Defn: [Boolean product] : Let $A = [a_{ij}]$ be $m \times n$ zero-one matrix, $B = [b_{ij}]$ be $k \times n$ zero-one matrix. Then Boolean product of A and B denoted by $A \odot B$ is the $m \times n$ matrix with $(i,j)^{th}$ entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee (a_{i3} \wedge b_{3j}) \vee \cdots \vee (a_{ik} \wedge b_{kj})$$

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}_{3\times 2}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$. Find $A \odot B$

Ans:

$$A \odot B = \begin{bmatrix} (1\wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find $A \odot A$

Defn: Given a set A and a relation R on A, we define the powers of R recursively by as $R^1 = R$ and for $n \in Z^+$, $R^{n+1} = R \circ R^n$

Ex: If $A = \{1, 2, 3, 4\}$ and $R = \{(1,2), (1,3), (2,4), (3,2)\}$ then

$\underbrace{R^2}_{R \circ R} = \{(1,4), (1,2), (3,4)\}$, $\underbrace{R^3}_{R \circ R^2} = \{(1,4)\}$ and for $n \geq 4$,

$R^n = \phi$

Ex: Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,2), (1,3), (2,4), (3,2)\}$.
Find $M(R^2)$ (same as $(M(R))^2$)

$$M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M(R^2) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

or

$$M(R^2) = M(R) \underbrace{\odot}_{\text{Boolean product}} M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (0\wedge 0)\vee(1\wedge 0)\vee(1\wedge 0)\vee(0\wedge 0) & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\|^{ly}$ $$(M(R))^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: Let A be a set with $|A| = n$ and R be a relation on A. If $M(R)$ is relation matrix for R, then

a) $M(R) = 0$ (the matrix of all 0's) iff $R = \phi$

b) $M(R) = 1$ (the matrix of all 1's) iff $R = A \times A$

c) $M(R^m) = (M(R))^m$, for $m \in Z^+$

Defn: Let $A = [a_{ij}]$, $B = [b_{ij}]$ be $m \times n$ zero-one matrices. We say $A \leq B$ if $\forall\, i, j$ $a_{ij} \leq b_{ij}$

ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Here $A \leq B$ (since $\forall\, i, j$ $a_{ij} \leq b_{ij}$)

Defn: For $n \in Z^+$, $I_n = (\delta_{ij})_{n \times n}$ is zero-one matrix where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Operations on Relations:

1) Union and intersection: Let R_1, R_2 be relations from set A to set B, the union of R_1 and R_2 $(R_1 \cup R_2)$ is defined by $(a, b) \in R_1 \cup R_2$ iff $(a, b) \in R_1$ or $(a, b) \in R_2$. The intersection of R_1 and R_2 $(R_1 \cap R_2)$ is defined by $(a, b) \in R_1 \cap R_2$ iff $(a, b) \in R_1$ and $(a, b) \in R_2$.

2) Complement of a relation: Given a relation R from A to B the complement of R $(\overline{R})$ is defined with the property that $(a, b) \in \overline{R}$ iff $(a, b) \notin \overline{R}$. In other words, $\overline{R}$ is complement of R in universal set $A \times B$.

3) Inverse or converse of a relation (R^c): Let R be a relation from set A to set B, the converse of R^c defined with the property $(a, b) \in R^c$ iff $(b, a) \in R^c$

ex: $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$. $R_1 = \{(1, 1), (2, 2), (3, 3)\}$

and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$.

$R_1 \cup R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (1, 4)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$\overline{R_1} = (A \times B) - R_1$

$A \times B = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3),$
$(2,4), (3,1), (3,2), (3,3), (3,4)\}$

$\overline{R_1} = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4),$
$(3,1), (3,2), (3,4)\}$

$R_1 - R_2 = \{(2,2), (3,3)\}$

$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$ $(R_1 \cup R_2) - (R_1 \cap R_2)$

$R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1) = \{(1,2), (1,3), (1,4)$
$(2,2), (3,3)\}$

symmetric difference

$R_2^c = \{(1,1), (2,1), (3,1), (4,1)\}$

Representing relations using matrices

Defn: Given a set A with $|A| = n$ and a relation R on A, let $M(R)$ denote the relation matrix for R. Then

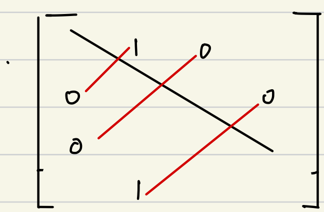
a) R is reflexive iff $\underbrace{m_{jj} = 1}_{\text{main diagonal entries are equal to 1}}$ $\forall j$ or $I_n \leq M(R)$

b) R is symmetric iff $m_{ij} = m_{ji}$ $\forall i, j$ or $M(R) = M(R^T)$ (symmetric matrix)

c) R is antisymmetric iff $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$ or $M(R) \cap M(R^T) \leq I_n$

meet of $M(R)$ and $M(R^T)$

(it has 1 in the position where both $M(R)$ and $M(R^T)$ has 1.)



ex:

$$M(R) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(R^T) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$M(R) \cap M(R^T) = M(R) \wedge M(R^T) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leq I_n$$

d) R is transitive iff $M(R^2) \leq M$ $\left[M_R^{[2]} \leq M \right]$

Ex: Let R be a relation on A.

a) $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ b) $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

c) $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Relation	a)	b)	c)
Reflexive	Yes	No	Yes
symmetric	Yes	No	No
Anti-symmetric	No	Yes	No
Transitive	Yes	Yes	No

a) $M(R^2) =$

$$R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$$

$$R^2 = \{(1,3), (1,1), (3,1), (3,3), (2,2)\}$$

$$M(R^2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \leq \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

or

$$M(R) \odot M(R) = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Ex: Let $M(R) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find matrix representation of a) R^{-1} b) $\overline{R}$

Ans: R^{-1} is inverse of R its matrix representation is $M(R^T) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$\overline{R}$ is complement of R its representation is

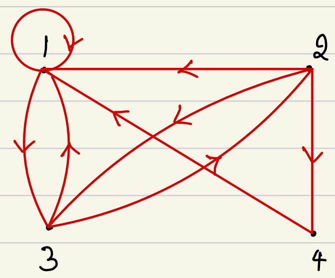
$$M(\overline{R}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Representing relations using digraphs

A directed graph or digraph consists of set V of vertices and set E of edges.
Here, $E = \{(a, b) \mid a, b \in V\}$

a is called initial vertex of edge (a, b), b is called terminal vertex

ex: Let $A = \{1, 2, 3, 4\}$. Let R be relation on A defined by $R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$



A directed graph on R.

Various properties of relation can be determined using digraph.

1) A relation is reflexive iff there is a loop at every vertex of graph

2) Symmetric iff for every edge b/w 2 distinct vertices in its digraph there is an edge in opposite direction

3) Antisymmetric iff there are never 2 edges

4) Transitive iff there is an edge from vertex x to a vertex z whenever there is an edge from x to y and an edge from y to z (for some y)

Ex: Determine if the relations for directed graphs shown are reflexive, symmetric, antisymmetric, transitive

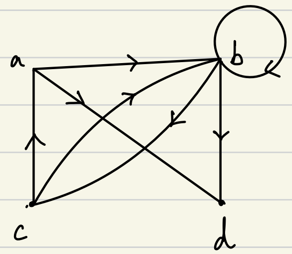


fig 1.

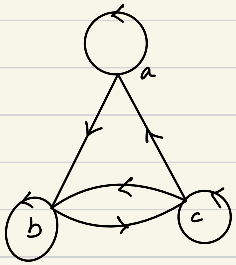


fig 2

Property	fig 1	fig 2
Reflexive	No	Yes
symmetric	No	No
antisymmetric	No	No
transitive	No	No

(fig 1, transitive:) $c R a$, $a R d$, but $c \not R d$

(fig 2, transitive:) since $(a,b) \in R$ and $(b,c) \in R$, but $(a,c) \notin R$

Representing union and intersection of relations

If R_1 and R_2 are relations on a set A represented by matrices $M(R_1)$ and $M(R_2)$ respectively. If $M_{R_1 \cup R_2}$ is the matrix representing the union of R_1 and R_2, it has 1 in the position where either M_{R_1} or M_{R_2} has a 1.

If $M_{R_1 \cap R_2}$ is the matrix representing the intersection of these relations, then it has 1 in the position where both M_{R_1} and M_{R_2} has a 1.

$\therefore \; M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$ (Join of M_{R_1} and M_{R_2})

$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$ (Meet of M_{R_1} and M_{R_2})

Ex: If the relations R_1 and R_2 on a set A are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Ans: $$M(R_1 \cup R_2) = M(R_1) \vee M(R_2) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M(R_1 \cap R_2) = M(R_1) \wedge M(R_2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Ex. For a fixed integer $n > 1$, P.T. the relation "congruent modulo n" is an equivalence relation on set of all integers $\mathbb{Z}$.

Sol: aRb means $a \equiv b \pmod{n}$, $a-b$ is multiple of n

(This means $n \mid (a-b)$ or $a-b = kn$ for some integer k, a and b leave same remainder when divided by n) ex: $17 \equiv 5 \pmod{12}$

1) Reflexive : For every $a \in Z$

$a - a = 0$ is a multiple of n

$\Rightarrow a \equiv a \pmod{n}$

$\Rightarrow aRa$

$\therefore R$ is reflexive

2) Symmetric : $\forall\ a, b \in Z$, $aRb \Rightarrow a \equiv b \pmod{n}$

$\Rightarrow (a-b)$ is a multiple of n

$\Rightarrow (b-a)$ is -11-

$\Rightarrow b \equiv a \pmod{n}$

$\Rightarrow bRa$

$\therefore R$ is symmetric

3) Transitive: $\forall\ a, b, c \in Z$

aRb and bRc $\Rightarrow a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$

$\Rightarrow (a-b)$ and $(b-c)$ are multiples of n

$\Rightarrow (a-b) + (b-c) = a-c$ is multiple of n

$\Rightarrow a \equiv c \pmod{n}$

$\Rightarrow aRc$

$\therefore R$ is transitive

Hence R is an equivalence relation.

Ex: Let $A = \{1, 2, 3, 4, 5\}$. Define a relation R on $A \times A$ by $(x_1, y_1)\ R\ (x_2, y_2)$ iff $x_1 + y_1 = x_2 + y_2$.

Verify that R is an equivalence relation on $A \times A$.

Closure of relations:

Let R be a relation of a set A, let P be some property (such as reflexive, symmetric, transitive) that R may or may not have. The relation S is said to be closure of R wrt P if S is the smallest relation with the property P with $R \subseteq S$.

Ex: Find reflexive closure, given $R = \{(1,2), (2,3), (3,4)\}$ on set $A = \{1, 2, 3, 4\}$

sol: Add all pairs (a, a) that are missing from R $\forall$ $a \in A$.

Reflexive closure of $R = R \cup \{(1,1), (2,2), (3,3), (4,4)\}$

$$= \{(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,4)\}$$

Ex: Find symmetric closure, given $R = \{(1,2), (2,3), (1,3)\}$ on set $A = \{1, 2, 3\}$.

sol: Add (b, a) for every (a, b) in R for symmetric closure.

$$S = R \cup \underbrace{\{(2,1), (3,2), (3,1)\}}_{R^{-1}} = R \cup R^{-1}$$

$$= \{(1,2), (2,3), (1,3), (2,1), (3,2), (3,1)\}$$

Ex: Let A be set of all Z^+. $R = \{(a,b) \mid a < b\}$.
Find its reflexive closure.

sol: $S = R \cup \{(a,a) \mid a \in Z\}$

$$S = \{(a,b) \mid a \leq b\}$$

Ex: Let A be set of all Z^+. $R = \{(a,b) \mid a > b\}$

Find symmetric closure.

Sol:

$$R^{-1} = \{(a,b) \mid \overbrace{b > a}^{\to a < b}\}$$

$$S = \{(a,b) \mid a > b \text{ or } a < b\} = \{(a,b) \mid a \neq b\}$$

Equivalence class

Let R be an equivalence relation on a set A and $a \in A$. The set of all those elements x of A which are related to a by R is called equivalence class of a wrt R. This equivalence class is denoted by $R(a)$ or $[a]$ or $\bar{a}$ or $[a]_R$

i.e. $\bar{a} = [a] = \{x \in A \mid \underbrace{(x,a) \in R}_{\text{or } x \sim a}\}$

$x \sim a$ means $(x,a) \in R$

Ex: Let $A = \{1, 2, 3, 4, 5\}$, $R = \{(a,b) \mid a+b \text{ is even}\}$

Find the equivalence class of all elements of A.

Sol: (Check that R is equivalence relation)

$[1] = \{1, 3, 5\}$, $[2] = \{2, 4\}$, $[3] = \{1, 3, 5\}$

$[4] = \{2, 4\}$, $[5] = \{1, 3, 5\}$.

Equivalence class of elements 1, 3, 5 are same and

equivalence class of elements 2 and 4 are same.
Any element out of 1, 3 and 5 can be chosen as a representative of equivalence class $\{1, 3, 5\}$. |||ly for equivalence class $\{2, 4\}$

$$\therefore [1] = \{1, 3, 5\}, \quad [2] = \{2, 4\}$$

Thm: Let R be an equivalence relation on a set A. The statements for elements a and b of A are equivalent.

1) $a R b$
2) $[a] = [b]$
3) $[a] \cap [b] \neq \phi$

Ex: Let m be an integer $m > 1$, S.T. the relation $R = \{(a, b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation on Z. Also find the equivalence class of 1 (consider $m = 4$).

Sol.
$$[1] = \{x \mid x \equiv 1 \pmod{4}\}$$

$$= \{4k + 1 \mid k \in Z\}$$

$$= \{\ldots -3, 1, 5, 9, 13 \ldots\}$$

for $k = -1$ $k = 0$ . . . - -

|||ly
$$[2] = \{x \mid x \equiv 2 \pmod{4}\}$$

$$= \{4k + 2 \mid k \in Z\} = \{\ldots -2, 2, 6, 10, \ldots\}$$

$[3] = \{ 4k+3 \mid k \in \mathbb{Z} \}$

$[4] = \{ 4k \mid k \in \mathbb{Z} \} = [0]$

$[5] = \{ 4k+5 \mid k \in \mathbb{Z} \} = \{ 4k+1 \mid k \in \mathbb{Z} \}$
$= [1]$

$4k+4+1$
$= 4(k+1)+1$
$= 4k'+1$

$[6] = [2]$

Observe: $\mathbb{Z} = \{ -4, -3, -2, -1, 0, 1, 2, 3, 4 \ldots \}$

$= [1] \cup [2] \cup [3] \cup [4] \rightsquigarrow \bigcup_{a \in A} [a] = A$

union covers all of A.

Also, $[1] \cap [2] = \phi$, $[3] \cap [4] = \phi$

$[2] \cap [3] = \phi$, $[1] \cap [4] = \phi$ pairwise disjoint

(Each equivalence class is like a 'piece' of the set)

Partition of a set :

Let A be a non-empty set. Suppose there exists non empty subsets $A_1, A_2, \ldots A_k$ of A such that the following 2 conditions hold

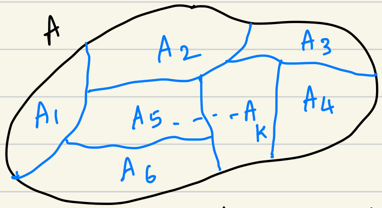
1) A is the union of $A_1, A_2, \ldots A_k$

i.e. $A = A_1 \cup A_2 \cup \ldots A_k$

$$A = \bigcup_{i=1}^{k} A_i$$

2) Any 2 of the subsets $A_1, A_2, \ldots A_k$ are disjoint i.e. $A_i \cap A_j = \phi$ for $i \neq j$

Then the set $\{A_1, A_2, \ldots A_k\}$ is called partition of A. Also $A_1, A_2, \ldots A_k$ are called the blocks or cells of partition.



Partition of set A.

Ex: Let $S = \{1, 2, 6, 7, 9, 11\}$, $A_1 = \{1, 2\}$, $A_2 = \{6\}$, $A_3 = \{9, 11, 7\}$ be subsets of S.

Clearly A_1, A_2, A_3 are disjoint subsets and

$$S = A_1 \cup A_2 \cup A_3$$

Thus $\{A_1, A_2, A_3\}$ is partition of S.

Thm: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of set S, there is an equivalence relation R that has the sets A_i, $i \in I$ as its equivalence classes.

WKT $\bigcup_{a \in S} [a] = S$, Also 2 equivalence classes are equal or disjoint. Thus we say that equivalence classes partition S.

Ex: Congruence modulo 4.

$$\mathbb{Z} = [0] \cup [1] \cup [2] \cup [3]$$

Ex: Let $S = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$

$A_3 = \{6\}$. List the ordered pairs in equivalence relation R.

Sol: $R = \{(1,1), (1,2), (1,3), (2,2), (2,1), (2,3), (3,1), (3,2),$

$(3,3), (4,4), (4,5), (5,4), (5,5), (6,6)\}$

Ex: Consider the set $A = \{1, 2, 3, 4, 5\}$ and the equivalence relation $R = \{(1,1), (2,2), (2,3), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$ defined on A. Find the partition of A induced by R.

Sol. $[1] = \{1\}$, $[2] = \{2, 3\}$, $[3] = \{2, 3\}$,

$[4] = \{4, 5\}$, $[5] = \{4, 5\}$

From these equivalence classes only $[1], [2], [4]$ are distinct.

$\therefore$ The partition P of A induced by R is

$$P = \{[1], [2], [4]\}$$

Partial orders:

A relation R on set A is said to be a partial ordering relation or a partial order on A if

i) R is reflexive

ii) R is antisymmetric

iii) R is transitive on A.

A set A together with a partial order R is called a partially ordered set or poset and is denoted by (A, R).

not less than or equal to

Notation: Let R be a partial order on set A.

For $a, b \in A$ aRb is denoted by $a \preceq b$ and poset (A, R) is written as $(A, \preceq)$

symbol to denote relation in any poset

Defn: 2 elements a and b of a poset are called comparable iff either aRb or bRa.

Defn: If (A, R) is a poset and every 2 elements are comparable then A is called total ordered set. or linearly ordered set

A total ordered set is also called chain.

ex: The poset $(\mathbb{Z}, \leq)$ is totally ordered. because $a \leq b$ or $b \leq a$ whenever a and b are integers.

Ex: Let R be a relation on set $\mathbb{Z}^+$,

$R = \{(a, b) : a \mid b\}$. Is $\{\mathbb{Z}^+, |\}$ a poset?

Ans: Yes (verify)

Is this poset totally ordered?

Ans: No $(\because 5 \nmid 7$ or $7 \nmid 5)$

<u>Hasse diagram</u>: (simpler graphical representation of digraph for partial order)

Procedure to construct Hasse diagram:

1) Write down all the ordered pairs which satisfy the relation R.

2) Each element is represented as a dot and all the edges are pointed upwards with no directions of edges. (smaller elements at bottom and larger ones at top)

3) Remove unnecessary edges or connections: Remove all loops and edges that appear due to transitivity.

Ex: Let $A = \{1, 2, 3, 4, 6, 12\}$. The relation R defined on A by aRb iff a divides b. P.T. R is a partial order on A. Draw the Hasse diagram for this relation.

Sol: $R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,12), (2,2), (2,4), (2,6), (2,12), (3,3), (3,6), (3,12), (4,4), (4,12), (6,6), (6,12), (12,12)\}$

a) Reflexive: $\forall a \in A, (a,a) \in R$

$\therefore$ R is reflexive.

b) Antisymmetric : If $a|b$ and $b|a$ then $a = b$ or If $a \neq b$ then either $a \nmid b$ or $b \nmid a$

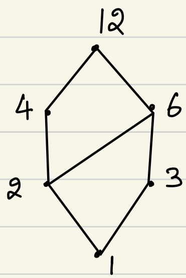
$\therefore$ R is antisymmetric

c) Transitive : For $(a, b) \in R$ and $(b, c) \in R$ i.e. if $a|b$ and $b|c$ then $a|c$ i.e. $(a, c) \in R$

$\therefore$ R is transitive

$\therefore$ R is a partial order on A.

<u>Hasse diagram</u>

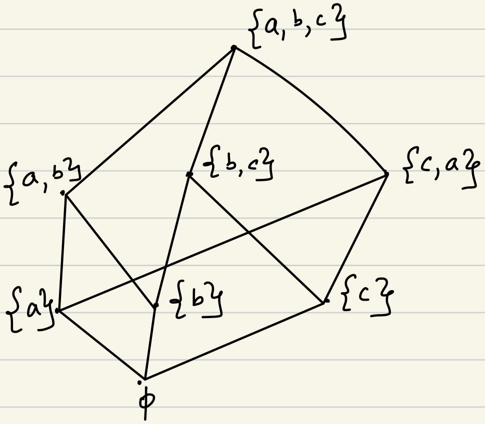


$R = \{(1,2), (1,3), (2,4), (2,6), (3,6), (4,12), (6,12)\}$

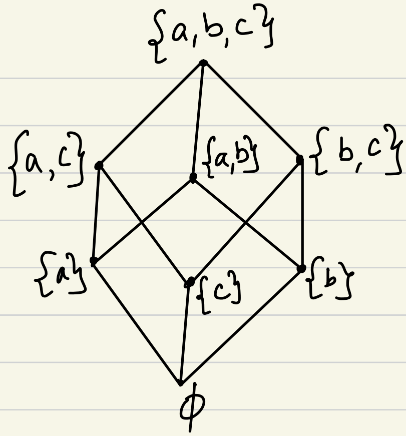
Ex: Draw the Hasse diagram for partial ordering $\{(A, B) \mid A \subseteq B\}$ on $P(S)$ where $S = \{a, b, c\}$

Sol:

$P(S) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}\}$

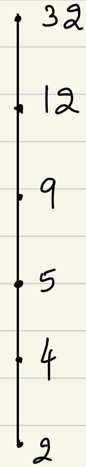


OR



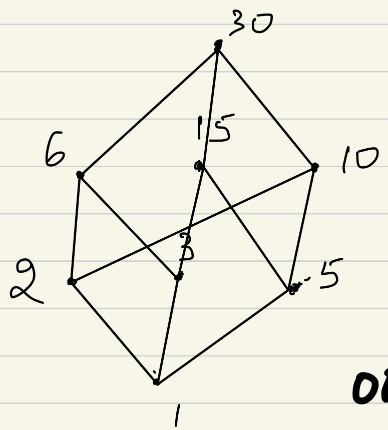
Ex: R be a relation on set $A = \{2, 4, 5, 9, 12, 32\}$ and $R = \{(a, b) \mid a \leq b\}$

Sol: Clearly $(A, \leq)$ is a poset. In particular this is totally ordered.

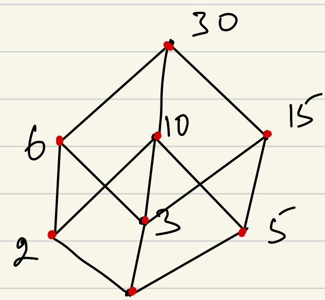


EX: Consider the partial order of divisibility on set A. Draw the Hasse diagram for the poset and determine if the poset is totally ordered. Given $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Sol:

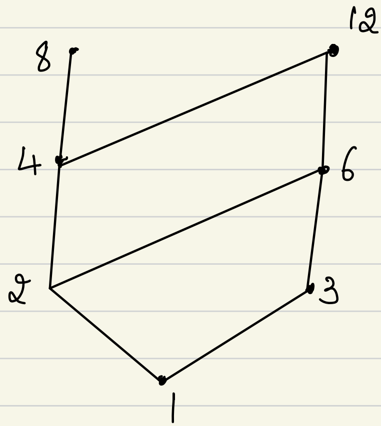


OR



Ex: Draw Hasse diagram representing the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $A = \{1, 2, 3, 4, 6, 8, 12\}$

Sol:



Defn: upper bound and lower bound

Let (S, R) be a poset such that S is an arbitrary set and R is a partial order defined on set S. Also, let $T \subseteq S$.

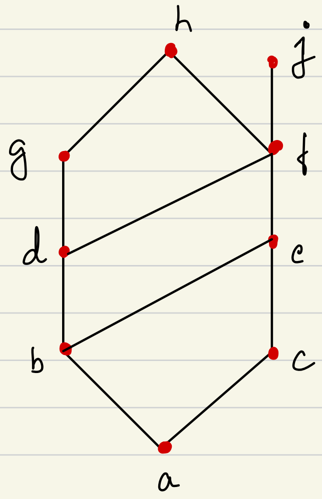
Lower bound: An element $x \in S$ is a lower bound of set T if $\forall y \in T$ $(x, y) \in R$

or $x \leq y$ $\forall y \in T$

upper bound: An element $x \in S$ is an upper bound of set T if $\forall y \in T$ $(y, x) \in R$.

or $x \geq y$ $\forall y \in T$

Ex: Find lower & upper bounds of subsets $\{a,b,c\}$, $\{j,h\}$ and $\{a,c,d,f\}$ in the poset with Hasse diagram shown below:



b is not U.B as it is not related c

sol:

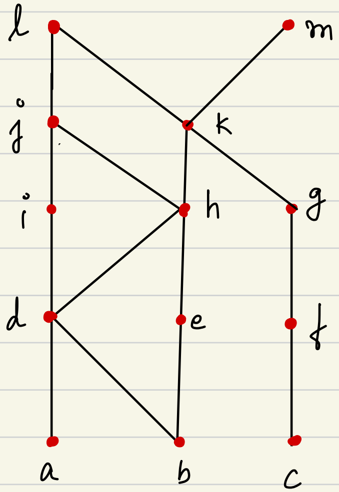
Subsets	Lower bound	upper bound
$\{a,b,c\}$	a	e, f, j, h
$\{j,h\}$	f, e, c, a, b, d	ϕ
$\{a,c,d,f\}$	a	f, j, h

there is path from a, b, c

① g is not U.B. of $\{a,b,c\}$ as $c \not R g$ and there is no path from c to g.
Every element of subset must be related to g
III^ly d is not U.B.

(2) g is not lower bound since g is not related to j.

Ex: Find all upper bounds of $\{a, b, c\}$ and all lower bounds of $\{f, g, h\}$



upper bounds of $\{a, b, c\}$:- k, l, m

lower bounds of $\{f, g, h\}$:- ϕ

Ex: Find all upper bounds of $\{2, 9\}$ and all lower bounds of $\{60, 72\}$ for the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$

largest element $\leq$ all elements in subset
(highest element below all others)

Def: Greatest lower bound (glb) or infimum:

The element x is called glb of a subset A if

i) x is a lower bound of A

ii) if y is any other lower bound of A, then $y \leq x$.

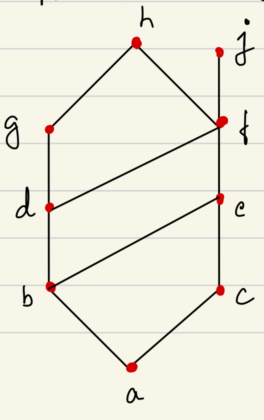
smallest element $\geq$ all elements in subset

Def: Least upper bound (lub) or supremum:
The element x is called lub of subset A if

i) x is an upper bound of A.

ii) if y is any other upper bound of A then $x \leq y$.

Ex: Find glb and lub of $\{b, d, g\}$ if they exist in the below poset



sol: lower bounds of $\{b, d, g\}$ are b, a.

closest below $\leftarrow$ glb is b $(\because b > a)$

upper bounds of $\{b, d, g\}$ are g, h

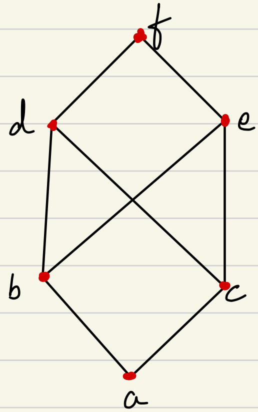
lub is g $(\because g < h)$

$\downarrow$

closest above

Ex: Find lub of $\{b, c\}$

sol:



upper bounds of $\{b, c\}$ are e, d, f

but lub does not exist

Because $d \not\leq e$ and $e \not\leq d$