CIE - II

# Scheme and Solutions

Date: 22-07-24

Duration: 120 Min

Sub: Discrete Mathematical Structures and Combinatorics

Sub. Code: CS241AT

## PART - A

1.1 No. of antisymmetric relations $= 2^4 \cdot 3^{\frac{1}{2}(4^2-4)}$

$= 11,664$ — 1 Mark

No. of relations which are neither reflexive nor irreflexive $= 2^{16} - 2 \cdot 2^{(16-3)}$

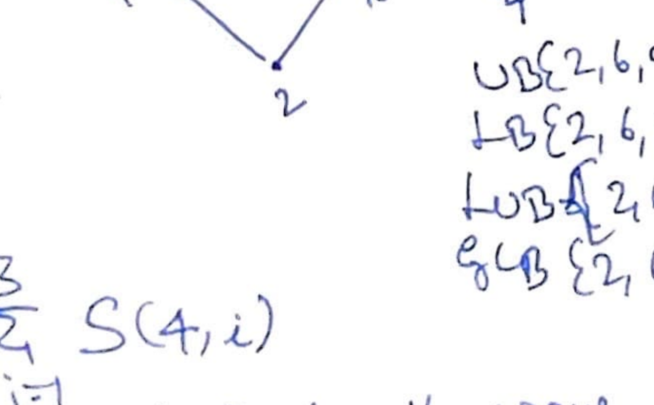
$= 57,344$ — 1 Mark

1.2 $R^4 = R \circ R \circ R \circ R$

$= \{(1,1), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (3,5), (4,1), (4,3), (4,4), (4,5), (5,1), (5,2), (5,3), (5,4), (5,5)\}$.

— 2 Marks.

1.3




UB{2, 6, 9, 18} = {18, 36, 72}

LB{2, 6, 9, 18} = Nil

LUB{2, 6, 9, 18} = 18 — $\frac{1}{2} \times 4 = 2$ Marks

GLB{2, 6, 9, 18} = Nil

1.4 $\sum_{i=1}^{3} S(4, i)$

$= 1 + 7 + 6 = 14$ ways — 1 Mark.

1.5 Let $|A| = m = 5$

$|B| = n$

$\therefore {}^nP_m$ possible injective functions from $A \to B$

$${}^nP_5 = 6720 \qquad \frac{n!}{(n-5)!} = 6720$$

$$n = 8 \qquad \frac{8!}{3!} = \frac{40320}{3!} = \frac{40320}{6} = 6720$$

$\therefore |B| = 8$ 1 mark

1.6 Truth Value of $\forall x \exists y \, P(x,y)$ is TRUE

Truth Value of $\forall x \forall y \, P(x,y)$ is FALSE

$1/2 + 1/2 = 1$ mark

1.7 $\exists x \forall y \, (P(x,y) \wedge \sim Q(x,y))$ 1 mark

PART-B

2.a) Statement:

$\forall x [(x^2+4x-21>0) \rightarrow [(x>3) \vee (x<-7)]]$

$\Rightarrow$ TRUE

Converse:

$\forall x [[(x>3) \vee (x<-7)] \rightarrow (x^2+4x-21>0)]$

$\Rightarrow$ TRUE

Inverse:

$\forall x [(x^2+4x-21 \leq 0) \rightarrow [(x \leq 3) \wedge (x \geq -7)]]$

$\Rightarrow$ TRUE

Contrapositive:

$\forall x [[(x \leq 3) \wedge (x \geq -7)] \rightarrow (x^2+4x-21 \leq 0)]$

$\Rightarrow$ TRUE

$1+1+1+1 = 4$

Truth Value $= 1$

2b) Let $P(x)$: $x$ is an integer
$Q(x)$: $x$ is a rational number — 1 Mark
$R(x)$: $x$ is a power of 7
Therefore the given argument is translated into the following.

$$\begin{array}{l} \exists x\,(Q(x) \to R(x)) \\ \forall x\,(P(x) \to Q(x)) \\ \hline \therefore \exists x\,(P(x) \to R(x)) \end{array}$$

— 2 Marks

To verify the validity of this proced as follows

1. $\exists x(Q(x) \to R(x))$ ; premise
2. $Q(a) \to R(a)$ ; Rule of ES - 1
3. $\forall x\,(P(x) \to Q(x))$ ; premise
4. $P(a) \to Q(a)$ ; Rule of US - 3
5. $P(a) \to R(a)$ ; Rule of syllogism 4,2.
6. $\therefore \exists x\,(P(x) \to R(x))$ ; Rule of EG - 5.

— 2 Marks

3.a)

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^2 = M_R \circ M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1 - Mark

$$M_{R^3} = M_R^2 \odot M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ 1-Mark

$$M_{R^4} = M_R^3 \odot M_R = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ 1- Mark

$$M_{R^\infty} = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore R^\infty = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)\}$

1- Mark.

3.b)(i) If $aRb$ then $[a] = [b]$.

Proof: Let $x \in [a]$. So $xRa$, by definition of class. But $aRb$, by hypothesis. Thus by transitivity of $R$ $xRb$ $\therefore x \in [b]$, by definition of class. $\therefore [a] \subseteq [b]$ 2-Marks

But $x \in [b]$, then $xRb$, by definition of class. Now $aRb$ by hypothesis. Since $R$ is symmetric $bRa$ also. Then, since $R$ is transitive $xRb$, & $bRa$ $\therefore xRa \Leftrightarrow x \in [a]$ $\therefore [b] \subseteq [a]$. Hence $[a] = [b]$

(ii) $[a] \cap [b] = \phi$ or $[a] = [b]$

Proof: If $A$ is a set, $R$ is an equivalence relation on $A$, and $a$ and $b$ are elements of $A$, then $[a] \cap [b] = \phi$.

Let $[a] \cap [b] \neq \phi$, Then there exists an element $x \in A$ such that $x \in [a] \cap [b]$.

By definition of intersection $x \in [a]$ and $x \in [b]$ and so $xRa$ and $xRb$, by definition of class then $aRx$.

Since $R$ is symmetric and $xRa$ and so, $aRx$, $xRb$, But $R$ is transitive, and so, $aRb$

Thus $a, b$ satisfies the proof in (i) $[a] = [b]$.

— 2 Marks

(iii) Let $x \in A$. By reflexivity of $R$, $xRx$

$x \in [x]$ and $x$ must be in one of the distinct equivalence classes $A_1, A_2, \ldots A_n$

$$\therefore x \in \bigcup_{i=1}^{n} A_i$$

$$A \subseteq \bigcup_{i=1}^{n} A_i \quad — (1)$$

Now, let $x \in \bigcup_{i=1}^{n} A_i$, $x \in A_i$ for some $i = 1, 2, \ldots n$

but each $A_i$ is an equivalence class of $A$.

$\therefore A_i \subseteq A$ and so $x \in A$

$$\therefore \bigcup_{i=1}^{n} A_i \subseteq A \quad — (2)$$

From (1) & (2) we get

— 2 Marks

$$A = \bigcup_{i=1}^{n} A_i$$

$\text{lly}$ $A_i \cap A_j = \phi$ from (ii)

4.a) Definition of POSET.

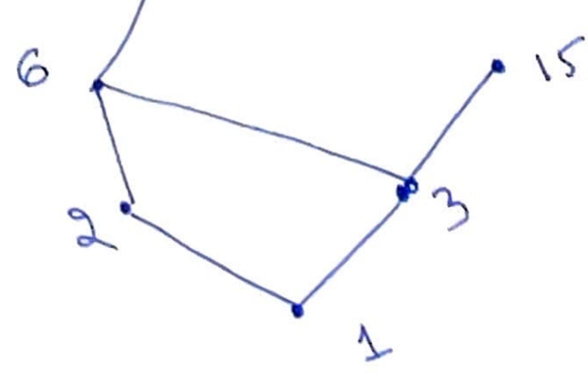
If Any set A under the given relation R where R is a partial ordered relation, then (A,R) is called as POSET. — 1 Mark

(A, |) = { (1,1), (1,2), (1,3), (1,6), (1,12), (1,15), (1,24), (1,36) (1,48), (2,2) (2,6), (2,12) (2,24), (2,36), (2,48), (3,3), (3,6), (3,12), (3,15), (3,24), (3,36), (3,48), (6,6) (6,12), (6,24), (6,36) (6,48), (12,12), (12,24), (12,36), (12,48), (15,15), (24,24), (24,48), (36,36), (48,48) }.

This is reflexive, antisymmetric & transitive — 2 Marks

∴ It is a POSET.

Hasse diagram:




— 2 Marks

4.b) a) 4 number of upperbounds of B
b) In total 15 upperbounds of B
c) LUB(B) = {1, 2, 3}
d) Only one lowerbound of B
e) glb(B) = ∅ $\qquad 1 \times 5 = 5$ Marks.

5.a) i) $(g \circ f)(x) = g(f(x)) = g(x^3)$
$= x^3 - 1 \quad \forall x \in R$
$(f \circ g)(x) = f(g(x)) = f(x-1) = (x-1)^3 \quad \forall x \in R$
— 1 Mark
— 1 Mark

$(g \circ f)(x) \neq (f \circ g)(x)$
For instance, let $x = 2$
$(g \circ f)(2) = 2^3 - 1 = 8 - 1 = 7$
$(f \circ g)(2) = (2-1)^3 = 1$

ii) $(g \circ f)(x) = g(f(x))$
$= g(ax+b)$ — 1 Mark
$= 1 - (ax+b) + (ax+b)^2$
$= a^2x^2 + (2ab - a)x + (b^2 - b + 1)$ — ①

It is given that — ②
$(g \circ f)(x) = 9x^2 - 9x + 3$

Comparing eqn ① and ②, we get
$a^2 = 9$ — 1 Mark
$2ab - a = -9$
$b^2 - b + 1 = 3$
$\therefore a = 3, \; b = 1$

5.b) Since $f$ and $g$ are invertible functions, both are bijective. Consequently $g \circ f$ is bijective.
$\therefore g \circ f$ is invertible. — 1 Mark

Now, the inverse $f^{-1}$ of $f$ is a function from B to A and the inverse $g^{-1}$ of $g$ is a function from C to B.
$\therefore h = f^{-1} \circ g^{-1}$ then $h$ is a function from C to A.
— 1 Mark

we find that

$$(g \circ f) \circ h = (g \circ f) \circ (f^{-1} \circ g^{-1})$$
$$= g \circ (f \circ f^{-1}) \circ g^{-1}$$
$$= g \circ I_B \circ g^{-1}$$
$$= g \circ g^{-1} = I_C$$

— 1½ Mark

and

$$h \circ (g \circ f) = (f^{-1} \circ g^{-1}) \circ (g \circ f)$$
$$= f^{-1} \circ (g^{-1} \circ g) \circ f$$
$$= f^{-1} \circ I_B \circ f$$
$$= f^{-1} \circ f = I_A$$

— 1½ Mark

Therefore, $h$ is the inverse of $g \circ f$

i.e $h = (g \circ f)^{-1}$ — 1 Mark

$$\therefore (g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

6. a) Proof (i):

Let $a_1, a_2 \in A$ with $(g \circ f)(a_1) = (g \circ f)(a_2)$, then

$(g \circ f)(a_1) = (g \circ f)(a_2) \Rightarrow g(f(a_1)) = g(f(a_2))$

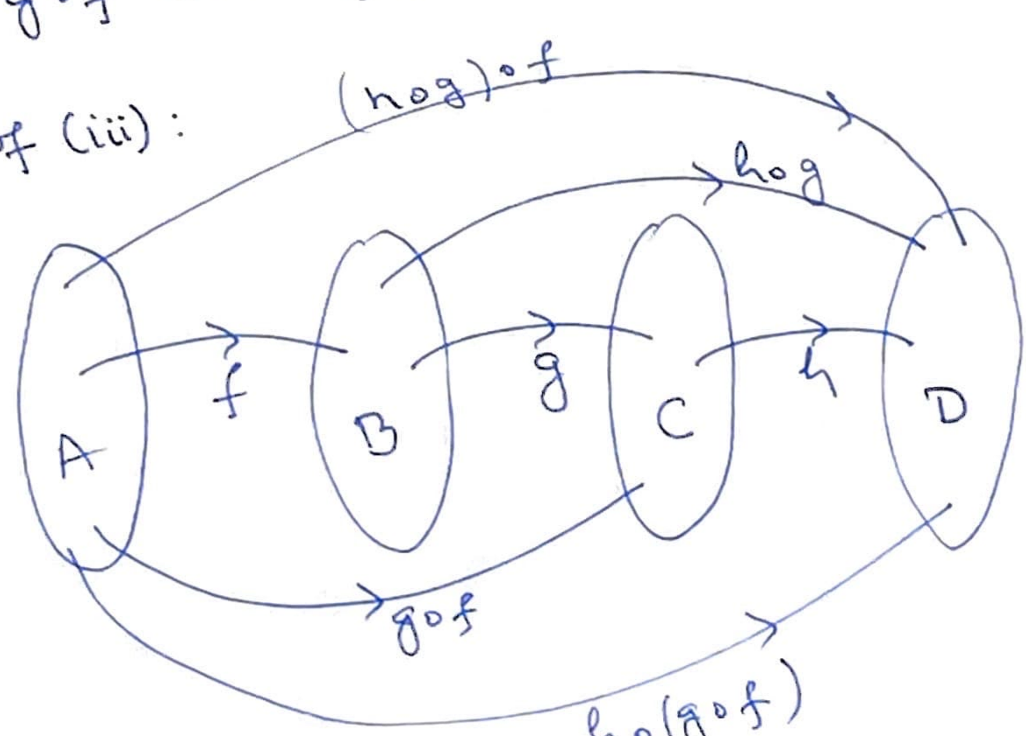
$\Rightarrow f(a_1) = f(a_2)$.

because $g$ is one to one, $a_1 = a_2$ Consequently $g \circ f$ is one to one. — 2 Marks

Proof (ii):

For $g \circ f : A \to C$, let $z \in C$.
Since $g$ is onto, there exists $y \in B$ with $g(y) = z$. with $f$ onto, there exists $x \in A$ with $f(x) = y$. Hence $z = g(y) = g(f(x)) = (g \circ f)(x)$,
so the range of $g \circ f = C =$ Co-domain of $g \circ f$ and $g \circ f$ is onto. — 2 Marks

Proof (iii):




$\therefore (h \circ g) \circ f = h \circ (g \circ f)$ is a fn from A to D.
— 2 Marks

6.b) i) $|A| = 5$ , $|B| = 7$

$|B|^{|A|} = 7^5$ —— 1 Mark

$= 16,807$ functions from A to B

iii) $P(|B|, |A|)$

$= (7, 5) = \frac{7!}{(7-5)!} = \frac{7!}{2!}$ —— 1 Mark

$= 2520$ one to one functions from A to B

iii) $3^2 \cdot 4^3 = 576$ functions. —— 2 Marks

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