Exponential Distribution

Many experiments involve the measurement of the duration of time x between an initial point of time and the occurance of some phenomenon of interest. Exponential distribution deals with such type of continuous random variable x.

The probability density function of an Exponential distribution is given by

$$f(x) = \lambda e^{-\lambda x}, \quad \text{for } \lambda > 0 \text{ and } x \geq 0$$

examples:

time between two successive job arrivals,
duration of telephone calls,
life time of a component or a product,
server time at a server in a queue.

Mean of an Exponential Distribution

$$\text{Mean} = \int_0^\infty x f(x)\,dx$$

$$= \int_0^\infty x \lambda e^{-\lambda x}\,dx$$

$$= \lambda \left[x \times \frac{e^{-\lambda x}}{-\lambda} - 1 \times \frac{e^{-\lambda x}}{(-\lambda)^2} \right]_0^\infty$$

$$= \lambda \left[-0 - 0 + 0 + \frac{1}{\lambda^2} \right]$$

$$= \lambda \times \frac{1}{\lambda^2}$$

$$\text{mean } \mu = \frac{1}{\lambda}$$

Variance of a Exponential Distribution

$$\text{Variance} = \int_0^\infty x^2 f(x)\,dx - \mu^2$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx - \mu^2$$

$$= \lambda \left[x^2 \frac{e^{-\lambda x}}{-\lambda} - 2x \frac{e^{-\lambda x}}{(-\lambda)^2} + 2 \frac{e^{-\lambda x}}{(-\lambda)^3} \right]_0^\infty - \frac{1}{\lambda^2}$$

$$= \lambda \left[-0 - 0 - 0 + 0 + 0 + \frac{2}{\lambda^3} \right] - \frac{1}{\lambda^2}$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\text{variance } \sigma^2 = \frac{1}{\lambda^2}$$

* Let the mileage (in thousands of miles) of a particular tyre be a random variable X having the probability density $f(x) = \begin{cases} \frac{1}{20} e^{-\frac{x}{20}} &, x>0 \\ 0 &, x\leq 0 \end{cases}$

Find the probability that one of these tyres will last (i) at most 10,000 miles (ii) anywhere between 16,000 to 24,000 miles (iii) atleast 30,000 miles. Also find (a) mean & (b) variance of the given probability density function.

Sol^n:

(i) $P(X \leq 10) = \int_0^{10} f(x)\, dx$

$= \int_0^{10} \frac{1}{20} e^{-\frac{x}{20}}\, dx$

$= \frac{1}{20} \times \left. \frac{e^{-\frac{x}{20}}}{\left(-\frac{1}{20}\right)} \right]_0^{10}$

$= \left[-e^{-\frac{x}{20}} \right]_0^{10}$

$= -e^{-\frac{1}{2}} + e^{0}$

$= 1 - e^{-\frac{1}{2}}$

$= 0.3934$

(ii) $P(16 \leq X \leq 24)$

$= \int_{16}^{24} f(x)\, dx$

$= \int_{16}^{24} \frac{1}{20} e^{-\frac{x}{20}}\, dx$

$= \left[-e^{-\frac{x}{20}} \right]_{16}^{24}$

$= -e^{-\frac{24}{20}} + e^{-\frac{16}{20}}$

$= 0.148$

(iii) $P(X \geq 30) = \int_{0}^{30} f(x)\,dx$

$= \int_{0}^{30} \frac{1}{20} e^{-\frac{x}{20}}\,dx$

$= \left[-e^{-\frac{x}{20}} \right]_{0}^{30}$

$= 0.2231$

(a) Mean $= \frac{1}{\lambda} = \frac{1}{\frac{1}{20}} = 20$

(b) Variance $= \frac{1}{\lambda^2} = \frac{1}{\left(\frac{1}{20}\right)^2} = 20^2$

*The length of time for one person to be served at a cafeteria is a random variable X having an exponential distribution with a mean of 4 minutes. Find the probability that a person is served in less than 3 minutes on atleast 4 of the next 6 days.

Sol^n.

Let $P(X<3) = 1 - P(X \geq 3)$

Given mean $= 4$

ie, $\frac{1}{\lambda} = 4$

$\Rightarrow \lambda = \frac{1}{4}$

$f(x) = \lambda e^{-\lambda x}$

$= \frac{1}{4} e^{-\frac{1}{4}x}$

$$= 1 - \int_3^{\infty} f(x)\,dx$$

$$= 1 - \int_3^{\infty} \frac{1}{4} e^{-\frac{1}{4}x}\,dx$$

$$= 1 - \left(-e^{-\frac{1}{4}x}\right)_3^{\infty}$$

$$= 1 - \left(0 + e^{-\frac{3}{4}}\right)$$

$$= 1 - e^{-\frac{3}{4}} = 0.5276$$

Let D represent the no of days on which a person is served in less than 3 minutes. Then using the binomial distribution, the probability that a person is served in less than 3 minutes on at least 4 of next 6 days is

$$P(D \geq 4) = \cancel{\sum_{x=4}^{6}} P(D=4) + P(D=5) + P(D=6)$$

$$= {}^6C_4\left(1-e^{-\frac{3}{4}}\right)^4\left(e^{-\frac{3}{4}}\right)^2 + {}^6C_5\left(1-e^{-\frac{3}{4}}\right)^5\left(e^{-\frac{3}{4}}\right)^1 + {}^6C_6\left(1-e^{-\frac{3}{4}}\right)^6\left(e^{-\frac{3}{4}}\right)^0 = 0.3968$$

* The life (in years) of a certain electrical switch has an exponential distribution with an average life of 2 year. If 100 of these switches are installed in different systems, find the probability that at most 30 fail during the first year. (4)

Sol^n:

~~RA~~ Given mean = 2, i.e, $\frac{1}{\lambda} = 2 \Rightarrow \lambda = \frac{1}{2}$.

$$P(T > 1) = \int_1^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx = \left[-e^{-\frac{1}{2}x}\right]_1^\infty = e^{-\frac{1}{2}} = 0.606$$

prob that at most 30 fail during first year $\Big\} = P(X \le 30) = \sum_{x=0}^{30} {}^nC_x\, p^x q^{n-x}$

$$= \sum_{x=0}^{30} {}^{100}C_x (0.606)^x (0.394)^{100-x}$$

* The increase in sales per day in a shop is exponentially distributed with Rs 800 as the average. If sales tax is paid at the rate of 6%, find the probability that increase in sales tax return from that shop will exceed Rs 30 per day.

Sol^n

Given mean = 800

ie $\frac{1}{\lambda} = 800$

$\Rightarrow \lambda = \frac{1}{800}$

$\Rightarrow f(x) = \frac{1}{800} e^{-\frac{1}{800}x}$

let X - denote the sales per day.

Total sales tax on X item $= \frac{6}{100}X$.

~~probability of sales tax exceeding 30 Rs~~

Given total sales tax exceeds Rs 30 per day.

ie, $\frac{6}{100}X > 30 \Rightarrow X > 500$.

$\left.\begin{array}{l}\text{Probability of sales tax} \\ \text{exceeding Rs 30}\end{array}\right\} \equiv$ probability of sales per day exceeding 500

$= P(X > 500)$

$= 1 - P(X \leq 500)$

$= 1 - \int_0^{500} f(x)\,dx$

$= 1 - \int_0^{500} \frac{1}{800} e^{-\frac{1}{800}x}\,dx = 0.5353$

* The sales per day in a shop is exponentially distributed with average sale amounting to Rs 100 and net profit is 8%. Find the probability that net profit exceeds Rs 30 on 2 consecutive days. (5)

Sol^n: Given mean = 100

ie $\frac{1}{\cancel{\lambda}} = 100$

$\Rightarrow \cancel{\lambda} = \frac{1}{100}$

$f(x) = \frac{1}{100} e^{-\frac{1}{100}x}$

let ~~x~~ represent the sales per day.

net profit ~~of~~ of X items $= \frac{8}{100} X$.

net profit exceeds Rs 30

$\Rightarrow \frac{8}{100} X \geq 30$

$\Rightarrow X > 375$

Prob of net profit exceeds Rs 30 $\equiv$ ~~prob~~ prob^l of sales per day exceeding 375

$= P(X > 375)$

$= 1 - P(X \leq 375)$

$= 1 - \int_0^{375} \frac{1}{100} e^{-\frac{1}{100}x} dx$

$= e^{-3.75} \approx 0.0235$

prob^l that it repeats on 2 consecutive days.

let $D = 2$, $P(D=2) = {}^nC_x p^x q^{n-x}$

$n = 2$, $p = e^{-3.75}$, $x = 2$

$= {}^2C_2 (e^{-3.75})^2 (1 - e^{-3.75})^{2-2}$

$= (e^{-3.75})^2 = 0.00055$

* After the appointment of a new sales manager, the sales in a 2 wheeler showroom is exponentially distributed with mean 4. If 2 days are selected at random what is the probability that

(i) On both days the sales is over 5 units

(ii) The sales is over 5 times atleast 1 of 2 days.

Soln: Given mean $= 4$

$\frac{1}{\cancel{\lambda}} = 4$

$\Rightarrow \lambda = \frac{1}{4}$

$f(x) = \frac{1}{4} e^{-\frac{1}{4}x}$

~~proof~~ let X represent the sales per day.

$$P(X > 5) = \int_5^\infty \frac{1}{4} e^{-\frac{1}{4}x} dx = -e^{-\frac{1}{4}}\Big]_5^\infty = e^{-5/4} = 0.2865.$$

(i) let D = no of days on which sales is over 5 units.

(i) $P(D = 2) = {}^nC_x \, p^n \, q^{n-x}$

$n = 2$

$x = 2$

$$= {}^2C_2 \left(e^{-5/4}\right)^2 \left(1 - e^{-5/4}\right)^{2-2}$$

$$= 0.0586$$

(i) $P(D = \text{atleast } 1 \text{ of } 2 \text{ days}) = \underset{n=2,\, x=1}{P(D=1)} + \underset{n=2,\, x=2}{P(D=2)}$

$$= {}^2C_1 \left(e^{-5/4}\right)^1 \left(1 - e^{-5/4}\right)^{2-1} + {}^2C_2 \left(e^{-5/4}\right)^2 \left(1 - e^{-5/4}\right)^{2-2}$$

$=$

* Jobs are sent to a printer at an average rate of 3 jobs per hour.

(a) what is the expected time between jobs?

(b) what is the probability that the next job is sent within 5 minutes?

Sol^n

$$f(x) = 3e^{-3x} \qquad \lambda = 3$$

(a) $E[x] = \frac{1}{\lambda} = \frac{1}{3}$ hours $= \frac{60}{3} = 20$ minutes.

(b) $P(X < 5\text{min}) = P(X < \frac{5}{60}\text{hour}) = P(X < \frac{1}{12}\text{hour})$

$$= \int_0^{1/12} 3e^{-3x}\,dx$$

$$= -e^{-3x}\Big]_0^{1/12}$$

$$= -e^{-3/12} + e^{0}$$

$$= 0.2212$$

* A computer processes tasks in the order they are received. Each task takes an exponential amount of time with the average of 2 minutes. Compute the probability that a package is processed in less than 8 minutes.

Sol^n $\mu = 2$ min $\Rightarrow \frac{1}{\lambda} = 2 \Rightarrow \lambda = \frac{1}{2}$

$$\therefore f(x) = \frac{1}{2}e^{-\frac{1}{2}x}$$

$$p(x<8) = \int_0^8 \frac{1}{2}e^{-\frac{1}{2}x}\,dx = -e^{-\frac{1}{2}x}\Big]_0^8$$

$$= -e^{-4} + 1$$

$$= 0.9817$$

* On the average, it takes 25 seconds to download a file from the internet. If it takes an exponential amount of time to download one file, then what is the probability that it will take more than 70 seconds to download a file?

Soln $\mu = 25 \Rightarrow \frac{1}{\lambda} = 25 \Rightarrow \lambda = \frac{1}{25}$

$$f(x) = \frac{1}{25} e^{-\frac{1}{25}x}.$$

$$p(x > 70) = \int_{70}^{\infty} \frac{1}{25} e^{-\frac{1}{25}x} dx$$

$$= - e^{-\frac{1}{25}x} \Big]_{70}^{\infty}$$

$$= -0 + e^{-2.8}$$

$$= 0.0608$$

Normal Distribution

Any quantity whose variation depends on random causes is distributed according to the normal law.

A continuous random variable X follows a normal distribution if its probability density function is defined as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

for $-\infty < x < \infty$, $-\infty < \mu < \infty$ & $0 < \sigma < \infty$.

where μ is the mean of X

and σ^2 is the variable of X. We say $X \sim N(\mu, \sigma^2)$

example:

Marks scored by students.

Life span of a product

Note:

The limiting form of the binomial distribution for large values of n with neither p nor q is very small, is the normal distribution.

Properties of Normal Distribution

1. All normal curves are bell-shaped.
2. All normal curves are symmetric about the mean μ.
3. The area under an entire normal curve is 1.
4. All normal curves are positive for all x. That is, $f(x) > 0$ for all x.
5. The limit of $f(x)$ as x goes to infinity is 0, and the limit of $f(x)$ as x goes to negative infinity is 0. i.e. $\lim_{x \to \infty} f(x) = 0$ & $\lim_{x \to -\infty} f(x) = 0$
6. The height of any normal curve is maximized at $x = \mu$.
7. The shape of any normal curve depends on its mean μ and the standard deviation σ.

Theorem:-

If $X \sim N(\mu, \sigma^2)$, then $Z = \dfrac{X - \mu}{\sigma}$ is called the standardized (or standard) normal distribution.

The probability or the cummulative distribution function is given by,

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

The probability of a normal distribution function is given by

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\,dt$$

Also, $$F(-z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} f(t)\,dt = 1 - F(z)$$

The probabilities ~~can be~~ are computed numerically and recorded in a special table called the normal distribution table(The probabilities can also be computed using a standard calculator).

We can used the following results for the calculation of probabilities.

* $P(a \le X \le b) = F(b) - F(a)$
* $P(a < X < b) = F(b) - F(a)$
* $\cancel{P(a \le X)}$ $P(a<X) = 1 - P(X \le a) = 1 - F(a)$
* $F(-b) = 1 - F(b)$, where b is positive.

* Find the (a) mean and (b) standard deviation of an examination in which grades 70 and 88 correspond to standard scores of -0.6 and 1.4 respectively.

<u>Solⁿ</u>

standard variable $z = \dfrac{x-\mu}{\sigma}$

here $\quad -0.6 = \dfrac{70-\mu}{\sigma} \Rightarrow \mu - 0.6\sigma = 70$

and $\quad 1.4 = \dfrac{88-\mu}{\sigma} \Rightarrow \mu + 1.4\sigma = 88$

solving, we get $\mu = 75.4$ and $\sigma = 9$.

* The marks X obtained in mathematics by 1000 students is normally distributed with mean 78% and S.D 11%. Determine how many students got marks above 90%.

<u>Solⁿ</u> Here $z = \dfrac{x-\mu}{\sigma} = \dfrac{X - 0.78}{0.11}$

$X = 0.9, \Rightarrow z = \dfrac{0.9 - 0.78}{0.11} = 1.09$

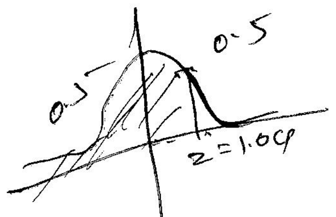
$P(X > 0.9) = \cancel{P(Z > 1.09)}\ 1 - P(X \le 0.9)$

$= \cancel{1 - F(1.09)}\ 1 - P(Z \le 1.09)$

$= \cancel{1 - 0.81594}$

$= 1 - 0.86214$ ✓

$= 0.13786$ ✓



* X is a normal variate with mean 30 and S.D 5. Find the probabilities that (i) $26 \le X \le 40$, (ii) $X \ge 45$ and (iii) $|X-30|>5$. (8)

Sol^n Given $\mu = 30$ and $\sigma = 5$

$$\therefore\ z = \frac{X-\mu}{\sigma} = \frac{X-30}{5}$$

(i) when $X = 26$, $z = \frac{26-30}{5} = -0.8$

when $X = 40$, $z = \frac{40-30}{5} = 2$

$$\therefore\ P(26 \le X \le 40) = P(-0.8 \le z \le 2)$$

$$= F(2) - F(-0.8)$$

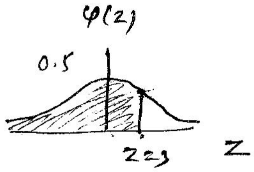
$$= 0.97725 - 0.21186$$

$$= 0.76539$$

(ii) when $X = 45$, $z = \frac{45-30}{5} = 3$

$$\therefore\ P(X \ge 45) = \cancel{P(z \ge 3)}\ 1 - P(X \le 45)$$

$$= 1 - P(z \le 3)$$



$$= 1 - F(3)$$

$$= 1 - 0.99865$$

$$= 0.00135$$

(iii) $P(|X-30| > 5) = 1 - P(|X-30| \le 5)$

$$= 1 - P(-5 \le X-30 \le 5)$$

$$= 1 - P(25 \le X \le 35)$$

$$= 1 - P(-1 \le z \le 1)$$

$$= 1 - (F(1) - F(-1))$$

$$= 1 - (0.84134 - 0.15866) = 0.31732$$

at $X = 25$, $z = -1$

at $X = 35$, $z = 1$

* In a test of 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and standard deviation of 60 hours. Estimate the number of bulbs likely to burn for (a) more than 2150 hours.
(b) less than 1950 hours and
(c) more than 1920 hours but less than 2060 hours

Sol^n: let X = life of an electric bulb (in hours)

Given $\mu = 2040$ hrs & $\sigma = 60$ hours.

(a) $P(X > 2150) = 1 - P(X \le 2150)$

$= 1 - P(z \le 1.8333)$

$= 1 - F(1.8333)$

$= 1 - 0.9664$

$= 0.0336$

$z = \frac{X-\mu}{\sigma} = \frac{2150-2040}{60} = 1.8333$

No of bulbs likely to burn more than 2150hr $= 0.0336 \times 2000 \approx 67$ bulbs

(b) $P(X < 1950)$ $\quad z = \frac{1950-2040}{60}$

$= P(z < -1.5)$

$= F(-1.5)$

$= 1 - F(1.5)$

$= 1 - 0.9332$

$= 0.0668$

no of bulbs to burn less than 1950hr $= 0.0668 \times 2000 \approx 134$ bulbs

(c) $P(1920 < X < 2060)$

$= P(\cancel{0.3333}\ {-2} < z < 0.3333)$

$= F(0.3333) - F(-2)$

$= F(0.3333) - (1 - F(2))$

$= F(0.3333) - 1 + F(2)$

$= 0.6293 - 1 + 0.9774$

$= 0.6065$

no of bulbs having life between 1920hr & 2060hrs $= 0.6065 \times 2000 \approx 1213$ bulbs.

* Assume that the reduction of a person's oxygen consumption during a period of Transcendental Meditation(T.M) is a continuous random variable X normally distributed with mean 37.6 cc/min and S.D 4.6 cc/min. Determine the probability that during a period of T.M a person's oxygen consumption will be reduced by (a) at least 44.5 cc/min (b) at most 35.0 cc/min (c) anywhere from 30.0 to 40.0 cc/min

Sol^n let X = reduction of oxygen consumption during T.M (in cc/min).

Given mean μ = 37.6 cc/min, S.D σ = 4.6 cc/min

(a) $P(X \geq 44.5)$ $\quad z = \frac{44.5 - 37.6}{4.6} = 1.5$

$1 - P(X \leq 44.5)$

$= \cancel{P(z \geq 1.5)}$

$= 1 - P(z \leq 1.5)$

$= 1 - F(1.5)$

$= 1 - 0.9332$

$= 0.0668$

(b) $P(X \leq 35.0)$ $\quad z = \frac{35 - 37.6}{4.6} = -0.5652$

$= P(z \leq -0.5652)$

$= F(-0.5652)$

$= 1 - F(0.5652)$

$= 1 - 0.7123$

$= 0.2877$

(c) $P(30 \leq X \leq 40)$ $\quad z = \frac{30 - 37.6}{4.6} = \cancel{0.5217}$ $= -1.6522$

$z = \frac{40 - 37.6}{4.6} = 0.5217$

$= P(-1.6522 \leq z \leq 0.5217)$

$= F(0.5217) - F(-1.6522)$

$= F(0.5217) - (1 - F(1.6522))$

$= 0.6985 - 1 + 0.9505$

$= 0.6490$

* An analog signal received at a detector (measured in micro volts) may be modeled as a Gaussian random variable $N(200, 256)$ at a fixed point in time. what is the probability that the ~~micro~~ signal ~~volt~~ will exceed 240 micro volts? what is the probability that the signal is larger than 240 micro volts, given that it is larger than 210 micro volts?

Sol^n: Let X be the signal detected by a detector (in micro volts)

Given mean $\mu = 200$ and ~~variance~~ $\sigma^2 = 256$

$\Rightarrow \sigma = \sqrt{256} = 16$

(a) $P(X > 240) = 1 - P(X \le 240)$ $\qquad z = \frac{240-200}{16} = 2.5$

$= 1 - P(z \le 2.5)$

$= 1 - F(2.5)$

$= 1 - 0.9938$

$= 0.0062$

(b) $P(X > 240 / X > 210) = \frac{P(X > 240 \text{ and } X > 210)}{P(X > 210)}$

$= \frac{P(X > 240)}{P(X > 210)}$

$= \frac{1 - P(X \le 240)}{1 - P(X \le 210)}$ $\qquad z = \frac{210-200}{16} = 0.625$

$= \frac{1 - P(z \le 2.5)}{1 - P(z \le 0.625)}$

$= \frac{1 - F(2.5)}{1 - F(0.625)}$

$= \frac{1 - 0.9939}{1 - 0.73401}$

$= 0.02335$

* Suppose that the average household income in some country is 900 coins, and the standard deviation is 200 coins. Assuming the Normal distribution of incomes, compute the proportion of "the middle class", whose income is between 600 and 1200 coins.

Sol^n $\mu = 900$, $\sigma = 200$ $z = \frac{x-\mu}{\sigma}$

$$P(600 < X < 1200) = P\left(\frac{600-900}{200} < z < \frac{1200-900}{200}\right)$$
$$= P(-1.5 < z < 1.5)$$
$$= F(1.5) - F(-1.5)$$
$$= 0.9332 - 0.0668$$
$$= 0.8664.$$

* The average height of professional basketball players is around 6 feet 7 inches and the standard deviation is 3.89 inches. Assuming Normal distribution of heights within this group (i) what percent of professional basketball players are taller than 7 feet?, (ii) if your favourite player is within the tallest 20% of all players, what can his height be?

Sol^n. $\mu = 6 \times 12 + 7 = 79$ inches, $\sigma = 3.89$ inches $z = \frac{x-\mu}{\sigma}$

(i) $P(X > 7 \times 12) = P(X > 84) = P(z > 1.2853) = 0.09934$.
ie, 9.9%.

(ii) $P(z > f.pl) = 0.20$, $P(f.pl < z) = 0.80 \Rightarrow z = 0.84162$

$\therefore \frac{x-\mu}{\sigma} = z \Rightarrow \frac{x-79}{3.89} = 0.84162 \Rightarrow x = 82.27$
ie, f.pl $\simeq$ 6 feet 10 inches

* The average household incomes in some country follow normal distribution with $\mu = 900$ coins and $\sigma = 200$ coins. (i) A recent economic reform made households with the income below 640 coins qualify for a free bottle of milk at every breakfast. What ~~prop~~ portion of the population qualifies for a free bottle of milk? (ii) Moreover, households with an income within the lowest 5% of population are entitled to a free sandwich. What income qualifies a household to receive free sandwiches?

Sol^n $\mu = 900$, $\sigma = 200$ $\quad z = \frac{x-\mu}{\sigma}$

(i) $P(x < 640) = P(z < -1.3) = 0.0968$

(ii) $P(z < h.i) = 0.05 \Rightarrow z = -1.64485$

$$\frac{x-\mu}{\sigma} = z \Rightarrow \frac{x-900}{200} = -1.64485 \Rightarrow x = 571.03$$

$x \approx 571$ coins

Uniform distribution

A continuous random variable is said to have a uniform distribution if all the values belonging to an interval have the same probability density.

Uniform distribution is used in any situation when a value is picked "at random" from a given interval; that is, without any preference to lower, higher or medium values.

example: locations of errors in a program

On the interval (a,b), the density function of the uniform distribution is given by $f(x) = \frac{1}{b-a}, a<x<b$.

Expectation, $E[X] = \int_a^b x \times \frac{1}{b-a}\, dx = \frac{1}{b-a} \times \left[\frac{x^2}{2}\right]_a^b = \frac{1}{2(b-a)}(b^2-a^2)$

$$\underline{E[X] = \frac{a+b}{2}}$$

$$\text{Variance, } Var[X] = \int_a^b x^2 \times \frac{1}{b-a}\, dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{b-a} \times \left[\frac{x^3}{3}\right]_a^b - \frac{(a+b)^2}{4}$$

$$= \frac{1}{3(b-a)}(b^3-a^3) - \frac{(a+b)^2}{4}$$

$$= \frac{1}{3(b-a)} \times (b-a)(b^2+ab+a^2) - \frac{(a+b)^2}{4}$$

$$= \frac{4(b^2+ab+a^2) - 3(a^2+2ab+b^2)}{12}$$

$$= \frac{b^2-2ab+a^2}{12}$$

$$\underline{Var[X] = \frac{(b-a)^2}{12}}$$

Standard uniform distribution

The uniform distribution with $a=0$ and $b=1$ is called standard uniform distribution. The standard uniform density is $f(x)=1$, for $0<x<1$.

* The amount of time, in minutes, that a person must wait for a bus is uniformly distributed between 0 and 15 minutes, inclusive. (i) what is the probability that a person waits fewer than 12.5 minutes? (ii) On an average, how long must a person wait? (iii) what is the standard deviation? (iv) Find the 90th percentile (i.e, ninety percent of the time, the time a person must wait falls below what value?

Sol^n. $f(x) = \frac{1}{15}$, $a=0$, $b=15$

(i) $P(x<12.5) = \int_0^{12.5} \frac{1}{15} dx = \left.\frac{x}{15}\right]_0^{12.5} = \frac{12.5}{15} = 0.8333$ min

(ii) $\mu = \frac{a+b}{2} = \frac{0+15}{2} = 7.5$ min

(iii) $\sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{15^2}{12}} = 4.33$ min

(iv) $P(x<k) = 0.90 \Rightarrow \int_0^k \frac{1}{15} dx = 0.9 \Rightarrow \left.\frac{x}{15}\right]_0^k = 0.9 \Rightarrow \frac{k}{15} = 0.9$
$\Rightarrow k = 13.5$ min

* A distribution is given as $X \sim U(0,20)$. what is $P(2<x<18)$? Find the 90th percentile.

Sol^n $a=0$, $b=20$ $f(x) = \frac{1}{20}$

(i) $P(2<x<18) = \int_2^{18} \frac{1}{20} dx = \left.\frac{x}{20}\right]_2^{18} = \frac{18-2}{20} = 0.8$.

(ii) $P(x<k) = 0.9 \Rightarrow \int_0^k \frac{1}{20} dx = 0.9 \Rightarrow \left.\frac{x}{20}\right]_0^k = 0.9 \Rightarrow \frac{k}{20} = 0.9$
$\Rightarrow k = 18$

* The total duration of baseball games in the major league in 2011 season is uniformly distributed between 447 hours and 521 hours inclusive. (i) What is the mean and standard deviation? (ii) What is the probability that the duration of games for a team for the 2011 season is between 480 and 500 hours? (iii) What is the 65th percentile for the duration of games for a team for the 2011 season?

Sol^n $a=447$, $b=521$, $f(x)=\frac{1}{74}$.

(i) $\mu = \frac{521+447}{2} = 484$ hrs, $\sigma = \sqrt{\frac{(521-447)^2}{12}} = 21.36$ hrs

(ii) $P(480 < x < 500) = \int_{480}^{500} \frac{1}{74}dx = \left.\frac{x}{74}\right]_{480}^{500} = 0.2703$ hrs

(iii) $P(x<k) = 0.65 \Rightarrow \int_{447}^{k} \frac{1}{74}dx = 0.65 \Rightarrow \left.\frac{x}{74}\right]_{447}^{k} = 0.65$

$\Rightarrow \frac{(k-447)}{74} = 0.65 \Rightarrow k = 491.5$ hrs.

* Suppose the time it takes a nine-year old to eat a donut is between 0.5 and 4 minutes, inclusive. Let x = the time, in minutes, it takes a nine-year old child to eat a donut. Then $x = U(0.5, 4)$. The probability that a randomly selected nine-year old child eats a donut in at least two minutes is —.

Sol^n $a=0.5$, $b=4$, $f(x)=\frac{1}{3.5}$

$P(x \geq 2) = \int_{2}^{4} \frac{1}{3.5}dx = \left.\frac{x}{3.5}\right]_{2}^{4} = \frac{4-2}{3.5} = 0.5714$