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Ordered n-tuple.

An ordered pair of two elements consists of two elements say a and b, in a given fixed order and denoted by (a, b). Here 'a' is called the first element and 'b' is called a second element.

Two ordered pairs (a, b) and (c, d) are said to be equal if and only if $a = c$ and $b = d$. i.e., $(a, b) = (c, d) \Leftrightarrow a = c;\ b = d$

Similarly an ordered triplet is defined and is denoted as (a, b, c).

In general an ordered n-tuple is defined and denoted by $(x_1, x_2, x_3, \ldots, x_n)$.

Cartesian Product

Let A and B be any two sets. The cartesian product of the sets A and B, denoted by $A \times B$ is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.

i.e., $A \times B = \{(a, b) \mid a \in A,\ b \in B\}$

In particular, $A \times A = \{(a, a) \mid a \in A,\ a \in A\}$

Note

1. In general $A \times B \neq B \times A$
2. If the set A has n elements and the set B has m elements, then the set $A \times B$ has mn elements
3. If $A = \phi$ or $B = \phi$, then $A \times B = \phi$

Note

1. The product set $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers is called Euclidean plane and this set is also denoted by $\mathbb{R}^2$.

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2 The product set $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers is called Euclidean ~~plane~~ space of three dimension and is also denoted by $\mathbb{R}^3$.

example If $A=\{1,2\}$ and $B=\{3,1,4\}$,

then $A\times B=\{(1,3),(1,1),(1,4),(2,3),(2,1),(2,4)\}$

$B\times A=\{(3,1),(3,2),(1,1),(1,2),(4,1),(4,2)\}$

$A\times A=\{(1,1),(1,2),(2,1),(2,2)\}$

$B\times B=\{(3,3),(3,1),(3,4),(1,3),(1,1),(1,4),(4,3),(4,1),(4,4)\}$

Results related to Cartesian products

1. $A\times(B\cap C)=(A\times B)\cap(A\times C)$
2. $A\times(B\cup C)=(A\times B)\cup(A\times C)$
3. If $B\subset A$ then $B\times B=(B\times A)\cap(A\times B)$
4. Let A and B be two sets. ~~Show that this~~ The sets $A\times B$ and $B\times A$ have an element in common if and only if the set A and B have an element in common.

example If $A=\{1,2,3\}$, $B=\{4,3\}$, $C=\{1,2\}$

then $A\cap B=\{3\}$, $A\cap C=\{1,2\}$, $B\cap C=\phi$

$A\cup B=\{1,2,3,4\}$, $A\cup C=\{1,2,3\}$, $B\cup C=\{1,2,3,4\}$

$A\times B=\{(1,4),(1,3),(2,4),(2,3),(3,4),(3,3)\}$

$A\times C=\{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\}$

$A\times(B\cap C)=\phi=(A\times B)\cap(A\times C)$

$A\times(B\cup C)=\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4)\}$

$A\times(B\cup C)=(A\times B)\cup(A\times C)$

$C\subset A$, $C\times C=\{(1,1),(1,2),(2,1),(2,2)\}$

$(C\times A)=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$

$(C\times A)\cap(A\times C)=\{(1,1),(1,2),(2,1),(2,2)\}=C\times C$

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Relations

Let A and B be two sets. A relation R from A to B is a subset of $A \times B$.

If A is any set then a relation R on A is a subset of $A \times A$.

If R is a relation from A to B and $(a,b) \in R$, then we say "a is related to b under the relation R" or we say "a is R-related to b" and we denote it by aRb.

If $(a,b) \notin R$, then we write $a \not R b$ i.e., "a is not related to b under the relation R".

examples.

1. Let $A = \{1,2,3,4\}$ and $B = \{3,5,7\}$
then $R = \{(a,b) \mid a < b\}$ is a subset of $A \times B$,
i.e., $R = \{(1,3), (1,5), (1,7), (2,3), (2,5), (2,7), (3,5), (3,7), (4,5), (4,7)\}$
is a relation from A to B.
2. Let A be the set of all integers and B be the set of all non negative integers.
Then $R = \{(a, a^2) \mid a \in A\}$ is a relation from A to B

Domain and Range of a relation

Let $R \subseteq A \times B$ be a relation from A to B. Then the set $\{a \mid (a,b) \in R\}$ is called the domain of the relation R. The set $\{b \mid (a,b) \in R\}$ is called the range of the relation R.

example: Consider the relation R defined on the set N of natural numbers by $R = \{(a,b) \mid a,b \in N \text{ \& } 2a+b=7\}$

Then $R = \{(1,5), (2,3), (3,1)\}$

Thus Domain $= \{1,2,3\}$, Range $= \{5,3,1\}$.

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Types of Relations

1. Identity relation

The relation denoted by I_A on a set A is called the identity relation on A, if

$$I_A = \{(x,y) \mid x=y\} = \{(x,x) \mid x \in A\}$$

2. Universal relation

The relation $R = A \times A$ on A is called universal relation on A.

3. Void relation

Since the null set ϕ is also a subset of $A \times A$, the null set ϕ is also a relation of the set A and is called the void relation or null relation.

4. Inverse relation

Let R be a relation from the set A into set B. The relation denoted by R^{-1} from B to A and defined by $R^{-1} = \{(b,a) \mid (a,b) \in R\}$ is called the inverse relation of R.

5. Reflexive relation

A relation R on a set A is said to be reflexive, if $\forall x \in A$, $(x,x) \in R$.

6. Irreflexive relation

A relation R on a set A is said to be irreflexive if for every $x \in A$, $(x,x) \notin R$.

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7. Symmetric relation.
A relation R defined on a set A is called symmetric relation, if $(x,y) \in R \Rightarrow (y,x) \in R$.

8. Anti symmetric relation
A relation R on a set A is called anti-symmetric if $(a,b) \in R$ and $(b,a) \in R \Rightarrow a=b$

9. Transitive relation
A relation R on a set A is said to be transitive if $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$ for all $x,y,z \in A$.

10. Equivalence relation
A relation R defined on a set A is called an equivalence relation, if it is reflexive, symmetric and transitive.

* Let $A=\{1,2,3\}$. Find the nature of the relations on A given below:
(i) $R_1=\{(1,2),(2,1),(1,3),(3,1)\}$ (ii) $R_2=\{(1,1),(2,2),(3,3),(2,3)\}$
(iii) $R_3=\{(1,1),(2,2),(3,3)\}$ (iv) $R_4=\{(1,1),(2,2),(3,3),(2,3),(3,2)\}$
(v) $R_5=\{(1,3),(3,2)\}$ (vi) $R_6=\{(2,3),(3,4),(2,4)\}$

Soln

R_1 – symmetric and irreflexive
R_2 – reflexive and transitive
R_3 – reflexive and symmetric
R_4 – reflexive and Symmetric
R_5 – irreflexive
R_6 – irreflexive, transitive.

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* Give an example of a relation defined on a suitable set which is:
(i) reflexive, symmetric and transitive.
(ii) reflexive, symmetric but not transitive
(iii) reflexive, transitive but not symmetric
(iv) symmetric, transitive but not reflexive.
(v) reflexive but not symmetric and not transitive
(vi) symmetric but not reflexive and not transitive.
(vii) transitive but not reflexive and not symmetric
(viii) not reflexive, not symmetric and not transitive
(ix) reflexive, anti-symmetric and transitive.

Sol) Consider $A = \{a, b, c\}$
(i) ~~Consider~~ $R_1 = \{(a,a), (b,b), (c,c)\}$.
(ii) $R_2 = \{(a,a)$
(iii) $R_3 = \{(a,a), (b,b), (c,c), (a,b)\}$
(iv) $R_4 = \{(c,c)\}$
(v) $R_5 = \{(a,a), (b,b), (c,c), (a,b), (c,a)\}$.
(vi) $R_6 = \{(a,a), (c,c), (a,b), (b,a)\}$
(vii) $R_7 = \{(a,a), (b,b), (a,b)\}$
(viii) $R_8 = \{(a,b), (b,c)\}$
(ix) $R_9 = \{(a,a), (b,b), (c,c)\}$

* Let $A = \{1, 2, 3, 4, 6\}$ and R be the relation on A defined by $(a,b) \in R$ if and only if a is a multiple of b. Write R as a set of ordered pairs.

Sol) $R = \{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (4,4), (6,6)\}$

* A relation R on the set of all integers $\mathbb{Z}$ is defined recursively by (i) $(0,0) \in R$ and (ii) if $(s,t) \in R$, then $(s+1, t+7) \in R$. Find R as a set of ordered pairs.

Sol) $R = \{(0,0), (1,7), (2,14), (3,21), (4,28), \ldots\}$

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* Let $A = \{1,2,3,4,5,6,7,8,9,10,11,12\}$. On this set define the relation R by $(x,y) \in R$ if and only if $x-y$ is a multiple of 5. Verify that R is an equivalence relation.

Sol) For any $x \in A$, $x-x=0$ which is a multiple of 5.
$\therefore (x,x) \in R$, hence R is reflexive.
For any $x, y \in A$, if $(x,y) \in R$ then $x-y=5k$ for some integer k.
Consequently $y-x = 5(-k)$
$\therefore (y,x) \in R$, hence R is symmetric.
For any $x,y,z \in A$, if $(x,y) \in R$ and $(y,z) \in R$, then $x-y=5k_1$ and $y-z=5k_2$ for some integers k_1 and k_2.
Consequently $x-z = (x-y)+(y-z)$
$= 5k_1 + 5k_2$
$= 5(k_1+k_2)$.
$\therefore (x,z) \in R$, hence R is transitive.
Therefor R is an equivalence relation

* Let S be the set of all non-zero integers and $A = S \times S$. On A define the relation R by $(a,b) R (c,d)$ if and only if $ad=bc$. Show that R is an equivalence relation.

Sol) $(a,a)R(a,a)$ as $aa=aa$. $\therefore R$ is reflexive
Suppose $(a,b)R(c,d)$.
Then $ad=bc \Rightarrow cb=da \Rightarrow (c,d)R(a,b)$
$\therefore R$ is symmetric.
Suppose $(a,b)R(c,d)$ and $(c,d)R(e,f)$
Then $ad=bc$ and $cf=de$
$\Rightarrow d=\frac{bc}{a}$, then $cf=\frac{bc}{a}e \Rightarrow af=be$
$\Rightarrow (a,b)R(e,f)$
$\therefore R$ is transitive
Therefor R is an equivalence relation.

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* For a fixed integer $n>1$, prove that the relation "congruent modulo n" is an equivalence relation on the set of all integers, $\mathbb{Z}$.

Sol) "a is congruent to b modulo n" written as
$a \equiv b \pmod{n} \Rightarrow n \mid a-b$ or $a-b=kn$, for some $k\in\mathbb{Z}$.

As $n \mid a-a \Rightarrow a \equiv a \pmod{n}$ i.e. aRa, hence R is transitive.

Let aRb,
then $a \equiv b \pmod{n} \Rightarrow n \mid a-b$ or $a-b=kn$
$\Rightarrow b-a=(-k)n \Rightarrow b \equiv a \pmod{n}$
i.e. bRa, hence R is symmetric

Let aRb and bRc
~~then~~ $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$
$\Rightarrow n \mid a-b$ and $n \mid b-c$
$\Rightarrow a-b=k_1 n$ and $b-c=k_2 n$
$\Rightarrow a-b+b-c=k_1 n+k_2 n$
$\Rightarrow a-c=(k_1+k_2)n$
$\Rightarrow n \mid a-c$
$\Rightarrow a \equiv c \pmod{n}$
i.e. aRc, hence R is transitive.

Therefore R ["congruent modulo n] is an equivalence relation.

*

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Let R_1 and R_2 be relations on a set A. Prove that (i) If R_1 and R_2 are reflexive, so are $R_1 \cap R_2$ and $R_1 \cup R_2$,
(ii) If R_1 and R_2 are symmetric, so are $R_1 \cap R_2$ and $R_1 \cup R_2$
(iii) If R_1 and R_2 are antisymmetric, so is $R_1 \cap R_2$.
(iv) If R_1 and R_2 are transitive, so is $R_1 \cap R_2$.

Solⁿ (i) Suppose R_1 and R_2 are reflexive
$\Rightarrow (a,a) \in R_1,\ (a,a) \in R_2 \ \forall a \in A$.
Consiquently $(a,a) \in R_1 \cap R_2$ and $(a,a) \in R_1 \cup R_2$
$\therefore R_1 \cap R_2$ and $R_1 \cup R_2$ are reflexive.

(ii) Suppose R_1 and R_2 are symmetric.
Take any $(a,b) \in R_1 \cap R_2$
then $(a,b) \in R_1$ and $(a,b) \in R_2$
$\therefore (b,a) \in R_1$ and $(b,a) \in R_2$ [$\because R_1$ & R_2 are symmetric]
consequently $(b,a) \in R_1 \cap R_2$
$\therefore R_1 \cap R_2$ is symmetric.
Now let $(a,b) \in R_1 \cup R_2$.
Then $(a,b) \in R_1$ or $(a,b) \in R_2$
$\therefore (b,a) \in R_1$ or $(b,a) \in R_2$ [$\because R_1$ & R_2 are symmetric]
Consequently $(b,a) \in R_1 \cup R_2$
$\therefore R_1 \cup R_2$ is symmetric.

(iii) Suppose R_1 and R_2 are antisymmetric
Take any $(a,b), (b,a) \in R_1 \cap R_2$
Then $(a,b), (b,a) \in R_1$ and $(a,b), (b,a) \in R_2$
$\Rightarrow a=b$ in R_1 and $a=b$ in R_2
$\therefore a=b$ in $(R_1 \cap R_2)$
$\therefore R_1 \cap R_2$ is antisymmetric

(iv) Suppose R_1 and R_2 are transitive.
Take $(a,b), (b,c) \in R_1 \cap R_2$
$\Rightarrow (a,b)\ (b,c) \in R_1$ and $(a,b), (b,c) \in R_2$
$\Rightarrow (a,c) \in R_1$ and $(a,c) \in R_2$ ($\because R_1$ & R_2 are transitive)
$\Rightarrow (a,c) \in R_1 \cap R_2$. $\therefore R_1 \cap R_2$ is transitive.

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* Let $A = \{1,2,3,4\}$ and
$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$
be a relation on A. Verify that R is an equivalence relation.

Solⁿ ~~Given~~ $(1,1),(2,2),(3,3),(4,4) \in R$,
$\Rightarrow (a,a) \in R \ \forall a \in A$. $\therefore$ R is reflexive.

(ii) $(1,2),(2,1),(3,4),(4,3) \in R$
$\Rightarrow$ ~~if~~ $(a,b) \in R$, $(b,a) \in R$ for $a,b \in A$
$\therefore$ R is symmetric

(iii) $(1,2)(2,1),(1,1) \in R$, $(2,1),(1,2),(2,2) \in R$,
$(4,3),(3,4),(4,4) \in R$, $(3,4),(4,3),(3,3) \in R$
~~Thus~~ if $(a,b),(b,c) \in R$ then $(a,c) \in R$ for $a,b,c \in A$
$\therefore$ R is transitive.

Thus R is an equivalence relation.

* Let $A = \{1,2,3,4\}$
and $R = \{(1,1),(1,2)(2,1),(2,2),(3,1)(3,3),(1,3),(4,1)(4,4)\}$
be a relation on A. Is R an equivalence relation?

Solⁿ (i) $(a,a) \in R \ \forall a \in A$. $\therefore$ R is reflexive.
(ii) $(4,1) \in R$ but $(1,4) \notin R$. $\therefore$ R is not symmetric
Hence R is not an equivalence relation.

* If $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{1,2\}$, $A_2 = \{2,3,4\}$ and $A_3 = \{5\}$, define the relation R on A by xRy if and only if x and y are in the same set A_i. Is R an equivalence relation?

Solⁿ Since x and x belong to the same A_i, xRx. $\therefore$ R is reflexive
If $x,y \in A_i$ then $y,x \in A_i$. $\therefore xRy$ and yRx. $\therefore$ R is symmetric
$(1,2) \in R$ ($\because$ 1 and $2 \in A_1$) and $(2,3) \in R$ ($\because$ 2 and $3 \in A_2$)
but $(1,3) \notin R$, as 1 and 3 are not ~~in~~ same A_i.
$\therefore$ R is not transitive.
Hence R is not an equivalence relation

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Union and Intersection of relations

Let R_1 and R_2 be relations from set A to set B. The union of R_1 and R_2 i.e, $R_1 \cup R_2$ is defined by $(a,b) \in R_1 \cup R_2$ if and only if $(a,b) \in R_1$ or $(a,b) \in R_2$.
The intersection of R_1 and R_2 i.e., $R_1 \cap R_2$ is defined by $(a,b) \in R_1 \cap R_2$ if and only if $(a,b) \in R_1$ and $(a,b) \in R_2$.

Complement of a relation

Given a relation R from A to B the complement of R, i.e, $\overline{R}$ is defined with the property that $(a,b) \in \overline{R}$ if and only if $(a,b) \notin R$. $(a,b) \in A \times B, (a,b) \notin R$

Converse of a relation

Let R be a relation from a set A to set B, the converse of R, i.e, R^c is defined with the property that $(a,b) \in R^c$ if and only if $(b,a) \in R$.

Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$
$R_1 = \{(a,1), (b,1), (c,2), (c,3)\}$, $R_2 = \{(a,1), (a,2), (b,1), (b,2)\}$,
find $\overline{R_1}$, R_2^c, $R_1 \cup R_2$, $R_1 \cap R_2$.

Solⁿ.

$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\}$

Then $\overline{R_1} = \{(a,2), (a,3), (b,2), (b,3), (c,1)\}$

$R_2^c = \{(1,a), (2,a), (1,b), (2,b)\}$

$R_1 \cup R_2 = \{(a,1), (a,2), (b,1), (b,2), (c,2), (c,3)\}$

$R_1 \cap R_2 = \{(a,1), (b,1)\}$

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Equivalence classes

Let R be an equivalence relation on a set A and $a \in A$. Then the set of all those elements of A which are related to a by R is called the equivalence class of a with respect to R. The equivalence class is denoted by $R(a)$ or $[a]$ or $\bar{a}$.

Thus $\bar{a} = [a] = R(a) = \{x \in A \mid (x, a) \in R\}$.

example: let $A = \{1, 2, 3\}$

and the equivalence relation $R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$

The elements x of A for which $(x, 1) \in R$ are $x = 1, x = 3$.

$\therefore \{1, 3\}$ is the equivalence class of 1. ie, $[1] = \{1, 3\}$.

Similaryly $[2] = \{2\}$, $[3] = \{1, 3\}$.

example

Consider the relation R on $\mathbb{Z}$ defined by xRy if and only if $x - y$ is even, where R is an equivalence relation,

For any $a \in \mathbb{Z}$, the equivalence class of a (w.r.t. R) is

$[a] = \{x \in \mathbb{Z} \mid xRa\} = \{x \in \mathbb{Z} \mid (x-a) \text{ is even}\}$

$= \{x \in \mathbb{Z} \mid (x-a) = 2k, k \in \mathbb{Z}\} = \{x \in \mathbb{Z} \mid x = a + 2k, k = 0, \pm 1, \pm 2 \ldots\}$

Thus $[0] = \{x \in \mathbb{Z} \mid x = 2k, k = \ldots -2, -1, 0, 1, 2, \ldots\}$

$\therefore [0] = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

Similarly $[1] = \{x \in \mathbb{Z} \mid x = 1 + 2k, k = \ldots -2, -1, 0, 1, 2 \ldots\}$

$\therefore [1] = \{\ldots -3, -1, 1, 3, \ldots\}$.

and $[2] = \{\ldots -4, -2, 0, 2, 4, \ldots\}$

Properties of equivalence classes

1. Let R be an equivalence relation on a set A and $a \in A$, then $a \in [a]$
2. Let R be an equivalence relation on a set A, and let $a, b \in A$, Then aRb if and only if $[a] = [b]$
3. Let R be an equivalence relation on a set A, and let $a, b \in A$. If $[a] \cap [b] \neq \phi$, then $[a] = [b]$

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Partition of a set.

Let A be a nonempty set. Suppose there exist nonempty subsets $A_1, A_2, \ldots A_k$ of A such that the following two conditions hold.

(i) A is the union of $A_1, A_2, \ldots A_k$

i.e. $A = A_1 \cup A_2 \cup \ldots \cup A_k$

(ii) Any two of the subsets $A_1, A_2, \ldots, A_k$ are disjoint

i.e. $A_i \cap A_j = \phi$ for $i \neq j$.

Then the set $P = \{A_1, A_2, \ldots, A_k\}$ is called a partition of A. Also $A_1, A_2, \ldots, A_k$ is called the blocks or cells of the partition.

examples

* Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and its following subsets $A_1 = \{1, 3, 5, 7\}$, $A_2 = \{2, 4\}$, $A_3 = \{6, 8\}$. Observe that $A = A_1 \cup A_2 \cup A_3$. Also any two of the subsets A_1, A_2, A_3 are disjoint. Therefore, $P = \{A_1, A_2, A_3\}$ is a partition of A with A_1, A_2, A_3 as the blocks (cells) of the partition.
* Suppose $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and its subsets are given as $A_1 = \{1, 3, 5\}$ $A_2 = \{2, 4\}$ $A_3 = \{6, 8\}$. Then $P = \{A_1, A_2, A_3\}$ is not a partition of the set A because although the subsets A_1, A_2, A_3 are disjoint A is not the union of the subsets
* Suppose $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and its subsets are given as $A_1 = \{1, 3, 5, 7\}$ $A_2 = \{2, 4\}$, $A_3 = \{5, 6, 8\}$, then $P = \{A_1, A_2, A_3\}$ is not a partition of the set A because although A is the union of A_1, A_2, A_3, the sets A_1 and A_3 are not disjoint.

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Note: If A is a non-empty set, then:
(i) Any equivalence relation R on A induces a partition of A.
(ii) Any partition of A gives rise to an equivalence relation R on A.

* For the equivalence relation $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$ defined on the set $A = \{1,2,3,4\}$, determine the partition induced.

Sol^n^ The equivalence classes of the elements of A w.r.t R are: $[1] = \{1,2\}$, $[2] = \{1,2\}$, $[3] = \{3,4\}$, $[4] = \{3,4\}$
Of these equivalence classes $[1]$ and $[3]$ are distinct. These two distinct equivalence classes constitute the partition $P = \{[1], [3]\}$
i.e. $P = \{\{1,2\}, \{3,4\}\}$

* For the set $A = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ and the relation R ~~on A~~ defined by $(x,y) \in R$ if and only if $x - y$ is a multiple of 5, find the partition of A induced by R.

Sol^n^ The equivalence classes of R are

$$[1] = \{1,6,11\} = [6] = [11]$$
$$[2] = \{2,7,12\} = [7] = [12]$$
$$[3] = \{3,8\} = [8]$$
$$[4] = \{4,9\} = [9]$$
$$[5] = \{5,10\} = [10]$$

Therefore the partition of A induced by R is
$P = \{\{1,6,11\}, \{2,7,12\}, \{3,8\}, \{4,9\}, \{5,10\}\}$

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\# Let $A = \{a, b, c, d, e\}$ Consider the partition $P = \{\{a,b\}, \{c,d\}, \{e\}\}$ of A. Find the equivalence relation inducing this partition.

Sol: Since a, b belong to same block, we have the ordered pairs $(a,a), (a,b), (b,a), (b,b)$

Since c, d belong to same block, we have the ordered pairs $(c,c), (c,d), (d,c), (d,d)$

For the block $\{e\}$ we have (e,e)

Hence the equivalence relation inducing this partition is

$$R = \{(a,a), (b,b), (c,c), (d,d), (e,e), (a,b), (b,a), (c,d), (d,c)\}$$

\# On the set $\mathbb{Z}$ of all integers, a relation R is defined by aRb if and only if $a^2 = b^2$. Verify that R is an equivalence relation. Determine the partition induced by this relation

Sol: (i) For any $a \in \mathbb{Z}$ $a^2 = a^2$, ie, $(a,a) \in R$. Hence R is reflexive.

(ii) Let $(a,b) \in R \Rightarrow a^2 = b^2 \Rightarrow b^2 = a^2 \Rightarrow (b,a) \in R$.

Hence R is symmetric.

(iii) Let $(a,b) \in R$ and $(b,c) \in R \Rightarrow a^2 = b^2$ and $b^2 = c^2$.

$\Rightarrow a^2 = c^2 \Rightarrow (a,c) \in R$. Hence R is transitive.

So. Hence R is an equivalence relation

For any $a \in \mathbb{Z}$, $[a] = \{x \in \mathbb{Z} \mid (x,a) \in R\}$

$= \{x \in \mathbb{Z} \mid x^2 = a^2\}$

$= \{x \in \mathbb{Z} \mid x = \pm a\}$

Thus $[0] = \{0\}$.

$[n] = \{n, -n\}$ for $n \in \mathbb{Z}^+$

also $[-n] = \{-n, n\}$ for $n \in \mathbb{Z}^+$

$\therefore$ The distinct partitions are $[0]$ and $[n]$

Hence $P = \{[0], [n]\}$, where $n \in \mathbb{Z}^+$.

Topic ______ Date ______

zero-one matrices

Consider the sets $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ of orders m and n respectively. Then $A \times B$ consists of all ordered pairs of the form (a_i, b_j), $1 \le i \le m$, $1 \le j \le n$, which are mn in number. Let R be a relation from A to B so that R is a subset of $A \times B$.

Let $m_{ij} = (a_i, b_j)$ and assign the values 1 or 0 to m_{ij} according to the following rule:

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

The $m \times n$ matrix formed by these m_{ij}'s is called the matrix of the relation R, or the relation matrix for R and is denoted by M_R or $M(R)$. Since M_R contains only 0 and 1 as its elements, it is also called the zero-one matrix for R.

example

* Consider the sets $A = \{0, 1, 2\}$ and $B = \{p, q\}$ and the relation R from A to B defined by $R = \{(0, p), (1, q), (2, p)\}$, then the matrix relation of R is

$$M_R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} p & q \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \quad \text{or} \quad M_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

* Consider the set $A = \{1, 2, 3, 4\}$ and the relation R defined by $R = \{(1,2), (1,3), (2,4), (3,2)\}$ then the matrix of the relation R is

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

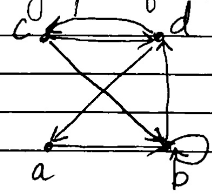
Topic ______ Date ______

Digraph of a relation

Let R be a relation on a finite set A. Then R can be represented pictorially as described. Draw a small circle or a bullet for each element of A and label the circle (bullet) with the corresponding element of A. These circles (bullets) are called vertices or nodes. Draw an arrow, called an edge, from a vertex x to a vertex y if and only if $(x, y) \in R$. The resulting pictorial representation of R is called a directed graph or digraph of R.

If a relation is pictorially represented by a digraph, a vertex from which an edge leaves is called the origin or the source for that edge, and a vertex where an edge ends is called the terminus for that edge. A vertex which is neither a source nor a terminus of any edge is called an isolated vertex. An edge for which the source and terminus are one and the same vertex is called a loop. The number of edges (arrows) terminating at a vertex is called the in-degree of the vertex and the number of edges (arrows) leaving a vertex is called the out-degree of that vertex.

Eg Consider $A = \{a, b, c, d\}$ and the relation $R = \{(a,b), (b,b), (b,d), (c,b), (c,d), (d,a), (d,c)\}$ defined on A. The digraph of this relation is as shown below:

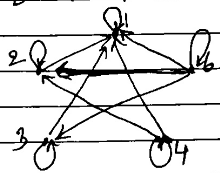


Topic ______________________ Date __________

* Let $A = \{1,2,3,4,6\}$ and R be a relation on A defined by aRb if and only if a is a multiple of b. Represent the relation R as a matrix, draw its digraph and determine the in-degrees and out-degrees of the vertices in the digraph.

Sol^n $R = \{(1,1),(2,1),(2,2),(3,1),(3,3),(4,1),(4,2),(4,4),(6,1),(6,2),(6,3),(6,6)\}$

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$



vertex	indegree	out degree
1	5	1
2	3	2
3	2	2
4	1	3
6	1	4

* Determine the relation R from a set A to a set B as described by the following matrix. ~~Also write the digraph~~

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Sol^n. Since the order of M_R is 4×3, $|A| = 4$ and $|B| = 3$

Let $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3\}$

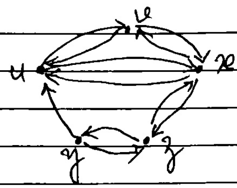
Then $R = \{(a_1,b_1),(a_1,b_3),(a_2,b_1),(a_2,b_2),(a_3,b_3),(a_4,b_1)\}$

Topic ______ Date ______

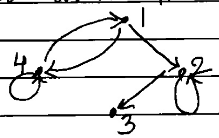
* Let $A = \{u, v, x, y, z\}$ and R be a relation on A whose matrix is as given below. Determine R and also draw the associated digraph.

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Sol^n. $R = \{(u,v), (u,x), (v,u), (v,x) (x,u), (x,v), (x,z), (y,u), (y,z), (z,x), (z,y)\}$



* Find the relation represented by the digraph given below. Also write its matrix



Sol^n

$R = \{(1,2), (1,4), (2,2), (2,3) (4,1), (4,4)\}$

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

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Composition of relations

Consider a relation R_1 from a set A to a set B and a relation R_2 from the set B to a set C. With these relations, we can define a new relation called the product or the composition of R_1 and R_2, denoted by $R_1 o R_2$.

Note. Composition of relations is not commutative but associative.

examples

Q) Let $A = \{1,2,3,4\}$, $B = \{w,x,y,z\}$, $C = \{5,6,7\}$.
Also let R_1 be a relation from A to B and R_2, R_3 be relations from B to C, defined by,

$R_1 = \{(1,x),(2,x),(3,y),(3,z)\}$

$R_2 = \{(w,5),(x,6)\}$, $R_3 = \{(w,5),(w,6)\}$

Then $R_1 o R_2 = \{(1,6),(2,6)\}$ ~~and $R_1 o R_3 = \{\}$~~

$\because (1,x) \in R_1$ & $(x,6) \in R_2 \Rightarrow (1,6) \in R_1 o R_2$

$(2,x) \in R_1$ & $(x,6) \in R_2 \Rightarrow (2,6) \in R_1 o R_2$

Since the second element of an order pair in R_1 is not the first element of an order pair in R_3, ~~the~~ an order pair in R_1 can't compose with an order pair in R_3. Hence $R_1 o R_3 = \{\ \}$.

Q) Let $R = \{(1,2),(3,4),(2,2)\}$ and $R_2 = \{(4,2),(2,5),(3,1),(1,3)\}$ be relations on the set $A = \{1,2,3,4,5\}$.
Find $R_1 o (R_1 o R_2)$, $R_1 o (R_2 o R_1)$, $R_2 o (R_1 o R_2)$, $R_2 o (R_2 o R_1)$

Sol^n

$R_1 o R_2 = \{(1,5),(3,2),(2,5)\}$

$R_2 o R_1 = \{(4,2),(3,2),(1,4)\}$

$R_1 o (R_1 o R_2) = \{(1,5),(2,5)\}$

$R_1 o (R_2 o R_1) = \{(3,2)\}$

$R_2 o (R_1 o R_2) = \{(4,5),(3,5),(1,2)\}$

$R_2 o (R_2 o R_1) = \{(3,4),(1,2)\}$

Topic ______ Date ______

* For the relation R_1 and R_2 where $R_1 = \{(1,x),(2,x),(3,y),(3,z)\}$ and $R_2 = \{(w,5),(x,6)\}$, defined on $A=\{1,2,3,4\}$, $B=\{w,x,y,z\}$, $C=\{5,6,7\}$, find M_{R_1}, M_{R_2}, $M_{R_1 o R_2}$. Also verify that $M_{R_1 o R_2} = M_{R_1} \cdot M_{R_2}$

Sol^n)

$R_1 o R_2 = \{(1,6),(2,6)\}$

$$M_{R_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad M_{R_1 o R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{R_1} \cdot M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = M_{R_1 o R_2}$$

Hence verified

* Let $A = \{a,b,c\}$ and R_1 & R_2 be relations on A, whose matrices are given below. Find the composition relations R_1^2, R_2^2 and their matrices.

$$M(R_1) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad M(R_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Sol^n)

$R_1 = \{(a,a),(a,c),(b,a),(b,b),(b,c),(c,b)\}$

$R_2 = \{(a,a),(b,b),(b,c),(c,a),(c,c)\}$.

$R_1^2 = R_1 o R_1 = \{(a,a),(a,c),(a,b),(b,a),(b,c),(b,b),(c,a),(c,b),(c,c)\}$

$R_2^2 = R_2 o R_2 = \{(a,a),(b,b),(b,c),(b,a),(c,a),(c,c)\}$

$$M(R_1)^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad M(R_2^2) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Topic Date

Partial Orders

A relation R on set A is said to be a partial ordering relation or a partial order on A if (i) R is reflexive, (ii) R is antisymmetric, and (iii) R is transitive on A.

A set A with a partial order R defined on it is called a partially ordered set or an ordered set or a poset, and is denoted by the pair (A, R)

eg 1. The relation "less than or equal to", on the set $\mathbb{Z}$ of all integers is a partial order on $\mathbb{Z}$. $(\mathbb{Z}, \leq)$ is a poset.

2. The "divisibility relation" on the set $\mathbb{Z}^+$ defined by a divides b for all $a, b \in \mathbb{Z}^+$ is a partial order on $\mathbb{Z}^+$.

3. The "subset relation" defined on the power set of a set S is a partial order on S. $(P(S), \subseteq)$ is a poset

4. The relation "is less than" and "is greater than" are not partial orders on $\mathbb{Z}$, because, these are not reflexive.

5. The relation "congruent modulo n" defined on the set of all integers $\mathbb{Z}$ is not a partial order, because this relation is not antisymmetric.

Total order

Let R be a partial order on a set A. Then R is called a total order on A if for all $x, y \in A$, either xRy, or yRx. In this case, the poset (A, R) is called a totally ordered set.

eg 1. The partial order relation "less than or equal to" is a total order on the set $\mathbb{R}$, because for any $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$. Thus $(\mathbb{R}, \leq)$ is a totally ordered set.

2. Consider the divisibility relation on the set $A = \{1, 2, 4, 6, 8\}$. R is a partial order on A, but it is not a total order on A, since neither 4 divides 6 nor 6 divides 4.

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Hasse Diagrams

Since a partial order is a relation on a set, we can think of the digraph of a partial order if the set is finite. The following rules are applied while drawing the digraph of a partial order.

(i) Since a partial order is reflexive at every vertex in the digraph of a partial order there would be a cycle of length 1. In view of this, while drawing the digraph of a partial order, we need not exhibit such cycles explicitly.

(ii) If, in the digraph of a partial order, there is an edge from a vertex a to a vertex b and there is an edge from the vertex b to a vertex c, then there should be an edge from a to c (because of transitivity). As such we need not exhibit an edge from a to c explicitly.

(iii) To simplify the format of the digraph of a partial order, we represent the vertices by dots (bullets) and draw the digraph in such a way that all edges point upward. With this convention, we need not put arrows in the edges.

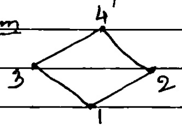
The diagraph of a partial order drawn by adopting the conventions indicated in the above points is called a poset diagram or the Hasse Diagram for the partial order.

Topic ______ Date ______

* Let $A = \{1, 2, 3, 4\}$
and $R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$
Verify that R is a partial order on A. Also write the Hasse diagram for R.

Sol^n $(x,x) \in R \; \forall \; x \in A \; \therefore R$ is reflexive.
R doesnot contain ordered pairs of the form (x,y) and (y,x) with $y \neq x$. $\therefore R$ is antisymmetric.
Also $(x,z) \in R \; \forall (x,y)$ and $(y,z) \in R$. $\therefore R$ is transitive.
Hence R is a partial order on A.

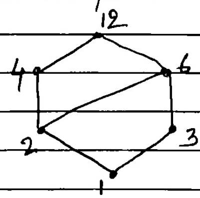
Hasse Diagram



* Let $A = \{1, 2, 3, 4, 6, 12\}$. On A, define the relation R by aRb if and only if a divides b. Prove that R is a partial order on A. Draw the Hasse Diagram from this relation.

Sol^n $R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,12), (2,2), (2,4), (2,6), (2,12), (3,6), (3,12), (4,12), (6,12), (3,3), (4,4), (6,6), (12,12)\}$
$\because (a,a) \in R \; \forall \; a \in A$, R is reflexive.
$\because$ R doesnot contain ordered pairs of the form (a,b) and (b,a) with $b \neq a$, R is antisymmetric.
$\because$ the elements of the form $(a,c) \in R$, whenever (a,b) and $(b,c) \in R$, R is transitive.
Hence R is a partial order on A.

Hasse Diagram



Topic ______ Date ______

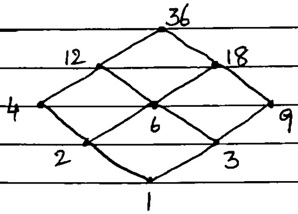
* Draw the Hasse diagram representing the positive divisors of 36, and R is the divisibility relation

Sol. $D = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

R will have the ordered pairs such that:

1 is related to 1, 2, 3, 4, 6, 9, 12, 18, 36;
2 is related to 2, 4, 6, 12, 18, 36;
3 is related to 3, 6, 9, 12, 18, 36;
4 is related to 4, 12, 36; 6 is related to 6, 12, 18, 36;
9 is related to 9, 18, 36; 12 is related to 12, 36;
18 is related to 18, 36; 36 is related to 36.

Hasse Diagram



* Let $S = \{1, 2, 3\}$ and $P(S)$ be the power set of S. On $P(S)$, define the relation R by XRY if and only if $X \subseteq Y$. Show that the relation is a partial order on $P(S)$. Draw its Hasse Diagram.

Sol. $\because A \subseteq A$, $(A, A) \in R$ $\therefore$ R is reflexive.

$\because A \subseteq B$ & $B \subseteq A \Rightarrow A = B$, $(A, B) \in R$ and $(B, A) \in R \Rightarrow B = A$.

$\therefore$ R is antisymmetric

$\because A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$, (A, B) and $(B, C) \in R \Rightarrow (A, C) \in R$

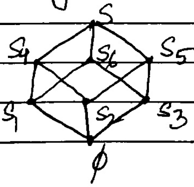
$\therefore$ R is transitive

Hence ~~P(S)~~ R is a partial order on $P(S)$

Hasse Diagram $P(S) = \{\phi, S_1, S_2, S_3, S_4, S_5, S_6, S\}$

where $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$

$S_4 = \{1, 2\}$, $S_5 = \{2, 3\}$, $S_6 = \{3, 1\}$

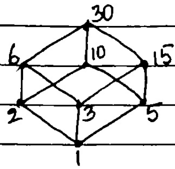


Topic ____ Date ____

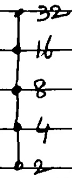
* In the following cases, consider the partial order of divisibility on the set A. Draw the Hasse Diagram for the poset and determine whether the poset is totally ordered or not.

(i) $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$, (ii) $A = \{2, 4, 8, 16, 32\}$

Sol (i)



(ii)



(i) is not a totally ordered set as $(x, y) \notin R\ \forall x, y \in A$

example neither $(2,3)$, nor $(3,2) \in R$.

(ii) is a totally ordered set as $(a,b) \in R\ \forall a, b \in A$.

* Prove that the set of all positive integers is not totally ordered by the relation of divisibility.

Sol For a set A to be totally ordered by a partial order R, we should have aRb or bRa, for every $a, b \in A$.

If R is the divisibility relation on Z^+, aRb or bRa need not hold for every $a, b \in Z^+$.

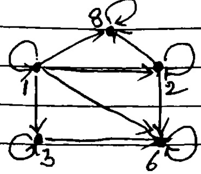
For example, if $a = 2$ and $b = 3$, then

a does not divide b and b does not divide a.

Therefore, Z^+ is not totally ordered by the relation of divisibility.

Topic ______ Date ______

* The digraph for a relation on the set $A = \{1, 2, 3, 6, 8\}$ is as shown below:

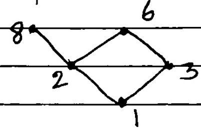


Verify that (A, R) is a poset and write its Hasse Diagram

Sol: $R = \{(1,1), (1,2), (1,3), (1,6), (1,8), (2,2), (2,6), (2,8), (3,3), (3,6), (6,6), (8,8)\}$

R is reflexive, antisymmetric and transitive,
Hence R is a partial order on A. i.e. (A, R) is a poset.

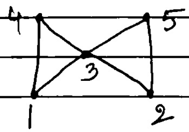
Hasse Diagram



* Draw the Hasse Diagram of the relation R on $A = \{1, 2, 3, 4, 5\}$ whose matrix is as given below!

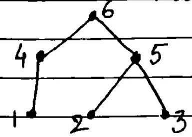
$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Sol: $R = \{(1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (5,5)\}$

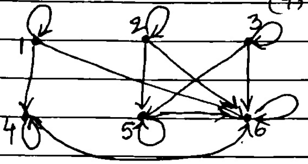


Topic ______ Date ______

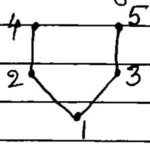
* The Hasse diagram of a partial order R on the set $A = \{1, 2, 3, 4, 5, 6\}$ is as given below. Write R as a subset of $A \times A$. Construct its digraph.



Sol^n $R = \{(1,1), (1,4), (1,6), (2,2), (2,5), (2,6), (3,3), (3,5), (3,6), (4,4), (4,6), (5,5), (5,6), (6,6)\}$



* Determine the matrix of the partial order whose Hasse diagram is given below:



Sol^n $R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,4), (3,3), (3,5), (4,4), (5,5)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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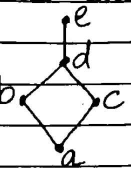
* For $A = \{a, b, c, d, e\}$, the Hasse diagram for the poset (A, R) is as shown below:

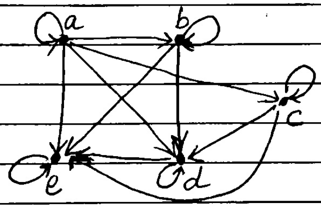


(i) Determine the relation matrix for R
(ii) Construct the digraph for R.

Soln.

$R = \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,b), (b,d), (b,e), (c,c), (c,d), (c,e), (d,d), (d,e), (e,e)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



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Extremal elements in Posets

Consider a poset (A, R).

1. An element $a \in A$ is called a maximal element of A if there exists no element $x \neq a$ in A such that aRx. In other words, $a \in A$ is a maximal element of A if whenever there is $x \in A$ such that aRx then $x = a$.
This means that a is a maximal element of A if and only if in the Hasse diagram of R no edge starts at a.
2. An element $a \in A$ is called a minimal element of A if there exists no element $x \neq a$ in A such that xRa. In other words, a is a minimal element of A if whenever there is $x \in A$ such that xRa, then $x = a$.
This means that a is a minimal element of A if and only if in the Hasse diagram of R no edge terminates at a.
3. An element $a \in A$ is called a greatest element of A if xRa for all $x \in A$.
4. An element $a \in A$ is called a least element of A if aRx for all $x \in A$.
5. An element $a \in A$ is called an upper bound of a subset B of A if xRa for all $x \in B$.
6. An element $a \in A$ is called a lower bound of a subset B of A if aRx for all $x \in B$.
7. An element $a \in A$ is called the least upper bound (LUB) of a subset B of A if (i) a is an upper bound of B (ii) If a' is an upper bound of B then aRa'. LUB is also called supremum
8. An element $a \in A$ is called the greatest lower bound (GLB) of a subset B of A if (i) a is a lower bound of B (ii) If a' is a lower bound of B then $a'Ra$. GLB is also called infimum.

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The following theorems contain some results on extremal elements:

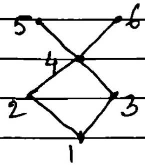
Theorem: If (A, R) is a poset and A is finite, then A has atleast one maximal element and at least one minimal element.

Theorem: Every poset has at most one greatest element and atmost one least element.

Theorem: If (A, R) is a poset and $B \subseteq A$, then B has atmost one LUB and at most one GLB.

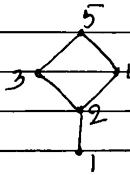
examples

(i)



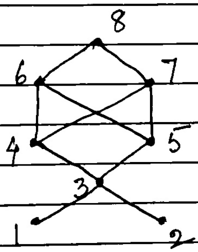
5, 6 are maximal elements
1 is the minimal element
1 is the least element
there is no greatest element

(ii)



5 is the maximal as well as the greatest element.
1 is the minimal as well as the least element.

(iii)



let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$
$B_1 = \{1, 2\}$. $B_2 = \{3, 4, 5\}$

(i) Since $1R3$, $2R3$, 3 is an upper bound of B_1, Similarly 4, 5, 6, 7, 8 are also upper bounds of B_1.

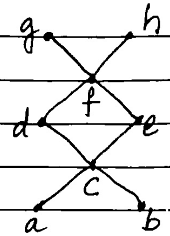
(ii) Since 3 is the least among all the upper bounds of B_1, $LUB(B_1) = 3$.

(iii) Since there is no element x such that $xR1$, $xR2$, B_1 and no lower bounds, and hence no greatest lower bound.

(iii) 6, 7, 8 are upper bounds of B_2 and $6 \not R 7$, hence no least upper bound.

(iv) 1, 2, 3 are lower bounds of B_2 and 3 is the greatest lower bound of B_2

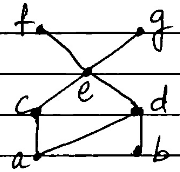
* Consider the Hasse diagram of a poset (A, R) given below:



If $B = \{c, d, e\}$, find (i) all upper bounds of B, (ii) all lower bounds of B, (iii) the least upper bound of B, (iv) the greatest lower bound of B.

Sol^n ~~upper~~ bounds of B are f, g, h
lower bound of B are a, b, c
the least upper bound of B is f
the greatest lower bound of B is **c**.

* Consider the poset whose Hasse diagram is shown below. Find LUB and GLB of $B = \{c, d, e\}$



Sol^n all upper bounds of B are e, f, g, hence $LUB = e$
all lower bounds ~~of~~ B are a, hence $GLB = a$

* Let $\mathbb{R}$ be the set of all real numbers with $\leq$ as the partial order. Also, let B be the open interval $(1, 2)$. Find (i) all upper bounds of B, (ii) all lower bounds of B, (iii) the LUB of B, (iv) the GLB of B

Sol^n (i) all upper bounds of B = every real number ≥ 2
(ii) all lower bounds of B = every real number ≤ 1.
(iii) $LUB(B) = 2$, (iv) $GLB(B) = 1$

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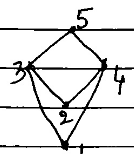
Lattice

Let (A, R) be a poset. This poset is called a lattice, if for all $x, y \in A$, the element $LUB\{x,y\}$ and $GLB\{x,y\}$ exist in A.

* Consider the set N of all natural numbers, and let R be the partial order "less than or equal to". Then for any $x, y \in N$, we note that $LUB\{x,y\} = \max\{x,y\}$ and $GLB\{x,y\} = \min\{x,y\}$ and both of these belong to N. Therefore, the poset $(N, \leq)$ is a lattice.
* Consider the poset whose Hasse diagram is shown below.



since the $GLB\{3,4\}$ does not exist, the above poset is not a lattice.