Chapter II
Trigonometric Identities and Equations

Anankan Manawichitwanit (Atom)

July 10, 2023

1 Identities

From the unit circle, the Pythagorean theorem could be applied that:

$$r^2 = x^2 + y^2$$

In case of the unit circle when the radius is equal to 1.

$$1 = x^2 + y^2$$

By the definition that $x = \cos\theta$ and $y = \sin\theta$ and by substitution of the above formula, we get the trigonometric identity.

$$\begin{aligned} 1 &= \cos^2\theta + \sin^2\theta && (1) \\ \sec^2\theta &= 1 + \tan^2\theta && (1)/\cos^2\theta \\ \csc^2\theta &= \cot^2\theta + 1 && (1)/\sin^2\theta \end{aligned}$$

1.1 Examples

1. Given that $\cos\theta = 3/5$, find the possible value of

$$\text{a) } \sin\theta \quad \text{b) } \tan\theta$$

Since quadrant is not given for θ From, $\sin^2\theta + \cos^2\theta = 1$, we get that:

$$\begin{aligned} \cos^2\theta &= (\frac{3}{5})^2 = \frac{9}{25} \\ \sin^2\theta + \frac{9}{25} &= 1 \\ \sin^2\theta &= 1 - \frac{9}{25} \\ &= \frac{16}{25} \\ \sin\theta &= \pm\sqrt{\frac{16}{25}} \\ &= \pm\sqrt{\frac{4}{5}} \end{aligned}$$

By the definition that $\tan^2\theta + 1 = \sec^2\theta$:

$$\begin{aligned}
\tan^2\theta + 1 &= 1/\cos^2\theta \\
&= (\frac{5}{3})^2 \\
&= \frac{25}{9} \\
\tan^2\theta &= \frac{25}{9} - 1 \\
&= \frac{16}{9} \\
\tan\theta &= \pm\sqrt{\frac{16}{9}} \\
&= \pm\frac{4}{3}
\end{aligned}$$

In conclusion, a) $\pm 4/5$ and b) $\pm 4/3$

---

2. Given that $x = 2\sin\theta$ and $y = \cos\theta + 1$. Show that $x^2 + 4(y-1)^2 = 4$.
From $x = 2\sin\theta$, $\sin\theta = x/2$ and $\cos\theta = y - 1$.

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 \\
(\frac{x}{2})^2 + (y-1)^2 &= 1 && \text{Substitution} \\
(x^2) + 4(y-1)^2 &= 4 && \times 4
\end{aligned}$$

Hence proven that $(x^2) + 4(y-1)^2 = 4$.

---

3. Simplify $(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)$ and deduce the value of $\sec\theta + \tan\theta$ if $\sec\theta - \tan\theta = 2$.
By the difference of square, we know that:

$$(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = \sec^2\theta - \tan^2\theta$$

Now, we know by trigonometric identity that:

$$\sec^2\theta = 1 + \tan^2\theta$$

By substitution we get that:

$$\begin{aligned}
\sec^2\theta - \tan^2\theta &= (1 + \tan^2\theta) - \tan^2\theta \\
&= 1 + (\tan^2\theta - \tan^2\theta) \\
&= 1
\end{aligned}$$

$$(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$$

Therefore:

$$\begin{aligned}\sec\theta - \tan\theta &= 1/(\sec\theta + \tan\theta)\\ &= 1/2\end{aligned}$$

---

4. Simplify $\dfrac{1-\csc^2 x}{(1-\sin x)(1+\sin x)}$.

By difference of squares:

$$\begin{aligned}\frac{1-\csc^2}{((1-\sin x)(1+\sin x))} &= \frac{1-\csc^2 x}{1-\sin^2 x}\\ &= \frac{1-\csc^2 x}{\cos^2 x} && \sin^2 x + \cos^2 x = 1\\ &= \frac{-(\csc^2 x - 1)}{\cos^2 x} && 1+\cot^2 x = \csc^2 x\\ &= \frac{-\cot^2 x}{\cos^2 x}\\ &= \frac{-\cos^2 x}{\sin^2 x}\frac{1}{\cos^2 x}\\ &= \frac{-1}{\sin^2 x}\\ &= -\csc^2 x\end{aligned}$$

---

5. Prove the following identities:

(a) $\sec x - \cos x \equiv \sin x \tan x$

(b) $(1+\tan\theta)^2 + (1-\tan\theta)^2 \equiv 2\sec^2\theta$

The proofs are as the following:

(a)

$$\begin{aligned}\sec x - \cos x &= \frac{1}{\cos x} - \cos x\\ &= \frac{1-\cos^2 x}{\cos x}\\ &= \frac{\sin^2 x}{\cos x} && \sin^2 x + \cos^2 x = 1\\ &= \sin x\frac{\sin x}{\cos x}\\ &= \sin x \tan x\\ \text{L.H.S.} &= \text{R.H.S}\end{aligned}$$

(b)

$$\begin{aligned}
(1+\tan\theta)^2+(1-\tan\theta)^2 &= 1+2\tan\theta+\tan^2\theta+1-2\tan\theta+\tan^2\theta \\
&= 2+2\tan^2\theta \\
&= 2(1+\tan^2\theta) \\
&= 2(\sec^2\theta) && \tan^2\theta+1=\sec^2\theta \\
&= 2\sec^2\theta \\
\text{L.H.S.} &= \text{R.H.S}
\end{aligned}$$