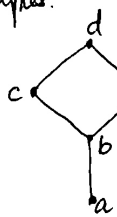
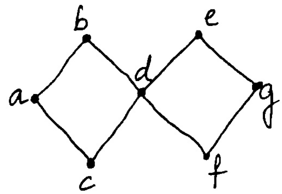
A circuit C in a graph G is called an <u>Eulerian circuit</u> if C contains every edge of G. Since no edge is repeated in a circuit, every edge appears exactly once in an Eulerian circuit.
A connected graph that contains an Eulerian circuit is called an <u>Eulerian graph</u>.
In a connected graph G, an open trail that contains every edge of G is an <u>Eulerian trail</u>.

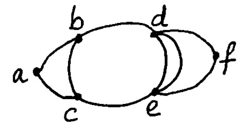
examples:



(a)



(b)

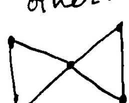


(c)

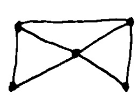
(a) has an Euler trail but no Euler circuit.
(b) has both Euler circuit and Euler trail.
(c) has an Euler trail but no Euler circuit.

<u>Theorem</u>: A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.

<u>Corollary</u>: A connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at the other.



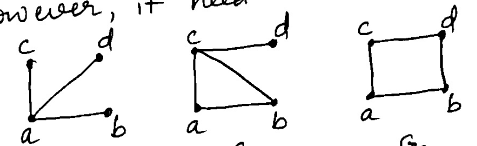
has both Euler circuit and Euler trail



has an Euler trail but no Euler circuit

A cycle in a graph G that contains every vertex of G is called a <u>Hamiltonian cycle</u> of G. Thus a Hamiltonian cycle of G is a spanning cycle of G.
A <u>Hamiltonian graph</u> is a graph that contains a Hamiltonian cycle. The graph C_n $(n \geq 3)$ is Hamiltonian. Also, for $n \geq 3$, the complete graph K_n is a Hamiltonian graph.

A path in a graph G that contains every vertex of G is called a <u>Hamiltonian path</u> in G. If a graph contains a Hamiltonian cycle, then it contains a Hamiltonian path. In fact, removing any edge from a Hamiltonian cycle produces a Hamiltonian path. If a graph contains a Hamiltonian path, however, it need not contain a Hamiltonian cycle.



G_1 G_2 G_3

The graph G_1 has no hamiltonian path (and no hamiltonian cycle)
The graph G_2 has hamiltonian path but no hamiltonian cycle.
The graph G_3 has both hamiltonian path and hamiltonian cycle.

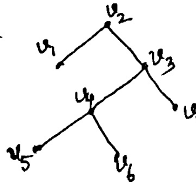
<u>Theorem</u>: Let G be a graph of order $n \geq 3$. If $\deg u + \deg v \geq n$ for each pair u, v of non adjacent vertices of G, then G is Hamiltonian (Converse need not be true).

<u>Corollary</u>: Let G be a graph of order $n \geq 3$. If $\deg v \geq \frac{n}{2}$ for every vertex v of G, then G is Hamiltonian. (Converse need not be true).

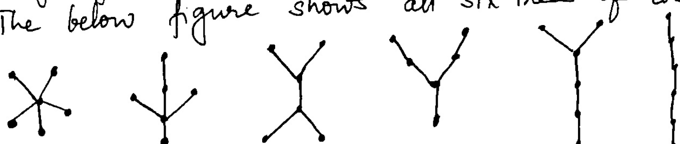
Tree.

A graph G is called acyclic if it has no cycles.
A tree is an acylic connected graph.

example



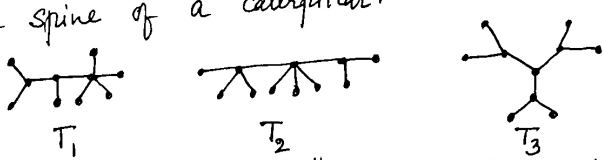
When dealing with trees, we often use T rather than G to denote a tree.
Indeed, we could define a tree as a connected graph, every edge of which is a bridge.
The below figure shows all six trees of order 6.



T_1 T_2 T_3 T_4 T_5 T_6.

The tree $T_1 = K_{1,5}$ is a star and $T_6 = P_6$ is a path.
The number of end vertices in the trees of the above figure ranges from 2 to 5. A tree containing exactly two vertices that are not end vertices is called a double star. The trees T_2 and T_3 in the above figure are double stars.

A caterpillar is a tree of order 3 or more, the removal of whose end-vertices produces a path called the spine of a caterpillar.

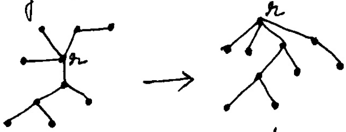


T_1 T_2 T_3

T_1 and T_2 are caterpillars but T_3 is not a caterpillar.

Sometimes it is convenient to select a vertex of a tree T under discussion and designate this vertex as the root of T. The tree T then becomes a rooted tree. Often the rooted tree T is drawn with the root r at the top and the other vertices of T drawn below, in levels according to their distance from r.

example



Acyclic graphs are also referred to as forests. Therefore, each component of a forest is a tree. One fact that distinguishes trees from forests is that a ~~trees~~ tree is required to be connected, while a forest is not required to be connected. Since a tree is connected, every two vertices in a tree are connected by a path.

Theorem:
A graph G is a tree if and only if every two vertices of G are connected by a unique path.

Proof:
First, let G be a tree. Then G is connected by definition. Thus every two vertices of G are connected by a path. Assume, to the contrary, that there are two vertices of G that are connected by two distinct paths. Then a cycle is produced from some or all of the edges of these two paths. This is a contradiction.
For the converse, suppose that every two distinct vertices of G are connected by a unique path. Certainly then, G is connected.
Assume, to the contrary, that G has a cycle C.
Let u and v be two distinct vertices of C.
Then C determines two distinct u-v paths, producing a contradiction. Thus G is acyclic and so G is a tree.

Theorem:
Every tree of order n has size $n-1$.

Proof
There is only one tree of order 1, namely K_1, which has size 0. Thus the result is true for $n=1$.
Assume for a positive integer k that the size of every tree of order k is $k-1$.
Let T be a tree of order $k+1$.
Every nontrivial tree T contains at least two end vertices.
Let v be one of them. Then $T' = T - v$ is a tree of order k.
By the induction hypothesis, the size of T' is $m = k-1$.
Since T has exactly one more edge than T', the size of T is $m+1 = (k-1)+1 = (k+1)-1$, as desired.

Problem: The degrees of the vertices of a certain tree T of order 13 are 1, 2 and 5. If T has exactly three vertices of degree 2, how many end-vertices does it have?

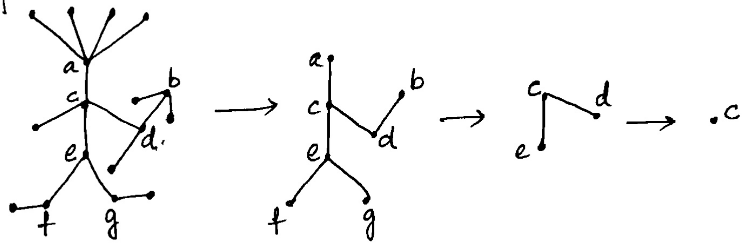
Ans: let x be the number of vertices of degree 1,
y be the number of vertices of degree 5,

then $x + 3 + y = 13 \Rightarrow y = 10 - x$.
Also $x \times 1 + 3 \times 2 + (10-x) \times 5 = 2 \times (13-1) \Rightarrow x = 8$.
Hence the tree has 8 end vertices.

Corollary
Every forest of order n with k components has size $n-k$.

Eccentricity
For a vertex v in a connected graph G, the eccentricity $e(v)$ of v is the distance between v and a vertex farthest from v in G. The center of G is a vertex having minimum eccentricity.
The centre of a tree can be obtained by removing the leaves of the tree continuously, until it reduces to a single edge or a single vertex.

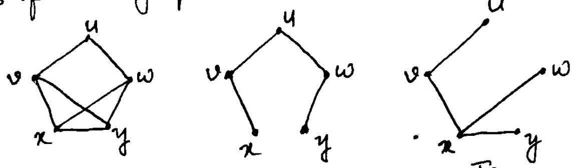
example



<u>Theorem:</u>
There are one or more centers in every tree; in the later case they are adjacent.

<u>Thearem:</u>
Let T be a tree on more than one vertex.
(i) If a longest path of T has an even length, then T has exactly one center, which is the mid-vertex of each longest path.
(ii) If a longest path of T has an odd length, then T has exactly two adjacent centers, which are the two mid most vertices of each longest path.

<u>Spanning Trees of a graph.</u>
Suppose G is a connected graph. We can produce trees T that are subgraphs of the given connected graph G such that $V(T) = V(G)$. Also it is a spanning subgraph of G. In the below figure T_1 and T_2 represent spanning trees of the graph G.



G $\qquad$ T_1 $\qquad$ T_2

Let G be a connected graph each of whose edge is assigned a number (called the cost or weight of the edge). We denote the weight of an edge e of G by $w(e)$. Such a graph is called a weighted graph. For each subgraph H of G, the weight $w(H)$ of H is defined as the sum of the weights of its edges, that is $w(H) = \sum_{e \in E(H)} w(e)$

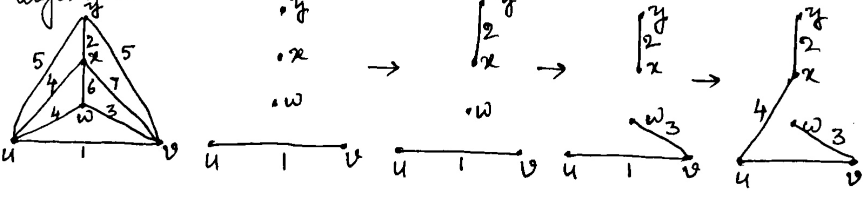
We seek a spanning tree of G whose weight is minimum among all spanning trees of G. Such a spanning tree is called a minimum spanning tree. The problem of finding a minimum spanning tree in a connected weighted graph is called the minimum spanning tree problem.

Kruskal's algorithm

For a connected weighted graph G, a spanning tree T of G is constructed as follows:

For the first edge e_1 of T, we select any edge of G of minimum weight and for the second edge e_2 of T, we select any remaining edge of G of minimum weight. For the third edge e_3 of T, we choose any remaining edge of G of minimum weight that does not produce a cycle with the previously selected edges. We continue in this manner until a spanning tree is produced.

The below figure shows on how a spanning tree of a connected weighted graph is constructed using Kruskal's algorithm.



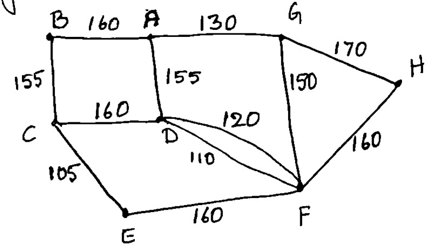
* Eight cities A, B, C, D, E, F, G and H are are required to be connected by a new railway network. The possible tracks and the cost involved to lay them (in crores of rupees) are summarized in the following table:

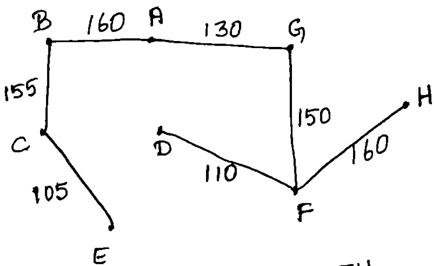
Between	Cost	Between	Cost	Between	Cost	Between	Cost
A and B	160	B and C	155	D and F	110	F and H	160
A and D	155	C and D	160	E and F	160	G and H	170
A and G	130	C and E	105	F and G	150	D and F	120

(i) Draw the weighted graph which represents the new railway network.

(ii) Further determine a railway network of minimal cost that connects all these cities using Kruskal's algorithm. Also mention the minimum cost.

Solⁿ





CE + DF + AG + FG + BC + AB + FH

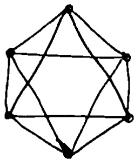
= 105 + 110 + 130 + 150 + 155 + 160 + 160

= 970

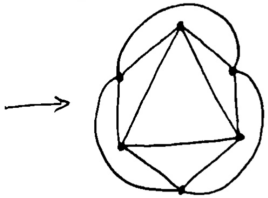
Planar Graphs

A graph G is called a planar graph if G can be drawn in the plane so that no two of its edges cross each other. A graph that is not planar is called nonplanar. A graph G is called a plane graph if it is drawn in the plane so that no two edges of G cross.

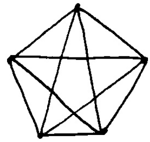
Eg



planar

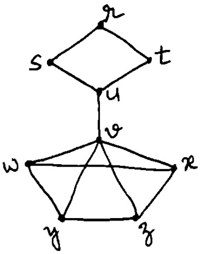


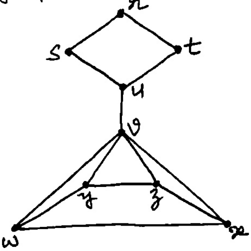
planar, plane.



non-planar

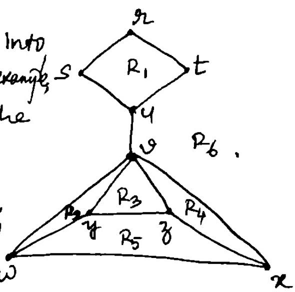
Consider the first graph shown in the below figure.



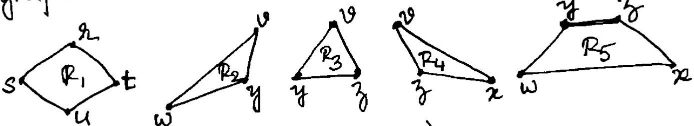


The graph is connected. But it is also planar, as we can see from the second graph, where it is drawn as a plane graph.

A plane graph divides the plane into connected pieces called regions. For example, in the case of the plane graph of the above figure, there are six regions. This graph is redrawn on the right, where the six regions are denoted by $R_1, R_2, R_3, R_4, R_5, R_6$



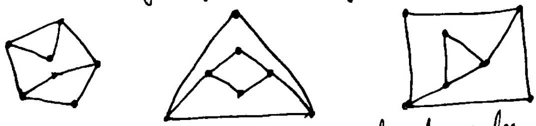
In every plane graph, there is always one region that is unbounded. This is the exterior region. For the previous graph, R_6 is the exterior region. The subgraph of a plane graph whose vertices and edges are incident with a given region R is the boundary of R. The boundaries of the six regions of the above graph are also shown in the below figure.



<u>Theorem</u>: (The Euler identity)
If G is a connected plane graph of order n, size m and having r regions, then $n-m+r=2$.

<u>Proof</u>
First, if G is a tree of order n, then $m=n-1$ and $r=1$; so $n-m+r=2$.
Therefore, we need only be concerned with connected graphs that are not trees.
Assume, to the contrary, that the theorem does not hold.
Then there exists a connected plane graph G of smallest size for which the Euler identity does not hold.
Suppose that G has order n, size m and r regions.
So $n-m+r\neq 2$.
Since G is not a tree, there is an edge e that is not a bridge.
Thus $G-e$ is a connected plane graph of order n and size $m-1$ having $r-1$ regions.
Because the size of $G-e$ is less than m, the Euler identity holds for $G-e$.
So $n-(m-1)+(r-1)=2$, but then $n-m+r=2$, which is a contradiction. Hence our assumption is incorrect.
Therefore $n-m+r=2$ holds.

The below figure shows a planar graph G and several ways of drawing G as a plane graph.



However, since G has a fixed order $n=7$ and fixed size $m=9$ and the Euler Identity holds $(n-m+r=7-9+r=2)$, each drawing of G as a plane graph always produces the same number of regions, namely $r=4$.

Theorem:
If G is a planar graph of order $n \geq 3$ and size m, then $m \leq 3n-6$.

The above theorem provides a necessary condition for a graph to be planar and so provides a sufficient condition for a graph to be nonplanar. In particular, the contrapositive of the above Theorem gives us the following.

If G is a graph of order $n \geq 3$ and size m such that $m > 3n-6$, then G is nonplanar.

Also,
Theorem:
If G is a planar graph of order $n \geq 3$ and size m, then $m \leq 2n-4$ if G has no 3-cycles.

The controposition,
If G is a graph with no triangles and $m > 2n-4$ then G is a nonplanar graph.

Corollary:
Every planar graph contains a vertex of degree 5 or less.

Proof:
Suppose that G is a graph, every vertex of which has degree 6 or more.
Let G have order n and size m.
~~Ce~~ Certainly, $n \geq 7$,
Then $2m = \sum_{v \in V(G)} \deg v \geq 6n$.

Thus $m \geq 3n > 3n-6$.

Hence G is nonplanar.
Therefore, if G is a planar graph then G contains a vertex of degree 5 or less.

Corollary
The complete graph K_5 is nonplanar.

Proof:
The graph K_5 has order $n=5$ and size $m=10$.
Since $m=10>9=3n-6$, it follows that K_5 is nonplanar.

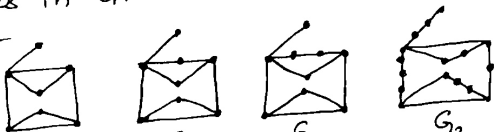
Corollary
The complete bipartite graph $K_{3,3}$ is nonplanar.

Proof
The graph $K_{3,3}$ has order $n=6$ and size $m=9$.
Since $m=9>8=2n-4$, it follows that $K_{3,3}$ is nonplanar

Note: There exists a graph of order $n \geq 3$ and size $m > 3n-6$ that contains neither K_5 nor $K_{3,3}$ as a subgraph.

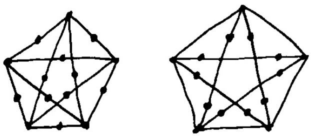
Let G be a graph and $e = uv$ an edge of G. A subdivision of e is the replacement of the edge e by a simple path $u_0, u_1, \ldots, u_k$, where $u_0 = u$ and $u_k = v$ are the only vertices of the path in $V(G)$.
We say that G' is a subdivision of G, if G' is obtained from G by a sequence of subdivisions of edges in G.

example

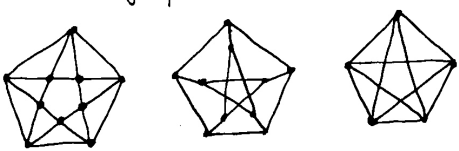


G $\quad G_1 \quad G_2 \quad G_3$

A graph G is said to be homeomorphic to G', if G' can be obtained from G by insertion or deletion of vertices of degree 2 between the edges of G.
The below graphs are homeomorphic to each other.



The below graphs are non-homeomorphic to eachother.



<u>Kuratowski's Theorem:</u>
A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

OR

A graph G is planar if and only if G does not contain any subgraph homeomorphic to K_5 or $K_{3,3}$.

Detection of planarity of a graph.

If a given graph G is planar or non planar is an important problem. We must have some simple and efficient criterion. We follow the following single steps to detect the planarity of a graph.

a. Since a disconnected graph is planar if and only if each of its component is planar, consider only one component at a time. Therefore, for the given arbitrary graph G, determine the components such that $G = G_1 \cup G_2 \cup \cdots \cup G_k$, where each G_i is a connected graph. Then check each component G_i for planarity.

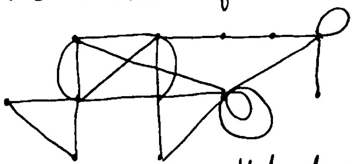
1. Since parallel edges do not affect planarity, eliminate the edges in parallel by removing all but one edge between every pair of vertices.
2. Since addition or removal of a selfloop does not affect planarity, remove all self-loops.
3. Elimination of a vertex of degree two by merging two edges in series does not affect planarity. Hence eliminate all edges in series.

Repeating the above steps yields one of the following graphs:

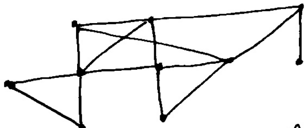
(i) A single edge.
(ii) A complete graph on four vertices.
(iii) $m > 3n-6$
(iv) A K_5 or $K_{3,3}$ subgraph.

If the graph reduces to (i) or (ii), the given graph is planar.
If the graph reduces to (iii) or (iv), the given graph is nonplanar.

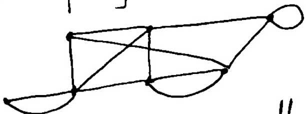
* Check the planarity of the following graph by the method of elementary deduction!



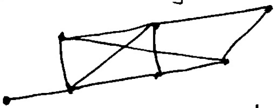
(i) removing parallel edges and self loops, eliminating edges in series



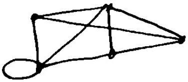
(ii) collapsing vertices of degree 1 and degree 2.



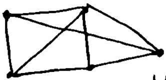
(iii) eliminating parallel edges and self loop



(iv) collapsing vertex of degree 1 and eliminating ~~vertex~~ edges in series



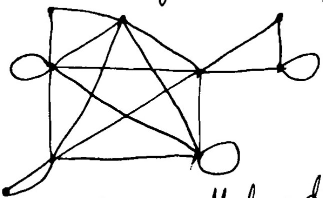
(v) removing the self loop



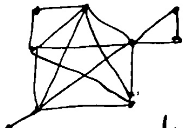
(vi) redrawing the graph

The graph is planar

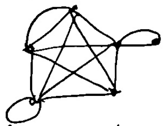
* Check the planarity of the following graph by the method of elementary deduction.



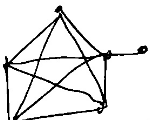
(i) eliminate parallel edges and self loops



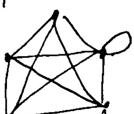
(ii) collapse vertices of degree 1 and degree 2



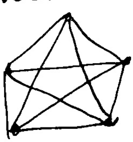
(iii) eliminate self loop and parallel edges



(iv) collapse vertex of degree 1



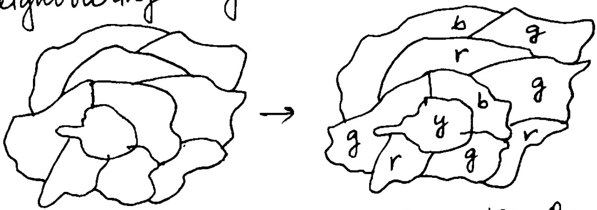
(v) eliminate self loop



The graph is k_5, which is non-planar

Vertex Colouring

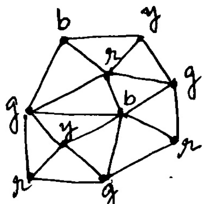
Consider the problem of colouring the regions on a map with different colors, such that no two neighbouring regions have the same color.



It can be seen that each region of the map can be assigned one of four given colours such that neighbouring regions are colored differently. Indeed, one such coloring is shown in the figure, where r, b, g and y denoted, red, blue, green and yellow respectively.

With each map, there is associated a graph G, called the dual of the map, whose vertices are the regions of the map and such that two vertices of G are adjacent if the corresponding regions are neighbouring regions.

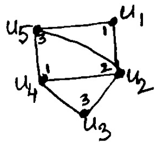
Coloring the regions of a map suggests coloring the vertices of its dual. Indeed, it suggests coloring the vertices of any graph. By a proper coloring (or, more simply, a coloring) of a graph G, we mean an assignment of colors (elements of some set) to the vertices of G, one color to each vertex, such that adjacent vertices are colored differently.



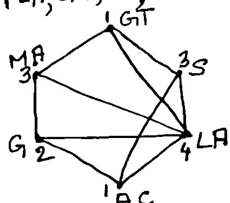
The smallest number of colors in any coloring of
a graph G is called the chromatic number of G and
is denoted by $\chi(G)$. If it is possible to color (the
vertices of) G from a set of k colors, then G is said to
be k-colorable. A coloring that uses k colors is called
a k-coloring. If $\chi(G) = k$, then G is said to be k-chromatic
and every k-coloring of G is a minimum coloring of G.
The following observations can be made w.r.t the
coloring of a graph.

1. A graph is 1-chromatic if and only if it is totally disconnected.
2. A graph having at least one edge is at least 2-chromatic (bichromatic)
3. A graph G having n vertices has $\chi(G) \leq n$.
4. If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.
5. $\chi(K_n) = n$ and $\chi(\overline{K_n}) = 1$.
6. $\chi(C_{2n}) = 2$ and $\chi(C_{2n+1}) = 3$.
7. If $G_1, G_2, \ldots, G_r$ are the components of a disconnected graph G, then $\chi(G) = \max\{\chi(G_1), \chi(G_2), \ldots \chi(G_r)\}$.
8. $\chi(K_{m,n}) = 2$.

A proper coloring of a graph naturally induces a partition of the vertices into different subsets. For example, the coloring in the following graph produces the partitioning, called the chromatic partitioning $V_1 = \{u_1, u_4\}$, $V_2 = \{u_2\}$, $V_3 = \{u_3, u_5\}$.



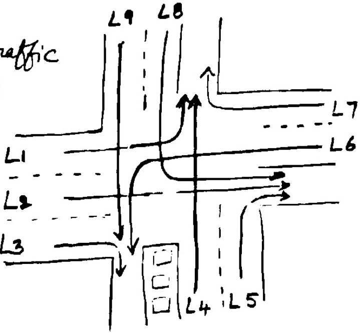
Problem.
The mathematics department of a certain college plans to schedule the classes Graph theory (GT), Statistics (S), Linear Algebra (LA), Advanced Calculus (AC), Geometry (G) and Modern Algebra (MA). Ten students have indicated the course they plan to take. With this information, use graph theory to determine the minimum number of time periods needed to offer these courses so that every two classes having a student in common are taught at different time periods during the day. Of course, two classes having no students in common can be taught during the same period.
Below is mentioned the student preferences.
A: LA, S ; B: MA, LA, G ; C: MA, G, LA ; D: G, LA, AC ; E: AC, LA, S ; F: G, AC ; G: GT, MA, LA ; H: LA, GT, S ; I: AC, S, LA ; J: GT, S.



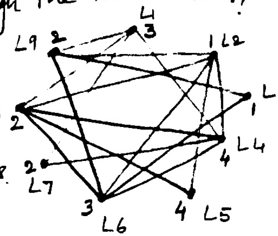
Sol^n The above situation can be represented by the following graph, where the vertices are the six subjects. Two vertices (subjects) are joined by an edge if some student is taking both classes.
The graph can be colored by 4 colors as shown.
Hence $\chi(H) = 4$. This also tells us one way to schedule these six classes during four time periods.
Period 1: GT, AC ; Period 2: G ; Period 3: S, MA; Period 4: LA

Problem

The following figure shows the traffic lanes L1, L2, L3, L4, L5, L6, L7, L8, L9 at the intersection of two busy streets. A traffic light is located at this intersection. During a certain phase of the traffic light, those cars in lane for which the light is green may proceed safely through the intersection. What is the minimum number of phases needed for the traffic light so that all cars may proceed through the intersection?



Sol^{n} Construct a graph G to model the situation, where, $V(G) = \{L1, L2, L3, L4, L5, L6, L7, L8, L9\}$ and two vertices (lanes) are joined by a edge if vehicles in these two lanes cannot safely enter the intersection at the same time, as there is a possibility of an accident.



Answering this question require determining the chromatic number of the graph. Notice that $\{L2, L4, L6, L8\} \cong K_4$. Since there exists a 4-coloring of G, as indicated in the graph, $\chi(G) = 4$.

So a minimum of 4 phases is needed for the traffic light so that all cars may proceed through the intersection.

Theorem (Greedy Algorithm): For every graph G, $\chi(G) \leq 1 + \Delta(G)$.

Brook's Theorem: For every connected graph G that is not an odd cycle or a complete graph, $\chi(G) \leq \Delta(G)$.

Five Color Theorem: For every planar graph G, $\chi(G) \leq 5$

The Four Color Theorem: The chromatic number of every planar graph is at most 4.

Coloring Enumeration

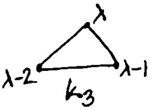
Let G be a graph and $\lambda \in N$. Define the number $P(G;\lambda)$ to be the number of proper λ-vertex colorings $c: V(G) \to \{1,2,3,\ldots,\lambda\}$. This property of a graph can be expressed by means of a polynomial. This polynomial is called the chromatic polynomial of G.

i.e, Let G be a labeled graph. A coloring of G from λ colors is a coloring of G which uses λ or fewer colors. Two colorings of G from λ colors will be considered different as atleast one of the labeled vertex is assigned different colors.

Note: (i) For each $\lambda < \chi(G)$, we have $P(G;\lambda) = 0$

(ii) For each $\lambda \geq \chi(G)$, we have $P(G:\lambda) > 0$

(iii) Indeed the smallest λ for which $P(G:\lambda) > 0$ is the chromatic number of G.



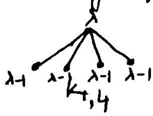
There are λ ways of coloring any given vertex of k_3. For a second vertex, any of $\lambda-1$ colors may be used, while there are $\lambda-2$ ways of coloring the remaining vertex.

Thus $P(k_3:\lambda) = \lambda(\lambda-1)(\lambda-2)$.

This can be generalized to any complete graph.

$P(k_n;\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$.

The corresponding polynomial of the totally disconnected graph (null graph) $\overline{K_n}$ is particularly easy to find, since each of its n vertices may be colored independently in any of λ ways. Thus $P(\overline{K_n}:\lambda) = \lambda^n$.

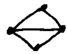


The central vertex of complete bipartite graph $K_{1,4}$ can be colored in λ ways, while each end vertex may be colored in any $\lambda-1$ ways.

Thus $P(K_{1,4}:\lambda) = \lambda(\lambda-1)^4$.

* A graph G with n vertices is a tree if and only if $P(G:\lambda) = \lambda(\lambda-1)^{n-1}$

Theorem! Let u and v be two non adjacent vertices in a graph G. $G+e$ be a graph obtained by adding an edge between u and v. Let $G.e$ be a single graph obtained from G by fusing the vertices u and v together and replacing sets of parallel edges with single edge. Then $P(G;\lambda) = P(G+e:\lambda) + P(G.e:\lambda)$

Problem: Find the chromatic polynomial of the graph 

Sol^n

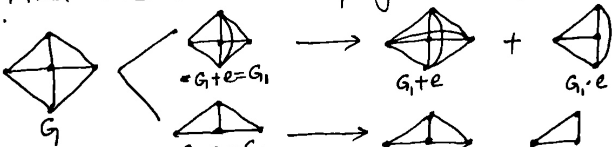
G G+e G.e

$\therefore P(G:\lambda) = P(G+e;\lambda) + P(G.e;\lambda) = P(K_4:\lambda) + P(K_3:\lambda)$
$= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) = \lambda(\lambda-1)(\lambda-2)(\lambda-3+1)$
$= \lambda(\lambda-1)(\lambda-2)^2$

Problem

Find the chromatic polynomial of the graph 

Sol^n.



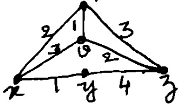
G $G+e=G_1$ G_1+e $G_1.e$

$G.e=G_2$ G_2+e $G_2.e$

$\therefore P(G:\lambda) = P(G_1+e:\lambda) + P(G_1.e:\lambda) + P(G_2+e:\lambda) + P(G_2.e:\lambda)$
$= P(K_5:\lambda) + P(K_4:\lambda) + P(K_4:\lambda) + P(K_3:\lambda)$
$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$
$= \lambda(\lambda-1)(\lambda-2)(\lambda^2-3\lambda-4\lambda+12+2\lambda-6+1)$
$= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$

Edge Coloring

A k-edge-coloring of G is a labeling $f: E(G) \to S$, where $|S| = k$. The labels are colors; the edges of one color from a color class. A k-edge-coloring is proper if incident edges have different labels; that is, if each color class is a matching. A graph is k-edge-colorable if it has a proper k-edge-coloring. The edge-chromatic number (chromatic index) $\chi'(G)$ of a loopless graph G is the least k such that G is k-edge-colorable.



example
The edge chromatic number of the following graph is four

For a graph G and any vertex $u \in V(G)$, all edges with u as an end vertex are adjacent and hence must receive different colors in a proper edge coloring of G. Hence, we note the obvious lower bound for the edge chromatic number of G $\chi'(G) \geq \Delta(G)$; the maximum degree of G.

Edge chromatic number of some basic graphs:

$$\chi'(K_n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even} \end{cases}, \quad \chi'(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}, \quad \chi'(P_n) = 2$$

Theorem: For a bipartite graph G, we have $\chi'(G) = \Delta(G)$.

Theorem: For the complete bipartite graph $K_{m,n}$, we have
$$\chi'(K_{m,n}) = \max\{m,n\}.$$

Theorem: If G is a simple graph, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$

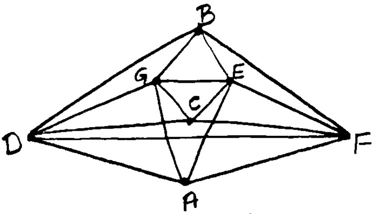
Vizing's Theorem: For a non empty graph G, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = 1 + \Delta(G)$.

Theorem: Let G be a graph of odd order n and size m. If $m > \frac{(n-1)\Delta(G)}{2}$, then $\chi'(G) = 1 + \Delta(G)$.

Problem:

Alvin (A) has invited three married couples to his summer house for a week: Bob (B) and Carrie (C) Hanson, David (D) and Edith (E) Irwin and Frank (F) and Gena (G) Jackson. Since all six guests enjoy playing tennis, he decides to set up some tennis matches. Each of his six guests will play a tennis match against every other guest except his/her spouse. In addition, Alvin will play a match against each of David, Edith, Frank and Gena. If no one is to play two matches on the same day, what is a schedule of matches over the smallest number of days?

Sol^n First, we construct a graph H whose vertices are the people at Alvin's summer house, so $V(H) = A, B, C, D, E, F, G$, where two vertices of H are adjacent if the two vertices (people) are to play a tennis match. To answer the question, we determine the chromatic index of H.



First, observe that $\Delta(H) = 5$.

Hence $\chi'(H) = 5$ or $\chi'(H) = 6$. Also, the order of H is $n = 7$ and its size is $m = 16$. Since $m = 16 > 15 = \frac{(7-1)\times 5}{2} = \frac{(n-1)\Delta(H)}{2}$, it follows that $\chi'(H) = 6$. And the following figure shows the 6-edge coloring of H, which provides a schedule of matches that take place over the smallest number of days (namely 6).

Day 1: B-G, C-E, D-F
Day 2: B-D, G-E, A-F
Day 3: B-F, A-E, C-G
Day 4: B-E, C-D, A-G
Day 5: D-G, E-F
Day 6: A-D, C-F