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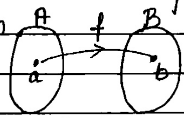
Functions

Let A and B be two non-empty sets. Then a function (or mapping) f from A to B is a relation from A to B such that for each a in A there is a unique b in B such that $(a,b)\in f$. Then we write $b=f(a)$. Here b is called the image of a, and a is called a preimage of b, under f. The element a is also called an argument of the function f, and $b=f(a)$ is then called the value of the function f for the argument a.

Pictorially f can be represented as follows:

The function f from A to B is denoted by $f: A\to B$



For the function $f: A\to B$, A is called the domain of f and B is called the codomain of f. The subset of B consisting of the images of all elements of A under f is called the range of f and is denoted by $f(A)$

The following observations are immediate consequences of the definition of a function $f: A\to B$ and other associated definitions given above:

1. Every a in A belongs to some pair $(a,b)\in f$, and if $f(a,b_1)\in f$ and $(a,b_2)\in f$, then $b_1=b_2$. That is every element of A has an unique image.
2. An element $b\in B$ need not have a preimage in A under f.
3. If an element $b\in B$ has a preimage $a\in A$ under f, the preimage need not be unique.
4. The statements $(a,b)\in f$, afb and $b=f(a)$ are equivalent.

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5. If g is a function from A to B, then $f = g$ if and only if $f(a) = g(a)$ for every $a \in A$.

6. The range of $f: A \to B$ is given by $f(A) = \{f(x) \mid x \in A\}$ and $f(A)$ is a subset of B.

7. For $f: A \to B$, if $A_1 \subseteq A$ and $f(A_1)$ is defined by $f(A_1) = \{f(x) \mid x \in A_1\}$, then $f(A_1) \subseteq f(A)$.

8. For $f: A \to B$, if $b \in B$ and $f^{-1}(b)$ is defined by $f^{-1}(b) = \{x \in A \mid f(x) = b\}$, then $f^{-1}(b) \subseteq A$.

9. For $f: A \to B$, if $B_1 \subseteq B$ and $f^{-1}(B_1)$ is defined by $f^{-1}(B_1) = \{x \in A \mid f(x) \in B_1\}$, then $f^{-1}(B_1) \subseteq A$.

* Let $A = \{1, 2, 3\}$ and $B = \{-1, 0\}$ and R be a relation from A to B defined by
$R = \{(1,-1), (1,0), (2,-1), (3,0)\}$. Is R a function from A to B?

Sol^n Since $1 \in A$ is related to two different element $-1, 0 \in B$, R is not a function.

* Let $A = \{1, 2, 3\}$ and $B = \{-1, 0\}$, and S be a relation from A to B defined by $S = \{(1,-1), (2,-1), (3,0)\}$.
Is S a function?

Sol^n Since each element of A is having a unique image in B, S is a function.

* Let $A = \{0, \pm 1, \pm 2, 3\}$. Consider the function $f: A \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, defined by $f(x) = x^3 - 2x^2 + 3x + 1$, for $x \in A$. Find the range of f.

Sol^n $f(0) = 1$, $f(1) = 3$, $f(-1) = -5$, $f(2) = 7$, $f(-2) = -21$, $f(3) = 19$

Hence range of f ie, $f(A) = \{1, 3, -5, 7, -21, 19\}$

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* Let a function $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2+1$. Determine the images of the following subsets of $\mathbb{R}$.
(i) $A_1 = \{2, 3\}$, (ii) $A_2 = \{-2, 0, 3\}$, (iii) $A_3 = (0,1)$, (iv) $A_4 = [-6, 3]$

Sol^n

(i) $f(A_1) = \{5, 10\}$
(ii) $f(A_2) = \{5, 1, 10\}$
(iii) $A_3 = \{x \in \mathbb{R} \mid 0 < x < 1\}$
then $f(A_3) = \{f(x) \mid 0 < x < 1\} = \{x^2+1 \mid 0 < x < 1\} = (1, 2)$
(iv) $f(A_4) = \{f(x) \mid -6 \le x \le 3\} = \{x^2+1 \mid -6 \le x \le 3\}$
$\left(f(-6) = 37,\ f(-5) = 26 \ldots f(0) = 1,\ - f(2) = 5,\ f(3) = 10\right) \quad = [1, 37]$

* Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 3x-5, & x > 0 \\ -3x+1, & x \le 0 \end{cases}$

(i) Determine $f(0)$, $f(-1)$, $f(5/3)$, $f(-5/3)$
(ii) Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-6)$,
(iii) What are $f^{-1}([-5,5])$ and $f^{-1}([-6,5])$?

Sol^n (i) $f(0) = -3\times(0)+1 = 1$, $f(-1) = -3\times(-1)+1 = 4$,
$f(5/3) = 3\times\frac{5}{3} - 5 = 0$, $f(-5/3) = -3\times\left(-\frac{5}{3}\right)+1 = 6$

(ii) let $f^{-1}(0) = x \Rightarrow f(x) = 0 \Rightarrow \begin{cases} 3x-5=0 \Rightarrow x = 5/3\ (>0) \\ \text{or } -3x+1=0 \Rightarrow x = 1/3\ (\not\le 0) \times \end{cases}$
$\therefore f^{-1}(0) = \{5/3\}$

$f^{-1}(1) = x \Rightarrow f(x) = 1 \Rightarrow \begin{cases} 3x-5=1 \Rightarrow x = 2\ (>0) \\ -3x+1=1 \Rightarrow x = 0\ (\le 0) \end{cases}$ $\therefore f^{-1}(1) = \{0, 2\}$

$f^{-1}(3) = x \Rightarrow f(x) = 3 \Rightarrow \begin{cases} 3x-5=3 \Rightarrow x = 8/3\ (>0) \\ -3x+1=3 \Rightarrow x = -2/3\ (\le 0) \end{cases}$ $\therefore f^{-1}(3) = \{8/3, -2/3\}$

$f^{-1}(-1) = x \Rightarrow f(x) = -1 \Rightarrow \begin{cases} 3x-5=-1 \Rightarrow x = 4/3\ (>0) \\ -3x+1=-1 \Rightarrow x = 2/3\ (\not\le 0) \end{cases}$ $\therefore f^{-1}(-1) = \{4/3\}$

$f^{-1}(-6) = x \Rightarrow f(x) = -6 \Rightarrow \begin{cases} 3x-5=-6 \Rightarrow x = -1/3\ (x \not> 0) \\ -3x+1=-6 \Rightarrow x = 7/3\ (x \not\le 0) \end{cases}$ $\therefore f^{-1}(-6) = \{\ \}$

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(iii) $f^{-1}([-5,5]) = \{x \in \mathbb{R} \mid f(x) \in [-5,5]\}$

$= \{x \in \mathbb{R} \mid -5 \le f(x) \le 5\}$

$-5 \le 3x-5 \le 5 \Rightarrow 0 \le 3x \le 10 \Rightarrow 0 \le x \le \frac{10}{3} \quad (x>0)$ ✓

$-5 \le -3x+1 \le 5 \Rightarrow -6 \le -3x \le 4 \Rightarrow -\frac{4}{3} \le x \le 2 \quad (x \le 0)$

$\Rightarrow -\frac{4}{3} \le x \le 0$ ($\because$ x should bee ≤ 0)

$\therefore f^{-1}([-5,5]) = \{x \in \mathbb{R} \mid -\frac{4}{3} \le x \le \frac{10}{3}\}$.

$= [-4/3, 10/3]$

$f^{-1}([-6,5]) = \{x \in \mathbb{R} \mid f(x) \in [-6,5]\}$

$= \{x \in \mathbb{R} \mid -6 \le f(x) \le 5\}$

$-6 \le 3x-5 \le 5 \Rightarrow -1 \le 3x \le 10 \Rightarrow -\frac{1}{3} \le x \le \frac{10}{3}$

$\Rightarrow 0 < x \le \frac{10}{3}$ ($\because$ x should be > 0)

$-5 \le -3x+1 \le 5 \Rightarrow -7 \le -3x \le 4 \Rightarrow -\frac{4}{3} \le x \le \frac{7}{3}$

$\Rightarrow -\frac{4}{3} \le x \le 0$ ($\because$ x should be ≤ 0).

$\therefore f^{-1}([-6,5]) = \{x \in \mathbb{R} \mid -\frac{4}{3} \le x \le \frac{10}{3}\}$

$= [-4/3, 10/3]$

(a) Let A and B be finite sets with $|A| = m$ and $|B| = n$. Find how many functions are possible from A to B?

(b) If there are 2187 functions from A to B and $|B| = 3$, what is $|A|$?

(a) Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$

Then a typical function $f: A \to B$ is of the form $f = \{(a_1, x), (a_2, x), \dots, (a_m, x)\}$ where x can be any of the b_j from the set B.

Therefore the total possible number of ~~choices~~ ordered pairs would be $n \times n \times \dots \times n$ (m times)

$= n^m$

Thus, there are $n^m = |B|^{|A|}$ possible functions from A to B

(b) given $n^m = 2187$ and $n = 3$.

$\Rightarrow 3^m = 2187 \Rightarrow m = \left(\frac{\ln 2187}{\ln 3}\right) \Rightarrow m = 7$

$\therefore |A| = 7$

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Types of functions

Identity function

A function $f: A \to A$ such that $f(a) = a$ for every $a \in A$ is called the identity function on A. The identity function defined on a set A is usually denoted as I_A.

Constant function

A function $f: A \to B$ such that $f(a) = c$ for every $a \in A$, where c is a fixed element of B, is called a constant function.

Onto function

A function $f: A \to B$ is said to be an onto function if for every b of B there is an element a of A such that $f(a) = b$.
An onto function is also called a surjective function.

One-to-One function

A function $f: A \to B$ is said to be a one-to-one function if different elements of A have different images in B under f. That is if whenever $a_1, a_2 \in A$ with $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$; or equivalently that whenever $f(a_1) = f(a_2)$ for $a_1, a_2 \in A$, then $a_1 = a_2$.
A one-to-one function is also called an injective function

One-to-One correspondence

A function which is both one-to-one and onto is called a one-to-one correspondence or a bijective function. If $f: A \to B$ is such a function, then every element of A has a unique image in B and every element in B has a unique preimage in A.

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* Find the nature of the following functions defined on $A = \{1,2,3\}$

(i) $f = \{(1,1),(2,2),(3,3)\}$ (ii) $g = \{(1,2),(2,2),(3,2)\}$

(iii) $h = \{(1,2),(2,2),(3,1)\}$ (iv) $p = \{(1,2),(2,3),(3,1)\}$

Sol^n (i) $\forall\, a \in A,\ (a,a) \in f$, hence f is a identity function.

(ii) $\forall\, a \in A\ (a,2) \in g$, hence g is a constant function.

(iii) Since distinct element of A dont have distinct images h is not one-to-one.

Since 3 doesnot have a preimage, h is not onto.

(iv) Since distinct elements of A have distinct image and all elements have preimages, p is both one-to-one and onto.

* The functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = 3x+7$ for all $x \in \mathbb{R}$ and $g(x) = x(x^3-1)$ for all $x \in \mathbb{R}$. Verify that f is one-to-one but g is not.

Sol^n Consider $f(x_1) = f(x_2)$

$\Rightarrow 3x_1 + 7 = 3x_2 + 7$

$\Rightarrow x_1 = x_2$

Hence f is one-to-one.

Consider $g(0) = 0$ and $g(1) = 0$

Since both 0 and 1 are mapped to the same element 0, g is not one-to-one.

* Let $A = \mathbb{R}$ and $B = \{x \mid x \text{ is real and } x \geq 0\}$. Is the function $f: A \to B$ defined by $f(a) = a^2$ an onto function? ~~a one-to-one function?~~

Sol^n let $f(a) = b \Rightarrow a^2 = b \Rightarrow a = \pm\sqrt{b}$,

ie $\exists\, +\sqrt{b}, -\sqrt{b} \in \mathbb{R}$ s.t $f(+\sqrt{b}) = b$ and $f(-\sqrt{b}) = b$

Hence f is onto.

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* let $A = \mathbb{R}$ and $B = \{x \mid x \text{ is real and } x \geq 0\}$.
Is the function $f: A \to B$ defined by $f(a) = a^2$
a one-to-one function?

Sol: Consider $f(a_1) = f(a_2)$

$\Rightarrow a_1^2 = a_2^2$

$\Rightarrow a_1 = \pm a_2$.

Since a_1 is not only equal to a_2, says that
both $+a_1$ and ~~+a~~ $-a_1$ are mapped to a_1^2,
hence f is not one-one.

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* Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(a) = a+1$ for $a \in \mathbb{Z}$.
Find whether f is one-to-one or onto.

Sol: Consider $f(a_1) = f(a_2)$

$\Rightarrow a_1 + 1 = a_2 + 1$

$\Rightarrow a_1 = a_2$

$\therefore f$ is one-to-one

let $f(a) = b \Rightarrow a+1 = b \Rightarrow a = b-1$.

i.e., $\exists\ b-1 \in \mathbb{Z}$ s.t $f(b-1) = b$.

Hence f is onto.

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Composition of functions

Consider three non-empty sets A, B, C and the functions $f: A \to B$ and $g: B \to C$. The composition (product) of these two functions is defined as the function $g \circ f: A \to C$ with $(g \circ f)(a) = g(f(a))$ for all $a \in A$

* Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$ and $C = \{\omega, x, y, z\}$ with $f: A \to B$ and $g: B \to C$ given by $f = \{(1,a), (2,a), (3,b), (4,c)\}$ and $g = \{(a,x), (b,y), (c,z)\}$. Find $g \circ f$.

Soln

$g \circ f(1) = g(f(1)) = g(a) = x$.
$g \circ f(2) = g(f(2)) = g(a) = x$.
$g \circ f(3) = g(f(3)) = g(b) = y$
$g \circ f(4) = g(f(4)) = g(c) = z$

Thus $g \circ f = \{(1,x), (2,x), (3,y), (4,z)\}$

* Consider the functions f and g defined by $f(x) = x^3$ and $g(x) = x^2 + 1$, $\forall x \in \mathbb{R}$. Find $g \circ f$, $f \circ g$, f^2 and g^2.

Sol.

$g \circ f(x) = g(f(x)) = g(x^3) = x^6 + 1$
$f \circ g(x) = f(g(x)) = f(x^2+1) = (x^2+1)^3$
$f^2(x) = (f \circ f)(x) = f(f(x)) = f(x^3) = x^9$
$g^2(x) = (g \circ g)(x) = g(g(x)) = g(x^2+1) = (x^2+1)^2 + 1$

* Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = ax + b$ and $g(x) = 1 - x + x^2$. ~~If go~~ If $(g \circ f)(x) = 9x^2 - 9x + 3$, determine a and b.

Sol

$(g \circ f)(x) = g(f(x)) = g(ax+b) = 1 - (ax+b) + (ax+b)^2$ — (1)

given $(g \circ f)(x) = 9x^2 - 9x + 3$ — (2)

Equating (1) & (2) $\Rightarrow$ $1 - ax - b + a^2x^2 + 2abx + b^2 = 9x^2 - 9x + 3$

$\Rightarrow 1 - b + b^2 = 3 \Rightarrow b^2 - b - 2 = 0 \Rightarrow b = -1$ & $b = 2$, $2ab - a = -9$ $\Rightarrow a = 3$, $a = -3$

Thus $a = 3, b = -1$ and $a = -3, b = 2$

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Some important results on composition of functions

Theorem: Let $f: A \to B$ and $g: B \to C$ be any two functions. Then the following are true:

(i) If f and g are one-to-one, so is $g \circ f$.
(ii) If $g \circ f$ is one to one, then f is one-to-one.
(iii) If f and g are onto, so is $g \circ f$.
(iv) If $g \circ f$ is onto, then g is onto.

Theorem: Let $f: A \to B$, $g: B \to C$ and $h: C \to D$ be three functions. Then $(h \circ g) \circ f = h \circ (g \circ f)$

* Let f, g, h be functions from Z to Z defined by $f(x) = x - 1$, $g(x) = 3x$,
$$h(x) = \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd} \end{cases}$$
Determine $(f \circ (g \circ h))(x)$ and $((f \circ g) \circ h)(x)$ and verify that $f \circ (g \circ h) = (f \circ g) \circ h$

Soln.
$$(g \circ h)(x) = g(h(x)) = 3h(x)$$
$$(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(3h(x)) = 3h(x) - 1$$
$$= \begin{cases} -1, & \text{if } x \text{ is even} \\ 2, & \text{if } x \text{ is odd} \end{cases} \quad \text{—(1)}$$

$$(f \circ g)(x) = f(g(x)) = f(3x) = 3x - 1.$$
$$((f \circ g) \circ h)(x) = \cancel{f(g(h(x)))}\ (f \circ g)(h(x)) = 3h(x) - 1$$
$$= \begin{cases} -1, & \text{if } x \text{ is even} \\ 2, & \text{if } x \text{ is odd} \end{cases} \quad \text{—(2)}$$

From (1) and (2) it follows that
$$f \circ (g \circ h) = (f \circ g) \circ h.$$

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Invertible functions

A function $f: A \to B$ is said to be invertible if there exists a function $g: B \to A$ such that $gof = I_A$ and $fog = I_B$, where I_A is the identity function on A and I_B is the Identity function on B.

* Let $A = \{1, 2, 3, 4\}$ and f and g be functions from A to A given by,
$f = \{(1,4), (2,1), (3,2), (4,3)\}$ and $g = \{(1,2), (2,3), (3,4), (4,1)\}$
Prove that f and g are inverses of each other.

Sol^n
$(gof)(1) = g(f(1)) = g(4) = 1 = I_A(1)$
$(gof)(2) = g(f(2)) = g(1) = 2 = I_A(2)$
$(gof)(3) = g(f(3)) = g(2) = 3 = I_A(3)$
$(gof)(4) = g(f(4)) = g(3) = 4 = I_A(4)$
$(fog)(1) = f(g(1)) = f(2) = 1 = I_A(1)$
$(fog)(2) = f(g(2)) = f(3) = 2 = I_A(2)$
$(fog)(3) = f(g(3)) = f(4) = 3 = I_A(3)$
$(fog)(4) = f(g(4)) = f(1) = 4 = I_A(4)$

Thus, for all $x \in A$, we have $(gof)(x) = I_A(x)$ and $(fog)(x) = I_A(x)$. Therefore, g is an inverse of f and f is an inverse of g.

* Consider the function $f: R \to R$ defined by $f(x) = 2x + 5$. Let a function $g: R \to R$ be defined by $g(x) = \frac{1}{2}(x-5)$. Prove that g is an inverse of f.

Sol^n Consider $(gof)(x) = g(f(x)) = g(2x+5) = \frac{1}{2}[(2x+5)-5]$
$= x = I_R(x)$

$(fog)(x) = f[g(x)] = f[\frac{1}{2}(x-5)] = 2 \times \frac{1}{2}(x-5) + 5$
$= x = I_R(x)$

These show that g is an inverse of f.
It also follows that f is an inverse of g.

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Some results on one-to-one, onto and invertible functions:

1. If a function $f: A \to B$ is invertible then it has a unique inverse. Further, if $f(a)=b$, then $f^{-1}(b)=a$.
2. A function $f: A \to B$ is invertible if and only if it is one-to-one and onto.
3. Let A and B be finite sets with $|A|=|B|$ and f be a function from A to B. Then the following statements are equivalent. (i) f is one-to-one, (ii) f is onto, (iii) f is invertible.
4. If $f: A \to B$ and $g: B \to C$ are invertible functions, then $g \circ f: A \to C$ is an invertible function and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

* Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, $\forall x \in \mathbb{R}$. Is f invertible?

Sol^n For any $a \in \mathbb{R}$, we have $f(a)=a^2$ and $f(-a)=a^2$.
Thus, both a and $-a$ have the same image a^2 under f.
Therefore, f is not one-to-one.
Consequently, f is not invertible.

* Let $A = \{x \mid x \text{ is real and } x \geq -1\}$, and $B = \{x \mid x \text{ is real and } x \geq 0\}$
Consider the function $f: A \to B$ defined by $f(a) = \sqrt{a+1}$ for all $a \in A$. Show that f is invertible and determine f^{-1}.

Sol^n $f(a_1) = f(a_2) \Rightarrow \sqrt{a_1+1} = \sqrt{a_2+1} \Rightarrow a_1+1 = a_2+1 \Rightarrow a_1 = a_2$ $\therefore f$ is one-to-one
$f(a) = b \Rightarrow \sqrt{a+1} = b \Rightarrow a+1 = b^2 \Rightarrow a = b^2-1$,
i.e. $\exists\, b^2-1 \in A$ such that $f(b^2-1) = b$ $\therefore f$ is onto
Hence f is invertible.
As $f(b^2-1) = b \Rightarrow f^{-1}(b) = b^2-1$ is the inverse of b.

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* Find the inverse (if any) of the function $f(x)=e^x$ defined from $R \to R^+$.

Sol: $f(x_1)=f(x_2) \Rightarrow e^{x_1}=e^{x_2} \Rightarrow x_1=x_2$ ∴ f is one to one

Let $f(x)=y \Rightarrow e^x=y \Rightarrow x=\ln y$

∴ $\exists \ln y \in R$ s.t. $f(\ln y)=y$. ∴ f is onto.

∴ f is invertible

As $f(\ln y)=y \Rightarrow f^{-1}(y)=\ln y$ is the inverse of y.

* Let $A=B=R$, the set of all real numbers and the functions $f: A\to B$ and $g: B\to A$ be defined by $f(x)=2x^3-1$, $\forall x\in A$; $g(y)=\left\{\frac{1}{2}(y+1)\right\}^{1/3}$, $\forall y\in B$. Show that each of f and g is the inverse of the other.

Sol: Consider $(g\circ f)(x) = g(f(x))$

$= g(2x^3-1)$

$= \cancel{2\left[\left\{\frac{1}{2}(x+1)\right\}^{1/3}\right]^3 - 1}\ \left[\frac{1}{2}(2x^3-1+1)\right]^{1/3}$

$= x$

Thus $(g\circ f)(x)=I_A$, hence g is the inverse of f.

Consider $(f\circ g)(y) = f(g(y))$

$= f\left(\left[\frac{1}{2}(y+1)\right]^{1/3}\right)$

$= 2\left(\left[\frac{1}{2}(y+1)\right]^{1/3}\right)^3-1$

$= y$

Thus $(f\circ g)\,y = I_B$, hence f is the inverse of g.

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Let $A = B = C = \mathbb{R}$ and $f : A \to B$ and $g : B \to C$ be defined by $f(a) = 2a+1$, $g(b) = \frac{1}{3}b$, $\forall a \in A$, $\forall b \in B$. Compute $g \circ f$ and show that $g \circ f$ is invertible. What is $(g \circ f)^{-1}$?

Sol^n. $(g \circ f)(a) = g(f(a)) = g(2a+1) = \frac{1}{3}(2a+1)$

$f(a_1) = f(a_2) \Rightarrow 2a_1 + 1 = 2a_2 + 1 \Rightarrow a_1 = a_2$ $\therefore f$ is one to one

$f(a) = b \Rightarrow 2a+1 = b \Rightarrow a = \frac{1}{2}(b-1)$

$\therefore \exists\ \frac{1}{2}(b-1) \in A$ s.t. $f(\frac{1}{2}(b-1)) = b$. $\therefore f$ is onto

Hence f is invertible. and $f^{-1}(b) = \frac{1}{2}(b-1)$

$g(b_1) = g(b_2) \Rightarrow \frac{1}{3}b_1 = \frac{1}{3}b_2 \Rightarrow b_1 = b_2$ $\therefore g$ is one to one.

$g(b) = c \Rightarrow \frac{1}{3}b = c \Rightarrow b = 3c$

$\therefore \exists\ 3c \in B$ s.t $g(3c) = c$. $\therefore g$ is onto.

Hence g is invertible. and $g^{-1}(c) = 3c$

Since f and g are invertible $g \circ f$ is invertible.

and $(g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c)$

$= f^{-1}(g^{-1}(c))$

$= f^{-1}[3c]$

$= \frac{1}{2}(3c-1)$

Topic ______ Date ______

Properties of functions

1. Let A and B be finite sets and f be a function from A to B. Then the following are true.
(i) If f is one-to-one, then $|A| \leq |B|$
(ii) If f is onto, then $|B| \leq |A|$.
(iii) If f is a one-to-one correspondence, then $|A| = |B|$.
(iv) If $|A| > |B|$, then at least two different elements of A have the same image under f.

Property 1(iv) can be interpreted as follows:
If every element of a set A with $|A| = n$ is assigned to a unique element of a set B with $|B| = m$ and $n > m$, then at least two different elements of A are assigned to the same element of B. In other words, if n objects (pigeons, say) are assigned to m places (pigeon holes), and if $n > m$, then at least two objects (pigeons) are assigned to the same place. This is known as the "Pigeonhole Principle".

2. ~~If~~ Suppose A and B are finite sets having the same number of elements, and f is a function from A to B. Then f is one-to-one if and only if f is onto.

3. A function f from a finite set A to a finite set B with $|A| = |B|$ is bijective if and only if f is one-to-one or onto.

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Number of one-to-one and onto functions

If A and B are finite sets with $|A| = m$ and $|B| = n$, where $m \geq n$, then:

(i) The number of one-to-one functions possible from A to B is given by the formula

$$\cancel{P(n,m) = \frac{n!}{(n-m)!}}$$

(ii) The number of onto functions possible from A to B is given by the formula

$$\cancel{p(n,m) = \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m}$$

* Let A and B be finite sets. If there are 60 one-to-one functions from A to B and $|A| = 3$, what is $|B|$?

Sol^n Given $m = 3$, $P(n,m) = 60$

$$\Rightarrow \frac{n!}{(n-3)!} = 60$$

$$\Rightarrow n(n-1)(n-2) = 60 \quad \Rightarrow \left(n^3 - 3n^2 + 2n - 60 = 0 \quad \Rightarrow n = 5 \right)$$

$$\Rightarrow n = 5.$$

i.e, $|B| = 5$

* Let $A = \{1,2,3,4,5,6,7\}$ and $B = \{w,x,y,z\}$. Find the number of onto functions from A to B

Sol^n Given $m = 7$, $n = 4$.

$$p(4,7) = \sum_{k=0}^{4} (-1)^k \binom{4}{4-k} (4-k)^7 = 4^7 - 4 \times 3^7 + 6 \times 2^7 - 4 = 8400.$$

* Let $A = \{1,2,3,4\}$ and $B = \{1,2,3,4,5,6\}$

(i) How many functions are there from A to B? How many of these are one-to-one? How many are onto?

(ii) How many functions are there from B to A? How may of these are one-to-one? How many are onto?

(i) possible function $6^4 = 1296$ one-to-one functions $\frac{6!}{2!} = 360$ onto functions $\rightarrow$ nil	(ii) possible functions $4^6 = 4096$ ~~one-to-one functions~~ nil onto functions $\sum_{k=0}^{4} (-1)^k \binom{4}{4-k} (4-k)^6 = 1560$

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Stirling number of the second kind.

For $m \geq n$ there are $\sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m$

ways to distribute m distinct objects into n numbered (but otherwise identical) containers with no container left empty. Removing the numbers on the containers, so that they are now identical in appearance, we find that one distribution into these n (nonempty) identical containers corresponds with $n!$ such distributions into the numbered containers. So the number of ways in which it is possible to distribute the m distinct objects into n identical containers, with no container left empty, is $\frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m$.

This will be denoted by $S(m,n)$ and is called a stirling number of the second kind. We note that for $|A| = m \geq n = |B|$, there are $n! \cdot S(m,n)$ onto functions from A to B.

Q A research chemist who have five laboratory assistants is engaged in a research project that calls for nine compounds that must be synthesized. In how many ways can the chemist assign these syntheses to the five assistants so that each is working on at least one synthesis.

Sol A = the 9 synthesis, B = the 5 assistants.
we need to find the number of onto functions from A to B, which is $\sum_{k=0}^{5} (-1)^k \binom{5}{5-k} (5-k)^9$

$= 5^9 - 5 \times 4^9 + 10 \times 3^9 - 10 \times 2^9 + 5 \times 1^9 = 834120$

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Growth of functions

Computer programmers are often concerned with two questions:

(i) How much time does an algorithm need to complete?
(ii) How much memory does an algorithm need for its computation?

Big O notation is a mathematical notation used in computer science to describe the upper bound of an algorithm's runtime complexity, or how its performance scales as the input size grows. It essentially provides a way to analyze the efficiency of an algorithm, focusing on its worst-case scenario.

The Big O notation is denoted as $O(f(n))$, where $f(n)$ is a function that represents the number of operations that an algorithm performs to solve a problem of size n.

Some common Big O notations include

* $O(1)$: The run time is independent of input size.
* $O(\log n)$: The runtime grows logarithmically with input size.
* $O(n)$: The runtime grows proportionally to input size.
* $O(n^2)$: The runtime grows with the square of input size.
* $O(2^n)$: The runtime grows exponentially with the input size.

The size of the input complexities ordered from smallest to largest:

$O(1)$, $O(\log n)$, $O(\sqrt{n})$, $O(n)$, $O(n\log n)$, $O(n^2)$, $O(n^3)$, $O(b^n)$ $(b>1)$, $O(n!)$

The asymptotic behaviour for large n should be determined by the most dominant term in the function for large n.

eg $f(x) = x^3 + 2x^2 - 2x$ for large x, is dominated by the term x^3.

$\therefore O(f(x)) = x^3$

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Properties of Big O notation

Suppose $f(x)$ is $O(F(x))$ and $g(x)$ is $O(G(x))$.

Then 1. $c.f(x)$ is $O(F(x))$

2. $f(x)+g(x)$ is $O(\max(F(x), G(x)))$

3. $f(x).g(x)$ is $O(F(x).G(x))$.

Find the Big O of each of the following:

~~ex~~ $f(x) = 2x^2 + 4x$.

Big O of $f(x)$ is $O(\max(x^2, x)) = O(x^2)$

$h(x) = (x^3+1).(x^2-x)$

Big O of $h(x)$ is $O(x^3 . x^2) = O(x^5)$

$g(x) = x^x(2^x + x^2)$

Big O of $g(x)$ is $O(x^x . 2^x)$

$p(x) = 5x! + 4x^3 \log x$.

Big O ~~of~~ $p(x)$ is $O(\max(x!, x^3 \log x))$

$= O(x!)$

$q(x) = (3x^3+x) 2^x + (x + x!) x^4$

Big O of $q(x)$ is $O(\max(x^3 . 2^x, x^4 . x!))$

$= O(x^4 x!)$