Compound propositions :-

$\sim, \wedge, \vee, \rightarrow, \leftrightarrow$

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Tautologies; contradictions :-

A compound proposition which is always true regardless of the truth values of its components is called tautology.

A compound proposition which is always false regardless of the truth values of its components is called contradiction.

A compound proposition that can be true or false, depending on the truth values of its components is called a contigency.

Ex: 1) $p \vee \sim p$ is tautology

2) $p \wedge \sim p$ is contradiction

3) Construct truth table for (i) $p \wedge \sim q$ (ii) $p \rightarrow \sim q$ (iii) $(p \wedge q) \rightarrow \sim r$ (iv) $p \wedge (q \rightarrow r)$

4) S.T. the truth values of the following statements are independent of the truth values of their components.

i) $(p \wedge (p \rightarrow q)) \rightarrow q$

ii) $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$

iii) $(\sim q \wedge (p \rightarrow q)) \rightarrow \sim p$

iv) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

Logical Equivalence, The laws of Logic :-

Definition: Two statements s_1, s_2 are said to be logically equivalent, and we write $s_1 \Leftrightarrow s_2$, when the statement s_1 is true (false) iff the statement s_2 is true (false).

Note that when $s_1 \Leftrightarrow s_2$ the statements s_1 and s_2 provide the same truth tables because s_1, s_2 have the same truth values for all choices of truth values for their primitive components.

p	q	$\sim p$	$\sim p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

table (1).

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

table (2)

From the results in tables (1) and (2), we have

$(p \to q) \Leftrightarrow \sim p \vee q$ &

$(p \leftrightarrow q) \Leftrightarrow (p \to q) \wedge (q \to p)$.

Prove the deMorgan's laws, over p & q :-

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F	T	F	F
T	F	F	T	F	T	T	T	F	F
F	T	F	T	T	F	T	T	F	F
F	F	F	T	T	T	T	F	T	T

$\therefore$ $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$ & $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$ } Demorgan's laws.

Similarly, the proof for the distributive laws, is shown below.

p	q	r	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
F	F	F	F	F	F	F
F	F	T	F	F	F	F
F	T	F	F	F	F	F
F	T	T	F	F	T	T
T	F	F	F	F	T	T
T	F	T	T	T	T	T
T	T	F	T	T	T	T
T	T	T	T	T	T	T

Using the concepts of logical equivalence, tautology, and contradiction, we state the following list of laws for the algebra of propositions.

Laws of logic

For any primitive statements p, q, r any tautology T_0 and any Contradiction F_0.

1. $\sim\sim p \Leftrightarrow p$

Law of double negation.

2. $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$

DeMorgan's Laws.

3. $p \vee q \Leftrightarrow q \vee p$

$p \wedge q \Leftrightarrow q \wedge p$

Commutative laws.

4. $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

Associative laws.

5. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Distributive laws.

6. $p \vee p \Leftrightarrow p$

$p \wedge p \Leftrightarrow p$

Idempotent laws.

7. $p \vee F_0 \Leftrightarrow p$

$p \wedge T_0 \Leftrightarrow p$

Identity laws.

8. $p \vee \sim p \Leftrightarrow T_0$

$p \wedge \sim p \Leftrightarrow F_0$

Inverse laws

9. $p \vee T_0 \Leftrightarrow T_0$

$p \wedge F_0 \Leftrightarrow F_0$

Domination laws

10. $p \vee (p \wedge q) \Leftrightarrow p$

$p \wedge (p \vee q) \Leftrightarrow p$

Absorption laws.

Definition: Let s be a statement, If s contains no logical connectives other than $\wedge$ and $\vee$, then the dual of s, denoted s^d, is the statement obtained from s by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$, respectively, and each occurrence of T_0 and F_0 by F_0 and T_0 respectively.

Ex: $s: (p \wedge \sim q) \vee (r \wedge T_0)$

$s^d: (p \vee \sim q) \wedge (r \vee F_0)$

Theorem: **principle of duality**:

Let s and t be statements that contains no logical connectives other than $\wedge$ and $\vee$.

If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

Proof of tautological statements:

$(r \wedge s) \rightarrow q \Leftrightarrow \sim(r \wedge s) \vee q$

$\therefore (r \wedge s) \rightarrow q \Longleftrightarrow \sim(r \wedge s) \vee q \Rightarrow$ is a tautology.

Converse, Inverse, contrapositive of an Implication

p	q	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$q \rightarrow p$	$\sim p \rightarrow \sim q$
0	0	1	1	1	1
0	1	1	1	0	0
1	0	0	0	1	1
1	1	1	1	1	1

$\therefore\ p \to q \iff \sim q \to \sim p$

$q \to p \iff \sim p \to \sim q$

$\sim q \to \sim p$ is contrapositive of $p \to q$.

$q \to p$ is converse of $p \to q$

$\sim p \to \sim q$ is inverse of $p \to q$

But $p \to q \not\Leftrightarrow q \to p$

Simplification of the compound proposition.

(1) Simplify $(p \vee q) \wedge \sim(\sim p \wedge q)$

$(p \vee q) \wedge \sim(\sim p \wedge q)$	Reason.
$\iff (p \vee q) \wedge (\sim\sim p \vee \sim q)$	Demorgan's law
$\iff (p \vee q) \wedge (p \vee \sim q)$	law of double negation.
$\iff [(p \vee q) \wedge p] \vee [(p \vee q) \wedge \sim q]$	Distributive law
$\iff p \vee (q \wedge \sim q)$	Inverse law
$\iff p \vee F_0$	Identity law.
$\iff p$	

$p \vee ((p \vee q) \wedge \sim q)$	; Absorption law
$p \vee [(p \wedge \sim q) \vee (q \wedge \sim q)]$	; Distributive law
$(p \vee (p \wedge \sim q)) \vee (q \wedge \sim q)$	; Absorption law
$p \vee (q \wedge \sim q)$	; ~~Identity~~ Inverse law

(2) Simplify $\sim[\sim[(p \vee q) \wedge r] \vee \sim q]$.

$$\sim[\sim[(p \vee q) \wedge r] \vee \sim q$$

$\Leftrightarrow \sim\sim[(p \vee q) \wedge r] \wedge \sim\sim q$; DeMorgan's law.

$\Leftrightarrow [(p \vee q) \wedge r] \wedge q$; Double Negation

$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$; Associative law.

$\Leftrightarrow (p \vee q) \wedge (q \wedge r)$; Commutative law

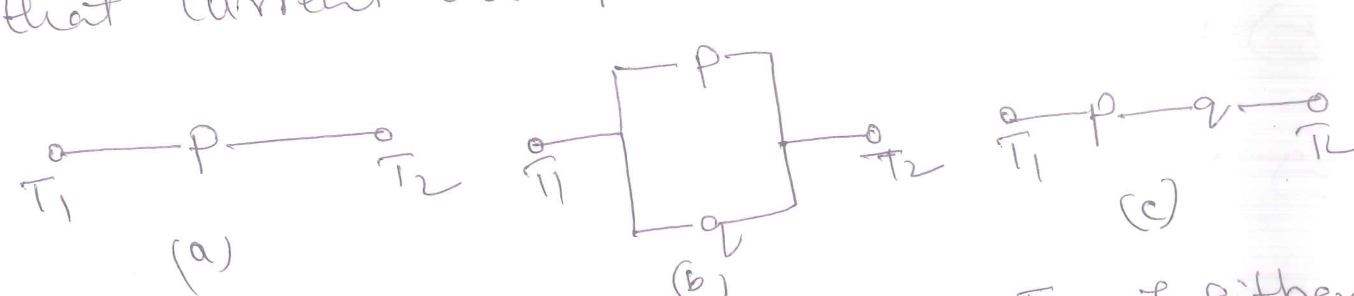
$\Leftrightarrow ((p \vee q) \wedge q) \wedge r$; Associative law

$\Leftrightarrow q \wedge r$; Absorption law

Consequently, $\sim[\sim[(p \vee q) \wedge r] \vee \sim q \Leftrightarrow q \wedge r$

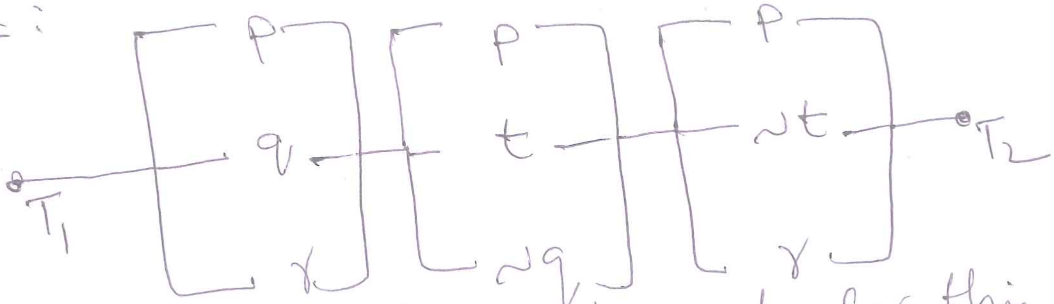
(3) A switching network is made up of wires and switches connecting two terminals T_1 and T_2. In such a n/w, each switch is either open (0), so that no current flows through it, or closed (1) so that current does flows through it.

Ex:



In n/w (b) the current flows from T_1 to T_2 if either of the switches p, q is closed. We call this a parallel n/w and represent it by $p \vee q$. The n/w (c) is called ~~sequential~~ serial n/w in which both p & q must be closed, hence represented by $p \wedge q$.

Ex:



The equivalent statement for this n/w is

$(p \vee q \vee r) \wedge (p \vee t \vee \sim q) \wedge (p \vee \sim t \vee r)$

upon simplification of this statement

$\Leftrightarrow p \vee [(q \vee r) \wedge (t \vee \sim q) \wedge (\sim t \vee q)]$; Distributive

$\Leftrightarrow p \vee [(q \vee r) \wedge (\sim t \vee r) \wedge (t \vee \sim q)]$; Commutative

$\Leftrightarrow p \vee [((q \wedge \sim t) \vee r) \wedge (t \vee \sim q)]$; Distributive

$\Leftrightarrow p \vee [((q \wedge \sim t) \vee r) \wedge (\sim\sim t \vee \sim q)]$; Double negation

$\Leftrightarrow p \vee [((q \wedge \sim t) \vee r) \wedge \sim(\sim t \wedge q)]$; DeMorgan's law

$\Leftrightarrow p \vee [\sim(\sim t \wedge q) \wedge ((\sim t \wedge q) \vee r)]$; Commutative

$\Leftrightarrow p \vee [(\sim(\sim t \wedge q) \wedge (\sim t \wedge q)) \vee (\sim(\sim t \wedge q) \wedge r)]$; Distributive

; $\sim s \wedge s \Leftrightarrow F_0$

$\Leftrightarrow p \vee [F_0 \vee (\sim(\sim t \wedge q) \wedge r)]$

; F_0 is identity for $\vee$

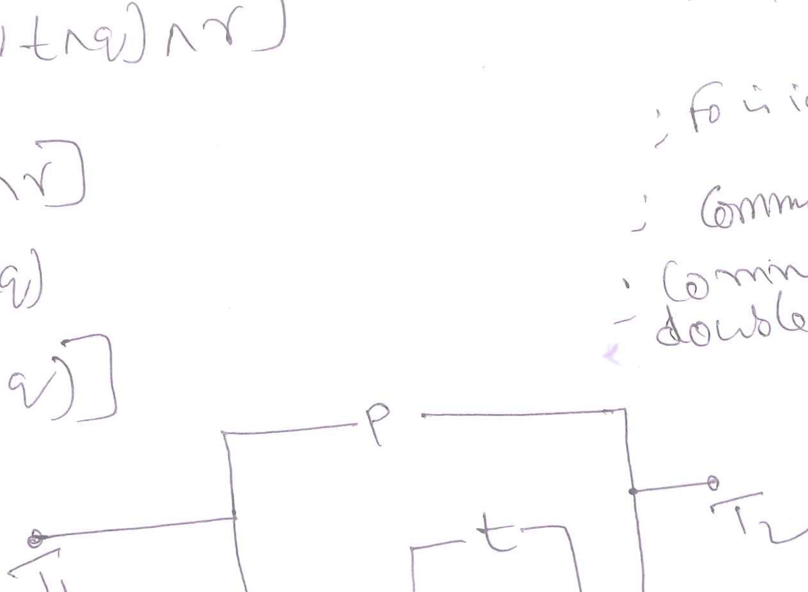
$\Leftrightarrow p \vee (\sim(\sim t \wedge q) \wedge r]$

; Commutative

$\Leftrightarrow p \vee (r \wedge \sim(\sim t \wedge q))$

; Commutative & double negation.

$\Leftrightarrow p \vee [r \wedge (t \vee \sim q)]$



Logical Implication: Rules of Inference

Natural deduction:

Consider the general form of an argument, One we wish to show is valid. So let us consider the implication

$$(P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n) \rightarrow q.$$

The statements $P_1, P_2, \ldots P_n$ are called the premises of the argument, and the statement q is the conclusion of the argument.

If this argument is said to be valid if whenever each of the premises $P_1, P_2, \ldots P_n$ is true, then the conclusion q is also true.

If any one of the premises are false then $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ becomes false, hence $(P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n) \rightarrow q$ become true, irrespective of the truth value of q.

Therefore, one way to establish the validity of the argument is to show that $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \rightarrow q$ is a tautology.

Ex ①: Let p, q, r denote the primitive statements given as

p: Roger studies.

q: Roger plays tennis.

r: Roger passes discrete mathematics.

Now, let P_1, P_2, P_3 denotes the premises

P_1: If Roger studies, then he will pass D.M.S.

Determine whether the argument $(P_1 \wedge P_2 \wedge P_3) \rightarrow q$ is valid.

$\Rightarrow$

$P_1 : p \rightarrow r$

$P_2 : \sim q \rightarrow p$

$P_3 : \sim r$

Therefore, the argument becomes $[(p \rightarrow r) \wedge (\sim q \rightarrow p) \wedge \sim r] \rightarrow q$, and we have to show that this is a tautology.

p	q	r	$p \rightarrow r$	$\sim q \rightarrow p$	$\sim r$	$(P_1 \wedge P_2 \wedge P_3) \rightarrow q$
0	0	0	1	0	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	0	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

$\boxed{\text{Definition}}$: If p, q are arbitrary statements such that $p \rightarrow q$ is a tautology, then we say that p logically implies q and we write $p \Rightarrow q$ to denote this situation.

Ex (2): $p \to r,\ r \to s,\ t \vee \sim s,\ \sim t \vee u,\ \sim u \vdash \sim p$

1. $p \to r$; premise
2. $r \to s$; premise
3. $p \to s$; Law of syllogism 1, 2
4. $t \vee \sim s$; premise
5. $\sim s \vee t$; commutative law on 4
6. $s \to t$; Equivalence $p \to q \Leftrightarrow \sim p \vee q$
7. $p \to t$; Rule of syllogism 3, 6
8. $\sim t \vee u$; premise
9. $t \to u$; Equivalence $p \to q \Leftrightarrow \sim p \vee q$
10. $p \to u$; Rule of syllogism 7, 9.
11. $\sim u$; premise
12. $\sim p$; Modus tollen 10, 11.

$\therefore [(p \to r) \wedge (r \to s) \wedge (t \vee \sim s) \wedge (\sim t \vee u) \wedge \sim u] \Rightarrow \sim p$

Rules of inferencing:

Modus phonen or Rule of detachment

$$p \to q$$
$$p$$
$$\overline{\therefore q}$$

$$((p \to q) \wedge p) \to q$$

p	q	$p \to q$	$p \to q \wedge p$	$((p \to q) \wedge p) \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Rule of syllogism:

$$[(p \to q) \wedge (q \to r)] \to (p \to r)$$

$$p \to q$$
$$q \to r$$
$$\overline{\therefore p \to r}$$

Give Truth table:

Ex (1):

$$p$$
$$p \to \sim q$$
$$\sim q \to \sim r$$
$$\overline{\therefore \sim r}$$

1. $p \to \sim q$; premise
2. $\sim q \to \sim r$; premise
3. $p \to \sim r$; Law of syllogism 1, 2.
4. p ; premise
5. $\sim r$; Rule of Detachment 3, 4.

Rule of Modus Tollens:

$$p \to q$$
$$\sim q$$
$$\overline{\therefore \sim p}$$

$$((p \to q) \wedge \sim q) \to \sim p$$

give truth table.

Other rules :

Rule - (4) : p
q
$\therefore p \wedge q$
// Rule of conjunction or $\wedge_i$

Rule - (5) : $p \vee q$
$\sim p$
$\therefore q$
$[(p \vee q) \wedge \sim p] \rightarrow q$ // Rule of disjunctive syllogism

Rule - (6) : $\sim p \rightarrow F_0$
$\therefore p$
$(\sim p \rightarrow F_0) \rightarrow p$ // Rule of contradiction

Rule - (7) : $p \wedge q$
$\therefore p$
$(p \wedge q) \rightarrow p$ // Rule of Constructive simplification or $\wedge_{e_1}$

Rule - (8) : p
$\therefore p \vee q$
$p \rightarrow (p \vee q)$ // Rule of disjunctive amplification.

Rule - (9) : $p \wedge q$
$p \rightarrow (q \rightarrow r)$
$\therefore r$
$((p \wedge q) \wedge (p \rightarrow (q \rightarrow r))) \rightarrow r$ // Rule of conditional proof.

Rule - (10) : $p \rightarrow r$
$q \rightarrow r$
$\therefore (p \vee q) \rightarrow r$
$[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow ((p \vee q) \rightarrow r)$ // Rule for proof by cases.

Rule - (11) : $p \rightarrow q$
$r \rightarrow s$
$p \vee r$
$\therefore q \vee s$
// Rule of constructive
// dilemma.

Rule - (12) : $p \rightarrow q$
$r \rightarrow s$
$\sim q \vee \sim s$
// Rule of destructive
// dilemma.

Ex (1) : Establish the validity of the argument
$p \to r, \sim p \to q, q \to s \vdash \sim r \to s$.

1. $p \to r$; premise
2. $\sim r \to \sim p$; $p \to r \Leftrightarrow \sim r \to \sim p$
3. $\sim p \to q$; premise
4. $\sim r \to q$; Law of syllogism 2, 3.
5. $q \to s$; premise
6. $\sim r \to s$; Law of syllogism 4, 5.

Another way to establish the proof.

1. $p \to r$; premise
2. $q \to s$; premise
3. $\sim p \to q$; premise
4. $p \vee q$; By equivalence $p \to q \Leftrightarrow \sim p \vee q$, & double negation
5. $r \vee s$; Law of constructive dilemma 1, 2, 4.
6. $\sim r \to s$; By equivalence $p \to q \Leftrightarrow \sim p \vee q$ & double negation.

Ex (2) : Establish the validity of the argument

$$\begin{array}{l} p \to q \\ q \to (r \wedge s) \\ \sim r \vee (\sim t \vee u) \\ \underline{p \wedge t} \\ \therefore u \end{array}$$

1. $p \to q$; premise
2. $q \to (r \wedge s)$; premise
3. $p \to (r \wedge s)$; Law of syllogism 1,2
4. $p \wedge t$; premise
5. p ; Conjunctive simplification on 4.
6. $r \wedge s$; MP on 3,5
7. r ; Conjunctive simplification on 6.
8. $\sim r \vee (\sim t \vee u)$; premise
9. $\sim(r \wedge t) \vee u$; DeMorgans & associative law.
10. t ; Conjunctive simplification (4)
11. $r \wedge t$; Rule of conjunction 7, 10.
12. $\therefore u$; Rule of disjunctive syllogism.

Ex (3): $(\sim p \vee \sim q) \to (r \wedge s)$

$$\begin{array}{l} (\sim p \vee \sim q) \to (r \wedge s) \\ r \to t \\ \sim t \\ \hline \therefore p \end{array}$$

1. $r \to t$; premise
2. $\sim t$; premise
3. $\sim r$; MT 1, 2
4. $\sim r \vee \sim s$; Viz
5. $\sim (r \wedge s)$; De Morgan law
6. $(\sim p \vee \sim q) \to (r \wedge s)$; premise
7. $\sim(\sim p \vee \sim q)$; MT 5, 6
8. $p \wedge q$; De Morgan & double negation
9. p

EX (4) $u \rightarrow r$

$(r \wedge s) \rightarrow (p \vee t)$

$q \rightarrow (u \wedge s)$

$\sim t$

q

$\therefore p$

1. q ;
2. $q \rightarrow (u \wedge s)$;
3. $(u \wedge s)$;
4. u ;
5. $u \rightarrow r$;
6. r ;
7. s ;
8. $r \wedge s$;
9. $(r \wedge s) \rightarrow (p \vee t)$;
10. $p \vee t$;
11. $\sim t$;
12. p ;

~~Prakash K (Maths)~~
~~944802096~~

[Pre]dicate logic : Use of quantifiers.

[A]ny sentence which involves a variable, such as x, [n]eed not be statements.

[F]or example, the sentence "The number $x+2$ is an even integer" is not necessarily true or false unless we know what value is substituted for x. If we restrict our choices to integers, then when x is replaced by -5, -1 or 3, for instance, the resulting statement is false. In fact it is false whenever x is replaced by an odd integer. When an even integer is substituted for x, however, the resulting statement is true.

The sentence which is of the form "The number $x+2$ is an even integer" is called as open statement.

Definition: A declarative sentence is an open statement if

1) it contains one or more variables, and
2) it is not a statement, but
3) it becomes a statement when the variables in it are replaced by certain allowable choices.

When we examine the sentence "The number $x+2$ is an even integer" in light of this definition, we find it is an open statement that contains the single variable x. With regard to the third element of the definition, in our earlier discussion we restricted the "certain allowable choices" to [illegible] choices constitute what is

while we dealing with the open statement, t
following notations are used.

The open statement "The number $x+2$ is an even integer" is denoted by $p(x)$ {or $q(x)$ or
Then $\sim p(x)$ may be read "The number x+ is not an even integer".

We can use $q(x,y)$ to represent the open statement consisting of two variables. For example.

$q(x,y)$: The numbers $y+2$, $x-y$ and $x+2y$ are even integers.

In the case of $q(x,y)$, there is more than One occurrence of each of the variables x, y. It is understood that when we replace one of the x's by a choice from our universe, we replace the other x by the same choice.

With $p(x)$ and $q(x,y)$ as above, and the universe is set of all integers as our only allowable choices, we get the following results when we make some replacements for the variables x, y.

$p(5)$: The number 7 $(=5+2)$ is an even integer (F)

$\sim p(7)$: The number 9 $(=7+2)$ is not an even integer (T)

$q(4,2)$: The numbers 4, 2 and 8 are even integers (T)

IIIly $q(5,2)$ and $q(4,7)$ are both false statements, whereas $\sim q(5,2)$ and $\sim q(4,7)$ are true.

Let $p(x)$ and $q(x, y)$ are as above, some substitution result in true statement and other results in false statements. Therefore we can make the following true statement. ③

1. For some x, $p(x)$.
2. For some x, y, $q(x, y)$.

Note that in this situation, the statements "For some x, $\sim p(x)$" and "For some x, y, $\sim q(x, y)$" are also true.

Quantifiers: The phrases "For some x" and "For some x, y" are said to quantify the open statements $p(x)$ & $q(x, y)$, respectively. There are two types of quantifiers: 1) Existential quantifier. 2) Universal quantifier.

The statement (1) use the existential quantifier "For some x", which can also be expressed as "For at least one x" or "There exist an x such that". This quantifier is written as $\exists x$ symbolically. ∴ The statement "For some x, $p(x)$" becomes $\exists x\, p(x)$.

Ily statement (2) becomes $\exists x \exists y\, q(x, y)$ in symbolic form. The notation $\exists_{x,y}$ can be used to abbreviate $\exists x \exists y\, q(x, y)$ to $\exists_{x,y}\, q(x, y)$

The universal quantifier is denoted by $\forall x$ and is read "For all x", "For any x", "For each x" or "For every x", "For all x, y", "For every x, y" ($\forall x \forall y$ or $\forall_{x,y}$)

Ex ① $\forall x\, p(x)$: For all x, the number $x+2$ is an even integer $\Rightarrow$ (F).

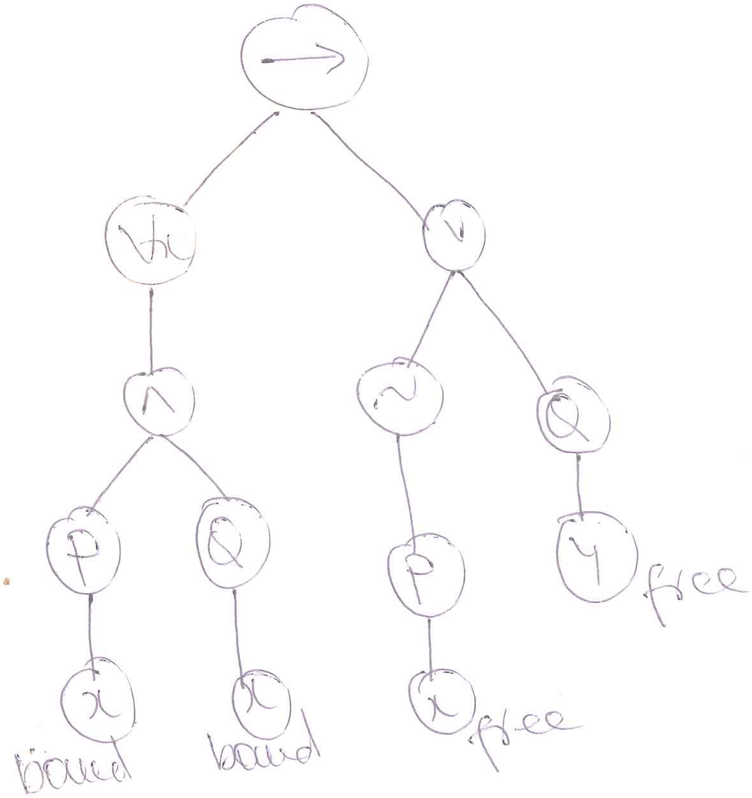
② Let $r(x)$: "$2x$ is an even integer"

Free and bound variables :-

(E) The variable x in each of open statements $p(x)$ and $r(x)$ is called a free variable. As x varies over the universe for an open statement the truth value of the statement may vary. For ex $x=5$, $p(5)$ is false, while $p(6)$ is true. The state $r(x)$ is true for every replacement of x.

In contrast with the open statement $p(x)$, $\exists x\, p(x)$ has the a fixed truth value - say true. In the symbolic representation $\exists x\, p(x)$, the variable x is said to be a bound variable. It is bound by the existential quantifier.

Construct a parse tree and find free & bound variables for the expression. $(\forall x(P(x) \wedge Q(x))) \rightarrow (\sim P(x) \vee Q(y))$



Ex (1) Here the universe comprises all real numbers.
(2) The open statements $p(x)$, $q(x)$, $r(x)$ and $s(x)$ are given by (5)

$p(x): x \geq 0$ $\quad r(x): x^2 - x - 4 = 0$

$q(x): x^2 \geq 0$ $\quad s(x): x^2 - 3 > 0$

Then the following statements are true:

(1) $\exists x (p[x] \wedge r[x])$; because $x = 4$, both $p(4)$ & $r(4)$ are true.

(2) $\forall x (p(x) \rightarrow q(x))$;

If we replace x in $p(x)$ by a negative real number a, then $p(a)$ is false, but $p(a) \rightarrow q(a)$ is true ~~regardless~~ of the truth value of $q(a)$. Replacing x in $p(x)$ by a non-negative real number b, we find that $p(b)$ and $q(b)$ are both true, as is $p(b) \rightarrow q(b)$.

40

This statement may be translated into any of the following.

a) For every real number x, if $x \geq 0$, then $x^2 \geq 0$.

b) Every nonnegative real number has a non-negative real square.

c) The square of any non-negative real number is a non negative real number.

d) All non-negative real numbers have non-negative squares.

$\exists x\, p(x)$: For some a in the universe, $p(a)$ is true

$\forall x\, p(x)$: For every replacement a from the universe, $p(a)$ is true.

$\exists x \sim p(x)$: For at least one choice a in the universe, $p(a)$ is false, so its negation $\sim p(a)$ is true.

$\forall x \sim p(x)$: For every replacement a from the universe, $p(a)$ is false and its negation $\sim p(a)$ is true.

Definition: Let $p(x)$ and $q(x)$ be open statement defined for a given universe. ② ⑥

The open statements $p(x)$ and $q(x)$ are called (logically) equivalent, and we write $\forall x\left[p(x) \Leftrightarrow q(x)\right]$ when the biconditional $p(a) \leftrightarrow q(a)$ is true for each replacement a from the universe.

If the implication $p(a) \rightarrow q(a)$ is true for each a in the universe, then we write $\forall x\left[p(x) \Rightarrow q(x)\right]$ and say that <u>$p(x)$ logically implies $q(x)$.</u>

Ex: For the universe of all triangles in the plane

Let $p(x)$: x is equiangular.
$q(x)$: x is equilateral

For every triangle a, we know that $p(a) \leftrightarrow q(a)$ is true
Consequently, $\forall x\left[p(x) \Leftrightarrow q(x)\right]$.

Definition: For open statements $p(x)$, $q(x)$ - defined for a prescribed universe - and the universally quantified statement $\forall x\left[p(x) \rightarrow q(x)\right]$ we define:

1) The contrapositive of this to be $\forall x\left[\neg q(x) \rightarrow \neg p(x)\right]$
2) The converse of this is $\forall x\left[q(x) \rightarrow p(x)\right]$.
3) The Inverse of this is $\forall x\left[\neg p(x) \rightarrow \neg q(x)\right]$

Ex ② (8): For the universe of quadrilaterals in the plane let

⑦

$s(x)$: x is a square $e(x)$: x is equilateral.

a) The statement $\forall x[s(x) \to e(x)]$ is true and is logically equivalent to contrapositive

$$\forall x[\sim e(x) \to \sim s(x)].$$

$$\therefore \forall x[s(x) \to e(x)] \Leftrightarrow \forall x[\sim e(x) \to \sim s(x)]$$

b) The statement $\forall x[e(x) \to s(x)]$ is false, and it is the converse of the true statement $\forall x[s(x) \to e(x)]$

The false statement $\forall x[\sim s(x) \to \sim e(x)]$ is the inverse of the given statement

$$\forall x[s(x) \to e(x)].$$

Ex ③: Let $p(x)$, $q(x)$, and $r(x)$ denotes open statements for a given universe. Find the logical equivalence of

(1) $\forall x[p(x) \wedge (q(x) \wedge r(x))] \Leftrightarrow \forall x[(p(x) \wedge q(x)) \wedge r(x)]$

Associative law.

(2) $\exists x[p(x) \to q(x)] \Leftrightarrow \exists x(\sim p(x) \vee q(x)]$

(3) $\forall x \sim\sim p(x) \Leftrightarrow \forall x\, p(x)$

(4) $\forall x \sim(p(x) \wedge q(x)) \Leftrightarrow \forall x[\sim p(x) \vee \sim q(x)]$

$\forall x \sim[p(x) \vee q(x)] \Leftrightarrow \forall x[\sim p(x) \wedge \sim q(x)]$

If we replace $\forall x$ by $\exists x$ in 1 & 4 they are still equivalent

(7) Negating quantifiers:

$\sim[\forall x\, p(x)] \Leftrightarrow \exists x \sim p(x)$

$\sim[\exists x\, p(x)] \Leftrightarrow \forall x \sim p(x)$

$\sim[\forall x \sim p(x)] \Leftrightarrow \exists x\, p(x)$

$\sim[\exists x \sim p(x)] \Leftrightarrow \forall x\, p(x)$.

Ex (4) Consider the universe as set all integers.

(1) Let $p(x)$ and $q(x)$ be given by

$p(x)$: x is odd

$q(x)$: x^2-1 is even.

The statement "If x is odd, then x^2-1 is even" can be symbolized as $\forall x\,[p(x) \to q(x)]$.

$\sim[\forall x\,(p(x) \to q(x))]$

$\Leftrightarrow \exists x\,[\sim p(x) \to q(x)]$

$\Leftrightarrow \exists x\,[\sim(\sim p(x) \vee q(x))]$

$\Leftrightarrow \exists x\,(\sim\sim p(x) \wedge \sim q(x)]$

$\Leftrightarrow \exists x\,(p(x) \wedge \sim q(x)]$

"There exists an integer x such that x is odd and x^2-1 is not even" (This is false).

Ex (1) $\forall t[p(t) \rightarrow q(t)], \forall t[q(t) \rightarrow r(t)], \sim r(c) \vdash \sim p(c)$

P

10

1. $\forall t[p(t) \rightarrow q(t)]$ premise
2. $p(c) \rightarrow q(c)$ Rule of universal simplification (1)
3. $\forall t[q(t) \rightarrow r(t)]$ premise
4. $q(c) \rightarrow r(c)$ Rule of US - (3)
5. $p(c) \rightarrow r(c)$ Law of syllogism 2, 4
6. $\sim r(c)$ premise
7. $\therefore \sim p(c)$ MT 5,6.

Rule of Universal Generalization:

If an open statement $p(x)$ is proved to be true when x is replaced by any arbitrarily chosen element c from our universe, then the universally quantified statement $\forall x\, p(x)$ is true. Furthermore, the rule extends beyond a single variable. So, if ~~for~~ ex, we have an open statement $q(x,y)$ that is proved to be true when x and y are replaced by arbitrarily elements from the same universe or their own universes, then the universally quantified statement $\forall x \forall y\, q(x,y)$ is true. Similar results holds for the case where three or more variables.

Ex(5): What is the negation of the following statement.

01

$$\forall x \exists y [(p(x,y) \wedge q(x,y)) \to r(x,y)]$$

$$\sim [\forall x \exists y ((p(x,y) \wedge q(x,y)) \to r(x,y))]$$

$$\Leftrightarrow \exists x \sim \exists y [(p(x,y) \wedge q(x,y)) \to r(x,y)]$$

$$\Leftrightarrow \exists x \forall y \sim [(p(x,y) \wedge q(x,y)) \to r(x,y)]$$

$$\Leftrightarrow \exists x \forall y \sim [\sim (p(x,y) \wedge q(x,y)) \vee r(x,y)]$$

$$\Leftrightarrow \exists x \forall y [\sim\sim (p(x,y) \wedge q(x,y)) \wedge \sim r(x,y)]$$

$$\Leftrightarrow \exists x \forall y [(p(x,y) \wedge q(x,y)) \wedge \sim r(x,y)]$$ //

Natural deduction in predicate logic:-

Rule of universal simplification:

If an open statement becomes true for all replacements by the members in a given universe, then that open statement is true for each specific individual member in that universe. (If $p(x)$ is an open statement for a given universe, and if $\forall x\, p(x)$ is true, then $p(a)$ is true for each a in the universe.

Ex (2)

$$\forall x\,(p(x) \to q(x))$$
$$\forall x\,(q(x) \to r(x))$$
$$\therefore \forall x\; p(x) \to r(x)$$

1. $\forall x\,[p(x) \to q(x)]$; premise
2. $p(c) \to q(c)$; US - 1
3. $\forall x\,[q(x) \to r(x)]$; premise
4. $q(c) \to r(c)$; US - 3
5. $p(c) \to r(c)$; Law of syllogism 2,4.
6. $\therefore \forall x\,[p(x) \to r(x)]$; UG - 5.

Ex (3):

$$\forall x\,[p(x) \wedge q(x)]$$
$$\forall x\,[(\sim p(x) \wedge q(x)) \to r(x)]$$
$$\therefore \forall x\; \sim r(x) \to p(x)$$

1. $\forall x\,[p(x) \vee q(x)]$ premise
2. $p(c) \vee q(c)$ US - 1
3. $\forall x\,[(\sim p(x) \wedge q(x)) \to r(x)]$ premise
4. $(\sim p(c) \wedge q(c)) \to r(c)$ US - 3
5. $\sim r(c) \to \sim[\sim p(c) \wedge q(c)]$ $s \to t \Leftrightarrow \sim t \to \sim s$
6. $\sim r(c) \to [p(c) \vee \sim q(c)]$ De Morgan & Double negation
7. $\sim r(c)$ Added premise.

12

8. $p(c) \vee \sim q(c)$ MP 7, 6

9. $[p(c) \vee q(c)] \wedge [p(c) \vee \sim q(c)]$ $\wedge i$ 2, 8

10. $p(c) \vee [q(c) \wedge \sim q(c)]$ Distributive law

11. $p(c)$ $q(c) \wedge \sim (q(c)) \to F_0$

12: $\forall x [\sim r(x) \to p(x)]$ 7 & 11 & UG.

er-5: Relations 3

Definition-1: For sets $A, B \subseteq \mathcal{U}$, the cartesian product, or cross product, of A and B is denoted by $A \times B$ and equals $\{(a,b) \mid a \in A, b \in B\}$.

We say that the elements of $A \times B$ are ordered pairs. For $(a,b), (c,d) \in A \times B$, we have $(a,b) = (c,d)$ iff $a = c$ & $b = d$.

If A and B are finite, it follows from the rule of product that $|A \times B| = |A| \cdot |B|$.

In general we will not have $A \times B = B \times A$, but we will have $|A \times B| = |B \times A|$.

Even though $A, B \subseteq \mathcal{U}$, but $A \times B$ need not be $\subseteq \mathcal{U}$.

$A_1 \times A_2 \times \cdots \times A_n$ equals $\{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, 1 \leq i \leq n\}$.

Ex: Let $\mathcal{U} = \{1, 2, 3, \ldots, 7\}$, $A = \{2, 3, 4\}$ & $B = \{4, 5\}$, then

a) $A \times B = \{(2,4), (2,5), (3,4), (3,5), (4,4), (4,5)\}$.

b) $B \times A = \{(4,2), (4,3), (4,4), (5,2), (5,3), (5,4)\}$.

c) $B^2 = B \times B = \{(4,4), (4,5), (5,4), (5,5)\}$.

d) $B^3 = B \times B \times B = \{(a,b,c) \mid a,b,c \in B\}$; for example $(4,5,5) \in B^3$

Definition-2: For sets $A, B \subseteq \mathcal{U}$, any subset of $A \times B$ is called a relation from A to B. Any subset of $A \times A$ is called as binary relation on A.

Ex: A, B & $\mathcal{U}$ are as in ex(1).

a) ϕ b) $\{(2,4)\}$ c) $\{(2,4), (2,5)\}$

d) $\{(2,4), (3,4), (4,4)\}$ e) $\{(2,4), (3,4), (4,5)\}$ f) $A \times B$

Since $|A \times B| = 6$, it follows form definition (2) that there are 2^6 possible relations from B to A or A to B.

In general, for finite sets A, B with $|A| = m$ and $|B| = n$, there are 2^{mn} relations from A to B, including the empty relation as well as the relation $A \times B$ itself. There are 2^{nm} $(= 2^{mn})$ relations from B to A, one of which is also ϕ and $B \times A$ itself. The reason we get the same number of relations from B to A as we have from A to B is that any relation R_1 from B to A can be obtained from a unique relation R_2 from A to B by simply reversing the components of each ordered pair in R_2 (and vice-versa).

Ex (3) Let $B = \{1, 2\} \subseteq N$, $U = P(B)$ and $A = U$ $= \{\phi, \{1\}, \{2\}, \{1, 2\}\}$. The following is an example of a binary relation on A: $R = \{(\phi, \phi), (\phi, \{1\}), (\phi, \{2\}), (\phi, \{1,2\}), (\{1\}, \{1\}), (\{1\}, \{1,2\}), (\{2\}, \{2\}), (\{2\}, \{1,2\}), (\{1,2\}, \{1,2\})\}$. We say that the relation R is the subset relation where $(C, D) \in R$ iff $C, D \subseteq B$ and $C \subseteq D$.

Theorem 1: For any sets $A, B, C \subseteq U$

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

proof: For $a, b \in U$, $(a, b) \in A \times (B \cap C)$

$\Leftrightarrow a \in A$ and $b \in (B \cap C)$

$\Leftrightarrow a \in A$ and $b \in B, C$

$\Leftrightarrow a \in A, b \in B$, and $a \in A, b \in C$

$\Leftrightarrow (a, b) \in A \times B$ and $(a, b) \in A \times C$

$\Leftrightarrow (a, b) \in (A \times B) \cap (A \times C)$.

Similarly, the followings are correct

1) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2) $(A \cap B) \times C = (A \times C) \cap (B \times C)$
3) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Definition - 3 : For nonempty sets A, B, a function, or mapping, f from A to B, denoted $f: A \to B$ is a relation from A to B in which every element of A appears exactly once as the first component of an ordered pair in the relation.

Ex : Let $A = \{1, 2, 3\}$
$B = \{w, x, y, z\}$
$f = \{(1, w), (2, x), (3, x)\}$ is a function from A to B.
$R_1 = \{(1, w), (2, x)\}$ and
$R_2 = \{(1, w), (2, w), (2, x), (3, z)\}$ are relations, but not functions, from A to B.

Representations of relations.

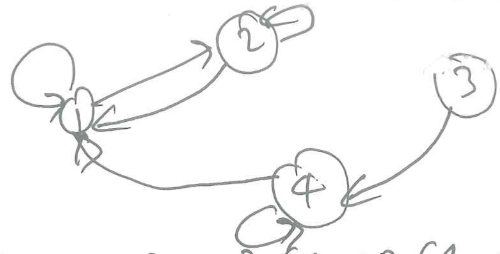
I Set notation
II Matrix form.
Consider the finite sets $A = \{a_1, a_2, \dots, a_m\}$ & $B = \{b_1, b_2, \dots, b_n\}$ of orders m and n respectively. Then $A \times B$ consists of all ordered pairs of the form (a_i, b_j), $1 \le i \le m$, $1 \le j \le n$. Let R be a relation from A to B then R is a subset of $A \times B$.
Define $m \times n$ matrix, m_{ij} as follows

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

Ex: $A = \{0, 1, 2\}$, $B = \{p, q\}$ & $R = \{(0, p), (1, q), (2, p)\}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Digraph:



R1

Ex: $A = \{1, 2, 3, 4\}$.

$R1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

$R2 = \{(1,1), (1,2), (2,1)\}$.

$R3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$

$R4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

$R5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

$R6 = \{(3,4)\}$.

Properties of relations.

1. Reflexive
2. Symetric
3. Antisymmimetric
4. Transitive.

A relation R on a set A is called reflexive if $(a,a) \in R$ for every elements $a \in A$.
Using quantifiers we see that the relation R on the set A is reflexive if $\forall a((a,a) \in R)$, where the universe of discourse is the set of all elements in A.

A relation R on a set A is called symmetric if $(b,a) \in R$ whenever $(a,b) \in R$, for all $a,b \in A$. A relation R on set A such that $(a,b) \in R$ and $(b,a) \in R$ only if $a=b$, for all $a,b \in A$, is called antisymmetric. Using quantifiers, we see that the relation R on the set A is symmetric if $\forall a \forall b((a,b) \in R \rightarrow (b,a) \in R)$. Similarly, the relation on the set A is antisymmetric if $\forall a \forall b(((a,b) \in R \wedge (b,a) \in R) \rightarrow (a=b))$.

A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Using quantifiers we see that the relation R on a set A is transitive if we have $\forall a \forall b \forall c (((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R)$.

Example: The relations R_3, R_5 above are reflexive since they both contain all pairs of the form (a,a) namely, $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4)$. The other relations are not reflexive since they do not contain all of these ordered pairs.

The relations R_2, and R_3 are symmetric, because in each case (b,a) belongs to the relation whenever (a,b) does. For R_2, the only thing to check is that both $(1,2)$ and $(2,1)$ are in the relation. For R_3, it is necessary to check that both $(1,2)$ and $(2,1)$ belongs to the relation, and $(1,4)$ and $(4,1)$ belongs to the relation.

R_4, R_5 and R_6 are all antisymmetric.

R_4, R_5 and R_6 are all transitive.

R_4 is transitive because $(3,2)$ and $(2,1)$, $(4,2)$ and $(2,1)$, $(4,3)$ and $(3,1)$, and $(4,3)$ and $(3,2)$ are the only such sets of pairs, and $(3,1)$, $(4,1)$ & $(4,2)$ belong to R_4.

R_1 is not transitive since $(3,4)$ and $(4,1)$ belongs to R_1, but $(3,1)$ does not. R_2 is not transitive since $(2,1)$ & $(1,2)$ belong to R_2, but $(2,2)$ does not. R_3 is not transitive since $(4,1)$ and $(1,2)$ belongs to R_3, but $(4,2)$ does not.

problem: Consider a relation "divides" on set of all integers. Is this relation reflexive, symmetric, antisymetric & transitive?

Soln: Since $a|a$ whenever a is a +ve integer, the divides relation is reflexive. Suppose that a divides b ~~and~~ then $b \nmid a$, for ep $1|2$ but $2 \nmid 1$, hence it is not symmetric, but it is antisymmetric, because if $a|b$ and $b|a$ iff $a=b$.

Suppose that $a|b$ and $b|c$. Then there are positive integers k & l such that $b=ak$ and $c=bl=a(kl)$ Hence a divides c. It follows that the relation "divides" is a transitive relation.

Operations on relations:—

Since relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Ex: Let $A=\{1,2,3\}$ and $B=\{1,2,3,4\}$. The relations $R_1=\{(1,1),(2,2),(3,3)\}$ and $R_2=\{(1,1),(1,2),(1,3),(1,4)\}$ can be combined to obtain

$R_1 \cup R_2=\{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$

$R_1 \cap R_2=\{(1,1)\}$

$R_1-R_2=\{(2,2),(3,3)\}$

$R_2-R_1=\{(1,2),(1,3),(1,4)\}$

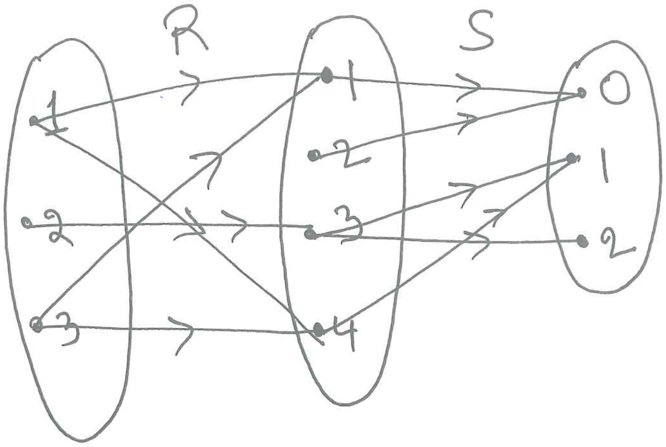
$R_1 \Delta R_2$ or $R_1 \oplus R_2=(R_1-R_2) \cup (R_2-R_1)=\{(1,2),(1,3),(1,4),(2,2),(3,3)\}$.

Composition of relations:—

Let R be a relation from a set A to a set B and S be a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a,c) where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. This is denoted by $S \circ R$

Ex(1): What is the composite of the relations R & S where R is the relation from {1,2,3} to {1,2,3,4} with $R=\{(1,1),(1,4),(2,3),(3,1),(3,4)\}$ and S is the relation from {1,2,3,4} to {0,1,2} with $S=\{(1,0),(2,0),(3,1),(3,2),(4,1)\}$?

Soln:



$$S \circ R = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$$

Ex(2): Composing the parent relation with itself. Let R be the relation on the set of all people such that $(a,b) \in R$ if person a is a parent of person b. Then $(a,c) \in R \circ R$ iff there is a person b such that $(a,b) \in R$ and $(b,c) \in R$, i.e iff there is a person b such that a is a parent of b and b is a parent of c. In other words, $(a,c) \in R \circ R$ iff a is a grandparent of c.

The powers of a relation R can be recursively defined from the definition of a composite of two relations.

$$R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R$$

$$\therefore R^2 = R \circ R,\ R^3 = R^2 \circ R = (R \circ R) \circ R, \text{ and so on.}$$

Ex(3): Let $R=\{(1,1),(2,1),(3,2),(4,3)\}$. Find R^2, R^3.

Soln: $R^2 = R \circ R = \{(1,1),(2,1),(3,1),(4,2)\}$.

$R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$.

Similarly $R^4 = R^5 = \cdots = R^n = R^3$.

Closures of Relations :-

The relation $R = \{(1,1), (1,2), (2,1), (3,2)\}$ on the set $A = \{1,2,3\}$ is not reflexive. How to produce a reflexive relation containing R that is as small as possible? This can be done by adding (2,2) and (3,3) to R, since these are the only pairs of the form (a, a) that are not in R, clearly this new relation contains R.

Any relation R' which is a reflexive relation and it contains the relation R is called as the Reflexive closure.

$\therefore \; R' = R \cup \Delta$, where $\Delta = \{(a,a) \mid a \in A\}$ is the diagonal relation.

Ex: what is the reflexive closure of the relation $R = \{(a,b) \mid a < b\}$ on the set of integers?

Soln: $R \cup \Delta = \{(a,b) \mid a < b\} \cup \{(a,a) \mid a \in Z\}$
$= \{(a,b) \mid a \leq b\}$.

The relation $\{(1,1), (1,2), (2,3), (2,2), (3,1), (3,2)\}$ on $\{1,2,3\}$ is not symmetric. How can we produce a symmetric relation that is as small as possible and contains R? To do this, we need only to add (2,1) and (1,3), since these are the only pairs of the form (b, a) with $(a,b) \in R$ that are not in R. This new relation is symmetric and contains R. furthermore, any symmetric relation that contains R must contain (2,1) and (1,3). This new relation is called the symmetric closure of R.

R^{-1} : inverse relation of $R = \{(a,b) \mid (b,a) \in R\}$

Ex: What is the symmetric closure of the Relation R $= \{(a,b) \mid a > b\}$ on the set of positive integers.

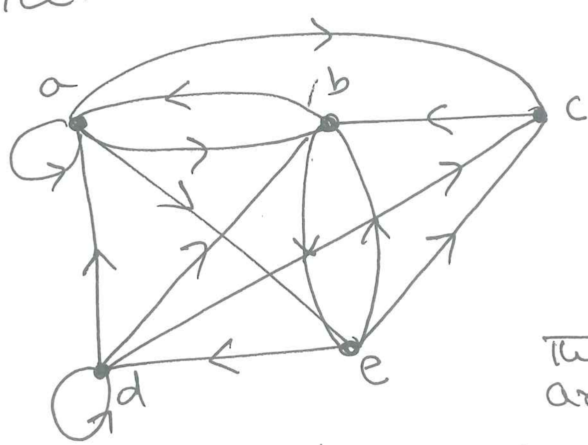
$R \cup R^{-1} = \{(a,b) \mid a > b\} \cup \{(b,a) \mid a > b\} = \{(a,b) \mid a \neq b\}$

Paths in directed graphs :-

Definition: A path from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a non-negative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has length n. Empty set of edges ~~from~~ can be viewed as a path from a to a. A path of length ≥ 1 that has its begin and end at the same vertex is called a circuit or cycle.

A path in a graph can pass through a vertex more than once and the edge can appear more than once.

Ex:



a, b, e, d : is a path of length 3

a, e, c, d, b : is not a path.

b, a, c, b, a, a, b : is a path of length 6.

c, b, a : is a path of length 2.

e, b, a, b, a, b, e : is a path of length 6.

The paths b, a, c, b, a, a, b & e, b, a, b, a, b, e are circuits.

The term path also applies to relations. Carrying over the definition from directed graphs to relations, there is a path from a to b in R if there is a sequence of elements $a, x_1, x_2, \ldots, x_{n-1}, b$ with $(a, x_1) \in R, (x_1, x_2) \in R, \ldots, (x_{n-1}, b) \in R$.

Theorem: Let R be a relation on set A. There is a path of length n, where n is a positive integer, from a to b iff $(a, b) \in R^n$.

Transitive closures:

Definition: Let R be a relation on a set A. The connectivity relation R^* consists of the pairs (a,b) such that there is a path of length at least one from a to b in R.

Since R^n consists of paths (a,b) of length n from a to b, it follows that R^* is the union of all R^n.

$$\boxed{R^* = \bigcup_{i=1}^{\infty} R^i} \qquad R^1 \cup R^2 \cup R^3 \cup R^4 \cup \ldots \cup R^{\infty} = R^*$$

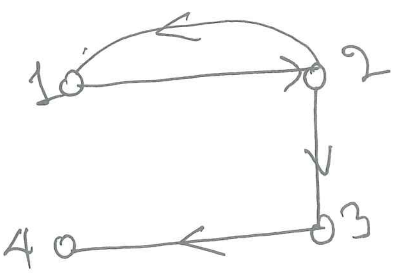
The transitive closure of the relation R equals the connectivity relation R^*.

Ex: Let $A = \{1,2,3,4\}$ and $R = \{(1,2),(2,3),(3,4),(2,1)\}$

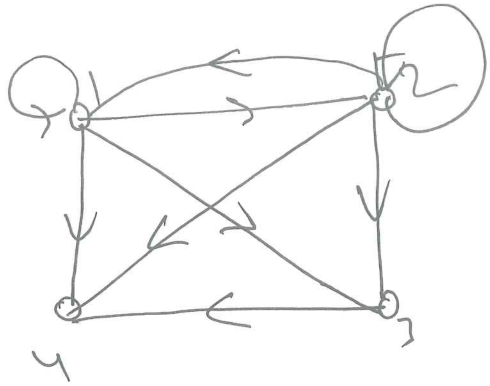
Find R^*.

$R^* = \{(1,2),(2,3),(1,3),(3,4),(2,4),(1,4),(2,1),(1,1),(2,2)\}$

Digraph of R:



Digraph of R^*:



Note: $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$ where $n = |A|$

Verify this with above example :-

$$R^* = R \cup R^2 \cup R^3 \cup R^4$$

$A = \{1, 2, 3, 4\}$ $|A| = 4$

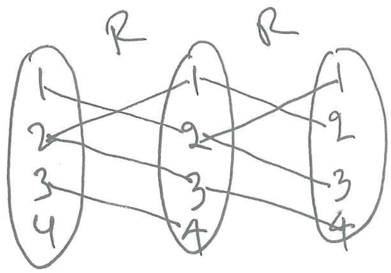
$R = \{(1,2), (2,3), (3,4), (2,1)\}$

$R^2 = R \circ R = \{(1,3), (2,4), (1,1), (2,2)\}$

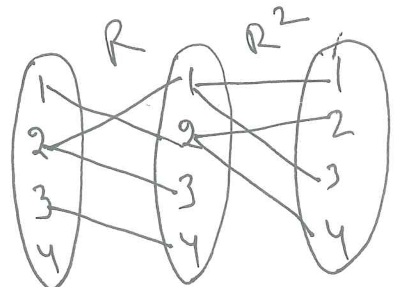
$R^3 = R \circ R^2 = \{(1,4), (2,3), (2,1), (1,2)\}$

$R^4 = R \circ R^3 = \{(1,3), (1,1), (2,4), (2,2)\}$

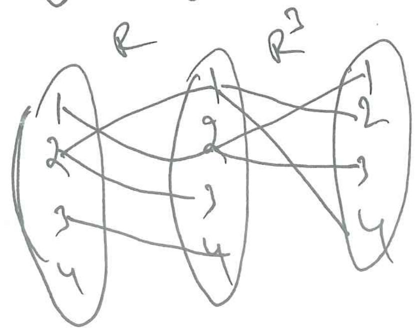
$R^* = R \cup R^2 \cup R^3 \cup R^4 = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)\}$



$= R \circ R = R^2 = \{(1,1), (1,3), (2,2), (2,4)\}$



$= R \circ R^2 = R^3 = \{(1,2), (1,4), (2,1), (2,3)\}$



$= R \circ R^3 = R^4 = \{(1,1), (1,3), (2,2), (2,4)\}$

Warshal's algorithm to compute transitive closure:

Let R be a relation on A.

Represent R by a relation matrix M_R.

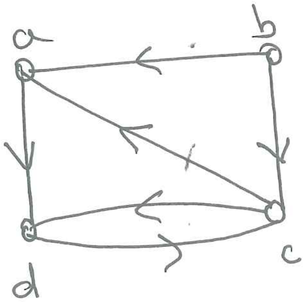
Compute Matrix M_{R^*} by filling the cells such that

if $M_{ij} = 1$ if $M_{R_{ij}} = 1$ or $M_{R_{ik}} = 1$ & $M_{R_{kj}} = 1$

else $M_{ij} = 0$.

```
for i ← 1 to n do
    for j ← 1 to n do
        for k ← 1 to n do
            M_ij = M_ij ∨ (M_ik ∧ M_kj)
```

Ex(1):



$R = \{(a,d), (b,a), (b,c), (c,a), (c,d), (d,c)\}$

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Find $M_{R^*} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ How ?

I find R^2, R^3, R^4 and their matrices and take union (or) of all these matrices to get M_{R^*}.

II List all paths of length 2, 3, 4 and their corresponding matrices and finally or ed them.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Equivalence Relations:

Definition: A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Two elements that are related by an equivalence relation are called equivalent.

Ex(1): Suppose that R is the relation on the set of strings of English letters such that aRb iff $l(a) = l(b)$, where $l(x)$ denotes the length of the string x. Is R an equivalence relation.

Soln: Since $l(a) = l(a)$, it follows that aRa whenever a is a string, so that R is reflexive. Suppose that aRb, so that $l(a) = l(b)$. Then bRa, since $l(b) = l(a)$. Hence, R is symmetric. Finally, suppose that aRb and bRc. Then $l(a) = l(b)$ and $l(b) = l(c)$. Hence, $l(c) = l(a)$, so that aRc. Consequently, R is transitive. It follows that R is an equivalence relation.

Ex(2): Let R be the relation on the set of real numbers such that aRb if and only if $a-b$ is an integer. Is R an equivalence relation?

Ex(3): Congruence Modulo m. Let m be a positive integer with $m > 1$. S.T. the relation $R = \{(a,b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation.

Ex(4): Which of these are equivalence relations on $\{0,1,2,3\}$

a) $\{(0,0), (1,1), (2,2), (3,3)\}$

b) $\{(0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3)\}$

c) $\{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3)\}$

d) $\{(0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

e) $\{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3)\}$

Equivalence classes and partition of set :-

If R is a equivalence relation on a set A and $a \in A$, then the R-relative set of a, namely $R(a)$ is defined as $[a] = R(a) = \{x \in A \mid aRx\}$, is called as equivalence ~~relation~~ class, determined by a w.r.t. R.

What is the maximum number of equivalence classes of R on A ?

If $|A| = n$ $\therefore$ There are n equivalence classes.

Theorem-1: Let R be an equivalence relation on a set A, and let $a, b \in A$. Then aRb iff $R(a) = R(b)$.

$\Rightarrow$ Suppose aRb. Take any $x \in R(a)$, then aRx. Since R is symmetric, it follows that xRa. Then we have xRa & aRb. Since, R is transitive, it follows that xRb. Since R is symmetric, it follows that bRx $\therefore$ $x \in R(b)$. Hence, $R(a) \subseteq R(b)$. Similarly, we find $R(b) \subseteq R(a)$. Therefore $R(a) = R(b)$.

Partition of a set :-

Let A be a non-empty set, and P be a family of non-empty subsets of A such that

i) Every element of A belongs to one of the sets in P.

ii) Any two distinct sets belongs to P are mutually distinct.

Then we call P as the partition, decomposition or quotient set of A. The sets in P are called as blocks or cells.

A partition of a set consisting of 6 blocks is shown in fig.

Ex: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and its subsets
$A_1 = \{1, 3, 5, 7\}$, $A_2 = \{2, 4\}$, $A_3 = \{6, 8\}$,
$A_4 = \{1, 3, 5\}$, $A_5 = \{5, 6, 8\}$.

Let $P = \{A_1, A_2, A_3\}$ is partition of A because Every element of A belongs one of the subsets A_1, A_2, A_3 and all these are disjoint.

Suppose if $P = \{A_2, A_3, A_4\}$ then P is not a partition of A because 7 not belongs to any of these three sets, even though they are all disjoint.

If $P = \{A_1, A_2, A_5\}$ is not a partion because 5 belongs to A_1 & A_5 hence these two are not disjoint.

Theorem: Let R be an equivalence relation on set A, and let P be the collection of all distinct R-relative sets in A. Then P is the partition of A, and R is the equivalence relation determined by P.

$\Rightarrow$ Here P is the collection of all distinct relative sets $R(a)$, $a \in A$.
We note that, for all $a \in A$, we have aRa, that is, $a \in R(a)$, because R is reflexive. Thus, every element of A belongs to one of the sets in P.
If $R(a)$ and $R(b)$ are distinct sets belongs to P, then $R(a) \cap R(b) = \phi$. This fact prove that P is a partition of the set A. This partition determines the relation R in the sence that aRb iff a and b belong to the same block of the partition and hence

Ex: Let $A = \{a, b, c, d\}$, and $P = \{\{a,b,c\},\{d\}\}$
Find the equivalence relation R induced by P.

$\Rightarrow$ Since a, b, c belongs to one block we have
$\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$
& d belongs to another block $\therefore (d,d) \in R$.

$\therefore R = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c),(d,d)\}$

Ex: Let $A = \{1,2,3,4,5,6,7\}$, and R be the relation
on A defined by aRb whenever $a-b$ is divisible
by 3. S.T R is an equivalence relation. Determine
the partition of A induced by R.

$\Rightarrow$ For any $a \in A$, we have $a - a = 0$ which is divisible
by 3, Therefore, aRa for $a \in A$. Accordingly, R is reflexive.
Next, suppose aRb, for $a, b \in A$. Then $a-b$ is divisible
by 3. This implies that $b-a$ is divisible by 3, so that
bRa, hence R is symmetric.
Lastly, suppose aRb and bRc, $a,b,c \in A$. Then $a-b$
is divisible by 3 and $b-c$ is divisible by 3, so that
aRc, hence R is transitive. $\therefore R$ is equivalence relation.

$R = \{(1,1),(1,4),(1,7),(2,2),(2,5),(3,3),(3,6),(4,1),(4,4),$
$(4,7),(5,2),(5,5),(6,3),(6,6),(7,1),(7,4),(7,7)\}$

$R(1) = \{1, 4, 7\} = R(4) = R(7)$.
$R(2) = \{2, 5\} = R(5)$
$R(3) = \{3, 6\} = R(6)$

$\therefore$ The partion of set A is $P = \{\{1,4,7\},\{2,5\},\{3,6\}\}$.

Partial Ordering :-

Definition: A relation R on a set S is called a
partial ordering or partial order if it is reflexive,
antisymmetric, and transitive. A set S together
with a partial ordering R is called a partially
ordered set, or poset, and is denoted by (S, R).

Ex (1): Show that the "greater than or equal" relation
$(\geq)$ is a partial ordering on the set of integers.

$\Rightarrow$ Since $a \geq a$ for every integer a, $\geq$ is reflexive.
If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric. finally, $\geq$ is transitive since $a \geq b$ and $b \geq c$
imply that $a \geq c$. It follows that $\geq$ is a partial
ordering on the set of integers and $(\mathbb{Z}, \geq)$ is a poset.

Ex (2): The relation "divides" on the set of positive
integers is a partial ordering as we saw earlier
that it is reflexive, antisymmetric and transitive.
Hence, $(\mathbb{Z}^+, |)$ is a poset.

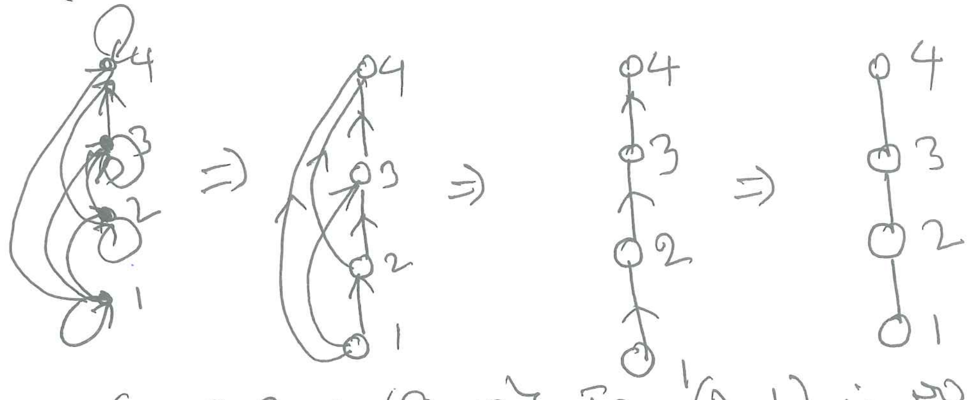
Ex (3): S.T the inclusion relation $\subseteq$ is a partial
ordering on the power set of a set S.

$\Rightarrow$ Since $A \subseteq A$ whenever A is a subset of S,
$\subseteq$ is reflexive. It is antisymmetric since $A \subseteq B$
and $B \subseteq A$ imply that $A = B$. finally, $\subseteq$ is
transitive, since $A \subseteq B$ and $B \subseteq C$ imply that
$A \subseteq C$. Hence, $\subseteq$ is a partial ordering on $P(S)$,
and $(P(S), \subseteq)$ is a poset.

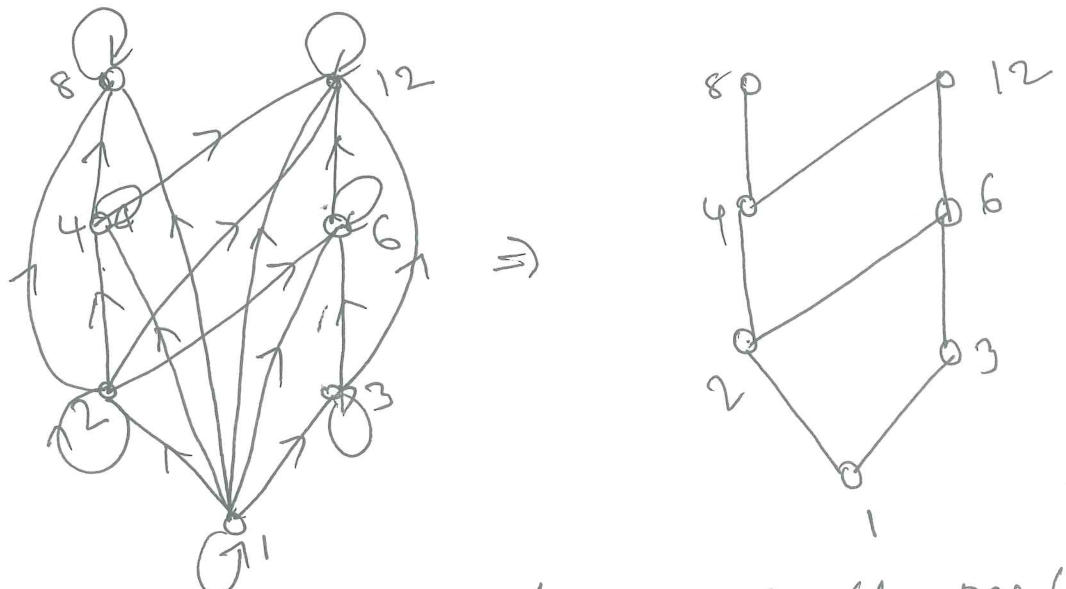
HASSE DIAGRAMS:

Many edges in the directed graph for a finite
poset do not have to be shown since they must
be present. i.e self loops and transitive edges need
not be shown in the graph. Moreover if will assume
all the edges are directed upwords then we can
drop the arrows or direction. This modified
drop ...

EX (1): Let $A = \{1, 2, 3, 4\}$ and the relation is $\leq$. Then $(A, \leq)$ is a poset. Its directed graph is

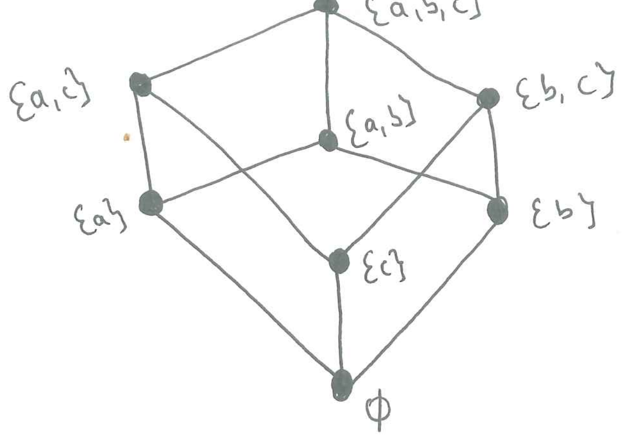


EX (2) Let $A = \{1, 2, 3, 4, 6, 8, 12\}$. Is $(A, |)$ is poset, if yes give its Hasse digram.



EX (3): Draw the Hasse diagram for the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set $P(S)$ where $S = \{a, b, c\}$.

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$



MAXIMAL AND MINIMAL ELEMENTS :-

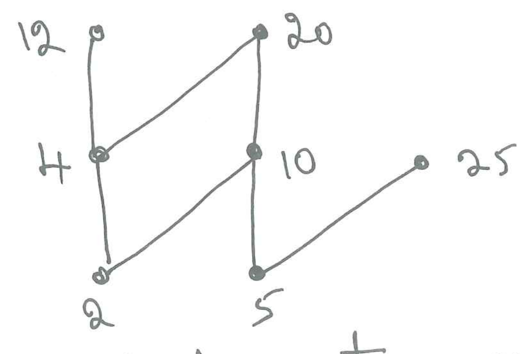
Elements of poset that have certain external properties are important for many applications.

An element of a poset is called <u>maximal</u> if it is not less than any element of the poset. i-e if a, is maximal in the poset (S, R) if there is no $b \in S$ such that aRb.

Similarly, an element of a poset is called <u>minimal</u> if it not greater than any element of the poset, i-e if a is said to be minimal if there is no $b \in S$ such that bRa.

These are the elements at top and bottom of a Hasse diagram of the poset.

<u>EX(1)</u>: which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal, and which are minimal.



<u>Soln</u>: Maximal elements are 12, 20, 25

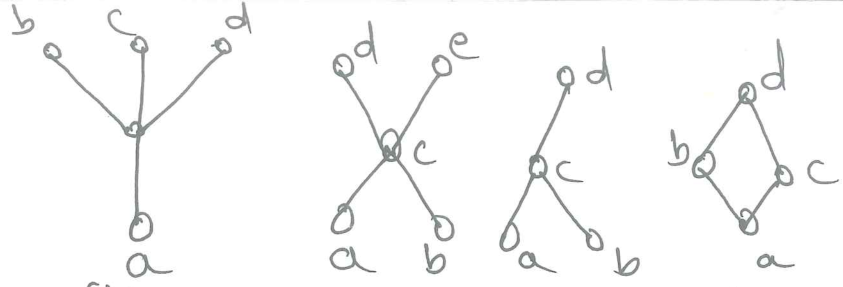
Minimal elements are 2, 5.

There fore, a poset can have more than one maximal and minimal elements.

<u>Greatest and Least element of a poset</u> :-

If there is an element in a poset that is greater than every other element. Such an element is called the greatest element. i-e if a is <u>greatest element</u> of the poset (S, R) if bRa, for all $b \in S$. The greatest element is unique if it exists. Likewise, an element is called the least element if it is less than all the

Ex(2):



(i) (ii) (iii) (iv)

Determine whether the posets given above have a greatest and least element.

Soln: (i) Has least element i.e a
This poset has no greatest element.
(ii) This poset has neither least nor greatest element.
(iii) This has the greatest element i.e d but no least element.
(iv) This has both greatest and least elements
greatest element is d
least element is a

Ex(3): Let S be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.

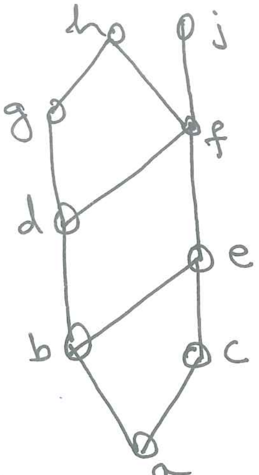
Soln: The least element is ϕ (null set) $\because \phi \subseteq$ all sets
The greatest element is the set S itself. $\because$ all subsets are subsets to S.

Ex(4): Is there a greatest element and a least element in the poset $(Z^+, |)$.

Soln: The integer 1 is the least element since $1|n$ whenever n is a positive integer. Since there is no integer that is divisible by all positive integers, there is no greatest element.

Sometimes it is possible to find an element that is greater than all the elements in a subset A of a poset (S, R). If u is an element of S such that aRu for all $a \in A$ then u is called as upper bound of A. Likewise, there may be an element less than all the elements in A. If l is an element of S such that l R a

Ex(5):



Find the lower bounds and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$.

Soln: $\{a, b, c\}$ upper bounds are e, f, h, j
lower bounds are only a

$\{h, j\}$ upper bounds are nil
lower bounds are a, b, c, d, e, f

$\{a, c, d, f\}$ upper bounds are h, j
lower bounds are a only.

The element x is called the least upper bound of the subset A if x is an upper bound that is less than every other upper bound of A. Since there is only one such element, if it exists, it makes sense to call this element the least upper bound.

Similarly, the element y is called the greatest lower bound of A if y is a lower bound of A and $z R y$ whenever z is a lower bound of A.

The least upper bound and the greatest lower bound is unique if it exists.

Ex(6): Find glb and lub of $\{b, d, g\}$, if they exist in the poset shown in Ex(5).

Soln: The upper bounds of $\{b, d, g\}$ are g & h
since g lower than h, g is the lub.
The lower bounds of $\{b, d, g\}$ are $\{a, b\}$
Since b is upper than a $\therefore$ glb$\{b, d, g\}$ is b.

Ex(7): $(Z^+, |)$, lub$\{3, 9, 12\} = 36 = \text{lcm}(3, 9, 12)$