Vector Spaces

Let F be a field, V be a non empty set.

For every ordered pair $\alpha, \beta \in V$, let there be defined uniquely a sum $\alpha+\beta$ and for every $\alpha \in V$, and $c \in F$ a scalar product $c.\alpha$ in V.

The set V is called a vector space over the field F, if the following axioms are satisfied, for every $\alpha, \beta, \gamma \in V$ and for every $c, c' \in F$.

(i) $\alpha + \beta \in V$. Closed under addition.

(ii) $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$. Associative w.r.t addition.

(iii) Identity element w.r.t addition exists.

ie, $\exists\ e \in V$ s.t $\alpha+e = e+\alpha = \alpha$

(iv) Inverse element w.r.t addition exists.

ie, $\exists\ \alpha^{-1} \in V$ s.t $\alpha+\alpha^{-1} = e = \alpha^{-1}+\alpha$

(v) $\alpha+\beta = \beta+\alpha$. Commutative w.r.t addition.

(vi) $c.(\alpha+\beta) = c.\alpha + c.\beta$

(vii) $(c+c').\alpha = c.\alpha + c'.\alpha$

(viii) $(c.c').\alpha = c.(c'.\alpha)$

(ix) $1.\alpha = \alpha$, $\forall\ \alpha \in V$, where 1 is the unit element of F.

Examples:

Let F be a field and n be a positive integer. Let $V_n(F)$ be the set of all ordered n tupples of the elements of the field F.

ie, $V_n(F) = \{(x_1, x_2, \dots, x_n) / x_i \in F\}$

Define addition and scalar multiplication as below:

(a) $\alpha + \beta = (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)$

$= (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$

(b) $c.\alpha = c.(x_1, x_2, \dots, x_n)$

$= (c.x_1, c.x_2, \dots, c.x_n), \forall c \in F.$

(i) $\alpha + \beta = (x_1+y_1, x_2+y_2, \dots, x_n+y_n) \in V_n(F)$

(ii) $(\alpha+\beta)+\gamma = (x_1+y_1, x_2+y_2, \dots, x_n+y_n) + (z_1, z_2, \dots, z_n)$

$= ((x_1+y_1)+z_1, (x_2+y_2)+z_2, \dots, (x_n+y_n)+z_n)$

$= (x_1+(y_1+z_1), x_2+(y_2+z_2), \dots, x_n+(y_n+z_n))$

$= (x_1, x_2, \dots, x_n) + (y_1+z_1, y_2+z_2, \dots, y_n+z_n)$

$(\alpha+\beta)+\gamma = \alpha + (\beta+\gamma)$

(iii) $\overset{\alpha}{(x_1, x_2, \dots, x_n)} \overset{+}{+} \overset{0}{(0, 0, 0, \dots, 0)} = (x_1, x_2, \dots, x_n)$

$= \underset{0}{(0, 0, \dots, 0)} \underset{+}{+} \underset{\alpha}{(x_1, x_2, \dots, x_n)}$

$\therefore (0, 0, \dots, 0)^{=0}$ is the additive identity.

(iv) $\alpha + (-\alpha) =$

$= (x_1, x_2 \dots x_n) + (-x_1, -x_2, \dots -x_n)$

$= (0, 0, \dots, 0)$

$= 0$

$= (-x_1, -x_2 \dots -x_n) + (x_1, x_2, \dots x_n)$

$= -\alpha + \alpha$

$\therefore -\alpha = (-x_1, -x_2 \dots -x_n)$ is the additive inverse of $\alpha = (x_1, x_2, \dots, x_n)$

(v) $\alpha + \beta = (x_1, x_2 \dots, x_n) + (y_1, y_2 \dots y_n)$

$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

$= (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n)$

$= (y_1, y_2, \dots, y_n) + (x_1, x_2, \dots, x_n)$

$= \beta + \alpha$

(vi) $c.(\alpha + \beta) = c.(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

$= (c.x_1 + c.y_1, c.x_2 + c.y_2, \dots, c.x_n + c.y_n)$

$= (cx_1, cx_2, \dots, cx_n) + (cy_1, cy_2 \dots, cy_n)$

$= c.(x_1, x_2, \dots x_n) + c.(y_1, y_2 \dots y_n)$

$= c.\alpha + c.\beta.$

(vii) $(c + c')\alpha = (c + c')(x_1, x_2 \dots, x_n)$

$= ((c+c')x_1, (c+c')x_2 \dots, (c+c')x_n)$

$= (cx_1 + c'x_1, cx_2 + c'x_2 \dots, cx_n + c'x_n)$

$= (cx_1, cx_2 \dots, cx_n) + (c'x_1, c'x_2 \dots, c'x_n)$

$= c(x_1, x_2, \dots x_n) + c'(x_1, x_2 \dots x_n)$

$= c\alpha + c'\alpha$

(viii) $(c.c').\alpha = (c.c')(x_1, x_2 \dots, x_n)$

$= ((c.c')x_1, (c.c')x_2, \dots (cc')x_n)$

$= (c.(c'.x_1), c.(c'.x_2), \dots, c.(c'.x_n))$

$= c.(c'.x_1, c'.x_2, \dots c'.x_n)$

$= c.(c'.(x_1, x_2 \dots, x_n))$

$= c.(c'.\alpha)$

(ix) $1.\alpha = 1.(x_1, x_2 \dots x_n)$

$= (1.x_1, 1.x_2 \dots, 1.x_n)$

$= (x_1, x_2 \dots, x_n)$

$= \alpha$

Thus $V_n(F)$ is a vector space over the field F.

Note:-

(i) With $F = \mathbb{R}$, $V_1(\mathbb{R})$, $V_2(\mathbb{R})$, $V_3(\mathbb{R})$ are all vector spaces. They are also denoted as $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, ~~which~~ respectively. The elements of $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$ are real numbers, plane vectors and space vectors respectively.

(ii) If $F = \mathbb{R}$, $V_n(\mathbb{R})$ is denoted as $\mathbb{R}^n$.

If $F = \mathbb{C}$, $V_n(\mathbb{C})$ is denoted as $\mathbb{C}^n$.

2. Show that $V = \{ a + b\sqrt{2} \mid a, b \in Q \}$, where Q is the set of all rationals, field Q, is a vector space under usual addition and scalar multiplication.

Let $\alpha = a_1 + b_1\sqrt{2}$, $\beta = a_2 + b_2\sqrt{2}$, $\gamma = a_3 + b_3\sqrt{2} \in V$

$c, c' \in Q$.

(i) $\alpha + \beta = (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) \in V$.

(ii) $(\alpha + \beta) + \gamma = ((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) + (a_3 + b_3\sqrt{2})$

$= \alpha + (\beta + \gamma)$

(iii) 0 is the additive identity,

as $0 + \alpha = \alpha = \alpha + 0$.

(iv) $-\alpha = -a_1 - b_1\sqrt{2}$ is the additive inverse of $\alpha = a_1 + b_1\sqrt{2}$, as $\alpha + (-\alpha) = 0 = (-\alpha) + \alpha$.

(v) $\alpha + \beta = (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = \beta + \alpha$

(vi) $c.(\alpha + \beta) = c.((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) = c.\alpha + c.\beta$

(vii) $(c + c')\alpha = (c + c')(a_1 + b_1\sqrt{2}) = c.\alpha + c'.\alpha$

(viii) $(c.c')\alpha = (c.c')(a_1 + b_1\sqrt{2}) = c.(c'.\alpha)$

(ix) $1.\alpha = 1.(a_1 + b_1\sqrt{2}) = a_1 + b_1\sqrt{2} = \alpha$

Thus V is a vector space over Q.

3. Let V be the set of all polynomials of degree $\leq n$, with coefficients in the field F, together with zero polynomial. Then Show that V is a vector space under addition of polynomials and scalar multiplication of polynomials with the scalar $c \in F$ defined by

$$c(a_0 + a_1 x + \dots + a_n x^n) = ca_0 + ca_1 x + \dots + ca_n x^n.$$

(i) sum of polynomials is again a polynomial.

(ii) sum of polynomials will be associative.

(iii) ~~The~~ '0' is the additive identity.

(iv) if $\alpha = a_0 + a_1 x + \dots + a_n x^n$, then $-\alpha = -a_0 - a_1 x - \dots - a_n x^n$ is the additive inverse.

(v) sum of polynomials is commutative.

(vi) $c(\alpha + \beta) = c.\alpha + c.\beta$ will hold

(vii) $(c+c')\alpha = c.\alpha + c'.\alpha$ will hold

(viii) $(c.c').\alpha = c.(c'.\alpha)$ will hold.

(ix) $1.\alpha = 1.(a_0 + a_1 x + \dots + a_n x^n) = a_0 + a_1 x + \dots + a_n x^n = \alpha$.

Thus V is a vector space over F.

4. Let $V = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \middle| \; x, y \in \mathbb{C} \right\}$, under usual addition and scalar multiplication, with field $\mathbb{C}$ of complex numbers. Show that V is a vector space

(i) let $\alpha = \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix}$, $\beta = \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix}$, $\gamma = \begin{pmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{pmatrix} \in V$

$$c_1 = a_1 + b_1 i \;, \quad c_2 = a_2 + b_2 i \in \mathbb{C}.$$

(i) $\alpha + \beta \in V$.

(ii) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

(iii) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \alpha = \alpha + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the additive identity

(iv) $-\alpha = \begin{pmatrix} -x_1 & -y_1 \\ y_1 & -x_1 \end{pmatrix}$ is the additive inverse.

(v) $\alpha + \beta = \beta + \alpha$.

(vi) $c.(\alpha + \beta) = c.\alpha + c.\beta$

(vii) $(c + c')\alpha = c.\alpha + c'.\alpha$

(viii) $(c \cdot c')\alpha = c.(c'.\alpha)$

(ix) $1.\alpha = 1.\begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} = \alpha$

Thus V is a vector space over $\mathbb{C}$.

5. Let R^+ be the set of all positive real numbers ~~integers~~.
Define the operations of addition and scalar multiplication as below:

$$\alpha + \beta = \alpha\beta \quad \forall \quad \alpha, \beta \in R^+$$

$$c.\alpha = \alpha^c \quad , \quad \alpha \in R^+ \text{ and } c \in R.$$

Show that R^+ is a vector space over the real field.

(i) $\alpha + \beta = \alpha\beta \in R^+$

(ii) $(\alpha+\beta)+\gamma = (\alpha\beta)+\gamma = (\alpha\beta)\gamma = \alpha(\beta\gamma) = \alpha + \beta\gamma = \alpha+(\beta+\gamma)$

(iii) $\alpha + 1 = \alpha.1 = \alpha = 1.\alpha = 1+\alpha$

$\therefore$ 1 is the additive identity

(iv) $\alpha + \frac{1}{\alpha} = \alpha.\frac{1}{\alpha} = 1 = \frac{1}{\alpha}.\alpha = \frac{1}{\alpha}+\alpha$

$\therefore \frac{1}{\alpha}$ is the additive inverse of α.

(v) $\alpha + \beta = \alpha\beta = \beta\alpha = \beta+\alpha$

(vi) $c.(\alpha+\beta) = c.(\alpha\beta) = (\alpha\beta)^c = \alpha^c\beta^c = \alpha^c + \beta^c = c.\alpha + c.\beta$

(vii) $(c+c')\alpha = \alpha^{(c+c')} = \alpha^c.\alpha^{c'} = \alpha^c + \alpha^{c'} = c.\alpha + c'.\alpha$

(viii) $(c.c')\alpha = \alpha^{(c.c')}$ ~~$= (\alpha^c)^{c'} = c'(\alpha^c)$~~

$$= \alpha^{(c'.c)} = (\alpha^{c'})^c = c(\alpha^{c'}) = c.(c'\alpha)$$

(ix) $1.\alpha = \alpha^1 = \alpha$, where 1 is the unit element of R^+.

Exercises

1. Consider the set $V = C[0,1]$, the set of all continuous functions defined over the interval $[0,1]$. The sum of any two elements $f, g \in V$ is defined by $(f+g)(x) = f(x) + g(x)$, $\forall x \in [0,1]$, and scalar multiplication is defined by

$$(c.f)(x) = c.f(x) \;\; \forall \; x \in [0,1], \forall c \in \mathbb{R}.$$

Show that V is a vector space.

2. Let $V = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$ with field $\mathbb{R}$.

Show that V is a vector space over the field $\mathbb{R}$, under usual addition and scalar multiplication.

3. Let $V = \{(x, y, z) \mid x, y, z \in \mathbb{Q}$ and $x + 2y = 3z\}$, with field $\mathbb{Q}$, under component wise addition and scalar multiplication.

Show that V is a vector space.

4. Let V be a set of all odd functions from $\mathbb{R}$ to $\mathbb{R}$, with field $\mathbb{R}$, under usual addition and scalar multiplication. Show that V is a vector space.

5. Let V be the set of all convergent sequences $\{a_n\}$ of real numbers. Then show that V is a vector space over the field $\mathbb{R}$ of real numbers, with the addition and scalar multiplication defined by $\{a_n\} + \{b_n\} = \{a_n + b_n\}$ and $c\{a_n\} = \{ca_n\}$, $\forall c \in \mathbb{R}$.

Subspace :-

A non empty subset W of a vector space V over a field F is called a subspace of V, if W is itself a vector space over F, under the same operations of addition and scalar multiplication as defined in V.

examples

(i) The set $\{0\}$ consisting of zero vector of V, is a subspace of V.

(ii) The whole vector space V, itself is a subspace of V.

These two subspaces are called trivial or improper subspaces of V.

Any subspace W of V different from $\{0\}$ and V is called a proper subspace of V.

Theorem 1. A non empty subset W of a vector space V over a field F is a subspace of V, if and only if (i) $\forall\ \alpha, \beta \in W$, $\alpha+\beta \in W$ (ii) $\forall\, c \in F, \alpha \in W$, $c.\alpha \in W$.

Proof: Suppose W is a subspace of V.

The W is a vector space over F under the same operation of addition and scalar multiplication as defined in V. Hence the conditions (i) & (ii) hold good.

Conversely, suppose W satisfies (i) & (ii).

We shall show that W is a subspace of V.

(i) $\Rightarrow$ '+' is a binary operation on W.

as '+' is asscociative in V, so is in W.

As $W \neq \phi$, $\exists\ \alpha \in W$.

By (ii), $\forall\ c \in F$ & $\alpha \in W \Rightarrow c.\alpha \in W$

In particular $0 \in F$, $\alpha \in W \Rightarrow 0.\alpha = \mathbf{0} \in W$, which acts as the identity element w.r.t addition.

Again, $-1 \in F$, $\alpha \in W \Rightarrow -1.\alpha = -\alpha \in W$, which acts as the additive inverse of α.

As '+' is commutative in V, so is in W.

Thus $(W,+)$ is an abelian group.

The other axioms (2, 3, 4) of the vector space hold in W, as they hold in the whole space V.

Hence W is a vector space over F and therefore a subspace of V.

* Verify whether $W = \{ f(x) \mid 2f(0) = f(1) \}$ over $0 \leq x \leq 1$, is a subspace of $V = \{\text{all functions}\}$ over the field $\mathbb{R}$.

Sol^n Let $f_1, f_2 \in W$. To show $f_1 + f_2 \in W$

Thus $2f_1(0) = f_1(1)$ & $2f_2(0) = f_2(1)$.

Consider,
$$2(f_1+f_2)(0) = 2[f_1(0) + f_2(0)]$$
$$= 2f_1(0) + 2f_2(0)$$
$$= f_1(1) + f_2(1)$$
$$= (f_1+f_2)(1)$$

Thus, $f_1 + f_2 \in W$. i.e, W is closed under vector addition

To show $cf_1 \in W$

Consider,
$$2(cf_1)(0) = (2c).f_1(0)$$
$$= c.2f_1(0)$$
$$= c.f_1(1)$$
$$= (cf_1)(1)$$

Thus $cf_1 \in W$. i.e, W is closed under scalar multiplication.

Hence W is a subspace.

* Is the subset $W = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$ of $V_3(\mathbb{R})$ is a subspace of $V_3(\mathbb{R})$?

H.W

* Verify whether $W = \{(x, y, z) \mid \sqrt{2}x = \sqrt{3}y\}$ is a subspace of $\mathbb{R}^3$.

* Verify whether $W = \{(a+2b, 0, 2a-b, b) \mid a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^4$.

* Show that the subset

$$W = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$$

of the vector space $V_3(R)$ is a subspace of $V_3(R)$.

Sol:

Let $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3)$ be any two elements of W.

$\therefore$ $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$.

Consider,

$$c_1 \cdot \alpha + c_2 \cdot \beta = c_1(x_1, x_2, x_3) + c_2(y_1, y_2, y_3)$$

$$= (c_1 x_1, c_1 x_2, c_1 x_3) + (c_2 y_1, c_2 y_2, c_2 y_3)$$

$$= (c_1 x_1 + c_2 y_1, c_1 x_2 + c_2 y_2, c_1 x_3 + c_2 y_3)$$

To show that $c_1\alpha + c_2\beta \in W$, we have to show that the sum of the components of $c_1\alpha + c_2\beta$ is zero.

$\therefore$ consider $c_1 x_1 + c_2 y_1 + c_1 x_2 + c_2 y_2 + c_1 x_3 + c_2 y_3$

$$= c_1(x_1 + x_2 + x_3) + c_2(y_1 + y_2 + y_3)$$

$$= c_1 \cdot 0 + c_2 \cdot 0$$

$$= 0$$

$\therefore$ $c_1\alpha + c_2\beta \in W$, hence W is a subspace of $V_3(R)$.

* Verify whether $W = \{$polynomials of degree three$\}$ defined on $0 \le x \le 1$ is a subspace of the vector space $V = \{$all polynomials$\}$ over $\mathbb{R}$. ~~is a subspace~~

Sol^n^:

The set of all polynomials of degree three is not a subspace, as the sum of two polynomials of degree three need not to be of degree three.

$$\because \quad f_1(x) = 3x^3 - 4x^2 + 2x + 1, \quad f_2(x) = -3x^3 + 3x^2 + 2x + 5$$

$$\Rightarrow \quad f_1(x) + f_2(x) = -x^2 + 4x + 6$$

which is not a polynomial of degree three.

Thus the set is not closed under vector addition.

* Verify whether $W = \{$polynomials of degree less than five$\}$ defined on $0 \le x \le 1$ is a subspace of the vector space $V = \{$all polynomials$\}$ over $\mathbb{R}$. ~~is a subspace~~

Corollary:

A non empty subset W is a subspace of a vector space V over F, if and only if

$$c_1\alpha + c_2\beta \in W, \ \forall\ \alpha,\beta \in W,\ c_1, c_2 \in F.$$

Proof:

Let W be a subspace of V.

Let $c_1, c_2 \in F$ and $\alpha, \beta \in W$.

By Theorem 1, $c_1\alpha,\ c_2\beta \in W$ and hence $c_1\alpha + c_2\beta \in W$.

Conversely, let $c_1\alpha + c_2\beta \in W$, $\forall\ \alpha, \beta \in W$, $c_1, c_2 \in W$.

let $c_1 = 1$, $c_2 = 1$, then $1\cdot\alpha + 1\cdot\beta = \alpha + \beta \in W$, $\forall\ \alpha, \beta \in W$,

$\therefore$ W is closed under vector addition.

Now take $\beta = 0$, then $c_1\cdot\alpha + c_2\cdot 0 = c_1\alpha \in W$, $\forall\ \alpha \in W$ and $c_1 \in F$.

$\therefore$ W is closed under scalar multiplication.

Hence W is a subspace of V.

* Let $V = \mathbb{R}^3$, the vectorspace of all ordered triplets of real numbers, over the field of real numbers. Show that the subset $W = \{(x,0,0) \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Sol^n: The element $\mathbf{0} = (0,0,0) \in W$.

Thus W is non empty.

Let $\alpha_1 = (x_1, 0, 0)$ & $\alpha_2 = (x_2, 0, 0)$ be any two elements of W.

Then $\alpha_1 + \alpha_2 = (x_1, 0, 0) + (x_2, 0, 0) = (x_1 + x_2, 0, 0) \in W$.

$\therefore$ W is closed under addition.

Again, for any scalar $c \in \mathbb{R}$,

$$c \cdot \alpha_1 = c(x_1, 0, 0) = (cx_1, 0, 0) \in W$$

$\therefore$ W is closed under scalar multiplication.

Hence W is a subspace of $\mathbb{R}^3$.

* Similarly $W = \{(0, x, 0) \mid x \in \mathbb{R}\}$

$W = \{(0,0,x) \mid x \in \mathbb{R}\}$

$W = \{(x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\}$

$W = \{(0, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$

$W = \{(x_1, 0, x_3) \mid x_1, x_3 \in \mathbb{R}\}$

are subspaces of $\mathbb{R}^3$.

Linear Combination

Let V be a vector space over the field F and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any n vectors of V. The vector of the form,

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$$

where $c_1, c_2, \ldots, c_n \in F$, is called a linear combination of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Consider the vectors $\alpha_1 = (-1, 3, -1)$, $\alpha_2 = (-1, 2, 3)$ and $\alpha_3 = (1, 0, 1)$ of the vector space $\mathbb{R}^3$.

Then the vector $\alpha = 2\alpha_1 - 3\alpha_2 - \alpha_3$

$$= 2(-1, 3, -1) - 3(-1, 2, 3) - (1, 0, 1)$$

$$= (-2, 6, -2) - (-3, 6, 9) - (1, 0, 1)$$

$$\alpha = (0, 0, -12)$$

is a linear combination of the vectors α_1, α_2 and α_3.

By choosing different set of scalars, different linear combinations of $\alpha_1, \alpha_2, \alpha_3$ can be formed.

Linear span of S

Let S be a non empty subset of a vectorspace $V(F)$.
The set of all linear combinations of finite number of elements of S is called the linear span of S and is denoted by $L[S]$.

i.e, $L[S] = \{c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n \mid c_i \in F,\ \alpha_i \in S,\ i = 1, 2, \dots n$ & n is any positive integer$\}$

If $\alpha \in L[S]$, then α is of the form,

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$, for some scalars, $c_1, c_2 \dots c_n \in F$.

Theorem:-
Let S be a nonempty subset of a vectorspace $V[F]$.
Then (i) $L[S]$ is a subspace of V

(ii) $S \subseteq L[S]$

(iii) $L[S]$ is the smallest subspace of V containing S.

* Show that the vector $(2,-5,3) \in V_3(R)$ is not in $L[S]$, where $S = \{(1,-3,2), (2,-4,-1), (1,-5,7)\}$ (11)

Sol^n: If $(2,-5,3) \in L[S]$, then

$$(2,-5,3) = c_1(1,-3,2) + c_2(2,-4,-1) + c_3(1,-5,7)$$
$$= (c_1 + 2c_2 + c_3,\ -3c_1 - 4c_2 - 5c_3,\ 2c_1 - c_2 + 7c_3)$$

$$c_1 + 2c_2 + c_3 = 2$$
$$-3c_1 - 4c_2 - 5c_3 = -5$$
$$2c_1 - c_2 + 7c_3 = 3$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{array}\right]$$

$$\begin{array}{l} R_2+3R_1 \\ R_3-2R_1 \end{array} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{array}\right]$$

$$2R_3+5R_2 \left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

The system is inconsistent. Hence has no solution.

$\therefore$ $(2,-5,3)$ cannot be expressed as linear combination of the elements of S and therefore $(2,-5,3) \notin L[S]$.

* Determine whether the polynomial $3x^2+x+5$ is the linear span of the set $S = \{x^3, x^2+2x, x^2+2, 1-x\}$ of the vector space of all polynomials over the field R.

Sol^n $3x^2+x+5 = c_1 x^3 + c_2(x^2+2x) + c_3(x^2+2) + c_4(1-x)$
$$= c_1x^3 + (c_2+c_3)x^2 + (2c_2 - c_4)x + (2c_3 + c_4)$$

$\Rightarrow \boxed{c_1 = 0}$, $c_2 + c_3 = 3$, $2c_2 - c_4 = 1$, $2c_3 + c_4 = 5$

$\Rightarrow \boxed{c_1 = 0}$, $\boxed{c_2 = 3}$, $\boxed{c_3 = 0}$, $\boxed{c_4 = 5}$

$\therefore$ $3x^2+x+5 = 0x^3 + 3(x^2+2x) + 0(x^2+2) + 5(1-x)$

$\therefore$ $3x^2+x+5 \in L[S]$.

* Find the subspace spanned by the set $S = \{(2,0,0), (0,0,-2)\}$ in the vector space $V_3(R)$.

<u>Sol</u>: The subspace spanned by S is $L[S]$.

Any element $\alpha \in L[S]$ is of the form

$$\alpha = c_1(2,0,0) + c_2(0,0,-2)$$

$$\alpha = (2c_1, 0, -2c_2)$$

$$\therefore L[S] = \{\alpha \,/\, \alpha = (2c_1, 0, -2c_2);\ c_1, c_2 \in R\}$$

* In $V_3(R)$ show that the plane $x_3 = 0$ may be spanned by the pair of vectors $(2,2,0)$ & $(4,1,0)$.

Soln:

Linear Dependence and Independence

A set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of vectors of a vector space $V[F]$ is said to be linearly dependent if there exists scalars $c_1, c_2, \ldots, c_n \in F$, not all zero such that $c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0$.

A set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of vectors of a vector space $V[F]$ is said to be linearly independent if ~~it is not be~~

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

* Show that the vectors $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $e_n = (0, 0, 0, \ldots, 0)$ of the vector space $V_n(\mathbb{R})$ are linearly independent.

Sol^n: let $c_1, c_2, \ldots, c_n \in \mathbb{R}$

Consider $c_1 e_1 + c_2 e_2 + \cdots + c_n e_n = 0$.

$\Rightarrow c_1(1, 0, 0, \ldots, 0) + c_2(0, 1, 0, \ldots, 0) + \cdots + c_n(0, 0, 0, \ldots, 1) = (0, 0 \ldots 0)$

$\Rightarrow (c_1, c_2, \ldots, c_n) = (0, 0, 0, \ldots, 0)$

$\Rightarrow c_1 = 0, \quad c_2 = 0, \ldots, c_n = 0$

i.e. $e_1, e_2, e_3, \ldots, e_n$ are linearly independent.

* Show that the set $S = \{(1,0,1), (1,1,0), (-1,0,-1)\}$ is linearly dependent in $V_3(\mathbb{R})$.

Solⁿ Consider $c_1(1,0,1) + c_2(1,1,0) + c_3(-1,0,-1) = (0,0,0)$

$\Rightarrow (c_1 + c_2 - c_3,\ c_2,\ c_1 - c_3) = (0,0,0)$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 - c_3 &= 0 \\ c_2 &= 0 \\ c_1 - c_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= c_3 \\ c_2 &= 0 \end{aligned}$$

let $c_3 = 1$ then $c_1 = 1$

Thus there exists, not all zero, scalars, such that

$c_1(1,0,1) + c_2(1,1,0) + c_3(-1,0,-1) = (0,0,0)$

$\therefore$ S is linearly dependent.

* The set $\{(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)\}$ of vectors of the vector space $V_3(\mathbb{R})$ is linearly dependent iff

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$$

* Two vectors $\alpha, \beta \in V_2(\mathbb{R})$ are linearly dependent iff $\alpha = k\beta$ for some non zero $k \in \mathbb{R}$.

* A set of vectors of V, containing the zero vector is linearly dependent.

* The set consisting of a single vector α of V is linearly independent iff $\alpha \neq 0$.

Basis:-

A subset B of a vector space V[F] is called a basis of V if

(i) B is a linearly independent set

(ii) $L[B] = V$.

That is a basis of a vector space V[F] is a linearly independent subset which spans the whole space.

* Finite dimensional space :

A vector space V[F] is said to be a finite dimensional space if it has a finite basis.

*Note:- The zero vector 0 can not be an element of a basis of a vector space because a set of vectors with zero vector is always linearly dependent.

example:- Show that the vectors $e_1 = (1,0,0,\dots,0)$, $e_2 = (0,1,0,\dots,0)$, $e_n = (0,0,0,\dots,1)$ of the vector space $V_n(\mathbb{R})$ form a basis of $V_n(\mathbb{R})$.

Sol^n:- ~~The~~ Consider, $S = \{e_1, e_2, \dots, e_n\}$

$$c_1 e_1 + c_2 e_2 + \dots + c_n e_n = \mathbf{0}$$

$$\Rightarrow c_1(1,0,\dots,0) + c_2(0,1,\dots,0) + \dots + c_n(0,0,\dots 1) = (0,0,\dots 0)$$

$$\Rightarrow (c_1, c_2, \dots c_n) = (0,0,\dots 0)$$

$$\Rightarrow c_1 = 0, \quad c_2 = 0, \quad c_3 = 0 \quad \dots, c_n = 0.$$

$\therefore$ S is linearly independent.

Further, any vector $(x_1, x_2, \dots, x_n) \in V_n(\mathbb{R})$ can be expressed as a linear combination of the elements of S, as

$$(x_1, x_2, \dots, x_n) = x_1 e_1 + x_2 e_2 + \dots + x_n e_n.$$

Hence $L[S] = V_n(\mathbb{R})$.

$\therefore$ S is a basis of $V_n(\mathbb{R})$.

* Standard basis

The basis $S = \{e_1, e_2, \dots, e_n\}$ of the vector space $V_n(\mathbb{R})$ is called the standard basis.

eg The vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ of $V_3(\mathbb{R})$ form a basis of $V_3(\mathbb{R})$, and is called the standard basis.

Example: Show that the set $B = \{(1,1,0), (1,0,1), (0,1,1)\}$ is a basis of the vectorspace $V_3(\mathbb{R})$.

Sol^n: Let $B = \{(1,1,0), (1,0,1), (0,1,1)\}$

Consider, $c_1(1,1,0) + c_2(1,0,1) + c_3(0,1,1) = (0,0,0)$

$$\Rightarrow (c_1 + c_2, c_1 + c_3, c_2 + c_3) = (0,0,0)$$

$$\Rightarrow c_1 + c_2 = 0, \quad c_1 + c_3 = 0, \quad c_2 + c_3 = 0$$

$$\Rightarrow c_1 = 0, \quad c_2 = 0, \quad c_3 = 0.$$

$\therefore$ B is linearly independent.

Let $(x_1, x_2, x_3) \in V_3(\mathbb{R})$ be arbitrary

let $c_1, c_2, c_3 \in \mathbb{R}$, such that

$$(x_1, x_2, x_3) = c_1(1,1,0) + c_2(1,0,1) + c_3(0,1,1)$$

$$(x_1, x_2, x_3) = (c_1 + c_2, c_1 + c_3, c_2 + c_3)$$

$$\Rightarrow \left.\begin{aligned} x_1 &= c_1 + c_2 \\ x_2 &= c_1 + c_3 \\ x_3 &= c_2 + c_3 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 - x_2 &= c_2 - c_3 \\ x_3 &= c_2 + c_3 \end{aligned}$$

$$\Rightarrow \boxed{\frac{x_1 - x_2 + x_3}{2} = c_2}$$

$$\Rightarrow x_1 = c_1 + \frac{x_1 - x_2 + x_3}{2} \Rightarrow \boxed{c_1 = \frac{x_1 + x_2 - x_3}{2}}$$

$$\Rightarrow x_2 = \frac{x_1 + x_2 - x_3}{2} + c_3 \Rightarrow \boxed{c_3 = \frac{-x_1 + x_2 + x_3}{2}}$$

$$\therefore (x_1, x_2, x_3) = \frac{x_1 + x_2 - x_3}{2}(1,1,0) + \frac{x_1 - x_2 + x_3}{2}(1,0,1) + \frac{-x_1 + x_2 + x_3}{2}(0,1,1)$$

$\therefore L[B] = V_3(\mathbb{R})$

Theorem :-
Any two bases of a finite dimensional vector space V have the same finite number of elements.

Dimension of a vector space V
The dimension of a finite dimensional vector space V over F is the number of elements in any basis of V and is denoted by d[V].
eg $V_n(R)$ is a n dimensional space.
$V_3(R)$ is a three dimensional space.

~~Theo~~
* A vector space which is not finitely generated may be called an infinite dimensional space.

Theorem :-
In an n dimensional vector space V(F)
(i) any n+1 elements of V are linearly dependent.
(ii) no set of n-1 elements can span V.

Theorem :-
In a n dimensional vector space V(F) any set of n linearly independent vector is a basis.

Theorem :-
Any linearly independent set of elements of a finite dimensional vector space V is a part of a basis

Theorem :-
For n vectors of n-dimensional vector space V, to be a basis, it is sufficient that they span V or that they are L.I.

Example :-
Let $A = \{(1,-2,5), (2,3,1)\}$ be a linearly independent subset of $V_3(\mathbb{R})$. Extend this to a basis of $V_3(\mathbb{R})$

Sol^n let $\alpha_1 = (1,-2,5)$, $\alpha_2 = (2,3,1)$

Let S be the subspace spanned by $\{\alpha_1, \alpha_2\}$

$\therefore S = \{ c_1\alpha_1 + c_2\alpha_2 \mid c_1, c_2 \in \mathbb{R}\}$

$$c_1\alpha_1 + c_2\alpha_2 = c_1(1,-2,5) + c_2(2,3,1)$$
$$= (c_1 + 2c_2, -2c_1 + 3c_2, 5c_1 + c_2)$$

$\therefore S = \{(c_1 + 2c_2, -2c_1 + 3c_2, 5c_1 + c_2) \mid c_1, c_2 \in \mathbb{R}\}$

Choose a vector of $V_3(\mathbb{R})$ outside of S.

$(1,0,0) \notin S$

$\therefore$ the set $A = \{(1,-2,5), (2,3,1), (1,0,0)\}$ is a basis of $V_3(\mathbb{R})$.

Example :-
Given two linearly independent vectors $(2,1,4,3)$ & $(2,1,2,0)$, find a basis of $V_4(\mathbb{R})$ that includes these two vectors.

Sol^n $\alpha_1 = (2,1,4,3)$, $\alpha_2 = (2,1,2,0)$

$S = \{c_1\alpha_1 + c_2\alpha_2 \mid c_1, c_2 \in \mathbb{R}\}$

Choose $\alpha_3 = (1,0,0,0)$ & $\alpha_4 = (0,1,0,0) \notin S$

$\therefore \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis of $V_4(\mathbb{R})$.

* The non-zero rows of a row-reduced echelon form of a matrix are linearly independent.
* Let A be a matrix of the given vectors. E be the row reduced echelon matrix of A.

* The

Example:

Test the following set of vectors for linear dependente in $V_3(\mathbb{R})$. $\{(1,0,1), (0,2,2), (3,7,1)\}$.
Do they form a basis?

Sol^n:- Consider the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 7 & 1 \end{pmatrix}$.

$$|A| = 1(2-14) - 0(0-6) + 1(0-6) = -18 \neq 0.$$

$\therefore$ The given set is linearly independent.
Any three vectors which are linearly independent, is a basis of $V_3(\mathbb{R})$.

Example:

Does the set $S = \{(1,2,3), (3,1,0), (-2,1,3)\}$ form a basis of $\mathbb{R}^3$.

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 1 & 3 \end{bmatrix}$,

$$|A| = 1(3-0) - 2(9+0) + 3(3+2) \Rightarrow 0$$

$\therefore S$ is linearly dependent and hence ~~doesnot~~ is not a basis of $\mathbb{R}^3$.

Example :-

Show that the vectors $(1,1,2,4)$, $(2,-1,-5,2)$, $(1,-1,-4,0)$ and $(2,1,1,6)$ are linearly dependent in $\mathbb{R}^4$ and extract a linearly independent subset. Also find the dimension and a basis of the subspace spanned by them.

Sol^n:

Consider $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{pmatrix}$

$R_2 - 2R_1$; $R_3 - R_1$; $R_4 - 2R_1$

$A \approx \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$

$R_2 \div -3$ $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$

$R_3 + 2R_2$
$R_4 + R_2$ $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The final matrix is in echelon form, and the rank of A is 2. $\therefore$ The given vectors are linearly dependent.

The corresponding non-zero rows of the initial matrix are $(1,1,2,4)$ & $(2,-1,-5,2)$ which are L.I.

The dimension of the subspace spanned by these vectors is $\underline{2}$. These two vectors form a basis of the subspace.

* Let S be the subspace of $\mathbb{R}^3$ defined by $S = \{(a,b,c) \mid a+b+c=0\}$. Find a basis and dimension of S.

Sol^n:-

$S \neq \mathbb{R}^3$ $[\because (1,2,3) \in \mathbb{R}^3$ but $(1,2,3) \notin S]$
(as $1+2+3 \neq 0$)

$\alpha = (1,0,-1)$ & $\beta = (1,-1,0) \in S$,

and further they are independent.

$\therefore$ $d[S] = 2$ & hence $\{\alpha, \beta\}$ form a basis of S.

* Show that the field $\mathbb{C}$ of complex numbers is a vector space over the field $\mathbb{R}$ of reals. What is its dimension?

Sol^n ~~for $c_1, c_2 \in$~~ $\mathbb{C} = \{a+ib \mid a,b \in \mathbb{R}\}$.

$\mathbb{C}$ is closed under '+'.
$\mathbb{C}$ is associative under '+'
$0+0i$ is the identity w.r.t '+'.
$-a-ib$ is the inverse of $a+ib$.
$\mathbb{C}$ is commutative

} $(\mathbb{C}, +)$ is an abelian gp'.

$c(a_1+ib_1 + a_2+ib_2) \in \mathbb{C}$
$(c_1+c_2)(a_1+ib_1) \in \mathbb{C}$
$c.(a_1+ib_1) \in \mathbb{C}$
'1' is the unity

$\therefore$ $\mathbb{C}$ is a vector space, over $\mathbb{R}$.

Let $\alpha \in \mathbb{C}$, $\alpha = a+ib$, $\ni a,b \in \mathbb{R}$.

$\therefore$ $\alpha = 1.a + i.b = a.1 + b.i$

i.e, every element of $\mathbb{C}$ is a linear combination of the element $1, i$. That is $\{1,i\}$ generates $\mathbb{C}$.

Further $c_1.1 + c_2.i = 0 \Rightarrow c_1 = 0$ & $c_2 = 0$. $\therefore \{1,i\}$ is L.I.

$\therefore$ $\{1,i\}$ is a basis of $\mathbb{C}$. & $d[\mathbb{C}] = 2$.

* Let V be the vector space of 2×2 symmetric matrices over the field F. Show that $d[V]=3$.

Sol^n Let $A=\begin{bmatrix} a & b \\ b & c \end{bmatrix} \in V$, $a,b,c \in F$.

set $a=1,\ b=0,\ c=0$; $a=0,\ b=1,\ c=0$; $a=0,\ b=0,\ c=1$

We get three matrices:

$$E_1=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad E_2=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_3=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We shall show that these elements of V form a basis. Let $A=\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in V$ be arbitrary.

Then,
$$A=\begin{pmatrix} a & b \\ b & c \end{pmatrix}=a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}+b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus $\{E_1, E_2, E_3\}$ generates V.

Suppose $c_1E_1+c_2E_2+c_3E_3=0$, $c_1, c_2, c_3 \in F$

$$\Rightarrow c_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}+c_2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+c_3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow c_1=c_2=c_3=0$$

$\therefore \{E_1, E_2, E_3\}$ is linearly independent.

Hence $\{E_1, E_2, E_3\}$ is a basis of V.

and $d[V]=3$.

* Show that the set

$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ form a basis of the vector space V of all 2×2 matrices over $\mathbb{R}$.

* Find the basis and dimension of the subspace spanned by the vectors $(1,2,0), (1,1,1), (2,0,1)$ of the vector space $V_3(Z_3)$.

* Find the basis and dimension of the subspace spanned by the subset.

$S = \left\{ \begin{pmatrix} 1 & -5 \\ -4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 5 \end{pmatrix}, \begin{pmatrix} 2 & -4 \\ -5 & 7 \end{pmatrix} \begin{pmatrix} 1 & -7 \\ -5 & 1 \end{pmatrix} \right\}$ of the vector space of all 2×2 matrices over $\mathbb{R}$.

Soln: Let $\alpha, \beta, \gamma, \delta$ are the matrices of S.

Then the coordinates of α, β, γ & δ w.r.t standard basis are $(1,-5,-4,2)$, $(1,1,-1,5)$, $(2,-4,-5,7)$, $(1,-7,-5,1)$.

Consider $\begin{pmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{pmatrix}$

$\begin{array}{l} R_2-R_1 \\ R_3-2R_1 \\ R_4-R_1 \end{array} \begin{pmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{pmatrix}$

$\begin{array}{l} R_3-R_2 \\ 3R_4+R_2 \end{array} \begin{pmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

The final matrix has two nonzero rows.

$\therefore$ d(subspace) $= 2$.

Further the matrices corresponding to the non-zero rows, in the final matrix are $\begin{pmatrix} 1 & -5 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 0 & 6 \\ 3 & 3 \end{pmatrix}$

* In a vector space $V_3(\mathbb{R})$, let $\alpha = (1,2,1)$, $\beta = (3,1,5)$ & $\gamma = (-1,3,-3)$. Show that the subspace spanned by $\{\alpha, \beta\}$ & $\{\alpha, \beta, \gamma\}$ are the same.

<u>Sol^n</u> Consider, $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \\ -1 & 3 & -3 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \\ -1 & 3 & -3 \end{vmatrix} = 1(-3-15) - 2(-9+5) + 1(9+1)$$
$$= -18 + 8 + 10 = 0$$

$\therefore \{\alpha, \beta, \gamma\}$ is L.D.

let $\gamma = c_1\alpha + c_2\beta$

$(-1,3,-3) = c_1(1,2,1) + c_2(3,1,5)$

$\Rightarrow (-1,3,-3) = (c_1 + 3c_2, 2c_1 + c_2, c_1 + 5c_2)$

$\Rightarrow c_1 + 3c_2 = -1$, $2c_1 + c_2 = 3$, $c_1 + 5c_2 = -3$.

Solving these equ^s, we get $c_1 = 2$, $c_2 = -1$.

$\therefore \gamma \in$ subspace spanned by $\{\alpha, \beta\}$.

$\therefore$ the subspace spanned by $\{\alpha, \beta\}$ & $\{\alpha, \beta, \gamma\}$ are same.

Null space

The null space of a $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $Ax = 0$.

Theorem:-

The null space of a $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $Ax = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

example:

Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and $u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$.

Determine if u belongs to the null space of A.

soln: $Au \cancel{\neq 0}$

$$\Rightarrow \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5-9+4 \\ -25+27-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ u is in Nul A.

example

Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Sol^n: Consider $Ax = 0$

$\Rightarrow$ Reducing A to echelon form

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

$$\begin{array}{l} 3R_2+R_1 \\ 3R_3+2R_1 \end{array} \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 13 & 26 & -26 \end{bmatrix}$$

$$\begin{array}{l} R_2 \div 5 \\ R_3 \div 13 \end{array} \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$$R_3 - R_2 \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\Rightarrow \begin{cases} -3x_1 + 6x_2 - x_3 + x_4 - 7x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 2x_2 - \frac{1}{3}x_3 + \frac{1}{3}x_4 - \frac{7}{3}x_5 \\ \boxed{x_3 = -2x_4 + 2x_5} \end{cases}$$

$$\Rightarrow x_1 = 2x_2 - \frac{1}{3}(-2x_4 + 2x_5) + \frac{1}{3}x_4 - \frac{7}{3}x_5$$

$$\Rightarrow \boxed{x_1 = 2x_2 + x_4 - 3x_5}$$

with x_2, x_4, x_5 as free variables.

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2 u + x_4 v + x_5 w$$

Every linear combination of u, v and w is an element of Nul A. Thus $\{u, v, w\}$ is a spanning set for Nul A.

Column space:-

The Column space of a $m \times n$ matrix A, written as $\text{Col } A$, is the set of all linear combinations of the columns of A.
If $A = [a_1, \ldots, a_n]$, then $\text{Col } A = \text{span}\{a_1, \ldots, a_n\}$

Theorem:-
The column space of a $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

example:-
Find a matrix A such that $W = \text{Col } A$.

$$W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

sol^n:- W can be written as

$$W = \left\{ a\begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Using the vectors in the spanning set as the columns of A, w get $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$.

Then $W = \text{Col } A$ as desired.

Note:- The column space of a $m \times n$ matrix A is all of $\mathbb{R}^m$ iff the equation $Ax = b$ has a solution for each b in $\mathbb{R}^m$.

example:

let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}_{3\times 4}$

(a) If the column space of A is a subspace of $\mathbb{R}^k$, what is k?

(b) If the null space of A is a subspace of $\mathbb{R}^k$, what is k?

sol^n: (a) $m = 3$, Col A is a subspace of $\mathbb{R}^m$, where $m=3$.

(b) $n = 4$, Nul A is a subspace of $\mathbb{R}^n$, where $n=4$.

example:-

let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, find a nonzero vector in Col A and a nonzero vector in Nul A.

sol^n: any column of $A \in$ Col A, eg $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \in$ Col A

Consider $Ax = 0$.

$$A = \left[\begin{array}{cccc:c} 2 & 4 & -2 & 1 & 0 \\ -2 & -5 & 7 & 3 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{array}\right]$$

$$\begin{array}{l} R_2+R_1 \\ 2R_3-3R_1 \end{array} \left[\begin{array}{cccc:c} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 2 & -10 & 9 & 0 \end{array}\right]$$

$$R_3+2R_2 \left[\begin{array}{cccc:c} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 17 & 0 \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{r} 2x_1 + 4x_2 - 2x_3 + x_4 = 0 \\ -x_2 + 5x_3 + 4x_4 = 0 \\ 17x_4 = 0 \end{array}\right\}$$

$x_4 = 0$

$\Rightarrow \cancel{2}x_1 = -2x_2 + x_3$

$\boxed{x_2 = 5x_3}$

$\Rightarrow \boxed{x_1 = -9x_3}$

x_3 is a free variable.

let $x_3 = 1$, then $x_1 = -9$, $x_2 = 5$, $x_4 = 0$

the vector $x = (-9, 5, 1, 0) \in$ Nul A.

$x_1 = \cancel{-2x_2 + 7x_3}$ $-9x_3$

$x_2 = 5x_3$

$x_3 = x_3$

$x_4 = 0$

$x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$

* Determine if $w = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in Nul A, where

$$A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}.$$

* Determine if $w = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ is in Nul A, where

$$A = \begin{bmatrix} -5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}.$$

* Find an explicit description of Nul A, by listing vectors that span the null space.

  * $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$

  * $A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

* Find A such that the given set is Col A.

  * $\left\{ \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} : r, s, t \text{ real} \right\}$ $\quad A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$

  * $\left\{ \begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} : b, c, d \text{ real} \right\}$

* Find (a) k such that Nul A is a subspace of $\mathbb{R}^k$

(b) k such that Col A is a subspace of $\mathbb{R}^k$.

$Ax = 0$

* $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ +3 & -9 \end{bmatrix}$ 4×2

* $A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 0 & -5 \\ 0 & -5 & 7 \\ -5 & 7 & -2 \end{bmatrix}$

* $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$

* $A = \begin{bmatrix} 1 & -3 & 9 & 0 & -5 \end{bmatrix}$

* With $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ +3 & -9 \end{bmatrix}$, find a nonzero vector in Nul A and a nonzero vector in Col A.

* With $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$, find a nonzero vector in Nul A and a nonzero vector in Col A

* Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Determine if w is in Col A. Is w in Nul A?

$Ax = 0$

* Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Determine if w is in Col A. Is w in Nul A?

example:-

With $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}_{3\times 4}$ $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$ (22)

(a) Determine if u is in Nul A. $\mathbb{R}^4$
Could u be in Col A? $\mathbb{R}^3$

(b) Determine if v is in Col A.
Could v be in Nul A? $\mathbb{R}^4$

Sol^n:

(a) Consider $Au = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ $u \notin$ Nul A.

$\because$ u has 4 entries, and Col A is subspace of $\mathbb{R}^3$,
$u \notin$ Col A.

(b) Consider $[A : v]$

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix}$$

$$\begin{matrix} \\ R_2+R_1 \\ 2R_3-3R_1 \end{matrix} \begin{bmatrix} 2 & 4 & -2 & 1 & : & 3 \\ 0 & -1 & 5 & 4 & : & 2 \\ 0 & 2 & -10 & 9 & : & -6 \end{bmatrix}$$

$$R_3+2R_2 \begin{bmatrix} 2 & 4 & -2 & 1 & : & 3 \\ 0 & -1 & 5 & 4 & : & 2 \\ 0 & 0 & 0 & 17 & : & 0 \end{bmatrix}$$

$\Rightarrow$ $Ax = v$ is consistent.

$\therefore$ v is in Col A.

$\because$ v has 3 entries and Nul A is a subspace of $\mathbb{R}^4$,
$v \notin$ Nul A.

* Find the bases for the null spaces of the matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix}$$

* Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x+2y+z=0$. [$Ax=0$] $A = [1 \;\; 2 \;\; 1]_{1\times 3}$

* Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = 5x$. [$Ax=0$] $A = [5 \;\; -1]_{1\times 2}$

* Find the bases for Nul A and Col A. Assume that A is row equivalent to B.

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}. \qquad B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A^{ech}

$(1, 0, 0)$ $\quad$ $(1, 0, 0)$

$(0, 1, 0)$ $\quad$ $(3, 0, 0)$

$(0, 0, 1)$

Row Space

If A is a $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$. The set of all linear combinations of the row vectors is called the row space of A and is denoted by Row A. Each row has n entries, so Row A is a subspace of $\mathbb{R}^n$.

Since the rows of A are identified with the columns of A^T, we could also write Col A^T in place of Row A.

example:-

let

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}_{4 \times 5}$$

Sol^n

$$r_1 = (-2, -5, 8, 0, -17)$$

$$r_2 = (1, 3, -5, 1, 5)$$

$$r_3 = (3, 11, -19, 7, 1)$$

$$r_4 = (1, 7, -13, 5, -3)$$

The row space of A is the subspace of $\mathbb{R}^5$ spanned by $\{r_1, r_2, r_3, r_4\}$. That is, Row A = Span $\{r_1, r_2, r_3, r_4\}$

Theorem:

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, then nonzero rows of B form a basis for the row space of A as well as for that of B.

Find the basis for the row space, ~~the~~ ~~the column space~~, ~~and the null space~~ and the ~~left null space of the~~ matrix:

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Soln:

To find the basis for the row space, reduce A to echelon form.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ -2 & -5 & 8 & 0 & -17 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$\begin{array}{l} R_2+2R_1 \\ R_3-3R_1 \\ R_4-R_1 \end{array} \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix}$$

$$\begin{array}{l} R_3-2R_2 \\ R_4-4R_2 \end{array} \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 20 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4 \quad B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three rows of B form a basis for the row space of A ~~(as well as for the rows space of B)~~.

Basis for Row A

$= \{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)\}$

* Find the basis for the ~~matrix~~ row space of

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$$

Sol^n

$$\begin{array}{l} R_2+R_1 \\ R_3-2R_1 \\ R_4+R_1 \end{array} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{bmatrix}$$

$$\begin{array}{l} R_3+R_2 \\ R_4-3R_2 \end{array} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{array}{l} R_3 \leftrightarrow R_4 \\ B \sim \end{array} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first 3 rows of B form the basis for the rowspace of A.

Row A $= \{(2,-3,6,2,5), (0,0,3,-1,1), (0,0,0,1,3)\}$

Left null space

The left null space of a $m \times n$ matrix A, written as $\text{Nul}(A^T)$, is the set of all solutions to the homogeneous equation $A^T y = 0$.

Theorem:-

The left null space of a $m \times n$ matrix A is a subspace of $\mathbb{R}^m$. Equivalently, the set of all solutions to a system $A^T y = 0$ of n homogeneous linear equations in m unknowns is a subspace of $\mathbb{R}^m$.

example:-

let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and $v = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Determine if v belongs to the left null space of A.

sol^n:

$$A^T v = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3+3 \\ -6+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\therefore$ v is in $\text{Nul}\, A^T$.

Find a spanning set for the left null space of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Sol^n:- $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$A^T y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $y_1 = 0$

y_2 is a free variable.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ y_2 \end{bmatrix} = y_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = y_2 u.$$

$\therefore$ Every linear combination of u is an element of Nul A^T. Thus $\{u\}$ is a spanning set of Nul A^T.

Find the dimension and basis for the four fundamental subspaces of the matrix.

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$ 3×4

Sol^n

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & \cancel{3} & \cancel{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$Ax = 0 \Rightarrow$

$x_1 + 3x_2 + 3x_3 + 2x_4 = 0$

$\cancel{3}x_3 + \cancel{3}x_4 = 0$

$\cancel{3x_4 = 0}$

$\Rightarrow \cancel{x_4 = 0}$ $\quad \cancel{6x_3 + 0 = 0}$

$x_3 = -x_4$ $\quad \cancel{x_3 = 0}$

$\cancel{x_1 + 3x_2 = 0}$ $\cancel{x_1 = -3x_2 + 5x_4}$

$\cancel{x_1 = -3x_2}$ $\quad x_1 = -3x_2 + x_4$

x_2, x_4 are a free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore \{(-3, 1, 0, 0), (1, 0, -1, 1)\}$ form a basis of Nul A

and $\dim(\text{Nul} A) = 2$.

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & \cancel{3} & \cancel{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} R_2 \div 3 \\ \cancel{R_3 \div 3} \end{array} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{(1, 2, -1), (3, 9, 3), \cancel{(2, 7, 4)}\}$ form a basis of Col A

and $\dim(\text{Col} A) = 2$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & \cancel{3} & \cancel{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{(1, 3, 3, 2)\ (0, 0, 3, 3), \cancel{(0, 0, 0, 0)}\}$ form a basis of Row A

$\dim(\text{Row } A) = 2$

$\longrightarrow$

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 3 & 9 & 3 \\ 2 & 7 & 4 \end{bmatrix}$$

$$\begin{array}{l} R_2 - 3R_1 \\ R_3 - 3R_1 \\ R_4 - 2R_1 \end{array} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{bmatrix}$$

$$R_4 - R_3 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^T y = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y_1 + 2y_2 - y_3 = 0$

$3y_2 + 6y_3 = 0$

$\Rightarrow y_2 = -2y_3$

$\Rightarrow y_1 = -2y_2 + y_3$

$= 4y_3 + y_3$

$y_1 = 5y_3$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5y_3 \\ -2y_3 \\ y_3 \end{bmatrix}$$

$$= y_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore \{(5, -2, 1)\}$ forms a basis of $Nul(A^T)$.

$\dim(Nul\, A^T) = 1$.

* ~~The rank of A is~~ The dimension of the column space of A, is called the rank of A.
* Since Row A is the same as Col A^T, The dimension of the row space of A is the rank of A^T.
* The dimension of the null space is called the nullity of A.

$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$

The Rank-Nullity Theorem.

For a $m \times n$ matrix A,

$$\text{rank } A + \text{nullity } A = n.$$

$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

example!

* If A is a 7×9 matrix with two dimensional null space, what is the rank of A?

$$\text{rank} + \text{nullity} = 9$$
$$\text{rank} + 2 = 9$$
$$\Rightarrow \text{rank} = 7$$

* ~~If~~ Could a 6×9 matrix have a two-dimensional null space?

$\text{Col } B \subseteq \mathbb{R}^6$

$$\text{rank} + \text{nullity} = 9$$
$$\text{rank} + 2 = 9$$
$$\Rightarrow \text{rank} = 7$$ which contradicts that basis of Col A is a subspace of $\mathbb{R}^6$.

$\therefore$ a 6×9 matrix cannot have a two-dimensional null space.

Linear Transformations

Consider the matrix equation $Ax = b$, where A ~~...~~ is $m \times n$ matrix, x - $n \times 1$ matrix, and b is $m \times 1$ matrix.

In other words x is a vector in $\mathbb{R}^n$ and b is a vector in $\mathbb{R}^m$.

Solving the equation $Ax = b$ amounts to finding all vectors x in $\mathbb{R}^n$ that are transformed into the vector b in $\mathbb{R}^m$ under the action of multiplication by A.

The correspondence from x to Ax is a function from one set of vectors to another. This concept generalizes the common notation of a function as a rule that transforms one real number into another.

A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the domain of T and $\mathbb{R}^m$ is called the codomain of T. The notation $T: \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.

For x in $\mathbb{R}^n$, the vector $T(x)$ in $\mathbb{R}^m$ is called the image of x. The set of all images $T(x)$ is called the range of T.

Matrix Transformations

For each x in $\mathbb{R}^n$, T(x) is computed as Ax, where A is an $m \times n$ matrix. It is also denoted ~~by~~ the matrix transformation $x \mapsto Ax$.

Observe that the domain of T is $\mathbb{R}^n$ when A has n columns and codomain of T is $\mathbb{R}^m$ when each column of A has m entries.

The range of T is the set of all linear combinations of the columns of A, because each image T(x) is of the form Ax.

ex Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and

define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$,

So that $T(x) = Ax = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$.

a. Find T(u), the image of u under the Transformation T.

b. Find an x in $\mathbb{R}^2$ whose image under T is b.

c. Is there more than one ~~vec~~ x whose image under T is b?

d. Determine if c is in the range of the transformation T.

Solⁿ

a. $T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

b. $T(x) = b \Rightarrow Ax = b \Rightarrow \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

$\longrightarrow$

which can be written in matrix form as

$$\begin{bmatrix} 1 & -3 & : & 3 \\ 3 & 5 & : & 2 \\ -1 & 7 & : & -5 \end{bmatrix}$$

Reducing to echelon form as below:

$R_2 = R_2 - 3R_1$, $R_3 = R_3 + R_1$
$$\sim \begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 4 & : & -2 \end{bmatrix}$$

$R_3 = 14R_3 - 4R_2$
$$\sim \begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 0 & : & 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} x_1 - 3x_2 = 3 \\ 14x_2 = -7 \end{array}\right\} \Rightarrow \begin{array}{l} x_2 = -1/2 \\ x_1 = 3/2 \end{array} \quad \text{Hence } x = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

c. From b. we can see that, the vector x is unique.

d. Let $T(x) = c$ i.e, $Ax = c$ i.e, $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} 1 & -3 & : & 3 \\ 3 & 5 & : & 2 \\ -1 & 7 & : & 5 \end{bmatrix}$$

$R_2 = R_2 - 3R_1$, $R_3 = R_3 + R_1$
$$\sim \begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 4 & : & 8 \end{bmatrix}$$

$R_3 = 14R_3 - 4R_2$
$$\sim \begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 0 & : & 140 \end{bmatrix}$$

The third row show that $0 = 140$. (which is invalid)

Hence the system is inconsistent.

Hence c is not in the range of the transformation.

ex Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$.

Find the images under T of $u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$

Sol^n $T(u) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

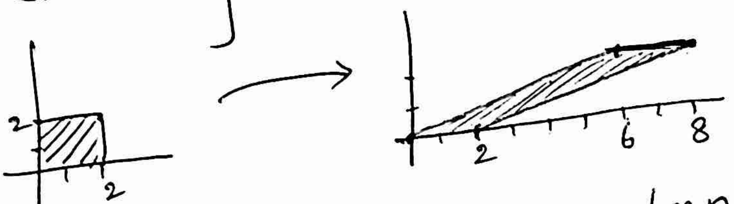
$T(v) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$

ex. let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Then the transformation $T(x) = Ax$, transforms the square with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$ to.

$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

sketching the above transformation, we see

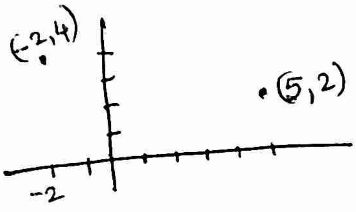


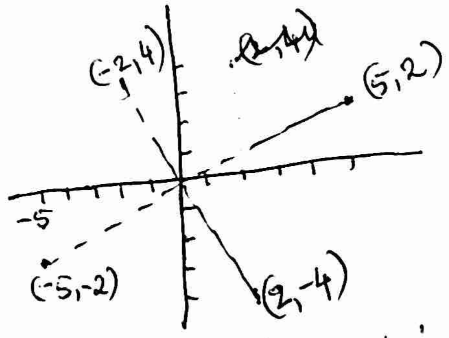
that the square has been transformed to a parallelogram. [In other words, the square has been stretched to a parallelogram, keeping the base intact].

ex. let $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

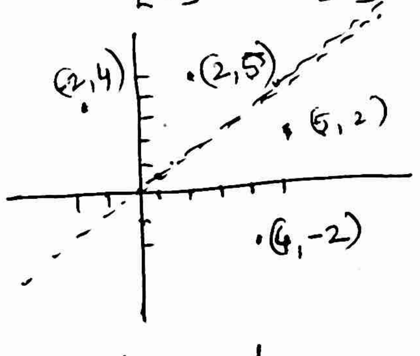
with $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $Au = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$ $Av = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

with $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $Au = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $Av = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$





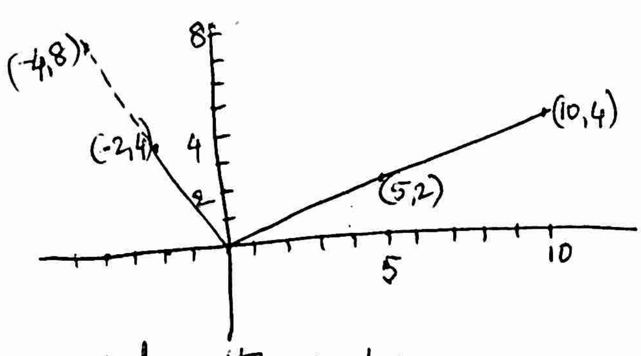
reflected about origin



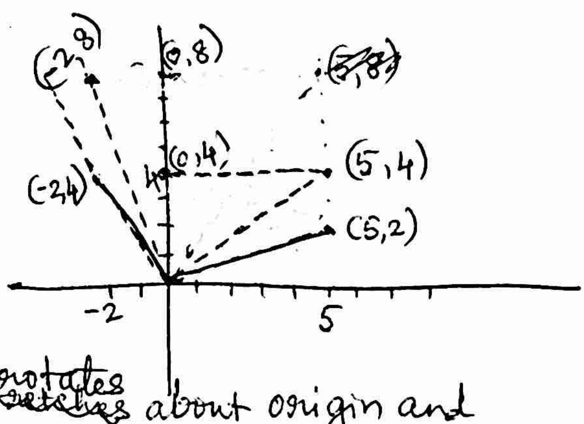
reflects about $y=x$.

with $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $Au = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$, $Av = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$

with $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ $Au = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $Av = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$



stretches the vector



~~Stretches~~ rotates about origin and projects onto ~~x~~ y-axis

If A is an $m\times n$ matrix, then the transformation $x \mapsto Ax$ has the properties $A(u+v) = Au + Av$ and $A(cu) = cAu$ for all u, v in $\mathbb{R}^n$ and all scalars c.

These properties, written in function notation, identify the most important class of transformations in Linear Algebra.

The transformation (or mapping) $T: V \to W$ is linear if:

(i) $T(u+v) = T(u) + T(v)$ for all u, v in the domain of T;

(ii) $T(cu) = cT(u)$ for all u and all scalars c.

Note: * Every matrix transformation is a linear transformation.

* Linear transformations preserve the operations of vector addition and scalar multiplication.

Q. If T is a mapping from $V_3(\mathbb{R})$ into $V_3(\mathbb{R})$ defined by $T(x_1, x_2, x_3) = (0, x_2, x_3)$, show that T is a linear transformation.

Sol: Let $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3)$,

Such that $T(\alpha) = (0, x_2, x_3)$, $T(\beta) = (0, y_2, y_3)$.

Consider (i) $T(\alpha+\beta) = T(x_1+y_1, x_2+y_2, x_3+y_3)$

$$= (0, x_2+y_2, x_3+y_3)$$

$$= (0, x_2, x_3) + (0, y_2, y_3)$$

$$= T(\alpha) + T(\beta)$$

(ii) $T(c\alpha) = T(cx_1, cx_2, cx_3)$

$$= (0, cx_2, cx_3)$$

$$= c(0, x_2, x_3)$$

$$= cT(\alpha)$$

From (i) & (ii) T is a linear Transformation.

ex. If T is a mapping from $V_2(\mathbb{R})$ into $V_2(\mathbb{R})$ defined by $T(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, show that T is a linear Transformation.

Solⁿ: Let $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2) \in V_2(\mathbb{R})$,

such that $T(\alpha) = (x_1\cos\theta - y_1\sin\theta, x_1\sin\theta + y_1\cos\theta)$

$T(\beta) = (x_2\cos\theta - y_2\sin\theta, x_2\sin\theta + y_2\cos\theta)$.

Consider,

(i) $T(\alpha+\beta) = T(x_1+x_2, y_1+y_2)$

$= ((x_1+x_2)\cos\theta - (y_1+y_2)\sin\theta, (x_1+x_2)\sin\theta + (y_1+y_2)\cos\theta)$

$= (x_1\cos\theta - y_1\sin\theta, x_1\sin\theta + y_1\cos\theta)$
$+ (x_2\cos\theta - y_2\sin\theta, x_2\sin\theta + y_2\cos\theta)$

$= T(\alpha) + T(\beta)$.

(ii) $T(c\alpha) = T(cx_1, cy_1)$

$= (cx_1\cos\theta - cy_1\sin\theta, cx_1\sin\theta + cy_1\cos\theta)$

$= c(x_1\cos\theta - y_1\sin\theta, x_1\sin\theta + y_1\cos\theta)$

$= c\,T(\alpha)$

From (i) & (ii) T is a linear transformation

*. If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(1,0) = (1,1)$ & $T(0,1) = (-1,2)$, show that T maps the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ & $(0,1)$, into a parallelogram.

* Let $M(\mathbb{R})$ be the vector space of all 2×2 matrices over $\mathbb{R}$ and B be a fixed non-zero element of $M(\mathbb{R})$. Show that the mapping $T: M(\mathbb{R}) \to M(\mathbb{R})$, defined by $T(A) = AB - BA$, $\forall A \in M(\mathbb{R})$ is a linear map.

* If $T: V_1(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ is defined by $T(x) = (x, x^2, x^3)$, verify whether T is linear or not.

Sol^n let $x, y \in V_1(\mathbb{R})$

Then $T(x) = (x, x^2, x^3)$, $T(y) = (y, y^2, y^3)$.

Consider $T(x+y) = (x+y, (x+y)^2, (x+y)^3)$

$$= (x+y, x^2+y^2+2xy, x^3+y^3+3x^2y+3xy^2)$$

$$\neq (x, x^2, x^3) + (y, y^2, y^3)$$

$$\neq T(x) + T(y).$$

$\therefore$ T is not a linear transformation.

* Find a linear transformation $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, such that $f(1,0) = (1,1)$ and $f(0,1) = (-1,2)$.

Sol^n let $(x, y) \in \mathbb{R}^2$.

Then $(x, y) = x(1,0) + y(0,1)$

and $T(x,y) = x\,T(1,0) + y\,T(0,1)$

$$= x(1,1) + y(-1,2)$$

$$\underline{T(x,y) = (x-y, x+2y)}$$

* Find the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $T(1,1) = (0,1,2)$ and $T(-1,1) = (2,1,0)$.

Sol^n let $(x,y) \in \mathbb{R}^2$

Then $(x,y) = c_1(1,1) + c_2(-1,1) \Rightarrow (x,y) = (c_1 - c_2, c_1 + c_2)$

$\Rightarrow c_1 - c_2 = x$, $c_1 + c_2 = y$ $\Rightarrow$ $c_1 = \frac{x+y}{2}$ & $c_2 = \frac{y-x}{2}$.

Hence $(x,y) = \left(\frac{x+y}{2}\right)(1,1) + \left(\frac{y-x}{2}\right)(-1,1)$

Then $T(x,y) = \left(\frac{x+y}{2}\right)T(1,1) + \left(\frac{y-x}{2}\right)T(-1,1)$

$$= \left(\frac{x+y}{2}\right)(0,1,2) + \left(\frac{y-x}{2}\right)(2,1,0)$$

$$T(x,y) = (y-x, y, x+y)$$

Range and kernel of a Linear Transformation

Definition :-

Let $T: V \to W$ be a linear transformation.
The range of T is the set $R(T) = \{T(\alpha) \mid \alpha \in V\}$

Definition :-

Let $T: V \to W$ be a linear transformation.
The kernel (or null space) of T is the set
$N(T) = \{\alpha \in V \mid T(\alpha) = 0\}$, where 0 is the zero vector of W.

Note :-

* For the identity map $I: V \to V$ the range is the entire space V and the kernel is the zero subspace of V.
* For the zero linear map $T: V \to W$ defined by $T(\alpha) = 0$ $\forall\, \alpha \in V$, the range $R(T) = \{0\}$ = zero space of V and the null space $N(T) = V$.

Theorem :-

Let $T: V \to W$ be a linear transformation.
Then (a) $R(T)$ is a subspace of W.
(b) $N(T)$ is a subspace of V
(c) T is one-one iff $N(T) = \{0\}$,
where 0 is the zero vector of W.

Definition :-

Let $T: V \to W$ be a linear transformation. The dimension of the range space $R(T)$ is called the rank of the linear transformation T and is denoted by $r(T)$. The dimension of the nullspace $N(T)$ is called the nullity of the linear transformation T and is denoted by $n(T)$.

Theorem :-

Let $T : V \to W$ be a linear transformation. If the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ generates V, then the vectors $T\alpha_1, T\alpha_2, \ldots, T\alpha_n$ generates $R(T)$.

Theorem : (Rank - nullity theorem)

Let $T : V \to W$ be a linear transformation and V be a finite dimensional vector space.

Then $r(T) + n(T) = d[V]$

or $d[R(T)] + d[N(T)] = d[V]$

or rank + nullity = dimension of the domain.

example :-

Let $T : V \to W$ be a linear transformation defined by $T(x, y, z) = (x+y,\ x-y,\ 2x+z)$.

Find the range, null space, rank, nullity and hence verify the rank-nullity theorem.

Sol^n

$T(e_1) = T(1,0,0) = (1,1,2) = \alpha_1$.

$T(e_2) = T(0,+1,0) = (1,-1,0) = \alpha_2$

$T(e_3) = T(0,0,1) = (0,0,1) = \alpha_3$

$\{\alpha_1, \alpha_2, \alpha_3\}$ generates $R[T]$.

Consider $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$|A| = -2 \neq 0$

$\therefore \{\alpha_1, \alpha_2, \alpha_3\}$ is L.I, thus it is a basis of $R[T]$.

$d[R(T)] = 3$

Let $\alpha \in R(T)$

Then $\alpha = c_1(\alpha_1) + c_2(\alpha_2) + c_3(\alpha_3)$

$= c_1(1,1,2) + c_2(1,-1,0) + c_3(0,0,1)$

$= (c_1 + c_2,\ c_1 - c_2,\ 2c_1 + c_3)$

$\therefore R(T) = \{(c_1+c_2,\ c_1-c_2,\ 2c_1+c_3) / c_1, c_2, c_3 \in \mathbb{R}\}$

Suppose $T(x,y,z) = (0,0,0)$

$\Rightarrow (x+y,\ x-y,\ 2x+z) = (0,0,0)$

$\Rightarrow x+y=0,\ x-y=0,\ 2x+z=0$

$\Rightarrow x=0,\ y=0,\ z=0$

$\therefore N(T) = \{(0,0,0)\}$

$\therefore d[N(T)] = 0$

rank + nullity $= 3+0 = 3 = d[V_3(\mathbb{R})]$.

* Find the range, nullspace, rank and nullity of the linear transformation $T: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ defined by $T(x,y,z) = (y-x,\ y-z)$ and also verify Rank-nullity theorem.

Soln:
$T(1,0,0) = (-1,0) = \alpha_1$

$T(0,1,0) = (1,1) = \alpha_2$

$T(0,0,1) = (0,-1) = \alpha_3$

$R(T) = L\{\{\alpha_1, \alpha_2, \alpha_3\}\}$

Consider $\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore d[R(T)] = 2$ Basis of $R(T) = \{(-1,0), (1,1)\}$

Let $\alpha \in R(T)$

$\alpha = c_1\alpha_1 + c_2\alpha_2$

$= c_1(-1,0) + c_2(1,1)$

$= \{(-c_1+c_2, c_2)\}$

$\therefore R(T) = \{(-c_1+c_2, c_2) \mid c_1, c_2 \in \mathbb{R}\}$.

Suppose $\cancel{N(T)=} T(x,y,z) = (0,0)$

$\Rightarrow (y-x, y-z) = (0,0)$

$\Rightarrow y = x,\ y = z \Rightarrow x = y = z$.

$\therefore N(T) = \{(a,a,a) \mid a \in \mathbb{R}\}$.

Basis of $N(T) = \{(1,1,1)\}$.

$\therefore d[N(T)] = 1$.

Rank + Nullity $= 2 + 1 = 3 = d[V_3(\mathbb{R})]$

* If T is a linear transformation from $V_3(\mathbb{R})$ into $V_4(\mathbb{R})$ defined by $T(1,0,0) = (0,1,0,2)$, $T(0,1,0) = (0,1,1,0)$, $T(0,0,1) = (0,1,-1,4)$. Find the range, nullspace, rank, nullity of T.

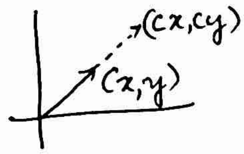
* Find the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by, $T(e_1) = e_1 - e_2$, $T(e_2) = 2e_1 + e_3$, $T(e_3) = e_1 + e_2 + e_3$. Also find the range, null space, rank and nullity of T.

Suppose x is an n-dimensional vector. when A multiplies x, it transforms that vector into a new vector Ax. This happens at every point x of the n dimensional space $\mathbb{R}^n$. The whole space is transformed or "mapped into itself" by the matrix A.

Stretch:

A multiple of the identity matrix, $A = cI = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ stretches every ~~no~~ vector by the same factor c. The whole space expands $(c>1)$ or contracts $(0<c<1)$ or goes through the origin and out the opposite side $(c<0)$.

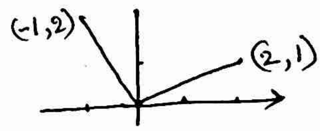
$$Ax = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$$



Rotation:

A rotation matrix turns the whole space around the origin.

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$



$\theta = 90° = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightarrow$ Rotation about $90°$

$Q_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \rightarrow$ Rotation in backwards through θ.

$Q_{-\theta} = (Q_\theta)^T$

$Q_\theta \cdot Q_{-\theta} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$Q_\theta \cdot Q_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = Q_{2\theta}$

$Q_\theta \cdot Q_\phi = Q_{\theta+\phi}$; $Q_\theta^{-1} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = (Q_\theta)^T$

Projection:-

A projection matrix takes the whole space onto a lower-dimensional subspace (not invertible).

$$P = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

$|P| = 0 \Rightarrow$ inverse doesnot exist.

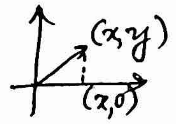
$$P^2 = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = P$$

$\Rightarrow$ projection twice on the θ-line is the same as projecting once on θ-line.

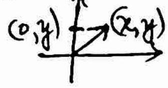
points on the Y-axis is projected to $(0,0)$.

points on θ-line projected onto itself.

$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ transforms each vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in the plane to nearest point $\begin{bmatrix} x \\ 0 \end{bmatrix}$ on the horizontal axis. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$



$P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ transforms $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} 0 \\ y \end{bmatrix}$

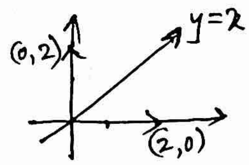


Reflection:-

A reflection matrix transforms every vector into its image on the opposite side of a mirror.

$$H = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$H^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$



$\Rightarrow$ Two reflections bring back the original.

$H^{-1} = H \Rightarrow$ self inverse.

$\theta = 45^\circ$ $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ~~transfo~~ gives the reflection through $y = x$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ gives the reflection through x axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$.