Unit 2 - Mathematical Logic

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1 Propositional Logic: Basics

1.1 Basic Definitions

A **proposition** is a declarative sentence that is either true or false, but not both.

1.2 Examples

The following are examples of propositions because they are declarative sentences with a definite truth value:

1. $2 + 2 = 4$
   Explanation: This is a proposition with truth value 'True (1)'.
2. The sky is blue.
   Explanation: This is a declarative sentence that is true under normal daylight conditions.
3. $5 > 10$
   Explanation: This is a proposition with truth value 'False (0)'.
4. All birds can fly.
   Explanation: This is a proposition with truth value 'False (0)'.

The following are **not propositions** because they do not have a definite truth value:

1. What time is it?
   Explanation: This is a question, not a declarative sentence.
2. $x + 1 = 5$
   Explanation: This is not a proposition because it contains a variable x, and its truth value depends on the value of x.
3. Let's go to the park!
   Explanation: This is an imperative sentence, not a declarative sentence.
4. This statement is false.
   Explanation: This is a paradox and cannot be classified as true or false.

Propositions are denoted by letters such as p, q, r.

1.3 Logical connectives

- **Negation** ($\neg p$): "Not p".
- **Conjunction** ($p \wedge q$): "p and q".
- **Disjunction** ($p \vee q$): "p or q".
- **Conditional** ($p \rightarrow q$): "If p, then q".
- **Biconditional** ($p \leftrightarrow q$): "p if and only if q".

1.3.1 Truth Tables

Truth tables are used to determine the truth value of compound propositions based on the truth values of their components.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	0	0
1	1	0	1	1	1	1

1.4 Tautologies and Contradictions

- A **tautology** (T_0 / 1) is a proposition that is always true (e.g., $p \vee \neg p$).
- A **contradiction** (F_0 / 0) is a proposition that is always false (e.g., $p \wedge \neg p$).
- A **contingency** is a proposition that is neither a tautology nor a contradiction.

1.4.1 Problems

1. Identify whether the following sentences are propositions. If they are, determine their truth value.

   (a) $12 + 3 = 4$

   (b) $x - 27 = 13$

   (c) $13 \times 5 = 65$

   (d) How beautiful the sunset is!

   (e) This statement is false.

2. Construct the truth table for the following compound propositions and determine whether they are tautologies, contradictions, or contingencies:

   (a) $p \vee \neg p$
   Solution:

p	$\neg p$	$p \vee \neg p$
0	1	1
1	0	1

$\therefore p \vee \neg p$ is a tautology.

(b) $\neg p \vee q$

Solution:

p	q	$\neg p$	$\neg p \vee q$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

$\therefore \neg p \vee q$ is a contingency.

(c) $(p \to q) \leftrightarrow (\neg q \to \neg p)$

Solution:

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	1
1	1	1	0	0	1	1

$\therefore (p \to q) \leftrightarrow (\neg q \to \neg p)$ is a tautology.

(d) $p \wedge (q \vee \neg r)$

Solution:

p	q	r	$\neg r$	$q \vee \neg r$	$p \wedge (q \vee \neg r)$
0	0	0	1	1	0
0	0	1	0	0	0
0	1	0	1	1	0
0	1	1	0	1	0
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	1	1

$\therefore p \wedge (q \vee \neg r)$ is a contingency.

3. Exercise:

(a) Show that $p \wedge \neg p$ is a contradiction.

(b) Check if $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology, contradiction, or contingency.

2 Logical Equivalence

Two propositions p and q are **logically equivalent** ($p \iff q$) if they have the same truth values in all cases. Examples:

$$p \to q \iff \neg p \lor q$$

$$p \leftrightarrow q \iff (p \to q) \land (q \to p)$$

2.0.1 Problems

(a) Show that $p \leftrightarrow q$ is logically equivalent to $(p \to q) \land (q \to p)$ using a truth table.
Solution:

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \land (q \to p)$
0	0	1	1	1	1
0	1	0	1	0	0
1	0	0	0	1	0
1	1	1	1	1	1

Since the columns for $p \leftrightarrow q$ and $(p \to q) \land (q \to p)$ are identical, the two expressions are **logically equivalent**.

(b) Verify whether $(p \land q) \lor r$ is logically equivalent to $(p \lor r) \land (q \lor r)$ using a truth table.
Solution:

p	q	r	$p \land q$	$(p \land q) \lor r$	$p \lor r$	$q \lor r$	$(p \lor r) \land (q \lor r)$
0	0	0	0	0	0	0	0
0	0	1	0	1	1	1	1
0	1	0	0	0	0	1	0
0	1	1	0	1	1	1	1
1	0	0	0	0	1	0	0
1	0	1	0	1	1	1	1
1	1	0	1	1	1	1	1
1	1	1	1	1	1	1	1

Since the columns for $(p \land q) \lor r$ and $(p \lor r) \land (q \lor r)$ are identical, the two expressions are **logically equivalent**.

(c) Show that $\neg(p \lor q \lor r)$ is logically equivalent to $\neg p \land \neg q \land \neg r$ (De Morgan's Law for three propositions) using a truth table.
Solution:

p	q	r	$p \vee q \vee r$	$\neg(p \vee q \vee r)$	$\neg p$	$\neg q$	$\neg r$	$\neg p \wedge \neg q \wedge \neg r$
0	0	0	0	1	1	1	1	1
0	0	1	1	0	1	1	0	0
0	1	0	1	0	1	0	1	0
0	1	1	1	0	1	0	0	0
1	0	0	1	0	0	1	1	0
1	0	1	1	0	0	1	0	0
1	1	0	1	0	0	0	1	0
1	1	1	1	0	0	0	0	0

Since the columns for $\neg(p \vee q \vee r)$ and $\neg p \wedge \neg q \wedge \neg r$ are identical, the two expressions are **logically equivalent**.

2.1 Exercise

(a) Determine whether $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$ (De Morgan's Law) using truth table.

(b) Check if $p \to (q \wedge r) \iff (p \to q) \wedge (p \to r)$ using truth table.

2.2 Principle of Duality

The **Principle of Duality** states that any Boolean expression or logical statement remains valid if:

- The logical operators $\wedge$ (AND) and $\vee$ (OR) are interchanged.
- The identity elements 1 (True) and 0 (False) are interchanged.

2.2.1 Examples of Duality

- Distributive law: $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$
  Dual Distributive law: $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
- Original Identity: $p \wedge 1 \iff p$
  Dual Identity: $p \vee 0 \iff p$

3 Laws of Logic

The laws of logic are fundamental rules that help simplify and manipulate logical expressions. These laws are used to prove logical equivalence and to transform complex logical statements into simpler forms.

(a) **Identity Laws**:

$$p \wedge \text{True} \iff p \quad \text{and} \quad p \vee \text{False} \iff p$$

(b) **Domination Laws**:

$$p \lor \text{True} \iff \text{True} \quad \text{and} \quad p \land \text{False} \iff \text{False}$$

(c) **Idempotent Laws**:

$$p \lor p \iff p \quad \text{and} \quad p \land p \iff p$$

(d) **Double Negation Law**:

$$\neg(\neg p) \iff p$$

(e) **Commutative Laws**:

$$p \lor q \iff q \lor p \quad \text{and} \quad p \land q \iff q \land p$$

(f) **Associative Laws**:

$$(p \lor q) \lor r \iff p \lor (q \lor r) \quad \text{and} \quad (p \land q) \land r \iff p \land (q \land r)$$

(g) **Distributive Laws**:

$$p \lor (q \land r) \iff (p \lor q) \land (p \lor r) \quad \text{and} \quad p \land (q \lor r) \iff (p \land q) \lor (p \land r)$$

(h) **De Morgan's Laws**:

$$\neg(p \land q) \iff \neg p \lor \neg q \quad \text{and} \quad \neg(p \lor q) \iff \neg p \land \neg q$$

(i) **Absorption Laws**:

$$p \lor (p \land q) \iff p \quad \text{and} \quad p \land (p \lor q) \iff p$$

(j) **Negation Laws**:

$$p \lor \neg p \iff \text{True} \quad \text{and} \quad p \land \neg p \iff \text{False}$$

(k) **Conditional**:

$$p \to q \iff \neg p \lor q$$

3.1 Proving Logical Equivalence Using Laws of Logic

To prove that two logical expressions are equivalent, we can use the laws of logic to transform one expression into the other.

3.1.1 Prove $p \to q \iff \neg q \to \neg p$ (Contrapositive)

Solution:

$$\begin{aligned} p \to q &\iff \neg p \lor q \quad \text{(Conditional)} \\ &\iff q \lor \neg p \quad \text{(Commutative Law)} \\ &\iff \neg(\neg q) \lor \neg p \quad \text{(Double Negation)} \\ &\iff \neg q \to \neg p \quad \text{(Conditional)} \end{aligned}$$

Thus, $p \to q \iff \neg q \to \neg p$.

3.1.2 Prove $p \leftrightarrow q \iff (p \wedge q) \vee (\neg p \wedge \neg q)$

Solution:

$$\begin{aligned}
p \leftrightarrow q &\iff (p \to q) \wedge (q \to p) \quad \text{(Biconditional)} \\
&\iff (\neg p \vee q) \wedge (\neg q \vee p) \quad \text{(Conditional)} \\
&\iff (\neg p \wedge \neg q) \vee (\neg p \wedge p) \vee (q \wedge \neg q) \vee (q \wedge p) \quad \text{(Distributive Law)} \\
&\iff (\neg p \wedge \neg q) \vee \text{False} \vee \text{False} \vee (q \wedge p) \quad \text{(Negation Law)} \\
&\iff (\neg p \wedge \neg q) \vee (p \wedge q) \quad \text{(Identity Law)} \\
&\iff (p \wedge q) \vee (\neg p \wedge \neg q) \quad \text{(Commutative Law)}
\end{aligned}$$

Thus, $p \leftrightarrow q \iff (p \wedge q) \vee (\neg p \wedge \neg q)$.

3.1.3 Prove $\neg(p \vee (q \wedge r)) \iff \neg p \wedge (\neg q \vee \neg r)$

Solution:

$$\begin{aligned}
\neg(p \vee (q \wedge r)) &\iff \neg p \wedge \neg(q \wedge r) \quad \text{(De Morgan's Law)} \\
&\iff \neg p \wedge (\neg q \vee \neg r) \quad \text{(De Morgan's Law)}
\end{aligned}$$

Thus, $\neg(p \vee (q \wedge r)) \iff \neg p \wedge (\neg q \vee \neg r)$.

3.1.4 Prove $(p \to q) \wedge (q \to r) \iff p \to r$ (Hypothetical Syllogism)

Solution:

$$\begin{aligned}
(p \to q) \wedge (q \to r) &\iff (\neg p \vee q) \wedge (\neg q \vee r) \quad \text{(Conditional)} \\
&\iff (\neg p \wedge \neg q) \vee (\neg p \wedge r) \vee (q \wedge \neg q) \vee (q \wedge r) \quad \text{(Distributive Law)} \\
&\iff (\neg p \wedge \neg q) \vee (\neg p \wedge r) \vee \text{False} \vee (q \wedge r) \quad \text{(Negation Law)} \\
&\iff (\neg p \wedge \neg q) \vee (\neg p \wedge r) \vee (q \wedge r) \quad \text{(Identity Law)} \\
&\iff \neg p \wedge (\neg q \vee r) \vee (q \wedge r) \quad \text{(Distributive Law)} \\
&\iff \neg p \vee (q \wedge r) \quad \text{(Simplification)} \\
&\iff \neg p \vee r \quad \text{(Absorption Law)} \\
&\iff p \to r \quad \text{(Conditional)}
\end{aligned}$$

Thus, $(p \to q) \wedge (q \to r) \iff p \to r$.

3.1.5 Prove $\neg(p \leftrightarrow q) \iff (p \wedge \neg q) \vee (\neg p \wedge q)$

Solution:

$$\begin{aligned}
\neg(p \leftrightarrow q) &\iff \neg\big((p \to q) \wedge (q \to p)\big) \quad \text{(Biconditional)} \\
&\iff \neg(p \to q) \vee \neg(q \to p) \quad \text{(De Morgan's Law)} \\
&\iff \neg(\neg p \vee q) \vee \neg(\neg q \vee p) \quad \text{(Conditional)} \\
&\iff (p \wedge \neg q) \vee (q \wedge \neg p) \quad \text{(De Morgan's Law)} \\
&\iff (p \wedge \neg q) \vee (\neg p \wedge q) \quad \text{(Commutative Law)}
\end{aligned}$$

Thus, $\neg(p \leftrightarrow q) \iff (p \wedge \neg q) \vee (\neg p \wedge q)$.

3.1.6 Exercise: Prove $(p \vee q) \to r \iff (p \to r) \wedge (q \to r)$

3.1.7 Exercise: Prove $(p \to q) \vee (p \to r) \iff p \to (q \vee r)$

3.1.8 Exercise: Prove $(p \to q) \wedge (r \to s) \iff (p \vee r) \to (q \vee s)$

4 Converse, Inverse, and Contrapositive

Given a conditional statement $p \to q$:

- The **converse** is $q \to p$.
- The **inverse** is $\neg p \to \neg q$.
- The **contrapositive** is $\neg q \to \neg p$.

4.1 Problems: Write the converse, inverse and contrapositive for the following conditionals

4.1.1 If a number is even, then it is divisible by 2.

(a) **Converse**: "If a number is divisible by 2, then it is even."

(b) **Inverse**: "If a number is not even, then it is not divisible by 2."

(c) **Contrapositive**: "If a number is not divisible by 2, then it is not even."

4.1.2 If a shape is a square, then it is a rectangle.

(a) **Converse**: "If a shape is a rectangle, then it is a square."

(b) **Inverse**: "If a shape is not a square, then it is not a rectangle."

(c) **Contrapositive**: "If a shape is not a rectangle, then it is not a square."

4.1.3 Exercise: If you study hard, then you will pass the exam.

4.1.4 Exercise: If it rains, then the ground becomes wet.

5 Rules of Inference

Rules of inference are logical tools used to derive conclusions from premises, in other words, to check the validity of an argument. They are fundamental in constructing valid arguments and proofs in mathematics and computer science.

An **argument** is a sequence of statements where one or more statements, called **premises**, are used to support another statement, called the **conclusion**. The goal is to determine whether the conclusion logically follows from the premises.

5.1 Structure of an Argument

- **Premises**: The statements that provide evidence or reasons.
- **Conclusion**: The statement that is being argued for.

$$\begin{array}{c} p_1 \\ p_2 \\ . \\ . \\ p_n \\ \hline \therefore q \end{array}$$

Equivalently, $(p_1 \wedge p_2 \wedge .. \wedge p_n) \implies q$

5.2 Validity of an Argument

An argument is **valid** if the conclusion is true whenever all the premises are true. If there is even one case where the premises are true and the conclusion is false, the argument is **invalid**.

The validity of an argument depends on its **logical form**, not the specific content of the statements. Hence, the arguments are generally written in symbolic form and *then* checked for their validity.

5.2.1 Example of a Valid Argument

- Premise 1: If it rains, then the ground will be wet.
- Premise 2: It is raining.
- Conclusion: The ground is wet.

The above argument can be symbolically written as follows:

$$\begin{array}{l} p \to q \\ \underline{p \qquad\quad} \\ \therefore q \end{array}$$

5.2.2 An Invalid Argument

- Premise 1: If it rains, then the ground will be wet.
- Premise 2: The ground is wet.
- Conclusion: It is raining.

5.3 Rules of inference

Rule of Inference	Related Logical Implication	Name of Rule
$p \wedge q$	$(p \wedge q) \implies p$	Conjunctive Simplification
$\therefore p$		
p	$p \implies p \vee q$	Disjunctive Amplification
$\therefore p \vee q$		
$p \to q$ p	$[p \wedge (p \implies q)] \to q$	Modus Ponens
$\therefore q$		
$p \to q$ $\neg q$	$[(p \to q) \wedge \neg q] \implies \neg p$	Modus Tollens
$\therefore \neg p$		
$p \to q$ $q \to r$	$[(p \to q) \wedge (q \to r)]$ $\implies (p \to r)$	Rule of Syllogism
$\therefore p \to r$		
p q		Rule of Conjunction
$\therefore p \wedge q$		
$p \vee q$ $\neg p$	$[(p \vee q) \wedge \neg p] \implies q$	Disjunctive Syllogism
$\therefore q$		
$\neg p \to F_0$	$(\neg p \to F_0) \implies p$	Rule of Contradiction
$\therefore p$		

5.4 Direct examples

(a) Use Modus Ponens to derive the conclusion from the premises:

If it rains, then the ground will be wet. It is raining.

Solution:

Therefore, the ground is wet.

(b) Use Modus Tollens to derive the conclusion from the premises:

If the dog barks, then someone is at the door. No one is at the door.

Solution:

Therefore, the dog did not bark.

(c) Use Disjunctive Syllogism to derive the conclusion from the premises:

Either the light is on or the switch is broken. The light is not on.

Solution:

Therefore, the switch is broken.

(d) Use Hypothetical Syllogism to derive the conclusion from the premises:

If I study hard, I will pass the exam. If I pass the exam, I will get a job.

Solution:

Therefore, if I study hard, I will get a job.

(e) Use Constructive Dilemma to derive the conclusion from the premises:
If I go to the party, I will have fun. If I stay home, I will study. Either I go to the party or I stay home.
Solution:

Therefore, either I will have fun or I will study.

(f) Use Conjunction to derive the conclusion from the premises:

The sky is blue. The grass is green.

Solution:

Therefore, the sky is blue and the grass is green.

5.5 Problems

(a) Prove that the following argument is valid using rules of inference:

If it snows, then I will stay home.
If I stay home, I will watch a movie.
It is snowing.

$\therefore$ I will watch a movie.

Solution: The argument can be symbolized as follows:

$$\begin{array}{l} p \to q \\ q \to r \\ p \\ \hline \therefore r \end{array}$$

1. $p \to q$ premise)
2. $q \to r$ (premise)
3. p (premise)
4. q (From 1 and 3 using Modus Ponens.)
5. r (From 2 and 4 using Modus Ponens.)

Therefore, the argument is valid.

(b) Prove the following argument using rules of inference:

If the meeting is canceled, then I will go to the gym.
If I go to the gym, I will feel tired.
I do not feel tired.

$\therefore$ The meeting was not canceled.

Solution: The argument can be symbolized as follows:

$$\begin{array}{l} p \to q \\ q \to r \\ \neg r \\ \hline \therefore \neg p \end{array}$$

1. $p \to q$ (premise)
2. $q \to r$ (premise)
3. $\neg r$ (premise)
4. $\neg q$ (From 2 and 3 using Modus Tollens.)
5. $\neg p$ (From 1 and 4 using Modus Tollens.)

Therefore, the argument is valid.

(c) Check if $\neg Q$ follows logically from the premises:

i. $P \to (Q \to R)$
ii. $\neg R$
iii. P

Solution:

i. $P \to (Q \to R)$ Premise
ii. $\neg R$ Premise
iii. P Premise

iv. $Q \to R$ Modus Ponens (from 1 and 3)
v. $\neg Q$ Modus Tollens (from 4 and 2)

Therefore the argument is valid.

(d) Prove that the conclusion s follows logically from the premises $p \vee q,\ \neg p,\ q \to r,\ r \to s$.
Solution:

1. $p \vee q$ (Premise 1)
2. $\neg p$ (Premise 2)
3. $q \to r$ (Premise 3)
4. $r \to s$ (Premise 4)
5. q (From 1 and 2 using Disjunctive Syllogism)
6. r (From 3 and 5 using Modus Ponens)
7. s (From 4 and 6 using Modus Ponens)

Thus, the argument is valid.

(e) Prove the following argument using rules of inference:
Either I will go to the party or I will stay home. If I go to the party, I will have fun. If I stay home, I will study. Therefore, either I will have fun or I will study.
Solution:
The argument can be symbolized as follows:

$$\begin{array}{l} p \vee q \\ p \to r \\ q \to s \\ \hline \therefore r \vee s \end{array}$$

1. $p \vee q$ (Either I will go to the party or I will stay home.)
2. $p \to r$ (If I go to the party, I will have fun.)
3. $q \to s$ (If I stay home, I will study.)
4. $\neg(\neg p) \vee q$ (law of double negation on 1)
5. $\neg p \to q$ (conditional on 4)
6. $\neg p \to s$ (rule of syllogism on 5 and 3)
7. $\neg s \to p$ (contrapositive on 6)
8. $\neg s \to r$ (rule of syllogism on 7 and 2)
9. $\neg(\neg s) \vee r$ (conditional)
10. $s \vee r$ (double negation on 9)

Therefore, the argument is valid.