Exercise 10: ML and MAP Estimators

Background reading

The background material for this exercise is Section 7.4 of Ljung (*System Identification: Theory for the User*, 2nd Ed., Prentice-Hall, 1999).

Problem 1:

Consider the model structure

$$\hat{y}(k|\theta) \;=\; -ay(k-1) + bu(k-1)$$

where $\theta = [a, b]^T$ is the parameter to be estimated. Assume that the true system is given by

$$y(k) - 0.9y(k-1) \;=\; u(k-1) + e(k)$$

where $e(k)$, $k = 1, ..., N$ is a sequence of independent identically distributed normal random variables with zero mean and unit variance, $e(k) \sim \mathcal{N}(0, 1)$.

(a) Obtain an expression for the maximum likelihood estimator of the parameter θ in terms of the data $u(k)$, $k = 0, \ldots, N-1$, and $y(k)$, $k = 0, \ldots, N$.

(b) Determine the Cramér-Rao bound for the estimation of a and b.

(c) How does the Cramér-Rao bound for this problem depend on the properties of u?

Problem 2:

Consider the following finite impulse response (FIR) model structure

$$\hat{y}(k|\theta) \;=\; \sum_{i=1}^{n} g_i u(k-i)$$

where $\theta = (g_i)_{i=1}^n$ is the parameter to be estimated. Assume that the true system is given by

$$y(k) \;=\; \sum_{i=1}^{n} g_i u(k-i) + e(k)$$

where $e(k)$, $k = 1, ..., N$ is a sequence of independent identically distributed normal random variables with zero mean and unit variance, $e(k) \sim \mathcal{N}(0, 1)$.

Suppose the parameter θ is characterized by the following distribution,

$$\theta \sim \mathcal{N}(0, S),$$

where S is the prior covariance of θ.

(a) Obtain an expression for the maximum a posteriori (MAP) estimator of the parameter θ in terms of the data $u(k)$, $k = -n+1, \dots, N-1$, and $y(k)$, $k = 1, \dots, N$.

(b) What can be a reasonable choice of S?

MATLAB **exercises:**

Please download the data file `SysID_Exercise_10.mat` from Piazza to finish the following exercises. Write a MATLAB function of the following form:

```
[theta_ml,theta_map] = HS2019_SysID_Exercise_10_LegiNumber()
```

that solves the following exercises. In this exercise, you are NOT allowed to use any toolbox. The only file should be used is the provided `SysID_Exercise_10.mat`.

In this exercise, we will also test a sanity check script that will be introduced in the final exam, in response to a suggestion posted on Piazza. The sanity check script will test if your script can run, its dependencies on other files and toolboxes, as well as the basic format of your output. To run the sanity check script, download the file `SysID_Exercise_10_check.p` from Piazza and put it in the same folder as your solution function. The sanity check is performed by running

```
SysID_Exercise_10_check(LegiNumber)
```

The result of the check will be displayed in the command window.

If you have any problem or suggestion on this sanity check, please feel free to post it on Piazza. You can also voluntarily submit your solution to `myin@control.ee.ethz.ch` by **23:55, December 9, 2019**. The subject of the email must be `SystemID HS2019 Exercise 10`. The email should contain only `HS2019_SysID_Exercise_10_LegiNumber.m` in the attachment. We will not grade your solution but do the same check to see if your solution is valid.

1. Consider the problem of estimating θ in the following system,

$$y(k) = \theta u(k-1) + v(k) + \beta_1 v(k-1) + \beta_2 v(k-2)\,,$$

where $\beta_1 = 0.5$, $\beta_2 = 0.2$ and $v(k)$ is sampled from the normal distribution, $v(k) \sim \mathcal{N}(\mu, \lambda)$, with $\mu = 1.5$ and $\lambda = 1.2$. Moreover, assume that the system is initially at rest, i.e., there is no noise, input and output for $k \leq 0$,

$$u(k) = y(k) = v(k) = 0 \text{ for all } k \leq 0\,.$$

Using the maximum likelihood (ML) method and the provided input and output data (`u1` and `y1` in `SysID_Exercise_10.mat`) as experimental data to estimate the value of θ.

2. Consider the problem of estimating θ in the following system,

$$y(k) = \theta u(k-1) + 2\theta u(k-2) + \theta u(k-3) + v(k)\,,$$

where $v(k)$ comes from the normal distribution $v(k) \sim \mathcal{N}(\mu, \lambda)$ with $\mu = 1.5$ and $\lambda = 1.2$. Assume that the system is initially at rest and that there is no noise for $k \leq 0$,

$$u(k) = y(k) = v(k) = 0 \text{ for all } k \leq 0\,.$$

The prior knowledge of θ is characterized by the following distribution,

$$\theta \sim \mathcal{N}(1.5, 0.5^2).$$

Calculate the maximum a posteriori (MAP) estimate of θ, for the provided input and output data (`u2` and `y2` in `SysID_Exercise_10.mat`).