Ray Optics Problem Sheet

Thales Swanson, Anango Prabhat

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1 Question 1

1.1 Part i

$n_{air} \approx 1.00$.
Critical angle for glass to air interface is $\arcsin(\frac{1.00}{1.50}) = 41.81°$.
Critical angle for glass to water interface is $\arcsin(\frac{1.34}{1.50}) = 63.30°$.

1.2 Part ii

$\lambda_1 n_1 = \lambda_2 n_2$.
$\lambda_{ice} = \frac{\lambda_{air} n_{air}}{n_{ice}} = \frac{520\times10^{-9}}{1.31} = 3.969 \times 10^{-7}$.
This is $\frac{5\times10^{-2}}{3.969\times10^{-7}} = 126000$ wavelengths into the ice.

1.3 Part iii

First let's consider the line going through the centre of the lens. The line goes through the origin, and has gradient $\frac{b}{a}$ and so the equation is $y = \frac{b}{a}x$.
Now let's consider the line going from the top of the lens to the green star. This line has gradient $\frac{b}{f}$ and goes through the point $(0, b)$ so has equation $y = \frac{b}{f}x + b$.
We are looking for the point where the lines meet. This will occur when $\frac{b}{a}x = \frac{b}{f}x + b$. Solving we get $(\frac{b}{a} - \frac{b}{f})x = b$ and so $x = \frac{b}{\frac{b}{a} - \frac{b}{f}} = \frac{1}{\frac{1}{a} - \frac{1}{f}} = -(\frac{1}{f} - \frac{1}{a})^{-1}$.
Note that this is the x-coordinate the green star, which is simply the value of a' and so $a' = -(\frac{1}{f} - \frac{1}{a})^{-1}$.
Subbing in $x = -(\frac{1}{f} - \frac{1}{a})^{-1}$ into the equation $y = \frac{b}{a}x$ gives $y = -\frac{b}{a}(\frac{1}{f} - \frac{1}{a})^{-1}$, and since this is the y-coordinate of the green star, and so simply the value of b' we have $b' = -\frac{b}{a}(\frac{1}{f} - \frac{1}{a})^{-1}$

Now let's talk about the image of Christiaan Huygens. Its coordinates are given by $a = 5$ and $b = 1.6$, and the focal length is given by $f = 20 \times 10^{-3}$.
Now let's use our equation $a' = -(\frac{1}{f} - \frac{1}{a})^{-1}$, so $a' = -(\frac{1}{20\times10^{-3}} - \frac{1}{5})^{-1} = -\frac{5}{249}$.
This means the new image is $-\frac{5}{249} = -0.0201$m from the lens, so 0.0201m from the lens in the x-direction on the opposite side from the original object.

Now let's use our equation $b' = -\frac{b}{a}(\frac{1}{f} - \frac{1}{a})^{-1}$, so $b' = -\frac{1.6}{5}(\frac{1}{20\times10^{-3}} - \frac{1}{5})^{-1} = -\frac{8}{1245}$. This means the new image is $-\frac{8}{1245} = -0.00643$m from the lens, so 0.00643m in the y-direction in the opposite direction from the original objects height.

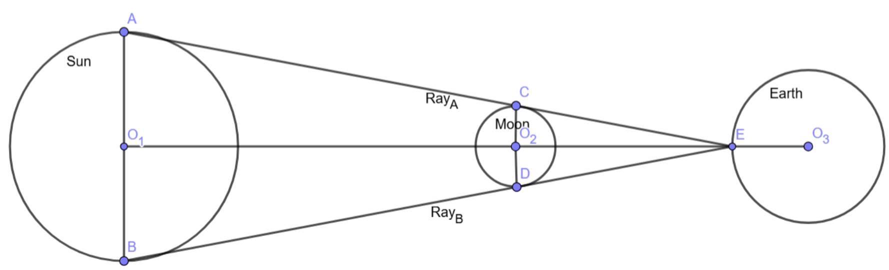
1.4 Part iv

First let's find the coordinates of a' and b' is, we will use the same process as the previous question to do this.
First let's consider the line going through the centre of the lens. The line goes through the origin, and has gradient $\frac{b}{a}$ and so the equation is $y = \frac{b}{a}x$.
Now let's consider the line going from the top of the lens to the green star. This line has gradient $\frac{b}{f}$ and goes through the point $(0, b)$ so has equation $y = \frac{b}{f}x + b$.
We are looking for the point where the lines meet. This will occur when $\frac{b}{a}x = \frac{b}{f}x+b$. Solving we get $(\frac{b}{a} - \frac{b}{f})x = b$ and so $x = \frac{b}{\frac{b}{a}-\frac{b}{f}} = \frac{1}{\frac{1}{a}-\frac{1}{f}} = -(\frac{1}{f} - \frac{1}{a})^{-1}$.
Note that this is the x-coordinate the green star, which is simply the value of a' and so $a' = -(\frac{1}{f} - \frac{1}{a})^{-1}$.
Subbing in $x = -(\frac{1}{f} - \frac{1}{a})^{-1}$ into the equation $y = \frac{b}{a}x$ gives $y = -\frac{b}{a}(\frac{1}{f} - \frac{1}{a})^{-1}$, and since this is the y-coordinate of the green star, and so simply the value of b' we have $b' = -\frac{b}{a}(\frac{1}{f} - \frac{1}{a})^{-1}$

The magnification factor is given by $\frac{b'}{b} = \frac{-\frac{b}{a}(\frac{1}{f}-\frac{1}{a})^{-1}}{b} = -\frac{1}{a}(\frac{1}{f} - \frac{1}{a})^{-1} = -\frac{1}{a(\frac{1}{f}-\frac{1}{a})} = \frac{1}{a(\frac{1}{a}-\frac{1}{f})} = \frac{1}{1-\frac{a}{f}} = \frac{f}{f-a}$. Hence we have shown $M = \frac{f}{f-a}$

Now let's think about Sherlock Holmes' mould. We can use the equation $M = \frac{f}{f-a}$, with $M = 5$ and $a = 8$, this gives $5 = \frac{f}{f-8}$ so $5f - 40 = f$, so $4f = 40$ and $f = 10$cm. Hence the focal length is 10cm = 0.1m.

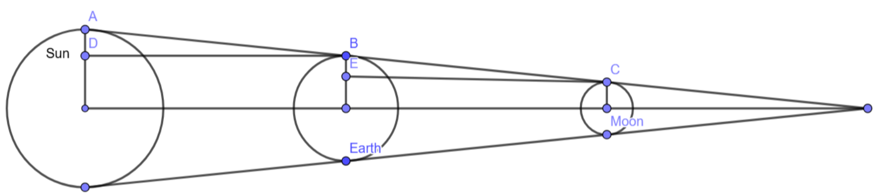
1.5 Part v



Above is a diagram showing the situation as described. (Not to scale)
Now notice that AO_1E and CO_2E are both equal to 90^o, and trivially $AEO_1 = CEO_2$, hence we can see that triangles AEO_1 and CEO_2 are similar. As this two triangles are similar we have $\frac{AO_1}{O_1E} = \frac{CO_2}{O_2E}$
EO_3 is the radius of the earth so $EO_3 = 6371$km. O_1O_3 is the distance from

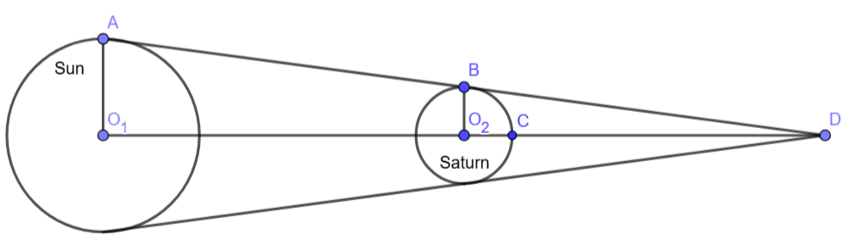
the sun to the earth so $O_1O_3 = 1.496 \times 10^8$km. Hence $O_1E = O_1O_3 - EO_3 =$ 149593629km. AO_1 is the radius of the sun so $AO_1 = 696340$km. CO_2 is the radius of the moon so $CO_2 = 1737.1$km.
From our previous equation: $\frac{AO_1}{O_1E} = \frac{CO_2}{O_2E}$ we can get $O_2E = \frac{CO_2 \times O_1E}{AO_1} = \frac{1737.1 \times 149593629}{696340} = 373178.4659$km. To find the distance between centres of the moon and the earth we add the radius of the earth to see that it is $373178.4659 + 6371 = 379549.4659\text{km} \approx 379549$km. This means the moon must get $379549 - 356500 = 23049$km further than it is currently. At a rate of 3.8cm per year $= 3.8 \times 10^{-5}$km per year this will take $\frac{23049}{3.8 \times 10^{-5}} = 606552631.6$ years
This gives us a time of roughly 600 million years.

1.6 Part vi



Above is a diagram showing the situation as described. (Not to scale)
Notice that the line connecting the centres of the Sun, Earth and Moon is parallel to EC and to BD, and perpendicular to AD and BE. This means $ADB = BEC = 90^o$ and $ABD = BCE$. This means triangles ABD and BCE are similar. Hence we must have $\frac{AD}{BD} = \frac{BE}{CE}$.
AD is the radius of the sun minus the radius of the earth so $AD = 696340 - 6371 = 689969$km. BD is the earth-sun distance, so $BD = 1.496 \times 10^8$km. BE is the radius of the earth minus the radius of the moon so $BE = 6371 - 1737.1 = 4633.9$km.
From our previous equation: $\frac{AD}{BD} = \frac{BE}{CE}$ we can get $CE = \frac{BE \times BD}{AD} = \frac{4633.9 \times 1.496 \times 10^8}{689969} =$ 1004728.386km.
This gives us our maximum earth-moon distance of roughly 1 million kilometers.

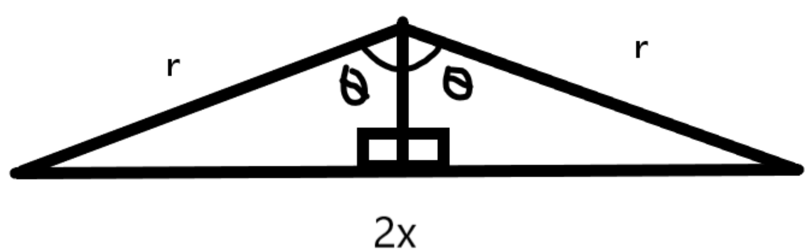
1.7 Part vii



Above is a diagram showing the situation as described. (Not to scale)
$AO_1D = BO_2D = 90^o$ and trivially $ADO_1 = BDO_2$ (as they are the same angle) so this means triangles AO_1D and BO_2D are similar. Hence we must have $\frac{AO_1}{O_1D} = \frac{BO_2}{O_2D}$
AO_1 is the radius of the sun so $AO_1 = 696340$km. BO_2 is the radius of Saturn so $BO_2 = 58232$km. O_1O_2 is the distance between Saturn and the sun so $O_1O_2 = 9.957\text{AU} = 9.957 \times 1.496 \times 10^8\text{km} = 1.4895672 \times 10^9$km. O_2C is the radius of Saturn so $O_2C = 58232$km. We do not know the distance CD so we will let $CD = X$. Then $O_1D = O_1O_2 + O_2C + CD = 1.48962543 \times 10^9 + X$ km and $O_2D = O_2C + CD = 58232 + X$ km.
Hence we have $\frac{696340}{1.48962543 \times 10^9 + X} = \frac{58232}{58232 + X}$ we can rearrange to make X the subject to get $(696340 - 58232)X = 58232 \times 1.48962543 \times 10^9 - 58232 \times 696340$ so then $X = \frac{58232 \times 1.48962543 \times 10^9 - 58232 \times 696340}{696340 - 58232} = 135875618.1$km
This gives us our maximum distance Cassini can be from the surface of Saturn as roughly $136,000,000$km.

1.8 Part viii

By Snell's law of refraction $n_1 \sin\theta_1 = n_2 \sin\theta_2$, for the critical angle we can set $\theta_2 = 90°$ so $\theta_1 = \arcsin(\frac{n_2}{n_1}) = \arcsin(\frac{1.4440}{1.4475}) = 86.01°$. Hence we have $\theta_c = 86.01°$



Hence we have $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{r}$, and so $r = \frac{x}{\sin\theta}$
The circumference of the earth is $2\pi R_{earth} = 2\pi \times 6371 = 40030.17359$km. We know $r = \frac{x}{\sin\theta}$, and θ is the critical angle of $86.01°$ so the distance that the light travels is $\frac{40030.17359}{\sin 86.01} = 40127.43407$km. The speed of light in a vacuum is 3×10^5km/s, so the speed of light in the fibre is $\frac{3\times10^5}{1.4475} = 207253.886$km/s so the time taken is $\frac{40127.43407}{207253.886} = 0.193614869$s. So the time taken to travel around the Earth is roughly 0.194 seconds.
Now let's think about the message from London to Sydney. The only change is that the distance is now 16983km instead of 40030.17359km, so the new time should be $\frac{16983}{40030.17359} \times 0.193614869 = 0.08214206998$. So the time taken from London to Sydney is roughly 0.082 seconds.

1.9 Part ix

By Snell's law of refraction we have $n_1 \sin\theta_i = n_2 \sin\theta_t$ and so $\theta_t = \arcsin(\frac{n_1 \sin\theta_i}{n_2}) = \arcsin(\frac{\sin 42}{1.5}) = 26.49291025°$. Substituting our values for $n_1, n_2, \theta_i, \theta_t$ into our equation gives us $|r_\perp|^2 = (\frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t})^2 = (\frac{\cos 42 - 1.5\cos 26.49291025}{\cos 42 + 1.5\cos 26.49291025})^2 = 0.0825793864$. This means 8.25793864% of light power is reflected and so $100 - 8.25793864 = 91.74206136\%$ of light power is transmitted through the glass-air interface. Hence we have our answer of roughly 91.7%.

2 Question 2

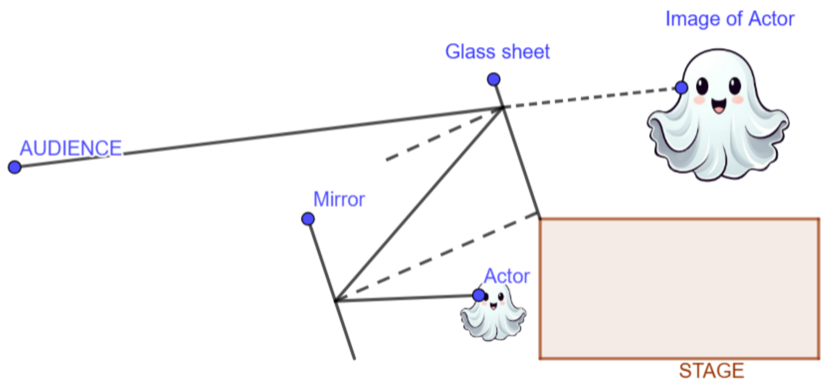
2.1 Part i

First let's consider the line going through the centre of the lens. The line goes through the origin, and has gradient $\frac{b}{a}$ and so the equation is $y = \frac{b}{a}x$.
Now let's consider the line going from the top of the lens to the green star. This line has gradient $-\frac{b}{f}$ and goes through the point $(0, b)$ so has equation $y = -\frac{b}{f}x + b$.
We are looking for the point where the lines meet. This will occur when $\frac{b}{a}x = -\frac{b}{f}x + b$. Solving we get $(\frac{b}{a} + \frac{b}{f})x = b$ and so $x = \frac{b}{\frac{b}{a}+\frac{b}{f}} = \frac{1}{\frac{1}{a}+\frac{1}{f}} = (\frac{1}{a} + \frac{1}{f})^{-1}$.
Note that this is the x-coordinate the green star, which is simply the value of a' and so $a' = (\frac{1}{a} + \frac{1}{f})^{-1}$.
Subbing in $x = a'$ into the equation $y = \frac{b}{a}x$ gives $y = \frac{b}{a}a'$, and since this is the y-coordinate of the green star, and so simply the value of b' we have $b' = \frac{b}{a}a'$

2.2 Part ii

Magnification factor is $\frac{b'}{b}$. Substituting $a' = (\frac{1}{a} + \frac{1}{f})^{-1}$ into $b' = \frac{b}{a}a'$ gives $b' = \frac{b}{a}(\frac{1}{a} + \frac{1}{f})^{-1}$, and substituting this into $M = \frac{b'}{b}$ gives us $M = \frac{\frac{b}{a}(\frac{1}{a}+\frac{1}{f})^{-1}}{b} = \frac{1}{a}(\frac{1}{a} + \frac{1}{f})^{-1} = \frac{1}{a(\frac{1}{a}+\frac{1}{f})} = \frac{1}{1+\frac{a}{f}} = \frac{f}{f+a}$. Hence we have shown $M = \frac{f}{f+a}$, note this results in $M < 1$ meaning the object is demagnified.

3 Question 3



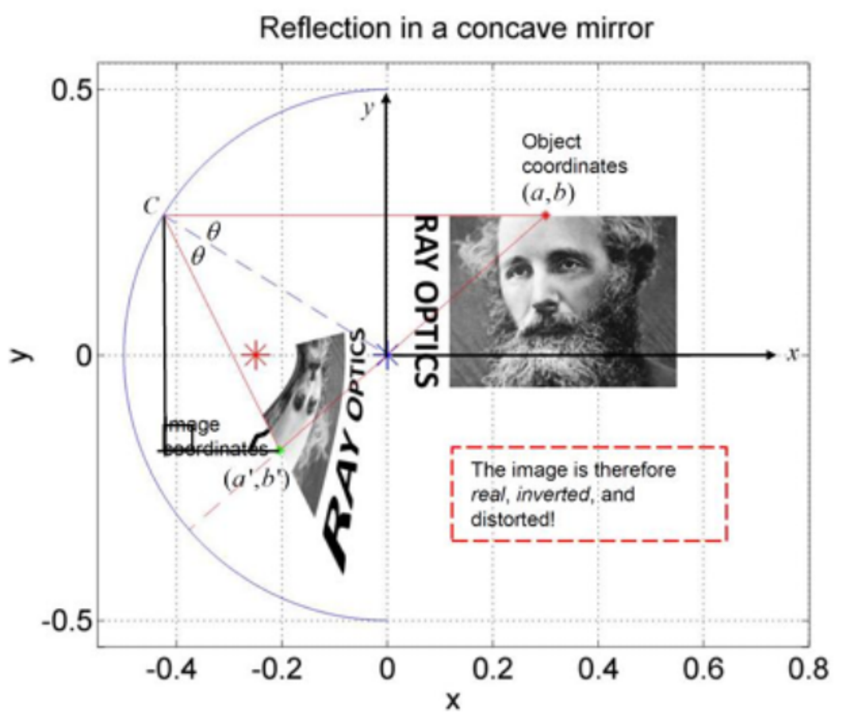
Above is a ray diagram that shows the Pepper's Ghost effect. Notice that the glass sheet is angled so that the light from the audience does not reflect directly back, preventing them from seeing their own reflection. This makes the glass less apparent.
The light from the actor is directed toward the angled glass by the mirror. When this light strikes the glass, part of it is reflected towards the audience, forming a virtual image. The reflected rays converge in such a way that the image appears to float in a specific area of space. Since the glass is semi-transparent, it allows some of the surrounding light to pass through. This make the reflected image blend in more naturally with the background.

4 Question 4

4.1 Part i

The circle is centred at the origin so its equation is $x^2 + y^2 = R^2$. The reflection point is the point on the circumference of the circle when $y = b$ so $x^2 + b^2 = R^2$, and so $x^2 = R^2 - b^2$ and so $x = \pm\sqrt{R^2 - b^2}$. However as the mirror is solely to the left of the origin we must have $x = -\sqrt{R^2 - b^2}$. So the coordinates are $(-\sqrt{R^2 - b^2}, b)$
$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{-\sqrt{R^2-b^2}}$ and so $\theta = \arctan(\frac{b}{-\sqrt{R^2-b^2}})$

4.2 Part ii



Consider drawing this triangle as shown above. Note that the angle at the bottom right of this triangle must equal 2θ as it and the angle at C are alternate. Now consider the line connecting C to (a', b'), this must have gradient of $\frac{\triangle y}{\triangle x} = \frac{-1\times\text{opposite}}{\text{adjacent}} = -\tan 2\theta = -m$. Additionally we know this line goes through the point $C = (-\sqrt{R^2 - b^2}, b)$, so the equation of the line is given by $y - b = -m(x - -\sqrt{R^2 - b^2})$ and so $y = -mx + b - m\sqrt{R^2 - b^2}$
Now let's consider the line connecting (a, b) to (a', b'). This line has gradient $\frac{b}{a}$ and goes through the origin, so has equation $y = \frac{b}{a}x$.
The coordinates (a', b') is the point where these two lines intersect. We can find this point by setting $\frac{b}{a}x = -mx + b - m\sqrt{R^2 - b^2}$ and simplifying to get $(m + \frac{b}{a})x = b - m\sqrt{R^2 - b^2}$ and so $x = \frac{b-m\sqrt{R^2-b^2}}{m+\frac{b}{a}} = -\frac{m\sqrt{R^2-b^2}-b}{m+\frac{b}{a}}$. This gives us $a' = -\frac{m\sqrt{R^2-b^2}-b}{m+\frac{b}{a}}$. We can find the y-coordinate by substituting $x = -\frac{m\sqrt{R^2-b^2}-b}{m+\frac{b}{a}}$ into $y = \frac{b}{a}x$ to get that $y = -\frac{b}{a}\frac{m\sqrt{R^2-b^2}-b}{m+\frac{b}{a}}$. And so we have $b' = -\frac{b}{a}\frac{m\sqrt{R^2-b^2}-b}{m+\frac{b}{a}}$

5 Question 5

5.1 Part i

By Snell's law of refraction $\sin\theta = n\sin\phi$ and so $\phi = \sin^{-1}\frac{\sin\theta}{n}$. Substituting this into $\epsilon = 4\phi - 2\theta$ for single internal reflection and $\epsilon = \pi - 6\phi + 2\theta$ for a double internal reflection gives $\epsilon = 4\sin^{-1}(\frac{\sin\theta}{n}) - 2\theta$ for a single internal reflection and $\epsilon = \pi - 6\sin^{-1}(\frac{\sin\theta}{n}) + 2\theta$ for a double internal reflection

5.2 Part ii

When $\frac{d\epsilon}{d\theta} = 0$ there is a stationary point which results in a similar value of ϵ for many values of θ. This higher concentration of light results in a greater light intensity at ϵ when $\frac{d\epsilon}{d\theta} = 0$. This means these ϵ values would be where a rainbow appears

5.3 Part iii

By Snell's law we have $\sin\phi = \frac{\sin\theta}{n}$.
Taking the derivative with respect to θ on both sides gives.
$\frac{d}{d\theta}\sin\phi = \frac{d}{d\theta}\frac{\sin\theta}{n}$ and so
$\frac{d\phi}{d\theta}\frac{d}{d\phi}\sin\phi = \frac{\cos\theta}{n}$ and so
$\frac{d\phi}{d\theta}\cos\phi = \frac{\cos\theta}{n}$
$(\frac{d\phi}{d\theta})^2\cos^2\phi = \frac{\cos^2\theta}{n^2}$
$(\frac{d\phi}{d\theta})^2(1 - \sin^2\phi) = \frac{1-\sin^2\theta}{n^2}$ We can now substitute in $\sin\phi = \frac{\sin\theta}{n}$
$(\frac{d\phi}{d\theta})^2(1 - \frac{\sin^2\theta}{n^2}) = \frac{1-\sin^2\theta}{n^2}$
$(\frac{d\phi}{d\theta})^2 = \frac{\frac{1-\sin^2\theta}{n^2}}{1-\frac{\sin^2\theta}{n^2}}$
$(\frac{d\phi}{d\theta})^2 = \frac{1-\sin^2\theta}{n^2-\sin^2\theta}$

Now let's think about the primary bow. For the primary bow $\frac{d\epsilon}{d\theta} = 0$, and we know that $\epsilon = 4\phi - 2\theta$, so by taking the derivative with respect to θ on the second equation we get $\frac{d\epsilon}{d\theta} = 4\frac{d\phi}{d\theta} - 2$ and so we get $0 = 4\frac{d\phi}{d\theta} - 2$. Now we can make $\frac{d\phi}{d\theta}$ the subject in order to get $\frac{d\phi}{d\theta} = \frac{1}{2}$, and so $(\frac{d\phi}{d\theta})^2 = \frac{1}{4}$.
We can now substitute in $(\frac{d\phi}{d\theta})^2 = \frac{1-\sin^2\theta}{n^2-\sin^2\theta}$ in order to get $\frac{1-\sin^2\theta}{n^2-\sin^2\theta} = \frac{1}{4}$. This can be simplified to give us $3\sin^2\theta = 4 - n^2$ and so $\sin\theta = \sqrt{\frac{4-n^2}{3}}$. This means $\theta = \sin^{-1}\sqrt{\frac{4-n^2}{3}}$ as expected.

Now let's think about the secondary bow, we will use a similar process as we did for the primary bow. For the secondary bow $\frac{d\epsilon}{d\theta} = 0$, and we know that $\epsilon = \pi - 6\phi + 2\theta$, so by taking the derivative with respect to θ on the second equation we get $\frac{d\epsilon}{d\theta} = -6\frac{d\phi}{d\theta} + 2$ and so we get $0 = -6\frac{d\phi}{d\theta} + 2$. Now we can make $\frac{d\phi}{d\theta}$ the subject in order to get $\frac{d\phi}{d\theta} = \frac{1}{3}$, and so $(\frac{d\phi}{d\theta})^2 = \frac{1}{9}$.

We can now substitute in $(\frac{d\phi}{d\theta})^2 = \frac{1-\sin^2\theta}{n^2-\sin^2\theta}$ in order to get $\frac{1-\sin^2\theta}{n^2-\sin^2\theta} = \frac{1}{9}$. This can be simplified to give us $8\sin^2\theta = 9 - n^2$ and so $\sin\theta = \sqrt{\frac{9-n^2}{8}}$. This means $\theta = \sin^{-1}\sqrt{\frac{9-n^2}{8}}$ as expected.

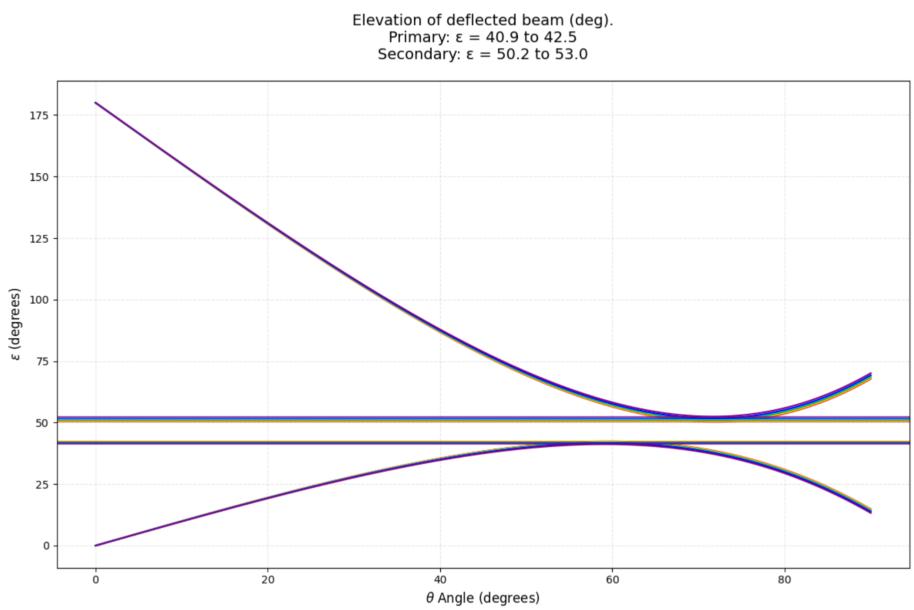
5.4 Part iv

Shown below is code which plots ϵ vs θ for single and double internal reflections over a range of optical frequencies.

```
import matplotlib.pyplot as plt
import numpy as np

plt.figure(figsize=(12, 8))

Theta = np.linspace(0, np.pi/2, 10000)
def convert(frequency, color_name):
    n = (1 + ((1 / (1.731 - 0.261*((frequency / 1000)**2))) ** 0.5)
    ) ** 0.5
    Epsilon1 = np.pi - 6*np.arcsin(np.sin(Theta) / n) + 2*Theta
    Epsilon2 = 4*np.arcsin(np.sin(Theta) / n) - 2*Theta

    SpecialTheta1 = np.arcsin(((9-n**2)/8)**0.5)
    SpecialTheta2 = np.arcsin(((4-n**2)/3)**0.5)
    SpecialEpsilon1 = np.pi - 6*np.arcsin(np.sin(SpecialTheta1) / n
    ) + 2*SpecialTheta1
    SpecialEpsilon2 = 4*np.arcsin(np.sin(SpecialTheta2) / n) - 2*
    SpecialTheta2
    return {
        'Epsilon1': np.rad2deg(Epsilon1),
        'Epsilon2': np.rad2deg(Epsilon2),
        'SpecialEpsilon1': np.rad2deg(SpecialEpsilon1),
        'SpecialEpsilon2': np.rad2deg(SpecialEpsilon2),
        'color': color_name,
        'frequency': f"{frequency} THz"
    }

colors = {
    'Red': convert(442.5, 'red'),
    'Orange': convert(495, 'orange'),
    'Yellow': convert(520, 'yellow'),
    'Green': convert(565, 'green'),
    'Cyan': convert(610, 'cyan'),
    'Blue': convert(650, 'blue'),
    'Violet': convert(735, 'purple')
}

Theta_deg = np.rad2deg(Theta)

for name, data in colors.items():
    plt.plot(Theta_deg, data['Epsilon1'], color=data['color'],
    linewidth=1.5,
             label=f"{name} {data['frequency']} (epsilon1)")
    plt.plot(Theta_deg, data['Epsilon2'], color=data['color'],
    linewidth=1.5,
             label=f"{name} {data['frequency']} (epsilon2)")
```

```python
    plt.axhline(y=data['SpecialEpsilon1'], color=data['color'], alpha=0.7)
    plt.axhline(y=data['SpecialEpsilon2'], color=data['color'], alpha=0.7)

plt.xlabel('$\\theta$ Angle (degrees)', fontsize=12)
plt.ylabel('$\epsilon$ (degrees)', fontsize=12)
plt.title('Elevation of deflected beam (deg).\nPrimary: epsilon = {:.1f} to {:.1f}\nSecondary: epsilon = {:.1f} to {:.1f}'.format(
    convert(790, '')['SpecialEpsilon2'],
    convert(405, '')['SpecialEpsilon2'],
    convert(405, '')['SpecialEpsilon1'],
    convert(790, '')['SpecialEpsilon1']
), fontsize=14, pad=20)
plt.grid(True, which='both', linestyle='--', alpha=0.3)
plt.tight_layout()
plt.show()
```

Shown below is the output of this code.

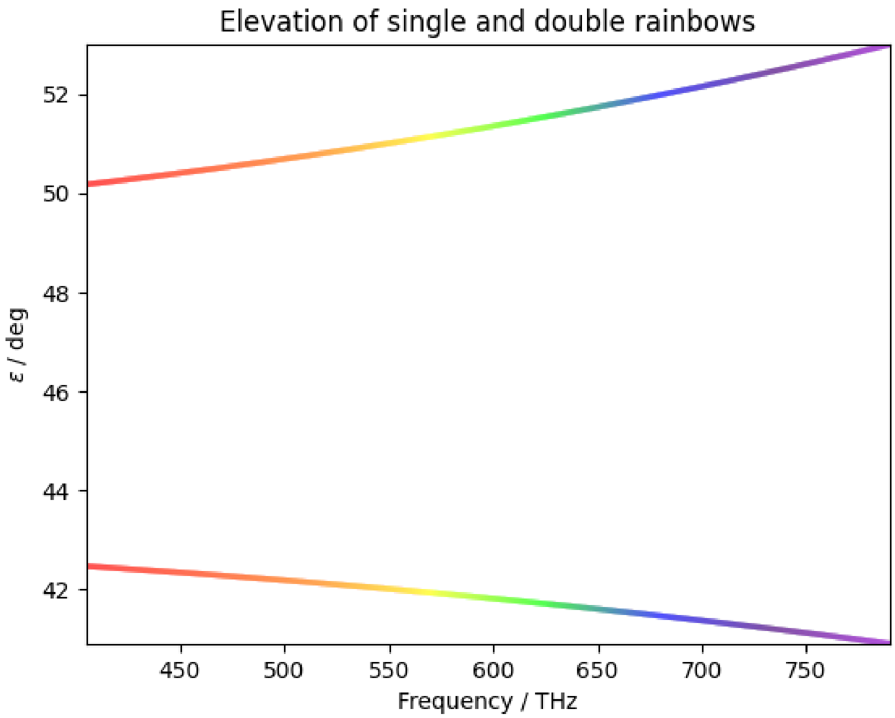


Shown below is code used to plot rainbow elevation angles against light frequency. It has been coloured based on the frequency.

```python
import matplotlib.pyplot as plt
import numpy as np
from matplotlib.collections import LineCollection
from matplotlib.colors import LinearSegmentedColormap

rainbow = [(1,0,0),(1,0.2,0),(1,0.5,0),(1,1,0),(0,1,0),(0,0,1),(0.29,0,0.51),(0.58,0,0.83)]
colourmap = LinearSegmentedColormap.from_list('colours', rainbow)
```

```python

frequency = np.linspace(405, 790, 10000)
RefractiveIndex = (1 + ((1 / (1.731 - 0.261*((frequency / 1000)**2)
    )) ** 0.5)) ** 0.5
Theta1 = np.arcsin(((4-RefractiveIndex**2)/3)**0.5)
Theta2 = np.arcsin(((9-RefractiveIndex**2)/8)**0.5)
Epsilon1 = 4*np.arcsin(np.sin(Theta1)/RefractiveIndex) - 2*Theta1
Epsilon2 = np.pi - 6*np.arcsin(np.sin(Theta2)/RefractiveIndex) + 2*
    Theta2
Epsilon1, Epsilon2 = np.rad2deg(Epsilon1), np.rad2deg(Epsilon2)

points1 = np.array([frequency, Epsilon1]).T.reshape(-1, 1, 2)
points2 = np.array([frequency, Epsilon2]).T.reshape(-1, 1, 2)
lines1 = np.concatenate([points1[:-1], points1[1:]], axis=1)
lines2 = np.concatenate([points2[:-1], points2[1:]], axis=1)
ColourLines1 = LineCollection(lines1, cmap=colourmap, linewidth
    =2.5)
ColourLines2 = LineCollection(lines2, cmap=colourmap, linewidth
    =2.5)
ColourLines1.set_array(frequency)
ColourLines2.set_array(frequency)

fig, ax = plt.subplots()
ColourLines1 = ax.add_collection(ColourLines1)
ColourLines2 = ax.add_collection(ColourLines2)
ax.set_xlim(405, 790)
ax.set_ylim(Epsilon1.min(), Epsilon2.max())

plt.title("Elevation of single and double rainbows")
plt.xlabel('Frequency / THz')
plt.ylabel('$\\epsilon$ / deg')
```

Shown below is the output of this code.

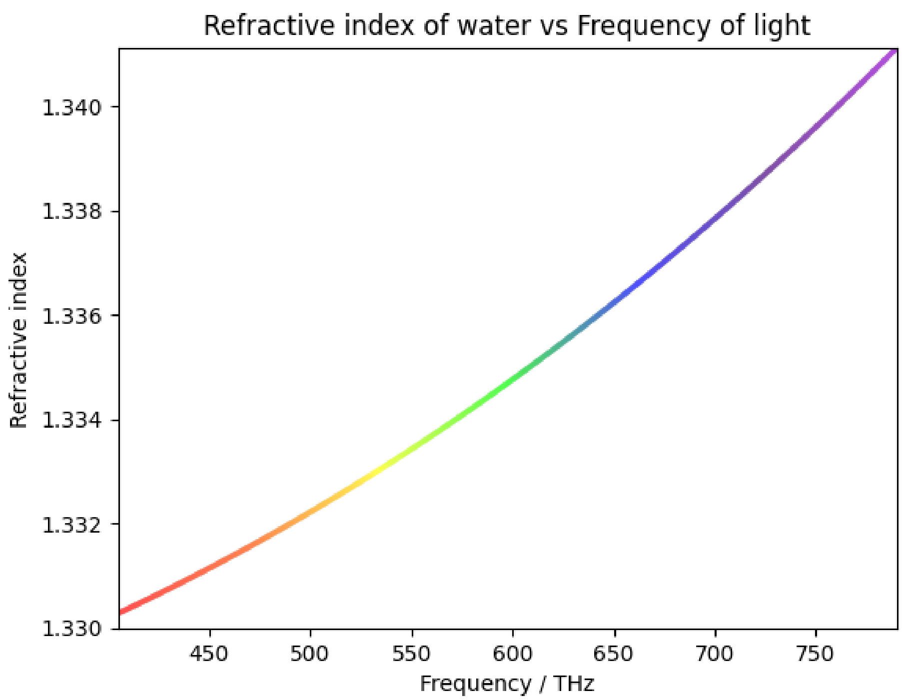


Shown below is code used to plot the refractive index of water against frequency. It has been coloured based on the frequency.

```python
import matplotlib.pyplot as plt
import numpy as np
from matplotlib.collections import LineCollection
from matplotlib.colors import LinearSegmentedColormap

rainbow = [(1,0,0),(1,0.3,0),(1,1,0),(0,1,0),(0,0,1),(0.29,0,0.51),(0.58,0,0.83)]
colourmap = LinearSegmentedColormap.from_list('colours', rainbow)

frequency = np.linspace(405, 790, 10000)
RefractiveIndex = (1 + ((1 / (1.731 - 0.261*((frequency / 1000)**2))) ** 0.5)) ** 0.5

points = np.array([frequency, RefractiveIndex]).T.reshape(-1, 1, 2)
lines = np.concatenate([points[:-1], points[1:]], axis=1)
ColourLines = LineCollection(lines, cmap=colourmap, linewidth=2.5)
ColourLines.set_array(frequency)

fig, ax = plt.subplots()
ColourLines = ax.add_collection(ColourLines)
ax.set_xlim(405, 790)
ax.set_ylim(1.33, RefractiveIndex.max())

plt.title("Refractive index of water vs Frequency of light")
```

```python
plt.xlabel('Frequency / THz')
plt.ylabel('Refractive index')
```

Shown below is the output of this code.



6 Question 6

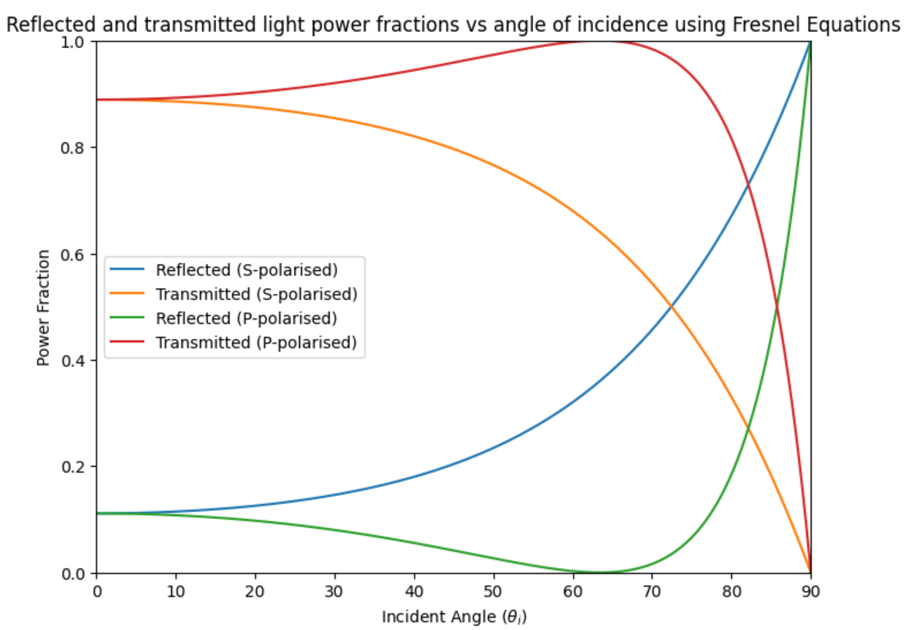
6.1 Part i

Show below is python code used to perform this task.

```python
import matplotlib.pyplot as plt
import numpy as np

def getAngle(n1, n2, theta):
    return np.arcsin(n1*np.sin(theta)/n2)

def getPerp(n1, n2, thetaI, thetaT):
    Reflected = ((n1*np.cos(thetaI) - n2*np.cos(thetaT)) / (n1*np.
    cos(thetaI) + n2*np.cos(thetaT)))**2
    Transmitted = 1 - Reflected
    return Reflected, Transmitted

def getPara(n1, n2, thetaI, thetaT):
    Reflected = ((n1*np.cos(thetaT) - n2*np.cos(thetaI)) / (n1*np.
    cos(thetaT) + n2*np.cos(thetaI)))**2
```

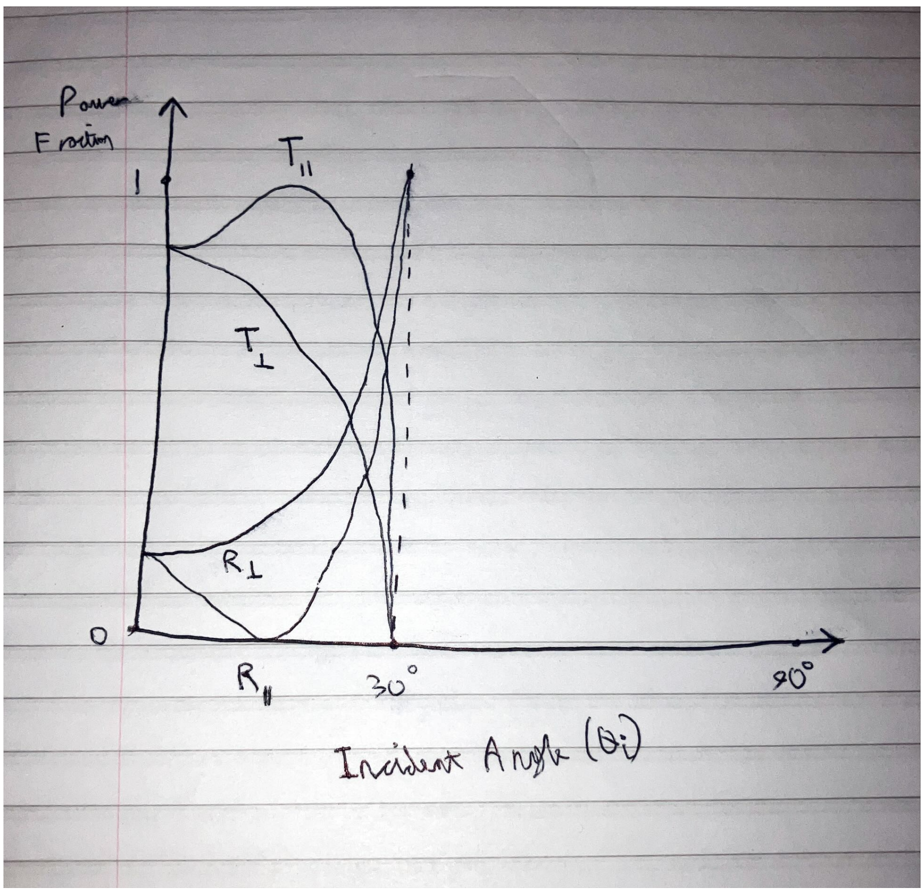
```python
    Transmitted = 1 - Reflected
    return Reflected, Transmitted

n1, n2 = 1, 2
thetaI = np.linspace(0, np.pi/2, 10000)
thetaT = getAngle(n1, n2, thetaI)
Rperp, Tperp = getPerp(n1, n2, thetaI, thetaT)
Rpara, Tpara = getPara(n1, n2, thetaI, thetaT)

plt.figure(figsize=(8, 6))
plt.plot(np.rad2deg(thetaI), Rperp, label='Reflected (S-polarised)')
plt.plot(np.rad2deg(thetaI), Tperp, label='Transmitted (S-polarised)')
plt.plot(np.rad2deg(thetaI), Rpara, label='Reflected (P-polarised)')
plt.plot(np.rad2deg(thetaI), Tpara, label='Transmitted (P-polarised)')
plt.title('Reflected and transmitted light power fractions vs angle of incidence using Fresnel Equations')
plt.xlabel('Incident Angle ($\\theta_i$)')
plt.ylabel('Power Fraction')
plt.xlim(0, 90)
plt.ylim(0, 1)
plt.legend()
plt.show()
```

Shown below is the output.



6.2 Part ii

The significance of the zero at 63.4° is that at this point there is no P-polarised light reflected, it is all transmitted. So all light reflected at this point would have to be S-polarised.
Now let's think about plotting the new graph.
First notice that by Snell's law of refraction $\theta_t = \arcsin(\frac{n_1 \sin\theta}{n_2})$. As $\arcsin(x)$ is undefined when $x > 1$ we must have $\frac{n_1 \sin\theta}{n_2} \leq 1$, and so $\sin\theta \leq \frac{1}{2}$, and so $\theta \leq 30°$.



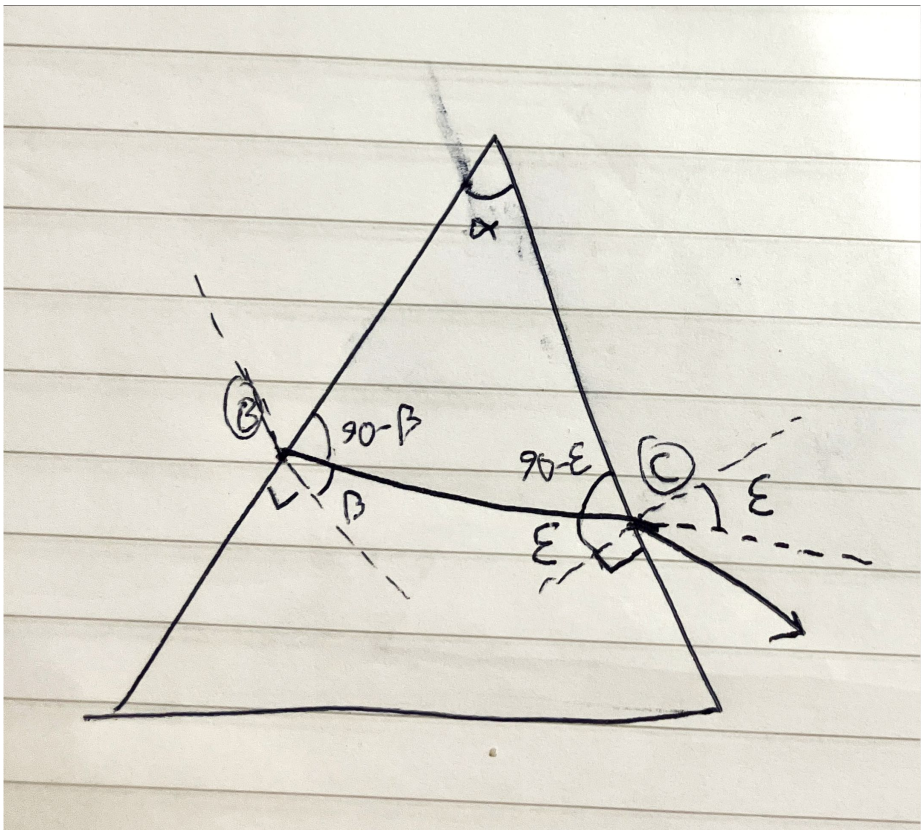
6.3 Part iii

When $|r_{\parallel}|^2 = 0$ we must have $(\frac{n_1\cos\theta_t - n_2\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t})^2 = 0$ and so
$n_1\cos\theta_t = n_2\cos\theta_i$ so
$n_1^2\cos^2\theta_t = n_2^2\cos^2\theta_i$ so
$n_1^2(1-\sin^2\theta_t) = n_2^2(1-\sin^2\theta_i)$.

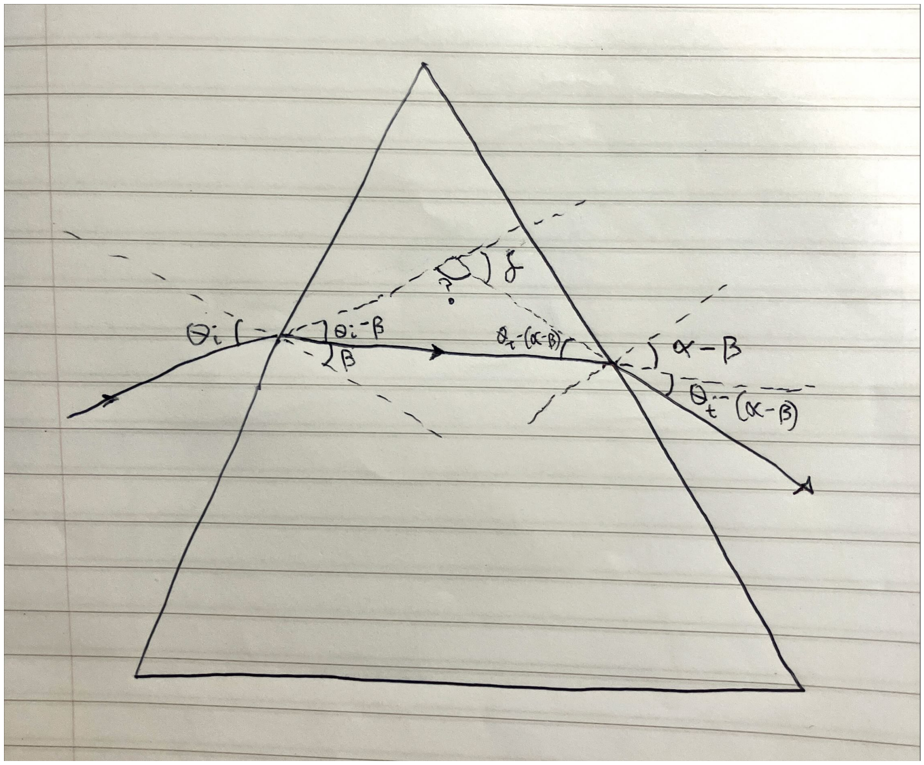
Now from Snell's law of refraction we have $n_1 \sin\theta_i = n_2 \sin\theta_t$ hence
$\sin\theta_t = \frac{n_1 \sin\theta_i}{n_2}$ so
$\sin^2\theta_t = \frac{n_1^2 \sin^2\theta_i}{n_2^2}$. We substitute this into our earlier equation:
$n_1^2(1 - \frac{n_1^2 \sin^2\theta_i}{n_2^2}) = n_2^2(1 - \sin^2\theta_i)$
$(n_2^2 - \frac{n_1^4}{n_2^2})\sin^2\theta_i = n_2^2 - n_1^2$ and so
$\sin^2\theta_i = \frac{n_2^2 - n_1^2}{n_2^2 - \frac{n_1^4}{n_2^2}}$
Now we know that $1 + \cot^2\theta = \text{cosec}^2\theta$ so we know
$1 + \frac{1}{\tan^2\theta} = \frac{1}{\sin^2\theta}$. We can use this identity with $\sin^2\theta_i$ to get
$\frac{n_2^2 - \frac{n_1^4}{n_2^2}}{n_2^2 - n_1^2} = 1 + \frac{1}{\tan^2\theta_i}$ so
$\frac{n_2^2 - \frac{n_1^4}{n_2^2} - n_2^2 + n_1^2}{n_2^2 - n_1^2} = \frac{1}{\tan^2\theta_i}$
$\frac{n_2^2 - n_1^2}{n_1^2 - \frac{n_1^4}{n_2^2}} = \tan^2\theta_i$ and so
$\tan^2\theta_i = \frac{\frac{n_2^2}{n_1^2}(n_1^2 - \frac{n_1^4}{n_2^2})}{n_1^2 - \frac{n_1^4}{n_2^2}}$ and so
$\tan^2\theta_i = \frac{n_2^2}{n_1^2}$, now since $n_1 \geq 1$ and $n_2 \geq 1$ this means that $\frac{n_2}{n_1}$ must be positive, and since $0 \leq \theta_i \leq 90$, this means that $\tan\theta_i$ must be positive. Hence $\tan\theta_i = -\frac{n_2}{n_1}$ is not possible, and so we must have $\tan\theta_i = \frac{n_2}{n_1}$ which gives us $\theta_i = \arctan\frac{n_2}{n_1}$. Hence we have shown that the Brewster angle $\theta_B = \tan^{-1}\frac{n_2}{n_1}$

7 Question 7

7.1 Part i



Let the angle shown above be ϵ, as shown it then follows that $\alpha + 90 - \beta + 90 - \epsilon = 180$, and so $\alpha - \beta - \epsilon = 0$, and so $\epsilon = \alpha - \beta$



By the diagram above shown it is clear the angle marked ? is given by $180-(\theta_i-\beta)-(\theta_t-(\alpha-\beta)) = 180-\theta_i+\beta-\beta-\theta_t+\alpha = 180-(\theta_i+\theta_t-\alpha)$. Hence $\delta = 180-(180-(\theta_i+\theta_t-\alpha)) = \theta_i+\theta_t-\alpha$ and so we have finished.

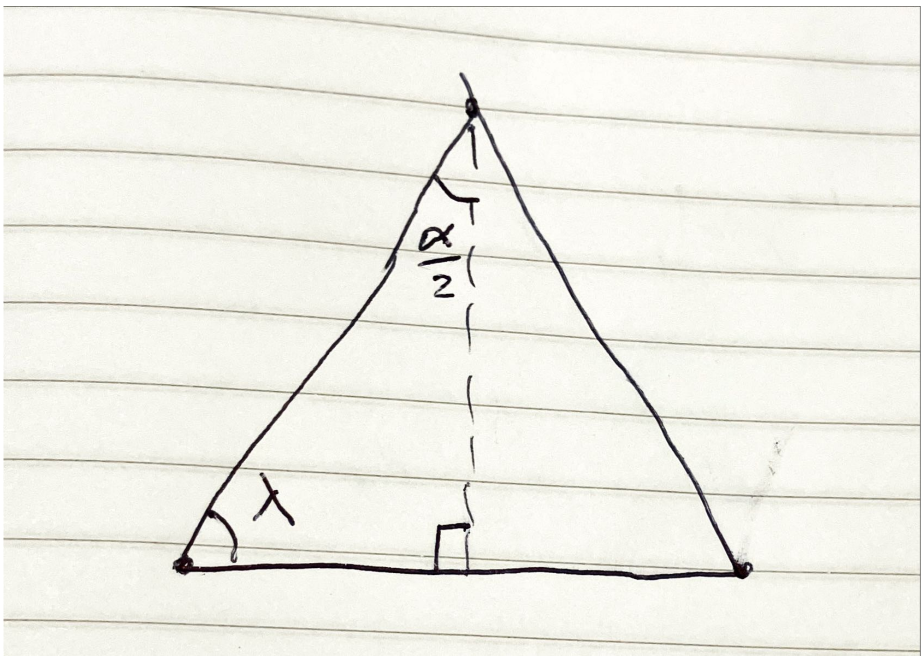
7.2 Part ii

By Snell's law of refraction we can gain 2 separate equations, one as the light enters the glass, and one as the light leaves the glass. These are: $n_a \sin\theta_i = n_g \sin\beta$, and $n_g \sin(\alpha-\beta) = n_a \sin\theta_t$, by the second equation we get $\sin\theta_t = \frac{n_g \sin(\alpha-\beta)}{n_a}$. Now $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$, and so we get $\sin\theta_t = \frac{n_g(\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta))}{n_a}$.
We know $\sin^2\beta + \cos^2\beta = 1$, and so $\cos\beta = \sqrt{1-\sin^2\beta}$, we are not worried about negatives, as we know $0° < \beta < 90°$, and so $0 < \cos\beta < 1$. By substituting this into our equation we get $\sin\theta_t = \frac{n_g}{n_a}(\sin(\alpha)\sqrt{1-\sin^2\beta} - \cos(\alpha)\sin(\beta))$. Additionally by our first equation we know $\sin\beta = \frac{n_a \sin\theta_i}{n_g}$, by substituting this in we get $\sin\theta_t = \frac{n_g}{n_a}(\sin(\alpha)\sqrt{1-(\frac{n_a\sin\theta_i}{n_g})^2} - \cos(\alpha)\frac{n_a\sin\theta_i}{n_g}) = \sin(\alpha)\sqrt{\frac{n_g^2}{n_a^2} - \sin^2\theta_i} - \cos(\alpha)\sin\theta_i$. The refractive index of air is approximately 1, so we have $\frac{n_g^2}{n_a^2} \approx n^2$.

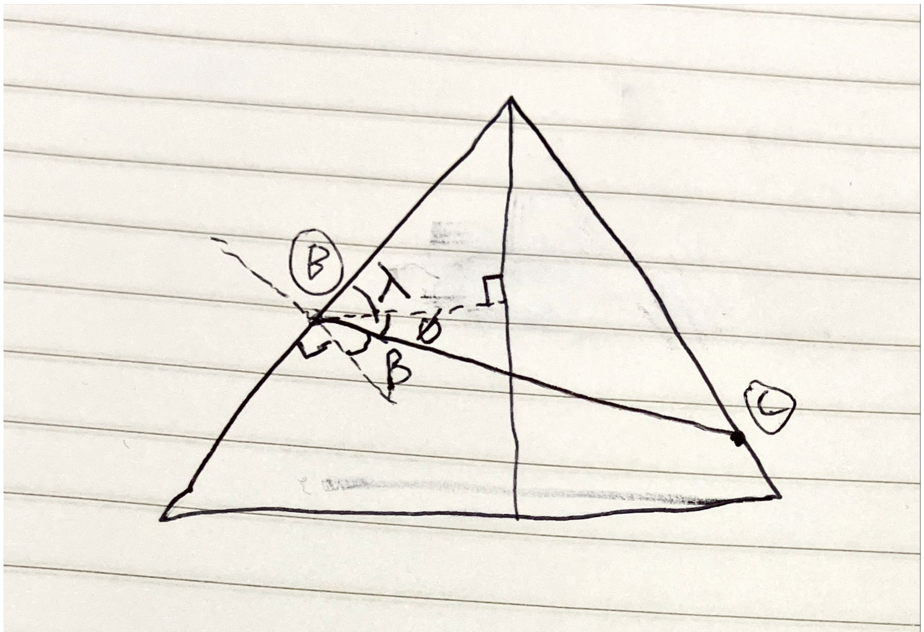
Hence we have shown $\sin\theta_t = \sqrt{n^2 - \sin^2(\theta_i)}\sin\alpha - \sin(\theta_i)\cos\alpha$ as required.

7.3 Part iii

First let's try to find the coordinates of point C.
By Snell's law of refraction we know $n_a \sin\theta_i = n_g \sin\beta$, and so we have $\beta = \arcsin(\frac{n_a \sin\theta_i}{n_g})$.

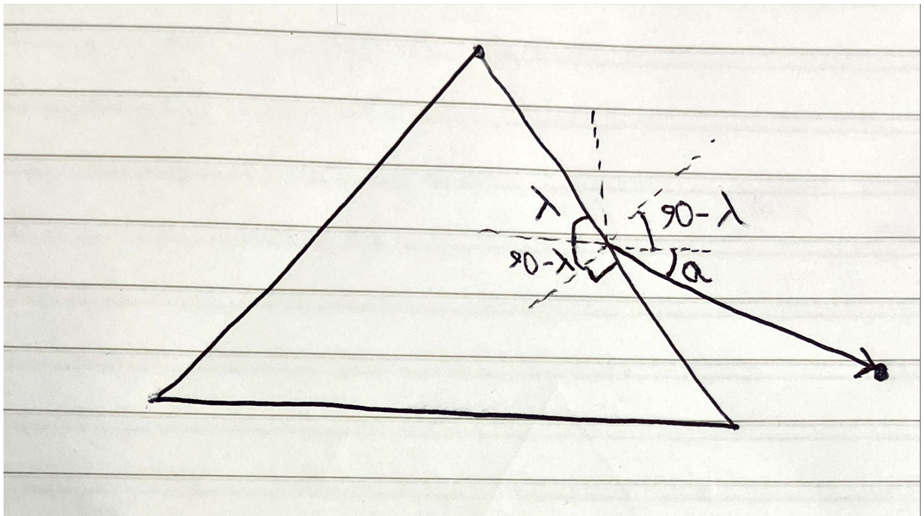


We define λ as shown, it then follows that $\lambda = 90 - \frac{\alpha}{2}$.



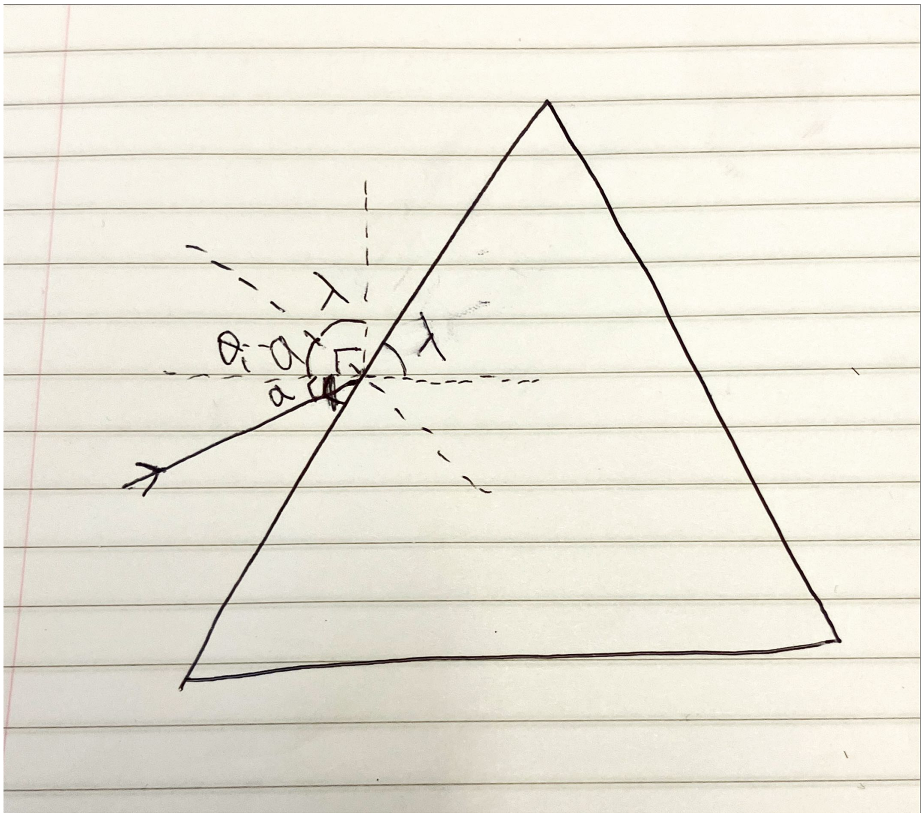
We define ϕ as shown, it then follows that $\phi = 90 - \lambda - \beta = \frac{\alpha}{2} - \beta$.
Now let's consider the line which connects B to C. The gradient of this line is $\frac{\triangle y}{\triangle x} = -\frac{\text{opposite}}{\text{adjacent}} = -\tan\phi$. Now let the origin of these axes be the midpoint of the base of the triangle. If we let the top two sides of the triangle have length 1, then the coordinates of point B are $(-\frac{1}{2}\cos\lambda, \frac{1}{2}\sin\lambda)$. The equation of this line is then given by $y - \frac{1}{2}\sin\lambda = -\tan\phi(x - -\frac{1}{2}\cos\lambda)$, which simplifies to $y = -\tan(\phi)x - \frac{1}{2}\cos\lambda\tan\phi + \frac{1}{2}\sin\lambda$.
Now let's consider the right side of the triangle, this line has gradient $\frac{\triangle y}{\triangle x} = -\frac{\text{opposite}}{\text{adjacent}} = -\tan\lambda$. This line goes through the top of the triangle which has coordinates $(0, \sin\lambda)$. This means the equation of this line is given by $y - \sin\lambda = -\tan(\lambda)x$, which simplifies to $y = -\tan(\lambda)x + \sin\lambda$.
The point C is the intersection of these two lines, and so we can find its coordinates: $-\tan(\lambda)x + \sin\lambda = -\tan(\phi)x - \frac{1}{2}\cos\lambda\tan\phi + \frac{1}{2}\sin\lambda$, we can simplify to $(\tan\lambda - \tan\phi)x = \sin\lambda + \frac{1}{2}\cos\lambda\tan\phi - \frac{1}{2}\sin\lambda$, and so $x = \frac{\frac{1}{2}\sin\lambda + \frac{1}{2}\cos\lambda\tan\phi}{\tan\lambda - \tan\phi}$, by using the second line we see $y = -\tan(\lambda)x + \sin\lambda$. Now we have our coordinates of point C, we will call these X_C and Y_C.

Now let's find where the light will be after it leaves the prism.



From the diagram above we can see that $\theta_t = 90 - \lambda + a$, and so $a = \theta_t + \lambda - 90$, and $\lambda = 90 - \frac{\alpha}{2}$, substituting this in we see $a = \theta_t - \frac{\alpha}{2}$. If we let the length of the line be 1 (in order to simplify calculations) we see that the new x coordinate is $X_C + \frac{\cos(\theta_t - \frac{\alpha}{2})}{1} = X_C + \cos(\theta_t - \frac{\alpha}{2})$, and the new y coordinate is $Y_C - \frac{\sin(\theta_t - \frac{\alpha}{2})}{1} = Y_C - \sin(\theta_t - \frac{\alpha}{2})$.

Now let's find where the light will be before it goes into the prism, we will use this in order to plot the white ray of light.



From the diagram above we can see that $\theta_i - a + \lambda = 90$, and so $a = \theta_i + \lambda - 90$, now we know that $\lambda = 90 - \frac{\alpha}{2}$, substituting this in gives $a = \theta_i + 90 - \frac{\alpha}{2} - 90 = \theta_i - \frac{\alpha}{2}$. If we let the length of the line be 1 (in order to simplify calculations) we see that the new x coordinate is $-\cos(\theta_i - \frac{\alpha}{2}) - \frac{1}{2}\cos\lambda$, and the new y coordinate is $-\sin(\theta_i - \frac{\alpha}{2}) + \frac{1}{2}\sin\lambda$.

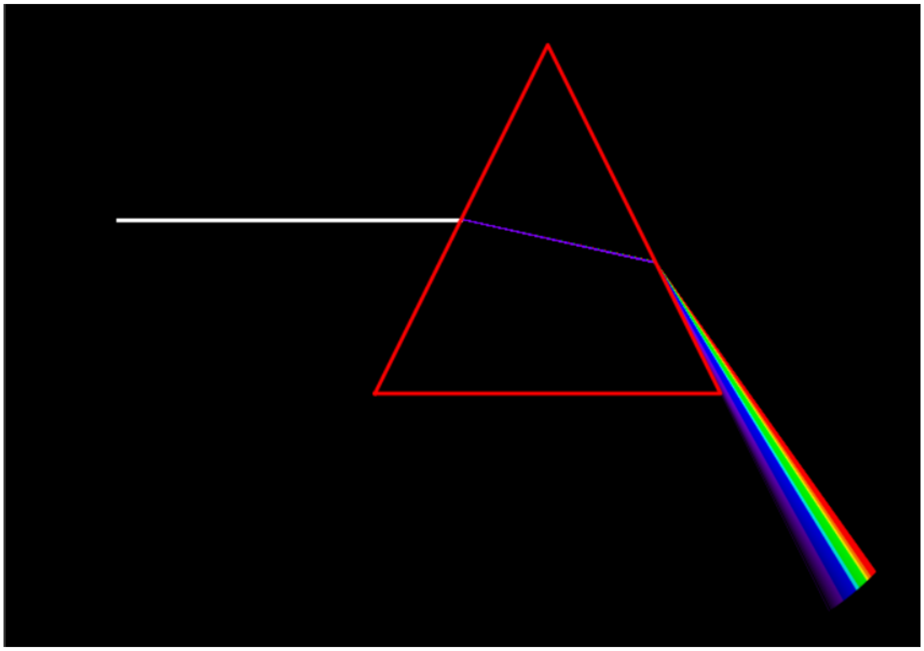
We can use the coordinates we have now found in order to plot the rays of light, code is shown below:

```
import matplotlib.pyplot as plt
from matplotlib.collections import LineCollection
import numpy as np
import ipywidgets as widgets
from ipywidgets import interact

frequency = np.linspace(405*10**12, 790*10**12, 500)
wavelength = 3*10**8 / frequency

def color(f):
    if 405e12 <= f < 480e12:
        return (1,0,0)
    elif 480e12 <= f < 510e12:
        return (1,127/255,0)
```

```python
    elif 510e12 <= f < 530e12:
        return (1,1,0)
    elif 530e12 <= f < 600e12:
        return (0,1,0)
    elif 600e12 <= f < 620e12:
        return (0,1,1)
    elif 620e12 <= f < 680e12:
        return (0,0,1)
    else:
        return (137/255,0,1)

def triangle(ax, alpha):
    base_length = 2 * np.sin(alpha / 2)
    left_vertex = (-base_length/2, 0)
    right_vertex = (base_length/2, 0)
    height = np.cos(alpha / 2)
    top_vertex = (0, height)
    plt.figure()
    plt.gca().set_aspect('equal', adjustable='box')
    vertices = [left_vertex, top_vertex, right_vertex, left_vertex]
    x_coords, y_coords = zip(*vertices)
    ax.plot(x_coords, y_coords, 'r-', linewidth=2)

def GetRefractIndex(wavelength):
    x = wavelength * 10**6
    a = np.array([1.03961212, 0.231792344, 1.01146945])
    b = np.array([0.00600069867, 0.0200179144, 103.560653])
    y = np.zeros(x.size)
    for i in range(len(a)):
        y += (a[i] * (x**2)) /  ((x**2) - b[i])
    return (1+y)**(0.5)

def GetCcoords(Lambda, phi):
    x = ((np.sin(Lambda) + np.cos(Lambda)*np.tan(phi))/2) / (np.tan
    (Lambda) - np.tan(phi))
    y = -np.tan(Lambda)*x + np.sin(Lambda)
    return x, y

def GetDcoords(ThetaT, alpha, x1, y1):
    x2 = x1 + np.cos(ThetaT - alpha/2)
    y2 = y1 - np.sin(ThetaT - alpha/2)
    return x2, y2

def GetAcoords(ThetaI, alpha, x1, y1):
    x2 = -np.cos(ThetaI - alpha/2) + x1
    y2 = -np.sin(ThetaI - alpha/2) + y1
    return x2, y2

def UpdateCanvas(ThetaI, alpha):
    ThetaI, alpha = np.deg2rad(ThetaI), np.deg2rad(alpha)
    Lambda = (np.pi/2) - alpha/2
    n = GetRefractIndex(wavelength)
    ThetaT = np.arcsin(((np.square(n) - np.square(np.sin(ThetaI)))
    **0.5)*np.sin(alpha) - np.sin(ThetaI)*np.cos(alpha))
    beta = np.arcsin(np.sin(ThetaI) / n)
    phi = alpha/2 - beta

```

```
    Xb = np.linspace(-np.cos(Lambda)/2, -np.cos(Lambda)/2, 500)
    Yb = np.linspace(np.sin(Lambda)/2, np.sin(Lambda)/2, 500)
    Xa, Ya = GetAcoords(ThetaI, alpha, Xb, Yb)
    Xc, Yc = GetCcoords(Lambda, phi)
    Xd, Yd = GetDcoords(ThetaT, alpha, Xc, Yc)

    colors = [color(f) for f in frequency]
    segments = np.array([[[x1, y1], [x2, y2]] for x1, y1, x2, y2 in zip(Xc, Yc, Xd, Yd)])
    segments2 = np.array([[[x1, y1], [x2, y2]] for x1, y1, x2, y2 in zip(Xb, Yb, Xc, Yc)])
    segments3 = np.array([[[x1, y1], [x2, y2]] for x1, y1, x2, y2 in zip(Xa, Ya, Xb, Yb)])
    lc = LineCollection(segments, colors=colors, alpha=0.5, linewidths=0.25)
    lc2 = LineCollection(segments2, colors=colors, alpha=0.5, linewidths=0.25)
    lc3 = LineCollection(segments3, colors="white", alpha=0.5, linewidths=1)
    fig, ax = plt.subplots(figsize=(8, 6), facecolor="black")
    ax.set_facecolor('black')
    ax.add_collection(lc)
    ax.add_collection(lc2)
    ax.add_collection(lc3)
    triangle(ax,alpha)

interact(UpdateCanvas,
         ThetaI=widgets.IntSlider(min=0, max=90, step=1, value=45, description="theta_i"),
         alpha=widgets.IntSlider(min=1, max=90, step=1, value=45, description="alpha"))
```

This code allows you to change α and θ_i between $1°$ and $90°$ using sliders. The output with $\alpha = 60°$, $\theta_i = 30°$ is shown below



8 Question 8

First let's think about Snell's law of reflection. By Pythagoras $AS = \sqrt{y^2 + x^2}$ and $SB = \sqrt{y^2 + (L-x)^2}$, so the total distance travelled by the wave is $\sqrt{y^2 + x^2} + \sqrt{y^2 + (L-x)^2}$. If we let the wave speed be v then the total time taken is $\frac{\sqrt{y^2+x^2}+\sqrt{y^2+(L-x)^2}}{v}$.

Note that we are trying to see how time taken changes as we change x. Sketching time against x makes it clear that the minimum time will be when $\frac{dt}{dx} = 0$. Hence we are looking for when

$\frac{d}{dx}\frac{\sqrt{y^2+x^2}+\sqrt{y^2+(L-x)^2}}{v} = 0$ so

$\frac{d}{dx}(\frac{1}{v}\sqrt{y^2 + x^2}) + \frac{d}{dx}(\frac{1}{v}\sqrt{y^2 + (L-x)^2}) = 0$ so

$\frac{1}{v}\frac{2x}{2\sqrt{y^2+x^2}} + \frac{1}{v}\frac{-2(L-x)}{2\sqrt{y^2+(L-x)^2}} = 0$

$\frac{x}{\sqrt{y^2+x^2}} - \frac{L-x}{\sqrt{y^2+(L-x)^2}} = 0$

Now notice that $\sin\theta = \frac{x}{\sqrt{y^2+x^2}}$ and also $\sin\phi = \frac{L-x}{\sqrt{y^2+(L-x)^2}}$, by substituting both of these into the previous equation we get $\sin\theta - \sin\phi = 0$, and so we must have $\phi = \theta$ and so Snell's law of reflection holds.

Now let's think about Snell's law of refraction.

$AS = \sqrt{x^2 + y^2}$, and the speed of the light during this portion of the journey is $\frac{c}{n_1}$. This means the travel time for this portion is $\frac{\sqrt{x^2+y^2}}{\frac{c}{n_1}} = \frac{n_1\sqrt{x^2+y^2}}{c}$

$SB = \sqrt{(L-x)^2+y^2}$, and the speed of the light during this portion of the journey is $\frac{c}{n_2}$. This means the travel time for this portion is $\frac{\sqrt{(L-x)^2+y^2}}{\frac{c}{n_2}} = \frac{n_2\sqrt{(L-x)^2+y^2}}{c}$. This makes the total travel time $\frac{n_1\sqrt{x^2+y^2}}{c} + \frac{n_2\sqrt{(L-x)^2+y^2}}{c}$
Note that we trying to see how time changes as we change x. Sketching time against x makes it clear that the minimum time will be when $\frac{dt}{dx} = 0$. Hence we are looking for when

$\frac{d}{dx}(\frac{n_1\sqrt{x^2+y^2}}{c} + \frac{n_2\sqrt{(L-x)^2+y^2}}{c}) = 0$ so

$\frac{d}{dx}(n_1\sqrt{x^2+y^2} + n_2\sqrt{(L-x)^2+y^2}) = 0$ so

$\frac{2xn_1}{2\sqrt{x^2+y^2}} + \frac{-2(L-x)n_2}{2\sqrt{(L-x)^2+y^2}} = 0$

$\frac{xn_1}{\sqrt{x^2+y^2}} - \frac{(L-x)n_2}{\sqrt{(L-x)^2+y^2}} = 0$

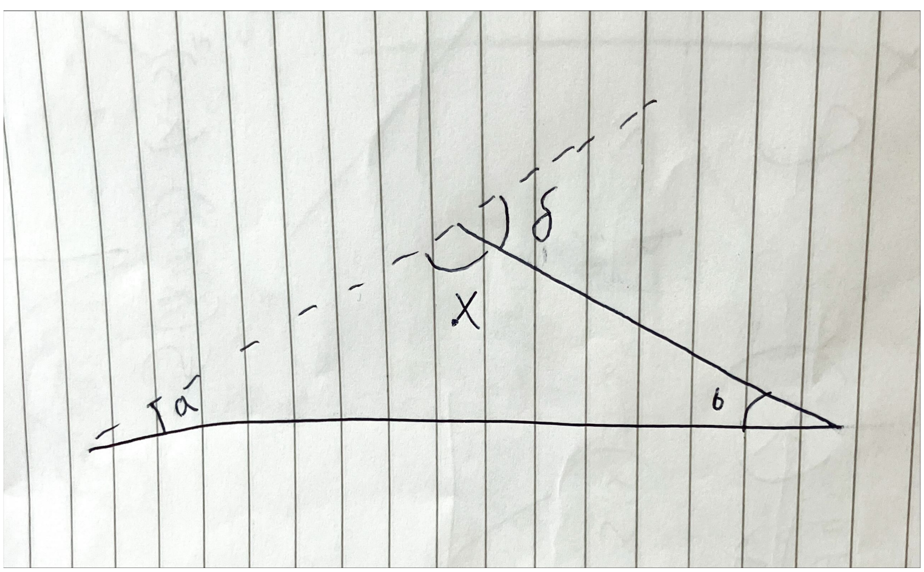
Now notice that we have $\sin\theta = \frac{x}{\sqrt{x^2+y^2}}$ and so $x = \sqrt{x^2+y^2}\sin\theta$, and we also have $\sin\phi = \frac{L-x}{\sqrt{(L-x)^2+y^2}}$ and so $L-x = \sqrt{(L-x)^2+y^2}\sin\phi$. Substituting both of these into the previous equation gives

$\frac{\sqrt{x^2+y^2}\sin(\theta)n_1}{\sqrt{x^2+y^2}} - \frac{\sqrt{(L-x)^2+y^2}\sin(\phi)n_2}{\sqrt{(L-x)^2+y^2}} = 0$

$n_1\sin\theta = n_2\sin\phi$.

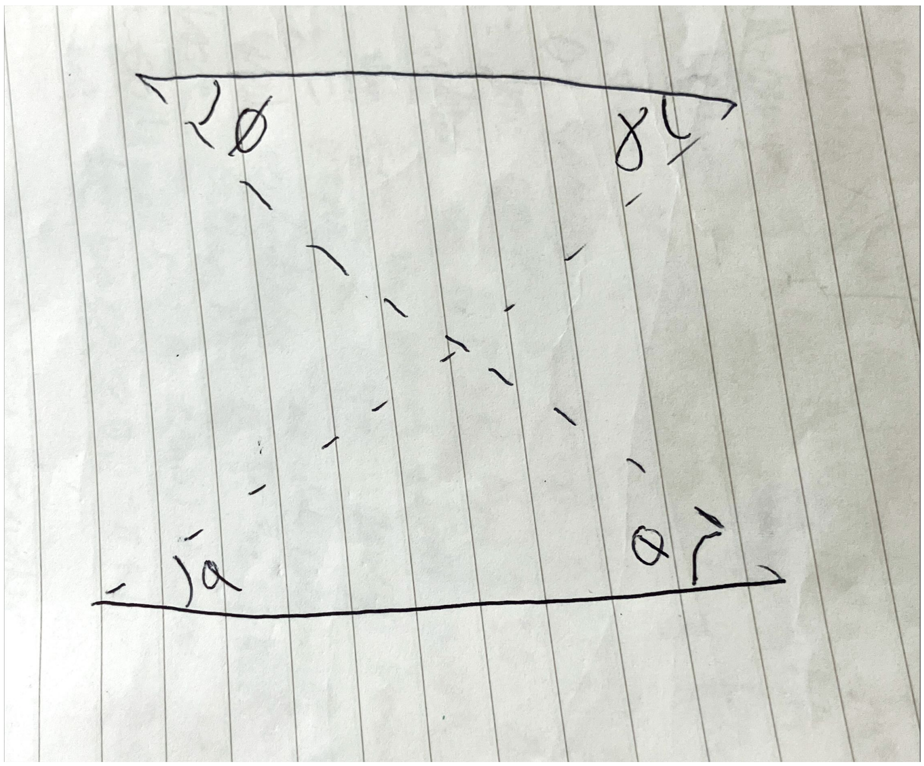
Hence we have shown that the travel time of light is least when $n_1\sin\theta = n_2\sin\phi$ and so by Fermat's principle Snell's law of refraction must hold and we are finished.

9 Question 9



By considering the diagram shown above we can see that $x = 180 - a - b$ and $x = 180 - \delta$, and so we have $180 - a - b = 180 - \delta$, this gives us our first equation.

$$a + b = \delta \tag{1}$$



By considering the digram shown above we can see that the two centre angles must be equal, so $180 - \phi - \gamma = 180 - a - \theta$, this gives us our second equation.

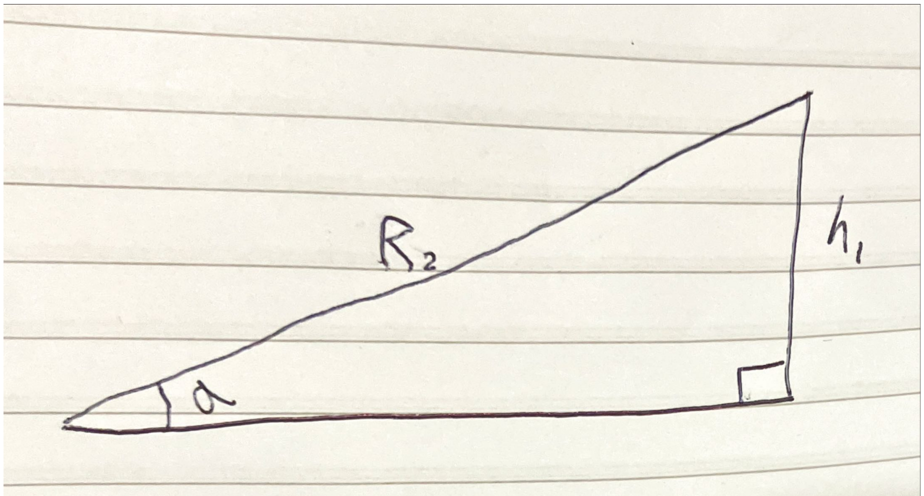
$$\phi + \gamma = a + \theta \tag{2}$$

By Snell's law as the light enters the glass we know $\sin\theta = n \sin\phi$, and so by using small angle approximations we obtain our third equation.

$$\theta \approx n\phi \tag{3}$$

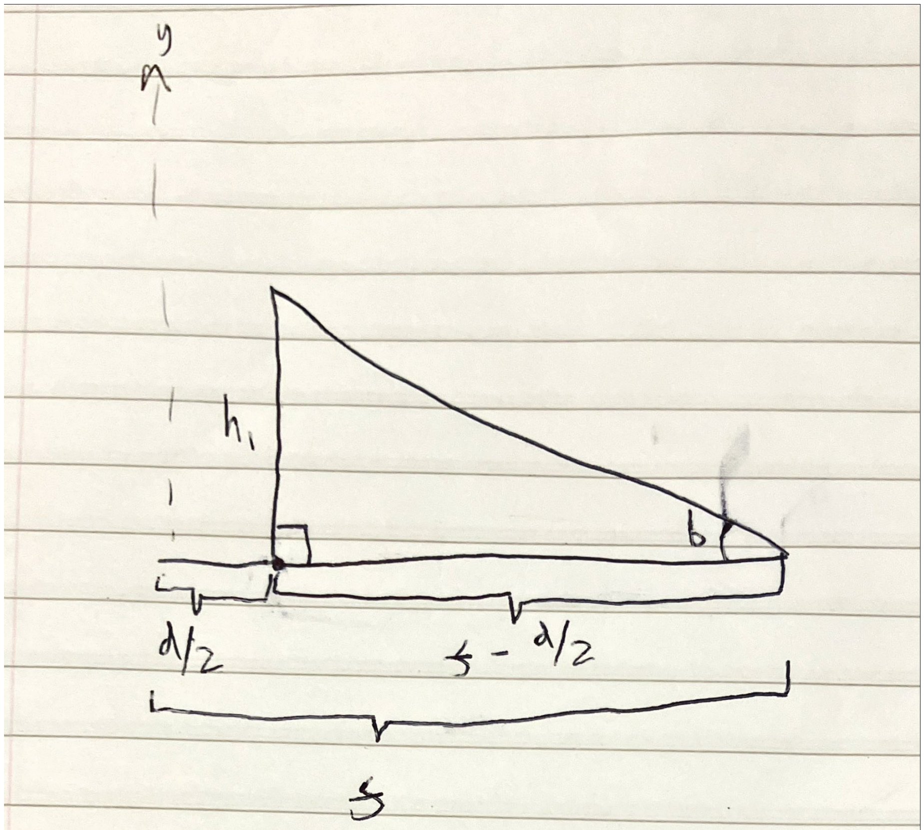
By Snell's law as the light leaves the glass we know $\sin\delta = n \sin\gamma$, and so by using small angle approximations we obtain our fourth equation.

$$\delta \approx n\gamma \tag{4}$$



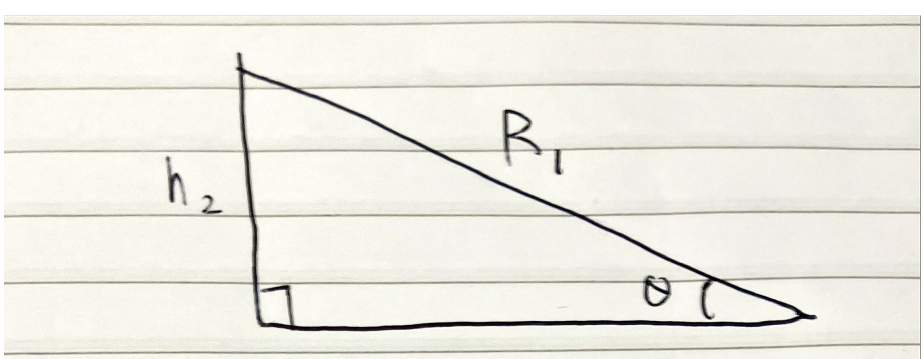
By considering the diagram above we know that $\sin a = \frac{h_1}{R_2}$, and so by rearranging and using the small angle approximations we get our fifth equation

$$aR_2 = h_1 \tag{5}$$



By considering the diagram above we know that $\tan b = \frac{h_1}{f-\frac{d}{2}}$, and so by rearranging, using the small angle and thin lens approximations we can obtain our sixth equation.

$$bf = h_1 \tag{6}$$



By considering the diagram above we know that $\sin\theta = \frac{h_2}{R_1}$ and so by rearranging and using the small angle approximations we get our seventh equation.

$$h_2 = R_1\theta \tag{7}$$

Because we know $h_1 \approx h_2$ by substituting in equations 5 and 7 we can see that we get $R_1\theta = R_2 a$, and so we get our eight equation.

$$a = \frac{R_1\theta}{R_2} \tag{8}$$

We have now found the eight equations we will use in order to derive the formula, to summarise these are.

$$a + b = \delta \tag{1}$$
$$\phi + \gamma = a + \theta \tag{2}$$
$$\theta \approx n\phi \tag{3}$$
$$\delta \approx n\gamma \tag{4}$$
$$aR_2 = h_1 \tag{5}$$
$$bf = h_1 \tag{6}$$
$$h_2 = R_1\theta \tag{7}$$
$$a = \frac{R_1\theta}{R_2} \tag{8}$$

By substituting (3), (4) and (8) into (2) we get that $\frac{\theta}{n} + \frac{\delta}{n} = \frac{R_1\theta}{R_2} + \theta$, and now by substituting in (1) we see that $\frac{\theta}{n} + \frac{a+b}{n} = \frac{R_1\theta}{R_2} + \theta$. Now by using (5) and (6) to see $bf = h_1 = aR_2$ we can rearrange to get $b = \frac{aR_2}{f} = \frac{\frac{R_1\theta}{R_2}R_2}{f} = \frac{R_1\theta}{f}$, by substituting this, and (8) into the equation $\frac{\theta}{n} + \frac{a+b}{n} = \frac{R_1\theta}{R_2} + \theta$ gives us $\frac{\theta}{n} + \frac{\frac{R_1\theta}{R_2} + \frac{R_1\theta}{f}}{n} = \frac{R_1\theta}{R_2} + \theta$. Now we multiply by $\frac{n}{R_1\theta}$
$\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{f} = \frac{n}{R_2} + \frac{n}{R_1}$, and so we have that $\frac{1}{f} = \frac{n}{R_1} - \frac{1}{R_1} + \frac{n}{R_2} - \frac{1}{R_2}$, and so we see that $\frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2})$

As shown in the solution it is possible to obtain the formula $\frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2} + \frac{d(n-1)}{R_1^2 n})$. However this is in fact less accurate than the original one which we proved, $\frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2})$. The reasoning is as follows, firstly we take note of the fact that the true formula is $\frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2} - \frac{(n-1)d}{nR_1R_2})$. Now since $n > 1$ and $R_1, R_2, d > 0$, this must mean that $\frac{(n-1)d}{nR_1R_2} > 0$, and so $\frac{1}{f} < (n-1)(\frac{1}{R_1} + \frac{1}{R_2})$. This means that the result for $\frac{1}{f}$ is always going to be strictly less than the result we would obtain by using the formula we proved. Now consider $\frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2} + \frac{d(n-1)}{R_1^2 n})$, as $n > 1$ and $d, R_1 > 0$, this must mean that $\frac{(n-1)d}{R_1^2 n} > 0$, and so $\frac{1}{f} > (n-1)(\frac{1}{R_1} + \frac{1}{R_2})$. This means that the result for $\frac{1}{f}$ that we obtain from this equation is always going to be strictly greater

than the result we would obtain by using the formula we proved.
This means that the result we obtain from using $\frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2})$ are always going to be more accurate than the result we obtain from $\frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2} + \frac{d(n-1)}{R_1^2 n})$.