227-0689-00L System Identification **18 November 2018**

Exercise 8: Prediction error methods

Background reading

The background material for this exercise is Sections 3.2, 4.2, and 7.2 of Ljung (*System Identification; Theory for the User*, 2nd Ed., Prentice-Hall, 1999).

Problem 1: Correlations in a first-order ARMAX model

Consider the model

$$y(k) + ay(k-1) = bu(k-1) + e(k) + ce(k-1),$$

where the error $e(k)$ is white zero-mean noise with variance λ^2, and the input is a zero-mean white signal $u(k) \sim \mathcal{N}(0, \sigma^2)$, such that $u(k)$ is uncorrelated with $\{e(\tau), \tau \leq k\}$. Show that the following hold:

(i) $\mathbb{E}[y(k)u(k)] = 0$

(ii) $\mathbb{E}[y(k)y(k-1)] = \frac{-ab^2\sigma^2+(c-a)(1-ac)\lambda^2}{1-a^2}$

(iii) $\mathbb{E}[y(k)u(k-1)] = b\sigma^2$.

Problem 2: Optimal predictor for a first-order ARMAX model

Consider the model

$$y(k) + ay(k-1) = bu(k-1) + e(k) + ce(k-1),$$

where $|c| < 1$ and $e(k) \sim \mathcal{N}(0, \lambda^2)$. The parameter vector is given by $\theta = \begin{bmatrix} a & b & c \end{bmatrix}^\top$.

1. Show that an optimal predictor $\hat{y}(t|t-1, \theta)$ which minimizes the prediction error variance σ is given by
$$\hat{y}(k|k-1, \theta) = -ay(k-1) + bu(k-1) + ce(k-1).$$
2. What difficulties can be encountered when implementing the optimal predictor? How can this be avoided?
3. Find a practical recursive implementation as a function of the innovation of the optimal predictor.

MATLAB exercises

Consider the ARMAX model

$$A(z)y(k) = B(z)u(k) + C(z)e(k),\ k = 1, \dots, N,$$

where $N = 10^4$ with the matrices defined by

$$\begin{aligned} A(z) &= 1 - 1.5z^{-1} + 0.7z^{-2} \\ B(z) &= 1.0z^{-1} + 0.5z^{-2} \\ C(z) &= 1 - 1.0z^{-1} + 0.2z^{-2}, \end{aligned}$$

where $e(k) \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, 1)$. Fix an input sequence $u(k)$ from the following ARMAX process

$$u(k) = 0.1u(k-1) + 0.12\, u(k-2) + e_u(k-1) + 0.2\, e_u(k-2).$$

where $e_u \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, 1)$.

Assuming that the transfer function $C(z)$ is exactly known:

1. Obtain least-squares (LS) estimates $\hat{A}_{LS}(z)$ and $\hat{B}_{LS}(z)$ for $A(z)$ and $B(z)$ respectively.
2. Plot the predicted values along with the true response $y(k)$ for validation set.
3. Repeat (1) for a different set of realizations of $e(k)$ and plot the histograms of the parameters.