y_n + \frac{(2p+1)}{2!}\nabla^2 y_n + \frac{(3p^2+6p+1)}{3!}\nabla^3 y_n + \frac{(4p^3+18p^2+22p+6)}{4!}\nabla^4 y_n\right]$

$p = \frac{8-9}{2} = -0.5$

$= \frac{1}{2}\left[-6.2 + 0.3\times 0 + \left(\frac{-5}{4}\times\frac{-0.7}{6}\right) + \left(\frac{-1}{24}\times 1.6\right)\right]$

$= -3.06$

Q Find minimum & maximum value of f(x)

x	1	3	5	7	9
y	9	11	13	63	209

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	9				
		2			
3	11		0		
		2		48	
5	13		48		0
		50		48	
7	63		96		
		146			
9	209				

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$\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + (2p-1)\frac{\Delta^2 y_0}{2!} + (3p^2-6p+2)\frac{\Delta^3 y_0}{3!}\right]$

$\Rightarrow 0 = \frac{1}{2}\left[2 + (2p-1)0 + (3p^2-6p+2)\frac{48}{6}\right]$

$\Rightarrow 0 = \frac{1}{2}\left[2 + (3p^2-6p+2)8\right]$

$\Rightarrow 0 = 2 + 24p^2 - 48p + 16$

$\Rightarrow 24p^2 - 48p + 18 = 0$

$\Rightarrow 4p^2 - 8p + 3 = 0$

$\therefore p = 0.5, 1.5$

$\Rightarrow x_{p_1} = x_0 + ph$
$= 1 + 0.5 \times 2 = 2$

$\Rightarrow$ ~~x_{p_2}~~ $x_{p_2} = x_0 + ph$
$= 1 + 1.5 \times 2 = 4$

When $p = 0.5$,

$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0\right]$

$= \frac{1}{4}\left[0 + (0.5-1) \times 48\right]$

$= \frac{-48 \times 0.5}{4} = \underline{-6} < 0$ so maximum

When $p = 1.5$,

$\left.\frac{d^2y}{dx^2}\right|_{x=4} = \frac{1}{4}\left[0 + (1.5-1) \times 48\right] = 6 > 0$ so minimum

$\therefore$ At $x = 2$ $f(x)$ will be minum & at $x = 4$ $f(x)$ will be maximum.

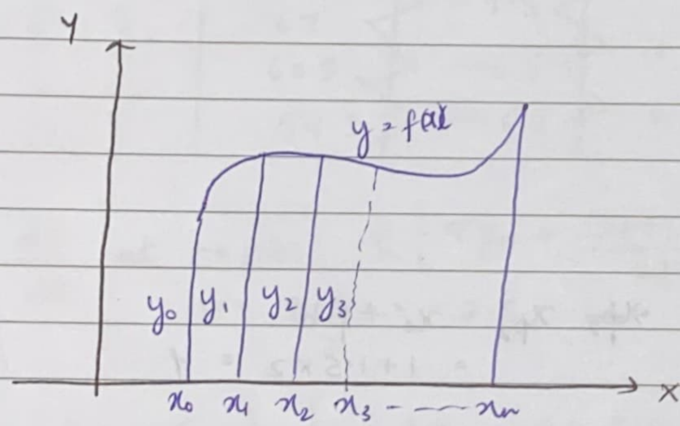
NUMERICAL INTEGRATION

It is the process of evaluating a definite integral from a set of taperated value of the integrant $f(x)$

- Quadrature

Numerical integration process when applied to a function of a single variant is known as Quadrature. (Numerical Quadrature)

- NEWTON QUADRATURE FORMULA



$x_1 = x_0 + h$

$x_2 = x_0 + 2h$

$x_3 = x_0 + 3h$

$\therefore \boxed{x_n = x_0 + nh}$

Let $I = \int_a^b f(x)\,dx$ where $f(x) = y_0, y_1, y_2 \ldots y_n$
for $x = x_0, x_2, x_3 \ldots x_n$

$= \int_{x_0}^{x_0+nh} f(x)\,dx$

put $x = x_0 + ph$

$dx = h\,dp$

$x = x_0,\ p = 0$

$x = x_0 + nh,\ p = n$

$$I = \int_{p=0}^{n} f(x_0+ph)h\,dp$$

$$= h\int_{p=0}^{n} [y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots]\,dp$$

$$= h\left\{y_0 p + \frac{p^2 \Delta y_0}{2} + \frac{1}{2!}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \cdots\right]_0^n$$

$$= h\left[ny_0 + \frac{n^2 \Delta y_0}{2!} + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} \cdots\right]$$

so,

$$\boxed{\int_{x=a}^{b} f(x) = h\left[ny_0 + \Delta y_0 \cdot \frac{n^2}{2!} + \frac{\Delta^2 y_0}{2!}\left(\frac{n^3}{3} - \frac{n^2}{2}\right) + \cdots\right]} \quad (1)$$

eq^n (1) is known as Newton's Cotes Quadrature formula

- TRAPEZOIDAL RULE

put $n=1$ in eq^n(1), then we get

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3 \cdots y_{n-1})\right]$$

- Simpson's 1/3rd rule:

put $n=2$ in eq^n (1)

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+y_6+\cdots y_{n-2}) + 4(y_1+y_3+\cdots y_{n-1})\right]$$

- Simpson's 3/8th rule: put $n=3$ in eq^n(1)

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 2(y_3+y_6+\cdots y_{n-3}) + 3(y_1+y_2+y_4+y_5+y_7+\cdots+y_{n-1})\right]$$

NOTE :

1. Trapezoidal rule can be applied for any no. of sub intervals.

2. Simpson's $1/3$rd rule can be applied for no. of sub intervals multiple of 2.

3. Simpson's $3/8$th rule can be applied for n is a multiple of 3.

4. If there are n ordinates, there must be $(n-1)$ equal divisions (sub intervals)

5. $h = \frac{b-a}{n}$

PROBLEMS

Q $\int_0^6 \frac{dx}{1+x^2}$ by using
1. Trapezoidal rule
2. Simpson's $1/3$rd rule
3. Simpson's $3/8$th rule

By dividing into 6 parts

$h = \frac{b-a}{n}$

$h = \frac{6-0}{6} = 1$

Let $I = \int_0^6 \frac{dx}{1+x^2}$

Let us divide $(0,6)$ into 6 equal parts by taking $h=1$.

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.058	0.038	0.027

y_0 y_1 y_2

(i) Trapezoidal rule

$$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3 - \text{---})\right]$$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{1}{2}\left[\left(1+\frac{1}{37}\right) + 2(0.5+0.2+0.1+0.058+0.03+0.027)\right]$$

$$= 1.437$$

(ii) Simpson's $\frac{1}{3}$ rule

$$\int_a^b f(x)dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4 \text{---}) + 4(y_1+y_3+ \text{---})\right]$$

$$= \frac{1}{3}\left[(1+1/37) + 2\left(\frac{1}{5}+\frac{1}{17}+\frac{1}{37}\right) + 4\left(\frac{1}{10}+\frac{1}{2}+\frac{1}{26}\right)\right]$$

$$= 1.44$$

(iii) Simpson's $3/8$th rule

$$\int_a^b f(x)dx = \frac{3h}{8}\left[(y_0+y_n) + 2(y_3+y_6 \text{----}) + 3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8}\left[(1+1/37) + 2\left(\frac{1}{10}+\frac{1}{37}\right) + 3\left(\frac{1}{2}+\frac{1}{5}+\frac{1}{17}+\frac{1}{26}\right)\right]$$

$$= 1.377$$

Q Using Simpson's $\frac{1}{3}$rd rule to evaluate $\int_0^1 \frac{dx}{1+x^2}$ considering 7 ordinates & hence find the value of π

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

x	0	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	$6/6 = 1$
$f(x)$	1	$36/37$	$36/40$	$36/45$	$36/52$	$36/61$	$1/2$

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Simpson's 1/3rd rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4)+4(y_1+y_3 \ldots)\right]$$

$$\Rightarrow \int_0^1 \frac{dx}{1+x^2} = \frac{1}{18}\left[\left(1+\frac{1}{2}\right)+2\left(\frac{36}{40}+\frac{36}{52}\right)+4\left[\frac{36}{37}+\frac{36}{45}+\frac{36}{61}\right]\right]$$

$$= 0.785$$

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \pi/4$$

$$\frac{\pi}{4} = 0.785 \Rightarrow \pi = 0.785 \times 4 = 3.14$$

Q Using Simpson's 3/8 rule obtain the approximate value $\int_0^{0.3}(1-8x^3)^{1/2}\,dx$ by considering 7 ordinates

$$h = \frac{0.3-0}{6} = \frac{1}{20}$$

x	0	1/20	2/20	3/20	4/20	5/20	6/20
$f(x)$	1	0.9994	0.9959	0.9864	0.9674	0.9354	0.8854

Simpson's 3/8 th rule,

$$\int_0^{0.3}(1-8x^3)^{1/2} = \frac{1}{20\times 3}\left[(1+0.8854)+2(0.9959+0.9674)+4(0.9994+0.9864+0.9354)\right]$$

$$= 0.2916$$

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Q Find the approximate value of $\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta$

$n=6$

$h = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$

x	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
$f(x)$	1	0.983	0.9306	0.841	0.707	0.5087	0

Simpson's 3/8 th rule,

$$\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta = \frac{\pi}{12 \times 3}\left[1+0+2(0.9306+0.707) + 4(0.983+0.841+0.5087)\right]$$

$$= \frac{\pi}{36}\left[1+3.2752+9.33\right]$$

$$= \underline{1.186}$$

Q ~~Fint~~ Find by Simpson's $\frac{1}{3}$rd rule the velocity of particle at distance from a point on its path is given by the table. Estimate the time taken to travel 60ft by using Simpson's 1/3rd rule.

s (in ft)	0	10	20	30	40	50	60
V (in ft/s)	47	58	64	65	61	52	38

$v = \frac{ds}{dt}$

$dt = \frac{1}{v} ds$

$t = \int_0^{60} \frac{1}{v}\, ds$

Reconstructing the table,

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$x(s)$	0	10	20	30	40	50	60
$f(x)(t)$	$1/47$	$1/58$	$1/64$	$1/65$	$1/61$	$1/52$	$1/38$

Simpson's rule,

$$h = \frac{60-0}{6} = 10$$

$$\int_0^{60} \frac{1}{v}\, ds = \frac{10}{3}\left[\left(\frac{1}{47}+\frac{1}{38}\right)+2\left(\frac{1}{64}+\frac{1}{61}\right)+4\left(\frac{1}{58}+\frac{1}{65}+\frac{1}{52}\right)\right]$$

$t = 1.06355$

Q Find an approximate value of log 2 applying Simpson's $1/3^{rd}$ rule to the integral $\int_0^1 \frac{1}{1+x}\,dx$ using 11 ordinates

$$h = \frac{1-0}{10} = \frac{1}{10}$$

x	0	$1/10$	$2/10$	$3/10$	$4/10$	$5/10$	$6/10$	$7/10$	$8/10$	$9/10$	1
$f(x)$	1	$10/11$	$10/12$	$10/13$	$10/14$	$10/15$	$10/16$	$10/17$	$10/18$	$10/19$	$1/2$

Simpson's $1/3^{rd}$ rule

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{1}{10\times 3}\left[1+\frac{1}{2}+2\left(\frac{10}{12}+\frac{10}{14}+\frac{10}{16}+\frac{10}{18}\right)+4\left(\frac{10}{11}+\frac{10}{13}+\frac{10}{15}+\frac{10}{17}+\frac{10}{19}\right)\right]$$

$$\int_0^1 \frac{1}{1+x}\,dx = 0.693$$

$\log 2 = 0.693$