Topic Fundamentals of Logic Date

Logic is the science dealing with the method of reasoning. A symbolic language has been developed over the past two centuries to express the principles of logic in precise and unambiguous terms. Logic expressed in such a language has come to be called "Symbolic logic" or "Mathematical logic".

## Propositions

A proposition is a statement (declaration), which, in a given context, can be said to be either true or false, but not both.

eg Bengaluru is in karnataka. → true
Three is a prime number. → true
Seven is divisible by 3. → false
Every rectangle is a square. → false

Propositions are usually represented by small letters as $p, q, r, s, \ldots$ The truth or the falsity of a proposition is called its truth value.

If the proposition is true, we will indicate its truth value by the symbol 1 and if it is false by the symbol 0.

eg If "Three is a prime number" is denoted as $p$, then the truth value of $p$ is 1.

If "Every rectangle is a square" is denoted as $q$, then the truth value of $q$ is 0.

Topic ____________ Date ______

## Logical connectives

New propositions are obtained by starting with given propositions with the aid of words or phrases like 'not', 'and' 'if... then', and 'if and only if'. Such words or phrases are called logical connectives. The new propositions obtained by the use of connectives are called compound propositions. The original propositions from which a compound proposition is obtained are called the components or the primitives of the compound proposition. Propositions which do not contain any logical connective are called simple propositions.

## Negation

A proposition obtained by inserting the work 'not' at an appropriate place in a given proposition is called the negation of the given proposition.

The negation of a proposition $p$ is denoted by $\neg p$ ('not $p$'), the symbol $\neg$ denoting the word not.

ex If "3 is a prime number" is denoted by $p$, then $\neg p$ is "3 is not a prime number"

## Conjunction:

A compound proposition obtained by combining two given propositions by inserting the word "and" in between them is called the conjunction of the given proposition.

The conjunction of two propositions $p$ and $q$

Topic ______________________ Date ________

is denoted by $p \wedge q$ ("p and q"), the symbol $\wedge$ denoting the word and.

ex. If p is "$\sqrt{2}$ is an irrational number" and q is "9 is a prime number", then $p \wedge q$ is "$\sqrt{2}$ is an irrational number and 9 is a prime number".

Disjunction

A compound proposition obtained by combining two give propositions by inserting the word 'or' in between them is called the disjunction of the given propositions.
The disjunction of two propositions p and q is denoted by $p \vee q$ ("p or q"), the symbol $\vee$ denoting the word or.

ex If p is "all triangles are equilateral", and q is "$2+5=7$", then $p \vee q$ is "all triangles are equilateral or $2+5=7$".

Conditional

A compound proposition obtained by combining two given propositions by using the words 'if' and 'then' at appropriate places is called a conditional.
The conditional of "If p, then q" is denoted $p \to q$ and the conditional "If q, then p" is denoted $q \to p$.

ex If p is "2 is a prime number" and q is "6 is a perfect square" then $p \to q$ is "If 2 is a prime number then 6 is a perfect square"

Topic ______________________ Date ________

## Biconditional

A proposition obtained by inserting the words 'if and only if' ~~at an appropriate place i.e. between the~~ in between them is called the biconditional of the given propositions.

The biconditional ~~$p \leftrightarrow q$~~ '$p$ if and only if $q$' or '$q$ if and only if $p$' are denoted as $p \Leftrightarrow q$, or $q \leftrightarrow p$.

[The conjunction of the conditional $p \to q$ and $q \to p$ is called the biconditional of $p$ and $q$].

ex. if $p$ is "2 is a prime number" and $q$ is "6 is a perfect square", then $p \Leftrightarrow q$ is "2 is a prime number if and only if 6 is a perfect square".

## Truth Tables

Truth Tables are used to determine the truth value of compound propositions based on the truth values of their components.

## Truth table for negation

| $p$ | $\neg p$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

If $p$ is true, then $\neg p$ is false

If $p$ is false, then $\neg p$ is true.

~~Truth table for conjunction~~

~~$p$~~

Topic ____________ Date ________

## Truth Table for conjunction

| p | q | $p \wedge q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

$p \wedge q$ is true only when p is true and q is true, in all other cases it is false.

## Truth Table for disjunction.

| p | q | $p \vee q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

$p \vee q$ is false only when p is false and q is false, in all other cases it is true.

## Truth Table for conditional

| p | q | $p \to q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

$p \to q$ is false only when p is true and q is false, in all other cases it is true.

## Truth Table for biconditional

| p | q | $p \to q$ | $q \to p$ | $p \leftrightarrow q$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 |

Topic ____________ Date ______

* let $p$: A circle is a conic,
$q$: $\sqrt{5}$ is a real number
$r$: Exponential series is convergent.

Express the following compound propositions in words:
(i) $p \wedge (\neg q)$, (ii) $(\neg p) \vee q$, (iii) $q \to (\neg p)$, (iv) $\neg p \leftrightarrow q$.

Sol$^n$ (i) $p \wedge (\neg q)$: A circle is a conic and $\sqrt{5}$ is not a real number

(ii) $(\neg p) \vee q$: A circle is not a conic or $\sqrt{5}$ is a real number

(iii) $q \to (\neg p)$: If $\sqrt{5}$ is a real number then a circle is not a conic.

(iv) $\neg p \leftrightarrow q$: A circle is not a conic if and only if $\sqrt{5}$ is a real number.

* Construct the truth tables for the following compound propositions:
(i) $p \wedge (\neg q)$, (ii) $(\neg p) \vee q$ (iii) $p \to (\neg q)$.

Sol$^n$

| $p$ | $q$ | $\neg q$ | $p \wedge (\neg q)$ | $\neg p$ | $(\neg p) \vee q$ | $p \to (\neg q)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 |

* Let $p$ and $q$ be primitive statements for which the conditional $p \to q$ is false. Determine the truth value of the following compound propositions (i) $p \wedge q$, (ii) $(\neg p) \vee q$, (iii) $q \to p$, (iv) $(\neg q) \to (\neg p)$

Sol$^n$ Since $p \to q$ is false, $p$ has to be true & $q$ has to be false.

∴ (i) $p \wedge q$ is 1 and 0 $\Rightarrow$ 0 is the truth value

(ii) $(\neg p) \vee q$ is 0 or 0 $\Rightarrow$ 0 is the truth value

(iii) $q \to p$ is $0 \to 1$ $\Rightarrow$ 1 is the truth value

(iv) $(\neg q) \to (\neg p) \Rightarrow 1 \to 0 \Rightarrow 0$ is the truth value

Topic ______ Date ______

Q Find the possible truth values of $p, q$ and $r$ in the following cases:
(i) $p \rightarrow (q \vee r)$ is false. (ii) $p \wedge (q \rightarrow r)$ is true.

Sol (i) $p \rightarrow (q \vee r)$ can be false only when $p$ is true and $q \vee r$ is false. Also $q \vee r$ is false only when both $q$ and $r$ are false. Hence the truth values of $p, q, r$ are 1, 0, 0 respectively.
(ii) $p \wedge (q \rightarrow r)$ can be true, only when $p$ is true and $q \rightarrow r$ is true. Also $q \rightarrow r$ is true, when $q$ is false and $r$ is false, $q$ is false and $r$ is true, $q$ is true and $r$ is true. Hence the possible values of $p, q, r$ are (a) 1, 0, 0
(b) 1, 0, 1
(c) 1, 1, 1

Q Construct the truth tables for the following compound propositions: (i) $(p \vee q) \wedge r$, (ii) $p \vee (q \wedge r)$

Sol

| $p$ | $q$ | $r$ | $p \vee q$ | $(p \vee q) \wedge r$ | $q \wedge r$ | $p \vee (q \wedge r)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

* Construct the truth tables for the following compound propositions:
(i) $(p \wedge q) \rightarrow (\neg r)$, (ii) $q \wedge ((\neg r) \rightarrow p)$

| $p$ | $q$ | $r$ | $p \wedge q$ | $\neg r$ | $p \wedge q \rightarrow (\neg r)$ | $\neg r \rightarrow p$ | $q \wedge ((\neg r) \rightarrow p)$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |

* If a proposition $q$ has the truth value 1, determine all truth value assignments for the primitive propositions $p$, $r$ and $s$ for which the truth value of the following compound proposition is 1. $[q \rightarrow [(\neg p \vee r) \wedge \neg s]] \wedge [\neg s \rightarrow (\neg r \wedge q)]$

Solⁿ: let $u \equiv q \rightarrow [(\neg p \vee r) \wedge \neg s]]$ and $v = [\neg s \rightarrow (\neg r \wedge q)]$
then given truth value of $u \wedge v$ is 1.
$\Rightarrow$ truth value of $u$ is 1 and truth value of $v$ is 1.
Since truth value of $q$ is 1 and truth value of $u$ is 1 it follows truth value of $(\neg p \vee r) \wedge \neg s$ is 1, consequently truth value of $\neg p \vee r$ is 1 and that of $\neg s$ is 1. consequently truth value of $s$ is 0.
~~consequently~~ since truth value of $\neg s$ is 1 and truth value of $v$ is 1, it follows that the truth value of $\neg r \wedge q$ is 1
since truth value of $\neg r \wedge q$ is 1 and that of $q$ is 1, it follows truth value of $\neg r$ is 1, hence truth value of $r$ is 0.
since truth value of $\neg p \vee r$ is 1 and truth value of $r$ is 0, it follows truth value of $\neg p$ is 1, hence truth value of $p$ is 0

Topic ______________________ Date ________

## Tautology, Contradiction, Contrapositive

A compound proposition which is true for all possible truth values of its components is called a tautology (or a logical truth).

A compound proposition which is false for all possible truth values of its components is called a contradiction or an absurdity.

A compound proposition that can be true or false (depending upon the truth values of its components) is called a contingency. In other words, a contingency is a compound proposition which is neither a tautology nor a contradiction.

* Prove that, for any proposition $p$, the compound proposition $p \lor \neg p$ is a tautology and the compound proposition $p \land \neg p$ is a contradiction.

| $p$ | $\neg p$ | $p \lor \neg p$ | $p \land \neg p$ |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 |

From the table, since $p \lor \neg p$ is always true, it's a tautology and since $p \land \neg p$ is always false, it's a contradiction.

* Show that for any propositions $p$ and $q$, the compound proposition $p \to (p \lor q)$ is a tautology and the compound proposition $p \land (\neg p \land q)$ is a contradiction.

Sol

| $p$ | $q$ | $p \lor q$ | $p \to (p \lor q)$ | $\neg p$ | $\cancel{p\land}(\neg p \land q)$ | $p \land (\neg p \land q)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 |

Topic ______________________ Date __________

* Prove that, for any propositions $p$ and $q$, the compound proposition $[(\neg q)\wedge(p\to q)]\to(\neg p)$ is a tautology.

sol)

| $p$ | $q$ | $p\to q$ | $\neg q$ | $(\neg q)\wedge(p\to q)$ | $\neg p$ | $[(\neg q)\wedge(p\to q)]\to(\neg p)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 1 |

Since the last column has all true values, $[(\neg q)\wedge(p\to q)]\to(\neg p)$ is a tautology.

* Prove that, for any propositions $p, q, r$, the compound proposition $[(p\to q)\wedge(q\to r)]\to(p\to r)$ is a tautology.

sol)

| $p$ | $q$ | $r$ | $p\to q$ | $q\to r$ | $(p\to q)\wedge(q\to r)$ | $(p\to r)$ | $[(p\to q)\wedge(q\to r)]\to(p\to r)$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Tautology

* Prove that for any propositions $p, q, r$, $[(p\vee q)\wedge[(p\to r)\wedge(q\to r)]]\to r$ is a tautology

| $p$ | $q$ | $r$ | $p\vee q$ | $p\to r$ | $q\to r$ | $(p\to r)\wedge(q\to r)$ | $(p\vee q)\wedge[(p\to r)\wedge(q\to r)]$ | $[(p\vee q)\wedge[(p\to r)\wedge(q\to r)]]\to r$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Tautology

Topic ______ Date ______

## Logical Equivalence

Two propositions $u$ and $v$ are said to be logically equivalent whenever $u$ and $v$ have the same truth value, or equivalently, the biconditional $u \leftrightarrow v$ is a tautology.

Then we write $u \Leftrightarrow v$. Here the symbol $\Leftrightarrow$ stands for "logically equivalent to". ~~When~~

When the propositions $u$ and $v$ are not logically equivalent, we write $u \not\Leftrightarrow v$.

Logically equivalent propositions are treated as identical propositions.

* Let $x$ be a specified positive integer. Consider the following propositions:

$p$: $x$ is an odd integer, $q$: $x$ is not divisible by 2

Are $p$ and $q$ logically equivalent?

Sol^n: We note that $p$ and $q$ have the same truth values. As such $p$ and $q$ are logically equivalent, i.e., $p \Leftrightarrow q$.

* For any two propositions $p, q$ prove that $(p \to q) \Leftrightarrow (\neg p) \vee q$.

| $p$ | $q$ | $p \to q$ | $\neg p$ | $\neg p \vee q$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 |

as $p \to q$ and $\neg p \vee q$ have the same truth values for all possible truth values of $p$ and $q$, $p \to q \Leftrightarrow (\neg p \vee q)$

Topic ______________________ Date ________

* Prove that $[(p \to q) \to r] \leftrightarrow [(\neg p \vee q) \to r]$ is a tautology.

sol

| $p$ | $q$ | $r$ | $p \to q$ | $(p \to q) \to r$ | $\neg p$ | $\neg p \vee q$ | $(\neg p \vee q) \to r$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |

as $(p \to q) \to r \Leftrightarrow [(\neg p \vee q) \to r]$, we can say $[(p \to q) \to r] \leftrightarrow [(\neg p \vee q) \to r]$ is a tautology.

* Prove that, for any three propositions $p, q, r$, $[p \to (q \wedge r)] \Leftrightarrow [(p \to q) \wedge (p \to r)]$

sol

| $p$ | $q$ | $r$ | $q \wedge r$ | $p \to (q \wedge r)$ | $p \to q$ | $p \to r$ | $(p \to q) \wedge (p \to r)$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

$\because$ $p \to (q \wedge r)$ and $(p \to q) \wedge (p \to r)$ have the same truth values, $(p \to (q \wedge r)) \Leftrightarrow [(p \to q) \wedge (p \to r)]$

Topic ______ Date ______

## The laws of logic

The following results, known as the laws of logic, follow from the definition of logical equivalence. In these laws, $T_0$ denotes a tautology and $F_0$ denotes a contradiction.

1. Law of Double negation: $(\neg\neg p) \Leftrightarrow p$
2. Idempotent law: (a) $(p \vee p) \Leftrightarrow p$, (b) $(p \wedge p) \Leftrightarrow p$
3. Identity law: (a) $(p \vee F_0) \Leftrightarrow p$, (b) $(p \wedge T_0) \Leftrightarrow p$
4. Inverse law: (a) $(p \vee \neg p) \Leftrightarrow T_0$, (b) $(p \wedge \neg p) \Leftrightarrow F_0$
5. Domination law: (a) $(p \vee T_0) \Leftrightarrow T_0$, (b) $(p \wedge F_0) \Leftrightarrow F_0$
6. Commutative law: (a) $(p \vee q) \Leftrightarrow (q \vee p)$, (b) $(p \wedge q) \Leftrightarrow (q \wedge p)$
7. Absorption law: (a) $[p \vee (p \wedge q)] \Leftrightarrow p$, (b) $[p \wedge (p \vee q)] \Leftrightarrow p$
8. DeMorgan's law: (a) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$, (b) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
9. Associative law: (a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$, (b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
10. Distributive law: (a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
    (b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Topic ______ Date ______

Prove the following logical equivalence using the laws of logic.

(i) $p \to q \Leftrightarrow \neg q \to \neg p$.

Sol^n Consider $p \to q \Leftrightarrow \neg p \vee q$ (conditional)

$\Leftrightarrow q \vee \neg p$ (commutative)

$\Leftrightarrow \neg(\neg q) \vee \neg p$ (double negation)

$\Leftrightarrow \neg q \to \neg p$. (conditional.

(ii) $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

Sol^n consider $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$ (biconditional)

$\Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$ (conditional)

$\Leftrightarrow [\neg p \wedge (\neg q \vee p)] \vee [q \wedge (\neg q \vee p)]$ (distributive)

$\Leftrightarrow [(\neg p \wedge \neg q) \vee (\neg p \wedge p)] \vee [(q \wedge \neg q) \vee (q \wedge p)]$ (distributive)

$\Leftrightarrow [(\neg p \wedge \neg q) \vee F_0] \vee [F_0 \vee (q \wedge p)]$ (Inverse)

$\Leftrightarrow (\neg p \wedge \neg q) \vee (q \wedge p)$ (Identity)

$\Leftrightarrow (\neg p \wedge \neg q) \vee (p \wedge q)$ (commutative)

$\Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$ (commutative)

(iii) $\neg(p \leftrightarrow q) \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$

Sol^n Consider $\neg(p \leftrightarrow q) \Leftrightarrow \neg((p \to q) \wedge (q \to p))$ (Biconditional)

$\Leftrightarrow \neg(p \to q) \vee \neg(q \to p)$ (De Morgan's)

$\Leftrightarrow \neg(\neg p \vee q) \vee \neg(\neg q \vee p)$ (Conditional)

$\Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$ (De Morgan's)

$\Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$ (Commutative)

(iv) $(p \vee q) \wedge [\neg(\neg p \vee q)] \Leftrightarrow p \wedge \neg q$

Consider $(p \vee q) \wedge \neg(\neg p \vee q) \Leftrightarrow (p \vee q) \wedge (p \wedge \neg q)$ (De Morgan's)

$\Leftrightarrow [(p \vee q) \wedge p] \wedge (\neg q)$ (Associative)

$\Leftrightarrow [p \wedge (p \vee q)] \wedge (\neg q)$ (Commutative

$\Leftrightarrow p \wedge (\neg q)$ (Absorption)

Topic ______ Date ______

(v) $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

Sol$^n$ consider $(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)$

$\Leftrightarrow \neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)$ [Conditional]

$\Leftrightarrow (p \wedge q) \vee (p \wedge q \wedge r)$ [De Morgan's]

$\Leftrightarrow (p \wedge q) \vee ((p \wedge q) \wedge r)$ (Associative)

$\Leftrightarrow p \wedge q$ (Absorption)

(vi) $(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)] \Leftrightarrow \neg(q \vee p)$

Sol$^n$ Consider $(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)]$

$\Leftrightarrow (p \rightarrow q) \wedge [\neg q \wedge (\neg q \vee r)]$ (commutative)

$\Leftrightarrow (p \rightarrow q) \wedge (\neg q)$ (Absorption)

$\Leftrightarrow \neg[\neg(p \rightarrow q) \vee q]$ [De Morgan's]

$\Leftrightarrow \neg[(p \rightarrow q) \rightarrow q]$ [Conditional]

$\Leftrightarrow \neg[\neg(p \rightarrow q) \vee q]$ [Conditional]

$\Leftrightarrow \neg[\neg(\neg p \vee q) \vee q]$ [Conditional]

$\Leftrightarrow \neg[(p \wedge \neg q) \vee q]$ [De Morgan's]

$\Leftrightarrow \neg[q \vee (p \wedge \neg q)]$ [Commutative]

$\Leftrightarrow \neg[(q \vee p) \wedge (q \vee \neg q)]$ [Distributive]

$\Leftrightarrow \neg[(q \vee p) \wedge T_0]$ [Inverse]

$\Leftrightarrow \neg(q \vee p)$ [Identity]

(vii) $\neg[[(p \vee q) \wedge r] \rightarrow \neg q] \Leftrightarrow \neg[\neg[(p \vee q) \wedge r] \vee \neg q] \Leftrightarrow q \wedge r$

Consider $\neg[[(p \vee q) \wedge r] \rightarrow \neg q]$ (1)

$\Leftrightarrow \neg[\neg[(p \vee q) \wedge r] \vee \neg q]$ (conditional) (2)

$\Leftrightarrow \neg\neg[[(p \vee q) \wedge r] \wedge q]$ (De Morgan's)

$\Leftrightarrow [(p \vee q) \wedge r] \wedge q$ (double negation)

$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$ (Associative)

$\Leftrightarrow (p \vee q) \wedge (q \wedge r)$ (Commutative)

$\Leftrightarrow [(p \vee q) \wedge q] \wedge r$ (Associative)

$\Leftrightarrow [q \wedge (p \vee q)] \wedge r$ (Commutative)

$\Leftrightarrow [q \wedge (q \vee p)] \wedge r$ (Commutative)

$\Leftrightarrow (q \wedge r)$ (Absorption) (3)

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is the required proof

Topic ______ Date ______

(viii) $[\neg p \wedge (\neg q \wedge r)] \vee (q \wedge r) \vee (p \wedge r)] \Leftrightarrow r$

Consider $\neg p \wedge (\neg q \wedge r) \vee (q \wedge r) \vee (p \wedge r)$

$\Leftrightarrow [(\neg p \wedge \neg q) \wedge r] \vee (q \wedge r) \vee (p \wedge r)$ (Associative)

$\Leftrightarrow [r \wedge (\neg p \wedge \neg q)] \vee (q \wedge r) \vee (p \wedge r)$ (Commutative)

$\Leftrightarrow [r \wedge \neg(p \vee q)] \vee (q \wedge r) \vee (p \wedge r)$ (De Morgans)

$\Leftrightarrow [r \wedge \neg(p \vee q)] \vee (r \wedge q) \vee (r \wedge p)$ (Commutative)

$\Leftrightarrow [r \wedge \neg(p \vee q)] \vee [r \wedge (q \vee p)]$ (Distributive)

$\Leftrightarrow r \wedge [\neg(p \vee q) \vee (p \vee q)]$ (Distributive)

$\Leftrightarrow r \wedge T_0$ (Inverse)

$\Leftrightarrow r$ (Identity)

4. Prove that $[(p \vee q) \wedge \neg[\neg p \wedge (\neg q \vee \neg r)]] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is a tautology.

Sol Let $w$ be the given proposition.

Then $w = u \vee v$,

where $u = (p \vee q) \wedge \neg[\neg p \wedge (\neg q \vee \neg r)]$

and $v = (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

Consider $u = (p \vee q) \wedge \neg[\neg p \wedge (\neg q \vee \neg r)]$

$\Leftrightarrow (p \vee q) \wedge (p \vee \neg(\neg q \vee \neg r))$ (De Morgans)

$\Leftrightarrow (p \vee q) \wedge (p \vee (q \wedge r))$ (De Morgan's)

$\Leftrightarrow p \vee [q \wedge (q \wedge r)]$ (Distributive)

$\Leftrightarrow p \vee [(q \wedge q) \wedge r]$ (Associative)

$\Leftrightarrow p \vee (q \wedge r)$ (Idempotent)

Consider $v = (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

$\Leftrightarrow \neg(p \vee q) \vee \neg(p \vee r)$ (De Morgan's)

$\Leftrightarrow \neg[(p \vee q) \wedge (p \vee r)]$ (De Morgan's)

$\Leftrightarrow \neg[p \vee (q \wedge r)]$ (Distributive)

$\Rightarrow v = \neg u$.

$\therefore w = u \vee v \Rightarrow w = u \vee \neg u$

$w = T_0$ (Inverse)

$\therefore$ The given proposition is a Tautology

Topic ______ Date ______

## Duality

Suppose $u$ is a compound proposition that contains the connectives $\wedge$ and $\vee$. Suppose we replace each occurence of $\wedge$ and $\vee$ in $u$ by $\vee$ and $\wedge$ respectively. Also, if $u$ contains $T_0$ and $F_0$ as components, suppose we replace each occurence of $T_0$ and $F_0$ by $F_0$ and $T_0$ respectively. Then the resulting compound proposition is called the dual of $u$ and is denoted by $u^d$.

eg If $u: p \wedge (q \vee \neg r) \vee (s \wedge T_0)$

then $u^d = p \vee (q \wedge \neg r) \wedge (s \vee F_0)$

Note

(i) $(u^d)^d \Leftrightarrow u$

(ii) If $u \Leftrightarrow v$, then $u^d \Leftrightarrow v^d$

(This is known as the Principle of Duality)

* Write the dual of the following propositions.

(i) $\neg(p \vee q) \wedge [p \vee \neg(q \wedge \neg s)]$

solⁿ The dual of given proposition is

$\neg(p \wedge q) \vee [p \wedge \neg(q \vee \neg s)]$

(ii) $u = [(p \vee T_0) \wedge (q \vee F_0)] \vee [(r \wedge s) \wedge T_0]$

solⁿ $u^d = [(p \wedge F_0) \vee (q \wedge T_0)] \wedge [(r \vee s) \vee F_0]$

(iii) $u = p \rightarrow q$

solⁿ $u = p \rightarrow q \Leftrightarrow \neg p \vee q$

$\therefore u^d = \neg p \wedge q$

(iv) $u = (p \rightarrow q) \rightarrow r$

solⁿ $u = (p \rightarrow q) \rightarrow r \Leftrightarrow \neg(p \rightarrow q) \vee r \Leftrightarrow \neg(\neg p \vee q) \vee r$

$\therefore u^d = \neg(\neg p \wedge q) \wedge r$

Topic ______________________________ Date ________

* Verify the principle of duality for the logical equivalence

$$[\neg(p \wedge q) \rightarrow \neg p \vee (\neg p \vee q)] \Leftrightarrow (\neg p \vee q)$$

sol^n

Let $u = [\neg(p \wedge q) \rightarrow \neg p \vee (\neg p \vee q)]$ & $v = (\neg p \vee q)$

Also

$u = [\neg\neg(p \wedge q) \vee \neg p \vee (\neg p \vee q)]$ conditional

$= (p \wedge q) \vee \neg p \vee (\neg p \vee q)$ (double negation)

Then $u^d = (p \vee q) \wedge \neg p \wedge (\neg p \wedge q)$

$\Leftrightarrow (p \vee q) \wedge (\neg p \wedge \neg p) \wedge q$ (Associative)

$\Leftrightarrow (p \vee q) \wedge (\neg p \wedge q)$ (Idempotent)

$\Leftrightarrow [p \wedge (\neg p \wedge q)] \vee [q \wedge (\neg p \wedge q)]$ (Distributive)

$\Leftrightarrow [(p \wedge \neg p) \wedge q] \vee [q \wedge (\neg p \wedge q)]$ (Associative)

$\Leftrightarrow (F_0 \wedge q) \vee [q \wedge (\neg p \wedge q)]$ (Inverse)

$\Leftrightarrow F_0 \vee [q \wedge (\neg p \wedge q)]$ (Domination)

$\Leftrightarrow F_0 \vee [q \wedge (q \wedge \neg p)]$ (Commutative)

$\Leftrightarrow F_0 \vee [(q \wedge q) \wedge \neg p]$ (Associative)

$\Leftrightarrow F_0 \vee (q \wedge \neg p)$ (Idempotent)

$u^d \Leftrightarrow q \wedge \neg p$ (Identity

Also $v^d = \neg p \wedge q$

$v^d \Leftrightarrow q \wedge \neg p$.

$\therefore u^d \Leftrightarrow v^d$

This verifies the principle of duality.

Topic ______ Date ______

## Converse, Inverse and Contrapositive; logical Implication

Consider the conditional $p \to q$.

Then (1) $q \to p$ is called the converse of $p \to q$.

(2) $\neg p \to \neg q$ is called the inverse of $p \to q$.

(3) $\neg q \to \neg p$ is called the contrapositive of $p \to q$.

ex $p$: 2 is an integer, $q$: 9 is a multiple of 3.

Then $p \to q$: If 2 is an integer, then 9 is a multiple of 3.

$q \to p$: If 9 is a multiple of 3, then 2 is an Integer.

$\neg p \to \neg q$: If 2 is not an integer, then 9 is not a multiple of 3.

$\neg q \to \neg p$: If 9 is not a multiple of 3, then 2 is not an integer.

The following table gives the truth value of $(p \to q)$, $(q \to p)$, $(\neg p \to \neg q)$, $(\neg q \to \neg p)$ for all possible truth values of two arbitrary propositions $p$ and $q$.

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \to q$ | $q \to p$ | $\neg p \to \neg q$ | $\neg q \to \neg p$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |

From this table, it is evident that $p \to q$ and $\neg q \to \neg p$ have the same truth values in all possible situations. Also, $q \to p$ and $\neg p \to \neg q$ have the same truth values in all possible situations.

We have two important results:

(1) A conditional and its contrapositive are logically equivalent: that is, for any propositions $p$ and $q$,

$$(p \to q) \Leftrightarrow (\neg q \to \neg p)$$

(2) The converse and the inverse of a conditional are logically equivalent; that is for any propositions $p$ and $q$,

$$(q \to p) \Leftrightarrow (\neg p \to \neg q)$$

Topic ______ Date ______

## logical implications

- Consider two propositions,
$p$: 6 is a multiple of 2, and $q$: 3 is a prime number
Then $p \to q$ : If 6 is a multiple of 2 then 3 is a prime number
Note that $p$ is true and $q$ is true, hence $p \to q$ is true.
This conditional makes no sence but it is logically true.
- Consider the propositions,
$p$: 4 is an odd number, and $q$; Bengaluru is not in Karnataka
Here both $p$ and $q$ are false,
The conditional $p \to q$ : If 4 is an odd number, then Bengaluru is not in Karnataka.
This conditional makes no sense but it is logically true.

We do not deal with conditionals such as the ones considered in the above two examples.

Our major interest lies in conditionals $p \to q$ where $p$ and $q$ are related in some way so that the truth value of $q$ depends upon the truth value of $p$ or vice versa. Such conditions are called hypothetical statements.

When a hypothetical statement $p \to q$ is such that $q$ is true whenever $p$ is true, we say that $p$ (logically) implies $q$. This is symbolically written as $p \Rightarrow q$, the symbol $\Rightarrow$ denoting the word implies.

When a hypothetical statement $p \to q$ is such that $q$ is not necessarily true whenever $p$ is true, we say that $p$ does not imply $q$. This is symbolically written as $p \not\Rightarrow q$, the symbol $\not\Rightarrow$ denoting the phrase does not imply.

Topic ______ Date ______

## Necessary and Sufficient Conditions

Consider two propositions $p$ and $q$ whose truth values are interrelated. Suppose that $p \Rightarrow q$. Then in order that $q$ may be true it is sufficient that $p$ is true. Also if $p$ is true then it is necessary that $q$ is true. In view of this interpretation, all of the following statements are taken to carry the same meaning: (i) $p \Rightarrow q$, (ii) $p$ is sufficient for $q$, (iii) $q$ is necessary for $p$.

For two propositions $p$ and $q$, the following situations are possible:

(i) $p \Rightarrow q$, but $q \not\Rightarrow p$.

(ii) $p \not\Rightarrow q$, but $q \Rightarrow p$.

(iii) $p \Rightarrow q$, and $q \Rightarrow p$

In the first of the above cases, $p$ is a sufficient but not a necessary condition for $q$.

In the second case, $p$ is a necessary but not a sufficient condition for $q$.

In the last case, $p$ is necessary and sufficient condition for $q$, and vice-versa.

Ex:
Let A denote a specified city.

Consider $p$: The city A is in Karnataka.

$q$: The city A is in India.

Here $p \Rightarrow q$, but $q \not\Rightarrow p$.

Accordingly, $p$ is a sufficient but not a necessary condition for $q$, and $q$ is a necessary but not a sufficient condition for $p$.

→

Topic ______ Date ______

* Consider a geometric object the quadrilateral.
Let $p$: A quadrilateral is a rectangle.
$q$: A quadrilateral is a square
Here $p \not\Rightarrow q$, but $q \Rightarrow p$.
Accordingly, $p$ is a necessary but not a sufficient condition for $q$, and $q$ is a sufficient but not a necessary condition for $p$.

* Consider a specified integer $x$.
Let $p$: The integer $x$ is even.
$q$: The integer $x$ is divisible by 2.
Here, $p \Rightarrow q$ and $q \Rightarrow p$.
Thus, here $p$ is a necessary and sufficient condition for $q$ and vice-versa.

* State the converse, inverse and contrapositive.
$q$: If a quadrilatoral is a parallelogram, then its diagonals bisect each other.
Soln converse: If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.
inverse: If ~~a~~ a quadrilateral is not a parallelogram, then its diagonals donot bisect each other.
contrapositive: If the diagonals of a quadrilateral do not bisect each other, then it is not a parallelogram.

* Write the converse, inverse and contrapositive of $p \to (q \to r)$, with no occurence of the connective $\to$
Soln converse: $(q \to r) \to p \Leftrightarrow (\neg q \vee r) \to p \Leftrightarrow \neg(\neg q \vee r) \vee p$
inverse: $\neg p \to \neg(q \to r) \Leftrightarrow \neg\neg p \vee \neg(q \to r) \Leftrightarrow p \vee (\neg q \vee r)$
contrapositive $\neg(q \to r) \to \neg p \Leftrightarrow \neg\neg(q \to r) \vee \neg p \Leftrightarrow (q \to r) \vee \neg p$
$\Leftrightarrow (\neg q \vee r) \vee \neg p$

Topic ______ Date ______

Prove the following

(i) $[p \wedge (p \to q)] \Rightarrow q$, (ii) $[(p \to q) \wedge \neg q] \Rightarrow \neg p$

(i)

| p | q | p→q |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

(i) we find that when both p and p→q are true then q is true.
Hence $[p \wedge (p \to q)] \Rightarrow q$

(ii)

| p | q | p→q | ¬q | ¬p |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 |

From the table, when both p→q and ¬q are true then ¬p is true
Hence $[(p \to q) \wedge \neg q] \Rightarrow \neg p$

(iii) $[p \wedge (p \to q) \wedge r] \Rightarrow [(p \vee q) \to r]$

| p | q | r | p→q | p∨q | (p∨q)→r |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 |

From the table, when p and p→q and r are true, then (p∨q)→r is true, Hence
$[p \wedge (p \to q) \wedge r] \Rightarrow [(p \vee q) \to r]$

Topic ______________________ Date __________

(iv) $\{[p \vee (q \vee r)] \wedge \neg q\} \Rightarrow p \vee r$

Sol$^n$

| $p$ | $q$ | $r$ | $q \vee r$ | $p \vee (q \vee r)$ | $\neg q$ | $p \vee r$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 | 1 |

From the table when $p \vee (q \vee r)$ and $\neg q$ are true, then $p \vee r$ is true, hence
$\{[p \vee (q \vee r)] \wedge \neg q\} \Rightarrow p \vee r$.

(v) $\{[(p \wedge q) \to r] \wedge (\neg q) \wedge (p \to \neg r)\} \Rightarrow \neg p \vee \neg q$

Sol$^n$

| $p$ | $q$ | $r$ | $p \wedge q$ | $(p \wedge q) \to r$ | $\neg q$ | $\neg r$ | $p \to \neg r$ | $\neg p$ | $\neg p \vee \neg q$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |

From the table when $(p \wedge q) \to r$, $\neg q$ and $p \to \neg r$ are true, then $\neg p \vee \neg q$ is true, hence
$\{[p \vee (q \vee r)] \wedge \neg q\} \Rightarrow p \vee r$.