Many of the principles or laws underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms the relations are equations and the rates are derivatives. Therefore, to understand and to investigate problems involving the motion of fluids, flow of current in electric circuits, the dissipation of heat in solid objects, the propagation and detection of seismic waves, increase or decrease of population, among many others, it is necessary to know something about differential equations.
Differential equations arise in a variety of applications in Physics, Chemistry, Biology etc. Hence an in depth study of differential equations has assumed prime importance in all modern scientific

Definition [Differential equation] : An equation containing the derivatives of one or more dependent variables (or unknown functions), with respect to one or more independent variables is called a differential equation.

Ex: 1) $\frac{dy}{dx}+5y=0$ 2) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=0$ 3) $\frac{dy}{dx}+4x\sin y=e^x$

4) $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$ 5) $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=2xy$ 6) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$

7) $\frac{dx}{dt}+\frac{dy}{dt}-2x-4y=e^t$

$\frac{dx}{dt}+\frac{dy}{dt}-y=e^{4t}$

Definition[Ordinary differential equation] : A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation (ODE). Examples (1)-(3) and (7) are ODEs.

Definition[Partial differential equation] : A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variables is called a partial differential equation (PDE). Examples (4)-(6) are PDEs.

Order of differential equation: It is defined as the highest derivative appearing in the equation.

Degree of the differential equation: It is defined as the positive integral power of the highest ordered derivative appearing in the equation.

Note: In order to obtain the degree of differential equation, the equation must be expressed in a form free from the radicals and fractional powers as far as derivatives are concerned. Degree is not defined if the equation cannot be expressed in the form of a polynomial in the derivative.

Exs:

1) $\frac{d^2y}{dx^2} + 16\frac{dy}{dx} + 2y = 0$ order – 2, deg – 1

2) $\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = \sin x$ order – 3, deg – 1

3) $x + \frac{dy}{dx} = \sqrt{1 + \frac{dy}{dx}}$ order – 1, degree – 2

(Square on B.S to remove fractional power of derivative)

4) $\left(\frac{d^2y}{dx^2} + 2y\right)^{3/2} = \frac{dy}{dx} + 2$ order – 2, deg – 3

5) $\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2} = \frac{d^2y}{dx^2}$ order – 2, deg – 2

6) $\frac{d^4y}{dx^4} + \tan\left(\frac{dy}{dx}\right) = 0$ order – 4, deg – not defined

Initial value problem (IVP):
Differential equation subject to the condition prescribed at one point is called an initial value problem.

Ex: 1) $\frac{dy}{dx} + x^2y = e^{2x}$; $y(0) = 1$.

2) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 6y = \sin x$; $y(1) = 1$, $y'(1) = 2$.

Boundary value problem (BVP):
Differential equation subject to the condition prescribed at more than one point is called boundary value problem.

Ex: 1) $\frac{d^2y}{dx^2} + y = \cos 2x$; $y(0)=1$, $y(\pi)=0$.

2) $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 6y = 2$; $y(0)=1$, $y(2)=\pi$.

Solution of DE

Consider a n^{th} order ode

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0 \quad \text{———} \quad (*)$$

where F is a fn of $(n+2)$ arguments $x, y, \frac{dy}{dx}, \dots, \frac{d^ny}{dx^n}$.

1) A function $y_1(x)$ defined on interval I, is called explicit soln of $(*)$ on I if

i) An n^{th} derivatives of $y_1(x)$ exist $\forall x \in I$ and are continuous

ii) $$F\left(x, y_1(x), y_1'(x), y_1''(x), \dots, y_1^{(n)}(x)\right) = 0$$

2) A relation $g(x,y)=0$ is called an implicit soln if this relation defines at least one real fn of $x \in I$ such that this fn is explicit soln of $(*)$.

Ex: Consider $$\frac{d^2y}{dx^2} + y = 0 \quad \text{———} \quad (1)$$

Claim that $y_1(x) = 2\sin x + 3\cos x$ is an explicit soln for all $x \in \mathbb{R}$ (1).

Observe that

$$y_1'(x) = 2\cos x - 3\sin x$$
$$y_1''(x) = -2\sin x - 3\cos x$$

Sub $y_1''(x)$ and $y_1(x)$ for $\frac{d^2y}{dx^2}$ and y in (1), we get

$$(-2\sin x - 3\cos x) + (2\sin x + 3\cos x) = 0 \quad \forall x \in \mathbb{R}$$

Thus $y_1(x)$ is an explicit soln of (1).

Ex: Consider $x + y\frac{dy}{dx} = 0$ ——— (2)

i) The relation $x^2+y^2-25=0$ is an implicit soln of (2) in the interval $(-5,5)$ └ (3)

From $x^2+y^2-25=0$, we define

$$y_1(x) = \sqrt{25-x^2}$$

and

$$y_2(x) = -\sqrt{25-x^2}.$$

Diff (3) wrt x,

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow x + y\frac{dy}{dx} = 0$$

Thus (3) satisfies (2).

Observe

$$y_1'(x) = \frac{1.(-\cancel{2}x)}{\cancel{2}\sqrt{25-x^2}}.$$

Sub. $y_1'(x)$ and $y_1(x)$ for $\frac{dy}{dx}$ and y in (2)

$$x + \cancel{\sqrt{25-x^2}}\left(\frac{-x}{\cancel{\sqrt{25-x^2}}}\right) = 0 \quad \text{or} \quad x - x = 0$$

$\forall x \in (-5,5)$.

Thus $x^2+y^2-25=0$ is an implicit soln of (2).

ii) Consider $x^2+y^2+25=0$ — (4)

$\Rightarrow$ (4) satisfies the DE (1), but it is not an implicit soln. (Why?)

diff (4) wrt x,

$$2x + 2y\frac{dy}{dx} = 0$$

or

$$x + y\frac{dy}{dx} = 0$$

Since $y^2 = -25-x^2$

$\Rightarrow y = \pm\sqrt{-25-x^2}$ are not real functions. Thus these are not explicit solns of (2).

Hence (4) is not an implicit soln.

We call (4) a formal soln of the DE (2).

Higher-order linear differential eqns

Defn: A linear diff eqn. of order n in the dependent Variable y and the independent var. x is an eqn that is in, or can be expressed in the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = F(x). \quad \circledast$$

where $a_i(x)$, $i = 0, 1, 2, \ldots n$ are continuous real valued fns on $a \le x \le b$ and $a_0(x) \ne 0$ $\forall x$.

$F(x)$ is called the nonhomogenous term.

If $F \equiv 0$, then $\circledast$ is called Homogenous LDE.

Otherwise, $\circledast$ is called a nonhomogenous LDE.

Linear independent and dependent functions

Defn [Linear dependent]: The n functions $f_1, f_2, \ldots, f_n$ are are called linear dependent on $a \le x \le b$ if there exist constants $c_1, c_2, \ldots, c_n$ not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \qquad \forall\ x \in [a,b].$$

In particular, two fns f_1 and f_2 are linearly dependent on $a \le x \le b$ if there exist c_1, c_2, not both zero, such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \qquad \forall\ x \in [a,b]$$

Exs: The fns $f_1(x) = x$ and $f_2(x) = 5x$ are linearly dependent because there exist c_1 and c_2, both not zero, such that

$$c_1 x + c_2(5x) = 0 \qquad \left(\text{put } \begin{array}{l} c_1 = -5 \\ c_2 = 1 \end{array}\right)$$

Defn [Linear independent]: The n functions $f_1, f_2, \ldots, f_n$ are called linear independent on $a \le x \le b$ if the relation

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \forall\ x \in [a,b]$$

$$\Rightarrow \quad c_1 = c_2 = c_3 = \cdots = c_n = 0$$

Ex: The fns $f_1(x) = e^x$ and $f_2(x) = e^{-x}$ are linearly independent because

$$c_1 e^x + c_2 e^{-x} = 0 \Rightarrow c_1 = c_2 = 0$$

Defn: Let $f_1, f_2, \dots, f_n$ be fns in $C^{(n-1)}[a,b]$, define the function $W[f_1, f_2, \dots, f_n]$ on $[a,b]$ by

$$W[f_1, f_2, \dots, f_n] = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

the function $W[f_1, f_2, \dots, f_n]$ is called the Wronskian of $f_1, f_2, \dots, f_n$

Theorem 5: Let $f_1, f_2, \dots, f_n$ be elements in $C^{(n-1)}[a,b]$. If $\exists\ x_0 \in [a,b]$ such that $W[f_1, f_2, \dots, f_n] \neq 0$, then $f_1, f_2, \dots, f_n$ are linearly independent.

Note: Converse of the above theorem is not true.

Ex 1: Let $f_1(x) = e^x$, $f_2(x) = e^{-x}$, $f_3(x) = e^{-2x}$.

Consider $W(f_1, f_2, f_3) = \begin{vmatrix} e^x & e^{-x} & e^{-2x} \\ e^x & -e^{-x} & -2e^{-2x} \\ e^x & e^{-x} & 4e^{-2x} \end{vmatrix}$

$$= e^x\left(-4e^{-3x} + 2e^{-3x}\right) - e^{-x}\left(4e^{-x} + 2e^{-x}\right) + e^{-2x}(1+1)$$

$$= -2e^{-2x} - 6e^{-2x} + 2e^{-2x}$$

$$= -6e^{-2x} \neq 0 \quad \text{for any } x$$

$\therefore$ f_1, f_2 and f_3 are linearly independent.

Ex 2: Check whether the fns are linearly depend. or indep.

i) $f_1(x) = \sin x$, $f_2(x) = \cos x$, $f_3(x) = x$ (Ans: LI)

ii) $f_1(x) = x^2$, $f_2(x) = x$, $f_3(x) = 2x - 3x^2$ (Ans: LD)

Thm: The n^{th}-order homogenous linear diff. eqn.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)\, y = 0 \quad \text{——— } (\#)$$

always possesses n linearly independent solns.

Further, if $f_1, f_2, f_3, \ldots, f_n$ are n LI solns, then every soln of (#) can be expressed as

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

$c_1, c_2, \ldots, c_n$ are arbitrary constants. It is called the general soln (GS) of (#).

Note: If $c_1, c_2, \ldots, c_n$ are given some particular values, then it is called particular soln of (#).

Homogeneous LDE with constant coefficients.

It is of the form

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = 0 \quad \text{——— } (\$)$$

where a_i's are constant.

Method to find solns of (\$).

Consider 2nd order homogenous LDE,

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0 \quad \text{——— } (₹)$$

Assume that $y = e^{mx}$ is a soln of (₹).

Then

$$\frac{d^2 e^{mx}}{dx^2} + a_1 \frac{d e^{mx}}{dx} + a_2 e^{mx} = 0$$

$$\Rightarrow m^2 e^{mx} + a_1 m e^{mx} + a_2 e^{mx} = 0$$

$$\Rightarrow m^2 + a_1 m + a_2 = 0 \quad \text{——— (a)} \quad \left(\because e^{mx} \neq 0, \forall x\right)$$

Thus, independent solns of (ii) depend on the nature of the roots of the above quadratic eqn.

The eqn (a) is called Auxillary eqn (AE) or characteristic Eqn

Case i : Roots are real and distinct $\left(m_1 \neq m_2, \text{ real}\right)$

if m_1 and m_2 are two distinct real roots, then linearly independent solns are

$$e^{m_1 x} \text{ and } e^{m_2 x}.$$

The G.S of (ii) is

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

Case ii : Roots are real and equal $\left(m_1 = m_2\right)$

Let m be the repeated root of the AE. To find independent solns.

(ii) is

Notation:

$$\frac{d}{dx} := D$$

$$D^2 y + a_1 Dy + a_2 y = 0$$

$$\Rightarrow \left(D^2 + a_1 D + a_2\right) y = 0$$

$$\Rightarrow \left((D-m)(D-m)\right) y = 0$$

$$\Rightarrow (D-m)(D-m) y = 0 \quad \text{——— (1)}$$

Let $(D-m) y = z$ ——— (2)

$$① \Rightarrow \quad (D-m)z = 0$$

$$\frac{dz}{dx} - mz = 0$$

$$\Rightarrow \quad z = c_1 e^{mx}$$

$$\frac{dz}{dx} = mz$$

$$\int \frac{dz}{z} = \int m\,dx$$

$$\log z = mx + c$$

$$\Rightarrow z = c_1 e^{mx} \quad \Big/ c_1 = e^c$$

Sub in ②,

$$(D-m)y = c_1 e^{mx}$$

$$\Rightarrow \quad \frac{dy}{dx} - my = c_1 e^{mx}$$

$$P = -m, \quad Q = c_1 e^{mx}$$

$$IF = e^{\int -m\,dx} = e^{-mx}$$

This is LDE.

G.S $\quad y\, IF = \int Q\, IF\, dx + C_2$

$$\Rightarrow \quad y\, e^{-mx} = \int c_1\, e^{mx} \cdot e^{-mx}\, dx + C_2$$

$$\Rightarrow \quad y\, e^{-mx} = \int c_1\, dx + c_2$$

$$\Rightarrow \quad y\, e^{-mx} = c_1 x + c_2$$

or

$$\boxed{y = c_1 x e^{mx} + c_2 e^{mx}}$$

Independent solns are

$$e^{mx}, \quad x e^{mx}$$

Case iii) Roots are complex $(m_1 = a+ib, \ m_2 = a-ib)$

Independent solns are $e^{m_1 x}$ and $e^{m_2 x}$ (complex valued fns)

Consider $e^{m_1 x} = e^{(a+ib)x} = e^{ax} \cdot e^{ibx}$

$$= e^{ax}(\cos bx + i \sin bx)$$

$$= e^{ax}\cos bx + i\, e^{ax} \sin bx$$

Thus, independent solns are $e^{ax}\cos bx, \ e^{ax}\sin bx$.

	Roots of AE	Independent solns	General soln
1)	$-1, \; 2$	$e^{-x}, \; e^{2x}$	$c_1 e^{-x} + c_2 e^{2x}$
2)	$1, \; 2, \; 2$	$e^{x}, \; e^{2x}, \; xe^{2x}$	$c_1 e^{x} + c_2 e^{2x} + c_3 x e^{2x}$
3)	$\pm i$	$\cos x, \; \sin x$	$c_1 \cos x + c_2 \sin x$
4)	$-2, \; -1 \pm 3i$	$e^{-2x}, \; e^{-x}\cos 3x, \; e^{-x}\sin 3x$	$c_1 e^{-2x} + c_2 e^{-x}\cos 3x + c_3 e^{-x}\sin 3x$
5)	$\sqrt{5} \pm 2$	$e^{(\sqrt{5}+2)x}, \; e^{(\sqrt{5}-2)x}$	$c_1 e^{(\sqrt{5}+2)x} + c_2 e^{(\sqrt{5}-2)x}$
6)	$\pm i, \; \pm i$	$\cos x, \; \sin x, \; x\cos x, \; x \sin x$	$c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$

Ex 1: Solve

$$2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3y = 0 \qquad \text{——— (1)}$$

Soln: Let $\frac{d}{dx} := D$ with this notation (1) is written as

$$2D^2 y - 5Dy - 3y = 0$$

or $$(2D^2 - 5D - 3)\, y = 0$$

AE is

$$2m^2 - 5m - 3 = 0$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solving AE we get two roots,

$$m = \frac{5 \pm \sqrt{25 + 24}}{4}$$

$$= \frac{5 \pm 7}{4}$$

Roots are $3, -\frac{1}{2}$ (Real and distinct)

independent solns are $e^{3x}, \; e^{-\frac{1}{2}x}$

$\therefore$ G.S is
$$y = c_1 e^{3x} + c_2 e^{-\frac{1}{2}x}.$$

Ex2: Solve the BVP

$$y'' - 2y' + 2y = 0, \text{ given } y(0) = 1, \; y'(\pi) = 1$$

Soln: Given DE is
$$(D^2 - 2D + 2)y = 0$$

AE: $m^2 - 2m + 2 = 0$

$$\Rightarrow m = \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$= \frac{2 \pm i2}{2}$$

Thus, roots are $1 \pm i$. (roots are complex no.s)

Independent solns
$$e^x \cos x, \quad e^x \sin x$$

G.S
$$\boxed{y = e^x (c_1 \cos x + c_2 \sin x)} \quad \text{——— (1)}$$

Given BC's $\quad \underbrace{y(0) = 1}_{(2)}, \quad \underbrace{y'(\pi) = 1}_{(3)}$

Apply (2) in (1),
$$\boxed{1 = c_1}$$

Diff (1) wrt x,
$$\frac{dy}{dx} = e^x(-c_1 \sin x + c_2 \cos x) + e^x(c_1 \cos x + c_2 \sin x) \quad \text{—— (4)}$$

Apply (3) in (4),
$$1 = e^\pi(-c_2) + e^\pi(-c_1)$$

$\Rightarrow \quad 1 = -e^{\pi}\left(c_2+1\right) \qquad (\because c_1 = 1)$

$\Rightarrow \quad -e^{-\pi} = c_2 + 1$

or $\quad c_2 = -1 - e^{-\pi}$

Sub for c_1 and c_2 in ①,

$$\boxed{y = e^{x}\left(\cos x + (-1-e^{-\pi})\sin x\right)}$$

is reqd particular soln.

Ex3: Solve

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$$

Soln: Given DE is

$$(D^3 - 3D^2 + 4)y = 0$$

AE:

$$m^3 - 3m^2 + 4 = 0.$$

By inspection we see that -1 is one of the roots.

By synthetic division method

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & 0 & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \underline{|0} \end{array}$$

Other two roots are the roots of the eqn

$$m^2 - 4m + 4 = 0$$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m = 2, 2 \quad \text{(repeated root)}$$

Thus, roots of the AE are $-1, 2, 2$

$\therefore$ Independent solns are e^{-x}, e^{2x}, xe^{2x}

The G.S is $\boxed{y = c_1 e^{-x} + c_2 e^{2x} + c_3 x e^{2x}.}$

Problems discussed in class room

Homogenous LDE

Solve the following

1) $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3y = 0$

2) $y'' - 2y' + 2y = 0, \quad y(0) = 1, \; y'(\pi) = 1$

3) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0.$

4) $y^{iv} - 16y = 0, \quad y(0) = 2, \; y'(0) = 0, \; y''(0) = 2, \; y'''(0) = 0$

5) $y''' - 6y'' + 11y' - 6y = 0, \quad y(0) = 0, \; y'(0) = 0, \; y''(0) = 2.$

6) $y^{iv} + y = 0$

7) $y^{vi} + 64y = 0$

8) $y''' + y = 0$

9) $y^{iv} + 8y'' + 16y = 0$

Non homogenous LDE with constant coefficients

In general,

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = F(x) \quad \text{\#}$$

Thm: The general soln of # is

$$y = y_c + y_p$$

y_p is called particular integral (PI) : It is the of # involving no arbitrary constants.

y_c is called complementary fn (CF) : It is the G.S of the corresponding homogenous DE of (#).

i.e $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0$.

Ex: $\frac{d^2 y}{dx^2} + y = x$

Here $y_c = c_1 \cos x + c_2 \sin x$

$y_p = x$

G.S is $y = y_c + y_p = c_1 \cos x + c_2 \sin x + x$.

We have different method to find P.I. We discuss the foll. methods

1) Inverse operator method
2) The method of variation of parameters

Non homogenous LDE with Constant coefficient, $\left(F(D)y=g(x)\right)$

General soln $\quad y = y_c + y_p$

where y_c is solution of associated homo DE, $F(D)y=0$ and is called complementary function (CF).

y_p is called particular integral, (PI)

Ex: $\quad \dfrac{d^2y}{dx^2} + y = x$

GS $\quad y = \underbrace{c_1 \sin x + c_2 \cos x}_{y_c} + \underbrace{x}_{y_p}$

Method to find particular integral (PI)

1) Inverse operator method
2) Method of variations of parameters

Inverse operator is useful when non homogenous term $g(x)$ is e^{ax}, $\sin ax$, $\cos ax$, x^n $(n \in \mathbb{Z})$ or products and sums of these fns.

Method to find y_p using inverse operator method

For the DE $\quad F(D)\, y = g(x)$

$$y_p = \frac{1}{F(D)} g(x),$$ where $\dfrac{1}{F(D)}$ is inverse operator of $F(D)$.

Case 1) when $g(x) = e^{ax}$.

$$y_p = \frac{1}{F(D)} e^{ax}$$

$$\Rightarrow \quad \boxed{y_p = \frac{1}{F(a)} e^{ax}, \quad \text{if } F(a) \neq 0}$$

If $F(a) = 0$, then

$$y_p = x \left(\frac{1}{F'(D)} e^{ax} \right)$$

$$\Rightarrow \quad \boxed{y_p = x \frac{e^{ax}}{F'(a)}, \quad \text{Provided } F'(a) \neq 0}$$

If $F'(a) = 0$, then

$$y_p = x^2 \left(\frac{1}{F''(D)} e^{ax} \right)$$

$$\Rightarrow \quad \boxed{y_p = x^2 \frac{1}{F''(a)} e^{ax}, \quad \text{Provided } F''(a) \neq 0}$$

$\vdots$

Case 2) $g(x) = \sin ax$ or $\cos ax$

$$y_p = \frac{1}{F(D)} \sin ax = \text{Im}\left(\frac{e^{iax}}{F(D)} \right)$$

$$y_p = \frac{1}{F(D)} \cos ax = \text{Re}\left(\frac{e^{iax}}{F(D)} \right)$$

Because $e^{iax} = \cos ax + i \sin ax \Rightarrow \text{Re}(e^{iax}) = \cos ax$

$\text{Im}(e^{iax}) = \sin ax$

Case 3) $g(x) = x^n \quad (n \in \mathbb{Z}^+)$

We use

i) Long division method or

ii) Power series expansion of $\frac{1}{F(D)} = (F(D))^{-1}$ in powers of D.

Note : 1)

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$$

$$(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$$

$$(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$$

$$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$$

2) $D^k(x^n) = 0$ if $k > n$

Case 4) $g(x) = e^{ax} V(x)$, where $V(x)$ is any fn of x.

$$y_p = \frac{1}{F(D)} e^{ax} V(x)$$

$$= e^{ax} \left(\frac{1}{F(D+a)} V(x) \right)$$

Ex: Solve: $y'' - y = x\sin 3x + \cos x$

Soln: Given $(D^2-1)y = x\sin 3x + \cos x$

G.S $\quad y = y_c + y_p$

<u>To find y_c</u>

Consider $(D^2-1)y = 0$

AE: $\quad m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

$$\therefore \boxed{y_c = c_1 e^{x} + c_2 e^{-x}} \quad \text{——— (1)}$$

<u>To find y_p</u>

$$y_p = \frac{1}{D^2-1}(x\sin 3x + \cos x)$$

$$= \frac{1}{D^2-1}(x\sin 3x) + \frac{1}{D^2-1}(\cos x)$$

$$= \text{Im}\left(\frac{1}{D^2-1} x e^{i3x}\right) + \text{Re}\left(\frac{1}{D^2-1} e^{ix}\right)$$

$$= \text{Im}\left(e^{i3x}\left(\frac{1}{(D+i3)^2-1} x\right)\right) + \text{Re}\left(\frac{e^{ix}}{i^2-1}\right)$$

$$= \text{Im}\left(e^{i3x}\left(\frac{x}{D^2+6iD-10}\right)\right) + \text{Re}\left(\frac{\cos x + i\sin x}{-2}\right)$$

$$= \text{Im}\left(\left(\cos 3x + i\sin 3x\right)\left(-\frac{x}{10} - \frac{6i}{100}\right)\right) - \frac{1}{2}\cos x$$

$$\boxed{y_p = -\frac{x\sin 3x}{10} - \frac{6}{100}\cos 3x - \frac{1}{2}\cos x} \quad \text{——— (2)}$$

$$\begin{array}{r|l} & \frac{x}{-10} + \frac{6i}{-100} \\ -10 + 6iD + D^2 & x \\ & x - \frac{6i}{10} \\ & (-)\ \ (+) \\ \hline & \frac{6i}{10} \\ & (-)\ \frac{6i}{10} \\ \hline & 0 \end{array}$$

$$\therefore \text{ GS } \quad y = c_1 e^{x} + c_2 e^{-x} - \frac{x\sin 3x}{10} - \frac{6}{100}\cos 3x - \frac{1}{2}\cos x$$

Ex2: $y'' - 4y' + 4y = x^2 e^{3x} + \sin^2 x$

Soln: GS $y = y_c + y_p$

To find y_c

Consider $(D^2 - 4D + 4)y = 0$

AE. $m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

roots $2, 2$

$$\therefore \boxed{y_c = c_1 e^{2x} + c_2 x e^{2x}}$$

To find y_p

$$y_p = \frac{1}{D^2 - 4D + 4}\left(x^2 e^{3x} + \sin^2 x\right)$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= \frac{1}{(D-2)^2} x^2 e^{3x} + \frac{1}{(D-2)^2}\left(\frac{1 - \cos 2x}{2}\right)$$

$$= e^{3x}\left(\frac{1}{(D+3-2)^2} x^2\right) + \frac{1}{2}\left(\frac{e^{0x}}{(D-2)^2}\right) - \frac{1}{2} \operatorname{Re}\left(\frac{1}{(D-2)^2} e^{i2x}\right)$$

$$= e^{3x}\left(\frac{x^2}{D^2 + 2D + 1}\right) + \frac{1}{8} - \frac{1}{2}\operatorname{Re}\left(\frac{e^{i2x}}{(i2-2)^2}\right)$$

$$= e^{3x}(x^2 - 4x + 6) + \frac{1}{8} - \frac{1}{2}\operatorname{Re}\left(\frac{\cos 2x + i\sin 2x}{4(\cancel{i^2} \cancel{+1} - 2i)}\right)$$

$$\begin{array}{r} x^2 - 4x + 6 \\ 1 + 2D + D^2 \overline{)\; x^2 \qquad\qquad} \\ (-)\; x^2 + 4x + 2 \\ \hline -4x - 2 \\ (+)\; -4x - 8 \\ \hline 6 \\ (-)\; 6 \\ \hline 0 \end{array}$$

$$= e^{3x}(x^2 - 4x + 6) + \frac{1}{8} - \frac{1}{16}\operatorname{Re}\left(i(\cos 2x + i\sin 2x)\right)$$

$$\boxed{y_p = e^{3x}(x^2 - 4x + 6) + \frac{1}{8} + \frac{1}{16}\sin 2x}$$

Non homogenous LDE

Solve the following

1) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1-e^x)^2$

2) $y'' + 4y' + 4y = 3\sin x + 4\cos x$, $y(0)=1$ and $y'(0)=0$

3) $y''' + 3y'' + 3y' + y = e^{-x}$

4) $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = 2\sin x\cos x$

5) $y'' + 9y = x\cos x$

6) $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

7) $y'' - 2y' + 2y = x + e^x\cos x$

8) $y'' - 3y' + 2y = xe^{3x} + \sin 2x$

9) $y'' - 4y = x\sin(hx)$

10) $y'' - y = x\sin 3x + \cos x$

11) $y'' - 2y' + y = xe^x\sin 2x$

12) $y'' - 4y = \cosh(2x-1) + 3^x$

13) $y'' - 4y' + 4y = x^2e^{3x} + \sin^2 x$

Exercise

Solve the following differential eqns:

1) $\frac{d^2y}{dx^2} + 8y = 0, \quad y(0)=1, \quad y'(0) = 2\sqrt{2}$

Ans: $y = \cos(2\sqrt{2}x) + \sin(2\sqrt{2}x)$.

2) $y'' - 7y' + 12y = 0$

Ans: $y = c_1 e^{3x} + c_2 e^{4x}$

3) $y''' - 6y'' + 11y' - 6y = 0$

Ans: $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

4) $\frac{d^4y}{dx^4} - 16y = 0$

Ans: $y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$

5) $y'' + 6y' + 9y = 0, \quad y(0)=2, \quad y'(0) = 3$

Ans: $y = (2+9x)e^{-3x}$

6) $y'' + 4y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = 1$

Ans: $y = \frac{e^{-2x}\sin 3x}{3}$

7) $y'' + 9y = x \sin x$

Ans: $y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{8} x \sin x - \frac{1}{32} \cos x$

8) $y'' + 3y' + 2y = x^2$

Ans: $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{4}(2x^2 - 6x + 7)$

9) $y'' + 4y = x^2 + \sin 2x$

Ans: $c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}(2x^2 - 1) - \frac{x}{4} \cos 2x$

10) $y'' - 10y' + 16y = e^{4x} \sin 2x$

Ans: $c_1 e^{8x} + c_2 e^{2x} + \frac{e^{4x}}{40}(\cos 2x - 3\sin 2x)$

8) $y'' + 3y' + 2y = x^2$

Ans: $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{4}(2x^2 - 6x + 7)$

9) $y'' + 4y = x^2 + \sin 2x$

Ans: $c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}(2x^2 - 1) - \frac{x}{4} \cos 2x$

10) $y'' - 10y' + 16y = e^{4x} \sin 2x$

Ans: $c_1 e^{8x} + c_2 e^{2x} + \frac{e^{4x}}{40}(\cos 2x - 3 \sin 2x)$

11) $(D^2 + 1)y = xe^{2x}$

Ans: $y = c_1 \cos x + c_2 \sin x + \frac{1}{25}(5x - 4)e^{2x}$

12) $y'' + 2y' + y = x \cos x$

Ans: $y = (c_1 + c_2 x)e^{-x} + \frac{1}{2}((x-1)\sin x + \cos x)$.

13 $y'' - y = e^x (1 + x^2)$

Ans: $y = c_1 e^x + c_2 e^{-x} + \frac{xe^x}{3}(9 - 3x + x^2)$

The method of variation of parameters

It is useful to find particular integral (P I) for an higher order linear differential equations.

(provided complementary fn the DE is known).

Consider a second order DE:

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\, y = F(x) \quad \text{———} \quad (*)$$

Let $y_1(x)$ and $y_2(x)$ be independent solns of the corresponding homo. DE,

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\, y = 0.$$

The C.F of $(*)$ is

$$y_c = c_1\, y_1(x) + c_2\, y_2(x) \quad \text{———} \quad (1)$$

We replace c_1 and c_2 in (1) by respective fns $V_1(x)$ and $V_2(x)$ (which will be determined later), we obtain

$$V_1(x)\, y_1(x) + V_2(x)\, y_2(x)$$

This will be P.I of the eqn $(*)$. (hence the name variation of parameters).

i.e

$$y_p = V_1(x)\, y_1(x) + V_2(x)\, y_2(x). \quad \text{———} \quad (2)$$

$$y_p' = V_1(x)\, y_1'(x) + V_1'(x)\, y_1(x) + V_2(x)\, y_2'(x) + V_2'(x)\, y_2(x)$$

$$\Rightarrow y_p' = V_1(x)\, y_1'(x) + V_2(x)\, y_2'(x) + V_1'(x)\, y_1(x) + V_2'(x)\, y_2(x)$$

Let us now impose a condition; we simplify y_p' by demanding that

$$V_1'(x)\, y_1(x) + V_2'(x)\, y_2(x) = 0. \quad \text{———} \quad (3)$$

With this condition,

$$y_p'(x) = v_1(x)\, y_1'(x) + v_2(x) y_2'(x) \quad \text{———} \quad (4)$$

diff wrt x,

$$y_p''(x) = v_1(x)\, y_1''(x) + v_2(x)\, y_2''(x) + v_1'(x)\, y_1'(x) + v_2'(x)\, y_2'(x) \quad \text{———} \quad (5)$$

Sub (5), (4) and (2) for y'', y' and y in (*),

$$a_0(x)\, y_p''(x) + a_1(x)\, y_p'(x) + a_2(x)\, y_p(x) = F(x)$$

$$\Rightarrow a_0(x)\left[v_1(x)\, y_1''(x) + v_2(x)\, y_2''(x) + v_1'(x)\, y_1'(x) + v_2'(x)\, y_2'(x)\right] + a_1(x)\left[v_1(x)\, y_1'(x) + v_2(x)\, y_2'(x)\right] + a_2(x)\left[v_1(x)\, y_1(x) + v_2(x)\, y_2(x)\right] = F(x).$$

$$\Rightarrow v_1(x)\left[a_0(x)\, y_1''(x) + a_1(x)\, y_1'(x) + a_2(x)\, y_1(x)\right] + v_2(x)\left[a_0(x)\, y_2''(x) + a_1(x)\, y_2'(x) + a_2(x)\, y_2(x)\right] + a_0(x)\left(v_1'(x) y_1'(x) + v_2'(x)\, y_2'(x)\right) = F(x) \quad \text{———} \quad (6)$$

Since y_1 and y_2 are indep. solns of the corres. homo. DE,

$$a_0(x)\, y_1''(x) + a_1(x)\, y_1'(x) + a_2(x)\, y_1(x) = 0$$
$$a_0(x)\, y_2''(x) + a_1(x)\, y_2'(x) + a_2(x)\, y_2(x) = 0$$

Thus, (6) becomes

$$v_1'(x)\, y_1'(x) + v_2'(x)\, y_2'(x) = \frac{F(x)}{a_0(x)} \quad \text{———} \quad (7)$$

From (3) and (7),

$$\left.\begin{aligned} v_1'(x)\, y_1(x) + v_2'(x)\, y_2(x) &= 0 \\ v_1'(x)\, y_1'(x) + v_2'(x)\, y_2'(x) &= \frac{F(x)}{a_0(x)} \end{aligned}\right\} \quad (\#)$$

We solve # for $v_1'(x)$ and $v_2'(x)$,

$$\cancel{v_1'(x)\, y_1(x)\, y_1'(x)} + v_2'(x)\, y_1'(x)\, y_2(x) = 0$$

$$\cancel{v_1'(x)\, y_1(x)\, y_1'(x)} + v_2'(x)\, y_1(x)\, y_2'(x) = \frac{F(x)}{a_0(x)}\, y_1(x)$$

$(-) \qquad\qquad (-)$

$$v_2'(x)\left(y_1'(x)\, y_2(x) - y_1(x)\, y_2'(x)\right) = \frac{F(x)\, y_1(x)}{a_0(x)}$$

or
$$v_2'(x) = \frac{F(x)\, y_1(x)}{a_0(x)\, W(y_1, y_2)}$$

and
$$v_1'(x) = -\frac{F(x)\, y_2(x)}{a_0(x)\, W(y_1, y_2)}$$

Wronskian of y_1 and y_2

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Hence
$$v_1(x) = \int \frac{-F(x)\, y_2(x)}{a_0(x)\, W(y_1, y_2)}\, dx$$

$$v_2(x) = \int \frac{F(x)\, y_1(x)}{a_0(x)\, W(y_1, y_2)}\, dx$$

Sub in 2, we get the reqd P.I.

Summary

Consider 2^{nd} order LDE

$$a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)\, y = F(x)$$

If CF : $y_c = c_1\, y_1(x) + c_2\, y_2(x)$, then

the PI : $y_p = v_1(x)\, y_1(x) + v_2(x)\, y_2(x)$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

where
$$v_1(x) = \int \frac{-F(x)\, y_2(x)}{a_0(x)\, W(y_1, y_2)}\, dx \; ; \quad v_2(x) = \int \frac{F(x)\, y_1(x)}{a_0(x)\, W(y_1, y_2)}\, dx .$$

Ex: Solve the foll. DEs by the method of variation of parameters

i) $\dfrac{d^2y}{dx^2} + y = \tan x$ ——— (1)

Soln: The G.S is

$$y = y_c + y_p$$

To find y_c:

Consider $\dfrac{d^2y}{dx^2} + y = 0$

$(D^2+1)y = 0$

AE: $m^2+1=0$

AE : $m^2+1 = 0$

Roots are $m = \pm\sqrt{-1}$

$= \pm i$

indep solns are $\cos x$, $\sin x$.

Thus, $\boxed{y_c = C_1 \cos x + C_2 \sin x}$ ——— (2)

To find y_p:

Assume $y_p = v_1(x)\cos x + v_2(x)\sin x$ ——— (3)

here $y_1(x) = \cos x$, $y_2(x) = \sin x$, $a_0(x) = 1$

non homo. term $F(x) = \tan x$,

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\therefore v_1(x) = \int \frac{-F(x)\, y_2(x)}{a_0(x)\, W(y_1, y_2)}$$

$$= \int -\frac{\tan x \cdot \sin x}{1}\, dx$$

$$= \int -\frac{\sin^2 x}{\cos x}\, dx$$

$$= \int -\frac{(1-\cos^2 x)}{\cos x}\, dx$$

$$= \int (-\sec x + \cos x)\, dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$V_2(x) = \int \frac{F(x)\, y_1(x)}{a_0(x)\, W(y_1, y_2)}\, dx$$

$$= \int \frac{\tan x \cos x}{1}\, dx$$

$$= \int \sin x\, dx = -\cos x$$

Sub for $V_1(x)$ and $V_2(x)$ in (3),

$$y_p = \left(-\log(\sec x + \tan x) + \sin x\right)\cos x - \cos x \sin x$$

$$\Rightarrow \boxed{y_p = -\log(\sec x + \tan x)\cos x} \quad \text{——— (4)}$$

The G.S is

$$y = y_c + y_p \quad (\text{sub.})$$

ii) $y'' - 4y' + 4y = (x+1)e^{2x}$ ——— (1)

Soln The G.S is

$$y = y_c + y_p.$$

To find y_c:

Consider $y'' - 4y' + 4y = 0$

AE: $m^2 - 4m + 4 = 0$

Roots are $m = 2, 2$ (repeated)

Indep. solns are e^{2x}, xe^{2x}

$\therefore$ $\boxed{y_c = c_1 e^{2x} + c_2 x e^{2x}}$ ——— 2

To find y_p:

$$y_p = v_1(x) e^{2x} + v_2(x) x e^{2x} \quad \text{——— 3}$$

Here

$y_1(x) = e^{2x}$
$y_2(x) = xe^{2x}$

$a_0(x) = 1$

$F(x) = (x+1) e^{2x}$

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + x\,2e^{2x} \end{vmatrix}$$

$$= e^{4x} + \cancel{x\,2e^{4x}} - \cancel{2x\,e^{4x}}$$

$$= e^{4x}$$

$$v_1(x) = \int -\frac{F(x)\, y_2(x)}{a_0\, W(y_1, y_2)}\, dx$$

$$= \int -\frac{(x+1)\cancel{e^{2x}}\, x\,\cancel{e^{2x}}}{\cancel{e^{4x}}}\, dx$$

$$= \int (-x^2 - x)\, dx$$

$$= -\frac{x^3}{3} - \frac{x^2}{2}$$

$$v_2(x) = \int \frac{F(x)\, y_1(x)}{W(y_1, y_2)\, a_0}\, dx$$

$$= \int \frac{(x+1)\cancel{e^{2x}}\, \cancel{e^{2x}}}{\cancel{e^{4x}}}\, dx$$

$$= \int (x+1)\, dx$$

$$= \frac{x^2}{2} + x$$

Sub for v_1 and v_2 in 3,

$$y_p = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right) e^{2x} + \left(\frac{x^2}{2} + x\right) x e^{2x}.$$

$$\Rightarrow \boxed{y_p = \left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right)e^{2x}}$$

The G.S

$$y = y_c + y_p \qquad (\text{sub.})$$

iii) $4y'' + 36y = \csc 3x$ ——— (1) | $\csc 3x := (\operatorname{cosec} 3x)$

Soln: The G.S is

$$y = y_c + y_p$$

To find y_c:

Consider $4y'' + 36y = 0$

$$\Rightarrow y'' + 9y = 0$$

AE: $m^2 + 9 = 0$

Roots are $m = \pm i3$

Indep. solns are $\cos 3x$, $\sin 3x$

$$\therefore \boxed{y_c = c_1 \cos 3x + c_2 \sin 3x} \text{ ——— (2)}$$

To find y_p:

$$y_p = v_1(x)\cos 3x + v_2(x)\sin 3x \text{ — (3)}$$

Here $y_1(x) = \cos 3x$

$y_2(x) = \sin 3x$

$a_0(x) = 4$

$F(x) = \csc 3x$

$$W(y_1, y_2) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x$$

$$= 3$$

$$v_1 = -\int \frac{F(x)\; y_2(x)}{W(y_1, y_2)\cdot a_0}\, dx$$

$$= -\int \frac{\csc 3x \cdot \sin 3x}{3 \cdot 4}\, dx$$

$$= -\int \frac{1}{12}\,dx \qquad = -x/12$$

$$V_2 = \int \frac{F(x)\; y_1(x)}{W(y_1, y_2)\, a_0(x)}\,dx$$

$$= \int \frac{\csc 3x \;\cos 3x}{4 \cdot 3}\,dx$$

$$= \frac{1}{12}\int \frac{\cos 3x}{\sin 3x}\,dx$$

$$= \frac{1}{12}\;\frac{\log(\sin 3x)}{3}.$$

Sub for V_1 and V_2 in (3),

$$\boxed{y_p = -\frac{x}{12}\cos 3x + \frac{1}{36}\log(\sin 3x)\,\sin 3x}.$$

$\therefore$ The G.S

$$y = y_c + y_p \qquad (\text{sub.})$$

Exercise:

Solve the following differential eqns by the method of variation of parameters.

1) $y'' - 2y' + y = x e^x \sin x$

Ans: $y = (c_1 + c_2 x)e^x - e^x(x \sin x + 2\cos x)$.

2) $y'' - 2y' = e^x \sin x$

Ans. $y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x$.

3) $y'' - 2y' + y = x e^x \log x$

Ans: $y = c_1 e^x + c_2 x e^x + \frac{1}{6} x^3 e^x \log x - \frac{5}{36} x^3 e^x$.

4) $y'' - 2y' + 2y = e^x \tan x$

Ans: $y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$.

5) $y'' + 4y = \cot 2x$

Ans : $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2} \log(\tan x)$.

6) $y'' - y = \cosh x$

Ans: $y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2} x \sinh x$.

Ex iv) $y'' + 16y = 32 \sec 2x$ ——— (1)

Soln: The G.S

$$y = y_c + y_p$$

To find y_c:

Consider $y'' + 16y = 0$

AE: $m^2 + 16 = 0$

Roots are $m = \pm\sqrt{-16}$

$\Rightarrow m = \pm 4i$

$$\therefore \boxed{y_c = c_1 \cos 4x + c_2 \sin 4x} \quad \text{——— (2)}$$

To find y_p:

$$y_p = v_1(x) \cos 4x + v_2(x) \sin 4x$$

Here

$F(x) = 32 \sec 2x$

$y_1 = \cos 4x$

$y_2 = \sin 4x$

$a_0(x) = 1$

$$W(y_1, y_2) = \begin{vmatrix} \cos 4x & \sin 4x \\ -4\sin 4x & 4\cos 4x \end{vmatrix}$$

$$= 4\cos^2 4x + 4\sin^2 4x$$

$$= 4$$

$$v_1(x) = \int \frac{-F(x)\, y_2}{a_0(x)\, W(y_1, y_2)}\, dx$$

$$= \int \frac{-32\sec 2x \cdot \sin 4x}{4}\, dx$$

$$= -8 \int \frac{1}{\cancel{\cos 2x}}\, 2\sin 2x\, \cancel{\cos 2x}\, dx$$

$$= -16 \left(-\frac{\cos 2x}{2}\right)$$

$$= 8\cos 2x$$

$$V_2(x) = \int \frac{F(x)\; y_1}{a_0(x)\, W(y_1, y_2)}\, dx$$

$$= \int \frac{32\sec 2x \quad \cos 4x}{4}\, dx$$

$$= 8 \int \frac{\cos 4x}{\cos 2x}\, dx \qquad\qquad \cos 2\theta = 2\cos^2\theta - 1$$

$$= 8 \int \frac{(2\cos^2 2x - 1)}{\cos 2x}\, dx$$

$$= 8 \int (2\cos 2x - \sec 2x)\, dx$$

$$= 8 \left[\sin 2x - \frac{1}{2}\log(\sec 2x + \tan 2x)\right]$$

$$\therefore\; y_p = 8\cos 2x \cos 4x + 8\left[\sin 2x - \tfrac{1}{2}\log(\sec 2x + \tan 2x)\right] \sin 4x$$

$$= 8(\cos 2x \cos 4x + \sin 2x \sin 4x) - 4\sin 4x \log(\sec 2x + \tan 2x)$$

$$= 8\cos 2x - 4\sin 4x \log(\sec 2x + \tan 2x).$$

The G.S

$$y = y_c + y_p \qquad (\text{sub})$$

2 The Cauchy - Euler Equation

In general, Cauchy - Euler equation is of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = F(x). \tag{2.1}$$

where a, b and $a_i's$ are constants.

Note that each term in the left member is a constant multiple of an expression of the form

$$(ax+b)^k \frac{d^k y}{dx^k}.$$

In order to solve the equation (2.1), we use suitable transformation to reduce the equation (2.1) to a linear differentail equation with constant coefficients.

Theorem 2.1. *The transformation $(ax+b) = e^t$ reduces the equation*

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = F(x).$$

to a linear differential equation with constant coefficients.

Throughtout, we use the following notation

$$D := \frac{d}{dx} \text{ and } D_1 := \frac{d}{dt}$$

We shall prove this theorem for the following 2nd order Cauchy - Euler differential equation

$$(ax+b)^2 \frac{d^2 y}{dx^2} + a_1(ax+b)\frac{dy}{dx} + a_2 y = F(x). \tag{2.2}$$

Letting

$$(ax+b) = e^t, \tag{2.3}$$

assuming $(ax+b) > 0$, we have

$$t = \log(ax+b). \tag{2.4}$$

Then, by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{a}{ax+b}\frac{dy}{dt} \tag{2.5}$$

which yields

$$(ax+b)\frac{dy}{dx} = a\frac{dy}{dt}$$

In the above notation

$$(ax+b)Dy = aD_1 y \tag{2.6}$$

Differentiating (2.5) w.r.t x,

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{a}{(ax+b)}\frac{dy}{dt}\right) \\
&= \frac{a}{(ax+b)}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dx}\left(\frac{a}{(ax+b)}\right) \\
&= \frac{a}{(ax+b)}\left(\frac{d^2y}{dt^2}\frac{dt}{dx}\right) + \frac{dy}{dt}\left(\frac{-a^2}{(ax+b)^2}\right) \\
&= \frac{a}{(ax+b)}\left(\frac{d^2y}{dt^2}\frac{a}{(ax+b)}\right) + \frac{dy}{dt}\left(\frac{-a^2}{(ax+b)^2}\right)
\end{aligned}$$

which yields

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) = a^2\frac{d}{dt}\left(\frac{d}{dt} - 1\right)y.$$

In the above notation

$$(ax+b)D^2y = a^2D_1(D_1-1)y \tag{2.7}$$

Substituting (2.3), (2.6) and (2.7) in (2.2), we get

$$a^2D_1(D_1-1)y + a_1aD_1y + a_2y = F((e^t-b)/a).$$

Now this is the differential equation with constant coefficients and independent variable t. One can use adove discussed method to find the general solution.

2.1 Summary of the method to solve Cauchy - Euler equation

Consider the equation

$$(ax+b)^n\frac{d^ny}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(ax+b)\frac{dy}{dx} + a_ny = F(x). \tag{2.8}$$

Use the substitution

$$(ax+b) = e^t, \tag{2.9}$$

with this substitution, we see that

$$\begin{aligned}
(ax+b)Dy &= aD_1y, && (2.10)\\
(ax+b)^2D^2y &= a^2D_1(D_1-1)y, && (2.11)\\
(ax+b)^3D^3y &= a^3D_1(D_1-1)(D_1-2)y. && (2.12)
\end{aligned}$$

In general

$$(ax+b)^nD^ny = a^nD_1(D_1-1)(D_1-2)(D_1-3)\ldots(D_1-(n-1))y \tag{2.13}$$

where

$$D := \frac{d}{dx} \text{ and } D_1 := \frac{d}{dt}.$$

By employing (2.9)-(2.13) in (2.8), the equation (2.8) reduces to a differential equation with constant coefficients. Then we can use either undetermined coefficient method or the method of variation of parameters to solve the resulting equation.

2.2 Examples:

Example 1 Solve by the method of undetermined coefficients:

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3. \tag{2.14}$$

Let

$$x = e^t. \tag{2.15}$$

Then assuming $x > 0$, we have

$$t = \log x, \tag{2.16}$$

and

$$xDy = D_1y, \tag{2.17}$$

$$x^2D^2y = D_1(D_1 - 1)y, \tag{2.18}$$

where

$$D := \frac{d}{dx} \quad \text{and} \quad D_1 := \frac{d}{dt}.$$

By substituting (2.15), (2.17) and (2.18) in (2.14), we obtain

$$D_1(D_1 - 1)y - 2D_1y + 2y = e^{3t}.$$

that is

$$D_1^2y - 3D_1y + 2y = e^{3t}. \tag{2.19}$$

General solution of (2.19),

$$y = y_c + y_p. \tag{2.20}$$

To find y_c, consider the AE of corresponding homogeneous DE of (2.19),

$$m^2 - 3m + 2 = 0.$$

Solving we get the roots $m_1 = 1$ and $m_2 = 2$.
Linearly independent solution are $y_1(x) = e^t$ and $y_2(x) = e^{2t}$. Thus

$$y_c = c_1e^t + c_2e^{2t}. \tag{2.21}$$

To find y_p, let us use inverse operator method.

From 2.19,

$$y_p = \frac{e^{3t}}{D_1^2 - 3D_1 + 2}$$

$$= \frac{e^{3t}}{3^2 - 3(3) + 2} = \frac{e^{3t}}{2}$$

$\therefore$ General of 2.19,

$$y = c_1e^t + c_2e^{2t} + \frac{e^{3t}}{2}$$

Now sub. $e^t = x$ to find G.S of 2.14,

$$y = c_1x + c_2x^2 + \frac{x^3}{2}$$

2.3 Exercises

Find the general solution of each of the following differential equations. In each case assume $x > 0$.

1. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 3y = 0.$ $[G.S. : y = c_1x + c_2x^3]$

2. $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = 4\log x.$ $[G.S. : y = c_1x^{-1} + c_2x^{-2} + 2\log x]$

3. $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 2\sin(\log x).$ $[G.S. : (c_1 + 1)\sin(\log x) + (c_2 + \log x)\cos(\log x)]$

Solve the initial - value problem

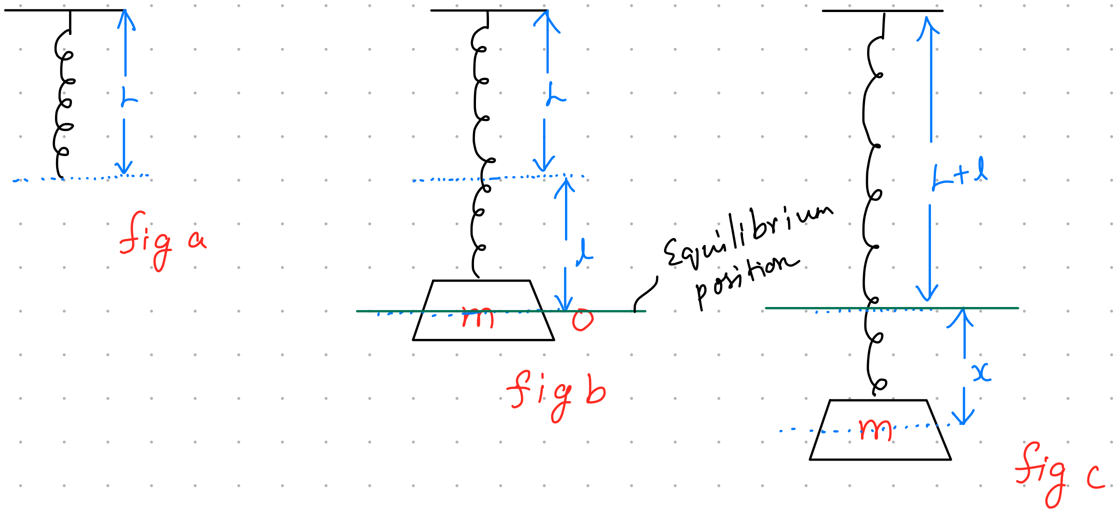
4. $x^2 \dfrac{d^2y}{dx^2} - 2y = 4x - 8,$ $y(1) = 4,$ $y'(1) = -1.$ $[P.S. : y = x^{-1} + x^2 - 2x + 4]$

Applications of 2^{nd} order LDE with constant coefficients

Mass and spring system

The differential equation of the vibrations of a mass on a spring.

The Basic problem:



- fig a shows natural length L of a spring, i.e, unstretch length of a spring.
- Now we attach a mass to the lower end of the spring and allow it to come to rest in an equilibrium position as given in fig b.
- The system is then set in motion. Let $x(t)$ denote the displacement of the mass from O along the line as shown in fig c.
- x is considered to be +ve, zero or −ve according to whether the mass is below, at or above its equilibrium position.

Problem is to determine the resulting motion of the mass on the spring.

We set up differential eqn to the problem, in order to do so we consider two laws of physics

1) Hooke's law
2) Newton's second law of motion

Hooke's law states that the magnitude of the force needed to produce certain elongation is directly proportional to the amount of its elongation

i.e $\boxed{|F| \propto l \Rightarrow |F| = kl,}$

where F is the force, l is the amount of elongation and k is a constant of proportionality called spring constant.

Newton's 2nd law states that net forces acting on an object is equal to the mass times acceleration of the object.

i.e $$\boxed{\sum F = m\frac{d^2x}{dt^2}.}$$

Here, we assume that forces tending to pull the mass downward are +ve, while those tending to pull it upward are negative.

In fig b The mass is in equibrium, Thus net forces on the mass is equal to zero.

Forces acting are i) Force of gravity (mg, g is acceleration due to gravity)

ii) Restoring force of the spring, equal to kl from Hooke's law

i.e $mg - kl = 0$ (mg acts downward (+ve) and kl acts upward (−ve))

$\Rightarrow \boxed{mg = kl}$ —— 1

In fig c, the mass is in motion, x is the displacement of the mass at time $t>0$

1) F_1, the force of gravity

$\boxed{F_1 = mg}$ 2 (it acts downward (+ve))

2) F_2, The restoring force of the spring

$F_2 = -k(x+l)$

$\Rightarrow \boxed{F_2 = -kx - mg}$ 3 ($\because kl = mg$ from 1)

3) F_3, the resisting force of the medium, called the damping force

$$\boxed{F_3 = -a\frac{dx}{dt}} \quad (4)$$

(−ve because it is against the motion of mass)

Here $a>0$, is called damping constant.

4) F_4, any external forces that act upon the mass.

Let $f(t)$ be the resultant of all external forces.

$$\boxed{F_4 = f(t)} \quad (5)$$

From Newton's law

$$F_1 + F_2 + F_3 + F_4 = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \cancel{mg} - kx - \cancel{mg} - a\frac{dx}{dt} + f(t) = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \boxed{m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = f(t)} \quad (6)$$

This is a second order non-homogenous LDE with constant coefficients.

This we take as the DE for the motion of the mass on the spring.

Note: If $a=0$ the motion is called undamped. Otherwise it is called damped.

If $f(t)=0 \; \forall t$ the motion is called free, otherwise it is called forced.

Free, undamped motion.

Eqn ⑥, reduces to

$$m\frac{d^2x}{dt^2} + Kx = 0. \quad \text{——} \quad (7)$$

where $m\ (>0)$ is the mass, $K\ (>0)$ is the spring constant.

Let $\frac{K}{m} = \lambda^2$, we write ⑦ in the form

$$\boxed{\frac{d^2x}{dt^2} + \lambda^2 x = 0} \quad \text{——} \quad (8)$$

Solve ⑧:

$$AE: \quad m^2 + \lambda^2 = 0$$
$$\Rightarrow \quad m = \pm\sqrt{-\lambda^2}$$

roots are $m = \pm \lambda i$

Independent soln of ⑧, $\cos\lambda t$ and $\sin\lambda t$.

Hence, the G.S is $\boxed{x(t) = C_1 \cos\lambda t + C_2 \sin\lambda t}$ —— ⑨

Let us assume that the mass was initially displaced a distance x_0 from equilibrium position and released from that pt with initial velocity V_0.

Then $\boxed{x(0) = x_0, \quad \frac{dx}{dt}(0) = V_0}$ are the IC's in addition to the DE.

Diff ⑨ wrt x,

$$\frac{dx}{dt} = -\lambda C_1 \sin\lambda t + \lambda C_2 \cos\lambda t \quad \text{——} \quad (10)$$

At $t=0$

⑨ $\Rightarrow$ $\boxed{x_0 = C_1}$

⑩ $\Rightarrow$ $V_0 = \lambda C_2 \Rightarrow \boxed{C_2 = \frac{V_0}{\lambda}}$

Sub for C_1 and C_2 in (9),

$$\boxed{x(t) = x_0 \cos\lambda t + \frac{V_0}{\lambda} \sin\lambda t} \quad \text{——— (10)}$$

put $x_0 = r\cos\phi$, $\frac{V_0}{\lambda} = -r\sin\phi$

$\Rightarrow r^2 = x_0^2 + \frac{V_0^2}{\lambda}$, ϕ is called phase angle or phase constant.

(10) $\Rightarrow$

$$x(t) = r\left(\cos\phi \cos\lambda t - \sin\phi \sin\lambda t\right)$$

or

$$x(t) = r\cos(\lambda t + \phi), \quad r = \sqrt{x_0^2 + \frac{V_0^2}{\lambda}}$$

It is a SHM.

Amplitude of the motion is (max. displacement)

$$\boxed{r = \sqrt{x_0^2 + \frac{V_0^2}{\lambda}}}$$

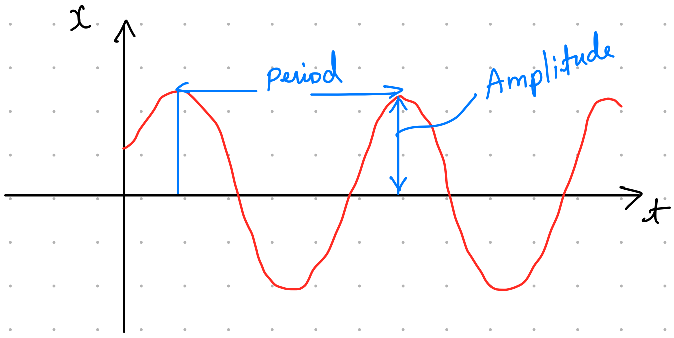
Frequency is (no. of oscillations per sec)

$$\boxed{f = \frac{\lambda}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}} \quad \left(\because \lambda^2 = \frac{k}{m}\right)$$

Period is (time interval b/w two successive maxima)

$$\boxed{T = \frac{2\pi}{\lambda} = 2\pi\sqrt{\frac{m}{K}}}$$

The graph:



Ex 1: An 8-lb weight is placed upon the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching the spring 6 in. The weight is then pulled down 3 in below its equilibrium position and released at $t=0$ with an initial velocity of 1 ft/sec, directed downward. Neglecting the resistance of the medium and assuming that no external forces are present, determine the displacement of the weight and hence determine amplitude, period and frequency of the resulting motion.

Soln: Weight $W = 8$ lb

8 lb weight streches the spring by 6 in $= \frac{1}{2}$ ft $(= l)$

If $x(t)$ is the displacement of the mass at time t,

$x(0) = 3$ in $= \frac{1}{4}$ ft,

$\frac{dx}{dt}(0) = 1$ ft/sec, (+ve because directed downward)

accelaration due to gravity, $g = 32$ ft/sec^2

$W = mg \Rightarrow m = \frac{W}{g} = \frac{8}{32} = \frac{1}{4}$

Given: $a = 0$, $f(t) = 0$ (external forces)

Spring constant $k = \frac{mg}{l} = \frac{8}{1/2} = 16$ lb/ft ($mg = kl$)

This is free and undamped motion, thus the DE is

$$m\frac{d^2x}{dt^2} + kx = 0$$

or $$\frac{1}{4}\frac{d^2x}{dt^2} + 16x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + 64x = 0 \qquad (\lambda^2 = 64)$$

AE : $m^2 + 64 = 0$

Roots are $m = \pm 8i$

Independent solns are $\cos 8t$, $\sin 8t$

G.S is $\boxed{x = c_1 \cos 8t + c_2 \sin 8t}$ —— (1)

IC's are $x(0) = \frac{1}{4}$ ft, $\frac{dx}{dt}(0) = 1$ ft/sec

Diff (1) wrt t,

$$\frac{dx}{dt} = -c_1 8 \sin 8t + 8 c_2 \cos 8t \quad \text{—— (2)}$$

At $t=0$,

(1) $\Rightarrow$ $\boxed{\frac{1}{4} = c_1}$

(2) $\Rightarrow$ $1 = 8c_2$ $\Rightarrow$ $\boxed{c_2 = \frac{1}{8}}$

Sub for c_1 and c_2 in (1),

$$\boxed{x(t) = \frac{1}{4} \cos 8t + \frac{1}{8} \sin 8t}$$

Amplitude is $\sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2} = \frac{\sqrt{5}}{8}$ ft.

frequency is $\frac{8}{2\pi} = \frac{4}{\pi}$ osc / sec.

Time period is $\frac{\pi}{4}$ sec.

Free, damped motion

Let $a>0$ be the damping constanst and $f(t)=0\ \forall t$. The basic DE reduces to

$$m\frac{d^2x}{dt^2}+a\frac{dx}{dt}+Kx=0$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2}+2b\frac{dx}{dt}+\lambda^2 x=0} \quad \text{--- (13)}$$

where $2b=\frac{a}{m}$, (for convenience)

$\lambda^2=\frac{k}{m}$

Note that b is +ve since a is positive.

AE of (1), $\gamma^2+2b\gamma+\lambda^2=0$ (γ is used $\because$ m is already used to denote mass)

$$\Rightarrow \gamma=\frac{-2b\pm\sqrt{4b^2-4\lambda^2}}{2}$$

$$\Rightarrow \gamma=-b\pm\sqrt{b^2-\lambda^2}$$

Depending upon the nature of these roots (in term depends on the sign of $b^2-\lambda^2$). Three distinct cases occur.

Case i : We consider $b^2-\lambda^2<0$ or $b<\lambda$ (damped oscillatory)

Then the roots of the AE are complex conjugates

$$-b\pm i\sqrt{\lambda^2-b^2}$$

indep. soln are $e^{-bt}\cos\left(\sqrt{\lambda^2-b^2}\,t\right)$, $e^{-bt}\sin\left(\sqrt{\lambda^2-b^2}\,t\right)$

The G.S is

$$x(t)=e^{-bt}\left(C_1\cos\left(\sqrt{\lambda^2-b^2}\,t\right)+C_2\sin\left(\sqrt{\lambda^2-b^2}\,t\right)\right)$$

put $C_1=R\cos\phi$, $C_2=-R\sin\phi$, we get

$$\boxed{x(t) = Re^{-bt} \cos\left(\sqrt{\lambda^2 - b^2}\, t + \phi\right)}$$

where $R = \sqrt{c_1^2 + c_2^2}$

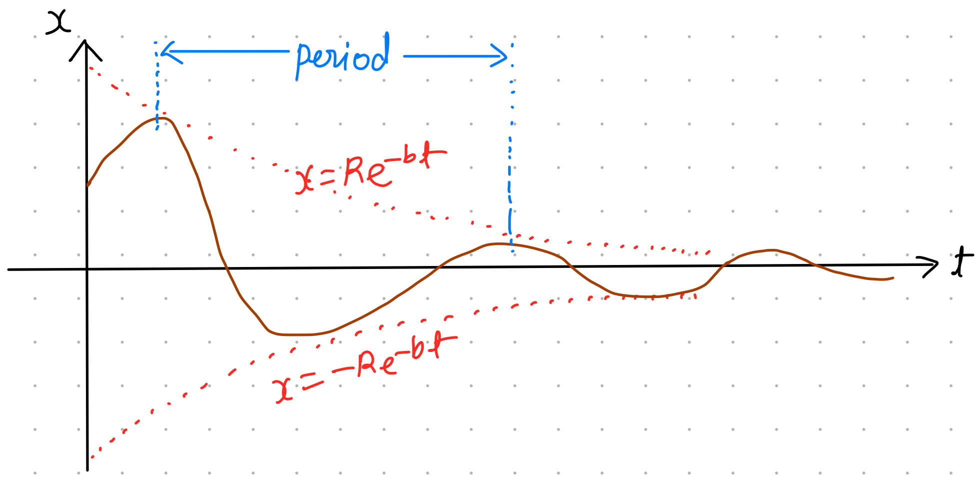
Re^{-bt} is called the damping factor. It tends to zero monotonically as $t \to \infty$

$\cos\left(\sqrt{\lambda^2 - b^2}\, t + \phi\right)$ is of a periodic oscillator character, it represents simple harmonic motion.

Thus RHS of (B) represents an oscillatory motion in which the oscillation becomes successively smaller and smaller.

The oscillation are said to be damped out and the motion is called damped oscillatory motion.

The graph



It is not periodic, but the time interval b/w two successive maximum displacement is called period.

In terms of m, a, k, G.S is

$$x(t) = R\, e^{-(a/2m\, t)} \cos\left(\sqrt{\frac{k}{m} - \frac{a^2}{4m^2}}\; t + \phi\right)$$

frequency of the osc. is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{a^2}{4m^2}}.$$

Period is $T = \frac{1}{f}$.

This motion is called damped, oscillatory motion.

If damping were not present, i.e $a = 0$ and the natural frequency would be

$$\frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Thus, frequency of oscillation is damped oscillatory motion is less than the frequency of the corresponding undamped system.

Case ii: We consider $b^2 - \lambda^2 = 0 \Rightarrow \lambda = b$ (Critical damping)

Roots of the AE is $-b$ (repeated), and the G.S is

$$x = (c_1 + c_2 t)\, e^{-bt} \qquad \text{—— } (14)$$

The motion is no longer oscillatory. Any slight decrease in the amount of damping, will change the situation back to that of Case i.

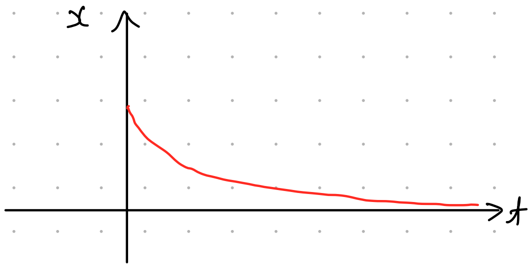
Case ii is border line case, Thus the motion is said to be critically damped.

From (14) we see that

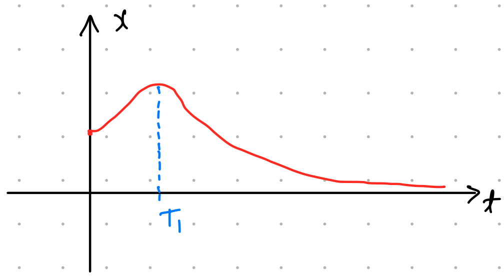
$$x \to 0 \quad \text{as } t \to \infty$$

i.e mass tends to its equilibrium position as $t \to \infty$. Depending on the initial conditions, we have the foll. possibilities.

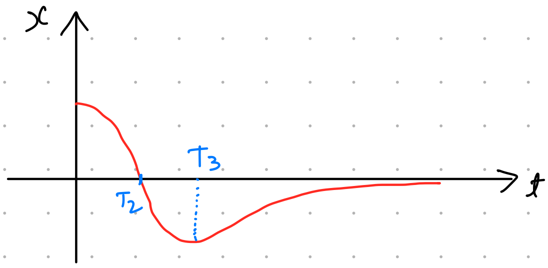
1) If initial velocity is zero, then the mass neither passes through its equilibrium position nor attains an extremum displacement.



2) If initial velocity is non-zero and directed downward, then mass does not pass through equilibrium but attains single extremum displacement from equilibrium at $t = T_1 > 0$. After this mass tends to equilibrium as $t \to \infty$.



3) If initial velocity is non-zero and directed upwards, then mass passes equilibrium position at $t = T_2 > 0$ and then attains an extreme displacement at $t = T_3 > T_2$, foll. which it tends to equilibrium as $t \to \infty$.



Case iii: Consider $b^2 - \lambda^2 > 0$ or $b > \lambda$ (over critical damping)

roots are $\quad r_1 = -b + \sqrt{b^2 - \lambda^2}$

$r_2 = -b - \sqrt{b^2 - \lambda^2}$ (real and distinct)

The G.S is

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

The damping is so great that no oscillation occur. Further, we cannot say that slight decrease in the damping will result in oscillation as in case ii.

Thus, the motion is called over critical damping.

Even here x tends to equilibrium monotonically as $t \to \infty$.

The three possible motion illustrated in case ii) also serve as possible motions in case iii).

Ex2: A 32 lb - weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching the spring 2 ft. The weight is then pulled down 6 in below its equilibrium position and released at $t=0$. No external forces are present; but the resistance of the medium in pounds is equal to $4\frac{dx}{dt}$, where $\frac{dx}{dt}$ is the instantaneous velocity in feet per sec.

Determine the resulting motion of the weight on the spring. Interpret the motion.

Soln: Given Weight $W = 32\,lb$

It stretches the spring by 2 ft. By Hooke's law

$W = kl$ or $32 = k\,2$ $\Rightarrow$ $k = 16\ lb/ft$

$\text{mass} = \frac{W}{g} = \frac{32}{32} = 1$ | $g = 32\ ft/sec^2$

$\Rightarrow$ $m = 1$

Damping constant $a = 4$

External forces $F(t) = 0$

The DE of the problem:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 16x = 0$$

where $x(t)$ is a displacement of the mass at time t.

IC's $x(0) = 1/2$ Since it is pulled $6\,in = 1/2\,ft$ below its equilibrium position at

$x'(0) = 0$ $t=0$ and released it with no initial velocity.

(observe that $2b = 4$ $\Rightarrow b = 2$ $\lambda = \sqrt{16} = 4$ i.e $\lambda > b$

$\therefore$ motion is damped oscillatory)

A E of the DE is

$$r^2 + 4r + 16 = 0$$

Roots are $\dfrac{-4 \pm \sqrt{16-64}}{2} = -2 \pm 2\sqrt{3}$

indep. solns are $e^{-2t} \cos 2\sqrt{3}t$, $e^{-2t} \sin 2\sqrt{3}t$

G.S is

$$\boxed{x = e^{-2t} \left(c_1 \cos(2\sqrt{3}t) + c_2 \sin(2\sqrt{3}t) \right)}$$

Given $x(0) = 1/2$

$$\therefore \quad \boxed{1/2 = c_1}$$

$$x'(t) = e^{-2t} \left(-2\sqrt{3}\, c_1 \sin(2\sqrt{3}t) + 2\sqrt{3}\, c_2 \cos(2\sqrt{3}t) \right)$$
$$+ (-2)\, e^{-2t} \left(c_1 \cos(2\sqrt{3}t) + c_2 \sin(2\sqrt{3}t) \right)$$

But $x'(0) = 0$

$$\therefore \quad 0 = 2\sqrt{3}\, c_2 - 2 c_1$$

$$\Rightarrow \quad \boxed{c_2 = \frac{1}{2\sqrt{3}}}$$

Sub for c_1 and c_2 in the G.S,

$$x(t) = e^{-2t} \left(\frac{1}{2} \cos(2\sqrt{3}t) + \frac{1}{2\sqrt{3}} \sin(2\sqrt{3}t) \right)$$

put $R \cos\phi = 1/2$, $R \sin\phi = 1/2\sqrt{3}$

$$\Rightarrow \quad R = \sqrt{\frac{1}{4} + \frac{1}{12}} = \frac{1}{\sqrt{3}} \quad \Bigg| \quad \frac{\sin\phi}{\cos\phi} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \tan\phi = 1/\sqrt{3}$$

$$\Rightarrow \quad \phi = \pi/6$$

Thus

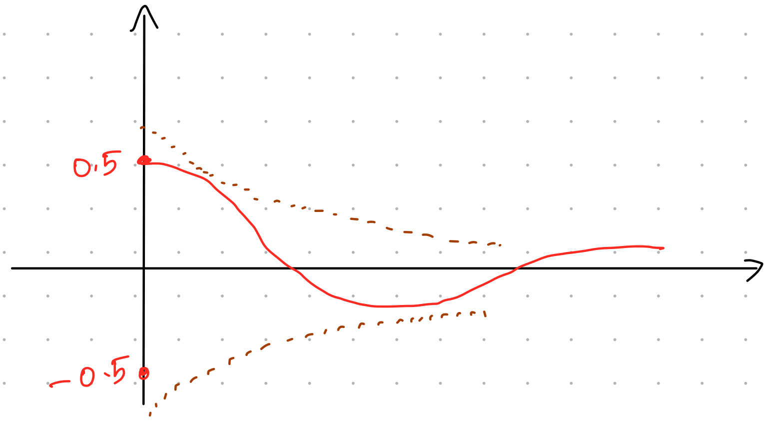
$$\boxed{x(t) = e^{-2t} \cdot \frac{1}{\sqrt{3}} \cos\left(2\sqrt{3}t - \pi/6 \right)}$$

Interpretation: Motion is damped oscillatory

damping factor is $\frac{1}{\sqrt{3}} e^{-2t}$ ft

Period is $\frac{2\pi}{2\sqrt{3}} = \frac{\pi}{\sqrt{3}}$ sec

Graph



Ex3: Same as example two, here we consider the resistance of the medium in pounds is equal to $8\frac{dx}{dt}$ instead of $4\frac{dx}{dt}$.

Soln: Formulation:

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0 \quad ; \quad x(0) = 1/2, \quad x'(0) = 0$$

(Here $2b = 8$ or $b = 4$, $\lambda = 4$

$\Rightarrow b = \lambda$, Thus it is critically damped)

AE: $m^2 + 8m + 16 = 0$

roots are $m = -4, -4$

G.S: $x = c_1 e^{-4t} + c_2 t e^{-4t}$

Applying IC's $x(0) = 1/2$ and $x'(0) = 0$,

$1/2 = c_1$

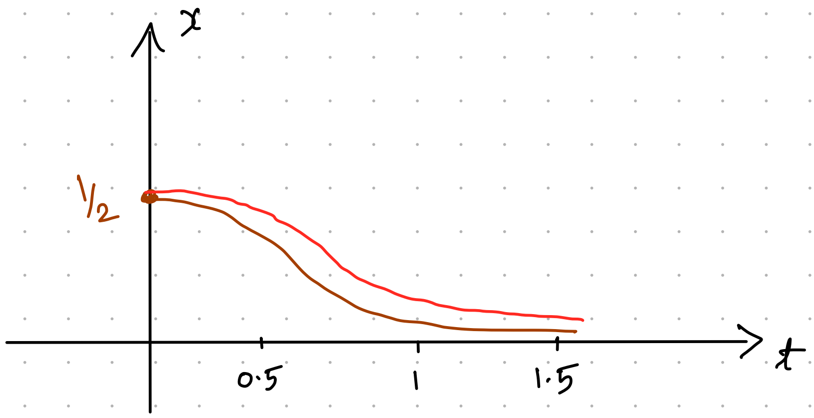
$x' = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t}$

$0 = -4c_1 + c_2$ or $c_2 = 2$

$$\therefore \quad \boxed{x = \tfrac{1}{2} e^{-4t} + 2t e^{-4t}}$$

interpretation: The motion is critically damped.
We see that $x=0$ iff $t = -\frac{1}{4}$. $\therefore$ $x \neq 0$ for $t \geq 0$
i.e mass does not pass through its equilibrium position.
The mass tends to equilibrium position as $t \to \infty$.
It is shown in brown curve below.

Graph



Ex 4: Same as example 2. Here resistance of the medium is $10\frac{dx}{dt}$ instead of $4\frac{dx}{dt}$.

Soln: DE is

$$\boxed{\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 16x = 0}$$

IC's $x(0) = \frac{1}{2}$
$x'(0) = 0$.

AE: $r^2 + 10r + 16 = 0$
Roots are $r = -2, -8$

G.S $x = c_1 e^{-2t} + c_2 e^{-8t}$

Apply IC's to find c_1 and c_2

$x' = -2c_1 e^{-2t} - 8c_2 e^{-8t}$

$x(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = c_1 + c_2$

$x'(0) = 0 \Rightarrow 0 = -2c_1 - 8c_2$

$\Rightarrow 0 = -c_1 - 4c_2$

Solving, we get $c_2 = -1/6, \ c_1 = 2/3$

Thus

$$\boxed{x = -\tfrac{1}{6} e^{-2t} + \tfrac{2}{3} e^{-8t}}$$

Interpretation: Motion is over damped. The graph is same as in above ex. but due to overdamping it reaches equilibrium position at a slower rate.

It is shown as the red curve in the fig.

Forced motion

Here we consider the effect of daming upon the mass on the spring and also external impressed $F(t)$ $\forall t > 0$.

Ex 5: A 16 lb weight is attached to the lower end of a coil spring suspended from the ceiling the spring constant of the spring being 10 lb/ft. The weight comes to rest in its equilibrium position. Beginning at $t=0$ an external force given by $F(t) = 5\cos 2t$ is applied to the system. Determine the resulting motion if the damping force in pounds is equal to $2\frac{dx}{dt}$, where $\frac{dx}{dt}$ is the instantaneous velocity in ft. per second.

Soln: Given $W = 16$ lb

$$\Rightarrow m = \frac{W}{g} = \frac{16}{32} = \frac{1}{2}$$

spring constant $k = 10$ lb/ft

and damping constant $a = 2$

External force $E(t) = 5\cos 2t$

The basic DE for the motion is

$$\frac{1}{2}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 5\cos 2t$$

or

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 20x = 10\cos 2t$$

IC's are $x(0) = 0$, $x'(0) = 0$

G.S of the DE is

$$x = x_c + x_p$$

To find x_c:

Consider $$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 20x = 0$$

AE : $r^2 + 4r + 20 = 0$

roots are $r = \dfrac{-4 \pm \sqrt{16-80}}{2}$

$$\Rightarrow \quad r = -2 \pm 4i$$

$$\therefore \quad \boxed{x_c = e^{-2t}\left(c_1 \cos 4t + c_2 \sin 4t\right)}$$

To find x_p: Use inverse operator method

$$x_p = \frac{10 \cos 2t}{D^2+4D+20}$$

$$= 10 \, Re\left(\frac{e^{i2t}}{D^2+4D+20}\right)$$

$$= 10 \, Re\left(\frac{e^{i2t}}{-4+8i+20}\right)$$

$$= 10 \, Re\left(\frac{e^{i2t}}{16+8i}\right)$$

$$= 10 \, Re\left(\frac{e^{i2t}(16-8i)}{(16+8i)(16-8i)}\right)$$

$$= 10 \, Re\left(\frac{(\cos 2t + i \sin 2t)(16-8i)}{256+64}\right)$$

$$= 10\left(\frac{16 \cos 2t + 8 \sin 2t}{320}\right)$$

Thus, $\boxed{x_p = \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t}$

The G.S

$$x = e^{-2t}\left(c_1 \cos 4t + c_2 \sin 4t\right) + \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t$$

and

$$x' = e^{-2t}\left(-4c_1 \sin 4t + 4c_2 \cos 4t\right) - 2e^{-2t}\left(c_1 \cos 4t + c_2 \sin 4t\right)$$
$$- \sin 2t + \frac{1}{2}\cos 2t$$

Applying IC's $x(0) = 0$ and $x'(0) = 0$, we get

$$0 = c_1 + 1/2 \Rightarrow \boxed{c_1 = -1/2}$$

and $$0 = 4c_2 - 2c_1 + 1/2 \Rightarrow \boxed{c_2 = -3/8}$$

$\therefore$ Particular soln is

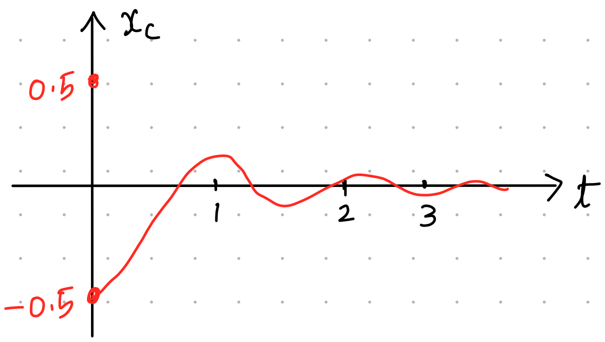
$$\boxed{x = e^{-2t}\left(-\tfrac{1}{2}\cos 4t - \tfrac{3}{8}\sin 4t\right) + \tfrac{1}{2}\cos 2t + \tfrac{1}{4}\sin 2t}$$

In phase angle form,

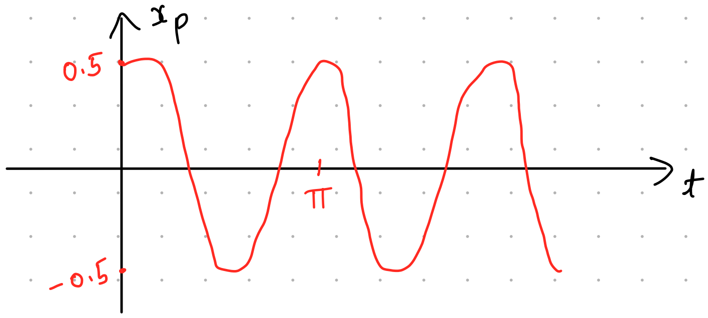
$$\boxed{x = \underbrace{-\tfrac{5}{8}e^{-2t}\cos(4t - 0.641)}_{x_c} + \underbrace{\tfrac{\sqrt{5}}{4}\cos(2t - 0.46)}_{x_p}}$$

Interpretation

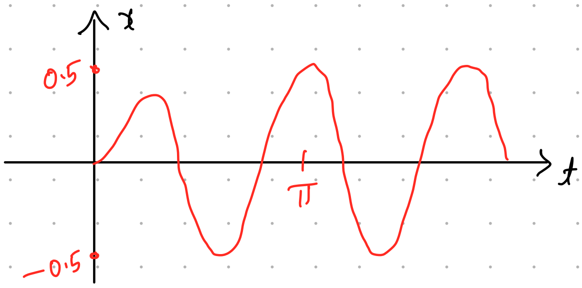
$x_c = -\frac{5}{8}e^{-2t}\cos(4t - 0.641)$ is the transient term, it represent damped oscillation. It becomes negligible in a short term.



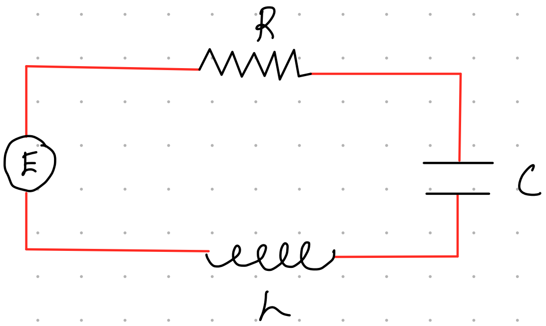
The term $x_p = \frac{\sqrt{5}}{4}\cos(2t - 0.46)$ is the steady-state term it represents simple harmonic motion of amplitude $= \frac{\sqrt{5}}{4}$, freq $= \frac{1}{\pi}$ and period $= \pi$



Graph of $x = x_c + x_p$ is



Electric circuit problems



Let E be emf, let i be the current in the circuit

Voltage drop across a resistor is iR

Voltage drop across an inductor is $L\frac{di}{dt}$

voltage drop across a capacitor is $\frac{1}{C}q = \frac{1}{C}\int i\,dt$

Kirchhoff's voltage law (KVL) :

The sum of the voltage drops across resistors, inductors and capacitors is equal to the total emf in a closed circuit.

That is $$L\frac{di}{dt} + iR + \frac{1}{C}q = E \quad \text{——} \ (1)$$

WKT $$\frac{dq}{dt} = i$$

Thus, (1) $\Rightarrow$ $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E$$

Also

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'$$