7/9/21

Propositional logic

*) Defn of proposition with Ex.

*) Logical operators

*) Compound proposition

*) Exs

1) Declarative sentence ✓
2) Interrogative ✓
3) Imperative ✓
4) Exclamatory ✓

Defn: Proposition

*) It is a declarative sentence.

*) It is either true or false, cannot be both.

Ex: 1) Bengaluru is capital of Karnataka. (It is a proposition)

2) $1+2=5$ (It is a proposition)

3) It is raining outside. (It is a proposition)

4) Happy birthday, Adarsh! (Not a proposition)

5) $x+1=5$ (It is not a proposition)

6) What time is it? (Not a proposition)

7) Drive the car carefully. (Not a proposition)

- We represent propositions by a variable called propositional variable

It is denoted by lower case letters.

$p, q, r, s, t, \ldots$

- The truth value of a proposition is true, denoted by T (or 1) If the proposition is true.

- If it is false, then truth value is false, denoted by F (or 0).

Proposition — Atomic propositions
Proposition — compound propositions.

Contruction of propositions using logical operators and existing propositions (Compound propositions)

logical operators

1) negation $\neg$ ($\sim$)
2) conjunction (AND) $\wedge$
3) disjunction (OR) $\vee$
4) Implication (Conditional) $\rightarrow$
5) biconditional $\leftrightarrow$

1) Negation:

Let p be a proposition

The negation of p, denoted by $\neg p$ (read "not p")

Ex: p : $1+2=5$ (F)

$\neg p$: $1+2 \neq 5$ (T)

Truth table

p	$\neg p$
1	0
0	1

2) Conjunction (And)

Let p and q be propositions.

Conjunction of p and q, denoted by $p \wedge q$ ("p and q")

$p \wedge q$ is true when both p and q are true and is false otherwise.

Truth table

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Ex: p : $1+2=3$ (T)

q : Arun smartphone has 16GB ROM. (F)

$p \wedge q$: $1+2=3$ and Arun smartphone has 16GB ROM. (F)

3) Disjunction (OR)

Let p and q be propositions.

Disjunction of p and q, denoted by $p \vee q$ (read "p or q")

$p \vee q$ is false when both p and q are false, otherwise true.

Truth table

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Ex: 1 p: Arun is a boy (T)

2) q: Sun rises in the ~~east~~ (F)

$p \vee q$: Arun is a boy or Sun rises in the ~~east~~ (T)

- Inclusive OR ($p \vee q$) : p or q

- Exclusive OR ($p \oplus q$) : p or q, but not both.

Truth table

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

p: Adi gets jamun in lunch.

q: Adi gets carrot halwa in lunch.

$p \vee q$: Adi gets jamun or carrot halwa.

$p \oplus q$: Adi get jamun or carrot halwa, but not both.

4) Conditional statements

Let p and q be propositions

hypothesis conclusion

Conditional stmt $p \to q$ (if p, then q)

$p \to q$ is false when p is true and q is false, and true otherwise.

Truth table

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Ex: p: Arun works hard $f(x) = x^2$ continuous (and) differentiable on $[0,1]$

q: Arun passes the exam. (F)

$p \Rightarrow q$: If Arun works hard, then he pass the exam. (T)

Different ways to express $p \Rightarrow q$

" if p, then q "

" if p, q "

" q if p "

" p implies q "

" p only if q

" p is sufficient of q "

" q is necessary for p "

"q when p"

"q whenever p"

" q follows from p"

Ex:

Converse, Inverse and Contrapositive of $p \to q$

1) Converse of $p \Rightarrow q$ is $q \to p$

Ex: The home team wins whenever it is raining

p: The home team wins
q: It is raining

$q \to p$: The home team wins whenever it is raining
(If it is raining, then the home team wins)

Converse of $q \Rightarrow p$ is $p \Rightarrow q$:

$p \Rightarrow q$: If the home team wins, then it is raining

2) Inverse of $p \to q$ is $\neg p \to \neg q$

In above ex. inverse of $q \to p$ is $\neg q \to \neg p$

$\neg q \to \neg p$: If it is not raining, then the home team does not win.

3) Contrapositive of $p \to q$ is $\neg q \to \neg p$

In the above ex, Contrapositive of $q \to p$ is $\neg p \to \neg q$

$\neg p \to \neg q$: If the home teams does not win, then it is not raining

The home team wins whenever it is raining

Truth table

p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$ (converse)	$\neg p \to \neg q$ (inverse)	$\neg q \to \neg p$ (contrapositive)
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

Fact: $p \to q$ is logically same as $\neg q \to \neg p$.

5) Biconditional

Let p and q be propositions

biconditional, $p \leftrightarrow q$ is true when both p and q have the same true value.

(read "p if and only if q") iff

("p is necessary and sufficient condition for q")

Truth table

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

More over, $p \leftrightarrow q$ is same as $(p \to q)$ and $(q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$		$p \leftrightarrow q$
0	0	1	1	1		1
0	1	1	0	0		0
1	0	0	1	0		0
1	1	1	1	1		1

Recollection

Proposition; Logical operator $\neg, \wedge, \vee, \to, \leftrightarrow$

p	q	$\neg p$	p and q $p \wedge q$	p or q $p \vee q$	$p \to q$	$(p \to q) \wedge (q \to p)$ $p \leftrightarrow q$
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	0	0
1	1	0	1	1	1	1

if p, then q | p if, and only if q
p implies q

Precedence of Logical operators

operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\to$	4
$\leftrightarrow$	5

Ex: 1) $\neg p \to q$ means $(\neg p) \to q$ rather than $\neg(p \to q)$

2) $p \wedge q \vee r$ means $(p \wedge q) \vee r$ rather than $p \wedge (q \vee r)$

3) $p \to q \vee r$

$(p \to q) \vee r$ ✗ $\quad p \to (q \vee r)$ ✓

4) $p \wedge q \wedge r$

$(p \wedge q) \wedge r$ is same as

$p \wedge (q \wedge r)$

1. Which of these are propositions? What are the truth values of those that are propositions?

a) Do not pass go.

b) What time is it?

c) There are no black flies in Maine.

d) $2^n \geq 100$

a) Not a proposition (Imperative sentence)

b) Not a proposition (Interogative ")

c) It is a proposition (No idea about Truth value)

d) Not a proposition,

2. Determine whether each of these conditional statements is true or false.

a) If $\underbrace{1+1=2}_{p}$, then $\underbrace{2+2=5}_{q}$. $p \rightarrow q$

b) If $1+1=3$, then $2+2=4$.

c) If $1+1=3$, then $2+2=5$.

d) If monkeys can fly, then $1+1=3$.

a) It is false $p: 1+1=2$, $q: 2+2=5$

p is true and q is false, $\therefore$ $p \rightarrow q$ is false.

b) $p: 1+1=3$, $q: 2+2=4$

p is false, $\therefore$ $p \rightarrow q$ is true.

c) $p: 1+1=3$, $q = 2+2=5$

p is false, $p \rightarrow q$ is true

d) p: Monkeys can fly, $q: 1+1=3$

p is false, $p \rightarrow q$ is true.

3. Let p, q, and r be the propositions

p : You have the flu.
q : You miss the final examination.
r : You pass the course.

Express each of these propositions as an English sentence.

a) $p \rightarrow q$
b) $\neg q \leftrightarrow r$
c) $q \rightarrow \neg r$
d) $p \vee q \vee r$
e) $(p \rightarrow \neg r) \vee (q \rightarrow \neg r)$
f) $(p \wedge q) \vee (\neg q \wedge r)$

a) If you the flu, then you miss the final examination.

b) You donot miss the final examination if, and only if you pass the course.

c) If you miss the final examination, then you donot pass the course.

d) You have the flu, or you miss the final examination, or you donot pass the course.

e) Either you donot pass the course whenever you have the flu or you donot pass the course whenever you miss the final examination.

f) You have the flu, and miss the final examination, or you donot miss the final examination and pass the course.

4. Let p and q be the propositions

p : You drive over 65 miles per hour.
q : You get a speeding ticket.

Write these propositions using p and q and logical connectives (including negations).

a) You do not drive over 65 miles per hour.
b) You drive over 65 miles per hour, but you do not get a speeding ticket.
c) You will get a speeding ticket if you drive over 65 miles per hour.
d) If you do not drive over 65 miles per hour, then you will not get a speeding ticket.
e) Driving over 65 miles per hour is sufficient for getting a speeding ticket.
f) You get a speeding ticket, but you do not drive over 65 miles per hour.
g) Whenever you get a speeding ticket, you are driving over 65 miles per hour.

a) $\neg p$

b) $p \wedge \neg q$

c) $p \rightarrow q$

d) $\neg p \rightarrow \neg q$

e) $p \rightarrow q$

f) $q \wedge \neg p$

g) $q \rightarrow p$

5. Let p, q, and r be the propositions

p : You get an A on the final exam.
q : You do every exercise in this book.
r : You get an A in this class.

Write these propositions using p, q, and r and logical connectives (including negations).

a) You get an A in this class, but you do not do every exercise in this book.
b) You get an A on the final, you do every exercise in this book, and you get an A in this class.
c) To get an A in this class, it is necessary for you to get an A on the final.
d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.
e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.
f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

a) $r \wedge \neg q$

b) $p \wedge q \wedge r$

c) $p \rightarrow r$

d) $p \wedge \neg q \wedge r$

e) $(p \wedge q) \rightarrow r$

f) $r \leftrightarrow (q \vee p)$

6. How many rows appear in a truth table for each of these compound propositions?

a) $(q \to \neg p) \vee (\neg p \to \neg q)$

b) $(p \vee \neg t) \wedge (p \vee \neg s)$

c) $(p \to r) \vee (\neg s \to \neg t) \vee (\neg u \to v)$

d) $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$

a) 2^2

b) 2^3

c) 2^6

d) 2^5

7. Construct a truth table for each of these compound propositions.

a) $p \wedge \neg p$

b) $p \vee \neg p$

c) $(p \vee \neg q) \to q$

d) $(p \vee q) \to (p \wedge q)$

e) $(p \to q) \leftrightarrow (\neg q \to \neg p)$

f) $(p \to q) \to (q \to p)$

e)

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	1
1	1	0	0	1	1	1

d)

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \to (p \wedge q)$
0	0	0	0	1
0	1	1	0	0
1	0	1	0	0
1	1	1	1	1

e) $(p \leftrightarrow q) \vee (\neg q \to r)$

p	q	r	$\neg q$	$p \leftrightarrow q$	$\neg q \to r$	$(p \leftrightarrow q) \vee (\neg q \to r)$
0	0	0	1	1	0	1
0	0	1	1	1	1	1
0	1	0	0	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	0	0
1	0	1	1	0	1	1
1	1	0	0	1	1	1
1	1	1	0	1	1	1

Logical Equivalence

Defn [Tautology]

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it. denoted by T

Ex: 1) $\neg p \vee p$

p	$\neg p$	$p \vee \neg p$
0	1	1
1	0	1

Defn [Contradiction]

A compound proposition that is always false, denoted by F.

Ex 2) $\neg p \wedge p$

p	$\neg p$	$p \wedge \neg p$
0	1	0
1	0	0

Defn [contingency]

A compound proposition that is neither tautology nor contradiction.

Ex 3) $p \vee q$

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Defn: Logical equivalence

Compound propositions p and q are called logically equivalent if they have same truth values in all possible values.
Denoted by $p \equiv q$

If p and q are logically equivalent, then $p \Leftrightarrow q$ is tautology.

Ex: $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$

i.e $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \wedge q$	$p \vee (\neg p \wedge q)$	$\neg(p \vee (\neg p \wedge q))$	$\neg p \wedge \neg q$
0	0	1	1	0	0	1	1
0	1	1	0	1	1	0	0
1	0	0	1	0	1	0	0
1	1	0	0	0	1	0	0

Laws of logic

1) $p \wedge T \equiv p$
2) $p \vee F \equiv p$

} Identity laws

3) $p \vee T \equiv T$
4) $p \wedge F \equiv F$

} Domination laws

5) $p \vee p \equiv p$
6) $p \wedge p \equiv p$

} Idempotent laws

7) $\neg(\neg p) \equiv p$ Double negation law

8) $p \vee q \equiv q \vee p$

9) $p \wedge q \equiv q \wedge p$

$\Big\}$ Commutative laws

10) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

11) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

$\Big\}$ Associative laws

12) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

13) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

$\Big\}$ De Morgan's laws

14) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

15) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$\Big\}$ Distributive laws

16) $p \vee (p \wedge r) \equiv p$

17) $p \wedge (p \vee r) \equiv p$

$\Big\}$ Absorption laws

18) $p \vee \neg p \equiv T$

19) $p \wedge \neg p \equiv F$

$\Big\}$ Negation laws

pfs of (12)

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

pf of (16)

p	r	$p \wedge r$	$p \vee (p \wedge r)$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

Logical equivalence involving implication

1) $p \to q \equiv \neg p \vee q$ (Verify using truth table)

2) $p \to q \equiv \neg q \to \neg p$

Logical equivalence involving Biconditional statements

1) $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ (Verify using truth table)

2) $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

Extended De Morgan's law:

1) $\neg(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \equiv \neg p_1 \vee \neg p_2 \vee \neg p_3 \vee \cdots \vee \neg p_n$

2) $\neg(p_1 \vee p_2 \vee p_3 \vee \cdots \vee p_n) \equiv \neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \cdots \wedge \neg p_n$

Examples:

1) S.T $\neg(p \to q) \equiv p \wedge \neg q$, using laws of logic.

pf: LHS $\neg(p \to q)$

$\equiv \neg(\neg p \vee q)$ ($\because p \to q \equiv \neg p \vee q$)

$\equiv \neg(\neg p) \wedge \neg q$ (De Morgan's law)

$\equiv p \wedge \neg q = RHS$ (double negation law)

2) S.T $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$, using laws of logic.

pf LHS : $\neg(p \vee (\neg p \wedge q))$

$\equiv \neg p \wedge \neg(\neg p \wedge q)$ (De Morgan's law)

$\equiv \neg p \wedge (p \vee \neg q)$ (De Morgan's and double negation law)

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ (distributive law)

$\equiv F \vee (\neg p \wedge \neg q)$ (Negation law)

$\equiv \neg p \wedge \neg q = RHS$ (Identity law)

Ex 3 : S.T $[(p\vee q)\wedge(\neg p\vee r)]\to(q\vee r)$ is a Tautology

pf: Consider

$$[(p\vee q)\wedge(\neg p\vee r)]\to(q\vee r)$$

$$\equiv \neg[(p\vee q)\wedge(\neg p\vee r)]\vee(q\vee r) \qquad (\because p\to q\equiv\neg p\vee q)$$

$$\equiv [(\neg p\wedge\neg q)\vee(p\wedge\neg r)]\vee(q\vee r) \qquad \text{(De Morgan's law)}$$

$$\equiv (\neg p\wedge\neg q)\vee[(p\wedge\neg r)\vee(q\vee r)] \qquad \text{(Associative law)}$$

$$\equiv (\neg p\wedge\neg q)\vee[(q\vee r)\vee(p\wedge\neg r)] \qquad \text{(commutative law)}$$

$$\equiv (\neg p\wedge\neg q)\vee[(q\vee r\vee p)\wedge(q\vee r\vee\neg r)] \qquad \text{(Distributive law)}$$

$$\equiv (\neg p\wedge\neg q)\vee[(q\vee r\vee p)\wedge T] \qquad \text{(Negation and Domination law)}$$

$$\equiv (\neg p\wedge\neg q)\vee[q\vee r\vee p] \qquad \text{(Identity law)}$$

$$\left.\begin{aligned}&\equiv (\neg p\wedge\neg q)\vee[(p\vee q)\vee r]\\ &\equiv [(\neg p\wedge\neg q)\vee(p\vee q)]\vee r\end{aligned}\right\} \text{Associative law}$$

$$\equiv [\neg(p\vee q)\vee(p\vee q)]\vee r \qquad \text{(De Morgan's law)}$$

$$\equiv T\vee r \qquad \text{(Negation law)}$$

$$\equiv T \qquad \text{(Domination law)}$$

Ex 4: S.T $p\to(q\vee r)\equiv\neg r\to(p\to q)\equiv(p\wedge\neg q)\to r$

Pf Consider

$$p\to(q\vee r)\equiv\neg p\vee(q\vee r)$$

$$\equiv\neg p\vee q\vee r \quad \text{— (1)} \quad \text{(associative law)}$$

Now,

$$\neg r\to(p\to q)\equiv r\vee(p\to q)$$

$$\equiv r\vee(\neg p\vee q)$$

$$\equiv\neg p\vee q\vee r \quad \text{— (2)} \quad \text{(associative law)}$$

Rules of inference

Argument: It is sequence of propositions.

- All propositions except the last one are called premises.
- Last proposition is called conclusion.

Ex: 'If you have a current password, then you can log onto the network'

'You have a current password'

Therefore,

'You can log onto the network'

Argument form: It is an argument involving propositional variables.

Ex:

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

In general argument is of the form:

$$\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \\ \hline \therefore q \end{array}$$

Here $P_1, P_2, P_3, \ldots, P_n$ are premises and q is conclusion.

It is valid if

$(P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n) \to q$ is always true. (a tautology)

Rules of inference:

Rule of inference	Tautology	Name
1) $p \to q$ p $\therefore q$	$[(p \to q) \land p] \to q$	Modus ponens (Rule of detachment)
2) $p \to q$ $\neg q$ $\therefore \neg p$	$[(p \to q) \land \neg q] \to \neg p$	Modus tollens

Ex: If it is snowing, then I will study math

I will not study math

∴ it is not snowing

3) $p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \land (q \to r)] \to p \to r$	Hypothetical syllogism

Ex: If it snows, then I will study math

If I study math, then I will get an A.

∴ If it snows, I will get an A.

4) $p \lor q$ $\neg p$ $\therefore q$	$[(p \lor q) \land \neg p] \to q$	Disjunctive syllogism

Ex: I will study math or I will study Physics

I will not study math.

∴ I will study physics.

5)	$p \wedge q$ $\therefore q$	$(p \wedge q) \rightarrow q$	Simplification
6)	p $\therefore p \vee q$	$p \rightarrow (p \vee q)$	addition
7)	p q $\therefore p \wedge q$	$(p) \wedge (q) \rightarrow p \wedge q$	Conjunction
8)	$\neg p \vee r$ $p \vee q$ $\therefore q \vee r$	$(\neg p \vee r) \wedge (p \vee q) \rightarrow q \vee r$	Resolution

Ex: I will not study math or I will study physics
I will study math or I will study Chemistry

$\therefore$ I will study chemistry or physics.

Ex1: Show that the premises

"It is not sunny this afternoon and it is cooler than yesterday",

"We will go swimming only if it is sunny".

"If we do not go swimming, then we will take a Canoe trip," and

"If we take a canoe trip, then we will be home by sunset" lead to the conclusion

"We will be home by sunset."

Soln: p: It is sunny this afternoon

q: It is cooler than yesterday

r: We will go swimming

s: We will take a canoe trip

t: We will be home by sunset

Argument form:

$$\begin{array}{l} \neg p \wedge q \\ r \rightarrow p \\ \neg r \rightarrow s \\ s \rightarrow t \\ \hline \therefore t \end{array}$$

Step	Reason
1) $\neg p \wedge q$	premise
2) $r \rightarrow p$	"
3) $\neg r \rightarrow s$	"
4) $s \rightarrow t$	"
5) $\neg p$	step 1, simplification
6) $\neg r$	step 2, 5, Modus tollens
7) s	step 3, 6, Modus ponens
8) t	Step 4 and 7, Modus ponens

Thus, the given argument is valid.

Ex2: Show that the premises
If you send me an e-mail msg, then I will finish writing the program, "If you do not send me an email msg, then I will go to sleep early", and If I go to sleep early, then I will wake up feeling refreshed"

lead to the conclusion

"If I do not finish writing the program, then I will wake up feeling refreshed."

Soln: p: You send me an e-mail msg
q: I will finish writing the program
r: I will go to sleep early

s: I will wake up feeling refreshed.

Argument form

$$\begin{array}{l} p \to q \\ \neg p \to r \\ \underline{r \to s} \\ \therefore \neg q \to s \end{array}$$

Step	Reason
1) $p \to q$	Premise
2) $\neg p \to r$	Premise
3) $r \to s$	"
4) $\neg p \to s$	Hypothetical syllogism (step 2 & 3)
5) $\neg q \to \neg p$	step 1, contrapositive
6) $\neg q \to s$	Hypothetical syllogism (step 4 & 5)

Thus, given argument is valid.

Rules of Inference for quantified statements

Rule of Inference	Name
$\dfrac{\forall x\ P(x)}{\therefore P(c)}$ where c is any element in the Universe	Universal instantiation
$\dfrac{P(c) \text{ for an abitrary } c}{\therefore \forall x\ P(x)}$	Universal generalisation
$\dfrac{\exists x\ P(x)}{\therefore P(c) \text{ for some element } c.}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c.}{\therefore \exists x\ P(x)}$	Existential generalisation