**227-0689-00L System Identification** **6 November 2019**

# Exercise 6: Frequency Domain Subspace and Closed-loop identification

## Background reading

The background material for this exercise is class lectures 7 and 8.

## Problem 1:

Consider the state space description of a discrete-time system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Here we examine several steps and assumptions in arriving at the results of Lecture 7.

(a) Consider the impulse response of the system. In the lecture notes, this was given as

$$g(k) = \begin{cases} 0, & \text{if } k < 0 \\ D, & \text{if } k = 0 \\ CA^{k-1}B & \text{if } k > 0 \end{cases}$$

Explain why $g(k)$ has the above form. Additionally, derive why $G(e^{j\omega}) = \sum_{k=0}^{\infty} g(k)e^{-j\omega k}$ can be written as $G(e^{j\omega}) = C\left(e^{j\omega}I - A\right)^{-1}B + D$.

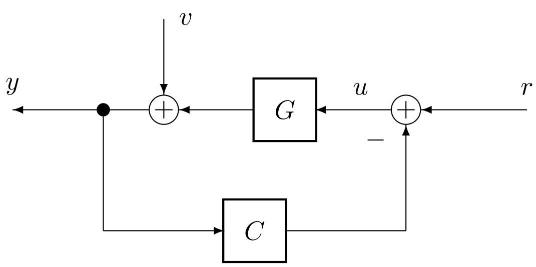
(b) Suppose $A \in \mathbb{R}^{n\times n}$ is nilpotent. How does this affect the impulse response, Hankel matrix, and singular value decomposition, compared to a full rank $A$?

(c) One of the potential pitfalls listed for the subspace identification method is that it can result in an unstable model for a stable system. Give a numerical example of a stable linear discrete-time system which, when the system order is truncated, results in an unstable system. Explain why this happened.

(d) Another disadvantage listed for the subspace identification method is that the truncated SVD of a Hankel matrix might no be longer Hankel. Show that the truncated SVD for any real-valued 2x2 Hankel matrix is still Hankel. Give an example of a 3x3 Hankel matrix which has a truncated SVD which is no longer Hankel. Why is having a non-Hankel truncated SVD a disadvantage?

## Problem 2

Consider a plant, $G$, operating in closed-loop with a controller, $C$, as illustrated in the following diagram.




We can choose the external excitation signal, $r$, but of course we cannot choose the noise signal $v$. Assume that these two signals have the spectra, $\Phi_r(\omega)$ and $\Phi_v(\omega)$ respectively. We can also assume that these two signals are independent and that $v$ is zero mean.

Now suppose we run an experiment and measure $y$ and $u$. We use a frequency domain window, $W_\gamma(\omega)$, with width parameter $\gamma$, to estimate the spectral densities $\widehat{\Phi}_y$ and $\widehat{\Phi}_u$ of the output, $y$, and input, $u$, signals respectively. Show that the filtered estimate of $G$,

$$\widehat{G}_N(e^{j\omega}) := \frac{\widehat{\Phi}^N_{yu}(\omega)}{\widehat{\Phi}^N_u(\omega)}$$

satisfies,

$$\lim_{N,\gamma\longrightarrow\infty} \widehat{G}_N(e^{j\omega}) \;=\; \frac{G(e^{j\omega})\,\widehat{\Phi}_r(\omega) \;-\; C(e^{-j\omega})\widehat{\Phi}_v(\omega)}{\widehat{\Phi}_r(\omega) \;+\; |C(e^{j\omega})|^2\widehat{\Phi}_v(\omega)}.$$

## Problem 3

Let $V$ be a zero-mean normally distributed random variable such that $V \sim \mathcal{N}(0,1)$. Consider the variable $Z = V^2$. Show that $Z$ is a stochastic variable with probability density function

$$f_Z(z) = \begin{cases} \frac{z^{-1/2}e^{-z/2}}{\sqrt{2}\,\Gamma(1/2)} & \text{if } z \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{6.1}$$

where $\Gamma(\cdot)$ is the Gamma function: $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$.

*Note: The distribution in (6.1) takes the name of chi-squared distribution (or $\chi^2$ distribution) with 1 degree of freedom.*

What is the expected value of $Z$? and its variance?

## Matlab **exercise:**

This exercise will look at identification in closed-loop.

Consider a continuous-time plant, $G(s)$, defined by,

$$G(s) \;=\; \frac{(s^2 + 2\zeta_z\omega_z s + \omega_z^2)}{(s^2 + 2\zeta_p\omega_p s + \omega_p^2)}\,\frac{5000}{(s+50)(s+200)},$$

where the pole and zero damping and modal frequency coefficients are,

$$\zeta_z = 0.1, \quad \omega_z = 3.0, \quad \zeta_p = 0.1, \quad \text{and} \quad \omega_p = 3.5.$$

At the input of this plant is a zero-order hold with a sample period of,

$$T_s \;=\; 0.02 \quad \text{seconds.}$$

Denote the discrete-time, zero-order hold equivalent of this plant by $G_d(z)$. This can be calculated in Matlab via the command:

```
>>   Gdz = c2d(Gs,Ts,'zoh');
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The input to $G_d(z)$ is $u(k)$. Unfortunately our measurement of the output has noise, $v(k)$, giving the measured output as,

$$y(k) \;=\; G_d(z)\,u(k) \;+\; v(k).$$

In the following we will simulate the noise as a normally distributed signal with variance, $\lambda = 0.1$.

Now implement a discrete-time PI controller for this system. The controller is,

$$C_d(z) \;=\; \frac{1.25z - 0.75}{z-1}.$$

We will implement this in a standard unity gain feedback configuration. The input to $C_d(z)$ is $r(k) - y(k)$, where $r(k)$ is a reference input to be tracked. The output of $C_d(z)$ is the control actuation, $u(k)$.

1. Calculate the discrete-time sensitivity function,

   $$S_d(z) \;=\; \frac{1}{1 + G_d(z)C_d(z)},$$

   and check that all of its poles are inside the unit circle. The complementary sensitivity function, $T_d(z)$ is the transfer function from the reference, $r(k)$, to the output, $y(k)$, and can be calculated by,

   $$T_d(z) \;=\; 1 - S_d(z).$$

   It has the same poles as $S_d(z)$.

2. Run the following series of identification experiments:

   a) Excite $r(k)$ with a PRBS signal of maximum amplitude $\pm 0.1$. Use a periodic signal and discard the first period. Measure $y(k)$ and $u(k)$. Use time-domain averaging to improve the signal to noise ratio.

b) Estimate the ETFE in the usual way (i.e. $Y_N(\omega)/U_N(\omega)$).

Compare (on a Bode plot), your estimate of $G_d(z)$ with its true frequency response.

3. Run the above experiment, again using a periodic excitation for $r(k)$ and measuring the "error signal",

$$e(k) = r(k) - y(k).$$

Discarding the first period and using time-domain averaging will improve the signal to noise ratio. Now use these measurements to estimate the sensitivity function, $S_d(z)$. Compare (on a Bode plot and an error magnitude plot) your estimate of $S_d(z)$ with its true frequency response.

4. For the next experiment we set $r(k) = 0$, but introduce another excitation signal, $w(k)$, at the output of the controller. In other words, the actuation signal going into the plant is now,

$$u(k) = w(k) + C_d(z)(r(k) - y(k)), \quad \text{with} \quad r(k) = 0.$$

Now run an experiment, this time specifying $w(k)$ as an excitation signal with the same properties as $r(k)$ in the previous experiment. Measure $w(k)$ and $y(k)$. Note that the relationship between these two signals (with $r(k) = 0$) is,

$$y(k) = \frac{G_d(z)}{1 + G_d(z)C_d(z)}\, w(k) + \frac{1}{1 + G_d(z)C_d(z)}\, v(k),$$

and so you will be able to estimate $G_d(z)S_d(z)$ from this experiment.

5. From your estimates of $S_d(z)$ and $G_d(z)S_d(z)$, divide one by the other to get an estimate for $G_d(z)$. Compare this (on a Bode plot and an error magnitude plot) with the true frequency response and your previous estimate of $G_d(z)$.