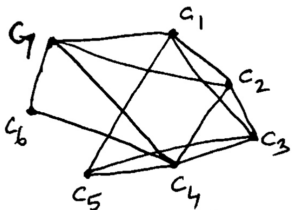
Graph Theory

A major publishing company has ten editors (referred to by 1, 2, ..., 10) in the scientific, technical and computing areas. These ten editors have a standard meeting time during the first Friday of every month and have divided themselves into seven committees to meet later in the day to discuss specific topics of interest to the company namely (i) advertising (ii) securing reviewers, (iii) contacting new potential authors, (iv) finances (v) used copies and new editions, (vi) competing textbooks and (vii) textbook representatives.

The ten editors have decided on the seven committees: $C_1 = \{1,2,3\}$, $C_2 = \{1,3,4,5\}$, $C_3 = \{2,5,6,7\}$, $C_4 = \{4,7,8,9\}$, $C_5 = \{2,6,7\}$, $C_6 = \{8,9,10\}$, $C_7 = \{1,3,9,10\}$. They have set aside three time periods for the seven committees to meet on those Fridays when all ten editors are present. Some pairs of committees cannot meet during the same period because one or two of the editors are on both committees. This situation can be modeled visually as shown in the below figure.

In this ~~grap~~ figure there are several small dots, representing the seven committees and a straight line segment is drawn between two circles if the committees they represent have at least one committee member in common. In other words, a straight line segment between two committees tells us that these two committees should be scheduled to meet at the same time. This gives us a picture or a model of the committees and the overlapping nature of their membership.

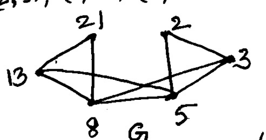


The above figure is called a graph.

Graph:
A graph is a figure consisting of finite nonempty set V of objects called vertices and a set E of 2-element subsets of V called edges. The sets V and E are the vertex set and edge set of G respectively. So, a graph G is a pair, $G = \{V, E\}$.

example
Consider the set $S = \{2, 3, 5, 8, 13, 21\}$. Let the edge set be those pair of distinct numbers of S such that the sum or difference also belongs to S. Construct the graph with S and $E = (x, y)$ s.t $|x-y| \in S$.

Sol^n vertex set $= \{2, 3, 5, 8, 13, 21\}$
edge set is $\{(2,3), (2,5), (3,5), (3,8), (5,8), (5,13), (8,13), (8,21), (13,21)\}$
∴ The graph is:

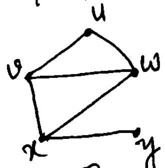


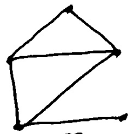
G

Order(n) and size(m) of a graph.
The number of vertices in the graph G is called the order of G, and the number of edges in the graph G is called the size of G.
The above graph G has order 6 and size 8.

* A graph with exactly one vertex is called a trivial graph, implying that the order of a nontrivial graph is at least 2.

A graph G with $V(G) = \{u, v, w, x, y\}$ and $E(G) = \{uv, uw, vw, vx, wx, xy\}$ is shown below.





(a) labelled graph   (b) unlabelled graph.

Here (a) is drawn with labelling and (b) is drawn without labelling. Hence the graph (a) is a labelled graph and (b) is an unlabelled graph.

Terminologies

* The two vertices u and v are end vertices of the edge (u, v).
* Edges that have the same end vertices are parallel edges.
* An edge of the form (v, v) is a loop.
* A graph is simple graph if it has no parallel edges or loops.
* A graph with no edges is empty graph.
* A graph with no vertices (and hence no edges) is a null graph.
* A graph with only one vertex is trivial graph.
* Two vertices u and v are adjacent vertices if they are connected by an edge, in other words (u, v) is an edge. Also u and v are referred to as neighbours of each other.
* Edges are said to be adjacent edges if they share a common end vertex (or they are incident with a common vertex).

Subgraphs

A graph H is called a <u>subgraph</u> of a graph G, written $H \subset G$, if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. We also say that G contains H as a subgraph.

If $H \subset G$ and either $V(H)$ is proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a <u>proper subgraph</u> of G.

If a subgraph of a graph G has the same vertex set a G, then it is a <u>spanning subgraph</u> of G.

A subgraph F of a graph G is called an <u>induced subgraph</u> of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well.

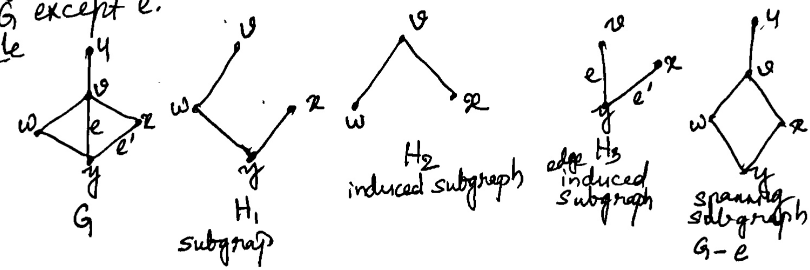
If S is a nonempty set of vertices of a graph G, then the subgraph of G induced by S is the induced subgraph with vertex set S. The induced subgraph is denoted by $\langle S \rangle$.

For a nonempty set X of edges, the subgraph induced by X has edge set X and consisting of all vertices that are incident with atleast one edge in X. This subgraph is called an <u>edge-induced subgraph</u> of G. It is denoted by $\langle X \rangle$.

Any proper subgraph of a graph G can be obtained by removing vertices and edges from G.

For an edge e of G, we write $G-e$ for the spanning subgraph of G whose edge set consists of all edges of G except e.

<u>example</u>



Walk, Trail, Path, Circuit, Cycle.

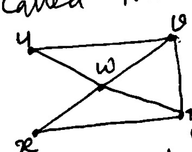
A *u-v walk* W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent, that is, we can express W as $W: u = v_0, v_1, \ldots, v_k = v$, where $k \geq 0$ and v_i and v_{i+1} are adjacent for $i = 0, 1, \ldots k-1$.

Each vertex v_i, $(0 \leq i \leq k)$ and each edge $v_i v_{i+1}$ $(0 \leq i \leq k-1)$ is said to lie on or belong to W.

If $u = v$, then the walk W is *closed*, ~~while~~ while if $u \neq v$, then W is *open*.

As we move from one vertex of W to the next, we are encountering or traversing edges of G, possibly traversing some edges of G more than once.

The number of edges encountered in a walk (including multiple occurrences of an edge) is called the length of the walk.



$W_1: x, y, w, y, v, w$ is a walk of length 5 also an open walk.

$W_2: w\,x\,y\,w$ is a closed walk of length 3

A walk of length 0 is a trivial walk. eg $W_3: v$ is a trivial walk.

A *u-v trail* in a graph G is a u-v walk in which no edge is traversed more than once.

A *u-v path* is a u-v walk in which no vertex is repeated. If no vertex in a walk is repeated, then no edge is repeated either. (Hence every path is a trail).

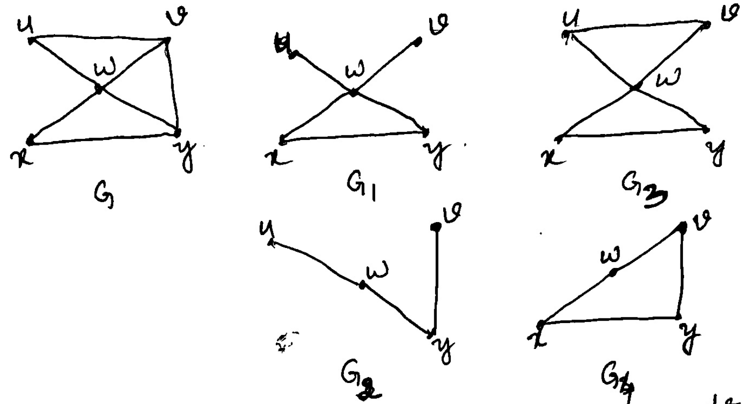
A *circuit* is a closed trail.

A *cycle* is a closed path.

A *k-cycle* is a cycle of length k.

A cycle of odd length is called an odd cycle.

A cycle of even length is called an even cycle.



Here the subgraphs G_1, G_2, G_3, G_4 are a trail, path, circuit and cycle respectively.

Connected and Disconnected Graphs.

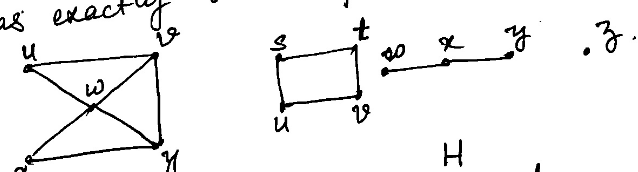
If G contains a u-v path, then u and v are said to be connected, and u is connected to v.

A graph G is connected if every two vertices of G are connected, that is, if G contains a u-v path for every pair u, v of distinct vertices of G.

A graph G that is not connected is called disconnected.

A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a component of G.

A graph G is then connected if and only if it has exactly one component.



G
connected.

H
disconnected.

[$= H_1 \cup H_2 \cup H_3$, where H_1, H_2, H_3 are called components of H]

Remark: If $x_1, x_2, \ldots, x_k \geq 1$ are real numbers and $x = x_1 + x_2 + \cdots + x_k$, then $\sum_{i=1}^{k} x_i^2 \leq x^2 - (k-1)(2x-k)$.

Proof: For each i let $x_i = y_i + 1$, where $y_i \geq 0$.

Let $y = y_1 + y_2 + \cdots + y_k$

Then $x^2 - (k-1)(2x-k)$

$$= (x_1 + x_2 + \cdots + x_k)^2 - (k-1)(2(x_1 + x_2 + \cdots + x_k)) - k$$
$$= (y_1 + 1 + y_2 + 1 + \cdots + y_k + 1)^2 - (k-1)(2(y_1 + 1 + y_2 + 1 + \cdots y_k + 1)) - k$$
$$= (y+k)^2 - (k-1)(2y + 2k - k)$$
$$= y^2 + 2yk + k^2 - 2yk - k^2 + 2y + k$$
$$= y^2 + 2y + k$$
$$= \left(\sum_{i=1}^{k} y_i\right)^2 + 2\sum_{i=1}^{k} y_i + k$$
$$\geq \sum_{i=1}^{k} y_i^2 + 2\sum_{i=1}^{k} y_i + k$$
$$= \sum_{i=1}^{k} (y_i^2 + 2y_i + 1)$$
$$= \sum_{i=1}^{k} (y_i + 1)^2$$
$$= \sum_{i=1}^{k} x_i^2$$

Hence $\sum_{i=1}^{k} x_i^2 \leq x^2 - (k+1)(2x+k)$.

Theorem:

For a simple graph G with n vertices and k components $m \leq \frac{(n-k)(n-k+1)}{2}$.

Proof. Let $H_1, \ldots, H_k$ be the components of G.

For $1 \leq i \leq k$, denote by n_i the number of vertices of H_i.

Note that $n = n_1 + n_2 + \cdots + n_k$.

Each H_i is a simple graph.

Each H_i can have at most $\frac{n_i(n_i-1)}{2}$ edges.

Hence, the maximum number of edges G has is

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i$$

$$\leq \frac{1}{2}\left[n^2 - (k-1)(2n-k)\right] - \frac{1}{2} \times n$$

$$= \frac{1}{2}\left[n^2 - 2nk + k^2 + 2n - k - n\right]$$

$$= \frac{1}{2}\left[n^2 - 2nk + k^2 + n - k\right]$$

$$= \frac{1}{2}\left[(n-k)^2 + (n-k)\right]$$

$$= \frac{(n-k)(n-k+1)}{2}$$

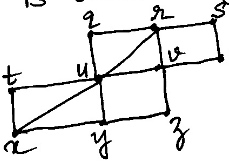
Distance between two vertices

Let G be a connected graph of order n and let u and v be two vertices of G. The <u>distance</u> between u and v is the smallest length of any u-v path in G and is denoted by $d(u,v)$.

Hence if $d(u,v)=k$, then there exists a u-v path $P: u=v_0 v_1, \ldots, v_k = v$ of length k in G, but no u-v path of smaller length exists in G.

A u-v path of length $d(u,v)$ is called a <u>u-v geodesic</u>.

The greatest distance between any two vertices of a connected graph G are called the diameter of G and is denoted by $\text{diam}(G)$.



Here $d(q,z)=3$, $d(u,v)=1$

$d(t,s)=3$, $d(u,x)=1$

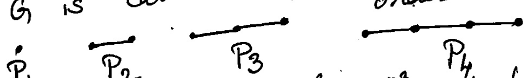
$d(x,v)=2$, $d(w,x)=3$

$d(y,r)=2$, $d(x,y)=1$

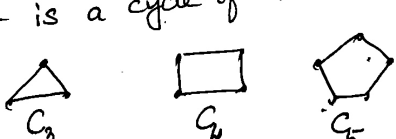
the diameter is 3.

Common classes of graphs.

If a graph G of order n can be labeled (or relabeled) as $v_1, v_2, \ldots, v_n$ so that its edges are $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$, then G is called a path. A graph that is a path of order n is denoted as P_n.



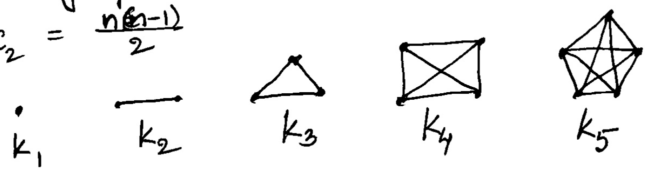
If a graph G of order $n \geq 3$ can be labeled (or relabeled) as $v_1, v_2, \ldots, v_n$ so that its edges are $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$, then G is called a cycle. A graph that is a cycle of order $n \geq 3$ is denoted as C_n.



A graph G is complete if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n.

Since every two distinct vertices of K_n are joined by an edge, K_n has the maximum possible size for a graph with n vertices and is equal to

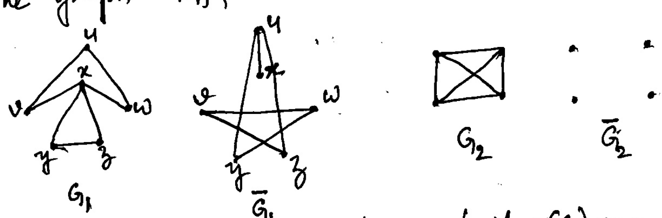
$${}^{n}C_2 = \frac{n(n-1)}{2}$$



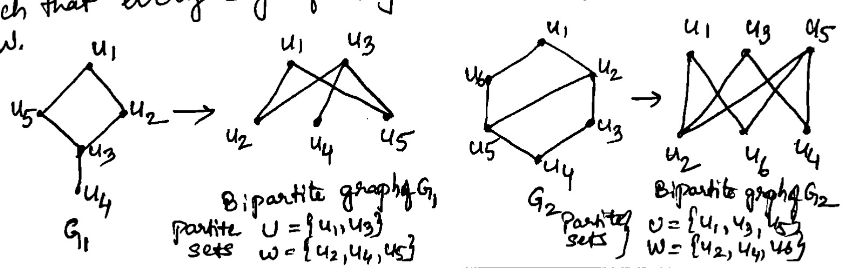
The complement $\bar{G}$ of a graph G is that graph whose vertex set is $V(G)$ such that for each pair u, v of vertices of G, uv is an edge of $\bar{G}$ if and only if uv is not an edge of G.

If G is a graph of order n and size m, then $\bar{G}$ is a graph of order n and size ${}^{n}C_2 - m$. ~~The~~

The graph $\bar{K}_n$, then has n vertices and no edges.

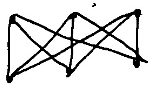


A graph G is a bipartite graph if $V(G)$ can be partitioned into two subsets U and W called partite sets, such that every edge of G joins a vertex of U and a vertex of W.



If for every $u \in U$ and every $w \in W$, the edge (u,w) is an edge of the graph G, where U and W are partite sets, then G is called complete bipartite graph and it is denoted by $K_{p,q}$. where p is the number of vertices in U and q is the number of vertices in W.

examples



$K_{3,3}$

$K_{2,3}$

* Number of vertices in $K_{p,q}$ is $n = p+q$.
* Number of edges in $K_{p,q}$ is $m = pq$

<u>Theorem:</u>
If a nontrivial, connected graph G is bipartite, then G contains no odd cycles.

<u>Proof:</u>
Suppose G is a nontrivial, connected bipartite graph. containing ~~and~~ odd cycle $v_1, v_2, \ldots, v_{2k+1}, v_1$, for some integer k.

Let V_0 and V_e be the partite sets.

Let $v_1 \in V_0$.

Since v_1v_2 is an edge, $v_2 \in V_e$.

Since v_2v_3 is an edge, $v_3 \in V_0$.

Continuing this reasoning $2k$ times we conclude that $v_{2k+1} \in V_0$.

Since $v_{2k+1}v_1$ is an edge, $v_1 \in V_e$.

Now we have $v_1 \in V_0$ and $v_1 \in V_e$, a contradiction

Hence our assumption is wrong.

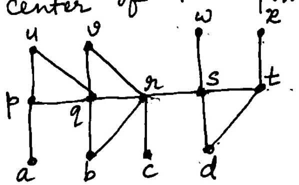
$\therefore$ If a nontrivial, connected graph G is bipartite then G contains no odd cycles.

[The converse: If G has no odd cycles then G is bipartite is also true
* We are not proving that here*]

Eccentricity of a vertex

For a vertex v in a connected graph G, the eccentricity $e(v)$ of v is the distance between v and a vertex farthest from v in G. The radius of a graph is the minimum eccentricity among all vertices. The diameter of a graph is the maximum eccentricity among all vertices. The center of G is a vertex having minimum eccentricity.

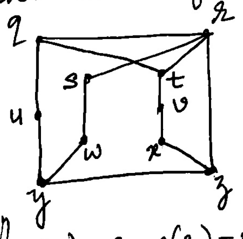
* Find the eccentricity of each vertex, radius, diameter and center of the following graph.



Sol^n^ $e(u)=5$, $e(v)=5$, $e(w)=5$, $e(x)=6$, $e(p)=5$, $e(q)=4$, $e(a)=6$, $e(b)=4$, $e(c)=4$, $e(d)=5$. $e(r)=3$, $e(s)=4$, $e(t)=5$,

radius $=3$, diameter $=6$, center is 3.

* Find the eccentricity of each vertex, radius, diameter and center of the following graph



Sol^n^ $e(q)=3$, $e(r)=2$, $e(s)=3$, $e(t)=3$, $e(u)=3$, $e(v)=4$, $e(w)=4$, $e(x)=3$, $e(y)=3$, $e(z)=2$

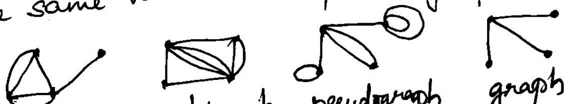
radius $=2$, diameter $=4$, centers are r and z.

<u>Multigraphs and Digraphs</u>

In a graph two vertices are either adjacent or they are not, that is, two vertices are joined by one edge or no edge.

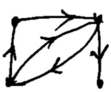
A <u>multigraph</u> M consists of a finite nonempty set V of vertices and a set E of edges, where every two vertices of M are joined by a finite number of edges (possibly zero). If two or more edges join the same pair of (distinct) vertices, then these edges are called <u>parallel edges.</u>

In a <u>pseudograph</u>, not only are parallel edges permitted but an edge is also permitted to join a vertex to itself. Such an edge is called a <u>loop</u>. If a loop e joins a vertex v to itself, then e is said to be a loop at v. There can be any finite number of loops at the same vertex in a pseudograph.



multigraph multigraph pseudograph graph

A <u>digraph</u> (or directed graph) D is a finite nonempty set V of objects called vertices together with a set E of ordered pairs of distinct vertices. The elements of E are called <u>directed edges</u> or <u>arcs</u>. If (u,v) is a directed edge, then we indicate this in a diagram representing D by drawing a directed line segment or curve from u to v. Then u is said to be adjacent to v and v is adjacent from u. Arcs (u,v) and (v,u) may both be present in some directed graph.



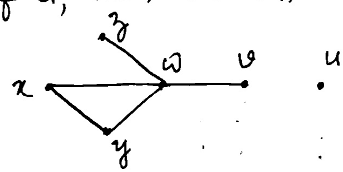
Degree of a vertex

The degree of a vertex v in a graph G is the number of edges incident with v and is denoted by $\deg v$. Also, $\deg v$ is the number of vertices adjacent to v. Two adjacent vertices are referred to as neighbours of each other. The set $N(v)$ of neighbours of a vertex v is called the neighbourhood of v. Thus $\deg v = |N(v)|$.

A vertex of degree 0 is referred to as an isolated vertex and a vertex of degree 1 is called a pendant vertex (end-vertex).

The minimum degree of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$, the maximum degree of G is denoted by $\Delta(G)$.

So if G is a graph of order n and v is any vertex of G, then $0 \le \delta(G) \le \deg v \le \Delta(G) \le n-1$.



Here $n=6$, $m=5$

u is an isolated vertex.

$\deg(u)=0$, $\deg(v)=1$, $\deg(w)=4$

$\deg(x)=2$, $\deg(y=2$, $\deg(z)=1$.

v and z are pendant vertices

$\delta(G)=0$, & $\Delta(G)=4$

The first theorem of graph theory

If G is a graph of size m, then $\sum_{v \in V(G)} \deg v = 2m$.

Proof:

When summing the degrees of the vertices of G, each edge of G is counted twice, once for each of its incident vertices.

Note: Suppose that G is a bipartite graph of size m with partite sets $U=\{u_1, u_2, \dots u_s\}$ and $W=\{w_1, w_2 \dots w_t\}$. Since every edge of G joins a vertex of U and a vertex of W it follows that adding degrees of the vertices in U (or W) gives the number of edges in G, i.e, $\sum_{i=1}^{s} \deg u_i = \sum_{i=1}^{t} \deg w_i = m$

Note; The vertex of even degree is called an even vertex, while a vertex of odd degree is an odd vertex

Corollary

Every graph has an even number of odd vertices

Proof; Let G be a graph of size m.

Divide $V(G)$ into two subsets V_1 and V_2, where V_1 consists of odd vertices of G and V_2 consists of the even vertices of G.

By First Theorem of graph theory,

$$\sum_{v \in V(G)} \deg(v) = \sum_{v \in V_1} \deg v + \sum_{v \in V_2} \deg v = 2m$$

The sum $\sum_{v \in V_2} \deg v$ is even since it is a sum of even integers

$2m$ is even.

$$\therefore \sum_{v \in V_1} \deg v = 2m - \sum_{v \in V_2} \deg v,$$

$$\Rightarrow \sum_{v \in V_1} \deg v = \text{even}.$$

Since each of the members $\deg v \in V_1$ is odd, the number of odd vertices of G is even.

Corollary

If G is a graph of order n with $\delta(G) \geq \frac{n-1}{2}$, then G is connected.

example

A certain graph G has order 14 and size 27. The degree of each vertex of G is 3, 4 or 5. There are six vertices of degree 4. How many vertices of G have degree 3 and how many have degree 5?

Sol^n: let x be the number of vertices of degree 3, and y of degree 5

Then $x + y + 6 = 14 \Rightarrow y = 8 - x$ $\quad x \times 3 + 6 \times 4 + (8-x) \times 5 = 2 \times 27 \Rightarrow x = 5$

i.e. 5 vertices ~~are~~ of degree 3 and 3 vertices are of degree 5. & $y = 3$

Regular graphs

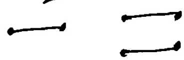
We know that $0 \le \delta(G) \le \Delta(G) \le n-1$ for every graph G of order n. If $\delta(G) = \Delta(G)$ then the vertices of G have the same degree and G is called regular.

If $\deg v = r$ for every vertex v of G, where $0 \le r \le n-1$, then G is r-regular or regular of degree r.

0-regular

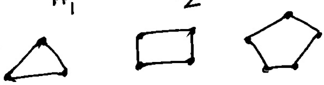
G_1 G_2 G_3

1-regular

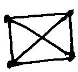


H_1 H_2

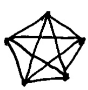
2-regular



3-regular

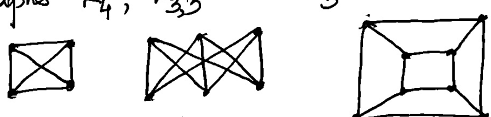


4-regular



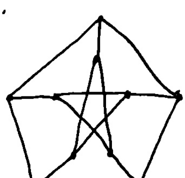
A 3-regular graph is also referred to as a cubic graph.
The graphs K_4, $K_{3,3}$ and Q_3 are cubic graphs

ex,



K_4 $K_{3,3}$ Q_3

The best known cubic graph is the Petersen graph, shown below.

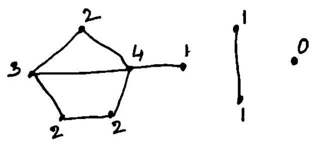


Theorem: Let r and n be integers with $0 \le r \le n-1$. There exists an r-regular graph of order n if and only if at least one of r and n is even.

Degree Sequences

It is typical for the vertices of a graph to have a variety of degrees. If the degrees of the vertices of a graph G are listed in a sequence s, then s is called a degree sequence of G.

For example, $s: 4,3,2,2,2,1,1,1,0$; $s': 0,1,1,1,2,2,2,3,4$; $s'': 4,3,2,1,2,2,1,1,0$. all of the sequences are degree sequences of the graph G of the below figure, each of whose vertices is labeled by its degree



The sequence s is non-increasing, s' is non-decreasing and s'' is neither.

Suppose that we are given a finite sequence s of nonnegative integers. This finite sequence of nonnegative integers is called graphical if it is a degree sequence of some graph.

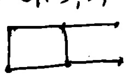
examples

Which of the following sequences are graphical.

(i) $S_1: 3,3,2,2,1,1$, (ii) $S_2: 6,5,5,4,3,3,3,2,2$, (iii) $S_3: 7,6,4,4,3,3,3$,

(iv) $S_4: 3,3,3,1$.

Sol. (i) $S_1: 3,3,2,2,1,1$, $\sum \deg v_i = 12$ (even) ✓, 4 vertices of odd degree. $n=6$, maximum deg ≤ 5 ✓ $\therefore$ graphical



(ii) $S_2: 6,5,5,4,3,3,3,2,2$, $\sum \deg v_i = 33$ not even $\therefore$ not graphical

(iii) $S_3: 7,6,4,4,3,3,3$, $\sum \deg v_i = 30$ (even) ✓ 4 vertices of odd degree. $n=7$, maximum deg $\not\le 6$ $\therefore$ not graphical.

(iv) $S_4: 3,3,3,1$, $\sum \deg v_i = 10$ (even) ✓ 4 vertices of odd degree ✓ $n=4$, max deg ≤ 3 ✓.

But we can not draw a graph with this sequence. 

Only 1 vertex can be of deg 3, other two have to be of deg 2. $\therefore$ not graphical

Isomorphic Graphs

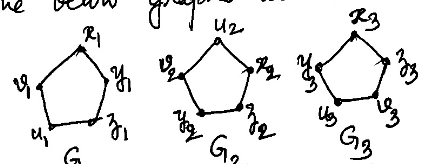
Two (labeled) graphs G_1 and G_2 are isomorphic (have the same structure), if there exists a one-to-one correspondence from $V(G_1)$ to $V(G_2)$ such that if $uv \in E(G_1)$ then $f(u)f(v) \in E(G_2)$ and if $uv \notin E(G_2)$ then $f(u)f(v) \notin E(G_2)$. In this case, f is called an isomorphism from G_1 to G_2.

Thus, if G_1 and G_2 are isomorphic graphs, then we say that G_1 is isomorphic to G_2 and we write $G_1 \cong G_2$.

If G_1 and G_2 are unlabeled then they are isomorphic if under any labeling of their vertices, they are isomorphic as labeled graphs.

If two graphs G_1 and G_2 are not isomorphic, then they are called non-isomorphic graphs and we write $G_1 \ncong G_2$.

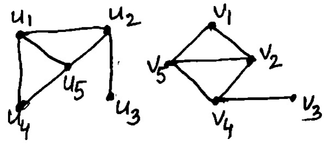
The below graphs are isomorphic.



The necessary conditions for two graphs to be isomorphic are:

1. Both graphs must have same number of vertices.
2. Both graphs must have same number of edges.
3. Both graphs must have equal number of vertices with the same degree.
4. Both the graphs must have the same degree sequence and same cycle vector $(c_1, c_2, \ldots, c_n)$, where c_i is the number of cycles of length i.

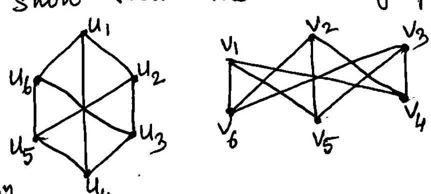
* Show that the below graphs are isomorphic.



Sol^n The order of both graphs is 5
The size of both graphs is 6
The degree sequence of both graphs is: 1, 2, 3, 3, 3
$\deg(u_3) = \deg(v_3) = 1$ ∴ u_3 can be mapped with v_3
$\deg(u_4) = \deg(v_1) = 2$ ∴ u_4 can be mapped with v_1
u_3 is adjacent to u_2 & v_3 is adjacent to v_4
∴ u_2 can be mapped with v_4

u_2 is also adjacent to u_1 and u_5.
v_4 is also adjacent to v_2 and v_5
∴ u_1 can be mapped with v_2 & u_5 can be mapped to v_5
∴ The one-to-one correspondence of the two graphs is: $u_1 \sim v_2$, $u_2 \sim v_4$, $u_3 \sim v_3$, $u_4 \sim v_1$, $u_5 \sim v_5$

* Show that the below graphs are isomorphic.



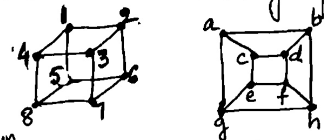
Sol^n $n=6$, $m=9$, degree sequence is: 3, 3, 3, 3, 3, 3.

(i) $u_1 < u_2, u_4, u_6$ & $v_1 < v_4, v_5, v_6$ ∴ $u_1 \sim v_1$, $u_2 \sim v_4$, $u_4 \sim v_5$, $u_6 \sim v_6$

(ii) $u_2 < u_1, u_3, u_5$ (from graph) & $v_4 < v_1$ (from mapping) & $v_4 < v_1, v_2, v_3$ (from graph) ∴ $u_3 \sim v_2$, $u_5 \sim v_3$

(iii) $u_3 < u_2, u_4, u_6$ (from graph) & $v_2 < v_4, v_5, v_6$ (from mapping) = $v_2 < v_4, v_5, v_6$ (from graph) ∴ verified. Hence the one-to-one correspondence is $u_1 \sim v_1$, $u_2 \sim v_4$, $u_3 \sim v_2$, $u_4 \sim v_5$, $u_5 \sim v_3$, $u_6 \sim v_6$

* Show that the below graphs are isomorphic.



sol^n $n=8$, $m=12$, degree sequence is 3, 3, 3, 3, 3, 3, 3, 3

(i) $1 \left\{ \begin{matrix} 2 \\ 4 \\ 5 \end{matrix} \right.$ $\quad a \left\{ \begin{matrix} b \\ c \\ g \end{matrix} \right.$ $\therefore$ $1 \sim a$, $2 \sim b$, $4 \sim c$, $5 \sim g$

(ii) $2 \left\{ \begin{matrix} 1 \\ 3 \\ 6 \end{matrix} \right.$ $\quad b \left\{ \begin{matrix} a \\ \\ \end{matrix} \right.$ $\quad$ & $\quad b \left\{ \begin{matrix} a \\ d \\ h \end{matrix} \right.$ $\quad \therefore 3 \sim d$, $6 \sim h$

from graph $\quad$ from mapping $\quad$ from graph

observe 3 is adjacent to 4 & d is adjacent to c

(iii) $3 \left\{ \begin{matrix} 2 \\ 4 \\ 7 \end{matrix} \right.$ $\quad d \left\{ \begin{matrix} b \\ c \\ \end{matrix} \right.$ $\quad$ & $\quad d \left\{ \begin{matrix} b \\ c \\ f \end{matrix} \right.$ $\quad \therefore 7 \sim f$

from graph $\quad$ from mapping $\quad$ from graph

(iv) $4 \left\{ \begin{matrix} 1 \\ 3 \\ 8 \end{matrix} \right.$ $\quad c \left\{ \begin{matrix} a \\ d \\ \end{matrix} \right.$ $\quad$ & $\quad c \left\{ \begin{matrix} a \\ d \\ e \end{matrix} \right.$ $\quad \therefore 8 \sim e$

from graph $\quad$ from mapping $\quad$ from graph

(v) $5 \left\{ \begin{matrix} 1 \\ 6 \\ 8 \end{matrix} \right.$ $\quad g \left\{ \begin{matrix} a \\ h \\ e \end{matrix} \right. = g \left\{ \begin{matrix} a \\ h \\ e \end{matrix} \right.$ $\quad \therefore$ verified.

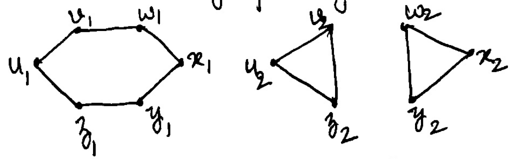
from graph $\quad$ from mapping = from graph

Hence the one-to-one correspondence is:

$1 \sim a$, $2 \sim b$, $3 \sim d$, $4 \sim c$, $5 \sim g$, $6 \sim h$, $7 \sim f$, $8 \sim e$.

Hence the given graphs are isomorphic.

Q. Are the two graphs given below isomorphic?



G_1 G_2

Sol^n $n=6$, $m=6$, degree sequence is 2, 2, 2, 2, 2, 2

$u_1 \langle {v_1 \atop z_1}$ from graph $u_2 \langle {v_2 \atop z_2}$ from graph ∴ $u_1 \sim u_2$, $v_1 \sim v_2$, $z_1 \sim z_2$

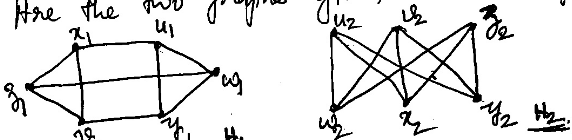
$v_1 \langle {u_1 \atop w_1}$ from graph $v_2 \langle {u_2 \atop }$ from mapping $v_2 \langle {u_2 \atop z_2}$ ∴ $w_1 \sim z_2$

but already $z_1 \sim z_2$

∴ one-to-one correspondence is not possible

∴ the graphs G_1 and G_2 are not isomorphic.

Q. Are the two graphs given below isomorphic?



Sol^n $n=6$, $m=6$, degree sequence is 2, 2, 2, 2, 2, 2.

$z_1 \langle x_1, w_1, v_1$ from graph $u_2 \langle w_2, x_2, y_2$ from graph. ∴ $z_1 \sim u_2$, $x_1 \sim w_2$, $w_1 \sim x_2$, $v_1 \sim y_2$

$x_1 \langle z_1, u_1, v_1$ from graph $w_2 \langle {u_2 \atop y_2}$ from mapping & $w_2 \langle u_2, v_2, z_2$ ∴ $x_1 \sim w_2$ & $v_1 \sim y_2$ & $x_1 v_1$ is an edge

but $w_2 y_2$ is not an edge

∴ one-to-one correspondence is not possible

H_1 & H_2 are complements of G_1 & G_2 respectively. ∴ the graphs are not isomorphic

Note! Two graphs G and H are isomorphic if and only if their complements are isomorphic.

Matrix representation of a graph.

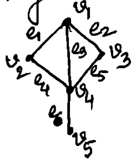
A graph G can be defined by two sets, namely its vertex set $V(G)$ and edge set $E(G)$ or by a diagram. A graph can also be described by a matrix and for some purposes this is especially useful.

Let G be a graph of order n and size m, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$..

The adjacency matrix of G is the $n \times n$ matrix $A = [a_{ij}]$, where $a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise} \end{cases}$

The incidence matrix of G is the $n \times m$ matrix $B = [b_{ij}]$, where $b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$

* Find the adjacency matrix and incidence matrix of the graph



sol^n

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

adjacency matrix   incidence matrix

* Find the adjacency matrix of K_4. 

sol^n

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

* Find the incidence matrix of $K_{3,2}$ 

sol^n

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$