227-0689-00L System Identification 26 November 2018

Exercise 9: Instrumental variable methods

Background reading

The background material for this exercise is Sections 7.5 and 7.6 of Ljung (*System Identification; Theory for the User*, 2nd Ed., Prentice-Hall, 1999).

Problem 1:

Suppose that a true description of a certain system is given by

$$y(k) + a_1 y(k-1) + \ldots + a_{n_a} y(k-n_a) = b_1 u(k-1) + \ldots + b_{n_b} u(k-n_b) + e(k) \tag{9.1}$$

where $\{e(k)\}$ is white noise independent of the input. An intuitive way to generate instruments is to use a similar structure to (9.1), as given in

$$\zeta(k) = K(q)[-x(t-1) \ \ -x(t-2) \ldots -x(t-n_a) \quad u(k-1) \ \ldots \ u(k-n_b)]^\intercal, \tag{9.2}$$

where $K(q)$ is a linear filter, $^\intercal$ is the transpose operator and $x(t)$ is generated from the input through the linear system

$$N(q)x(t) = M(q)u(t).$$

Here, the polynomials M and N are given by

$$\begin{aligned} M(q) &= m_0 + m_1 q^{-1} + \ldots + m_{n_m} q^{-n_m} \\ N(q) &= 1 + n_1 q^{-1} + \ldots + n_{n_n} q^{-n_n}, \end{aligned}$$

and the values of n_n and n_m are chosen to be equal to n_a and n_b respectively. Note that this choice requires the knowledge of the true system structure. Consider the alternative set of instrumental variables

$$\tilde{\zeta}(k) = \frac{K(q)}{N(q)}[u(k-1) \ \ u(k-2) \ \ \ldots \ \ u(k-n_a-n_b)]^\intercal. \tag{9.3}$$

Show that if the polynomials M and N have no common factor, then the instruments ζ and $\tilde{\zeta}$ give the same estimate of the system parameters. How does the choice of M affect the estimates?

Problem 2:

Consider the following system:

$$y(k) + ay(k-1) = b_1 u(k-1) + b_2 u(k-2) + v(k).$$

Parameters of the system should be estimated by using the instrumental variables method. To this end, it has been decided to use delayed inputs as instruments:

$$z(k) = [u(k-1)\, u(k-2)\, u(k-3)]^T .$$

Assuming that $u(k)$ is white noise with zero mean and unit variance and that it is uncorrelated with $v(k)$, find for which values of parameters a, b_1 and b_2, the instrumental variables are correlated with the regression variables (i.e. $E\{z(k)\varphi^T(k)\}$ is nonsingular).

Matlab exercise:

The dataset 'Data_ex9.mat' needed for this exercise can be downloaded from the resources page on Piazza.

Consider data, $y(k)$ and $u(k)$, collected from the system,

$$y(k) = \frac{B(z)}{A(z)}u(k) + C(z)e(k), \quad e(k) \sim \mathcal{N}(0, \lambda) .$$

The polynomials are of the form:

$$\begin{aligned} A(z) &= 1 + a_1 z^{-1} + a_2 z^{-2} , \\ B(z) &= b_1 z^{-1} , \\ C(z) &= 1 + c_1 z^{-1} . \end{aligned}$$

For the questions below, the following experimental data is provided:

- `ex9_u`: the input signal, $u(k)$, for $k = 1, 2, \ldots, K$,
- `ex9_y`: the corresponding output signal, $y(k)$, for $k = 1, 2, \ldots, K$.

The system is at rest and that there is no noise for $k \leq 0$,

$$u(k) = y(k) = e(k) = 0 \text{ for all } k \leq 0 .$$

1. Using pseudo-linear regression (PLR) over the entire data, estimate the problem parameters $\hat{\theta}_{\text{PLR}} = [\hat{a}_1\ \hat{a}_2\ \hat{b}_1\ \hat{c}_1]^T$.
   (a) Formulate a pseudo-linear regression that, when solved, estimates the parameters $\hat{\theta}_{\text{PLR}}$. Express the one-step-ahead prediction estimator in terms of the solution to the pseudo-linear regression.
   (b) For the given input and output sequence, `ex9_u` and `ex9_y`, solve the above problem for the estimated parameters $\hat{\theta}_{\text{PLR}}$. Generate the resulting vector of prediction errors, $\epsilon(k) = y(k) - \hat{y}(k|\hat{\theta}_{\text{PLR}})$, $k = 1, \ldots, K$.

2. Use an instrumental variable (IV) method to estimate $\hat{\theta}_{\mathrm{IV}} = [\hat{a}_1 \ \hat{a}_2 \ \hat{b}_1]^T$. Starting from a least-squares initialization, compute suitable instruments $\zeta(k)$.

   (a) Describe your choice of instrumental variables, $\zeta(k)$.

   (b) Estimate $\hat{\theta}_{\mathrm{IV}}$ using your chosen $\zeta(k)$.