Relations and Functions

- Sets (optional for recollection)
- Relations
- Functions

Sets – Recollection

Defn[Set]: A set is an unordered collection of objects

- denote by upper case letters
- members of sets are denoted by lowercase letter

A be a set, x is a member in A, we write $x \in A$.

We describe sets in the following ways

i) Roster method : Here all members are listed explicitly.

Ex : The set of all vowels, $V = \{a, e, i, o, u\}$

Ex : Set of all +ve integers less than 100

$$A = \{1, 2, 3, 4, 5, 6, \dots, 100\}$$

ii) Set builder method (Rule method):

We characterize elements in the set by stating their properties

Ex : The set of all odd integers less than 10.

$$A = \{x \mid x \text{ is an odd int. less than } 10\}$$

Ex: Set of Rational no.s.

$$\mathbb{Q} = \left\{\frac{p}{q} \;\middle|\; p, q \text{ are integers and } q \neq 0\right\}$$

Some important sets.

1) Set of all Natural no.s

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

2) Set of all integers

$$\mathbb{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$$

3) Set of all +ve integers

$$\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$$

4) Set of all nonnegative integers

$$\mathbb{Z}^+ \cup \{0\} = \{0, 1, 2, \dots\}$$

5) $\mathbb{R}$ be set of all real no.s

6) Set of all complex no.s

$$\mathbb{C} = \{ x + iy \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \}$$

Equal sets : Two sets A and B are said to be equal if they have same elements.

Ex : $A = \{1, 2, 3\}$, $B = \{2, 1, 3\}$, $C = \{a, b, c\}$

Here $A = B$, $A \neq C$.

The empty set : A set is called empty set if it has no elements, denoted by $\{ \}$ or ϕ.

Singleton set : A set with one element.

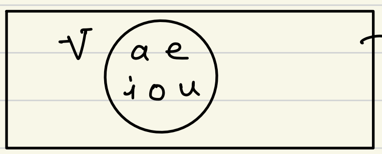
Ex : $A = \{0\}$, $B = \{\phi\}$

Venn diagram (graphical representation of sets)

Universal set U – Set containing all objects under consideration

Ex : V is a set of all vowels

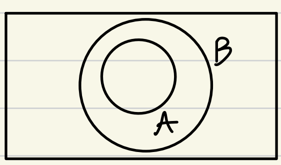
U



English Alphabet

Subset : We say that the set A is a subset of B if and only if every element of A is also an element of B.

Denoted by $A \subseteq B$.



Venn diagram of $A \subseteq B$

Symbolically,

$$\boxed{A \subseteq B \quad \text{iff} \quad \forall x\,(x \in A \rightarrow x \in B) \text{ is true}}$$

If A is not a subset of B, denoted by $A \nsubseteq B$.

$$\boxed{A \nsubseteq B \quad \text{iff} \quad \exists x\,(x \in A \text{ and } x \notin B) \text{ is true}}$$

Ex: Let $A = \{5, 6\}$, $B = \{1, 4, 5, 7, 8, 6\}$

$C = \{1, 3\}$

Here, $A \subseteq B$ and $C \nsubseteq B$.

Note: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

Thm: For every set S, i) $\phi \subseteq S$

ii) $S \subseteq S$

pf: We know that $A \subseteq B$ iff $\forall x\,(x \in A \Rightarrow x \in B)$ is true.

and $\forall x\,(x \in \phi \rightarrow x \in S)$ is true since $x \in \phi$ is false.

III^{ly} we can say that $S \subseteq S$.

proper subset: Set A is said to be proper subset of B iff A is a subset of B but $A \neq B$.

Denoted by $A \subset B$.

Symbollicaly

$$\boxed{A \subset B \quad \text{iff} \quad \forall x\,(x \in A \rightarrow x \in B) \text{ and } \exists x\,(x \in B \text{ and } x \notin A)}$$

Size of a set

1) **Finite set**: A set is finite if it has exactly n elements, where n is a nonnegative integer.

Ex: $A = \{1, 2, 3, 4, 9, 12\}$

2) **Infinite set**: If a set is not finite, then it is called infinite

Ex: Set of all even integers $E = \{2, 4, 6, 8, \dots\}$

Defn [Cardinality]: It is no. of elements in a set, if S is a set, then its cardinality is denoted by $|S|$.

Ex: If $S = \{a, b, c, d, g\}$, $|S| = 5$

Defn [Power set]: The power set of a set S is the set of all subsets of S, denoted by $\mathcal{P}(S)$.

Ex: If $S = \{0, 1, 2\}$

$\mathcal{P}(S) = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, \phi, S\}$

$|\mathcal{P}(S)| = 8 = 2^3$

Ex: If $A = \{a, b, c, d\}$

Subsets are ϕ, A, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a,b\}$, $\{b,c\}$, $\{a,d\}$
$\{a,c\}$, $\{b,d\}$, $\{c,d\}$, $\{a,b,c\}$, $\{b,c,d\}$, $\{c,d,a\}$, $\{a,b,d\}$

$|\mathcal{P}(A)| = 16 = 2^4$

In general, if $|A| = n$, then $|\mathcal{P}(A)| = 2^n$

Thm: If a set has n elements, then its power set has 2^n elements.

Defn : [ordered n-tuples].

It is ordered collection of n elements.

that is $(a_1, a_2, a_3, \dots, a_n)$ here a_i is ith element.

Ex : $(a_1, a_2) \neq (a_2, a_1)$

But $\{a_1, a_2\} = \{a_2, a_1\}$

and $(a_1, a_2) = (b_1, b_2)$ iff $a_1 = b_1$ and $a_2 = b_2$.

Defn [Cartesian product] :

Let $A_1, A_2, A_3, \dots, A_n$ be n sets. Then Cartesian product of $A_1, A_2, A_3, \dots, A_n$ is

$$A_1 \times A_2 \times A_3 \times \dots \times A_n = \{(a_1, a_2, a_3, \dots, a_n) \mid a_i \in A_i, \; i \in \{1, 2, \dots n\}\}$$

Ex : Let $A = \{1, 2\}$ and $B = \{a, b, c\}$

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,b)\}$

$B \times A = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

Note: $A \times B \neq B \times A$

But, $|A \times B| = |B \times A| = |A| \cdot |B|$.

Ex : $A = \{0, 1\}$

$A \times A = A^2 = \{(0,0), (0,1), (1,0), (1,1)\}$

In general

$$A^n = \underbrace{A \times A \times A \times \dots \times A}_{n \text{ times}}$$

Note : i) $A \times (B \times C) \neq A \times B \times C$

ii) $|A_1 \times A_2 \times \dots \times A_n| = |A_1| \cdot |A_2| \cdot \dots \cdot |A_n|$

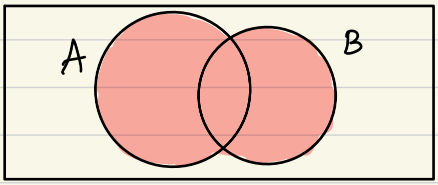
Set operations

Defn [Union]: Let A and B be sets. The union of A and B is

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

(i.e set that contains the elements that are either in A or B, or both)

Venn diagram



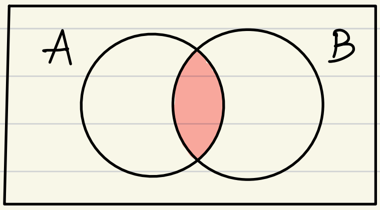
$A \cup B$ (shaded)

Defn [Intersection]: Let A and B be two sets. The intersection of A and B

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

(i.e $A \cap B$ is set of all those elements belong to A and A and B

Venn diagram



$A \cap B$ (shaded)

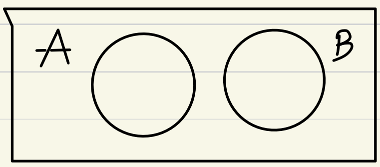
Ex: $A = \{1, 3, 5\}$, $B = \{1, 2, 3\}$

$A \cup B = \{1, 3, 5, 2\}$

$A \cap B = \{1, 3\}$

Defn [disjoint]: Two sets A and B are said to be disjoint if $A \cap B = \phi$.

Venn diagran



Ex: $A = \{1, 5, 9\}$, $B = \{2, 4\}$

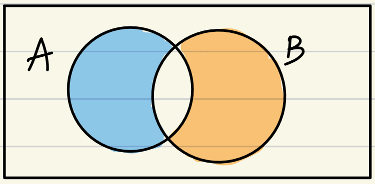
$A \cap B = \phi$

Defn [Difference]: Let A and B be sets. The difference of A and B is

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

It is also denoted by $A \setminus B$.

Venn diagram



$A - B$ (shaded, blue)

$B - A$ (shaded, brown)

Note: i) $A - B \neq B - A$

ii) $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$ (Clear from Venn diagram)

Ex: $A = \{1, 3, 5\}$, $B = \{1, 2, 3\}$

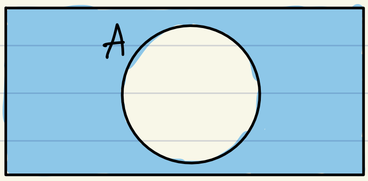
$A - B = \{1, 3, 5\} - \{1, 2, 3\}$
$= \{5\}$

$B - A = \{1, 2, 3\} - \{1, 3, 5\}$
$= \{2\}$

Defn (Complement): Let U be the universal set. The Complement of A is

$$A^c := \overline{A} = \{x \mid x \notin A\}$$

Venn diagram



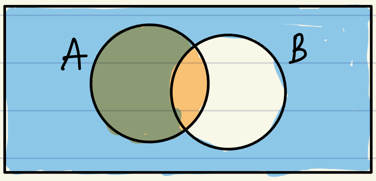
$\overline{A}$ (shaded)

Also, $\overline{A} = U - A$ $(U \setminus A)$

Note: i) $A - B$ is also called complement of B wrt A.

ii) $A - B = A \cap \overline{B}$

Venn diagram



$\overline{B}$ (shaded, blue)

$A - B = A \cap \overline{B}$ (shaded, blue and brown)

Set identities (Let A, B and C be sets)

Identity	Name
$A \cap U = A$ $A \cup \phi = A$	Identity laws
$A \cup U = U$ $A \cap \phi = \phi$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \phi$	Complement laws

Definition

Cartesian Product

Given two sets A and B, their Cartesian product, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$$

Example

Let $A = \{1, 2\}$ and $B = \{x, y\}$.

$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$$

- First element from A
- Second element from B

Number of Elements in Cartesian Product

Formula

If A and B are finite sets, then:

$$|A \times B| = |A| \times |B|$$

Example

Let $A = \{1, 2\}$ and $B = \{x, y, z\}$.

$$|A| = 2, \quad |B| = 3 \implies |A \times B| = 2 \times 3 = 6$$

The elements are: $(1, x), (1, y), (1, z), (2, x), (2, y), (2, z)$

Note

If either A or B is empty, then $|A \times B| = 0$.

What is a Relation?

Relation

- Let A and B be sets.
- A relationship between elements of A and B can be expressed using ordered pairs.
- A set of ordered pairs is called a **binary relation**.
- A binary relation from A to B is a subset of $A \times B$.
- A relation may have some, all, or none of the pairs from $A \times B$

Notation

- If R is a relation from A to B and $(a, b) \in R$, we write aRb. → a is related b
- If $(a, b) \notin R$, then $a \not{R}\, b$.

→ a is not related to b

Example 1:

Cities and States

- Let A = set of cities in South India.
- Let B = set of states in South India.
- Define $R = \{(y, x) \in B \times A : \text{city } x \text{ is in state } y\}$
- For instance
    - (Karnataka, Bengaluru)
    - (Tamil Nadu, Chennai)
    - (AP, Hyderabad)
    - (Telangana, Hyderabad)
    - (Kerala, Cochi)

  are some elements in R

Functions as Relations

- Let $f : A \to B$ be a function.
- The graph of f is the set $\{(a, b) \in A \times B : b = f(a)\}$.
- So, the graph of f is a relation from A to B.
- Hence, functions are special cases of relations.

Important Note

- Every function is a relation, but not every relation is a function.
- In a relation, one element of A may relate to multiple elements in B.

Relations on a Set

- A relation on a set A is a relation from A to A.
- That is, it is a subset of $A \times A$.

Example 2: Divides Relation

Let $A = \{1, 2, 3, 4\}$

$$R = \{(a, b) \mid a \text{ divides } b\}$$

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$$

Example 3:

Consider the relations on $\mathbb{Z}$:

- $R_1 = \{(a, b) \mid a \leq b\}$
- $R_2 = \{(a, b) \mid a > b\}$
- $R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$
- $R_4 = \{(a, b) \mid a = b\}$
- $R_5 = \{(a, b) \mid a = b + 1\}$
- $R_6 = \{(a, b) \mid a + b \leq 3\}$

Question:

Which of the relations in Example 3 contain each of the following pairs $(1,1),(1,2),(2,1),(1,-1)$ and $(2,2)$?

Answer:

- $(1,1)$: R_1, R_3, R_4, R_6
- $(1,2)$: R_1, R_6
- $(2,1)$: R_2, R_5, R_6
- $(1,-1)$: R_2, R_3, R_6
- $(2,2)$: R_1, R_3, R_4

Example 4:

How many relations are there on a set with n elements?

Example 4:

How many relations are there on a set with n elements?

Answer:

- Let $|A| = n$.
- A relation on A is a subset of $A \times A$ and $A \times A$ has n^2 elements
- Therefore, number of relations on A is: 2^{n^2}

Example 5:

How many relations are there from a set A to B, where $|A| = m$ and $|B| = n$?

Example 5:

How many relations are there from a set A to B, where $|A| = m$ and $|B| = n$?

Answer:

- Given $|A| = n$ and $|B| = n$.
- Therefore, $|A \times B| = mn$.
- Thus, number of relations from A to B is: 2^{mn}

Reflexive Relation

A relation R on a set A is called **reflexive** if $(a, a) \in R$ for all $a \in A$.

Using quantifiers:

$$\forall a \in A, (a, a) \in R$$

Example : The following relation on $\mathbb{Z}$ is reflexive.

$$R = \{(a, b) \mid a \leq b\}$$

Irreflexive Relation

A relation R on a set A is called **irreflexive** if $(a, a) \notin R$ for all $a \in A$.

Using quantifiers:

$$\forall a \in A, (a, a) \notin R$$

Example: The following relation on $\mathbb{Z}$ is irreflexive

$$R = \{(a, b) \mid a < b\}$$

Symmetric Relation

A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$.

Using quantifiers:

$$\forall a \forall b, ((a, b) \in R \implies (b, a) \in R)$$

Example : The following relation on $\mathbb{Z}$ is symmetric.

$$R = \{(a, b) \mid a = b\}$$

Properties of Relations

Antisymmetric Relation

A relation R on a set A is called **antisymmetric** if,

$$\text{for all } a, b \in A, \;\; \text{if } (a,b) \in R \text{ and } (b,a) \in R \text{ then } a = b.$$

or

$$\forall a \forall b \in A, \; ((a,b) \in R \wedge (b,a) \in R) \implies (a = b))$$

In otherwords,

$$\text{for all } a, b \in A, \;\; \text{if } a \neq b, \; \text{then } (a,b) \notin R \text{ or } (b,a) \notin R.$$

or

$$\forall a \forall b \in A, \; ((a \neq b) \implies ((a,b) \notin R \vee (b,a) \notin R))$$

Example : The following relation on $\mathbb{Z}$ is antisymmetric.

$$R = \{(a,b) \mid a \leq b\}$$

Properties of Relations

Asymmetric Relation

A relation R on a set A is called **asymmetric** if,

$$\text{for all } a, b \in A, \;\; \text{if } (a, b) \in R \text{ then } (b, a) \notin R.$$

Alternate for

$$\forall a \forall b \in A, \;\; ((a, b) \in R \implies (b, a) \notin R).$$

Example : The following relation on $\mathbb{Z}$ is asymmetric.

$$R = \{(a, b) \mid a \leq b\}$$

Note

Every asymmetric relation is antisymmetric.

Transitive Relation

A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Using quantifiers:

$$\forall a, b, c \in A, ((a, b) \in R \wedge (b, c) \in R) \implies (a, c) \in R$$

Example : The following relation on $\mathbb{Z}$ is transitive.

$$R = \{(a, b) \mid a < b\}$$

Example 5:

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$$
$$R_2 = \{(1,1),(1,2),(2,1)\}$$
$$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$$
$$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$$
$$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$$
$$R_6 = \{(3,4)\}$$

Question:

Which of the above relations are reflexive, irreflexive, symmetric, antisymmetric, asymmetric and transitive.

Reflexive:
R_3 and R_5 are reflexive.

Irreflexive
R_4 and R_6 are irreflexive.

Symmetric:
In Ex5, R_2 and R_3 are symmetric.

Antisymmetric:
Only R_4, R_5, R_6 are antisymmetric.

Asymmetric:
Only R_4, R_6 are asymmetric.

Transitive:
R_4, R_5, R_6 are transitive.

Example 6

Divides Relation

Let R be a relation on the set of all positive integers defined by:

$$R = \{(a, b) \mid a \text{ divides } b\}$$

Example 6

Divides Relation

Let R be a relation on the set of all positive integers defined by:

$$R = \{(a, b) \mid a \text{ divides } b\}$$

Properties:

- Reflexive: Yes, since $a \mid a$ for all a.
- Symmetric: No, $a \mid b$ does not imply $b \mid a$.
- Antisymmetric: Yes, if $a \mid b$ and $b \mid a$ then $a = b$.
- Transitive: Yes, if $a \mid b$ and $b \mid c$, then $a \mid c$.

Pf: Let $a|b$ and $b|c$ for all $a, b, c \in \mathbb{Z}^+$

$\Rightarrow$ $b = k_1 a$ and $c = k_2 b$ for some $k_1, k_2 \in \mathbb{Z}^+$

$\Rightarrow$ $c = k_2(k_1 a) \Rightarrow c = (k_2 k_1) a \Rightarrow a | c$

Combining Relations

Relations can be combined by Union, intersection, difference to obtain new relations.

Ex7: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relation $R_1 = \{(1,1), (2,2), (3,3)\}$ and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$.

$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$,

$R \cap R_2 = \{(1,1)\}$,

$R_1 - R_2 = \{(2,2), (3,3)\}$

$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

Ex 8: Let R_1 and R_2 be relations of set of real no.s.

$R_1 = \{(x,y) \mid x < y)$ and $R_2 = \{(x,y) / x > y\}$

$R_1 \cup R_2 = \{(x,y) / x < y$ or $x > y\}$

$= \{(x,y) \mid x \neq y\}$

$R \cap R_2 = \{(x,y) \mid x < y$ and $x > y\}$

$= \phi$

$R_1 - R_2 = R_1 - (R_1 \cap R_2) = R_1$

$R_2 - R_1 = R_2 - (R_1 \cap R_2) = R_2$

$R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1) = (R_1 \cup R_2) - (R_1 \cap R_2)$

Symmetric difference

$= R_1 \cup R_2$

Complement of a Relation: Let R be a relation from A to B. The complement of R, denoted by $\overline{R}$ is defined with the property that $(a,b) \in \overline{R}$ iff $(a,b) \notin R$

i.e $\quad \overline{R} = (A \times B) - R$

Converse of a relation: For a relation R from A to B, the converse of R, denoted by R^c or R^{-1} is defined with the property that $(a,b) \in R^c$ iff $(b,a) \in R$.

Ex 9: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Suppose $R = \{(a,1), (b,1), (c,2), (c,3)\}$. Find $\overline{R}$ and R^c.

Soln:

$\overline{R} = (A \times B) - R$

$= \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\} - R$

$= \{(a,2), (a,3), (b,2), (b,3), (c,1)\}$

$R^c = \{(1,a), (1,b), (2,c), (3,c)\}$

Ex 10: Let R be a relation on a set A. Prove that

i) R is reflexive iff $\overline{R}$ is irreflexive

ii) If R is reflexive, so is R^c

iii) If R is symmetric, so is R^c and $\overline{R}$

iv) If R is transitive, so is R^c

pf: i) Let R be reflexive. Then for all $a \in A$

$(a,a) \in R$

$\Leftrightarrow (a,a) \notin \overline{R}$

$\Rightarrow \overline{R}$ is irreflexive

ii) Let R be reflexive. Then for any $a \in A$

$(a,a) \in R \Rightarrow (a,a) \in R^c$. $\therefore R^c$ is reflexive.

iii) Take any $(a,b) \in R^c$ — ①

$\Rightarrow (b,a) \in R$

$\Rightarrow (a,b) \in R$ ($\because R$ is symmetric)

$\Rightarrow (b,a) \in R^c$ — ②

$\therefore R^c$ is symmetric from ① and ②

Take any $(a,b) \in \overline{R}$

$\Rightarrow (a,b) \notin R$

$\Rightarrow (b,a) \notin R$ ($\because R$ is symmetric)

$\Rightarrow (b,a) \notin \overline{R}$

$\therefore \overline{R}$ is symmetric.

iv) Let $(a,b) \in R^c$ and $(b,c) \in R^c$ for any $a,b,c \in A$

$\Rightarrow (b,a) \in R$ and $(c,b) \in R)$

$\Rightarrow (c,a) \in R$ ($\because R$ is transitive)

$\Rightarrow (a,c) \in R^c$

$\therefore R^c$ is transitive.

Ex 11: Let R and S be relation on a set A. Prove that

i) If R and S are reflexive, so are $R \cap S$ and $R \cup S$.

ii) If R and S are symmetric, so are $R \cap S$ and $R \cup S$.

iii) If R and S are antisymmetric, so is $R \cap S$

iv) If R and S are transitive, so is $R \cap S$

Pf i) Suppose R and S are reflexive,

$\Rightarrow (a,a) \in R, \quad (a,a) \in S \quad \forall a \in A$

$\Rightarrow (a,a) \in R \cup S$ and $(a,a) \in R \cap S$

Therefore, $R \cup S$ and $R \cap S$ are reflexive.

ii) Suppose R and S are symmetric

Take any $(a,b) \in R \cap S$

$\Rightarrow (a,b) \in R$ and $(a,b) \in S$

$\Rightarrow (b,a) \in R$ and $(b,a) \in S$ ($\because$ R, S are symm)

$\Rightarrow (b,a) \in R \cap S$

$\Rightarrow (b,a)$ is symm.

Now, take any $(a,b) \in R \cup S$

$\Rightarrow (a,b) \in R$ or $(a,b) \in S$

$\Rightarrow (b,a) \in R$ or $(b,a) \in S$

$\Rightarrow (b,a) \in R \cup S$

$\therefore$ $R \cup S$ is symm.

iii) Suppose R and S are antisymm.

Take any $(a,b) \in R \cap S$ and $(b,a) \in R \cap S$

$\Rightarrow (a,b), (b,a) \in R$ and $(a,b), (b,a) \in S$

$\Rightarrow a = b$ ($\because$ R and S are antisymm)

$\therefore$ $R \cap S$ is antisymm.

iv) Let R and S be transitive. For any $a, b, c \in A$

let $(a,b), (b,c) \in R \cap S$

$\Rightarrow$ $(a,b),(b,c) \in R$ and $(a,b),(b,c) \in S$

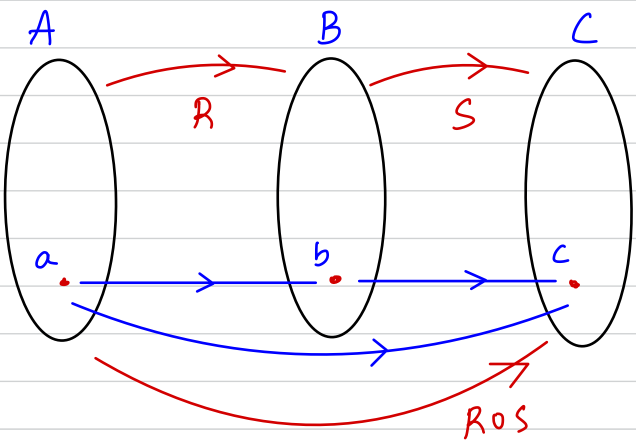
$\Rightarrow$ $(a,c) \in R$ and $(a,b) \in S$ ($\because$ R and S are transitive)

$\Rightarrow$ $(a,c) \in R \cap S$

$\therefore$ $R \cap S$ is transitive.

Composite relations

Let R be a relation from A to B and S a relation from B to C. The composite relation of R and S, denoted by $R \circ S$ consists of ordered pairs (a,c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$.



$R \circ S$ is a relation from A to C.

Ex12: Let $A = \{1,2,3\}$, $B = \{1,2,3,4)$, and $C = \{0,1,2\}$, let R be a relation from A to B and S be a relation from B to C.

$R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$

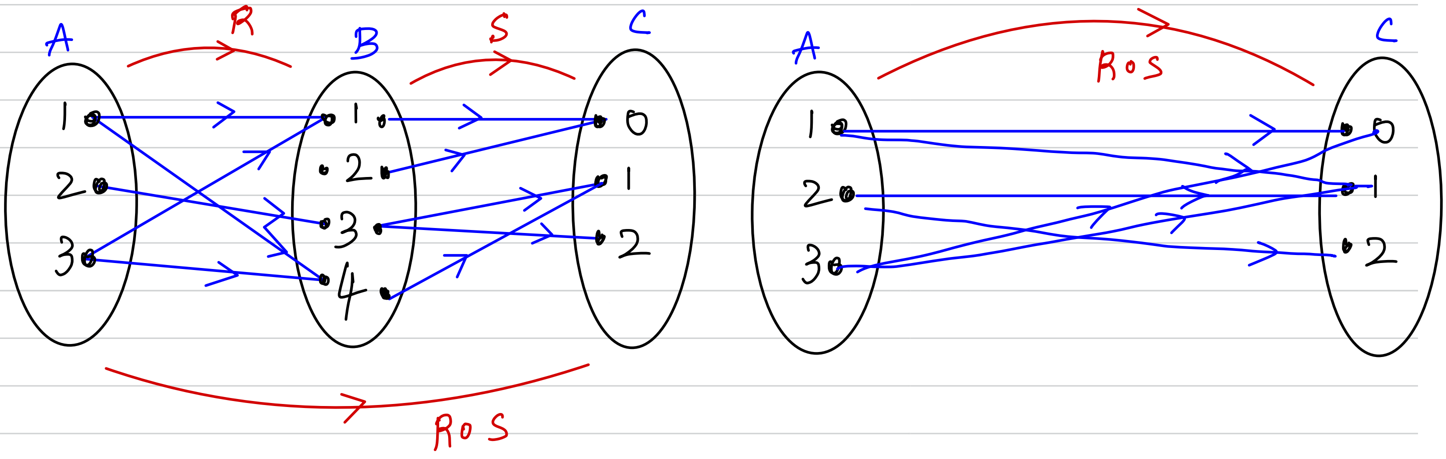
$S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$

Composite Relation of R and S is

$$R \circ S = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$$

|||ly Composite Relation of S and R is

$$S \circ R = \{(3,1),(3,4),(3,3),(4,1),(4,4)\}$$



Note: i) $R \circ S \neq S \circ R$

ii) For any relations R_1, R_2 and R_3,

$$R_1 \circ (R_2 \circ R_3) = (R_1 \circ R_2) \circ R_3$$

Ex. Let A be a set of all people. $R = \{(a,b) \mid a \text{ is a parent of } b\}$.

Then $(a,c) \in R \circ R$ iff there is a b such that $(a,b) \in R$ and $(b,c) \in R$.

In other words, $(a,c) \in R \circ R$ iff a is a grand parent of c.

Defn: Let R be a relation on the set A. The powers R^n, $n = 1,2,3,\ldots$ are defined recursively by

$R^1 = R$ and $R^{n+1} = R^n \circ R$.

Ex: Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$. Find R^n, $n = 1,2,3,\cdots$

Soln: $R^2 = R \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$

$R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$

$R^4 = R^3 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$

|||ly $R^n = R^3$ for $n \geq 4$.

Theorem: The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1, 2, 3, \cdots$

Pf: Let $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$. We show that R is transitive. In other words, we show that if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ for any a, b, c in A.

By the defn of composition, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R^2$. Since $R^2 \subseteq R$, we see that $(a,c) \in R$. Thus R is transitive.

Conversely, let R be transitive. We show by induction that $R^n \subseteq R$ for $n = 1, 2, 3, \cdots$

Basis step: Clearly $R^1 \subseteq R$. The result is true for $n = 1$.

Inductive step: Assume that $R^k \subseteq R$ for an arbitrary fixed integer $k \geq 1$. We shall show that $R^{k+1} \subseteq R$.

To show this, we assume that $(a,b) \in R^{k+1}$. Because $R^{k+1} = R^k \circ R$, there is an element $x \in A$ such that $(a,x) \in R$ and $(x,b) \in R^k$. Since $R^k \subseteq R$, implies that $(x,b) \in R$. Furthermore, because R is transitive, it follows that $(a,b) \in R$. This shows that $R^{k+1} \subseteq R$, completing the proof.

Zero – one matrices:

A matrix all of whose entries are 0 and 1 is called a zero-one matrix.

Ex: $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Boolean operations $\wedge$ and $\vee$ on pairs of bits, defined by

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Arithmetic of zero-one matrices is based on Boolean operations.

Defn: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices.

Join of A and B, denoted by $A \vee B$, is a zero-one matrix with $(i,j)^{th}$ entry $a_{ij} \vee b_{ij}$.

Meet of A and B, denoted by $A \wedge B$, is a zero-one matrix with $(i,j)^{th}$ entry $a_{ij} \wedge b_{ij}$.

Ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find $A \vee B$ and $A \wedge B$

Ans: $A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Defn: [Boolean product]: Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix, $B = [b_{ij}]$ be $k \times n$ zero-one matrix. Then

Boolean product of A and B, denoted by $A \odot B$, is the $m \times n$ matrix with $(i,j)^{th}$ entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee (a_{i3} \wedge b_{3j}) \vee \cdots \vee (a_{ik} \wedge b_{kj})$$

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find $A \odot B$.

Ans: $$A \odot B = \begin{bmatrix} (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \\ (0\wedge 1)\vee(1\wedge 0) & (0\wedge 1)\vee(1\wedge 1) & (0\wedge 0)\vee(1\wedge 1) \\ (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Defn: Let A be a square zero-one matrix, let r be a +ve integer.

The r^{th} Boolean power of A is

$$A^{[r]} = A \odot A \odot A \odot \cdots \odot A \quad (r \text{ times})$$

Ex: $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find $A^{[n]}$ for $n = 2, 3, 4, \cdots$.

Ans: $A^{[2]} = A \odot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ $A^{[3]} = A^{[2]} \odot A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$,

$A^{[4]} = A^{[3]} \odot A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $A^{[5]} = A^{[4]} \odot A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

And $A^{[n]} = A^{[5]}$ for all positive integers $n \geq 5$.

Defn: Let $A = [a_{ij}]$, $B = [b_{ij}]$ be $m \times n$ zero-one matrices. We say $A \leq B$ if $\forall\, i,j$ $a_{ij} \leq b_{ij}$

Ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Here $A \leq B$. (Since for all i, j $a_{ij} = b_{ij}$)

Representing relations

Representing relations using Matrices:

Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be finite sets. Let R be a relation from A to B.

A relation R can be represented using a zero-one matrix $M_R = [m_{ij}]$, where $m_{ij} = \begin{cases} 0 & \text{if } (a_i, b_j) \notin R \\ 1 & \text{if } (a_i, b_j) \in R \end{cases}$

Note: If $A \neq B$, we have to list A and B in a particular, but arbitrary, order.

If $A = B$, we have to use same ordering for A and B.

Ex: Let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$ and R is a relation defined by $R = \{(a,b) \mid a > b\}$. Write its matrix representation.

Ans: Here $R = \{(2,1), (3,1), (3,2)\}$

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

Let $a_1 = 1$, $b_1 = 1$
$a_2 = 2$, $b_2 = 2$
$a_3 = 3$

Ex: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R?

Given

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Soln: $R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$

Let R be a relation on a set A. Let $M_R = [m_{ij}]$ be the matix representing the relation R. Then

i) R is reflexive iff $m_{jj} = 1$ for all j. In other words R is reflexive iff main diagonal entries of M_R are equal to 1 and all elements off the diagonal can be 0 or 1.

The zero-one matrix for a Reflexive relation

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

From this it is clear that

R is reflexive iff $I_n \leq M_R$, I_n is identity matrix.

ii) R is symmetric iff $m_{ij} = m_{ji}$ for all i, j.

In other words, R is symmetric iff $M_R = M_R^T$ (that is M_R is symmetric matrix)

The zero-one matrix for a symmetric relation

$$\begin{bmatrix} & & 1 & \\ 1 & & & \\ & & & 0 \\ & 0 & & \end{bmatrix}$$

iii) R is antisymmetric iff either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$

The zero-one matrix for a antisymmetric relation

$$\begin{bmatrix} & 1 & 0 \\ 0 & & \\ 0 & & 0 \\ & 1 & \end{bmatrix}$$

Thus, R is antisymmetric iff $M_R \wedge M_R^T \leq I_n$.

iv) R is transitive if $M_R^{[2]} \leq M_R$,

Since we know that R is transitive iff $R^n \subseteq R$.

Ex: Suppose R is the relation represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Check whether R is reflexive, symmetric, antisymm and transitive.

Soln: • Clearly, $I_3 \leq M_R$ (or $m_{ii} = 1 \ \forall i$)

$\therefore$ it is reflexive

• It is symmetric, since $M_R = M_R^T$.

• It is not anti-symmetric.

Since $m_{12} = m_{21} = 1$ or $M_R \wedge M_R^T \nleq I_3$.

• Consider

$$M_R^{[2]} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$\Rightarrow M_R^{[2]} \nleq M_R$. $\therefore$ it is not transitive.

Ex: Let $M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find the matrix repre. of

i) R^{-1} ii) $\overline{R}$

Ans: R^{-1} is inverse relation of R, its Matrix representation is $M_R^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

$\overline{R}$ is complement of R, its representation is

$$M_{\overline{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Representing Union and Intersection of relations

Suppose R_1 and R_2 are relations on a set A represented by the matrices M_{R_1} and M_{R_2} respectively.

If $M_{R_1 \cup R_2}$ is the matrix representing the union of R_1 and R_2, it has 1 in the position where either M_{R_1} or M_{R_2} has a 1.

If $M_{R_1 \cap R_2}$ is the matrix representing the intersection of these relations, then it has 1 in the position where both M_{R_1} and M_{R_2} has a 1. Therefore

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} \quad (\text{Join of } M_{R_1} \text{ and } M_{R_2})$$

and

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} \quad (\text{Meet of } M_{R_1} \text{ and } M_{R_2})$$

Ex: Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$.

Ans: The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Representing Composite relations

Suppose R is a relation from A to B and S is a relation from B to C. Suppose $|A| = m$, $|B| = n$, $|C| = p$.

Let $M_R = [r_{ij}]$, $M_S = [s_{ij}]$ and $M_{R\circ S} = [t_{ij}]$

WKT $(a_i, c_j) \in R\circ S$ iff there is an element b_k such that $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$.

It follows that $t_{ij} = 1$ iff $r_{ik} = s_{kj} = 1$ for some k.

From the defn of the Boolean product, this means that

$$M_{R\circ S} = M_R \odot M_S.$$

Further, $M_{R^n} = M_{\underbrace{R\circ R\circ R\circ \cdots \circ R}_{n \text{ times}}} = \underbrace{M_R \odot M_R \odot M_R \odot \cdots \odot M_R}_{n \text{ times}} = M_R^{[n]}$

Ex: Let $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

Then $M_{R\circ S} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Ex: Let R be a relation on a set A.

i) $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ ii) $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ iii) $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Check which are reflexive, symmetric, anti symm. and transitive.

Questions	i)	ii)	iii)
Is this reflexive?	Yes	No	Yes
Is this Symmetric?	Yes	No	No
Is this anti-sym?	No	Yes	No
Is this transitive?	Yes	Yes	No

i) and ii) are transitive since in both cases $M_R^{[2]} = M_R$.

But, in iii) $M_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \not\leq M_R$

Representing relations using digraphs

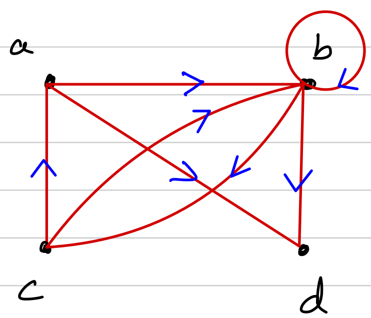
Defn: A directed graph or digraph, consist of set V of vertices and set E of edges.

Here $E = \{(a,b) \mid a, b \in V\}$

a is called initial vertex of the edge (a,b), b is called terminal vertex.

Ex: Let $V = \{a, b, c, d\}$

$E = \{(a,b), (a,d), (b,b), (b,d), (c,a), (c,b), (b,c)\}$



Note: any edge (a,a) is called a loop.

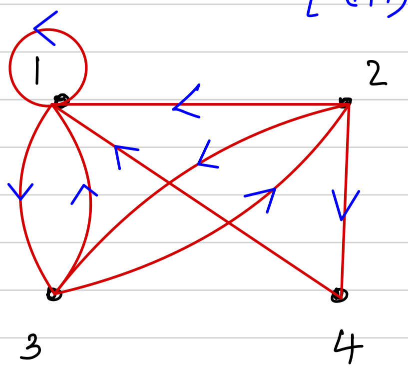
A directed graph.

Note:

The relation R on a set A can be represented using diagraph.

Ex: Let $A = \{1, 2, 3, 4\}$, let R be the relation on A defined by

$$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$$



A directed graph on R.

Various properties of a relation can be determined using its digraph.

i) A relation is reflexive iff there is a loop at every vertex of the directed graph.

ii) A relation is symmetric iff for every edge between two distinct vertices in its digraph there is an edge in the opposite direction.

iii) A relation is antisymmetric iff there are never two edges.

iv) A relation is transitive iff there is an edge from a vertex x to a vertex z whenever there is an edge from x to y and an edge from y to z (for some y).

Ex: Determine whether the relations for the directed graphs shown are reflexive, symmetric, antisymmetric and/or transitive.

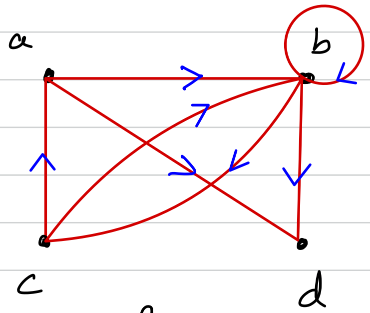


fig 1

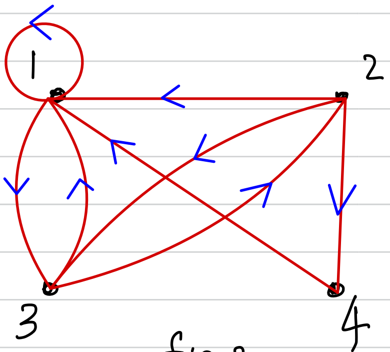


fig 2

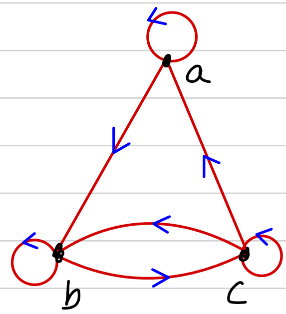


fig 3

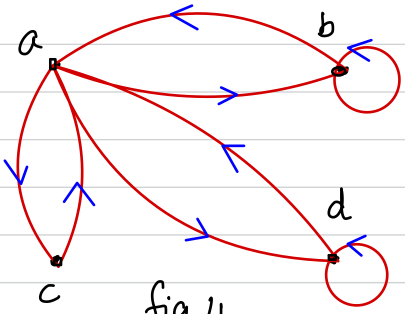


fig 4

Property	fig 1	fig 2	fig 3	fig 4
Reflexive	No	No	Yes	No
Symmetric	No	No	No	Yes
antisymmetric	No	No	No	No
transitive	No	No	No	No

Since $(a,b) \in R$ and $(b,c) \in R$, but $(a,c) \notin R$

Since $(c,a) \in R$, $(a,b) \in R$, but $(c,b) \notin R$.

Closures of relations

Let R be a relation of a set A, let P be some property (such as reflexive, symmetric, transitive, ...) that R may or may not have.

The relation S is said to be closure of R wrt P if S is the smallest relation with the property P with $R \subseteq S$.

S is said be the smallest relation with property P iff $S \subseteq T$ for any relation T with the property P.

If property P is reflexive, corresponding closure is called reflexive closure.

A relation S is said to be reflexive closure of R iff

i) $R \subseteq S$

ii) S is reflexive

iii) $S \subseteq T$ for any reflexive relation T containing R.

IIIly A relation S is said to be symmetric closure of R iff

i) $R \subseteq S$

ii) S is symmetric

iii) $S \subseteq T$ for any symmetric relation T containing R.

Ex: Let A be set of all the integers

$R = \{(a,b) \mid a < b\}$. Find its reflexive closure.

Soln: If S is the reflexive closure of R, then

$S = R \cup \{(a,a) \mid a \in \mathbb{Z}\}$

$S = \{(a,b) \mid a \leq b\}$

Ex: Let $A = \{1,2,3\}$, $R = \{(1,2), (3,1), (1,1), (2,3)\}$

Find its reflexive closure.

Soln: Reflexive closure of R is

$S = \{(1,2), (3,1), (1,1), (2,3), (2,2), (3,3)\}$

Note: $T = \{(1,2), (3,1), (1,1), (2,3), (2,2), (3,3), (3,2)\}$ is not the reflexive closure, since it is not the smallest relation with reflexive property and $R \subseteq T$.

Defn: $\Delta = \{(a,a) \mid a \in A\}$ is called the diagonal relation of A

Reflexive closure: If R is any relation on A, then its reflexive closure is $R \cup \Delta$.

Ex: Let $A = \{1,2,3\}$, $R = \{(1,2), (1,1), (3,2), (2,1)\}$

Find its symmetric closure.

Soln: Symm. closure of R is

$$S = \{(1,2), (1,1), (3,2), (2,1), (2,3)\}$$

Symmetric closure: Symm closure of R is $R \cup R^{-1}$

Ex: Find symm. closure of $R = \{(a,b) \mid a > b\}$ on the set of +ve ints.

Ans: Given $R = \{(a,b) \mid a > b\}$

$$R^{-1} = \{(a,b) \mid a < b\}$$

Thus symmetric closure is $R \cup R^{-1} = \{(a,b) \mid a > b \text{ or } a < b\}$

$$= \{(a,b) \mid a \neq b\}$$

The number of possible relations with certain properties on a set A such that $|A| = n$ are given below:

	Properties	# of possible relations
i)	Reflexive	2^{n^2-n}
ii)	Irreflexive	2^{n^2-n}
iii)	Neither reflexive nor irreflexive	$2^{n-1} \times 2^{n^2-n}$
iv)	Symmetric	$2^{(n^2+n)/2}$
v)	Antisymmetric	$2^n \times 3^{(n^2-n)/2}$
vi)	Asymmetric	$3^{(n^2-n)/2}$
vii)	Both symmetric and asymmetric	1
viii	Both reflexive and antisymmetric	$3^{(n^2-n)/2}$

Let M_R be matrix representation of a relation R on A

i) R is reflexive iff $m_{ii} = 1 \ \forall i$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1_{\ddots} & \\ & & & 1 \end{bmatrix}_{n\times n}$$

# of diagonal entries $= n$

# of off diagonal entries $= n^2 - n$

Diagonal entries are fixed and each off diagonal entries can be either 0 or 1.

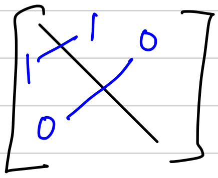
$\therefore$ We have 2^{n^2-n} possible reflexive relations.

ii) For irreflexive diagonal enteries are fixed to 0.

$$\begin{bmatrix} 0 & & \\ & 0_{\ddots} & \\ & & 0 \end{bmatrix}_{n\times n}$$

$\therefore$ We have 2^{n^2-n} possible irreflexive relations.

iv) R is symmetric iff $m_{ij} = m_{ji} \; \forall ij$



- Diagonal entries

  There are n diagonal enteries m_{ii}

  Each can be independently be 0 or 1 → 2 choices each

  So : Choices for diagonal enteries $= 2^n$

- Off - Diagonal entries

  There are $\frac{n(n-1)}{2}$ unique pairs

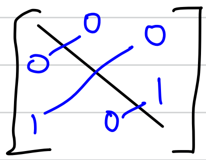
  Because the matrix is symm each pair has two choices

  → (0,0) or (1,1).

So : Choice for symmetric off-diagonal pairs $= 2^{n(n-1)/2}$

Total no. of symm relations $= 2^n \times 2^{n(n-1)/2} = 2^{(n+n^2)/2}$.

v)



- Choices for diagonal enteries $= 2^n$
- off diagonal enteries has $\frac{n(n-1)}{2}$ pair and

  each has 3 choices → (0,0), (1,0) and (0,1)

  ∴ Choices for anti-symm off-diagonal pairs $= 3^{n(n-1)/2}$

Total no. of anti symm relations $= 2^n \times 3^{n(n-1)/2}$.

Equivalence relation

Let R be a relation on A, R is said to be equivalence if

i) R is reflexive

ii) R is symmetric

iii) R is transitive

If aRb and R is equivalence, we say a is equivalent to b, denoted by $a \sim b$.

If $a \sim b$ means by some sense a is same as b.

Ex: Let $A = \{1,2,3,4,5\}$ and R be a relation on A, where

$$R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(2,3),(3,2),(4,5),(5,4)\}$$

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

From M_R it is clear that

R is reflexive, since $(a,a) \in R \ \forall \ a \in A$

R is symmetric, since $m_{ij} = m_{ji} \ \forall \ i,j$

R is transitive, since $M_R^2 \leq M_R$

$\therefore$ R is equivalence relation.

Ex: Equality $(=)$ on any set.

Ex: "Has the same absolute value" on the set of real numbers $\mathbb{R}$.

Equivalence class.

Let R be an equivalence relation on a set A, and let a be an element of A. Then equivalence class of a is

$$[a]_R = \{x \mid x \sim a\}$$ | $x \sim a$ means $(x, a) \in R$

i.e $[a]$ is set of all elements which are equivalent to a.

- any element $b \in [a]$ can be considered as representative.

Ex 1: Let R be the relation on the set of integers such that aRb iff $a = b$ or $a = -b$

Is this an equivalence relation?

Soln: i) for all $a \in \mathbb{Z}$ aRa, $\because a = a$

$\therefore R$ is reflexive

ii) Let $a, b \in \mathbb{Z}$ such that $aRb \Rightarrow a = b$ or $a = -b$

$\Rightarrow b = a$ or $b = -a$

$\Rightarrow bRa$

$\therefore R$ is symmetric

iii) Assume aRb and bRc, where $a, b, c \in \mathbb{Z}$

implies $a = b$ or $a = -b$ and $b = c$ or $b = -c$

$\Rightarrow a = c$ or $a = -c$

$\therefore R$ is transitive.

Thus, R is an equivalence relation.

For any $a \in \mathbb{Z}$,
$$\begin{aligned} [a] &= \{x \mid x \sim a\} \\ &= \{x \mid a = x \text{ or } a = -x\} \\ &= \{a, -a\} \end{aligned}$$

For instance, $[7] = \{-7, 7\}$, $[-3] = \{3, -3\}$, $[0] = \{0\}$, ...

Ex2: Let $A = \{1, 2, 3, 4\}$ and

$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$

be a relation on A. Verify that R is an equivalence relation.

Soln: Consider matrix representation of R,

$$M_R = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

i) It is reflexive since $m_{ii} = 1 \quad \forall i$

or $I_4 \leq M_R$

ii) Since $M_R = M_R^T$, we see that

R is symmetric.

iii) $$M_R^2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = M_R$$

$\because M_R^2 \leq M_R$

R is transitive

Thus, R is an equivalence relation

Equivalence classes:

$[1] = \{1, 2\} = [2]$ $\quad (\because 1 \sim 2$ and $3 \sim 4)$

$[3] = \{3, 4\} = [4]$

Ex 3: If $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{1, 2\}$ $A_2 = \{2, 3, 4\}$ and $A_3 = \{5\}$. Define the relation R on A by xRy iff x and y are in same set A_i, $i = 1, 2, 3$. Is this relation equivalence?

Soln: Relation is

$$R = \{(1,1), (2,2), (1,2), (2,1), (3,3), (3,4), (2,3), (3,2), (3,4), (4,3), (2,4), (4,2), (5,5)\}$$

i) It is clear that for any $a \in A$, $(a,a) \in R$

$\therefore$ R is reflexive.

ii) For any $a, b \in R$ if $(a,b) \in R$

$\Rightarrow a, b \in A_j$ for some i

$\Rightarrow (b,a) \in R$

ii) For any $a, b, c \in R$ if $(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow a, b \in A_i$ and $b, c \in A_j$

for some i, j

It is not necessary that $a, c \in A_k$ for some k

Counter example $(1,2) \in R$ and $(2,3) \in R$

$\Rightarrow 1, 2 \in A_1$ and $2, 3 \in A_2$

But $1, 3 \notin A_i$ for any i

$\therefore (1,3) \notin R$

$\therefore$ R is not transitive.

Thus R is not equivalence.

Ex 4: Let $A = \{1, 2, 3, 4, 5\}$. Define a relation R on $A \times A$ by $(x_1, y_1)\ R\ (x_2, y_2)$ iff $x_1 + y_1 = x_2 + y_2$. Verify that R is an equivalence relation.

Soln: i) For any $(a,b) \in A \times A$, $(a,b)\ R\ (a,b)$

$\therefore$ R is reflexive.

ii) For any $(a,b), (c,d) \in A \times A$

if $(a,b)\ R\ (c,d)$

$\Rightarrow a+b = c+d$

$\Rightarrow c+d = a+b$

$\Rightarrow (c,d)\ R\ (a,b)$

$\therefore$ R is Symmetric

iii) For any $(a,b), (c,d), (e,f) \in A \times A$

if $(a,b)\ R\ (c,d)$ and $(c,d)\ R\ (e,f)$

$\Rightarrow a+b = c+d$ and $c+d = e+f$

$\Rightarrow a+b = e+f$

$\Rightarrow (a,b)\ R\ (e,f)$

$\therefore$ R is transitive.

Thus, R is an equivalence relation.

Ex 5: Let R be a relations on set of all integers.

$R = \{ (a,b) \mid a-b \in \mathbb{Z} \}$. Is this relation R equivalence ?

Soln: i) Clearly $\forall a \in \mathbb{R}$ $(a,a) \in R$, because $a-a=0 \in \mathbb{Z}$,

$\therefore$ R is reflexive

ii) For any $a,b \in \mathbb{R}$ if $(a,b) \in R$, that is $a-b \in \mathbb{Z}$

$\Rightarrow b-a \in \mathbb{Z}$

$\Rightarrow (b,a) \in R$

$\therefore$ R is symmetric.

iii) Assume $(a,b) \in R$ and $(b,c) \in R$, where $a,b,c \in \mathbb{R}$

$\Rightarrow a-b \in \mathbb{Z}$ and $b-c \in \mathbb{Z}$

$\Rightarrow (a-b)+(b-c) \in \mathbb{Z}$

$\Rightarrow a-c \in \mathbb{Z}$

$\Rightarrow (a,c) \in R$

$\therefore$ R is transitive

Thus, R is an equivalence relation.

For any $a \in \mathbb{R}$

$$[a] = \{ b \mid b-a \in \mathbb{Z} \}$$

$$= \{ b \mid b-a = k, \text{ for some } k \in \mathbb{Z} \}$$

$$= \{ b \mid b = a+k, \ k \in \mathbb{Z} \}$$

$$= \{ a+k, \text{ for any } k \in \mathbb{Z} \}$$

In particular,

$[0] = \{ k, \text{ for all } k \in \mathbb{Z} \} = \{ \cdots, -3, -2, -1, 0, 1, 2, \cdots \}$

$[1.2] = \{ 1.2 + k, \; k \in \mathbb{Z} \} = \{ \cdots, -0.8, 0.2, 1.2, 2.2, 3.2, \cdots \}$

$[1.723] = \{ 1.723 + k, \; k \in \mathbb{Z} \} = \{ \cdots, -0.277, 0.723, 1.723, 2.723, \cdots \}$

Ex 6: Congruence modulo m. Let m be an integer $m \geq 1$ S.T the relation $R = \{ (a,b) \mid a \equiv b \pmod{m} \}$ is an equivalence relation on the set of integers.

Find all distinct equivalence classes of R on $\mathbb{Z}$.

Soln: We say $a \equiv b \pmod{m}$ if $m \mid a-b$

or $a - b = km$ for some $k \in \mathbb{Z}$.

i) For any $a \in \mathbb{Z}$, we have $(a,a) \in R$

because $a \equiv a \pmod{m}$. $\therefore$ R is reflexive

ii) Suppose for any $a, b \in \mathbb{Z}$ if $(a,b) \in R$

$\Rightarrow a \equiv b \pmod{m}$

$\Rightarrow m \mid (a-b)$

$\Rightarrow a = b + km$ for some $k \in \mathbb{Z}$

$\Rightarrow b = a + (-k)m$

$\Rightarrow b \equiv a \pmod{m}$

$\therefore$ R is symmetric.

iii) Suppose if $(a,b) \in R$ and $(b,c) \in R$, for any $a, b, c \in \mathbb{Z}$

$\Rightarrow a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

$\Rightarrow m \mid (a-b)$ and $m \mid (b-c)$

$\Rightarrow m \mid (a-b+b-c)$

$\Rightarrow m \mid (a-c)$

$\Rightarrow a \equiv c \pmod{m}$ or $(a,c) \in R$

$\therefore$ R is transitive.

Thus R is an equivalence relation.

Equivalence classes: Let $a \in \mathbb{Z}$ be any integer. Then

$[a]_m = \{x \mid x \sim a\}$

$[2] = 7k+2$

$= \{x \mid x \equiv a \pmod{m}\}$

$= \{x \mid x - a = mk,\ k \in \mathbb{Z}\}$

$= \{mk+a,\ k \in \mathbb{Z}\}$

In particular,

$[0]_m = \{mk,\ k \in \mathbb{Z}\} = \{\cdots, -2m, -m, 0, m, 2m, \cdots\}$

$[1]_m = \{mk+1,\ k \in \mathbb{Z}\} = \{\cdots, -2m+1, -m+1, 1, m+1, 2m+1, \cdots\}$

$[2]_m = \{mk+2,\ k \in \mathbb{Z}\} = \{\cdots, -2m+2, -m+2, 2, m+2, 2m+2, \cdots\}$

$\vdots$

$[m-1]_m = \{mk+m-1,\ k \in \mathbb{Z}\}$

$[m]_m = [0]$

$[m+1]_m = [1]$

$\therefore$ Distinct equivalence classes are

$[0], [1], [2], [3], \cdots, [m-1]$

when $m=5$, Equivalence classes are

$[0]_5 = \{5k,\ k \in \mathbb{Z}\}$

$[1]_5 = \{5k+1,\ k\in\mathbb{Z}\}$

$[2]_5 = \{5k+2,\ k\in\mathbb{Z}\}$

$[3]_5 = \{5k+3,\ k\in\mathbb{Z}\}$

$[4]_5 = \{5k+4,\ k\in\mathbb{Z}\}$

Note: 1) If $b\in[a]$ or $b\sim a$, then $[a]=[b]$

2) If a is not equivalent to b, then $[a]\cap[b]=\phi$

Ex 7: Let R be a relation defined on the integers.

$$R=\{(a,b)\ |\ a \text{ divides } b\}$$

Show that R is not equivalence relation

Soln: It is not symmetric (verify)

$\therefore$ it is not an equivalence relation

Ex 8: Let R be a relation on the real no.s

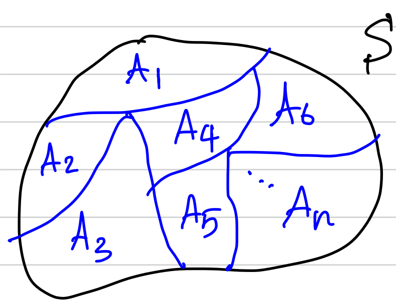
$R=\{(a,b)\ |\ |a-b|<1\}$. S.T R is not an equivalence relation

Exercise

Defn [Partition of a set]:

Let S be a set and $A_1, A_2, \dots, A_n$ be disjoint subsets of S i.e, $A_i\cap A_j=\phi$ when $i\neq j$.

Then this collection of subsets $\{A_1, A_2, \dots, A_n\}$ is called partition of S if $S=\bigcup_i A_i = A_1\cup A_2\cup A_3\cup\cdots\cup A_n$



Ex: Let $S = \{1, 2, 6, 7, 9, 11\}$ be a set and

$A_1 = \{1, 2\}$, $A_2 = \{6\}$, $A_3 = \{9, 11, 7\}$ be subsets of S.

Clearly A_1, A_2, A_3 are disjoint subsets, and

$$S = A_1 \cup A_2 \cup A_3$$

Thus, $\{A_1, A_2, A_3\}$ is called partition of S.

Note:

- Let R be an equivalence relation on a set A. Then

$$A = \bigcup_R [a]_R$$

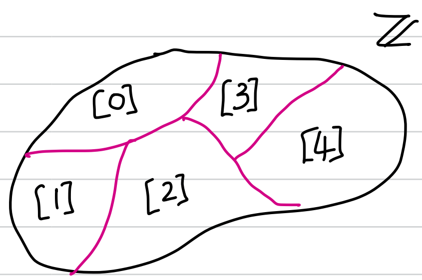
- We know that two equivalence classes are either disjoint or equal.
- $\therefore$ Equivalence classes partition A.
- Conversely, given a partition $\{A_1, A_2, \ldots, A_n\}$ of a set A, then there is a equivalence relation R that has the sets A_i as its equivalence classes.

Ex: Disjoint equivalence Classes of Congruence modulo 5 are

$[0]_5$, $[1]_5$, $[2]_5$, $[3]_5$, $[4]_5$ and

$$\mathbb{Z} = [0]_5 \cup [1]_5 \cup [2]_5 \cup [3]_5 \cup [4]_5$$

$\therefore$ We say $\{[0], [1], [2], [3], [4]\}$ is partition of $\mathbb{Z}$.



Ex: In the above Ex 2, the partition of $A = \{1,2,3,4\}$ under the given R is $P = \{[1], [3]\}$, sine $A = [1] \cup [3]$.

Ex: Let $A = \{1,2,3,4,5,6,7\}$ and R be the equivalence relation on A that induces the partition $A = \{1,2\} \cup \{3\} \cup \{4,5,7\} \cup \{6\}$. Find R.

Soln: Given partition of A has 4 blocks $\{1,2\}$, $\{3\}$, $\{4,5,7\}$ and $\{6\}$.

Let R be an equivalence relation. Then

$[1] = \{1,2\} = [2]$

$[3] = \{3\}$

$[4] = \{4,5,7\} = [5] = [7]$

$[6] = \{6\}$

$\therefore$ $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4), (5,5), (7,7),$
$(4,5), (5,4), (5,7), (7,5), (4,7), (7,4), (6,6)\}$

Partial ordering

We can use relations to order some or all of the elements of sets.

Ex1 : Let R be a relation on a set $A = \{1, 2, 4, -1, 3, 11\}$.

$$R = \{(a, b) \mid a \leq b\}$$

R is 'less than or equal' (denoted by $\leq$).

This relation R will order all elements of the set.

Observe that
i) R is reflexive. ($\because a \leq a \ \forall a \in A$)
ii) R is not symmetric ($\because$ if $a \leq b$, that $b \nleq a$)
iii) R is anti symmetric (if $a \leq b$ and $b \leq a$, then $a = b$)
iv) R is transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$)

Ex 2: Let R be a relation on a power set of a set S.

$$R = \{(A, B) \mid A \subseteq B\}$$

Observe that R is
i) Reflexive
ii) Antisymmetric
iii) transitive

Here R order some subset in the collection of $P(S)$

For instance, let $S = \{a, b, c\}$

$$P(S) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \phi, \{a, b, c\}\}$$

Clearly,

$$\phi \subseteq \{a\} \subseteq \{a, b\} \subseteq \{a, b, c\}, \quad \phi \subseteq \{a\} \subseteq \{a, c\} \subseteq \{a, b, c\}$$

$\phi \subseteq \{b\} \subseteq \{a,b\} \subseteq \{a,b,c\}$ | $\phi \subseteq \{c\} \subseteq \{a,c\} \subseteq \{a,b,c\}$

$\phi \subseteq \{b\} \subseteq \{b,c\} \subseteq \{a,b,c\}$ | $\phi \subseteq \{c\} \subseteq \{b,c\} \subseteq \{a,b,c\}$

Defn: A relation R on a set S is called a partial ordering or partial order if it is

i) Reflexive ii) Antisymmetric and iii) Transitive

Set S together with a partial ordering R is called Partial ordered set or POSET and is denoted by (S, R)

- Let R be a partial order on S. For $a, b \in S$, aRb is denoted by $a \preceq b$.
- POSET, (S, R) is written as $(S, \preceq)$
- Two elements a and b of a POSET $(S, \preceq)$ are called comparable iff either $a \preceq b$ or $b \preceq a$.

Defn: If $(S, \preceq)$ is a POSET and every two elements are comparable, S is called total ordered set.

A total ordered set is also called chain.

less than or equal to

Ex 3: The POSET $(\mathbb{Z}, \leq)$ is totally ordered.

Because $a \leq b$ or $b \leq a$ whenever a and b are integers.

Ex 4: Let R be a relation on set of +ve integers $\mathbb{Z}^+$,

$$R = \{(a,b) : a \mid b\}$$

Is $(\mathbb{Z}^+, |)$ a POSET or totally ordered set?

Note that a divide b i.e $a \mid b$ iff there exist $k \in \mathbb{Z}$ such that $b = ka$.

Soln: i) For any $a \in \mathbb{Z}^+$ $a \mid a$ i.e $(a,a) \in R$

$\therefore$ R is reflexive

ii) Suppose for $a, b \in \mathbb{Z}^+$

if $a \mid b$ and $b \mid a$

$\Rightarrow b = ak_1$ and $a = k_2 b$ for some $k_1, k_2 \in \mathbb{Z}^+$

$\Rightarrow b = k_1 k_2 b$

$\Rightarrow k_1 = k_2 = 1$ or $a = b$

$\therefore$ R is antisymmetric

iii) Suppose for $a, b, c \in \mathbb{Z}$ if

$a \mid b$ and $b \mid c$

$\Rightarrow b = k_1 a$ and $c = k_2 b$ for some $k_1, k_2 \in \mathbb{Z}^+$

$\Rightarrow c = k_2 (k_1 b)$

$\Rightarrow c = (k_1 k_2) b$

$\Rightarrow c = k_3 b$ $\quad k_3 = k_1 k_2 \in \mathbb{Z}^+$

$\Rightarrow b \mid c$

$\therefore$ R is transitive

Thus the relation $(\mathbb{Z}^+, \mid)$ is a POSET.

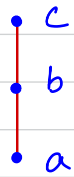
It is not totall ordered because $\exists$ $x, y \in \mathbb{Z}^+$ such that $x \nmid y$ and $y \nmid x$, for instance $5 \nmid 7$ and $7 \nmid 5$.

Remark: A partial ordering relation R on a set A or a POSET (A, R) can be represented by the Hasse diagram.

Steps involve in Hasse diagram:

- It is directed from bottom to top.
- If $a \preceq b$, then it is represented as

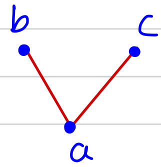
- If $a \preceq b$ and $b \preceq c$, then it is represented by



  Note: No edge must be drawn from a to c.

- In Hasse diagram, if there is a path from a to b, the we can say $a \preceq b$.

  If $a \preceq b$, $a \preceq c$ and $b \npreceq c$, then it is represented by



  Note: b and c are in same level.

Ex 1: Let R be a relation on a set $A = \{2, 5, 9, 12, 32, 4\}$

$$R = \{(a,b) \mid a \leq b\}$$ less than or equal to

Draw the Hasse diagram representing R on A.

Soln: Clearly $(A, \leq)$ is a POSET. In particular, a totally ordered set.

$R = \{(2,2), (2,5), (2,9), (2,12), (2,32), (2,4), (4,4), (4,5), (4,9),$
$(4,12), (4,32), (5,5), (5,9), (5,12), (5,32), (9,9), (9,12), (9,32),$
$(12,12), (12,32), (32,32)\}$

Clearly $2 \leq 4 \leq 5 \leq 9 \leq 12 \leq 32$

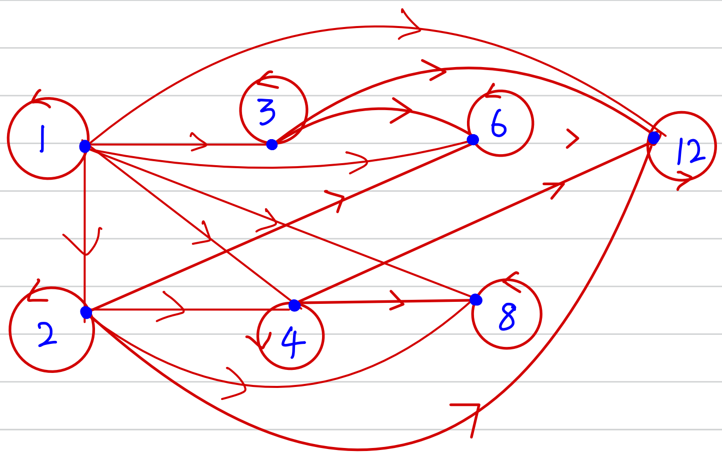
Hasse diagram:

Ex 2: Draw the Hasse diagram representing the partial ordering, $R = \{(a,b) \mid a \text{ divides } b\}$ on $A = \{1, 2, 3, 4, 6, 8, 12\}$

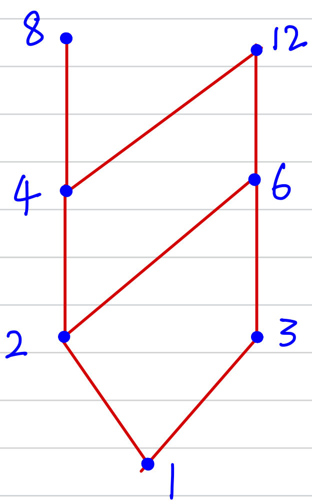
Soln: Partial order is

$\{(1,1), (1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (2,2), (2,4), (2,6), (2,8), (2,12), (3,3), (3,6), (3,12), (4,4), (4,8), (4,12), (6,6), (6,12), (8,8), (12,12)\}$

Digraph:



Hasse diagram:



Clearly,

$1 \preceq 2 \preceq 4 \preceq 8$

$1 \preceq 2 \preceq 6 \preceq 12$

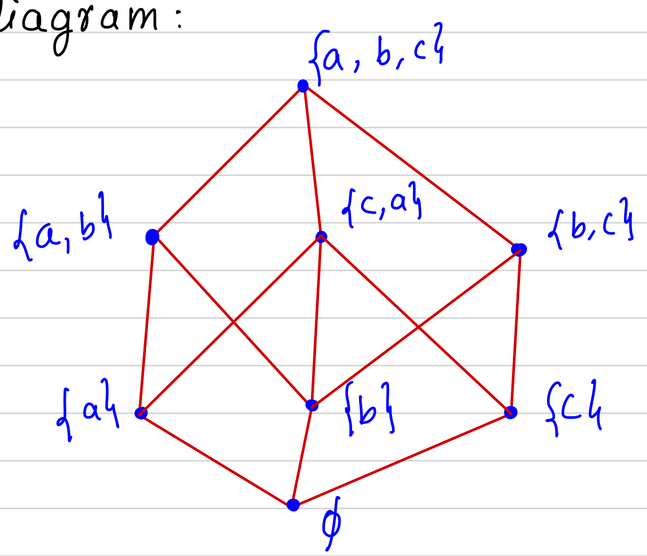
$1 \preceq 2 \preceq 4 \preceq 12$

$1 \preceq 3 \preceq 6 \preceq 12$

Ex 3: Draw Hasse diagram for the partial ordering $\{(A,B) \mid A \subseteq B\}$ on $P(S)$ where $S = \{a,b,c\}$.

Soln: $P(S) = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \phi, \{a,b,c\}\}$

Hasse diagram:



Ex 4: Consider the partial order of divisibility on the given set.

Draw the Hasse diagram for the POSET

Determine whether the POSET is totally ordered or not.

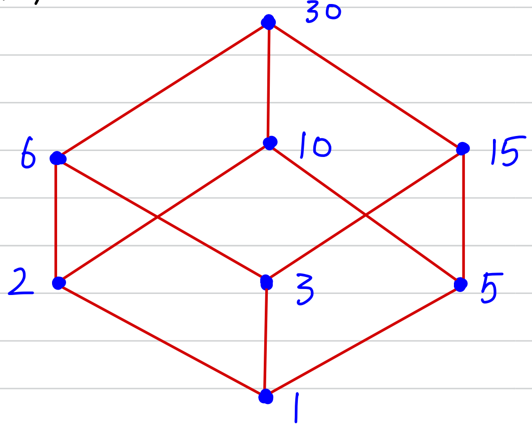
i) $A_1 = \{1, 2, 3, 5, 6, 10, 15, 30\}$

ii) $A_2 = \{1, 3, 6, 12, 24\}$

Soln: i) Partial order is

$R_1 = \{(1,1), (2,2), (3,3), (5,5), (6,6), (10,10), (15,15), (30,30),$
$(1,2), (1,3), (1,5), (1,6), (1,10), (1,15), (1,30), (2,6), (2,10),$
$(2,30), (3,6), (3,15), (3,30), (5,10), (5,15), (5,30),$
$(6,30), (10,30), (15,30)\}$

Hasse diagram :



It is not totally ordered, $\because$ $3 \nmid 10$.

ii) Partial order is

$R = \{(1,1), (3,3), (6,6), (12,12), (24,24), (1,3), (1,6), (1,12),$
$(1,24), (3,6), (3,12), (3,24), (6,12), (6,24), (12,24)\}$

It is totally ordered.

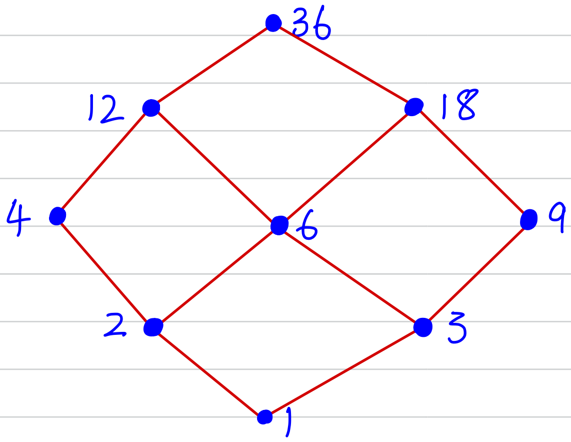
Ex 5 : Draw the Hasse diagram

i) Representing the divisibility relation on the positive divisors of 36

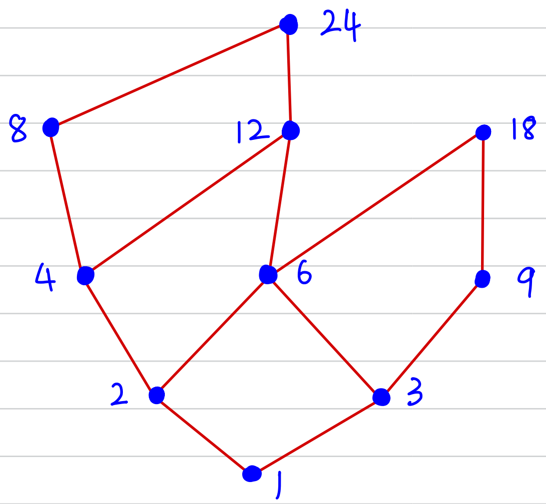
ii) For the divisibility relation on the set
$A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$

Soln : i) Divisors of 36 , $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

Hasse diagram



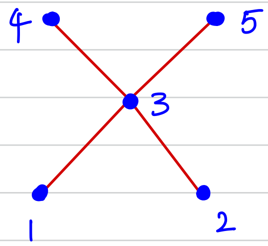
ii) Hasse diagram



Ex 6: Draw the Hasse diagram of the partial order R on $A = \{1, 2, 3, 4, 5\}$ whose matrix representation is

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Soln: $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5)\}$



Maximal, minimal, greatest and least element in a POSET. $(S, \preceq)$

Consider the following Hasse diagrams

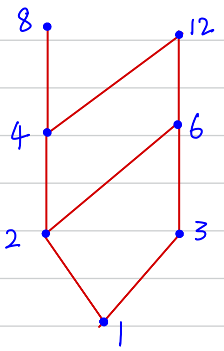


fig 1

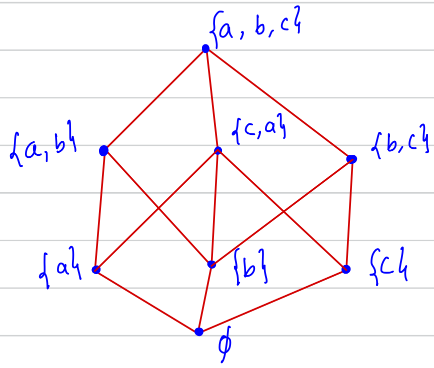


fig 2

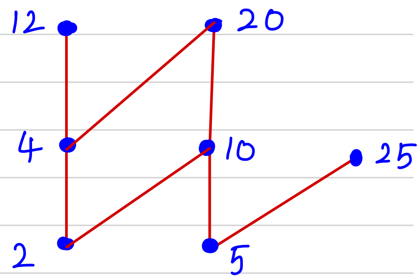


fig 3

Maximal element	An element a is maximal in the POSET $(S, \preceq)$ if there is no $b \in S$ $(b \neq a)$ such that $a \preceq b$	maximal els in fig1 : 8, 12 fig2 : $\{a, b, c\}$ fig 3 : 12, 20, 25
Minimal element	An element a is called minimal in the POSET $(S, \preceq)$ if there is no $b \in S$ $(b \neq a)$ such that $b \preceq a$ Maximal and minimal elements are not unique.	minimal els in fig1 : 1 fig2 : ϕ fig 3 : 2, 5
Greatest element	An element a is the greatest of the POSET $(S, \preceq)$ if $b \preceq a$ for all $b \in S$.	Greatest el. in fig1 : No greatest el. fig2 : $\{a, b, c\}$ fig3 : No greatest el.
Least element	An element a is the least element of the POSET $(S, \preceq)$ if $a \preceq b$ $\forall$ $a \in S$. Greatest and least el. if it exist, then it is unique.	least el. in fig1 : 1 fig2 : ϕ fig 3 : No least el.

Ex1: What is the greatest and least element in the POSET $(\mathbb{Z}^+, |)$?

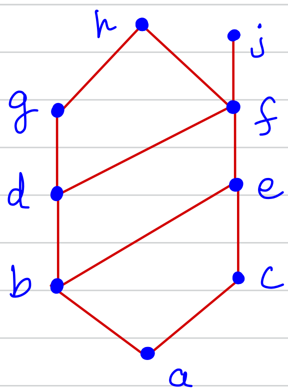
Answer: There is no greatest element.

Least element is 1.

Definition [Upper bound]: Let $(S, \leq)$ be a POSET and let T be a subset of S. For any $x \in S$, x is called upper bound of T if $u \leq x \;\; \forall\, u \in T$.

Defn [Lower bound]: Let $(S, \leq)$ be a POSET and let T be a subset of S, For any $x \in S$, x is called lower bound of T if $x \leq l \;\; \forall\, x \in T$.

Ex2: Let $S = \{a, b, c, d, e, f, g, h, j\}$ and $(S, \preceq)$ be a poset given in the below Hasse diagram.



Find the Lower and upper bound of

i) $\{a, b, c\}$ ii) $\{j, h\}$

Soln: i) Upper bounds are : e, f, h, j

Lower bounds are : only a

ii) upper bounds are : No elements

lower bounds are : $a, b, c, d, e,$ and f

Let $(S, \leq)$ be a POSET and let T be a subset of S.

Defn: [Least upper bound (lub)] : The element x is called lub of T if

i) x is an upper bound of T

ii) x is less than every other upper bound of T.

In other words, if y is any other upper bound of T, then $x \leq y$.

Defn: [Greatest lower bound (glb)] : The element x is called glb of T if

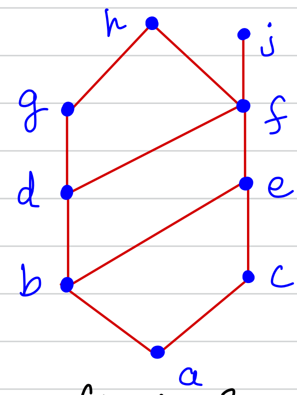
i) x is a lower bound of T.

ii) If y is any other lower bound of T, then $y \leq x$.

Note: lub and glb if they exist, then it is unique.

Ex3: Find the glb and lub of $\{b, d, g\}$, if they exist, in the POSET shown in EX2.

Ans: Ex2 is



lower bounds of $\{b, d, g\}$ are : a and b

glb is b. (Because $a < b$)

Upper bounds of $\{b, d, g\}$ are : g and h

lub is g (Since $g \preceq h$)

Ex 4 : Find the glb and lub of the sets $\{3, 9, 12\}$ if they exist in the POSE $(\mathbb{Z}^+, |)$

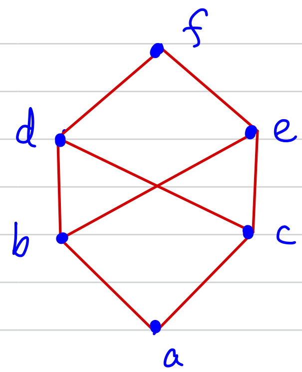
Soln : Lower bounds of $\{3, 9, 12\}$ are : 1 and 3 ~ divisors of $\{3,9,12\}$

glb is 3 ~ gcd of $\{3,9,12\}$

Upper bounds of $\{3, 9, 12\}$ are : $36, 72, 108, \ldots$ ~ multiples of $\{3,9,12\}$

lub is 36 ~ lcm of $\{3,9,12\}$

Ex 5 : Find lub of $\{b, c\}$



Soln : upper bounds of $\{b, c\}$ are : e, d and f

lub doesnot exist.

Because $d \not\preceq e$ and $e \not\preceq d$

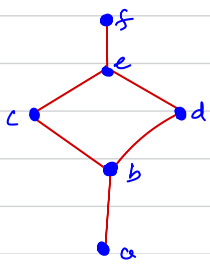
Lattices

A POSET in which every pair of elements has both a lub and a glb is called a lattice.

Ex : Is the poset $(\mathbb{Z}^+, |)$ a lattice?

Ans : It is a lattice. Because any two +ve integers a and b has gcd and lcm (that is, glb and lub).

Ex : Consider



Is this a lattice?

Ans : Yes (Verify)