1. Orthogonal Vectors :-

Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are orthogonal to each other if $\vec{u}\cdot\vec{v}=0$

ex $\vec{u}=(1,2)$ and $\vec{v}=(6,-3)$ are orthogonal in $\mathbb{R}^2$, as

$$\vec{u}\cdot\vec{v} = (1,2)\cdot(6,-3) = 6-6=0$$

A set of vectors $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ are mutually orthogonal if every pair of vectors is orthogonal.

ie, $\vec{v_i}\cdot\vec{v_j}=0$, for all $i\neq j$.

The set of vectors $(1,0,-1)$, $(1,\sqrt{2},1)$, $(1,-\sqrt{2},1)$ are mutually orthogonal, since

$$(1,0,-1)\cdot(1,\sqrt{2},1) = 1+0-1=0$$
$$(1,0,-1)\cdot(1,-\sqrt{2},1) = 1+0-1=0$$
$$(1,\sqrt{2},1)\cdot(1,-\sqrt{2},1) = 1-2+1=0$$

3. Orthogonal Subspaces

Subspace S is orthogonal to subspace T means: every vector is S is orthogonal to every vector in T.

ex: In a plane, the space containing only the zero vector and any line through the origin are orthogonal subspaces.

A line through the origin and the whole plane are never orthogonal subspaces.

Two lines through the origin are orthogonal subspaces if they meet at right angles.

The row space of a matrix is orthogonal to the nullspace, because $Ax = 0$ means the dot product of x with each row of A is 0.
But then the product of x with any combination of rows of A must be 0.

The columnspace is orthogonal to the left nullspace of A because the rowspace of A^T is perpendicular to the nullspace of A^T, as $A^Ty = 0$.

$A_{m\times n}$
$N(A) \to \mathbb{R}^n$
$C(A) \to \mathbb{R}^m$
$R(A) \to \mathbb{R}^n$
$N(A^T) \to \mathbb{R}^m$
$A^T_{n\times m}$

ex: $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}_{2\times 3}$

$R_2 - 2R_1$ $\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

Row space has dimension 1 with basis $\{(1, 2, 5)\}$

$Ax = 0 \Rightarrow x_1 + 2x_2 + 5x_3 = 0 \Rightarrow x_1 = -2x_2 - 5x_3$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$

Nullspace has dimension 2 with basis $\{(-2,1,0), (-5,0,1)\}$ which is orthogonal to the rowspace $\begin{Bmatrix} \because (-2,1,0)\cdot(1,2,5) = 0 \\ (-5,0,1)\cdot(1,2,5) = 0 \end{Bmatrix}$

Not only is the nullspace orthogonal to the rowspace, their dimensions add up to the dimension of the whole space. The nullspace and the rowspace are orthogonal complements in $\mathbb{R}^n$.
Similarly the column space and the left nullspace are orthogonal complements in $\mathbb{R}^m$

Orthogonal complement :-

Let V be a subspace of $\mathbb{R}^n$.

The set $V^{\perp} = \{ \vec{w} \mid \vec{w} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in V \}$

is called the orthogonal complement of V.

Note:-

* A vector $\vec{w}$ is in $V^{\perp}$ iff $\vec{w}$ is orthogonal to every vector in a set that spans V.
* $V^{\perp}$ is a subspace of $\mathbb{R}^n$.

2. Orthogonal Sets :-

A set of vectors $\{u_1, \cdots u_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if $u_i \cdot u_j = 0$ whenever $i \neq j$.

ex $\{u_1, u_2, u_3\}$ such that $u_1 = (3,1,1)$, $u_2 = (-1,2,1)$,

$u_3 = \left(-\frac{1}{2}, -2, \frac{7}{2}\right)$.

$$u_1 \cdot u_2 = (3,1,1) \cdot (-1,2,1) = -3 + 2 + 1 = 0$$

$$u_1 \cdot u_3 = (3,1,1) \cdot \left(-\frac{1}{2}, -2, \frac{7}{2}\right) = -\frac{3}{2} - 2 + \frac{7}{2} = 0$$

$$u_2 \cdot u_3 = (-1,2,1) \cdot \left(-\frac{1}{2}, -2, \frac{7}{2}\right) = \frac{1}{2} - 4 + \frac{7}{2} = 0$$

Each pair of distinct vectors is orthogonal, and so $\{u_1, u_2, u_3\}$ is an orthogonal set.

If $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Proof: If $c_1u_1 + c_2u_2 + \cdots + c_pu_p = 0$, for scalars ~~$u$~~, $c_1, \ldots, c_p$,

then $(c_1u_1 + c_2u_2 + \cdots + c_pu_p).u_1 = 0.u_1$

$\Rightarrow (c_1u_1).u_1 + (c_2u_2).u_1 + \cdots + (c_pu_p).u_1 = 0.u_1$

$\Rightarrow c_1(u_1.u_1) + c_2(u_2.u_1) + \cdots + c_p(u_p.u_1) = 0.u_1$

$\Rightarrow c_1(u_1.u_1) = 0$ $\left[\because u_2.u_1 = \cdots = u_p.u_1 = 0 \text{ as } \{u_1, \ldots u_p\} \text{ is an orthogonal set}\right]$.

$\Rightarrow c_1 = 0$

Similarly $c_2 = \cdots = c_p = 0$

$\therefore$ S is linearly independent.

4. Orthogonal basis:-

An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

ex: $S = \{u_1, u_2, u_3\}$, $u_1 = (3,1,1)$, $u_2 = (-1,2,1)$, $u_3 = (-\frac{1}{2}, -2, \frac{7}{2})$

is an orthogonal basis for $\mathbb{R}^3$ as (i) S is an orthogonal set and (ii) S forms a basis of $\mathbb{R}^3$.

$$\begin{vmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ -\frac{1}{2} & -2 & \frac{7}{2} \end{vmatrix} = 3(7+2) - 1\left(-\frac{7}{2} + \frac{1}{2}\right) + 1(2+1) = 27 + 3 + 3 = 33 \neq 0$$

3. Orthonormal Sets

A set $\{u_1, \dots, u_p\}$ is an orthonormal set if it is an orthogonal set of unit vectors. If W is the subspace spanned by such a set, then $\{u_1, \dots, u_p\}$ is an orthonormal basis for W, since the set is automatically linearly independent.

ex $\{e_1, \dots, e_n\}$, the standard basis for $\mathbb{R}^n$, is an orthonormal set.

Any nonempty subset of $\{e_1, \dots, e_n\}$ is orthonormal, too.

example:

5. Show that $\{v_1, v_2, v_3\}$ is an orthonormal basis of $\mathbb{R}^3$, where $v_1 = \left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$, $v_2 = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, $v_3 = \left(-\frac{1}{\sqrt{66}}, -\frac{4}{\sqrt{66}}, \frac{7}{\sqrt{66}}\right)$

$$v_1 \cdot v_2 = -\frac{3}{\sqrt{66}} + \frac{2}{\sqrt{66}} + \frac{1}{\sqrt{66}} = 0$$

$$v_1 \cdot v_3 = -\frac{3}{\sqrt{726}} - \frac{4}{\sqrt{726}} + \frac{7}{\sqrt{726}} = 0$$

$$v_2 \cdot v_3 = \frac{1}{\sqrt{396}} - \frac{8}{\sqrt{396}} + \frac{7}{\sqrt{396}} = 0$$

Thus $\{v_1, v_2, v_3\}$ is an orthogonal set.

$$v_1 \cdot v_1 = \frac{9}{11} + \frac{1}{11} + \frac{1}{11} = 1$$

$$v_2 \cdot v_2 = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = 1$$

$$v_3 \cdot v_3 = \frac{1}{66} + \frac{16}{66} + \frac{49}{66} = 1$$

which shows that v_1, v_2 and v_3 are unit vectors.

Thus $\{v_1, v_2, v_3\}$ is an orthonormal set.

Since the set is linearly independent, its three vectors form a basis for $\mathbb{R}^3$.

ex: Show that $\{u_1, u_2\}$, where $u_1 = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ &

$u_2 = \left(\frac{2}{\sqrt{5}}\ \frac{1}{\sqrt{5}}\right)$ is an orthonormal basis for $\mathbb{R}^2$.

6. Orthogonal matrix

A square matrix A with real entries and satisfying the condition $A^{-1} = A^T$ is called an orthogonal matrix.

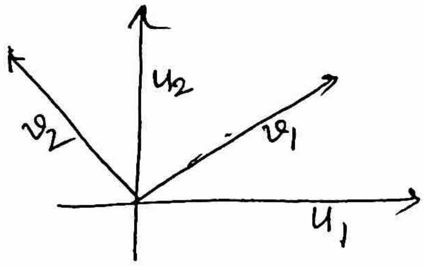
The vectors $u_1 = (1,0)$ and $u_2 = (0,1)$ form an orthonormal basis $B = \{u_1, u_2\}$.

Rotating the vectors u_1 and u_2 anticlockwise by an ange θ, we obtain ~~$v_1 =$~~ $\quad Q_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$

$v_1 = (\cos\theta, \sin\theta)$ and $v_2 = (-\sin\theta, \cos\theta)$.

Then $C = \{v_1, v_2\}$ is also an orthonormal basis

The transition matrix from the basis C to the basis B is given by



$$\cancel{P_{B\leftarrow C}} \left[\begin{array}{cc|cc} 1 & 0 & \cos\theta & -\sin\theta \\ 0 & 1 & \sin\theta & \cos\theta \end{array}\right]$$

$$P_{B\leftarrow C} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

~~Clearly~~ $P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\qquad P^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Clearly $P^{-1} = P^T$.

$\therefore$ P is an orthogonal matrix.

* Suppose that $B = \{u_1, \ldots, u_n\}$ and $C = \{v_1, \ldots, v_n\}$ are two orthonormal bases of a vectorspace V. Then the transition matrix P from the basis C to the basis B is an orthogonal matrix.

example:

The matrix $A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ is orthogonal,

Since $A^T A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The row vector of A, namely $(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$, $(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3})$ and $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ are orthonormal.

So are the column vectors of A.

* Suppose that A is an $n \times n$ matrix with real entries. Then (a) A is orthogonal iff the row vectors of A form an orthonormal basis of $\mathbb{R}^n$.

(b) A is orthogonal iff the column vectors of A form an orthonormal basis of $\mathbb{R}^n$.

ex:- Show that the matrix $U = \begin{bmatrix} \frac{3}{\sqrt{11}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{66}} \\ \frac{1}{\sqrt{11}} & \frac{2}{\sqrt{6}} & -\frac{4}{\sqrt{66}} \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{6}} & \frac{7}{\sqrt{66}} \end{bmatrix}$ is an orthogonal matrix.

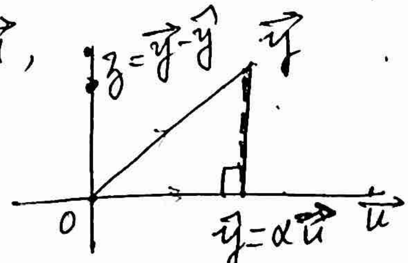
$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

7. Orthogonal Projections:

Given a nonzero vector $\vec{u}$ in $\mathbb{R}^n$, consider the problem of decomposing a vector $\vec{y}$ in $\mathbb{R}^n$ into the sum of two vectors, one a multiple of $\vec{u}$ and the other orthogonal to $\vec{u}$. We wish to write

$\vec{y} = \overset{\alpha\vec{u}}{\hat{y}} + \vec{z}$ — ① where $\hat{y} = \alpha\vec{u}$. for some scalar α and $\vec{z}$ is some vector orthogonal to $\vec{u}$.

Given any scalar α, let $\vec{z} = \vec{y} - \alpha\vec{u}$, so that ① is satisfied.



Then $\overset{\vec{z}}{\vec{y} - \hat{y}}$ is orthogonal to $\vec{u}$ iff

Finding α to make $y - \hat{y}$ orthogonal to $\vec{u}$.

$$0 = (\vec{y} - \alpha\vec{u}) \cdot \vec{u} = \vec{y} \cdot \vec{u} - (\alpha\vec{u}) \cdot \vec{u}$$
$$= \vec{y} \cdot \vec{u} - \alpha(\vec{u} \cdot \vec{u})$$

That is, ① is satisfied with $\vec{z}$ orthogonal to $\vec{u}$

iff $\alpha = \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$ and $\overset{\alpha\vec{u}}{\hat{y}} = \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$.

The vector $\alpha\vec{u} = \hat{y}$ (denoted as $\hat{y}$) is called the orthogonal projection of $\vec{y}$ onto $\vec{u}$, and the vector $\vec{z}$ is called the component of $\vec{y}$ orthogonal to $\vec{u}$.

ex:- Let $\vec{y} = (7,6)$ and $\vec{u} = (4,2)$.
Find the orthogonal projection of $\vec{y}$ onto $\vec{u}$.
Then write $\vec{y}$ as the sum of two orthogonal vectors,
one in Span$\{\vec{u}\}$ and one orthogonal to $\vec{u}$.

Solⁿ: $\vec{y}\cdot\vec{u} = (7,6).(4,2) = 28+12 = 40$

$\vec{u}\cdot\vec{u} = (4,2).(4,2) = 16+4 = 20.$

The orthogonal projection of $\vec{y}$ onto $\vec{u}$ is

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{40}{20}\vec{u} = 2\vec{u} = 2(4,2) = (8,4).$$

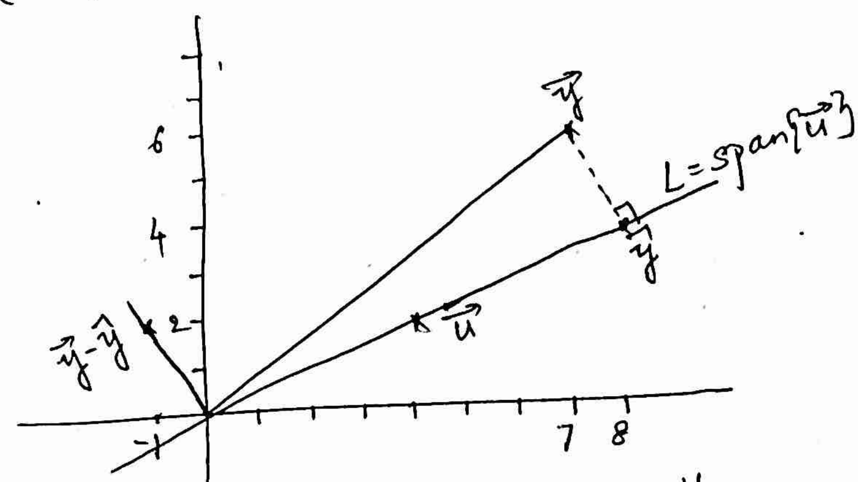
The component of $\vec{y}$ orthogonal to $\vec{u}$ is

$$\vec{y} - \hat{y} = (7,6) - (8,4) = (-1,2)$$

The component of $\vec{y}$ in span$\{\vec{u}\}$ is

$$\alpha\vec{u} = 2(4,2) = (8,4).$$

$$\therefore \quad \vec{y} = \alpha\vec{u} + (\vec{y} - \hat{y})$$
$$= (8,4) + (-1,2)$$

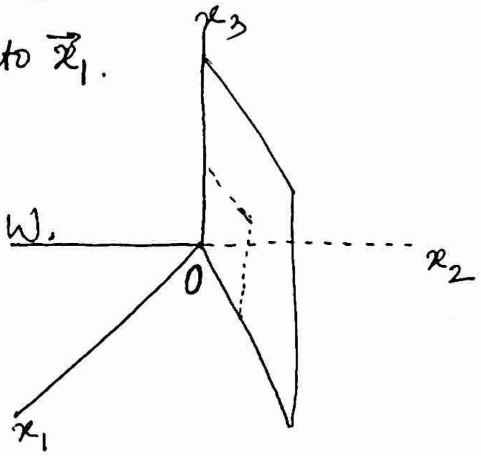


ex: Let $\vec{y} = (2,3)$ and $\vec{u} = (4,-7)$. Write $\vec{y}$ as the sum of two orthogonal vectors, one in span$\{\vec{u}\}$ and a vector orthogonal to $\vec{u}$.

Gram-Schmidt Orthogonalization.

The Gram-Schmidt process is a single algorithm for producing an orthogonal or orthonormal basis for any nonzero subspace of $\mathbb{R}^n$.

ex! Let $W = \text{Span}\{x_1, x_2\}$ where $x_1 = (3,6,0)$ and $x_2 = (1,2,2)$. Construct an orthogonal basis $\{v_1, v_2\}$ for W.



Let $\vec{p}$ be the projection of $\vec{x}_2$ onto $\vec{x}_1$.

The component of $\vec{x}_2$ orthogonal to $\vec{x}_1$ is $\vec{x}_2 - \vec{p}$, which is in W.

Let $\vec{v}_1 = \vec{x}_1$ and

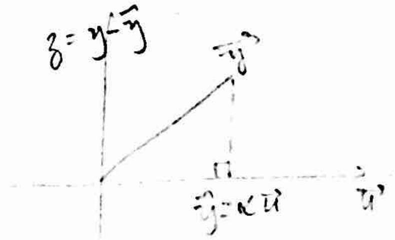
$$\vec{v}_2 = \vec{x}_2 - \vec{p}$$

$$= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1}\vec{x}_1$$

$$= (1,2,2) - \frac{(1,2,2)\cdot(3,6,0)}{(3,6,0)\cdot(3,6,0)}(3,6,0)$$

$$\vec{v}_2 = (0,0,2).$$

Then $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal set of nonzero vectors in W. Since $\dim W = 2$, the set $\{\vec{v}_1, \vec{v}_2\}$ is a basis in W.



$$\hat{y} = \frac{y \cdot u}{u \cdot u} u$$

ex: Let $W = \text{Span}\{v_1, v_2\}$ where $v_1 = (1,1)$ & $v_2 = (2,-1)$.
Construct an orthogonal basis $\{u_1, u_2\}$ for W.

Sol^n: Set $u_1 = v_1$
$$u_1 = (1,1)$$

and
$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$$

$$= (2,-1) - \frac{(2,-1)\cdot(1,1)}{(1,1)\cdot(1,1)}(1,1)$$

$$= \left(\frac{3}{2}, -\frac{3}{2}\right)$$

ex: Let $W = \text{Span}\{v_1, v_2, v_3\}$, where $v_1 = (0,1,2)$, $v_2 = (1,1,2)$, $v_3 = (1,0,1)$. Construct an orthogonal basis $\{u_1, u_2, u_3\}$ for W.

Sol^n: Set $u_1 = v_1$
$$u_1 = (0,1,2)$$

and
$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$$
$$= (1,1,2) - \frac{(1,1,2)\cdot(0,1,2)}{(0,1,2)\cdot(0,1,2)}(0,1,2)$$

$$u_2 = (1,0,0)$$

and
$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 \cancel{+} - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= (1,0,1) - \frac{(1,0,1)\cdot(0,1,2)}{(0,1,2)\cdot(0,1,2)}(0,1,2) - \frac{(1,0,1)\cdot(1,0,0)}{(1,0,0)\cdot(1,0,0)}(1,0,0)$$

$$= (1,0,1) - \frac{2}{5}(0,1,2) - (1,0,0)$$

$$u_3 = \left(0, -\frac{2}{5}, \frac{1}{5}\right)$$

The Gram-Schmidt Process [gram-shmit]

Given a basis $\{x_1, \ldots, x_p\}$ for a subspace W of $\mathbb{R}^n$, define

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

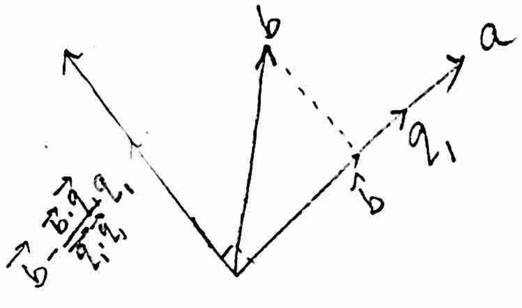
$$\vdots$$

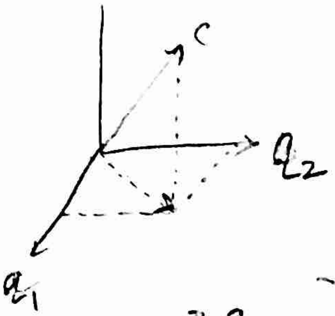
$$v_p = v_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}$$

Then $\{v_1, \ldots, v_p\}$ is an orthogonal basis for W.

In addition, $\text{Span}\{v_1, \ldots v_k\} = \text{Span}\{x_1, \ldots x_k\}$ for $1 \le k \le p$.

The construction, which converts a skewed set of axes into a perpendicular set, is known as Gram-Schmidt Orthogonalization.





$$\vec{c} - \left(\frac{\vec{c}\cdot q_1}{q_1 \cdot q_1} q_1 + \frac{\vec{c}\cdot q_2}{q_2 \cdot q_2} q_2\right)$$

QR Factorization

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for Col A and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

example:-

Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

sol^n: Constructing an ortho~~gonal~~normal basis for Col A.

The columns of A are the vectors $\{x_1, x_2, x_3\}$.

let $v_1 = x_1 = (1,1,1,1)$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= (0,1,1,1) - \frac{(0,1,1,1)\cdot(1,1,1,1)}{(1,1,1,1)\cdot(1,1,1,1)}(1,1,1,1)$$

$$= (0,1,1,1) - \frac{3}{4}(1,1,1,1)$$

$$v_2 = \left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= (0,0,1,1) - \frac{(0,0,1,1)\cdot(1,1,1,1)}{(1,1,1,1)\cdot(1,1,1,1)}(1,1,1,1) - \frac{(0,0,1,1)\cdot\left(-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)}{\left(-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\left(-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)}$$

$$\left(-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$$

$$= (0,0,1,1) - \frac{2}{4}(1,1,1,1) - \frac{\frac{2}{4}}{\frac{12}{16}}\left(-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$$

$$0 - \frac{2}{4} + \frac{2}{4}$$

$$0 - \frac{2\times3}{4\times3} - \frac{2}{12} = -\frac{8}{12}$$

$$\frac{1\times12}{12} - \frac{2\times3}{4\times3} - \frac{2}{12} = \frac{4}{12}$$

$$1 - \frac{2}{4} - \frac{2}{12} = \frac{4}{12}$$

$$v_3 = \left(0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$\therefore \{v_1, v_2, v_3\}$ forms an orthogonal basis of Col A.

$\{(1,1,1,1), (-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}), (0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3})\}$

$\sqrt{1+1+1+1} = \sqrt{4} = 2$

$\sqrt{\frac{9}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}} = \sqrt{\frac{12}{16}} = \frac{\sqrt{3}}{2}$

$\sqrt{0+\frac{4}{9}+\frac{1}{9}+\frac{1}{9}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{2}}{\sqrt{3}}$

and

$$\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(-\frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right), \left(0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right\}$$

forms an orthonormal basis of Col A.

$$\therefore \quad Q = \begin{bmatrix} \frac{1}{2} & -\frac{3}{\sqrt{12}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

To construct an upper triangular invertible matrix

We have, $A = QR \Rightarrow Q^T A = Q^T Q R \Rightarrow Q^T A = IR$

$\Rightarrow Q^T A = R$ ie, $R = Q^T A$

$$\therefore R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{3}{\sqrt{12}} & \frac{2}{\sqrt{12}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

example 1

Find a QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

Sol^n $\{x_1, x_2, x_3\}$ are the columns of the matrix A.

let $v_1 = x_1 = (1, -1, -1, 1, 1)$

$2-1-4-4+2=-5$
$1+1+1+1+1=5$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= (2, 1, 4, -4, 2) - \frac{(2,1,4,-4,2).(1,-1,-1,1,1)}{(1,-1,-1,1,1).(1,-1,-1,1,1)}(1,-1,-1,1,1)$$

$2+1,\ 1-1,\ 4-1, -4+1,\ 2+1$

$$=(2,1,4,-4,2) - \frac{(-5)}{(5)}(1,-1,-1,1,1)$$

$v_2 = (3, 0, 3, -3, 3)$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= (5,-4,-3,7,1) - \frac{(5,-4,-3,7,1).(1,-1,-1,1,1)}{(1,-1,-1,1,1).(1,-1,-1,1,1)}(1,-1,-1,1,1) - \frac{(5,-4,-3,7,1).(3,0,3,-3,3)}{(3,0,3,-3,3).(3,0,3,-3,3)}(3,0,3,-3,3)$$

$5+4+3+7+1$
$15+0-9-21+3$
$9+0+9+9+9$

$$=(5,-4,-3,7,1) - \frac{20}{5}(1,-1,-1,1,1) - \frac{(-12)}{36}(3,0,3,-3,3)$$

$5-4+1 \quad -3+4+1$
$-4+4+0 \quad 7-4-1$
$1-4+1$

$v_3 = (2, 0, 2, 2, -2)$

$\{(1,-1,-1,1,1), (3,0,3,-3,3), (2,0,2,2,-2)\}$ forms an orthogonal basis of Col A.

$\{(\frac{1}{\sqrt5}, -\frac{1}{\sqrt5}, -\frac{1}{\sqrt5}, \frac{1}{\sqrt5}, \frac{1}{\sqrt5}), (\frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$ forms an orthonormal basis of Col A.

$$\therefore \quad Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{5}} & 0 & 0 \\ -\frac{1}{\sqrt{5}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{5}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{5}} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{5}} & 0 & 0 \\ -\frac{1}{\sqrt{5}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{5}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{5}} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

$1+0+2+2+1$

$\frac{5-0-3-7+1}{2}$

$\frac{5+0-3+7-1}{2}$

$$R = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

example

Find an orthogonal basis for the column space of the matrix $\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$

Sol^n: The columns of A are the vectors $\{x_1, x_2, x_3\}$

Let $v_1 = (3, 1, -1, 3)$

$x_1 = (3,1,-1,3)$, $x_2 = (-5,1,5,-7)$

$x_3 = (1,1,-2,8)$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= (-5,1,5,-7) - \frac{(-5,1,5,-7)\cdot(3,1,-1,3)}{(3,1,-1,3)\cdot(3,1,-1,3)}(3,1,-1,3)$$

$$= (-5,1,5,-7) - \frac{(-40)}{(20)}(3,1,-1,3)$$

$-5+6 \mid 1+2$

$5-2 \mid -7+6$

$v_2 = (1, 3, 3, -1)$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1}(v_1) \mp \frac{x_3 \cdot v_2}{v_2 \cdot v_2}(v_2)$$

$$= (1,1,-2,8) - \frac{(1,1,-2,8)\cdot(3,1,-1,3)}{(3,1,-1,3)\cdot(3,1,-1,3)}(3,1,-1,3) - \frac{(1,1,-2,8)\cdot(1,3,3,-1)}{(1,3,3,-1)\cdot(1,3,3,-1)}(1,3,3,-1)$$

$$= (1,1,-2,8) - \frac{30}{20}(3,1,-1,3) - \frac{(-10)}{20}(1,3,3,-1)$$

$1 - \frac{9}{2} + \frac{1}{2} = -\frac{6}{2}$

$1 - \frac{3}{2} + \frac{3}{2} = 1$

$-2 + \frac{3}{2} + \frac{3}{2} = \frac{4}{2}$

$8 - \frac{9}{2} - \frac{1}{2} = \frac{6}{2}$

$v_3 = (-3, 1, 1, 3)$

$\{(3,1,-1,3), (1,3,3,-1), (-3,1,1,3)\}$ is an orthogonal basis for the column space of the given matrix.

example

Find an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$$

The columns of A are $\{x_1, x_2, x_3\}$,

where $x_1 = (-1,3,1,1)$, $x_2 = (6,-8,-2,-4)$, $x_3 = (6,3,6,-3)$

let $v_1 = (-1,3,1,1)$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= (6,-8,-2,-4) - \frac{(6,-8,-2,-4)\cdot(-1,3,1,1)}{(-1,3,1,1)\cdot(-1,3,1,1)}(-1,3,1,1)$$

$$= (6,-8,-2,-4) - \frac{(-\cancel{36}^{3})}{\cancel{12}}(-1,3,1,1)$$

$-6-24-2-4$

$1+9+1+1$

$6-3, -8+9$

$-2+3, -4+3$

$$v_2 = (3,1,1,-1)$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= (6,3,6,-3) - \frac{(6,3,6,-3)\cdot(-1,3,1,1)}{(-1,3,1,1)\cdot(-1,3,1,1)}(-1,3,1,1) - \frac{(6,3,6,-3)\cdot(3,1,1,-1)}{(3,1,1,-1)(3,1,1,-1)}(3,1,1,-1)$$

$$= (6,3,6,-3) - \frac{\cancel{6}}{\cancel{12}_2}(-1,3,1,1) - \frac{\cancel{30}^{5}}{\cancel{12}_2}(3,1,1,-1)$$

$-6+9+6-3$

$18+3+6+3$

$9+1+1-1$

$$= (6,3,6,-3) - \left(-\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{15}{2}, \frac{5}{2}, \frac{5}{2}, -\frac{5}{2}\right)$$

$6 + \frac{1}{2} - \frac{15}{2} = \cancel{\frac{12+1-15}{2}}$

$3 - \frac{3}{2} - \frac{5}{2}$

$6 - \frac{1}{2} - \frac{5}{2} = \frac{12-1-6}{2}$

$-3 - \frac{1}{2} + \frac{5}{2} = \cancel{-6+1+6}$

$$v_3 = \left(\cancel{-\frac{5}{2}}, \cancel{-\frac{3}{2}}, \cancel{\frac{5}{2}}, \cancel{-\frac{1}{2}}\right) \quad (-1,-1,3,-1)$$

$\{(-1,3,1,1), (3,1,1,-1), (-1,-1,3,-1)\}$ is an orthogonal basis for the column space of the given matrix.

Eigen values and Eigen vectors

If A is a square matrix of order n, we can find the matrix $A-\lambda I$, where I is the n^{th} order unit matrix. The determinant of this matrix equated to zero, i.e.,

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

is called the characteristic equation of A.

On expanding the determinant, the characteristic equation takes the form

$$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \cdots + k_n = 0 ,$$

where k^s are expressible in terms of the elements a_{ij}. The roots of this equation are called the characteristic roots or latent roots or eigen-values of the matrix A.

$\frac{dv}{dt} = 4v - 5w$, $v = 8$ at $t = 0$

$\frac{dw}{dt} = 2v - 3w$, $w = 5$ at $t = 0$

$u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$, $u(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$, $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

$\frac{du}{dt} = Au$, $u = u(0)$ at $t = 0$

$v(t) = e^{\lambda t} x_1$

$w(t) = e^{\lambda t} x_2$

$u(t) = e^{\lambda t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\lambda e^{\lambda t} x_1 = 4e^{\lambda t} x_1 - 5e^{\lambda t} x_2$

$\lambda e^{\lambda t} x_2 = 2e^{\lambda t} x_1 - 3e^{\lambda t} x_2$

$4x_1 - 5x_2 = \lambda x_1$

$2x_1 - 3x_2 = \lambda x_2$

$Ax = \lambda x$

$(A - \lambda I)x = 0$

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

then the linear transformation $Y = AX$ — ① carries the column vector X into the column vector Y by means of the square matrix A.

In practice it is often required to find such vectors which transform into themselves or to a scalar multiple of themselves.

Let X be such a vector which transforms into λX by means of the transformation ①.

Then, $\lambda X = AX$ or $AX - \lambda I X = 0$ or $[A - \lambda I]X = 0$ — ②

The matrix equation represents n homogeneous linear equations,

$$\left.\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n &= 0 \end{aligned}\right\} \text{ — ③}$$

which will have a non-trivial solution only if the coefficient matrix is singular.

i.e, if $|A - \lambda I| = 0$

This is called the characteristic equation of the transformation and is same as the characteristic equation of the matrix A.

It has n roots and corresponding to each root, the equation (2) (or equation (3)) will have a non-zero solution, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, which is known as the eigen vector or latent vector.

Observation 1:-

Corresponding to n distinct eigen values, we get n independent eigen vectors. But when two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated roots.

Observation 2:-

If X_i is a solution for a eigen value λ_i then it follows from (2) that cX_i is also a solution, where c is an arbitrary constant. Thus the eigen vector corresponding to an eigen value is not unique, but may be any one of the vectors cX_i

Problems

1. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Solution:-

The characteristic equation is $|A-\lambda I|=0$.

ie, $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix}=0$

$\Rightarrow \quad (5-\lambda)(2-\lambda)-4=0$

$\Rightarrow \quad 10-5\lambda-2\lambda+\lambda^2-4=0$

$\Rightarrow \quad \lambda^2-7\lambda+6=0$

$\Rightarrow \quad (\lambda-6)(\lambda-1)=0$

$\Rightarrow \quad \lambda=1,\ \lambda=6$ are the eigen values.

If $X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigen vector corresponding to the eigen value λ, then

$$[A-\lambda I]X=0$$

$$\Rightarrow \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda=1$, we have

$$\begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad \left.\begin{array}{l} 4x_1+4x_2=0 \\ x_1+x_2=0 \end{array}\right\}$

or $\quad x_1=-x_2$

let $x_2=k$, then $x_1=-k$.

$\therefore \quad X=\begin{bmatrix} -k \\ k \end{bmatrix}$ is the eigen vector corresponding to $\lambda=1$.

For $\lambda = 6$, we have

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x_1 + 4x_2 = 0$

or $\quad x_1 = 4x_2$

let $x_2 = k$, then $x_1 = 4k$

$\therefore \; X = \begin{bmatrix} 4k \\ k \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 6$.

2. Find the eigen values and eigenvectors of the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Solution:-

The characteristic equation is $|A - \lambda I| = 0$

is $\quad \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (1-\lambda)(4-\lambda) - 4 = 0$

$\Rightarrow \quad 4 - \lambda - 4\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \quad \lambda^2 - 5\lambda = 0$

$\Rightarrow \quad \lambda(\lambda - 5) = 0$

$\Rightarrow \quad \underline{\lambda = 0, \; \lambda = 5}$ are the eigen values.

$\longrightarrow$

If $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigen vector corresponding to the eigen value λ, then $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = 0$, we have

$$\begin{bmatrix} 1-0 & 2 \\ 2 & 4-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 + 2x_2 &= 0 \\ 2x_1 + 4x_2 &= 0 \end{aligned}\right\}$$

or $\cancel{x_1 + x_2}$

$x_1 + 2x_2 = 0$

$\therefore \dfrac{x_1}{2} = \dfrac{-x_2}{1}$

$\Rightarrow x_1 = 2, \; x_2 = -1$

or $x_1 = 2k, \; x_2 = -k$

$\therefore X = \begin{bmatrix} 2k \\ -k \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 0$

For $\lambda = 5$, we have

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -4x_1 + 2x_2 &= 0 \\ 2x_1 - x_2 &= 0 \end{aligned}\right\}$$

or $2x_1 - x_2 = 0$

$\therefore \dfrac{x_1}{-1} = \dfrac{-x_2}{2}$

$\Rightarrow \cancel{x_1 = -1, \; x_2 =}$

$\dfrac{x_1}{1} = \dfrac{x_2}{2}$

$\Rightarrow x_1 = 1, \; x_2 = 2$

or $x_1 = k, \; x_2 = 2k$

$\therefore X = \begin{bmatrix} k \\ 2k \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 5$

3. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:-

The characteristic equation is $|A - \lambda I| = 0$

ie, $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (1-\lambda)\left[(5-\lambda)(1-\lambda) - 1\right] - 1\left[1(1-\lambda) - 3\right] + 3\left[1 - 3(5-\lambda)\right] = 0$

$\Rightarrow \quad (1-\lambda)\left[5 - 5\lambda - \lambda + \lambda^2 - 1\right] - \left[1 - \lambda - 3\right] + 3\left[1 - 15 + 3\lambda\right] = 0$

$\Rightarrow \quad (1-\lambda)\left[\lambda^2 - 6\lambda + 4\right] - \left[-\lambda - 2\right] + 3\left[3\lambda - 14\right] = 0$

$\Rightarrow \quad \lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + \lambda + 2 + 9\lambda - 42 = 0$

$\Rightarrow \quad -\lambda^3 + 7\lambda^2 - 36 = 0$

$\Rightarrow \quad \lambda^3 - 7\lambda^2 + 36 = 0$

Solving, $\Rightarrow \quad \lambda = -2, 3, 6$.

If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigen vector corresponding to the eigen value λ, then $[A - \lambda I] X = 0$

$\longrightarrow \quad \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = -2$, we have

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{7\times 3 - 1\times 1} = \frac{-x_2}{1\times 3 - 3\times 1} = \frac{x_3}{1\times 1 - 3\times 7}$$

$$\Rightarrow \frac{x_1}{20} = \frac{-x_2}{0} = \frac{x_3}{-20}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$\therefore$ $X = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} k \\ 0 \\ -k \end{bmatrix}$ is the eigenvector corresponding to $\lambda = -2$.

For $\lambda = 3$, we have

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{-4-1} = \frac{-x_2}{-2-3} = \frac{x_3}{1-6}$$

$$\Rightarrow \frac{x_1}{-5} = \frac{-x_2}{-5} = \frac{x_3}{-5}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$\therefore$ $X = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} k \\ -k \\ k \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 3$

For $\lambda = 6$, we have

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{4} = \frac{-x_2}{-8} = \frac{x_3}{4}.$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$\therefore$ $X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} k \\ 2k \\ k \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 6$

4. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Solution:

The characteristic equation is $|A-\lambda I| = 0$

i.e, $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)[(1-\lambda)(2-\lambda)-0] - 1[0-0] + 1[0-(1-\lambda)] = 0$

$\Rightarrow (2-\lambda)(1-\lambda)(2-\lambda) - (1-\lambda) = 0$

$\Rightarrow (1-\lambda)[(2-\lambda)(2-\lambda) - 1] = 0$

$\Rightarrow (1-\lambda)(4-2\lambda-2\lambda+\lambda^2-1) = 0$

$\Rightarrow (1-\lambda)(\lambda^2-4\lambda+3) = 0$

$\Rightarrow (1-\lambda)(\lambda-1)(\lambda-3) = 0$

$\Rightarrow \lambda = 1, 1, 3$ are the eigen values.

If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the eigen vector corresponding to the eigen value λ, then $[A-\lambda I]X = 0$

$\Rightarrow \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = 1$, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 + x_2 + x_3 = 0$

$\Rightarrow \quad x_1 = -x_2 - x_3$

let $\quad x_2 = k_1$, $x_3 = k_2$

Then $\quad x_1 = -k_1 - k_2$

$\therefore \quad X = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$.

For $\lambda = 3$, we have

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad \frac{x_1}{2} = \frac{-x_2}{0} = \frac{x_3}{2}$

$\Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$

$\therefore \quad X = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} k \\ 0 \\ k \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$.

Diagonalization of a matrix

Suppose the n by n matrix A has n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix P, then $P^{-1}AP$ is a diagonal matrix D.

The eigenvalues of A are on the diagonal of D

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

* Any matrix with distinct eigenvalues can be diagonalized.
* The diagonalization matrix P is not unique.
* Not all matrices posses n linearly independent eigenvectors, so not all matrices are diagonalizable
* Diagonalizability of A depends on enough eigenvectors. Invertibility of A depends on nonzero eigenvalues.
* Diagonalization can fail only if there are repeated eigen values.
* The eigen values of A^k are $\lambda_1^k, \dots, \lambda_n^k$ and each eigen vector of A is still an eigen vector of A^k.
  $[D^k = (P^{-1}AP)(P^{-1}AP)\dots(P^{-1}AP) = P^{-1}A^kP]$.
* Diagonalizable matrices share the same eigenvector matrix iff $AB=BA$

example $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

$|A-\lambda I| = 0 \Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda = 0, 1$.

For $\lambda = 0$ $A - 0I = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \frac{1}{2}x_1 + \frac{1}{2}x_2 = 0 \Rightarrow x_1 = -x_2$

$\therefore X = \begin{bmatrix} k \\ -k \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = 1$ $A - I = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \Rightarrow -\frac{1}{2}x_1 + \frac{1}{2}x_2 = 0 \Rightarrow x_1 = x_2$

$\therefore X = \begin{bmatrix} k \\ k \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = -i, +i$

For $\lambda = -i$ $A + iI = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \Rightarrow ix_1 - x_2 = 0 \Rightarrow x_2 = ix_1$

$\therefore X = \begin{bmatrix} k \\ ik \end{bmatrix}$ or $\begin{bmatrix} 1 \\ i \end{bmatrix}$

For $\lambda = i$ $A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \Rightarrow -ix_1 - x_2 = 0 \Rightarrow x_2 = -ix_1$

$\therefore X = \begin{bmatrix} k \\ -ik \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\therefore P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$. $D = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$ $P^{-1} = \frac{1}{-2i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix}$

* Diagonalize the matrix $A=\begin{bmatrix}1&1&3\\1&3&1\\3&1&1\end{bmatrix}$

Sol^n

$|A-\lambda I|=0 \Rightarrow \lambda^3-5\lambda^2-4\lambda+20=0 \Rightarrow \lambda=5,2,-2$

For $\lambda=5$

$A-5I=\begin{bmatrix}-4&1&3\\1&-2&1\\3&1&-4\end{bmatrix} \Rightarrow \frac{x_1}{7}=\frac{-x_2}{-7}=\frac{x_3}{7} \Rightarrow X_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$

For $\lambda=2$

$A-2I=\begin{bmatrix}-1&1&3\\1&1&1\\3&1&-1\end{bmatrix} \Rightarrow \frac{x_1}{-2}=\frac{-x_2}{-4}=\frac{x_3}{-2} \Rightarrow X_2=\begin{bmatrix}1\\-2\\1\end{bmatrix}$

For $\lambda=-2$

$A+2I=\begin{bmatrix}3&1&3\\1&5&1\\3&1&3\end{bmatrix} \Rightarrow \frac{x_1}{14}=\frac{-x_2}{0}=\frac{x_3}{-14} \Rightarrow X_3=\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$\therefore P=\begin{bmatrix}1&1&1\\1&-2&0\\1&1&-1\end{bmatrix} \quad D=\begin{bmatrix}5&0&0\\0&2&0\\0&0&-2\end{bmatrix} \quad P^{-1}=\frac{1}{6}\begin{bmatrix}+2&-(-1)&+3\\-(-2)&+(-2)&-0\\+2&-(-1)&+(-3)\end{bmatrix}^T=\frac{1}{6}\begin{bmatrix}2&2&2\\1&-2&1\\3&0&-3\end{bmatrix}$

* Diagonalize the matrix $A=\begin{bmatrix}3&-2&4\\-2&6&2\\4&2&3\end{bmatrix}$

Sol^n $|A-\lambda I|=0 \Rightarrow \lambda^3-12\lambda^2+21\lambda+98=0 \Rightarrow \lambda=7,7,-2$

For $\lambda=7$

$A-7I=\begin{bmatrix}-4&-2&4\\-2&-1&2\\4&2&-4\end{bmatrix} \Rightarrow \frac{x_1}{0}=\frac{-x_2}{0}=\frac{x_3}{0}$ reducing to echelon form $\xrightarrow[R_3+R_1]{2R_2-R_1} \begin{bmatrix}-4&-2&4\\0&0&0\\0&0&0\end{bmatrix}$

$\Rightarrow -4x_1-2x_2+4x_3=0 \Rightarrow x_1=-\frac{1}{2}x_2+x_3 \quad \therefore X=\begin{bmatrix}-\frac{1}{2}x_2+x_3\\x_2\\x_3\end{bmatrix}=x_2\begin{bmatrix}-1/2\\1\\0\end{bmatrix}+x_3\begin{bmatrix}1\\0\\1\end{bmatrix}$

letting x_2, x_3 are free variables

$\therefore X_1=\begin{bmatrix}1\\-2\\0\end{bmatrix}$ and $X_2=\begin{bmatrix}1\\0\\1\end{bmatrix}$

For $\lambda=-2$

$A+2I=\begin{bmatrix}+5&-2&4\\-2&8&2\\4&2&+5\end{bmatrix} \Rightarrow \frac{x_1}{36}=\frac{-x_2}{-18}=\frac{x_3}{-36} \Rightarrow X_3=\begin{bmatrix}2\\1\\-2\end{bmatrix}$

$\therefore P=\begin{bmatrix}1&1&2\\-2&0&1\\0&1&-2\end{bmatrix} \quad D=\begin{bmatrix}7&0&0\\0&7&0\\0&0&-2\end{bmatrix} \quad P^{-1}=\frac{1}{-9}\begin{bmatrix}+(-1)&-4&+(-2)\\-(-4)&+(-2)&-1\\+1&-5&+2\end{bmatrix}^T$

$=\frac{1}{9}\begin{bmatrix}1&-4&-1\\4&2&5\\2&1&-2\end{bmatrix}$

Singular Value Decomposition

Any m by n matrix A can be factored into

$A = U\Sigma V^T =$ (orthogonal)(diagonal)(orthogonal)

The columns of U (m by m) are eigenvectors of AA^T, and the columns of V (n by n) are eigenvectors of A^TA. The r singular values on the diagonal of Σ (m by n) are the square roots of the nonzero eigenvalues of both AA^T and A^TA.

Remark

For positive definite matrices, Σ is Λ and $U\Sigma V^T$ is identical to $Q\Lambda Q^T$.

For other symmetric matrices, ~~Σ remains real~~ any negative eigenvalues in Λ become positive in Σ.

Remark.

U and V give orthonormal bases for all four fundamental subspaces:

first r columns of U : column space of A

last $m-r$ columns of U : left nullspace of A

first r columns of V : row space of A

last $n-r$ columns of V : nullspace of A.

The diagonal (but rectangular) matrix Σ has eigenvalues from A^TA. These positive entries (also called sigma) will be $\sigma_1, \dots, \sigma_r$. They are the singular values of A.

Remark:
When A multiplies a column v_j of V, it produces σ_j times a column of U. ($A = U\Sigma V^T \Rightarrow AV = U\Sigma$).

Remark:
Eigenvectors of AA^T and A^TA must go into the columns of U and V!

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = (U\Sigma V^T)(V\Sigma^T U^T) = U\Sigma\Sigma^T U^T.$$

$\therefore$ U must be the eigenvector matrix of AA^T.
The eigenvalue matrix $\Sigma\Sigma^T$ is an $m\times m$ matrix with $\sigma_1^2, \dots, \sigma_r^2$ on the diagonal.

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T\Sigma V^T$$

$\therefore$ V must be the eigenvector matrix of A^TA.
The diagonal matrix $\Sigma^T\Sigma$ has the same $\sigma_1^2, \dots \sigma_r^2$, but it is $n\times n$.

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \qquad D \rightarrow \underbrace{\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0}_{+ve}$$

Decompose $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ as $U\Sigma V^T$ where U and V are orthogonal matrices.

Sol^n:

$$AA^T = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}_{3\times 1} \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}_{1\times 3}$$

$$= \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$$

$$|AA^T - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -2 & -2 \\ -2 & 4-\lambda & 4 \\ -2 & 4 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(4-\lambda)^2 - 16\right] + 2\left[-2(4-\lambda)+8\right] - 2\left[-8 + 2(4-\lambda)\right] = 0$$

$$(1-\lambda)\left[16 - 8\lambda + \lambda^2 - 16\right] + 2\left[\cancel{-8} + 2\lambda + \cancel{8}\right] - 2\left[\cancel{-8} + \cancel{8} - 2\lambda\right] = 0$$

$$(1-\lambda)(\lambda^2 - 8\lambda) + 4\lambda + 4\lambda = 0$$

$$\lambda^2 \cancel{-8\lambda} - \lambda^3 + 8\lambda^2 + \cancel{8\lambda} = 0$$

$$-\lambda^3 + 9\lambda^2 = 0$$

$$-\lambda^2(\lambda - 9) = 0$$

$$\lambda_1 = 0, \ \lambda_2 = 0, \ \lambda_3 = 9$$

$\lambda = 9$

$$(AA^T - \lambda I)X = 0$$

$$\begin{bmatrix} -8 & -2 & -2 \\ -2 & -5 & 4 \\ -2 & 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\sim \begin{bmatrix} -8 & -2 & -2 \\ 0 & -18 & 18 \\ 0 & 18 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} -8x_1 - 2x_2 - 2x_3 &= 0 \\ -18x_2 + 18x_3 &= 0 \end{aligned}\right\} \quad x_2 = x_3, \quad x_1 = -\tfrac{1}{2}x_3$$

$$X_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

$\lambda = 0$

$$(AA^T - \lambda I)X = 0$$

$$\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\sim \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow x_1 - 2x_2 - 2x_3 = 0$$

$$\Rightarrow x_1 = 2x_2 + 2x_3$$

let ~~$x_2 = -\frac{1}{2}$~~ $x_2 = -1$, $x_3 = 2$ $\qquad X_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

let $x_2 = 2$, $x_3 = -1$ $\qquad X_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

$$\therefore U = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}$$

$$|A^TA - \lambda I| = 0 \Rightarrow |9 - \lambda| = 0 \Rightarrow \underline{\lambda = 9}$$

Then $(A^TA - \lambda I)X = 0$

$\Rightarrow \quad 0 \cdot x_1 = 0$

let $x_1 = 1$ $\quad \therefore X = [1]$

$\therefore V = [1]$ or $V^T = [1]$.

9 is an eigenvalue of both AA^T and A^TA.

And rank of $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ is $r = 1$.

$\therefore \Sigma$ has only $\sigma_1 = \sqrt{9} = 3$. $\quad \therefore \Sigma = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$$\therefore \text{The SVD of } A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

Obtain the SVD of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

Sol^n:

$$AA^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$|AA^T - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(1-\lambda) - 1 = 0 \Rightarrow \lambda^2 - 3\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{3 \pm \sqrt{5}}{2}$$

$\lambda_1 = \frac{3 - \sqrt{5}}{2}$

$(AA^T - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 2 - \left(\frac{3-\sqrt{5}}{2}\right) & 1 \\ 1 & 1 - \left(\frac{3-\sqrt{5}}{2}\right) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \frac{1+\sqrt{5}}{2} x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -\frac{2}{1+\sqrt{5}} x_2$$

let $x_2 = \frac{1+\sqrt{5}}{2} \Rightarrow x_1 = -1$

$$X = \begin{bmatrix} -1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \alpha \end{bmatrix}$$

$\alpha = \frac{1+\sqrt{5}}{2}$

$\lambda_2 = \frac{3+\sqrt{5}}{2}$

$(AA^T - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 2 - \left(\frac{3+\sqrt{5}}{2}\right) & 1 \\ 1 & 1 - \left(\frac{3+\sqrt{5}}{2}\right) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \frac{1-\sqrt{5}}{2} x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -\frac{2}{1-\sqrt{5}} x_2$$

let $x_2 = \frac{1-\sqrt{5}}{2} \Rightarrow x_1 = -1$

$$X = \begin{bmatrix} -1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \beta \end{bmatrix}$$

$\beta = \frac{1-\sqrt{5}}{2}$

$$U = \begin{bmatrix} \frac{-1}{\sqrt{1+\alpha^2}} & \frac{-1}{\sqrt{1+\beta^2}} \\ \frac{\alpha}{\sqrt{1+\alpha^2}} & \frac{\beta}{\sqrt{1+\beta^2}} \end{bmatrix}$$

As $A^TA = AA^T$ $\quad V^T = \begin{bmatrix} \frac{-1}{\sqrt{1+\alpha^2}} & \frac{\alpha}{\sqrt{1+\alpha^2}} \\ \frac{-1}{\sqrt{1+\beta^2}} & \frac{\beta}{\sqrt{1+\beta^2}} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}$

Obtain the SVD of $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}_{2\times 3}$

$$AA^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$|AA^T - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)^2 - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda-1)(\lambda-3) = 0$

$\lambda = 1, \ \lambda = 3$

$\lambda_1 = 3$ $\quad (AA^T - \lambda I)X = 0$

$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x_1 = -x_2$

Let $x_2 = 1$, $x_1 = -1$

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$

$(AA^T - \lambda I)X = 0$

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow x_1 = x_2$

Let $x_2 = 1$, $x_1 = 1$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}_{3\times 2} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$|A^TA - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)[(2-\lambda)(1-\lambda)-1] + 1[-(1-\lambda)] = 0$

$\Rightarrow (1-\lambda)(2 - 2\lambda - \lambda + \lambda^2 - 1) - 1 + \lambda = 0$

$(1-\lambda)(2 - 2\lambda - \lambda + \lambda^2 - 1) - 1 + \lambda = 0$

$(1-\lambda)(\lambda^2 - 3\lambda + 1) - 1 + \lambda = 0$

$\lambda^2 - 3\lambda + \cancel{1} - \lambda^3 + 3\lambda^2 - \cancel{\lambda} - \cancel{1} + \cancel{\lambda} = 0$

$-\lambda^3 + 4\lambda^2 - 3\lambda = 0$

$-\lambda(\lambda^2 - 4\lambda + 3) = 0$

$\lambda_1 = 0, \ \lambda_2 = 1, \ \lambda_3 = 3$

$\lambda_1 = 0$

$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$x_1 - x_2 = 0 \qquad x_1 = x_2$

$x_2 - x_3 = 0 \qquad x_2 = x_3$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 1$

$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\sim \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$-x_2 = 0 \qquad x_2 = 0$

$-x_1 + x_2 - x_3 = 0 \qquad x_1 = -x_3$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 3$

$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\sim \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$-2x_1 - x_2 = 0$

$x_2 + 2x_3 = 0$

$x_2 = -2x_1$

$\begin{bmatrix} 1 \\ -2 \\ +1 \end{bmatrix}$

$U = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}_{2\times 2}$

$V = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$ $\quad V^T = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}_{3\times 3}$

$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$