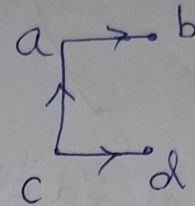
UNIT-5

Introduction to Graph Theory.

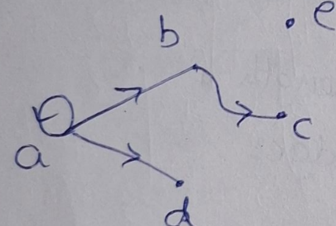
Graph: A graph $G=(V,E)$ where V is set of vertices or nodes and $E \subseteq V \times V$ called set of (directed) edges or arcs.

Ex:-



$V=\{a,b,c,d\}$
$E=\{(a,b),(c,a),(c,d)\}$.

If edges are unordered pair then it is called ~~unordered~~ directed graph. Here edges are represented in { }
Ex:- $\{a,b\}$ represent (a,b) & (b,a)



Here $(a,a) \rightarrow$ loop
e is called isolated vertex.
A vertex which has no incedent edge.

Walk:- Let x, y be (not necessarily distinct) vertices in an undirected graph $G=(V,E)$. An x-y walk in G is a (loop-free) finite alternating sequence

$$x = x_0, e_1, x_1, e_2, x_2, \ldots\ldots e_n, x_n = y.$$

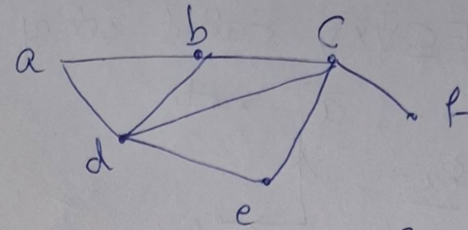
of vertices and edges from G starting at vertex x, and ending at vertex y & involving n edges $e_i = \{x_{i-1}, x_i\}$ where $1 \le i \le n$.

The length of this walk is n, the number of edges in the walk.

Any x-y walk where $x=y$ (& $n>1$) is called closed walk otherwise it is open walk.

Note that a walk may repeat both vertices and edges.

Ex:-



1) $\{a,b\}, \{b,d\}, \{d,c\}, \{c,e\}, \{e,d\}, \{d,b\}$ -
Here walk length is 6, vertices b, d are repeated, edges $\{b,d\}$ ($\{d,b\}$) is also repeated.

2) $b \to c \to d \to e \to c \to f$: walk length is 5 & vertex c is repeated.

3) $\{f,c\}$ $\{c,e\}$, $\{e,d\}$, $\{d,a\}$. walk of length 4.
no repetition of either vertices or edges.

Defn:-
Let x-y be a walk in an undirected graph $G=(V,E)$.

i) If no edge in ~~E~~ x-y walk is repeated, then walk is called an x-y trail.
A closed ~~walk~~ x-x trail is called a circuit.

ii) If no vertex of x-y walk occurs more than once, then the walk is called an x-y path.
When $x=y$, that closed path is called a cycle.

NOTE:- In dealing with circuit, there will be atleast one edge. When there is only one edge, then circuit is a loop.

A cycle will have atleast 3 distinct edges.

a) A b-f ~~pa~~ walk in 2 is a b-f trail, but not a b-f path because of repetition of vertex c. but f-a walk in 3 is both a path & a trail.

b) Edges {a,b}, {b,d}, {d,c}, {c,e}, {e,d} & {d,a} forms a-a circuit, but not a cycle as d is repeated.

c) Edges {a,b} {b,c}, {c,d} & {d,a} provide a-a cycle of length 4. and is a circuit.

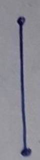
Name	Repeated Vertex	Repeated Edge	open	closed
Walk (open)	Yes	Yes	Yes	
Walk (closed)	Yes	Yes		Yes
Trail	Yes	No	Yes	
Circuit	Yes	No		Yes
Path	No	No	Yes	
Cycle	No	No		Yes

<u>Order</u> - # of vertices in a (finite) graph is called order.

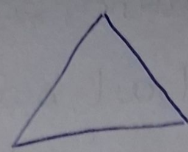
<u>Size</u> - # of edges is called Size.

A graph of order n, size m is called a (n,m) graph.

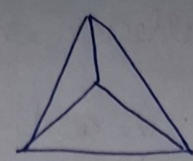
<u>Complete graph</u>:- A simple graph of order $n \geq 2$ in which there is an edge b/n every pair of vertices is called a complete graph (full graph), denoted by K_n.



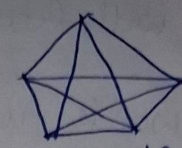
K_2



K_3

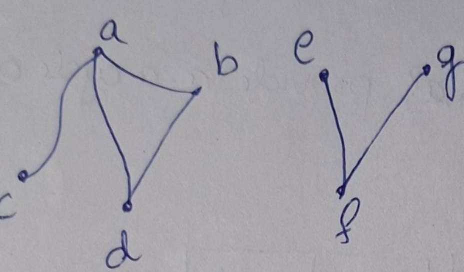


K_4



K_5 - Kuratowskis first graph.

Connected Components



$G=(V,E)$ is disconnected iff V can be partitioned into atleast two subsets V_1, V_2 such that there is no edge E of form (x,y) where $x \in V_1$ & $y \in V_2$.

A graph is connected iff it has one component.

The # of components of G is denoted by $K(G)$.

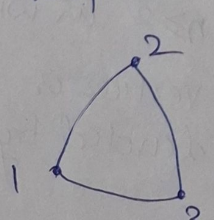
For above graph G, $K(G)=2$.

Simple Graph, Multigraph, General graph.

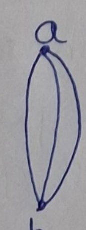
A graph which does not contain loops & multiple edges is called a simple graph.

A graph which contains multiple edges but no loops is called multigraph.

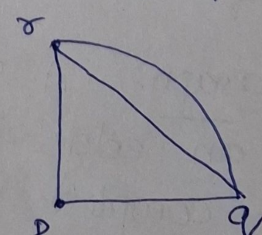
A graph which contains multiple edges or loops (or both) is called general graph.



Simple graph.



multigraph.
(multiplicity 3 for edge $\{a,b\}$)



General graph

Operations on Graphs

$G_1 = (V_1, E_1) \qquad G_2 = (V_2, E_2)$

$G_1 \cup G_2 = (V_1 \cup V_2 \,,\, E_1 \cup E_2)$

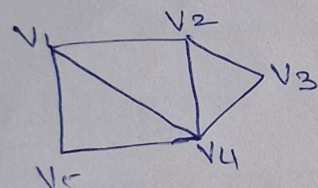
$G_1 \cap G_2 = (V_1 \cap V_2 \,,\, E_1 \cap E_2)$ if $V_1 \cap V_2 \neq \phi$

Ring sum of G_1, G_2 is

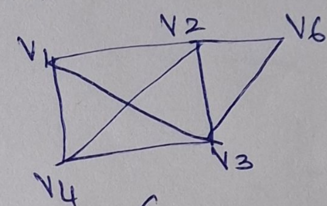
$G_1 \Delta G_2 = (V_1 \cup V_2 \,,\, E_1 \Delta E_2)$ where

$E_1 \Delta E_2 = (E_1 \cup E_2) - (E_1 \cap E_2)$

Ex:-

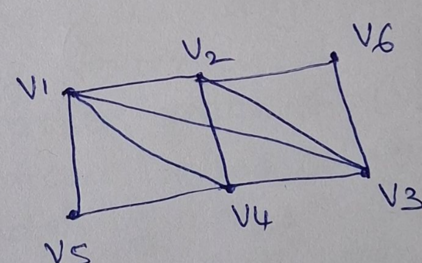


G_1.

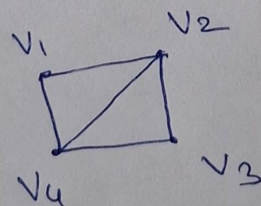


G_2

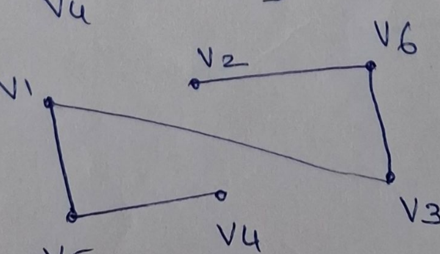
Then $G_1 \cup G_2$:



$G_1 \cap G_2$:



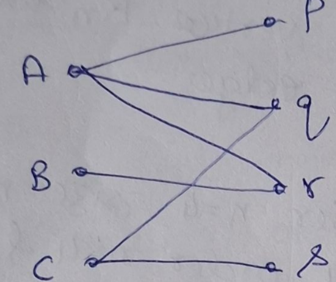
$G_1 \Delta G_2$:



Bipartite graph:

A graph G in which vertex set V is the union of two of its mutually disjoint non empty subsets V_1, V_2 which are such that each edge in G joins a vertex in V_1 and a vertex in V_2 is called a Bipartite graph denoted by $G = (V_1, V_2; E)$. V_1, V_2 are partitions or bipartites of V.

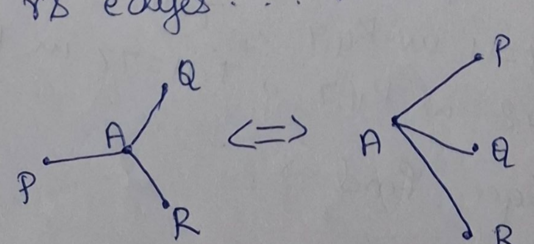
Ex:-



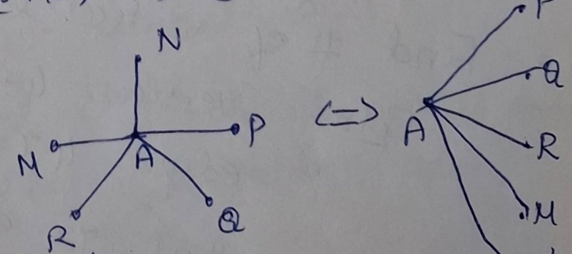
Complete bipartite graph

A bipartite graph $G = (V_1, V_2; E)$ is called complete bipartite if there is an edge b/n every vertex in V_1 and every vertex in V_2.

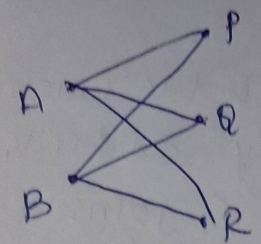
A complete bipartite graph $G = (V_1, V_2; E)$ in which V_1, V_2 contain r and s vertices resp, with $r \leq s$ denoted by $K_{r,s}$. In this graph, each of r vertices in V_1 is joined to each of s vertices in V_2. So, $K_{r,s}$ has $r+s$ vertices & rs edges. ∴ it is a $(r+s, rs)$ graph.



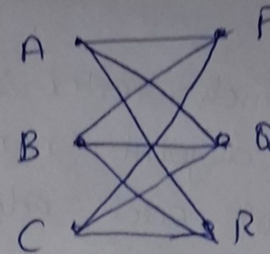
$K_{1,3}$



$K_{1,5}$



$K_{2,3}$



$K_{3,3}$
Kuratowskis's
second graph.

I For simple graph, $2|E| \le |V|^2 - |V|$ ie $2m \le n^2 - n$

II A complete graph with n vertices K_n will have $nC_2 = \frac{n!}{(n-2)!\,2!} = \frac{1}{2} \cdot n(n-1)$ ~~edges~~ edges.

1). S.T simple graph of order $n=4$ size $m=7$ and a complete graph of order $n=4$ & $m=5$ do not exists.

$n=4$.

$\frac{1}{2} \cdot n(n-1) = \frac{4(3)}{2} = 6$.

$\because m=7$ exceeds this number, a simple graph of order $n=4$, size $m=7$ does not exist.

III^ly $\because m=5$ is not equal to 6, so a complete graph of order 4 and size 5 does not exist.

2. Find # of vertices & edges in $K_{4,7}$ and $K_{7,11}$?

Soln:- vertices $= 4+7=11$ in $K_{4,7}$, $7+11=18$ in K_{18}
edges $= 4 \cdot 7 = 28$ in $K_{4,7}$, $7 \cdot 11 = 77$ in $K_{7,11}$.

3. If $K_{r,12}$ has 72 edges, find r.

$r = \frac{72}{12} = \underline{\underline{6}}$.

NOTE:- In a bipartite graph of $G(n,m)$,

$$4m \leq n^2.$$

S.T a simple graph of $n=4$ $m=5$ cannot be bipartite.

$$4 \cdot 5 = 20 \nleq 4^2$$

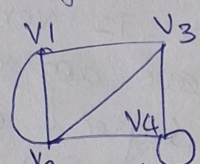
$\therefore$ cannot be bipartite.

Degree of a vertex v is the # of edges that join v to other vertices of G with loops counted twice.

Degrees of all vertices of graph arranged in non-decreasing order is called degree-sequence.

Minimum of degrees of vertices is called degree of graph. denoted by $d(v)$.

Ex:-



$d(v_1) = 3$ $d(v_3) = 3$

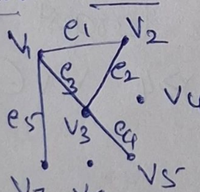
$d(v_2) = 4$ $d(v_4) = 4$.

Max degree $\Delta(G) = 4$, Min degree $(G) = \delta(G) = 3$

Isolated vertex - vertex whose degree is 0.

pendent vertex - vertex whose degree is 1.

Ex:-



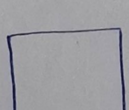
v_4, v_6 - isolated

v_5, v_7 - pendent vertices

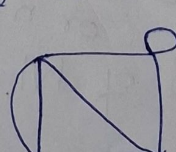
e_4, e_5 - pendent edges

Regular graph - A graph in which all vertices are of same degree k is called k-regular graph.

Ex:-



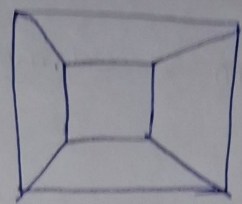
2-regular (Q_2)



4-regular

3-regular / peterson graph

10-vertices
15-edges.



Cubic graph $Q_3 \equiv$ hypercube
$2^3 = 8$ vertices
12 Edges.

A loop free k-regular graph with 2^k vertices is called k-dimensional hyper cube, Q_k.

Handshaking property:-

$$\sum d(v) = 2 \cdot |E| = 2m.$$

1) Can there be a graph with vertices A, B, C, D with $d(A) = 2 = d(C) = d(D)$, $d(B) = 3$?

Soln:- $\sum d(v) = 2+2+2+3 = 9$ which is not even.
So graph does not exist.

2) Can there be a graph with 12 vertices such that two of vertices have degree 3 each & remaining 10 vertices have degree 4 each?

Soln:- $\sum d(v) = 2\times 3 + 10 \times 4 = 46 = 2(23)$
$\therefore$ Graph exists with 23 edges.

3) For a graph $G = (V, E)$ what is largest possible value for $|V|$ if $|E| = 19$ & $d(v) \geq 4 \ \forall v \in V$?

Soln:- $\sum d(v) \geq 4n$, n is $|V|$

$4n = 2 \cdot |E|$

$\therefore \ 4n = 2 \cdot 19 = 38$ ie $2 \cdot 19 \geq 4n$

$n = \frac{38}{4} = 9.5$ or $n \leq 38/4 < 10$

$\therefore$ largest possible value of $|V|$ is $\underline{\underline{9}}$.

NOTE: A K-dimensional hypercube Q_k has $k \cdot 2^{k-1}$ edges. (5)

Find $|V|$ in

i) Cubic graph with 9 edges.

$$\sum d(v) = 3n = 2 \cdot 9 = 18 \Rightarrow \underline{\underline{n = 6}}.$$

ii) Regular graph with 15 edges

$$\sum d(v) = k \cdot n = 2 \cdot 15 = 30 \Rightarrow n = 30/k$$

$\therefore$ possible $n = 1, 2, 3, 5, 6, 10, 15, 30$.

iii) G has 10 edges with 2 vertices of degree 4 & all other vertices of degree 3.

$$\sum d(v) = \overset{2 \cdot |E|}{\cancel{2 \cdot 4 + 8 \cdot 3 = 32 = 2n \Rightarrow n = 16}}.$$

ie $2 \cdot 10 = 20 = 2 \cdot 4 + (n-2)3$

$12 = (n-2) \cdot 3$

$n - 2 = 4$

$\Rightarrow \underline{\underline{n = 6}}$.

Subgraph:-

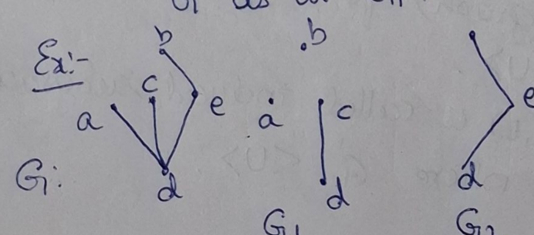
If $G = (V, E)$ is a graph, then $G_1 = (V_1, E_1)$ is called a subgraph of G if $\phi \neq V_1 \subseteq V$ and $E_1 \subseteq E$ where each edge in E_1 is incident with vertices in V_1.

ie G_1 is a subgraph of G if

i) all vertices & edges of G_1 are in G

ii) each edge of G_1 has same end vertices in G as in G_1.

Ex:-

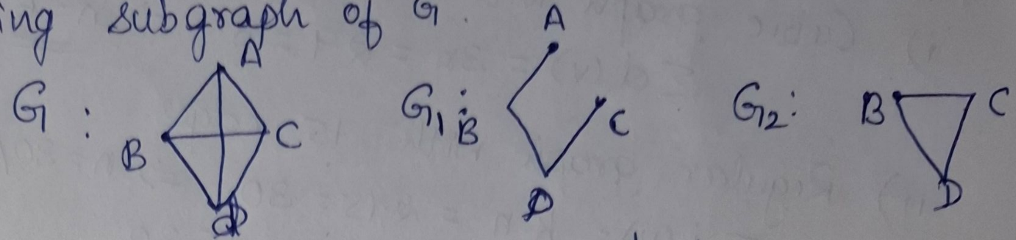


G_1, G_2 are Subgraphs of G.

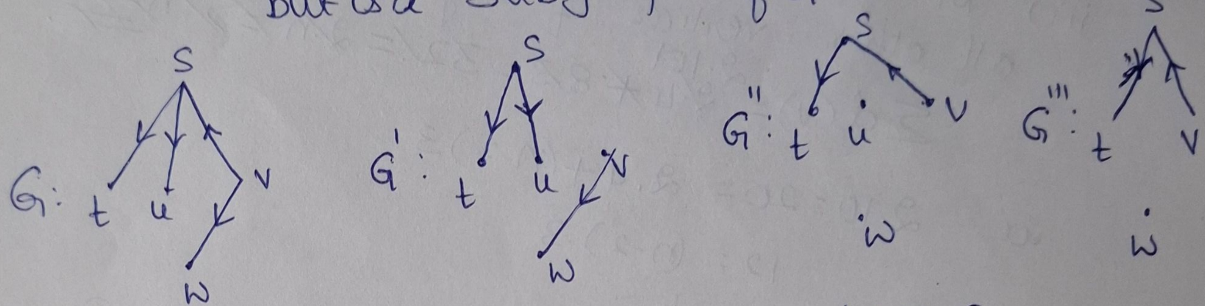
Spanning Subgraph

Given a graph $G=(V,E)$. Let $G_1=(V_1,E_1)$ be a subgraph of G such that $V_1=V$, then G_1 is called a spanning subgraph of G.

Ex:-



Here G_1 is spanning subgraph but G_2 is not spanning subgraph $\because V_2 \neq V_2$. but is a subgraph of G.



Here G', G'' are spanning subgraphs of G but G''' is not as $V_0''' \neq V_2$.

NOTE:- # spanning ~~trees~~ subgraphs with $m=4$ is 2^m as each edge may or maynot be included in spanning subgraph.

Induced Subgraph:

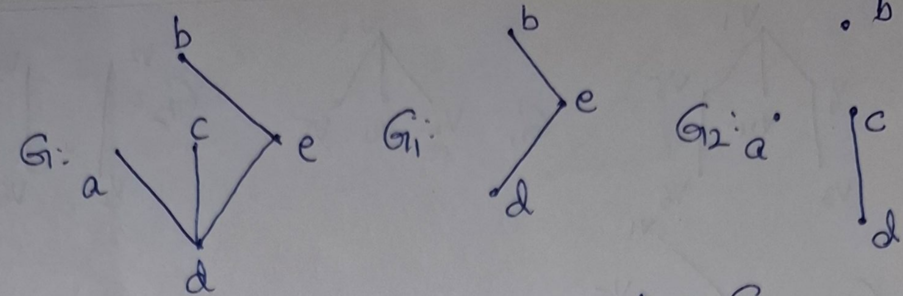
Let $G=(V,E)$ be graph. If $\phi \neq U \subseteq V$, the subgraph of G induced by U is the subgraph whose vertex set is U & which contains all edges (from G) of either of form (x,y) (for directed Graph) or $\{x,y\}$ (for undirected graph) for $x,y \in U$.

This is denoted by $\langle U \rangle$.

A subgraph G' of $G=(V,E)$ is called induced subgraph if there exists $\phi \neq U \subseteq V$, where $G' = \langle U \rangle$.

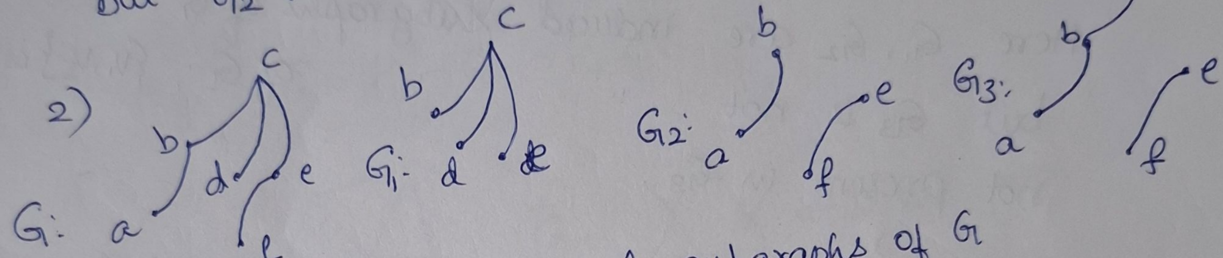
Ex:-

1)



Here G_1 is an induced subgraph of G
but G_2 is not an induced " of G $\because \{a,d\}$ is missing in G_2.

2)



Here G_1, G_2 are induced subgraphs of G
with $U_1 = \{b,c,d,e\}$ $U_2 = \{a,b,e,f\}$.
but G_3 is not an induced subgraph since $\{c,e\}$ is not in G_3

Edge disjoint & vertex disjoint subgraphs:

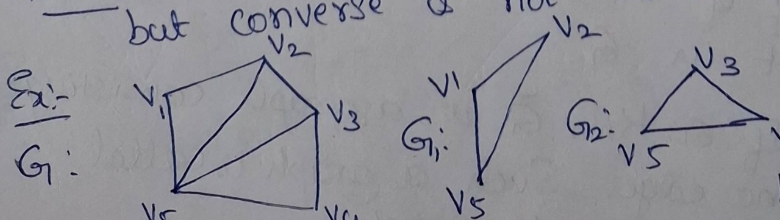
Let G_1, G_2 be subgraphs of G. Then

1) G_1, G_2 are said to be edge-disjoint if they do not have any edge in common.

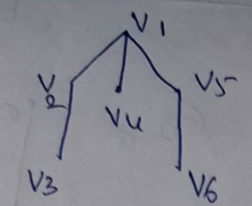
2) G_1, G_2 are said to be vertex-disjoint if they do not have any common edge & any common vertex.

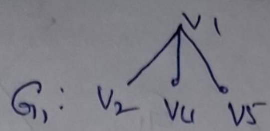
NOTE:- Vertex disjoint must be edge-disjoint also but converse is not necessarily true.

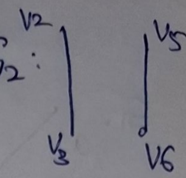
Ex:-

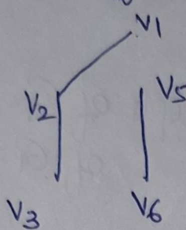


Here for G, G_1, G_2 are edge disjoint but not vertex disjoint since $V_5 \in G_1 \& G_2$.

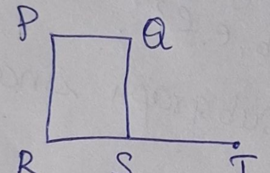
1) G: 

G_1: 

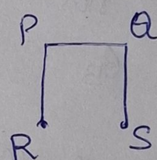
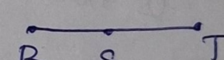
G_2: 

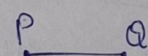
G_3: 

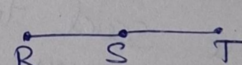
Here G_1, G_2 are induced subgraphs of G.
but G_3 is not " " of G ∵ $\{V_1, V_5\}$ is
not present in G_3.

2) G: 

Write two edge disjoint &
two vertex " subgraphs.

Ans:- Edge disjoint subgraphs:  

Vertex disjoint subgraphs: 

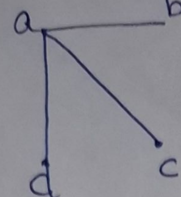


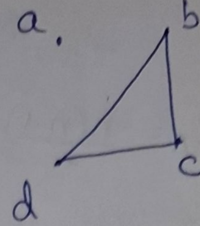
Complement of a graph

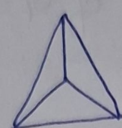
Let G be a loop free undirected graph on n vertices.
The complement of G denoted $\overline{G}$ is subgraph of
K_n consisting of n vertices in G & all edges
that are not in G.

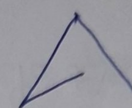
Null graph:- If $G = K_n$, $\overline{G}$ is a graph consisting of
n vertices & no edges. Such a graph is called
null graph.

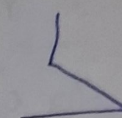
Ex:-

1) G: 

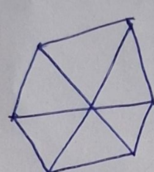
$\bar{G}$: 

2) ~~G~~ K_4 

G: 

$\bar{G}$: 

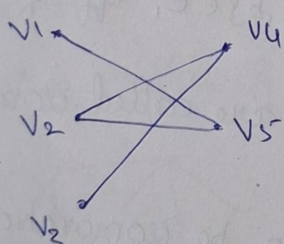
i.e., $\bar{G} = K_n - G = K_n \Delta G$

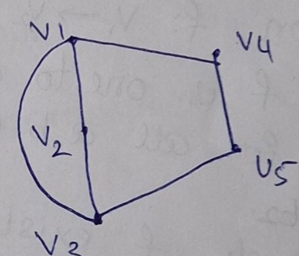
3) G: 

$\bar{G}$:  ← David graph.

1) S.T complement of a bipartite graph need not be a bipartite graph.

Soln:-

G: 

$\bar{G}$: 

Here G is bipartite graph but
$\bar{G}$ is not " "

2) Let G be a simple graph of order n. If size of G is 56 & size of $\bar{G}$ is 80, what is n.

Soln: $\bar{G} = K_n - G$

$\therefore$ size of $\bar{G}$ = size of K_n - size of G.

Size of $K_n = \frac{n(n-1)}{2} = 56$

$\therefore\ n^2 - n = 112$

Size of $\bar{G}$ = Size of K_n - Size of G

$80 = \cancel{112}\ \frac{n(n-1)}{2} - 56$

$\therefore\ 136 \times 2 = n^2 - n$

$n^2 - n = 272 = 17^2 - 17$

$\therefore\ \underline{n = 17}$.

Isomorphism:-

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two undirected graphs. A function $f: V_1 \rightarrow V_2$ is called graph isomorphism if

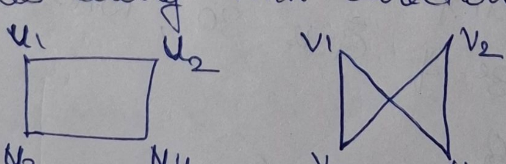
i) f is one to one & onto &

ii) for all $a, b \in V_1$, ~~if~~ $\{a,b\} \in E_1$ iff if $f(a), f(b) \in E_2$.

When such f exists, G_1, G_2 are called isomorphic graphs.

That is, G_1 and G_2 are said to be isomorphic if there is a one-to-one correspondence b/n their vertices & b/n their edges such that adjacency of vertices along with directions is preserved.

Ex:- 1)

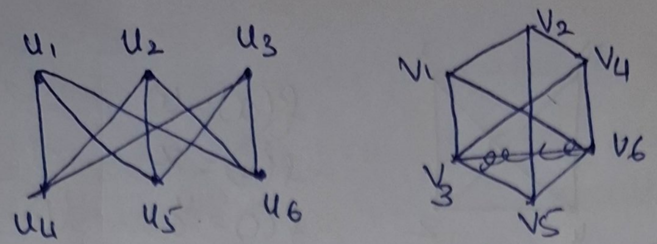


$u_1 \leftrightarrow v_1$, $u_2 \leftrightarrow v_4$, $u_3 \leftrightarrow v_3$, $u_4 \leftrightarrow v_2$

$\{u_1, u_2\} \leftrightarrow \{v_1, v_4\}$ $\{u_1, u_3\} \leftrightarrow \{v_1, v_3\}$ $\{u_2, u_4\} \leftrightarrow \{v_4, v_2\}$ &

$\{u_3 u_4\} \leftrightarrow \{v_3, v_2\}$

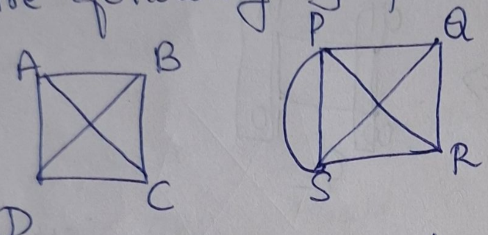
2)



Here every vertex in both graphs have degree 3. & has 9 edges.

$u_1 \leftrightarrow v_1$, $u_4 \leftrightarrow v_2$, $u_2 \leftrightarrow v_4$ $u_3 \leftrightarrow v_5$ $u_5 \leftrightarrow v_3$ $u_6 \leftrightarrow v_6$

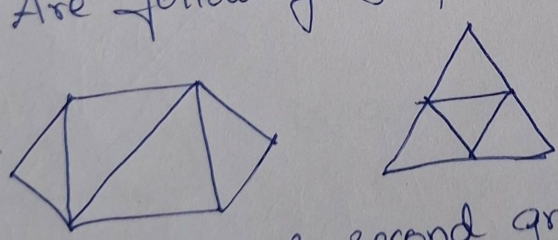
3) Are following graphs isomorphic?



Soln:- For first graph $n=4$, $m=6$
second graph $n=4$, $m=7$

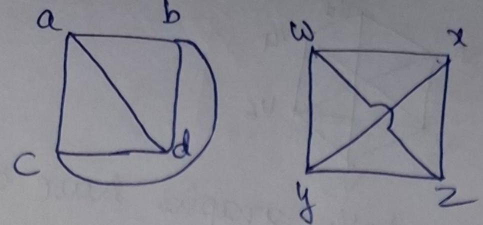
$\therefore$ There is no one-to-one correspondence possible
So they are not isomorphic.

4) Are following graphs isomorphic?

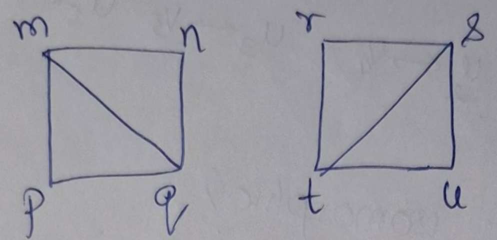


Soln:- Here first & second graph both have $n=6$, $m=9$

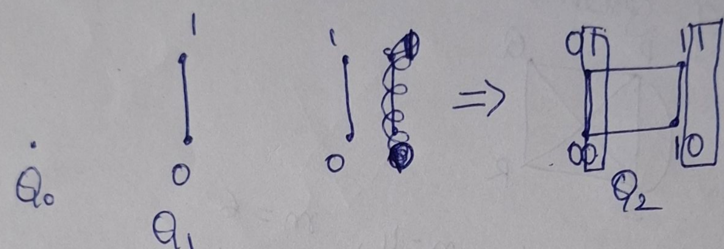
But in first graph there are two vertices of degree 4, & second graph has three vertices of degree 4. $\therefore$ there cannot be one to one correspondence b/n edges. $\therefore$ not isomorphic.

5) 

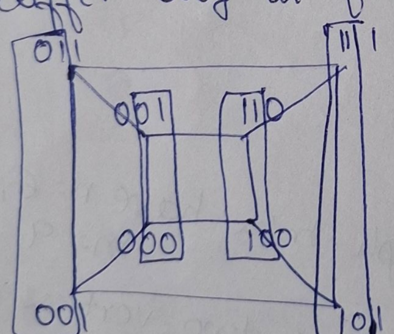
$f(a) = w$ $\quad d(a) = 3 = d(d)$
$f(b) = x$ $\quad d(w) = 3 = d(z)$
$f(c) = y$ $\quad d(b) = 3 = d(c)$
$f(d) = z$

6) 

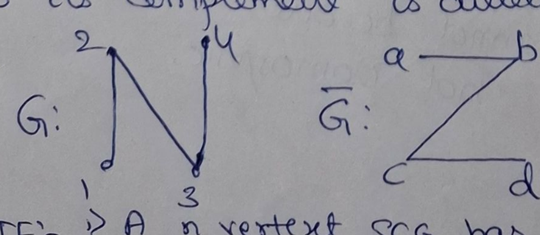
$f(m) = s$ $\quad f(n) = r$
$f(q) = t$ $\quad f(p) = u$



Prefix vertex labels of one copy of Q_n with 0 ($Q_{0,n}$) & those of the other with 1 ($Q_{1,n}$). For x in $Q_{0,n}$ & y in $Q_{1,n}$ draw an edge $\{x, y\}$ if labels of x, y differ only in first position.



<u>Self complementary graph</u>:- A graph which is isomorphic to its complement is called Self complementary graph.

<u>Ex</u>:- 

$f(1) = a$,
$f(2) = b$, $f(3) = c$, $f(4) = d$

<u>NOTE</u>:- 1) A n vertex S.C.G has $n(n-1)/4$ edges.
2) n must be congruent to 0 or 1 modulo 4.

Euler Circuit

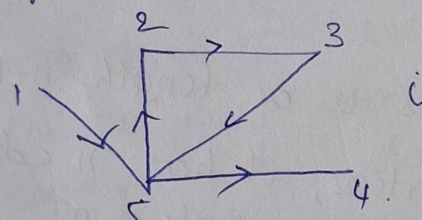
Defn:- Let $G=(V,E)$ be an undirected graph or multigraph with no isolated vertices. Then G is said to have an Euler circuit if there is a circuit in G that traverses every edge of G exactly once.

Euler Trail:-

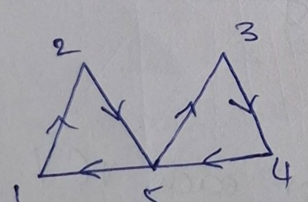
If there is an open trail from a to b in G & this trail traverses every edge in G exactly once, the trail is called an Euler trail.

Note:- If in an open walk, no edge appears more than once, then it is trail. A closed walk in which no edge appears more than once is a circuit.

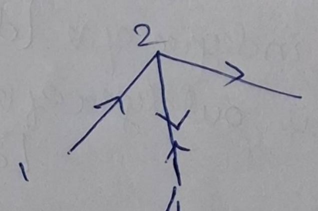
Ex:-



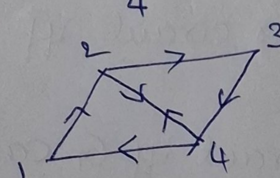
is a trail ∵ 1-5-2-3-5-4



is a ckt ∵ 1-2-5-3-4-5-1

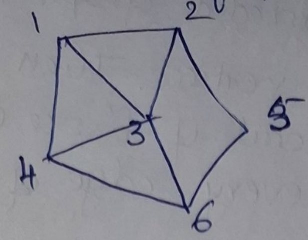


is not a trail



is not a ckt ∵ edge b/n 2,4 is repeated.

Determine # of diff paths of length 2.



Soln:- # of paths of length 2 that pass through 1 = # of pairs of edges incident on v_1 ∴ ${}^3C_2 = 3$.

IIIly passing through 2, 3, 4, 5, 6 are ${}^3C_2$, ${}^4C_2$, ${}^3C_2$, ${}^2C_2$, ${}^3C_2$.

∴ $3+3+6+3+1+3=19$.

In general if G is a simple graph of order n with d_i as degree of a vertex v_i for $i=1,2 \dots n$ then # of paths of length 2 in G

$$= \sum_{i=1}^{n} \binom{d_i}{2}.$$

NOTE:-

1. A path with n vertices is of length n-1
2. If a cycle has n vertices, it has n edges.
3. Degree of every vertex in a cycle is 2

Indegree (id) and Outdegree (od)

Let $G=(V,E)$ be a graph. For each $v \in V$.

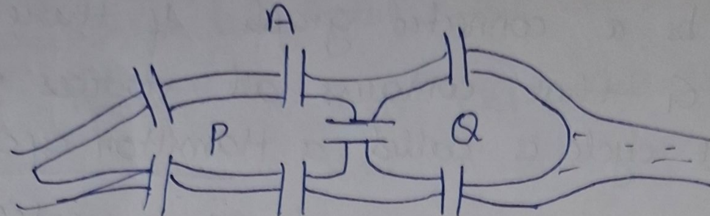
i) the # of edges enter into v is indegree of v. $[d^-(v)]$

ii) the # of edges going out from v is outdegree of v. $[d^+(v)]$

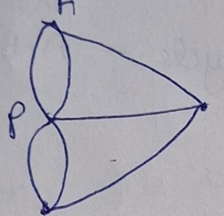
NOTE:- 1. A connected graph G has an Euler circuit iff all vertices of G are of even degree.

2. A connected graph G has an Euler circuit iff G can be decomposed into edge disjoint cycles.

Konigsberg Bridge problem



Land areas of city are P, Q, A B where A, B → banks of rivers ~~land areas~~
P, Q → islands.



4 areas → 4 vertices
7 bridges → 7 edges

$d(A) = d(B) = d(Q) = 3$ $d(P) = 5$. which are not even.

∴ Graph does not have Euler circuit ∴ it is not possible to walk over each of 7 bridges exactly once & return to starting pt.

NOTE:- 1. Let G be a connected graph, then we can construct an Euler trail iff G has exactly two vertices of odd degree.

2. Let $G = (V, E)$ be a directed graph with no isolated vertices. Then G has directed Euler circuit iff G is connected & $id(v) = od(v)$ $\forall v \in V$.

3. G is regular if $\Delta(G) = \delta(G)$
i.e G is K-regular if common degree is K.

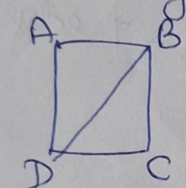
Hamiltonial cycles & Hamilton paths

Defn:- Let G be a connected graph. If there is a cycle in G that contains all vertices of G, then that cycle is called a Hamilton cycle.

A Hamilton cycle in a graph of n vertices consists of exactly n edges (need not include all edges).

A graph that consists of Hamilton cycle is called a Hamilton graph (or Hamiltonian graph).

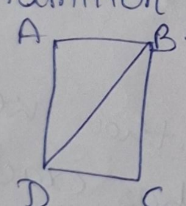
Ex:-



Hamilton cycle – A-B-C-D-A

Defn: A path in a connected graph which includes every vertex of graph is called a Hamilton path.

Ex:-



Hamilton path:- B–A–D–C

NOTE:-

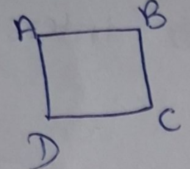
1. In a connected graph G with n vertices $(n \geq 3)$, the degree of every vertex is $\geq \frac{n}{2}$, then graph is Hamilton.
2. Every simple k regular graph with $2k-1$ vertices is a Hamilton graph.
3. $G=(V,E)$ loop free graph with $|V| = n \geq 2$. If $d(x)+d(y) \geq n-1$ $\forall\, x,y \in V$, $x \neq y$, then G has hamilton path.
4. $G=(V,E)$, $|V|=n \geq 2$, If $d(v) \geq (n-1)/2$ $\forall\, v \in V$, then G has a Hamilton path.

4. $G=(V,E)$, $|V|=n\geq 3$. If $d(x)+d(y)\geq n$ for all non adjacent $x,y\in V$, then G contains a Hamilton cycle.

Ex:-

5. $d(v)\geq n/2$ $\forall v\in V$, G has Hamilton cycle.

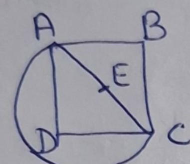
1) Graph which has both Euler ckt & Hamilton cycle



Euler ckt: A-B-C-D-A

Hamilton cycle: A-B-C-D

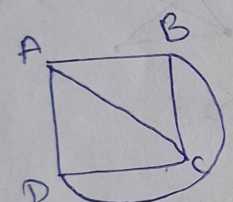
2) Euler ckt but no Hamilton cycle.



Euler ckt: A-B-C-A-E-C-D-A.

Here $n=4$ $\therefore \frac{n}{2}=2$

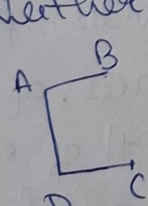
3) Hamilton cycle but no Euler ckt.



Hamilton cycle: A-B-C-D-A

Euler ckt exists iff all vertices are of even degree but here $d(A)=d(B)=d(C)=d(D)=3$.

4) Neither Hamilton cycle nor Euler ckt.



Graph is open so no cycle.

No Euler ckt $\because$ $d(B)=d(C)=1$ (not even)

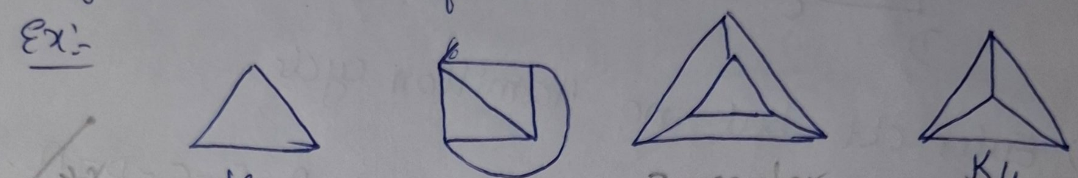
NOTE:- 1) # of Hamilton cycles in a complete Bipartite graph $K_{n,n}$ $(n\geq 2)$ is $\frac{1}{2}(n-1)!\,n!$

2) # of Hamilton paths in $K_{nn} = n!\cdot n! = (n!)^2$

Planar Graphs and Colouring

Defn:- A graph is called planar if G can be drawn in plane with its edges intersecting only at vertices of G.

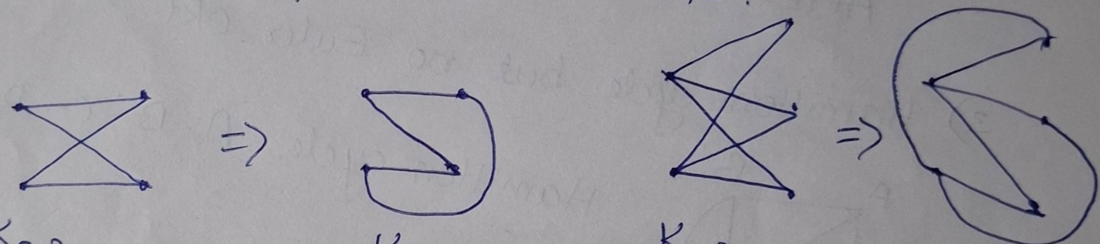
Ex:-



K_3 K_4 3-regular K_4

K_1, K_2, K_3, K_4 are planar but not K_5.

$K_{2,2}$, $K_{2,3}$ are planar but not $K_{3,3}$.



$K_{2,2}$ $K_{2,2}$ $K_{2,3}$ $K_{2,3}$

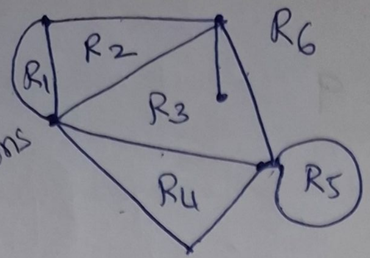
NOTE:- G is planar iff G does not contain K_5 or $K_{3,3}$.

Eulers Formula

Let G be planar graph. This divides plane into number of parts called regions or faces, of which exactly one part is unbounded. The # of edges that form boundary of region is called degree of region.

4. $G=(V,E)$, $|V|=n \geq 2$, if ...
has a Hamilton path.

Ex:-



R_1 to R_5 → Interior (bounded)

R_6 → Exterior (unbounded)

$n=6$
$m=10$
$m-n+2=6$ regions

$d(R_1)=2$ for pendent edge, count as 2.

$d(R_2)=3$

$d(R_3)=5$ (3 edges & pendent edge)

$d(R_5)=1$ (single loop)

$d(R_6)=6$. $d(R_4)=3$

$$\sum_{i=1}^{6} d(R_i) = 20 = 2|E|$$

Euler's Thm:- A connected planar graph G with n vertices & m edges has exactly $m-n+2$ regions.

i.e, $r = m-n+2$ or $n-m+r=2$.

NOTE:- 1) If G is connected, simple planar graph with $n(\geq 3)$ vertices and $m(>2)$ edges & r egions then i) $m \geq \frac{2}{3}r$ ($3m \geq 2r$ or $2r \leq 3m$)

and ii) $m \leq 3n-6$

1) P.T K_5 is non planar

Soln:- In K_5, $n=5$ $m=10$.

$r = m-n+2 = 10-5+2 = 7$.

Consider $m \leq 3n-6$ ie $10 \leq 3(5)-6$ ie $10 \not\leq 9$

∴ It is not planar.

2) P.T $K_{3,3}$ is not planar.

Soln:- Here $n = 6$ $m = 9$

Consider $m \le 3n - 6$

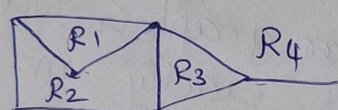
$9 \le 3(6) - 6$

$9 \le 12$ is True.

In $K_{3,3}$ each region is bounded by atleast 4 edges we have $4r \le 2m$. From Euler thm $r = m - n + 2$ $= 9 - 6 + 2 = 5$. so $4(5) \le 2(9)$ ie $20 \not\le 18$

$\therefore$ $K_{3,3}$ is not planar.

3)



Here $m = 9$

$n = 7$

$r = m - n + 2 = 9 - 7 + 2 = 4$.

$d(R_1) = 3$

$d(R_2) = 5$, $d(R_3) = 3$, $d(R_4) = 7$ (5 edges & 1 pendent edge)

$\therefore \sum_{i=1}^{4} d(R_i) = 18 = 2 \cdot 9$

4) G has 9 vertices of degrees 2, 2, 3, 3, 3, 4, 5, 6, 6 Find r.

Soln: $\sum d(v_i) = 2m$ Here $n = 9$

ie $2m = 34 \Rightarrow m = 17$

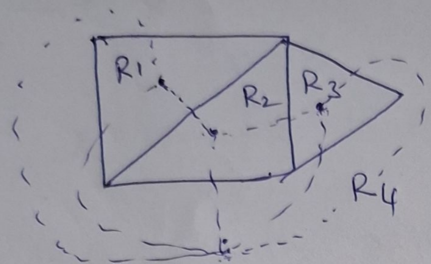
$\therefore$ By Eulers formula; $r = m - n + 2 = 17 - 9 + 2 = 10$

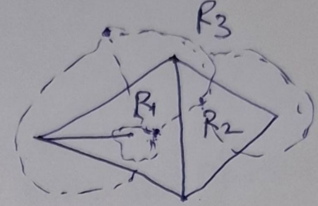
4. $G = (V, E)$ ...
has a Hamilton path.

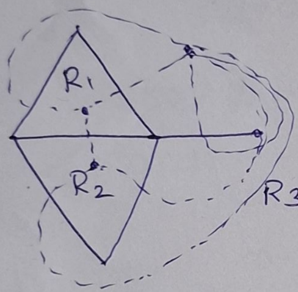
Dual of a planar graph.

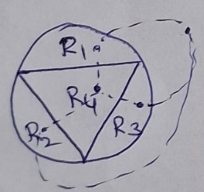
Dual of graph G is a graph that has a vertex corresponding to each region of G and an edge joining two neighboring regions for each edge in G.

Ex:-

1) 

5) 

2) 

3) 

4) 

5) Let G be 4-regular connected planar graph having 16 edges. Find r.

Soln:- $k=4 \quad m=16$

$$\Sigma d(v) = 4 \cdot n = 2m = 32$$

$$\therefore n = 8$$

$$r = m-n+2 = 16-8+2 = 10.$$

6) Can a G with $n=1000$ $m=3000$ be planar?

Soln:- G is planar if $m \leq 3n-6$.

i.e., $3000 \nleq 3000-6$

$\therefore$ not planar.

NOTE:- n-order m-size r-regions k-# of components

then $\underline{n-m+r = k+1}$

Graph Colouring:-

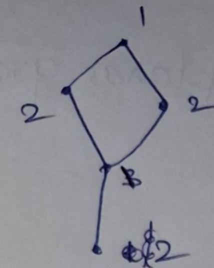
Defn:- Let $G=(V,E)$ be an undirected graph. A proper coloring of G occurs when we color vertices of G so that if $\{a,b\}$ is an edge in G, then a and b are colored with different color.

ie assigning colors to vertices such that adjacent vertices have different colors.

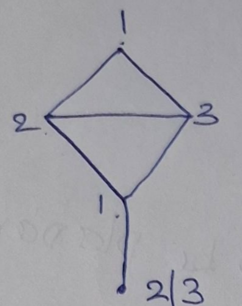
The minimum # of colors needed to properly color G is called $\underline{\text{Chromatic \# of G}}$, $\chi(G)$

Ex:-

G:



$\chi(G) = 2$



$\chi(G) = 3$

NOTE:-

1) For Null graph (containing only one isolated vertex), or with no edges
$\chi = 1$

2) for G with $m \geq 1$ is $\chi \geq 2$.

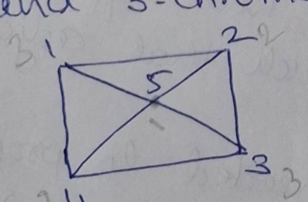
3) If G_1 is a subgraph of G, $\chi(G) \geq \chi(G_1)$

4) If G has n vertices, $\chi(G) \leq n$.

5) For K_n, $\chi(K_n) = n$. for all $n \geq 1$.

6) If G contain K_n as subgraph, $\chi(G) \geq n$.

7) A cycle with n vertices is 2 chromatic if n is even, and 3-chromatic if n is odd.



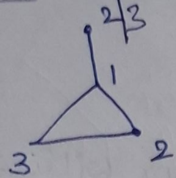
$\chi(G) = 3$ $\because$ u have cycle with 3 vertices

8) $\chi(G) \leq 1 + \Delta(G)$

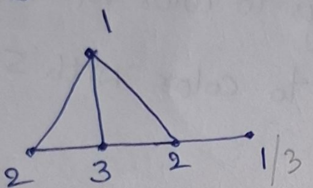
Thm:- If $\Delta(G)$ is max of degrees of vertices of a connected graph G, then

$$\chi(G) \le 1 + \Delta(G)$$

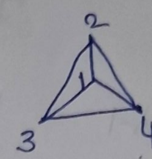
Find chromatic nos

1) 

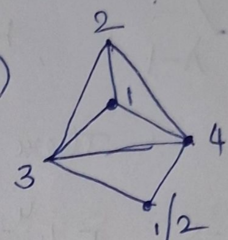
$\chi = 3$

2) 

$\chi = 3$

3) 

$\chi = 4$

4) 

$\chi = 4$

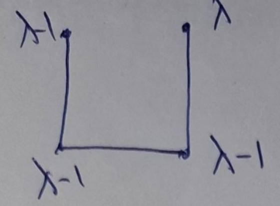
Chromatic Polynomial

Let G be an undirected graph, λ be number of colors available for properly covering vertices of G. Objective is to find a polynomial function $P(G, \lambda)$ is called chromatic polynomial of G which tells in how many diff ways we can properly color the vertices of G, using at most λ colors.

Ex:-

1) If $G = (V, E)$ with $|V| = n$ & $E = \phi$ then G contain n isolated points, then by product rule, $P(G, \lambda) = \lambda^n$ [$\because$ each point has λ options]

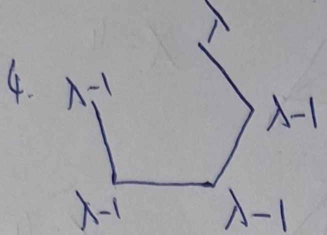
2) If $G = K_n$, then atleast n colors must be available for proper coloring. So by product rule,
$P(G, \lambda) = \lambda \cdot (\lambda - 1) \cdots (\lambda - n + 1)$ which is denoted by $\lambda^{(n)}$ $= {}^{\lambda}P_n$. For $\lambda < n$, $P(G, \lambda) = 0$.

3. 

with λ colors,

$$P(G,\lambda) = \lambda(\lambda-1)(\lambda-1)(\lambda-1)$$
$$= \lambda(\lambda-1)^3 \quad \text{with } \lambda=2,$$
$$2\cdot 1 = 2$$

There are two ways to color with 2 colors.

4. 

$$P(G,\lambda) = \lambda(\lambda-1)^4.$$

With $\lambda=2$,

$$P(G,\lambda) = 2.$$

$\therefore$ There are two ways to color with 2 colors.

With $\lambda=5$, $5\cdot 4^4 = 1280$ ways to color with 5 colors.

<u>Note</u>:-

1. If G is a path on n vertices, $P(G,\lambda) = \lambda\cdot(\lambda-1)^{n-1}$.
2. If G is made up of components $G_1, G_2, G_3 \ldots G_K$ then $P(G,\lambda) = P(G_1,\lambda).\, P(G_2,\lambda) \ldots \ldots P(G_K,\lambda)$.

<u>Decomposition thm of Chromatic polynomials</u>:-

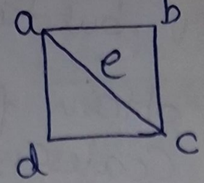
If $G=(V,E)$ is a connected graph and $e \in E$, then

$$P(G_e,\lambda) = P(G,\lambda) + P(G_e',\lambda).$$
$$\therefore P(G,\lambda) = P(G_e,\lambda) - P(G_e',\lambda)$$

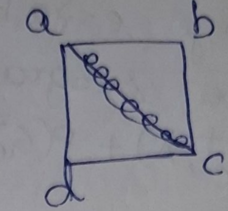
Let $G=(V,E)$ be an undirected graph. For $e=\{a,b\}\in E$, Let G_e denote subgraph of G by deleting e from G, without removing a and b; i.e., $G_e = G-e$. From G_e, G_e' is obtained by coalescing (or merging a and b).

Ex:-

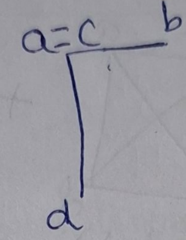
G:



$G_e = G - e$:

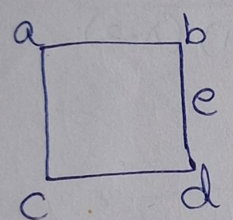


G_e' :

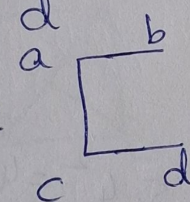


Ex:- Find $P(G,\lambda)$ for G using decomposition

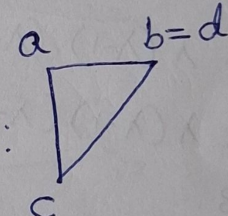
G:



Soln:- G_e :



$-$ G_e' :



$P(G_e) = \lambda(\lambda-1)^3 \qquad P(G_e',\lambda) = P(K_3,\lambda) = \lambda(\lambda-1)(\lambda-2)$

$\therefore P(G,\lambda) = P(G_e,\lambda) - P(G_e',\lambda)$

$= \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2)$

$= \lambda(\lambda-1)\left[(\lambda-1)^2 - (\lambda-2)\right]$

$= \lambda(\lambda-1)\left[\lambda^2 - 2\lambda + 1 - \lambda + 2\right]$

$= \lambda(\lambda-1)(\lambda^2-3\lambda+3) = (\lambda^2-\lambda)(\lambda^2-3\lambda-3)$

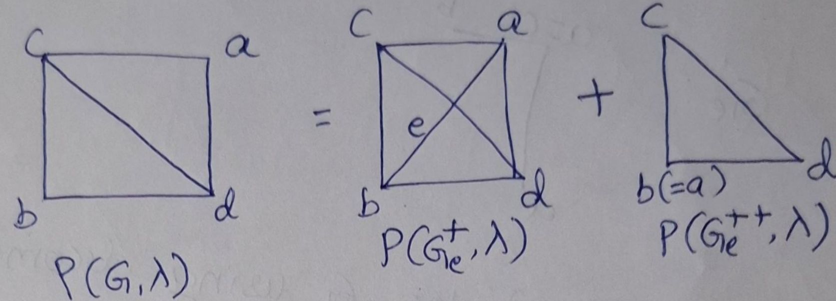
$= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$.

$\therefore P(G,1) = 0, \quad P(G,2) = 2 > 0 \qquad \chi(G) = 2.$

Thm:- Let $G=(V,E)$ with $a,b \in V$, but $\{a,b\} \notin E$.
3. We write G_e^+ for graph we obtain from G by adding edge $e=\{a,b\}$. Coalescing vertices a,b in G gives subgraph G_e^{++} of G. Then

$$P(G,\lambda) = P(G_e^+,\lambda) + P(G_e^{++},\lambda)$$

Ex:-



$\therefore P(G,\lambda) = \lambda^4 + \lambda^3$

$= \lambda.(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2).$

$= \lambda(\lambda-1)(\lambda-2)[\lambda-3+1]$

$= \lambda(\lambda-1)(\lambda-2)^2$

$\therefore \chi(G)=3$

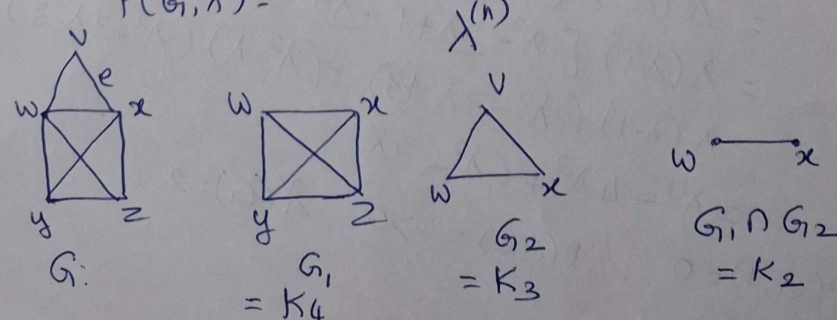
With $\lambda=6$,

G can be colored in $6.5.4^2 = 480$ ways.

Thm:- G be an undirected graph with subgraphs G_1, G_2. If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = K_n$ for some $n \in \mathbb{Z}^+$, then

$$P(G,\lambda) = \frac{P(G_1,\lambda).\,P(G_2,\lambda)}{\lambda^{(n)}}$$

Ex:-
1)



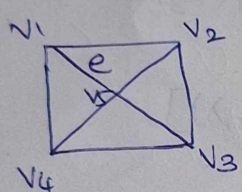
$$\therefore \quad P(G,\lambda) = \frac{P(G_1,\lambda)\cdot P(G_2\lambda)}{\lambda^{(2)}}$$

$$= \frac{\lambda^{(4)}\cdot\lambda^{(3)}}{\lambda^{(2)}} = \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)\,\lambda\cdot(\lambda-1)(\lambda-2)}{\lambda(\lambda-1)}$$

$$= \frac{\lambda^{\cancel{2}}(\lambda-1)^{\cancel{2}}(\lambda-2)^2(\lambda-3)}{\cancel{\lambda}\cancel{(\lambda-1)}}$$

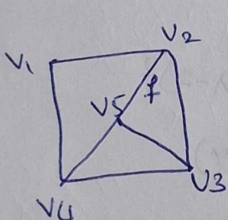
$$= \lambda(\lambda-1)(\lambda-2)^2(\lambda-3)$$

2) Find chromatic polynomial for G

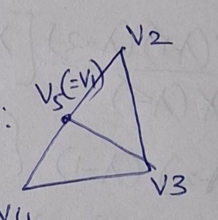


Soln:- Let $\{v_1, v_5\} = e$

G_e: $= H$

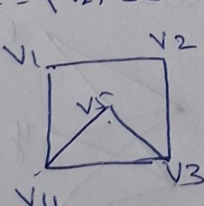


G'_e:

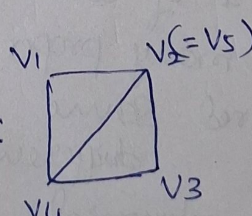


Let $f = \{v_2, v_5\}$

H_f:



H'_f:



Now for H'_f: it is union of K_3, K_3 & intersection is K_2 ($\{v_2, v_4\}$).

$$\therefore \quad P(H'_f, \lambda) = \frac{P(K_3,\lambda)\;P(K_3,\lambda)}{\lambda^{(2)}} = P(G'_e, \lambda) \rightarrow (1)$$

Also H_f is union of $C_4(v_1 v_2 v_3 v_4 v_1)$ & $K_3(v_5 v_3 v_4 v_5)$ & intersection is $K_2(\{v_3, v_4\})$.

$$\therefore P(H_f,\lambda) = \frac{P(C_4,\lambda)\cdot P(K_3,\lambda)}{\lambda^{(2)}} \rightarrow (2)$$

By decomposition thm, $H \equiv G_e$

$$P(G,\lambda) = P(G_e,\lambda) - P(G_e',\lambda)$$

$$= \underline{P(H,\lambda)} - P(G_e',\lambda)$$

$$= P(H_f,\lambda) - P(H_f',\lambda) - P(G_e',\lambda)$$

$$= \frac{1}{\lambda^{(2)}}\left[P(C_4,\lambda)\cdot P(K_3,\lambda) - 2\cdot P(K_3,\lambda)\cdot P(K_3,\lambda)\right]$$

$$= \frac{P(K_3,\lambda)}{\lambda^{(2)}}\left[P(C_4,\lambda) - 2P(K_3,\lambda)\right]$$

$$= \frac{\cancel{\lambda}\cancel{(\lambda-1)}(\lambda-2)}{\cancel{\lambda}\cancel{(\lambda-1)}}\left[\lambda\{(\lambda-1)^3 - (\lambda-1)(\lambda-2)\} - 2\lambda(\lambda-1)(\lambda-2)\right]$$

$$= \lambda(\lambda-2)(\lambda-1)\{(\lambda-1)^2 - 3(\lambda-2)\}$$

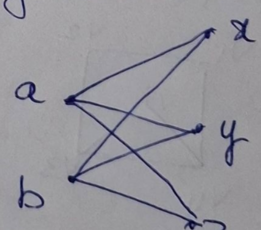
$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7)$$

3) In $K_{2,3}$ λ - # of colors available.

i) Find how many proper colorings are there when a, b coloured same

ii) " " differently

iii) Chromatic polynomial.



Ans: i) a has λ choices, b has 1 choice.

x, y, z have $(\lambda-1)$ choices. $\therefore$ proper colorings $= \lambda(\lambda-1)^3$

ii) a has λ choices, b has $\lambda-1$ choices

x, y, z have $(\lambda-2)$ choices. $\therefore$ proper colorings $= \lambda(\lambda-1)(\lambda-2)^3$

iii) $\therefore P(K_{2,3},\lambda) = \lambda(\lambda-1)^3 + \lambda(\lambda-1)(\lambda-2)^3$.

(16-1)

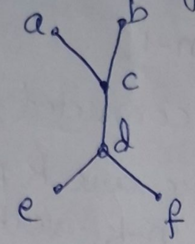
4) chromatic polynomial of $K_{2,n}$

is $P(K_{2,n}, \lambda) = \lambda(\lambda-1)^n + \lambda(\lambda-1)(\lambda-2)^n$.

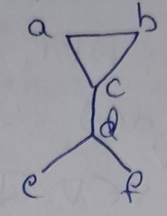
Trees

A graph G is said to be a tree if it is connected and has no cycles, represented as T.

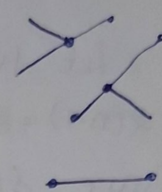
Ex:-



T



G not T
since cycle is present.



G not T
not connected
it is a Forest.

Thm:-

1. If a, b are distinct vertices in a tree $T=(V,E)$ then there exists a unique path that connects these vertices.

Proof:- Since T is connected, there is atleast one path in T that connects a and b. If there were more, then from two such paths some of edges would form cycle. But T has no cycles.

2. If $G=(V,E)$ is an undirected graph, then G is connected iff G has a spanning tree.

Proof:- If G has a spanning tree T, then for every pair a, b of distinct vertices in V a subset of edges in T provides path b/n a and b & so G is connected. Conversely if G is connected and G is not a tree, remove loops from G and form T.

3. In every tree $T = (V, E)$, $|V| = |E| + 1$.

4. For every tree $T = (V, E)$, if $|V| \geq 2$ then T has atleast two pendant vertices.

Proof:- Let $|V| = n \geq 2$. WKT, $|E| = n - 1$.

$\therefore$ $2(n-1) = 2 \cdot |E| = \sum_{v \in V} d(v)$ $\because$ T is connected, we have $d(v) \geq 1$ $\forall v \in V$. If there are k pendant vertices in T, then each of other n-k vertices has degree atleast 2 &

$$2(n-1) = 2|E| = \sum_{v \in V} d(v) \geq k + 2(n-k)$$

From this $2(n-1) \geq k + 2(n-k)$

$\Rightarrow \cancel{2n} - 2 \geq k + \cancel{2n} - 2k \Rightarrow k \geq 2$.

4. F is a forest with k components (trees). If n is # of vertices & m is # of edges in F, prove that $n = m + k$.

Proof:- $m_i = n_i - 1$ for $i = 1, 2 \ldots k$

$\Rightarrow m_1 + m_2 + m_3 + \ldots m_k = (n_1 - 1) + (n_2 - 1) + \ldots (n_k - 1)$

$= n_1 + n_2 + \ldots n_k - k$

$m = n - k$

$\therefore n = m + k$.

Ex:- 1) $T_1 = (V_1, E_1)$ $T_2 = (V_2, E_2)$ If $|E_1| = 19$ $|V_2| = 3|V_1|$ find $|V_1|, |V_2|, |E_2|$.

Soln:-

$|E_1| = 19 = |V_1| - 1$

$\therefore |V_1| = 20$.

$\because |V_2| = 3|V_1| = 60$

$|E_2| = |V_2| - 1 = 59$.

2) If a tree has 2020 vertices, find sum of degrees of the vertices.

Soln:- $|V| = 2020$ $\quad |E| = |V| - 1 = 2019$.

$\Sigma d(v) = 2 \cdot |E| = \underline{4038}$.

3) A tree T has 4 vertices of degree 2, 1 vertex of degree 3, 2 vertices of degree 4 and 1 vertex of degree 5. Find # of leaves in T.

Soln:- Let k be # of leaves (pendant) vertices in T.

Then $|V| = 4+1+2+1+k = 8+k$

$\Sigma d(v) = \cancel{8\times 2+k} \; (4\times 2)+(1\times 3)+(2\times 4)+(1\times 5)+(k\times 1)$

$= 24+k$

$\because$ T has $k+8$ vertices, it has $k+7$ edges.

$\therefore \quad 2(k+7) = 24+k$

$\therefore \quad 2k+14 = 24+k$

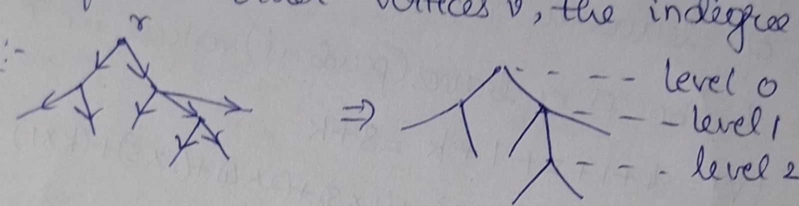
$k = \underline{10}$.

$\therefore$ T has 10 leaves.

Rooted Trees

Defn:- If G is a directed graph, then G is called a directed tree if the undirected graph associated with G is a tree. When G is a directed tree, G is called a rooted tree if there is a unique vertex r, called root in G with indegree of $r = id(r) = 0$ and for all other vertices v, the indegree of $v = id(v) = 1$.

Ex:-

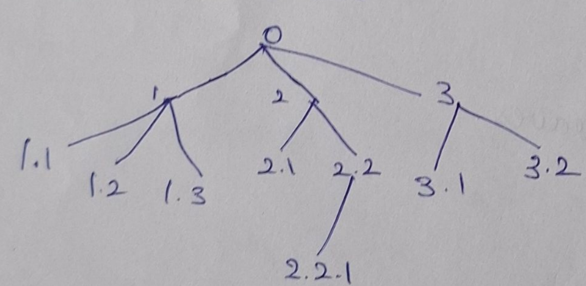


Vertices with common parent are called siblings.

For root, indegree is 0

For leaf, outdegree is 0.

Ordered rooted tree



m-ary Tree

If every internal vertex of T is of out degree $\leq m$. ie if every internal vertex of T has atmost m children.

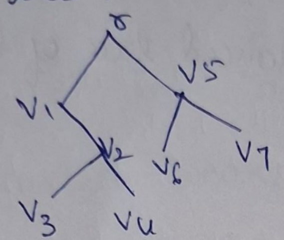
A rooted tree T is called complete m-ary tree if every internal vertex of T is of out-degree m. ie every internal vertex has exactly m children.

Binary tree

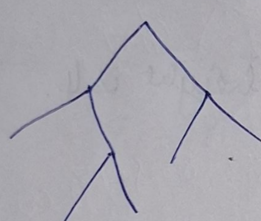
An m-ary tree with $m=2$ is called binary tree. ie if every vertex has atmost two children.

A complete m-ary tree with $m=2$ is called complete binary tree.

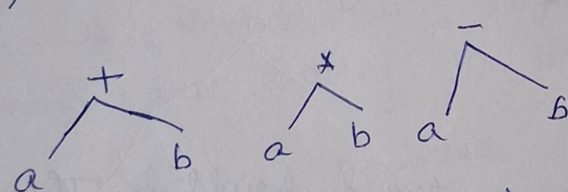
Ex:-



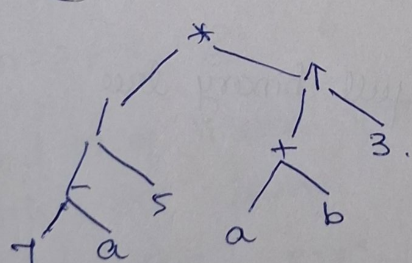
is a binary tree but not complete binary tree.



is a complete binary tree.



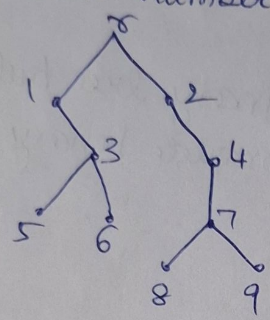
$((7-a)/5) * ((a+b)\uparrow 3)$



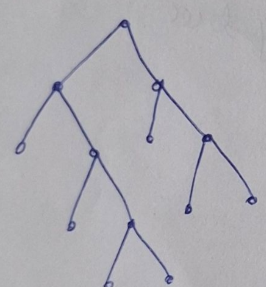
Balanced Tree

If T is a rooted tree and h is the largest level number of leaf, then T is said to have height h. A rooted tree of height h is said to be balanced if the level number of every leaf is h or h-1.

Ex:-



is balanced tree of height 4



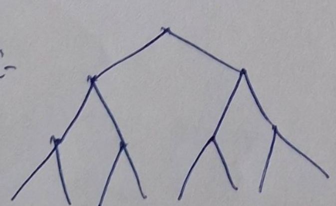
is not balanced of its height is 4.

not full binary tree.

Full binary Tree

T be a complete binary tree of height h. Then T is called a full binary tree if all leaves in T are at level h.

Ex:-



is full binary tree.

20

1) Let T be a complete m-ary tree of order n with p leaves and q internal vertices. Then

i) $n = mq + 1 = \frac{mp - 1}{m - 1}$

ii) $p = (m-1)q + 1 = \frac{(m-1)n + 1}{m}$

iii) $q = \frac{n-1}{m} = \frac{p-1}{m-1}$

2) In a complete binary tree,

$n = 2p - 1 = 2q + 1$

$p = \frac{2n+1}{2} = q + 1$

$q = p - 1 = \frac{n-1}{2}$.

3) Find # of vertices & # of leaves in a complete binary tree having 10 internal vertices.

Soln:- $q = 10$ p - leaves n = nodes

$p = q + 1 = 11$

$n = 2p - 1 = \underline{\underline{21}}$

4) Find # of internal vertices in a complete 5-ary tree with 817 leaves.

Soln:- $p = 817$ $m = 5$ $q = ?$

$q = \frac{p-1}{m-1} = \frac{816}{4} = \underline{\underline{204}}$.

5) Find # of leaves in a complete 6-ary tree of order 733.

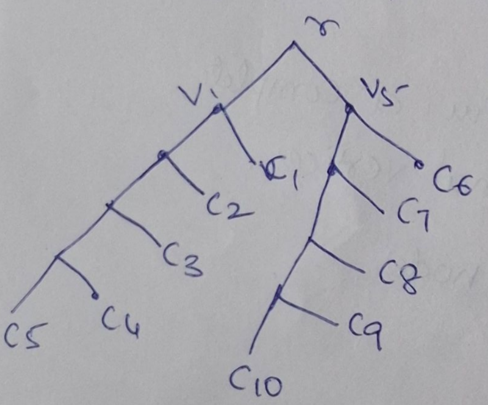
Soln:- $m = 6 \quad n = 733. \quad p = ?$

$$p = \frac{(m-1)n+1}{m} = \frac{5(733)+1}{6} = 611.$$

6) Computer Lab of School has 10 computers that are to be connected to a wall socket that has 2 outlets. Connections are made by using extension cords that have 2 outlets each. Find # of cords needed to get these computers set up for use.

Soln:- $p = 10, \quad q = p - 1 = 9.$

Comp – leaves, & internal nodes other than root – extension cord



Cords: $q - 1 = 8.$

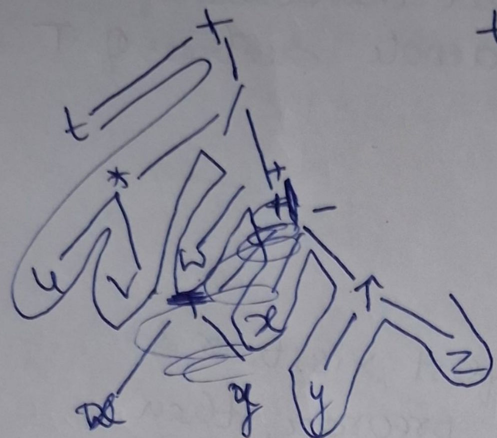
7) A complete binary tree has 20 leaves. How many vertices does it have?

Soln:- $p = 20 \quad n = ?$

$$p = \frac{n+1}{2} \quad \therefore n = 2p - 1 = 39.$$

Expression Trees:

$t+(uv)/(w+x-y^2)$



$t + (u * v) / (w + x - y \uparrow z)$

(6) (4) (5) (3) (2) (1)

To evaluate this exp computer should scan back & forth continuously. So it is converted into a notation which is independent of parentheses. This is known as Polish notation/ prefix notation.

$+ t / * u v + w - x \uparrow y z$

Now evaluation proceeds from right to left. when an operator encountered, it performs on two operands to its right. Ex:-

1) $+ 4 / * 2 3 + 1 - 9 \uparrow \underline{2 3}$

2) $+ 4 / * 2 3 + 1 - \underline{9 8}$

3) $+ 4 / * 2 3 \underline{+ 1 1}$

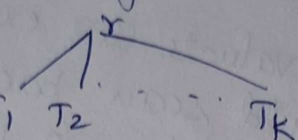
4) $+ 4 / \underline{* 2 3} 2$

5) $+ 4 \underline{/ 6 2}$

6) $\underline{+ 4 3}$

7) 7

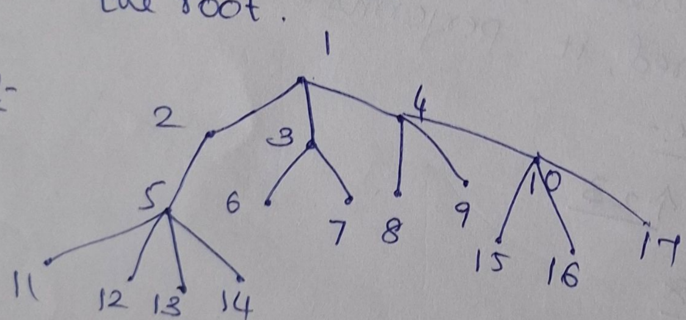
Defn:- Let $T = (V, E)$ be a rooted tree with root r. If T has no other vertices, then root by itself constitute preorder & post order traversals of T. If $|V| > 1$, let $T_1, T_2 \ldots T_k$ denote subtrees of T from left to right



a) Preorder traversal of T first visits r & traverses vertices of T_1 in preorder, then vertices of T_2 in preorder & so on until vertices of T_k are traversed in preorder.

b) Postorder traversal of T traverses in post order the vertices of $T_1, T_2 \ldots T_k$ and then visits the root.

Ex:-



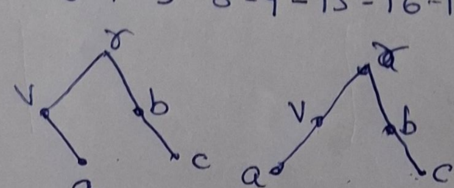
Pre order: rt-l-r

1-2-5-11-12-13-14-3-6-7-4-8-9-10-15-16-17

Post order: l-r-rt

11-12-13-14-5-2-6-7-3-8-9-15-16-17-10-4-1

Inorder: L-rt-r



v-a-r-b-c

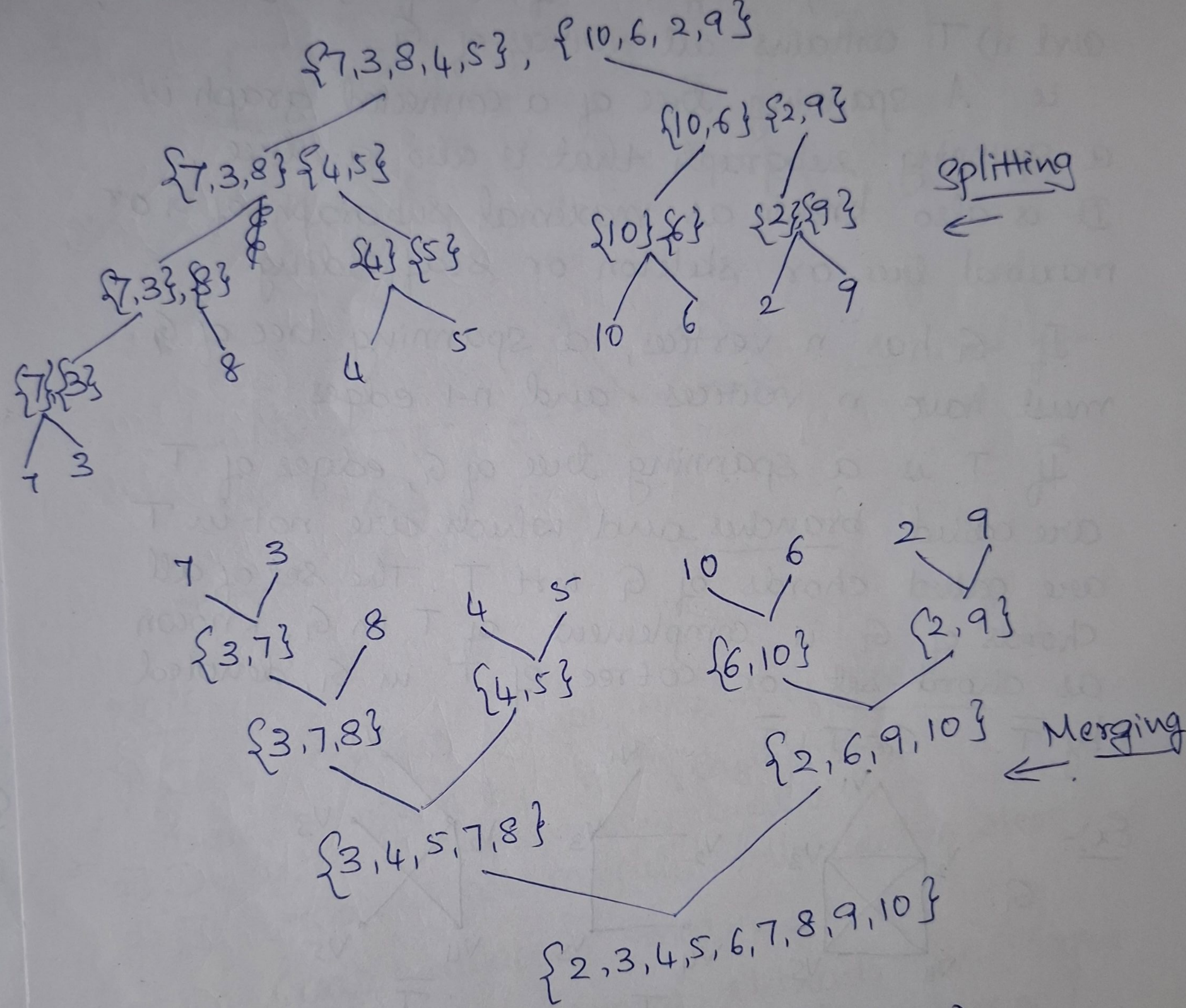
a-v-r-b-c

Preorder: r-v-a-b-c for both

Post order: ~~a-v~~-c-b-r for both.

Sorting - Merge sort.

7, 3, 8, 4, 5, 10, 6, 2, 9.



Worst case time complexity - O(n log n).

Spanning trees

Let G be a connected graph. A subgraph T of G is called a spanning tree of G if i) T is a tree, and ii) T contains all vertices of G.

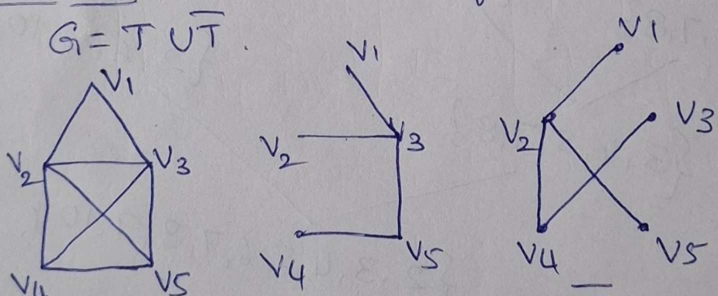
ie A spanning tree of a connected graph is a spanning subgraph that is also a tree. It is also known as maximal subgraph of G or maximal tree or skeleton or scaffolding.

If G has n vertices, a spanning tree of G must have n vertices and $n-1$ edges.

If T is a spanning tree of G, edges of T are called branches and which are not in T are called chords of G wrt T. The set of all chords of G is complement of T in G, known as chord-set or cotree of T in G, denoted by $\bar{T}$. $G = T \cup \bar{T}$.

Ex:-

G:

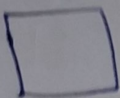


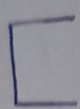
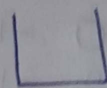
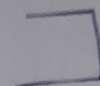
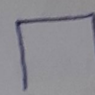
T $\qquad$ $\bar{T}$

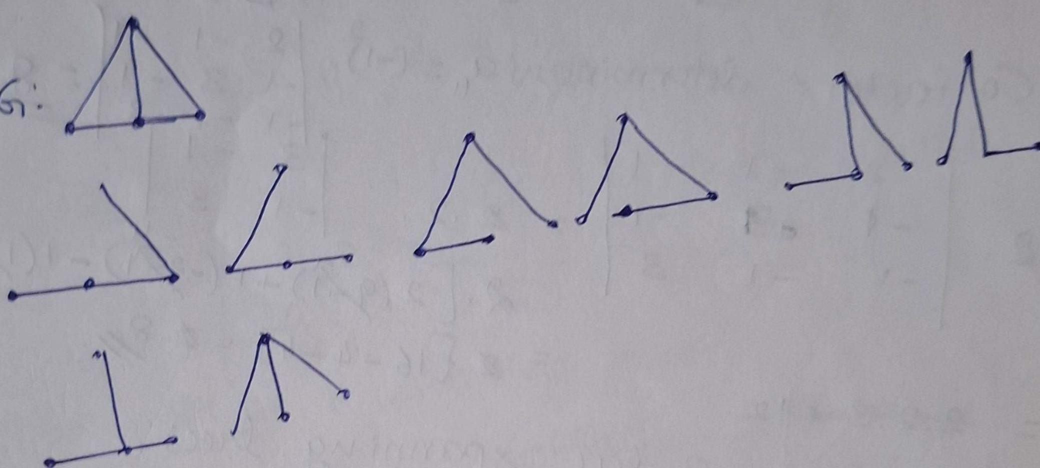
NOTE:-

1. A graph is connected iff it has spanning tree.
2. Wrt any of spanning tree of G with n vertices & m edges will have $n-1$ branches & $m-n+1$ edges

1) Find all spanning trees.

G: 

Soln:- T_1:  T_2:  T_3:  T_4: 

2) G: 

NOTE:-

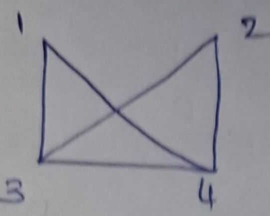
1) In a complete graph, the # of spanning trees is given by n^{n-2} [Cayley's Thm]

2] In cycle C_n, # of spanning trees is n.

2) If graph is not complete, follow below steps:
   1) Create adjacency matrix for G
   2) Replace all diagonal elements with degree of nodes. ie (1,1) replaced by degree of 1, (2,2) replaced by degree of 2 etc.
   3) Replace all non diagonals 1's with -1.
   4) Calculate co-factor for any element
   5) Co factor is # of spanning trees.

a_1

Ex:-



$d(1) = 2 = d(2)$
$d(3) = 3 = d(4)$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 1 & 1 & 1 & 0 \end{array} \Rightarrow \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Co factor determinant $a_{11} = (-1)^2 \cdot \begin{vmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = \underline{8}$

$$\cancel{2} \cdot \begin{vmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = \cancel{2} \cdot 2 \cdot \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

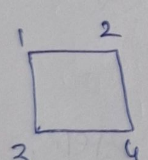
$$\cancel{2} \cdot [2(\cancel{9-1}^{\,8}) + 1(\cancel{-3-1}^{\,-4}) - 1(\cancel{1+3}^{\,4})]$$

$$= \cancel{2}\,[16 - 4 - 4] = \cancel{4} \cdot 8$$

$= \cancel{2.2.3 = 12}$

∴ There are 8 diff spanning trees.

2)

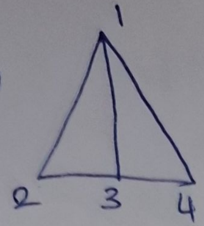


$d(1) = d(2) = d(3) = d(4) = 2.$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 0 & 1 & 1 & 0 \end{array} \Rightarrow \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 2(4-1) - 1(0+2) = 6 - 2 = \underline{4}$$

∴ There are 4 spanning trees possible.

3) 

$d(1) = 3 = d(3)$
$d(2) = d(4) = 2$

Soln:-

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \Rightarrow L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Laplacian matrix

Cofactor(L) $= 8$

$$\begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \cancel{2(6-1)+1(4)+0} = \cancel{2.5}$$

∴ There will be 8 spanning trees.

Laplacian matrix L is obtained as below

$L[i,i] = \text{degree}(i)$

$L[i,j] = \begin{cases} -1 & \text{if there is an edge b/n } i \,\&\, j \\ 0 & \text{.. no edge b/n } i \,\&\, j. \end{cases}$

To find cofactor

$a_{11} = (-1)^{4+1} \begin{vmatrix} -1 & -1 & -1 \\ 2 & -1 & 0 \\ -1 & 3 & -1 \end{vmatrix} + \cancel{0(-1)(-1)^{4+3} \begin{vmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & -1 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{vmatrix}}$

$\cancel{+ \left[3(-2)+1(-4)-1(6+1)\right] + 2\left[3(6-1)+1(-3-1)-1(1+2)\right]}$

$= -1(6-1) - 1(1+2)$
$-5-3 = -8 \times -1 = 8$

$a_{21} = (-1)^{2+1} \cdot \begin{vmatrix} -1 & -1 & -1 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = -1(6-1)+1(-2)-1(1)$
$= -5-2-1 = -8 \times -1$
$= 8$

Cofactor of any elements $= 8$.

∴

Ex:-

1. For a $K_{m,n}$ bipartite graph, there will be $m^{n-1} n^{m-1}$ spanning trees.

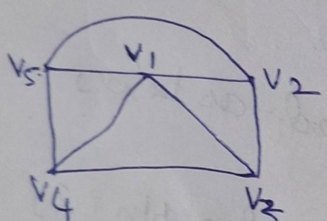
2. Chromatic polynomial of $K_{2,n}$ is

$$P(K_{2,n}, \lambda) = \underbrace{\lambda(\lambda-1)^n}_{\text{2 left nodes with same color}} + \underbrace{\lambda(\lambda-1)(\lambda-2)^n}_{\text{diff colors}}$$

3. Chromatic polynomial of C_n is

$$P(C_n, \lambda) = \lambda(\lambda-1)(\lambda-2) \cdots (\lambda-(n-1))$$
$$= \lambda(\lambda-1)(\lambda-2) \cdots (\lambda-n+1)$$
$$= (\lambda-1)^n + (-1)^n(\lambda-1)$$

4. Find Chromatic polynomial



Soln:- Here $\chi(G) = 3$ with $\lambda = 5$

Chromatic polynomial $= c_1 \cdot {}^5C_1 + c_2 \cdot {}^5C_2 + c_3 \cdot {}^5C_3 + c_4 \cdot {}^5C_4 + c_5 \cdot {}^5C_5$

Here $c_1 = 0$, $c_2 = 0$ $\because$ G cannot be colored with 1 and 2 colors.

$c_3 = 3! = 6$

$c_4 = 4!\,2! = 48$

$c_5 = 5! = 120$

$$P(G,5) = 6 \cdot {}^5C_3 + 48 \cdot {}^5C_4 + 120\,{}^5C_5$$
$$= 6 \cdot \frac{5!}{3!\,2!} + 48 \cdot \frac{5!}{4!} + 120$$
$$= 6(10) + 48(5) + 120$$
$$= 420.$$

With λ colors

$$P(G,\lambda) = \cancel{c_1 \lambda C_1} + \cancel{c_2 \lambda C_2} + c_3 \cdot \lambda C_3 + c_4 \cdot \lambda C_4 + c_5 \lambda C_5 = \sum_{i=1}^{n} c_i \cdot \lambda C_i$$

5 nodes.

$$= \lambda(\lambda-1)(\lambda-2) + 2(\lambda)(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

$$= \cancel{6} \cdot \frac{\lambda(\lambda-1)(\lambda-2)}{\cancel{6}} + \cancel{48}\,\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{\cancel{24}} + \cancel{120} \cdot \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{\cancel{120}}$$