Defn: [Boolean product] : Let $A = [a_{ij}]$ be $m \times n$ zero-one matrix, $B = [b_{ij}]$ be $k \times n$ zero-one matrix. Then Boolean product of A and B denoted by $A \odot B$ is the $m \times n$ matrix with $(i, j)^{th}$ entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee (a_{i3} \wedge b_{3j}) \vee \dots \vee (a_{ik} \wedge b_{kj})$$

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}_{3\times 2}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$. Find $A \odot B$

Ans:

$$A \odot B = \begin{bmatrix} (1\wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find $A \odot A$

Defn: Given a set A and a relation R on A, we define the powers of R recursively by as $R^1 = R$ and for $n \in Z^+$, $R^{n+1} = R \circ R^n$

Ex: If $A = \{1, 2, 3, 4\}$ and $R = \{(1,2), (1,3), (2,4), (3,2)\}$ then

$R^2 = \{(1,4), (1,2), (3,4)\}$, $R^3 = \{(1,4)\}$ and for $n \geq 4$,

$R \circ R$ $\quad$ $R \circ R^2$

$R^n = \phi$

Ex: Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,2), (1,3), (2,4), (3,2)\}$.
Find $M(R^2)$ (same as $(M(R))^2$)

$$M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M(R^2) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

or

$$M(R^2) = M(R) \odot M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Boolean product

$$= \begin{bmatrix} (0\wedge 0)\vee(1\wedge 0)\vee(1\wedge 0)\vee(0\wedge 0) & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$|||^{ly}$ $$(M(R))^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: Let A be a set with $|A| = n$ and R be a relation on A. If $M(R)$ is relation matrix for R, then

a) $M(R) = 0$ (the matrix of all 0's) iff $R = \phi$

b) $M(R) = 1$ (the matrix of all 1's) iff $R = A \times A$

c) $M(R^m) = (M(R))^m$, for $m \in Z^+$

Defn: Let $A = [a_{ij}]$, $B = [b_{ij}]$ be $m \times n$ zero-one matrices. We say $A \leq B$ if $\forall\ i, j\ \ a_{ij} \leq b_{ij}$

ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Here $A \leq B$ (since $\forall\ i, j\ \ a_{ij} \leq b_{ij}$)

Defn: For $n \in Z^+$, $I_n = (\delta_{ij})_{n \times n}$ is zero-one matrix where $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

Operations on Relations:

1) Union and intersection: Let R_1, R_2 be relations from set A to set B, the union of R_1 and R_2 $(R_1 \cup R_2)$ is defined by $(a,b) \in R_1 \cup R_2$ iff $(a,b) \in R_1$ or $(a,b) \in R_2$. The intersection of R_1 and R_2 $(R_1 \cap R_2)$ is defined by $(a,b) \in R_1 \cap R_2$ iff $(a,b) \in R_1$ and $(a,b) \in R_2$.

2) Complement of a relation: Given a relation R from A to B the complement of R $(\overline{R})$ is defined with the property that $(a,b) \in \overline{R}$ iff $(a,b) \notin \overline{R}$. In other words, $\overline{R}$ is complement of R in universal set $A \times B$.

3) Inverse or converse of a relation (R^c): Let R be a relation from set A to set B, the converse of R^c defined with the property $(a,b) \in R^c$ iff $(b,a) \in R^c$

ex: $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$. $R_1 = \{(1,1), (2,2), (3,3)\}$

and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$.

$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$\overline{R_1} = (A \times B) - R_1$

$A \times B = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3),$
$(2,4), (3,1), (3,2), (3,3), (3,4)\}$

$\overline{R_1} = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4),$
$(3,1), (3,2), (3,4)\}$

$R_1 - R_2 = \{(2,2), (3,3)\}$

$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$ $(R_1 \cup R_2) - (R_1 \cap R_2)$

$R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1) = \{(1,2), (1,3), (1,4)$
$(2,2), (3,3)\}$

symmetric difference

$R_2^c = \{(1,1), (2,1), (3,1), (4,1)\}$

Representing relations using matrices

Defn: Given a set A with $|A| = n$ and a relation R on A, let $M(R)$ denote the relation matrix for R. Then

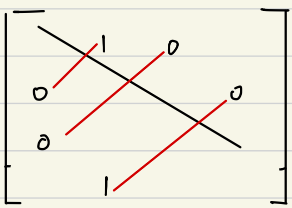
a) R is reflexive iff $m_{jj} = 1 \ \forall j$ or $I_n \leq M(R)$

main diagonal entries are equal to 1

b) R is symmetric iff $m_{ij} = m_{ji} \ \forall i, j$ or $M(R) = M(R^T)$ (symmetric matrix)

c) R is antisymmetric iff $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$ or $M(R) \cap M(R^T) \leq I_n$

meet of $M(R)$ and $M(R^T)$
(it has 1 in the position where both $M(R)$ and $M(R^T)$ has 1.)



ex:

$$M(R) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(R^T) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$M(R) \cap M(R^T) = M(R) \wedge M(R^T) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leq I_n$$

d) R is transitive iff $M(R^2) \leq M$ $\quad [M_R^{[2]} \leq M]$

Ex: Let R be a relation on A.

a) $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\quad$ b) $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

c) $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Relation	a)	b)	c)
Reflexive	Yes	No	Yes
symmetric	Yes	No	No
Anti-symmetric	No	Yes	No
Transitive	Yes	Yes	No

a) $M(R^2) =$

$R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$

$R^2 = \{(1,3), (1,1), (3,1), (3,3), (2,2)\}$

$M(R^2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \leq \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

or

$M(R) \circ M(R) = \begin{bmatrix} \; \end{bmatrix}$

Ex: Let $M(R) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find matrix representation of a) R^{-1} b) $\bar{R}$

Ans: R^{-1} is inverse of R its matrix representation is $M(R^T) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$\bar{R}$ is complement of R its representation is

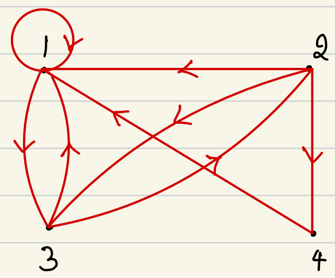
$$M(\bar{R}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Representing relations using digraphs

A directed graph or digraph consists of set V of vertices and set E of edges.
Here, $E = \{ (a, b) \mid a, b \in V \}$

a is called initial vertex of edge (a, b), b is called terminal vertex

ex: Let $A = \{1, 2, 3, 4\}$. Let R be relation on A defined by $R = \{ (1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1) \}$



A directed graph on R.

Various properties of relation can be determined using digraph.

1) A relation is reflexive iff there is a loop at every vertex of graph

2) Symmetric iff for every edge b/w 2 distinct vertices in its digraph there is an edge in opposite direction

3) Antisymmetric iff there are never 2 edges

4) Transitive iff there is an edge from vertex x to a vertex z whenever there is an edge from x to y and an edge from y to z (for some y)

Ex: Determine if the relations for directed graphs shown are reflexive, symmetric, antisymmetric, transitive

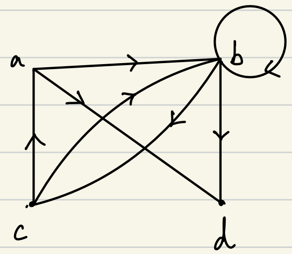


fig 1.

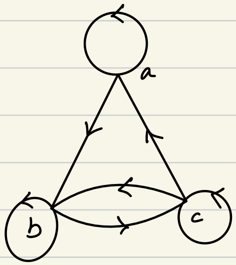


fig 2

Property	fig 1	fig 2
Reflexive	No	Yes
symmetric	No	No
antisymmetric	No	No
transitive	No	No

fig 1, transitive: cRa, aRd but $c \not R d$

fig 2, transitive: since $(a,b) \in R$ and $(b,c) \in R$, but $(a,c) \notin R$

Representing union and intersection of relations

If R_1 and R_2 are relations on a set A represented by matrices $M(R_1)$ and $M(R_2)$ respectively.
If $M_{R_1 \cup R_2}$ is the matrix representing the union of R_1 and R_2, it has 1 in the position where either M_{R_1} or M_{R_2} has a 1.

If $M_{R_1 \cap R_2}$ is the matrix representing the intersection of these relations, then it has 1 in the position where both M_{R_1} and M_{R_2} has a 1.

$\therefore \quad M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$ (Join of M_{R_1} and M_{R_2})

$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$ (Meet of M_{R_1} and M_{R_2})

Ex: If the relations R_1 and R_2 on a set A are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Ans:
$$M(R_1 \cup R_2) = M(R_1) \vee M(R_2) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M(R_1 \cap R_2) = M(R_1) \wedge M(R_2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Ex. For a fixed integer $n > 1$, P.T. the relation "congruent modulo n" is an equivalence relation on set of all integers $\mathbb{Z}$.

Sol: aRb means $a \equiv b \pmod{n}$, $a-b$ is multiple of n

(This means $n \mid (a-b)$ or $a-b = kn$ for some integer k, a and b leave same remainder when divided by n) ex: $17 \equiv 5 \pmod{12}$

1) Reflexive : For every $a \in Z$

$a - a = 0$ is a multiple of n

$\Rightarrow a \equiv a \pmod{n}$

$\Rightarrow aRa$

$\therefore R$ is reflexive

2) Symmetric : $\forall\, a, b \in Z$, $aRb \Rightarrow a \equiv b \pmod{n}$

$\Rightarrow (a-b)$ is a multiple of n

$\Rightarrow (b-a)$ is -11-

$\Rightarrow b \equiv a \pmod{n}$

$\Rightarrow bRa$

$\therefore R$ is symmetric

3) Transitive: $\forall\, a, b, c \in Z$

aRb and bRc $\Rightarrow a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$

$\Rightarrow (a-b)$ and $(b-c)$ are multiples of n

$\Rightarrow (a-b) + (b-c) = a-c$ is multiple of n

$\Rightarrow a \equiv c \pmod{n}$

$\Rightarrow aRc$

$\therefore R$ is transitive

Hence R is an equivalence relation.

Ex: Let $A = \{1, 2, 3, 4, 5\}$. Define a relation R on $A \times A$ by $(x_1, y_1) R (x_2, y_2)$ iff $x_1 + y_1 = x_2 + y_2$.

Verify that R is an equivalence relation on $A \times A$.

Closure of relations :

Let R be a relation of a set A, let P be some property (such as reflexive, symmetric, transitive) that R may or may not have. The relation S is said to be closure of R wrt P if S is the smallest relation with the property P with $R \subseteq S$.

Ex: Find reflexive closure, given $R = \{(1,2), (2,3), (3,4)\}$ on set $A = \{1, 2, 3, 4\}$

sol: Add all pairs (a, a) that are missing from R $\forall$ $a \in A$.

Reflexive closure of $R = R \cup \{(1,1), (2,2), (3,3), (4,4)\}$

$$= \{(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,4)\}$$

Ex: Find symmetric closure, given $R = \{(1,2), (2,3), (1,3)\}$ on set $A = \{1, 2, 3\}$.

sol: Add (b, a) for every (a, b) in R for symmetric closure.

$$S = R \cup \underbrace{\{(2,1), (3,2), (3,1)\}}_{R^{-1}} \qquad = R \cup R^{-1}$$

$$= \{(1,2), (2,3), (1,3), (2,1), (3,2), (3,1)\}$$

Ex: Let A be set of all Z^+. $R = \{(a,b) \mid a < b\}$. Find its reflexive closure.

sol: $S = R \cup \{(a,a) \mid a \in Z\}$

$$S = \{(a,b) \mid a \le b\}$$

Ex: Let A be set of all Z^+. $R = \{(a,b) \mid a > b\}$
Find symmetric closure.

Sol:

$$R^{-1} = \{(a,b) \mid \overbrace{b > a}^{a<b}\}$$

$$S = \{(a,b) \mid a > b \text{ or } a < b\} = \{(a,b) \mid a \neq b\}$$

Equivalence class

Let R be an equivalence relation on a set A and $a \in A$. The set of all those elements x of A which are related to a by R is called equivalence class of a wrt R. This equivalence class is denoted by $R(a)$ or $[a]$ or $\bar{a}$ or $[a]_R$

i.e. $\bar{a} = [a] = \{x \in A \mid \underbrace{(x,a) \in R}_{\text{or } x \sim a}\}$

$x \sim a$ means $(x,a) \in R$

Ex: Let $A = \{1, 2, 3, 4, 5\}$, $R = \{(a,b) \mid a+b \text{ is even}\}$
Find the equivalence class of all elements of A.

Sol: (Check that R is equivalence relation)

$$[1] = \{1,3,5\}, \quad [2] = \{2,4\}, \quad [3] = \{1,3,5\}$$

$$[4] = \{2,4\}, \quad [5] = \{1,3,5\}.$$

Equivalence class of elements 1, 3, 5 are same and equivalence class of elements 2 and 4 are same. Any element out of 1, 3 and 5 can be chosen as a representative of equivalence class $\{1, 3, 5\}$. Simly for equivalence class $\{2, 4\}$

$$\therefore [1] = \{1, 3, 5\}, \quad [2] = \{2, 4\}$$

Thm: Let R be an equivalence relation on a set A. The statements for elements a and b of A are equivalent.

1) aRb
2) $[a] = [b]$
3) $[a] \cap [b] \neq \phi$

Ex: Let m be an integer $m > 1$, S.T. the relation $R = \{(a, b) \mid a \equiv b \pmod m\}$ is an equivalence relation on Z. Also find the equivalence class of 1 (consider $m = 4$).

Sol.
$$[1] = \{x \mid x \equiv 1 \pmod 4\}$$

$$= \{4k+1 \mid k \in Z\}$$

$$= \{\ldots -3, 1, 5, 9, 13 \ldots\}$$

for $k = -1$ $\quad k=0$ . . . - -

Simly $\quad [2] = \{x \mid x \equiv 2 \pmod 4\}$

$$= \{4k+2 \mid k \in Z\} = \{\ldots -2, 2, 6, 10, \ldots\}$$

$$[3] = \{ 4k+3 \mid k \in \mathbb{Z} \}$$

$$[4] = \{ 4k \mid k \in \mathbb{Z} \} = [0]$$

$$[5] = \{ 4k+5 \mid k \in \mathbb{Z} \} = \{ 4k+1 \mid k \in \mathbb{Z} \} = [1]$$

$$4k+4+1 = 4(k+1)+1 = 4k'+1$$

$$[6] = [2]$$

Observe: $\mathbb{Z} = \{ -4, -3, -2, -1, 0, 1, 2, 3, 4 \ldots \}$

$$= [1] \cup [2] \cup [3] \cup [4] \rightsquigarrow \bigcup_{a \in A} [a] = A$$

union covers all of A.

Also, $[1] \cap [2] = \phi$, $[3] \cap [4] = \phi$,

$[2] \cap [3] = \phi$, $[1] \cap [4] = \phi$ $\rightsquigarrow$ pairwise disjoint

(Each equivalence class is like a 'piece' of the set)

Partition of a set :

Let A be a non-empty set. Suppose there exists non empty subsets $A_1, A_2, \ldots A_k$ of A such that the following 2 conditions hold

1) A is the union of $A_1, A_2, \ldots A_k$

i.e. $A = A_1 \cup A_2 \cup \ldots A_k$

$$A = \bigcup_{i=1}^{k} A_i$$