CHAPTER 10
RECURRENCE RELATIONS

Section 10.1

1. (a) $a_n = 5a_{n-1},\ n \geq 1,\ a_0 = 2$
(b) $a_n = -3a_{n-1},\ n \geq 1,\ a_0 = 6$
(c) $a_n = (2/5)a_{n-1},\ n \geq 1,\ a_0 = 7$

2. (a) $a_{n+1} = 1.5a_n,\ a_n = (1.5)^n a_0,\ n \geq 0.$
(b) $4a_n = 5a_{n-1},\ a_n = (1.25)^n a_0,\ n \geq 0.$
(c) $3a_{n+1} = 4a_n,\ 3a_1 = 15 = 4a_0,\ a_0 = 15/4$, so
$a_n = (4/3)^n a_0 = (4/3)^n(15/4) = 5(4/3)^{n-1},\ n \geq 0.$
(d) $a_n = (3/2)a_{n-1},\ a_n = (3/2)^n a_0,\ 81 = a_4 = (3/2)^4 a_0$ so $a_0 = 16$ and $a_n = (16)(3/2)^n,\ n \geq 0.$

3. $a_{n+1} - da_n = 0,\ n \geq 0$, so $a_n = d^n a_0$. $153/49 = a_3 = d^3 a_0,\ 1377/2401 = a_5 = d^5 a_0 \Longrightarrow a_5/a_3 = d^2 = 9/49$ and $d = \pm 3/7$.

4. $a_{n+1} = a_n + 2.5a_n,\ n \geq 0.$
$a_n = (3.5)^n a_0 = (3.5)^n(1000)$. For $n = 12$, $a_n = (3.5)^{12}(1000) \doteq 3,379,220,508.$

5. $P_n = 100(1 + 0.015)^n,\ P_0 = 100$
$200 = 100(1.015)^n \Longrightarrow 2 = (1.015)^n$
$(1.015)^{46} \doteq 1.9835$ and $(1.015)^{47} \doteq 2.0133$.
Hence Laura must wait $(47)(3) = 141$ months for her money to double.

6. $P_n = P_0(1.02)^n$
$7218.27 = P_0(1.02)^{60}$, so $P_0 = (7218.27)(1.02)^{-60} = \$2200.00$

7. (a) $19 + 18 + 17 + \ldots + 10 = 145$
(b) $9 + 8 + 7 + \ldots + 1 = 45$

8. (a) Suppose that for $i = k$, where $1 \leq k \leq n-2$, no interchanges result (for the first time) in the execution of the inner **for** loop. Up to this point the number of executions that have been made is $(n-1)+(n-2)+\ldots+(n-k)$. If we continue and execute the inner **for** loop for $k+1 \leq i \leq n-1$, then we make $[n-(k+1)]+[n-(k+2)]+\ldots+3+2+1 = (1/2)(n-k-1)(n-k)$ unnecessary comparisons. [Note: $(n-1)+(n-2)+\ldots+(n-k) = kn-(1+2+3+\ldots+k) = kn-(1/2)(k)(k+1)$ and $(1/2)(n-1)(n)-[kn-(1/2)(k)(k+1)] = (1/2)(n-k-1)(n-k)$]

(b) The input for the following procedure is an array A of n real numbers. The output is the reordered array A with $A[1] \leq A[2] \leq \ldots \leq A[n]$.

```
Procedure BubbleSort2(var A: array; n: integer);
Var
    Switch: boolean;   {The value of Switch is true if}
                       {an interchange actually takes place.}
    i,j: integer;
    temp: real;
Begin
    Switch := true;
    While Switch do
        Begin
            Switch := false;
            For i := 1 to n-1 do
              For j := n downto i+1 do
                If A[j] < A[j-1] then
                    Begin
                        temp := A[j-1];
                        A[j-1] := A[j];
                        A[j] := temp;
                        Switch := true
                    End {if}
        End {while}
End. {procedure}
```

(c) The best case occurs when the array A is already in nondecreasing order. When this happens the procedure is only processed for i:= 1 and j:= n down to 2. This results in $n-1$ comparisons so the best-case complexity is $O(n)$.

The worst case occurs when a Switch is made for all i := 1 to $n-1$. This results (as in Example 10.5) in $(n^2-n)/2$ comparisons, so the worst-case complexity is $O(n^2)$.

9. (a) 21345 (b) 52143, 52134, 25134
(c) 25134, 21534, 21354, 21345

10.

(a)

1,2,3	3,1,2
1,3,2	3,2,1

(b)

1,2,3,4	4,1,2,3
1,2,4,3	4,1,3,2
1,4,2,3	4,3,1,2
1,4,3,2	4,3,2,1

(c) The value of p_1 is either 1 or 5.

(d) Let $p_1, p_2, p_3, \ldots, p_n$ be an orderly permutation of $1, 2, 3, \ldots, n$. Then p_1 is either 1 or n. If $p_1 = 1$, then $p_2 - 1, p_3 - 1, \ldots, p_n - 1$ is an orderly permutation of $1, 2, 3, \ldots, n-1$. For $p_1 = n$ we find that $p_2, p_3, \ldots, p_n$ is an orderly permutation of $1, 2, 3, \ldots, n-1$. Since these two cases are exhaustive and have nothing in common we may write

$$a_n = 2a_{n-1}, \quad n \geq 3, \quad a_2 = 2.$$

Hence, $a_3 = 2a_2 = 2 \cdot 2 = 2^2$,

$$a_4 = 2a_3 = 2 \cdot 2^2 = 2^3,$$

and, in general,

$$a_n = 2^{n-1}, \quad n \geq 2.$$

Section 10.2

1. (a) $a_n = 5a_{n-1} + 6a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 3$.
Let $a_n = cr^n$, $c, r \neq 0$. Then the characteristic equation is $r^2 - 5r - 6 = 0 = (r-6)(r+1)$, so $r = -1, 6$ are the characteristic roots.
$a_n = A(-1)^n + B(6)^n$
$1 = a_0 = A + B$
$3 = a_1 = -A + 6B$, so $B = 4/7$ and $A = 3/7$.
$a_n = (3/7)(-1)^n + (4/7)(6)^n$, $n \geq 0$.

(b) $a_n = 4(1/2)^n - 2(5)^n$, $n \geq 0$.

(c) $a_{n+2} + a_n = 0$, $n \geq 0$, $a_0 = 0$, $a_1 = 3$.
With $a_n = cr^n$, $c, r \neq 0$, the characteristic equation $r^2 + 1 = 0$ yields the characteristic roots $\pm i$. Hence $a_n = A(i)^n + B(-i)^n = A(\cos(\pi/2) + i\sin(\pi/2))^n + B(\cos(\pi/2) + i\sin(-\pi/2))^n = C\cos(n\pi/2) + D\sin(n\pi/2)$.
$0 = a_0 = C$, $3 = a_1 = D\sin(\pi/2) = D$, so $a_n = 3\sin(n\pi/2)$, $n \geq 0$.

(d) $a_n - 6a_{n-1} + 9a_{n-2} = 0$, $n \geq 2$, $a_0 = 5$, $a_1 = 12$.
Let $a_n = cr^n$, $c, r \neq 0$. Then $r^2 - 6r + 9 = 0 = (r-3)^2$, so the characteristic roots are 3,3 and $a_n = A(3^n) + Bn(3^n)$.
$5 = a_0 = A$; $12 = a_1 = 3A + 3B = 15 + 3B$, $B = -1$.
$a_n = 5(3^n) - n(3^n) = (5-n)(3^n)$, $n \geq 0$.

(e) $a_n + 2a_{n-1} + 2a_{n-2} = 0$, $n \geq 2$, $a_0 = 1$, $a_1 = 3$.
$r^2 + 2r + 2 = 0$, $r = -1 \pm i$
$(-1+i) = \sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4))$
$(-1-i) = \sqrt{2}(\cos(5\pi/4) + i\sin(5\pi/4)) =$
$\sqrt{2}(\cos(-3\pi/4) + i\sin(-3\pi/4)) = \sqrt{2}(\cos(3\pi/4) - i\sin(3\pi/4))$
$a_n = (\sqrt{2})^n[A\cos(3\pi n/4) + B\sin(3\pi n/4)]$
$1 = a_0 = A$
$3 = a_1 = \sqrt{2}[\cos(3\pi/4) + B\sin(3\pi/4)] =$

$\sqrt{2}[(-1/\sqrt{2})+B(1/\sqrt{2})]$, so $3=-1+B,\ B=4$
$a_n=(\sqrt{2})^n[\cos(3\pi n/4)+4\sin(3\pi n/4)],\ n\geq 0$

2. (a) Example 10.14: $a_n=10a_{n-1}+29a_{n-2},\ n\geq 2,\ a_1=10,\ a_2=100.$
$r^2-10r-29=0,\ r=5\pm 6\sqrt{6}.$
$a_n=A(5+6\sqrt{6})^n+B(5-6\sqrt{6})^n$
$a_2=100=10a_1+29a_0=100+29a_0$, so $a_0=0$
$0=a_0=A+B$, so $B=-A$.
$a_n=A[(5+6\sqrt{6})^n-(5-6\sqrt{6})^n]$
$10=a_1=A[5+6\sqrt{6}-5+6\sqrt{6}]=12\sqrt{6}A,\ A=5/6\sqrt{6}.$
$a_n=(5/6\sqrt{6})[(5+6\sqrt{6})^n-(5-6\sqrt{6})^n],\ n\geq 0.$

Example 10.23: $a_n=c_1(2^n)+c_2n(2^n),\ n\geq 0,\ a_0=1,\ a_1=3.$
$a_0=1=c_1;\ a_1=3=2+c_2(2),\ c_2=1/2.$
$a_n=(2^n)[1+(n/2)],\ n\geq 0.$

(b) Example 10.16: $a_n=a_{n-1}+a_{n-2},\ n\geq 2,\ a_0=1,\ a_1=2.$
$r^2-r-1=0,\ r=(1\pm\sqrt{5})/2.$
$a_0=1=A+B$
$a_1=2=A[(1+\sqrt{5})/2]+B[(1-\sqrt{5})/2]$
$4=A(1+\sqrt{5})+B(1-\sqrt{5})=(A+B)+\sqrt{5}(A-B)=1+\sqrt{5}(A-B)$, so $3=\sqrt{5}(A-B)$ and $A-B=3/\sqrt{5}$.
$2A=(A+B)+(A-B)=1+3/\sqrt{5}=(3+\sqrt{5})/\sqrt{5},\ A=(3+\sqrt{5})/2\sqrt{5};\ B=1-A=(\sqrt{5}-3)/2\sqrt{5},$
$a_n=[(\sqrt{5}+3)/2\sqrt{5}][(1+\sqrt{5})/2]^n+[(\sqrt{5}-3)/2\sqrt{5}][(1-\sqrt{5})/2]^n,\ n\geq 0$

3. $(n=0):\ a_2+ba_1+ca_0=0=4+b(1)+c(0)$, so $b=-4.$
$(n=1):\ a_3-4a_2+ca_1=0=37-4(4)+c$, so $c=-21.$
$a_{n+2}-4a_{n+1}-21a_n=0$
$r^2-4r-21=0=(r-7)(r+3),\ r=7,-3$
$a_n=A(7)^n+B(-3)^n$
$0=a_0=A+B\Longrightarrow B=-A$
$1=a_1=7A-3B=10A$, so $A=1/10,\ B=-1/10$ and $a_n=(1/10)[(7)^n-(-3)^n],\ n\geq 0.$

4. $a_n=a_{n-1}+a_{n-2},\ n\geq 2,\ a_0=a_1=1$
$r^2-r-1=0,\ r=(1\pm\sqrt{5})/2$
$a_n=A((1+\sqrt{5})/2)^n+B((1-\sqrt{5})/2)^n$
$a_0=a_1=1\Longrightarrow A=(1+\sqrt{5})/2\sqrt{5},\ B=(\sqrt{5}-1)/2\sqrt{5}$
$a_n=(1/\sqrt{5})[((1+\sqrt{5})/2)^{n+1}-((1-\sqrt{5})/2)^{n+1}]$

5. For all three parts, let a_n, $n\geq 0$, count the number of ways to fill the n spaces under the condition(s) specified.
(a) Here $a_0=1$ and $a_1=2$. For $n\geq 2$, consider the *nth* space. If this space is occupied by a motorcycle – in one of two ways, then we have $2a_{n-1}$ of the ways to fill the n spaces.

Further, there are a_{n-2} ways to fill the n spaces when a compact car occupies positions $n-1$ and n. These two cases are exhaustive and have nothing in common, so

$$a_n = 2a_{n-1} + a_{n-2}, \quad n \geq 2, \quad a_0 = 1, \quad a_1 = 2.$$

Let $a_n = cr^n$, $c \neq 0$, $r \neq 0$. Upon substitution we have $r^2 - 2r - 1 = 0$, so $r = 1 \pm \sqrt{2}$ and $a_n = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n$, $n \geq 0$. From $1 = a_0 = c_1 + c_2$ and $2 = a_1 = c_1(1+\sqrt{2}) + c_2(1-\sqrt{2})$, we have $c_1 = \frac{2+\sqrt{2}}{4}$ and $c_2 = \frac{2-\sqrt{2}}{4}$. So $a_n = ((\sqrt{2}+2)/4)(1+\sqrt{2})^n + ((2-\sqrt{2})/4)(1-\sqrt{2})^n = (1/2\sqrt{2})[(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}]$, $n \geq 0$.
(b) Here $a_0 = 1$ and $a_1 = 1$. For $n \geq 2$, consider the nth space. This space can be occupied by a motorcycle in one way and accounts for a_{n-1} of the a_n ways to fill the n spaces. If a compact car occupies the $(n-1)st$ and nth spaces, then we have the remaining $3a_{n-2}$ ways to fill n spaces. So here $a_n = a_{n-1} + 3a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 1$.

Let $a_n = cr^n$, $c \neq 0$, $r \neq 0$. Upon substitution we have $r^2 - r - 3 = 0$, so $r = (1 \pm \sqrt{13})/2$, and $a_n = c_1[(1+\sqrt{13})/2]^n + c_2[(1-\sqrt{13})/2]^n$, $n \geq 0$. From $1 = a_0 = c_1 + c_2$ and $1 = a_1 = c_1[(1+\sqrt{13})/2] + c_2[(1-\sqrt{13})/2]$, we find that $c_1 = [(1+\sqrt{13})/2\sqrt{13}]$ and $c_2 = [(-1+\sqrt{13})/2\sqrt{13}]$. So $a_n = (1/\sqrt{13})[(1+\sqrt{13})/2]^{n+1} - (1/\sqrt{13})[(1-\sqrt{13})/2]^{n+1}$, $n \geq 0$.
(c) Comparable to parts (a) and (b), here we have $a_n = 2a_{n-1} + 3a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 2$. Substituting $a_n = cr^n$, $c \neq 0$, $r \neq 0$, into the recurrence relation, we find that $r^2 - 2r - 3 = 0$ so $(r-3)(r+1) = 0$ and $r = 3$, $r = -1$. Consequently, $a_n = c_1(3^n) + c_2(-1)^n$, $n \geq 0$. From $1 = a_0 = c_1 + c_2$ and $2 = a_1 = 3c_1 - c_2$, we learn that $c_1 = 3/4$ and $c_2 = 1/4$. Therefore, $a_n = (3/4)(3^n) + (1/4)(-1)^n$, $n \geq 0$.

6. For all three parts, let b_n, $n \geq 0$, count the number of ways to fill the n spaces under the condition(s) specified – including the condition allowing empty spaces.
(a) Here $b_0 = 1$, $b_1 = 3$, and $b_n = 3b_{n-1} + b_{n-2}$, $n \geq 2$. This recurrence relation leads us to the characteristic equation $r^2 - 3r - 1 = 0$, and the characteristic roots $r = (3 \pm \sqrt{13})/2$. Consequently, $b_n = c_1[(3+\sqrt{13})/2]^n + c_2[(3-\sqrt{13})/2]^n$, $n \geq 0$. From $1 = b_0 = c_1 + c_2$ and $3 = b_1 = c_1[(3+\sqrt{13})/2] + c_2[(3-\sqrt{13})/2]$, we find that $c_1 = (3+\sqrt{13})/2\sqrt{13}$ and $c_2 = (-3+\sqrt{13})/2\sqrt{13}$. So $b_n = (1/\sqrt{13})[(3+\sqrt{13})/2]^{n+1} - (1/\sqrt{13})[(3-\sqrt{13})/2]^{n+1}$, $n \geq 0$.
(b) For this part we have $b_n = 2b_{n-1} + 3b_{n-2}$, $n \geq 0$, $b_0 = 1$, $b_1 = 2$. Here the characteristic equation is $r^2 - 2r - 3 = 0$ and the characteristic roots are $r = 3$, $r = -1$. Therefore, $b_n = c_1(3^n) + c_2(-1)^n$, $n \geq 0$. From $1 = b_0 = c_1 + c_2$ and $2 = b_1 = 3c_1 - c_2$, we find that $c_1 = 3/4$, $c_2 = 1/4$. So $b_n = (3/4)(3^n) + (1/4)(-1)^n$, $n \geq 0$.
(c) Here $b_0 = 1$, $b_1 = 3$, and $b_n = 3b_{n-1} + 3b_{n-2}$, $n \geq 2$. The characteristic equation $r^2 = 3r + 3$ gives us the characteristic roots $r = (3 \pm \sqrt{21})/2$. So $b_n = c_1[(3+\sqrt{21})/2]^n + c_2[(3-\sqrt{21})/2]^n$, $n \geq 0$. From $1 = b_0 = c_1 + c_2$ and $3 = b_1 = c_1[(3+\sqrt{21})/2] + c_2[(3-\sqrt{21})/2]$, we have $c_1 = [(3+\sqrt{21})/2\sqrt{21}]$ and $c_2 = [(-3+\sqrt{21})/2\sqrt{21}]$. Consequently, $b_n = (1/\sqrt{21})[((3+\sqrt{21})/2)^{n+1} - ((3-\sqrt{21})/2)^{n+1}]$, $n \geq 0$.

7. (a)

$$\begin{aligned} F_1 &= F_2 - F_0 \\ F_3 &= F_4 - F_2 \\ F_5 &= F_6 - F_4 \\ \cdots & \quad\ \cdots \quad \cdots \\ F_{2n-1} &= F_{2n} - F_{2n-2} \end{aligned}$$

Conjecture: For all $n \in \mathbf{Z}^+$, $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n} - F_0 = F_{2n}$.
Proof: (By the Principle of Mathematical Induction).
For $n = 1$ we have $F_1 = F_2$, and this is true since $F_1 = 1 = F_2$. Consequently, the result is true in this first case (and this establishes the basis step for the proof).
Next we assume the result true for $n = k\ (\geq 1)$ – that is, we assume

$$F_1 + F_3 + F_5 + \cdots + F_{2k-1} = F_{2k}.$$

When $n = k + 1$ we then find that

$$F_1 + F_3 + F_5 + \cdots + F_{2k-1} + F_{2(k+1)-1} =$$

$$(F_1 + F_3 + F_5 + \cdots + F_{2k-1}) + F_{2k+1} = F_{2k} + F_{2k+1} = F_{2k+2} = F_{2(k+1)}.$$

Therefore the truth for $n = k$ implies the truth at $n = k + 1$, so by the Principle of Mathematical Induction it follows that for all $n \in \mathbf{Z}^+$

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}.$$

(b)

$$\begin{aligned} F_2 &= F_3 - F_1 \\ F_4 &= F_5 - F_3 \\ F_6 &= F_7 - F_5 \\ \cdots & \quad\ \cdots \quad \cdots \\ F_{2n} &= F_{2n+1} - F_{2n-1} \end{aligned}$$

Conjecture: For all $n \in \mathbf{N}$, $F_2 + F_4 + \cdots + F_{2n} = F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - F_1 = F_{2n+1} - 1$.
Proof: (By the Principle of Mathematical Induction)
When $n = 0$ we find that $0 = F_0 = F_1 - F_1 = 0$, so the result is true for this initial case, and this provides the basis step for the proof.
Assuming the result true for $n = k\ (\geq 0)$ we have $\sum_{i=0}^{k} F_{2i} = F_{2k+1} - 1$. Then when $n = k+1$ it follows that $\sum_{i=0}^{k+1} F_{2i} = \sum_{i=0}^{k} F_{2i} + F_{2(k+1)} = F_{2k+1} - 1 + F_{2k+2} = (F_{2k+2} + F_{2k+1}) - 1 = F_{2k+3} - 1 = F_{2(k+1)+1} - 1$. Consequently we see how the truth of the result for $n = k$ implies the truth of the result for $n = k + 1$. Therefore it follows that for all $n \in \mathbf{N}$.

$$F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1,$$

by the Principle of Mathematical Induction.

8. (a) $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{(1/\sqrt{5})[((1+\sqrt{5})/2)^{n+1}-((1-\sqrt{5})/2)^{n+1}]}{(1/\sqrt{5})[((1+\sqrt{5})/2)^{n}-((1-\sqrt{5})/2)^{n}]}$

$= \lim_{n\to\infty} \frac{[((1+\sqrt{5})/2)^{n+1}-((1-\sqrt{5})/2)^{n+1}]}{[((1+\sqrt{5})/2)^{n}-((1-\sqrt{5})/2)^{n}]}$

$= \lim_{n\to\infty} \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha^n-\beta^n}$ (where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$)

$= \lim_{n\to\infty} \frac{1-(\frac{\beta}{\alpha})^{n+1}}{(\frac{1}{\alpha})-(\frac{1}{\alpha})(\frac{\beta}{\alpha})^n}$

Since $|\beta| < 1$ and $|\alpha| > 1$, it follows that $|\frac{\beta}{\alpha}| < 1$ and $|\frac{\beta}{\alpha}|^n \to 0$ as $n \to \infty$.

Consequently, $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1}{(\frac{1}{\alpha})} = \alpha = \frac{1+\sqrt{5}}{2}$.

(b) (i) $AC/AX = \sin AXC/\sin ACX = \sin 108°/\sin 36° = 2\sin 36°\cos 36°/\sin 36° = 2\cos 36°$

(ii) $\cos 18° = \sin 72° = 2\sin 36°\cos 36° =$
$2(2\sin 18°\cos 18°)(1-2\sin^2 18°) \Longrightarrow 1 =$
$4\sin 18°(1-2\sin^2 18°) = 4\sin 18° - 8\sin^3 18°$.
$0 = 8\sin^3 18° - 4\sin 18° + 1$, so $\sin 18°$ is a root of $8x^3 - 4x + 1 = 0$.
$8x^3 - 4x + 1 = (2x-1)(4x^2+2x-1) = 0$.
The roots of $4x^2 + 2x - 1 = 0$ are $(-1 \pm \sqrt{5})/4$.
Since $0 < \sin 18° < \sin 30° = 1/2$, $\sin 18° = (-1+\sqrt{5})/4$.

(c) $(1/2)(AC/AX) = \cos 36° = 1 - 2\sin^2 18° = 1 - 2[(-1+\sqrt{5})/4]^2 = (1+\sqrt{5})/4$.
$AC/AX = 2(1+\sqrt{5})/4 = (1+\sqrt{5})/2$.

9. $a_n = a_{n-1} + a_{n-2}$, $n \geq 0$, $a_0 = a_1 = 1$

(Append '+1') (Append '+2')

$a_n = A[(1+\sqrt{5})/2]^n + B[(1-\sqrt{5})/2]^n$
$1 = a_0 = A + B$; $1 = a_1 = A(1+\sqrt{5})/2 + B(1-\sqrt{5})/2$ or
$2 = (A+B) + \sqrt{5}(A-B) = 1 + \sqrt{5}(A-B)$ and $A - B = 1/\sqrt{5}$.
$1 = A + B$, $1/\sqrt{5} = A - B \Longrightarrow A = (1+\sqrt{5})/2\sqrt{5}$, $B = (\sqrt{5}-1)/2\sqrt{5}$ and $a_n = (1/\sqrt{5})[((1+\sqrt{5})/2)^{n+1} - ((1-\sqrt{5})/2)^{n+1}]$, $n \geq 0$.

10. Here $a_1 = 1$ and $a_2 = 1$. For $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$, because the strings counted by a_n either end in 1 (and there are a_{n-1} such strings) or they end in 00 (and there are a_{n-2} such strings).

Consequently, $a_n = F_n$, the nth Fibonacci number, for $n \geq 1$.

11. a) The solution here is similar to that for part (b) of Example 10.16. For $n = 1$, there are two strings – namely, 0 and 1. When $n = 2$, we find three such strings: 00, 10, 01. For $n \geq 3$, we can build the required strings of length n (1) by appending '0' to each of the a_{n-1} strings of length $n-1$; or (2) by appending '01' to each of the a_{n-2} strings of length

$n-2$. These two cases have nothing in common and cover all possibilities, so

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 3, a_1 = 2, a_2 = 3.$$

We find that $a_n = F_{n+2} = (\alpha^{n+2} - \beta^{n+2})/(\alpha - \beta)$ where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.
b) Here $b_1 = 1$ since 0 is the only string of length 1 that satisfies both conditions. For $n = 2$, there are three strings: 00, 10, and 01 – so $b_2 = 3$. For $n \geq 3$, consider the bit in the nth position of such a binary string of length n.
(1) If the nth bit is a 0, then there are a_{n-1} possibilities for the remaining $n-1$ bits.
(2) If the nth bit is a 1, then the $(n-1)$st and 1st bits are 0, and so there are a_{n-3} possibilities for the remaining $n-3$ bits.

Hence $b_n = a_{n-1} + a_{n-3} = F_{n+1} + F_{n-1}$, from part (a). So

$$b_n = (F_n + F_{n-1}) + (F_{n-2} + F_{n-3}) = (F_n + F_{n-2}) + (F_{n-1} + F_{n-3}) = b_{n-1} + b_{n-2}.$$

The characteristic equation $x^2 - x - 1 = 0$ has characteristic roots $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, so $b_n = c_1\alpha^n + c_2\beta^n$. From $1 = b_1 = c_1\alpha + c_2\beta$ and $3 = b_2 = c_1\alpha^2 + c_2\beta^2$ we learn that $c_1 = c_2 = 1$. Hence $b_n = \alpha^n + \beta^n = L_n$, the nth Lucas number. [Recall that in Example 4.20 we showed that $L_n = F_{n+1} + F_{n-1}$.]

12. Let a_n count the number of ways to arrange n such chips with no consecutive blue chips. Let b_n equal the number of arrangements counted in a_n that end in blue; $c_n = a_n - b_n$.
Then $a_{n+1} = 3b_n + 4c_n = 3(b_n + c_n) + c_n = 3a_n + 3a_{n-1}$.
Hence $a_{n+1} - 3a_n - 3a_{n-1} = 0$, $n \geq 1$, $a_0 = 1$, $a_1 = 4$. This recurrence relation has characteristic roots $r = (3 \pm \sqrt{21})/2$ and $a_n = A((3+\sqrt{21})/2)^n + B((3-\sqrt{21})/2)^n$.
$a_0 = 1$, $a_1 = 4 \Longrightarrow A = (5+\sqrt{21})/2\sqrt{21}$, $B = (\sqrt{21}-5)/2\sqrt{21}$ and
$a_n = [(5+\sqrt{21})/(2\sqrt{21})][(3+\sqrt{21})/2]^n - [(5-\sqrt{21})/(2\sqrt{21})][(3-\sqrt{21})/2]^n, n \geq 0$.

13. For $n \geq 0$, let a_n count the number of words of length n in Σ^* where there are no consecutive alphabetic characters. Let $a_n^{(1)}$ count those words that end with a numeric character, while $a_n^{(2)}$ counts those that end with an alphabetic character. Then $a_n = a_n^{(1)} + a_n^{(2)}$.

$$\text{For } n \geq 1, \quad \begin{aligned} a_{n+1} &= 11a_n^{(1)} + 4a_n^{(2)} \\ &= [4a_n^{(1)} + 4a_n^{(2)}] + 7a_n^{(1)} \\ &= 4a_n + 7a_n^{(1)} \\ &= 4a_n + 7(4a_{n-1}) \\ &= 4a_n + 28a_{n-1}, \end{aligned}$$

and $a_0 = 1$, $a_1 = 11$.

Now let $a_n = cr^n$, where $c, r \neq 0$ and $n \geq 0$. Then the resulting characteristic equation is

$$r^2 - 4r - 28 = 0,$$

where $r = (4 \pm \sqrt{128})/2 = 2 \pm 4\sqrt{2}$.

Hence $a_n = A[2+4\sqrt{2}]^n + B[2-4\sqrt{2}]^n$, $n \geq 0$.

$$\begin{aligned} 1 = a_0 \Rightarrow 1 &= A + B, \quad \text{and} \\ 11 = a_1 \Rightarrow 11 &= A[2+4\sqrt{2}] + B[2-4\sqrt{2}] \\ &= A[2+4\sqrt{2}] + (1-A)[2-4\sqrt{2}] \\ &= [2-4\sqrt{2}] + A[2+4\sqrt{2}-2+4\sqrt{2}] \\ &= [2-4\sqrt{2}] + 8\sqrt{2}A, \end{aligned}$$

so $A = (9+4\sqrt{2})/(8\sqrt{2}) = (8+9\sqrt{2})/16$, and $B = 1 - A = (8-9\sqrt{2})/16$.

Consequently,

$$a_n = [(8+9\sqrt{2})/16][2+4\sqrt{2}]^n + [(8-9\sqrt{2})/16][2-4\sqrt{2}]^n, \quad n \geq 0.$$

14. Using the ideas developed in the prior exercise we find that $7k = 63$, or $k = 9$.

15. Here we find that
$a_0 = 1, \quad a_1 = 2, \quad a_2 = 2, \quad a_3 = 2^2, \quad a_4 = 2^3, \quad a_5 = 2^5, \quad a_6 = 2^8,$
and, in general, $a_n = 2^{F_n}$, where F_n is the nth Fibonacci number for $n \geq 0$.

16. $a_1 = 0$, $a_2 = 1$, $a_3 = 1$. For $n \geq 4$, let $n = x_1 + x_2 + \ldots + x_t$, where $x_i \geq 2$ for $1 \leq i \leq t$, and $1 \leq t \leq \lfloor n/2 \rfloor$. If $x_1 = 2$, then $x_2 + \ldots + x_t$ is counted in a_{n-2}. If $x_1 \neq 2$, then $x_1 > 2$ and $(x_1 - 1) + x_2 + \ldots + x_t$ is counted in a_{n-1}. Hence $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$, and $a_n = F_{n-1}$, the $(n-1)$-st Fibonacci number.

17. (a) From the previous exercise the number of compositions of $n+3$ with no 1s as summands is F_{n+2}.
(b) (i) The number that start with 2 is the number of compositions of $n+1$ with no 1s as summands. This is F_n.
(ii) F_{n-1}
(iii) The number that start with k, for $2 \leq k \leq n+1$, is the number of compositions of $(n+3) - k$ with no 1s as summands. This is $F_{(n+3)-k-1} = F_{n-k+2}$, $2 \leq k \leq n+1$.
(c) If the composition starts with $n+2$ then there is only one remaining summand – namely, 1. But here we are not allowed to use 1 as a summand, so there are no such compositions that start with $n+2$.

The one-summand composition '$n+3$' is the only composition here that starts (and ends) with $n+3$.
(d) These results provide a combinatorial proof that
$F_{n+2} = \sum_{k+2}^{n+1} F_{n-k+2} + 1 = (F_n + F_{n-1} + \cdots + F_2 + F_1) + 1$, or
$F_{n+2} - 1 = \sum_{i=1}^{n} F_i = \sum_{i=0}^{n} F_i$, since $F_0 = 0$.

18. From $x^2 - 1 = x$ we have $x^2 - x - 1 = 0$, so $x = (1 \pm \sqrt{5})/2$. Consequently, the points of intersection are $((1+\sqrt{5})/2, (1+\sqrt{5})/2) = (\alpha, \alpha)$ and $((1-\sqrt{5})/2, (1-\sqrt{5})/2) = (\beta, \beta)$.

19. From $1+\frac{1}{x}=x$ we learn that $x+1=x^2$, or $x^2-x-1=0$. So $x=(1\pm\sqrt{5})/2$ and the points of intersection are $((1+\sqrt{5})/2,\,(1+\sqrt{5})/2)=(\alpha,\alpha)$ and $((1-\sqrt{5})/2,\,(1-\sqrt{5})/2)=(\beta,\beta)$.

20. (a) $\alpha^2=[(1+\sqrt{5})/2]^2=(1+2\sqrt{5}+5)/4=(6+2\sqrt{5})/4=(3+\sqrt{5})/2=[(1+\sqrt{5})/2]+(2/2)=\alpha+1$.

(b) Proof: (By Mathematical Induction) For $n=1$, we have $\alpha^n=\alpha^1=\alpha=\alpha\cdot 1+0=\alpha F_1+F_0=\alpha F_n+F_{n-1}$, so the result is true in this case. This establishes the basis step. Now we assume for an arbitrary (but fixed) positive integer k that $\alpha^k=\alpha F_k+F_{k-1}$. This is our inductive step. Considering $n=k+1$, at this time, we find that

$$\begin{aligned}
\alpha^{k+1} &= \alpha(\alpha^k)=\alpha[\alpha F_k+F_{k-1}] \quad \text{(by the inductive step)}\\
&= \alpha^2F_k+\alpha F_{k-1}\\
&= (\alpha+1)F_k+\alpha F_{k-1} \quad \text{[by part (a)]}\\
&= \alpha(F_k+F_{k-1})+F_k\\
&= \alpha F_{k+1}+F_k.
\end{aligned}$$

Since the given result is true for $n=1$ and the truth for $n=k+1$ follows from that for $n=k$, it follows by the Principle of Mathematical Induction that $\alpha^n=\alpha F_n+F_{n-1}$ for all $n\in\mathbf{Z}^+$.

21. Proof (By the Alternative Form of the Principle of Mathematical Induction):

(a) $F_3=2=(1+\sqrt{9})/2>(1+\sqrt{5})/2=\alpha=\alpha^{3-2}$,
$F_4=3=(3+\sqrt{9})/2>(3+\sqrt{5})/2=\alpha^2=\alpha^{4-2}$,

so the result is true for these first two cases (where $n=3,4$). This establishes the basis step. Assuming the truth of the statement for $n=3,4,5,\ldots,k(\geq 4)$, where k is a fixed (but arbitrary) integer, we continue now with $n=k+1$:

$$\begin{aligned}
F_{k+1} &= F_k+F_{k-1}\\
&> \alpha^{k-2}+\alpha^{(k-1)-2}\\
&= \alpha^{k-2}+\alpha^{k-3}=\alpha^{k-3}(\alpha+1)\\
&= \alpha^{k-3}\cdot\alpha^2=\alpha^{k-1}=\alpha^{(k+1)-2}
\end{aligned}$$

Consequently, $F_n>\alpha^{n-2}$ for all $n\geq 3$ – by the Alternative Form of the Principle of Mathematical Induction.

(b) $F_3=2=(3+\sqrt{1})/2<(3+\sqrt{5})/2=\alpha^2=\alpha^{3-1}$,
$F_4=3=2+1<2+\sqrt{5}=\alpha^3=\alpha^{4-1}$,

so this result is true for these first two cases (where $n=3,4$). This establishes the basis step. Assuming the truth of the statement for $n=3,4,5,\ldots,k(\geq 4)$, where k is a fixed (but arbitrary) integer, we continue now with $n=k+1$:

$$\begin{aligned}
F_{k+1} &= F_k+F_{k-1}\\
&< \alpha^{k-1}+\alpha^{(k-1)-1}\\
&= \alpha^{k-1}+\alpha^{k-2}=\alpha^{k-2}(\alpha+1)\\
&= \alpha^{k-2}\cdot\alpha^2=\alpha^k=\alpha^{(k+1)-1}
\end{aligned}$$

Consequently, $F_n<\alpha^{n-1}$ for all $n\geq 3$ – by the Alternative Form of the Principle of Mathematical Induction.

22. (a) Since $a_{n+1} = 2a_n$ we have $a_n = c(2^n)$, $n \geq 1$. Then $a_1 = 2 \Rightarrow 2c = 2 \Rightarrow c = 1$, so $a_n = 2^n$. Consequently, for n even, the number of palindromes of n is counted by $a_{n/2} = 2^{n/2} = 2^{\lfloor n/2 \rfloor}$.

(b) Here $b_{n+1} = 2b_n$, $n \geq 1$, $b_1 = 1$. So $b_n = d(2^n)$, $n \geq 1$ and $b_1 = 1 \Rightarrow 2d = 1 \Rightarrow d = 1/2$, so $b_n = 2^{n-1}$. Hence, for n odd, the number of palindromes of n is counted by $b_{(n+1)/2} = 2^{[(n+1)/2]-1} = 2^{(n-1)/2} = 2^{\lfloor n/2 \rfloor}$.

23. Here we shall use auxiliary variables. For $n \geq 1$, let $a_n^{(0)}$ count the number of ternary strings of length n where there are no consecutive 1s and no consecutive 2s and the nth symbol is 0. We define $a_n^{(1)}$ and $a_n^{(2)}$ analogously. Then

$$\begin{aligned} a_n &= a_n^{(0)} + a_n^{(1)} + a_n^{(2)} \\ &= a_{n-1} + [a_{n-1} - a_{n-1}^{(1)}] + [a_{n-1} - a_{n-1}^{(2)}] \\ &= 2a_{n-1} + [a_{n-1} - a_{n-1}^{(1)} - a_{n-1}^{(2)}] \\ &= 2a_{n-1} + a_{n-1}^{(0)} = 2a_{n-1} + a_{n-2} \end{aligned}$$

Letting $a_n = cr^n$, $c \neq 0$, $r \neq 0$, we find that $r^2 - 2r - 1 = 0$, so the characteristic roots are $1 \pm \sqrt{2}$. Consequently, $a_n = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n$. Here $a_1 = 3$, for the three one-symbol ternary strings 0, 1, and 2. Since we cannot use the two-symbol ternary strings 11 and 22, we have $a_2 = 3^2 - 2 = 7$. Extending the recurrence relation so that we can use $n = 0$, we have $a_2 = 2a_1 + a_0$ so $a_0 = a_2 - 2a_1 = 7 - 2 \cdot 3 = 1$. With

$$\begin{aligned} 1 = a_0 &= c_1 + c_2, \text{ and} \\ 3 = a_1 &= c_1(1+\sqrt{2}) + c_2(1-\sqrt{2}) \\ &= (c_1 + c_2) + \sqrt{2}(c_1 - c_2), \end{aligned}$$

we now have $1 = c_1 + c_2$ and $\sqrt{2} = c_1 - c_2$, so $c_1 = (1+\sqrt{2})/2$ and $c_2 = (1-\sqrt{2})/2$. Consequently,

$$a_n = (1/2)(1+\sqrt{2})^{n+1} + (1/2)(1-\sqrt{2})^{n+1}, \quad n \geq 0.$$

24. Here $a_1 = 1$, for the case of one vertical domino, and $a_2 = 3$ – use (i) one square tile ; or (ii) two horizontal dominoes; or (iii) two vertical dominoes. For $n \geq 3$ consider the nth column of the chessboard. This column can be covered by
(1) one vertical domino – this accounts for a_{n-1} of the tilings of the $2 \times n$ chessboard;
(2) the right squares of two horizontal dominoes placed in the four squares for the $(n-1)$st and nth columns of the chessboard – this accounts for a_{n-2} of the tilings; and
(3) the right column of a square tile placed on the four squares for the $(n-1)$st and nth columns of the chessboard – this also accounts for a_{n-2} of the tilings.

These three cases account for all the possible tilings and no two cases have anything in common so

$$a_n = a_{n-1} + 2a_{n-2}, \quad n \geq 3, a_1 = 1, a_2 = 3.$$

Here the characteristic equation is $x^2 - x - 2 = 0$ which gives $x = 2$, $x = -1$ as the characteristic roots. Consequently, $a_n = c_1(-1)^n + c_2(2)^2$, $n \geq 1$. From $1 = a_1 = c_1(-1) +$

$c_2(2)$ and $3 = a_2 = c_1(-1)^2 + c_2(2)^2$ we learn that $c_1 = 1/3$, $c_2 = 2/3$. So $a_n = (1/3)[2^{n+1} + (-1)^n]$, $n \geq 1$. [The sequence $1, 3, 5, 11, 21, \ldots$, described here, is known as the *Jacobsthal* sequence.]

25. Let a_n count the number of ways one can tile a $2 \times n$ chessboard using these colored dominoes and square tiles. Here $a_1 = 4$, $a_2 = 4^2 + 4^2 + 5 = 37$, and, for $n \geq 3$, $a_n = 4a_{n-1} + 16a_{n-2} + 5a_{n-2} = 4a_{n-1} + 21a_{n-2}$. The characteristic equation is $x^2 - 4x - 21 = 0$ and this gives $x = 7$, $x = -3$ as the characteristic roots. Consequently, $a_n = c_1(7)^n + c_2(-3)^n$, $n \geq 1$.

Here $a_0 = (1/21)(a_2 - 4a_1) = 1$ can be introduced to simplify the calculations for c_1, c_2. From $1 = a_0 = c_1 + c_2$ and $4 = 7c_1 - 3c_2$ we learn that $c_1 = 7/10$, $c_2 = 3/10$, so $a_n = (7/10)(7)^n + (3/10)(-3)^n$, $n \geq 0$.

When $n = 10$ we find that the 2×10 chessboard can be tiled in $(7/10)(7)^{10} + (3/10)(-3)^{10} = 197{,}750{,}389$ ways.

26. Here $a_1 = 1$ (for the string 0) and $a_2 = 3$ (for the strings 00, 01 and 11). For $n \geq 3$, there are three cases to consider:
(1) The nth symbol is 0: There are a_{n-1} such strings.
(2) The $(n-1)$st and nth symbols are 0, 1, respectively: There are a_{n-2} such strings.
(3) The $(n-1)$st and nth symbols are both 1: Here there are also a_{n-2} strings.

These three cases include all possibilities and no two cases have anything in common. Consequently,

$$a_n = a_{n-1} + 2a_{n-2}, \quad a_1 = 1, \quad a_2 = 3.$$

The characteristic equation, $r^2 - r - 2 = 0$, yields the characteristic roots 2 and -1, so $a_n = c_1(2)^n + c_2(-1)^n$. From $1 = a_1 = 2c_1 - c_2$ and $3 = a_2 = 4c_1 + c_2$, we learn that $c_1 = 2/3$ and $c_2 = 1/3$. So

$$a_n = (2/3)(2)^n + (1/3)(-1)^n, \quad n \geq 1.$$

[So here we find another occurrence of the Jacobsthal numbers.]

27. There is $a_1 = 1$ string of length 1 (namely, 0) in A^*, and $a_2 = 2$ strings of length 2 (namely, 00 and 01) and $a_3 = 5$ strings of length 3 (namely, 000, 001, 010, 011 and 111). For $n \geq 4$ we consider the entry from A at the (right) end of the string.
(1) 0: there are a_{n-1} strings.
(2) 01: there are a_{n-2} strings.
(3) 011, 111: there are a_{n-3} strings in each of these two cases.

Consequently,

$$a_n = a_{n-1} + a_{n-2} + 2a_{n-3}, \quad n \geq 4, \quad a_1 = 1, \quad a_2 = 2, \quad a_3 = 5.$$

From the characteristic equation $r^3 - r^2 - r - 2 = 0$, we find that $(r-2)(r^2+r+1) = 0$ and the characteristic roots are 2 and $(-1 \pm i\sqrt{3})/2$. Since $(-1+i\sqrt{3})/2 = \cos 120^\circ + i \sin 120^\circ =$

$\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})$, we have

$$a_n = c_1(2)^n + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3}, \quad n \geq 1.$$

From

$$\begin{aligned} 1 &= a_1 = 2c_1 - c_2/2 + c_3(\sqrt{3}/2) \\ 2 &= a_2 = 4c_1 - c_2/2 - c_3(\sqrt{3}/2) \\ 5 &= a_3 = 8c_1 + c_2, \end{aligned}$$

we learn that $c_1 = 4/7$, $c_2 = 3/7$, and $c_3 = \sqrt{3}/21$, so

$$a_n = (4/7)(2)^n + (3/7)\cos(2n\pi/3) + (\sqrt{3}/21)\sin(2n\pi/3), \quad n \geq 1.$$

[Note that a_n also counts the number of ways one can tile a $1 \times n$ chessboard using 1×1 square tiles of one color, 1×2 rectangular tiles of one color, and 1×3 rectangular tiles that come in *two* colors.]

28. Here $a_1 = 1$ (for 0), $a_2 = 2$ (for 00,01), $a_3 = 4$ (for 000, 001, 010, 011), $a_4 = 9$ (for 0000, 0001, 0010, 0100, 0011, 0110, 0111, 1111, 0101), and for $n \geq 5$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2a_{n-4}.$$

The characteristic equation $r^4 - r^3 - r^2 - r - 2 = 0$ tells us that $(r-2)(r+1)(r^2+1) = 0$, so the characteristic roots are $2, -1, \pm i$. Consequently,

$$a_n = c_1(2)^n + c_2(-1)^n + c_3\cos(n\pi/2) + c_4\sin(n\pi/2), \quad n \geq 1.$$

From

$$\begin{array}{llllll} 1 = a_1 & = 2c_1 & -c_2 & & +c_4 \\ 2 = a_2 & = 4c_1 & +c_2 & -c_3 & \\ 4 = a_3 & = 8c_1 & -c_2 & & -c_4 \\ 9 = a_4 & = 16c_1 & +c_2 & +c_3 & \end{array}$$

we learn that $c_1 = 8/15$, $c_2 = 1/6$, $c_3 = 3/10$, and $c_4 = 1/10$, so $a_n = (8/15)(2)^n + (1/6)(-1)^n + (3/10)\cos(n\pi/2) + (1/10)\sin(n\pi/2)$, $n \geq 1$.

[Note that a_n also counts the number of ways one can tile a $1 \times n$ chessboard using red 1×1 square tiles, white 1×2 rectangular tiles, blue 1×3 rectangular tiles, black 1×4 rectangular tiles and green 1×4 rectangular tiles.]

29. $x_{n+2} - x_{n+1} = 2(x_{n+1} - x_n)$, $n \geq 0$, $x_0 = 1$, and $x_1 = 5$.

$x_{n+2} - 3x_{n+1} + 2x_n = 0$

For $n \geq 0$, let $x_n = cr^n$, where $c, r \neq 0$. Then we get the characteristic equation $r^2 - 3r + 2 = 0 = (r-2)(r-1)$, so $x_n = A(2^n) + B(1^n) = A(2^n) + B$.

$x_0 = 1 = A + B$

$x_1 = 5 = 2A + B$

Hence $4 = A$, $B = -3$, and $x_n = 4(2^n) - 3 = 2^{n+2} - 3$, for $n \geq 0$.

30. Expanding by row 1, $D_n = 2D_{n-1} - D$, where D is an $(n-1)$ by $(n-1)$ determinant whose value, upon expansion by its first column, is D_{n-2}. Hence $D_n = 2D_{n-1} - D_{n-2}$. This recurrence relation determines the characteristic roots $r = 1, 1$ so the value of $D_n = A(1)^n + Bn(1)^n = A + Bn$.

$$D_1 = |2| = 2 \qquad D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$2 = D_1 = A + B$; $3 = D_2 = A + 2B \Longrightarrow B = A = 1$ and $D_n = 1 + n$, $n \geq 1$.

31. Let $b_n = a_n^2$, $b_0 = 16$, $b_1 = 169$.

This yields the linear relation $b_{n+2} - 5b_{n+1} + 4b_n = 0$ with characteristic roots $r = 4, 1$, so $b_n = A(1)^n + B(4)^n$.

$b_0 = 16$, $b_1 = 169 \Longrightarrow A = -35$, $B = 51$ and $b_n = 51(4)^n - 35$. Hence $a_n = \sqrt{51(4)^n - 35}$, $n \geq 0$.

32. $a_n = c_1 + c_2(7)^n$, $n \geq 0$, is the solution of $a_{n+2} + ba_{n+1} + ca_n = 0$, so $r^2 + br + c = 0$ is the characteristic equation and $(r-1)(r-7) = (r^2 - 8r + 7) = r^2 + br + c$. Consequently, $b = -8$ and $c = 7$.

33. Since $\gcd(F_1, F_0) = 1 = \gcd(F_2, F_1)$, consider $n \geq 2$. Then

$F_3 = F_2 + F_1 (= 1)$

$F_4 = F_3 + F_2$

$F_5 = F_4 + F_3$

$\vdots$

$F_{n+1} = F_n + F_{n-1}$.

Reversing the order of these equations we have the steps in the Euclidean Algorithm for computing the gcd of F_{n+1} and F_n, for $n \geq 2$. Since the last nonzero remainder is $F_1 = 1$, it follows that $\gcd(F_{n+1}, F_n) = 1$ for all $n \geq 2$.

34.

```
Program Fibonacci (input, output);
Var
    number: integer; {the input}
    i: integer; {i is a counter}
    current: integer;
    Fibonacci: array [1..100] of integer;
Begin
    Write ('This program is designed to determine if ');
    Write ('a given nonnegative integer is a ');
    Writeln ('Fibonacci number.');
    Writeln ('What nonnegative integer n do you wish to test?');
    Write ('n= ');
```

```
Readln (number);
If number < 0 then
     Writeln ('Your input is not appropriate.')
Else if number = 0 then
     Writeln ('Your number is the 0-th Fibonacci number.')
Else if number = 1 then
     Writeln ('Your number is the 1-st Fibonacci number.')
Else {number ≥ 2}
     Begin
          Fibonacci [1] := 1;
          Fibonacci [2] := 1;
          current := 1;
          i := 3;
          While number > current do
               Begin
                    Fibonacci [i] := Fibonacci [i-1] + Fibonacci [i-2];
                    current := Fibonacci [i];
                    If number < current then
                         Writeln ('Your number is not a Fibonacci number.')
                    Else if number = current then
                         Writeln ('Your number is the ', i:0, '-th Fibonacci number.')
                    Else i := i + 1     {number > count}
               End {while}
     End {else}
End.
```

Section 10.3

1. (a) $a_{n+1} - a_n = 2n + 3,\ n \geq 0,\ a_0 = 1$

$a_1 = a_0 + 0 + 3$

$a_2 = a_1 + 2 + 3 = a_0 + 2 + 2(3)$

$a_3 = a_2 + 2(2) + 3 = a_0 + 2 + 2(2) + 3(3)$

$a_4 = a_3 + 2(3) + 3 = a_0 + [2 + 2(2) + 2(3)] + 4(3)$

$\vdots$

$a_n = a_0 + 2[1 + 2 + 3 + \ldots + (n-1)] + n(3) = 1 + 2[n(n-1)/2] + 3n = 1 + n(n-1) + 3n = n^2 + 2n + 1 = (n+1)^2,\ n \geq 0.$

(b) $a_n = 3 + n(n-1)^2,\ n \geq 0$

(c) $a_{n+1} - 2a_n = 5,\ n \geq 0,\ a_0 = 1$

$a_1 = 2a_0 + 5 = 2 + 5$

$a_2 = 2a_1 + 5 = 2^2 + 2 \cdot 5 + 5$

$a_3 = 2a_2 + 5 = 2^3 + (2^2 + 2 + 1)5$

$\vdots$

$a_n = 2^n + 5(1 + 2 + 2^2 + \ldots + 2^{n-1}) = 2^n + 5(2^n - 1) = 6(2^n) - 5,\ n \geq 0.$

(d) $a_n = 2^n + n(2^{n-1}),\ n \geq 0.$

2. $a_n = \sum_{i=0}^{n} i^2.$
$a_{n+1} = a_n + (n+1)^2,\ n \geq 0,\ a_0 = 0.$
$a_{n+1} - a_n = (n+1)^2 = n^2 + 2n + 1$
$a_n^{(h)} = A,\ a_n^{(p)} = Bn + Cn^2 + Dn^3$
$B(n+1) + C(n+1)^2 + D(n+1)^3 = Bn + Cn^2 + Dn^3 + n^2 + 2n + 1 \Longrightarrow$
$Bn + B + Cn^2 + 2Cn + C + Dn^3 + 3Dn^2 + 3Dn + D = Bn + Cn^2 + Dn^3 + n^2 + 2n + 1.$
By comparing coefficients on like powers of n we find that $C + 3D = C + 1$, so $D = 1/3$. Also $B + 2C + 3D = B + 2$, so $C = 1/2$. Finally, $B + C + D = 1 \Longrightarrow B = 1/6$.
So $a_n = A + (1/6)n + (1/2)n^2 + (1/3)n^3$. With $a_0 = 0$, it follows that $A = 0$ and
$a_n = (1/6)(n)[1 + 3n + 2n^2] = (1/6)(n)(n+1)(2n+1), n \geq 0.$

3. (a) Let $a_n =$ the number of regions determined by the n lines under the conditions specified. When the n-th line is drawn there are $n - 1$ points of intersection and n segments are formed on the line. Each of these segments divides a region into two regions and this increases the number of previously existing regions, namely a_{n-1}, by n.
$a_n = a_{n-1} + n,\ n \geq 1,\ a_0 = 1.$
$a_n^{(h)} = A,\ a_n^{(p)} = Bn + Cn^2$
$Bn + Cn^2 = B(n-1) + C(n-1)^2 + n$
$Bn + Cn^2 - Bn + B - Cn^2 + 2Cn - C = n.$
By comparing the coefficients on like powers of n we have $B = C = 1/2$ and $a_n = A + (1/2)n + (1/2)n^2$.
$1 = a_0 = A$ so $a_n = 1 + (1/2)(n)(n+1),\ n \geq 0.$

(b) Let $b_n =$ the number of infinite regions that result for n such lines. When the nth line is drawn it is divided into n segments. The first and nth segments each create a new infinite region. Hence $b_n = b_{n-1} + 2,\ n \geq 2,\ b_1 = 2$. The solution of this recurrence relation is $b_n = 2n,\ n \geq 1,\ b_0 = 1$.

4. Let p_n be the value of the account n months after January 1 of the year the account is started.
$p_0 = 1000$
$p_1 = 1000 + (.005)(1000) + 200 = (1.005)p_0 + 200$
$p_{n+1} = (1.005)p_n + 200,\ 0 \leq n \leq 46$
$p_{48} = (1.005)p_{47}$
$p_{n+1} - 1.005p_n = 200,\ 0 \leq n \leq 46$
$p_n^{(h)} = A(1.005)^n,\ p_n^{(p)} = C$
$C - 1.005C = 200 \Longrightarrow C = -40{,}000$
$p_0 = A(1.005)^0 - 40{,}000 = 1000$, so $A = 41{,}000$
$p_n = (41{,}000)(1.005)^n - 40{,}000$
$p_{47} = (41{,}000)(1.005)^{47} - 40{,}000 = \$11{,}830.90$

$p_{48} = (1.005)p_{47} = \$11,890.05$

5. (a) $a_{n+2} + 3a_{n+1} + 2a_n = 3^n,\ n \geq 0,\ a_0 = 0,\ a_1 = 1.$
With $a_n = cr^n$, $c, r \neq 0$, the characteristic equation $r^2 + 3r + 2 = 0 = (r+2)(r+1)$ yields the characteristic roots $r = -1, -2$.
Hence $a_n^{(h)} = A(-1)^n + B(-2)^n$, while $a_n^{(p)} = C(3)^n$.
$C(3)^{n+2} + 3C(3)^{n+1} + 2C(3)^n = 3^n \Longrightarrow 9C + 9C + 2C = 1 \Longrightarrow C = 1/20.$
$a_n = A(-1)^n + B(-2)^n + (1/20)(3)^n$
$0 = a_0 = A + B + (1/20)$
$1 = a_1 = -A - 2B + (3/20)$
Hence $1 = a_0 + a_1 = -B + (4/20)$ and $B = -4/5$. Then $A = -B - (1/20) = 3/4$.
$a_n = (3/4)(-1)^n + (-4/5)(-2)^n + (1/20)(3)^n,\ n \geq 0$
(b) $a_n = (2/9)(-2)^n - (5/6)(n)(-2)^n + (7/9),\ n \geq 0$

6. $a_{n+2} - 6a_{n+1} + 9a_n = 3(2)^n + 7(3)^n,\ n \geq 0,\ a_0 = 1,\ a_1 = 4.$
$a_n^{(h)} = A(3)^n + Bn(3)^n \qquad a_n^{(p)} = C(2)^n + Dn^2(3)^n.$
Substituting $a_n^{(p)}$ into the given recurrence relation, by comparison of coefficients we find that $C = 3,\ D = 7/18$.
$a_n = A(3)^n + Bn(3)^n + 3(2)^n + (7/18)n^2(3)^n$
$1 = a_0, 4 = a_1 \Longrightarrow A = -1, B = 17/18$, so
$a_n = (-2)(3)^n + (17/18)n(3)^n + (7/18)n^2(3)^n + 3(2)^n,\ n \geq 0.$

7. Here the characteristic equation is $r^3 - 3r^2 + 3r - 1 = 0 = (r-1)^3$, so $r = 1, 1, 1$ and $a_n^{(h)} = A + Bn + Cn^2,\ a_n^{(p)} = Dn^3 + En^4$.
$D(n+3)^3 + E(n+3)^4 - 3D(n+2)^3 - 3E(n+2)^4 + 3D(n+1)^3 + 3E(n+1)^4 - Dn^3 - En^4 = 3 + 5n \Longrightarrow D = -3/4,\ E = 5/24.$
$a_n = A + Bn + Cn^2 - (3/4)n^3 + (5/24)n^4,\ n \geq 0.$

8. $a_{n+1} = 3a_n + 3^n,\ a_0 = 1,\ a_1 = 4$. The term 3^n accounts for the sequences of length n that end in 3; $3a_n$ accounts for those sequences of length n that end in 0, 1, or 2.
$a_n^{(h)} = A3^n,\ a_n^{(p)} = Bn3^n$
$B(n+1)3^{n+1} = 3(Bn3^n) + 3^n \Longrightarrow 3B(n+1) = 3Bn + 1 \Longrightarrow 3B = 1 \Longrightarrow B = 1/3$
$a_n = A \cdot 3^n + n \cdot 3^{n-1}$
$1 = a_0 = A$, so $a_n = 3^n + n3^{n-1}, n \geq 0.$

9. From Example 10.29, $P = (Si)[1 - (1+i)^{-T}]^{-1}$, where P is the payment, S is the loan (\$2500), T is the number of payments (24) and i is the interest rate per month (1%).

$$P = (2500)(0.01)[1 - (1.01)^{-24}]^{-1} = \$117.68.$$

10. $a_{n+2} + b_1 a_{n+1} + b_2 a_n = b_3 n + b_4$
$a_n = c_1 2^n + c_2 3^n + n - 7$
$r^2 + b_1 r + b_2 = (r-2)(r-3) = r^2 - 5r + 6 \Longrightarrow b_1 = -5,\ b_2 = 6$

$a_n^{(p)} = n - 7$
$[(n+2)-7] - 5[(n+1)-7] + 6(n-7) = b_3 n + b_4 \Longrightarrow b_3 = 2,\ b_4 = -17.$

11. (a) Let $a_n^2 = b_n, n \geq 0$
$b_{n+2} - 5b_{n+1} + 6b_n = 7n$
$b_n^{(h)} = A(3^n) + B(2^n),\ b_n^{(p)} = Cn + D$
$C(n+2) + D - 5[C(n+1) + D] + 6(Cn + D) = 7n \Longrightarrow C = 7/2,\ D = 21/4$
$b_n = A(3^n) + B(2^n) + (7n/2) + (21/4)$
$b_0 = a_0^2 = 1,\ b_1 = a_1^2 = 1$
$1 = b_0 = A + B + 21/4$
$1 = b_1 = 3A + 2B + 7/2 + 21/4$
$3A + 2B = -31/3$
$2A + 2B = -34/4$
$A = 3/4,\ B = -5$
$a_n = [(3/4)(3)^n - 5(2)^n + (7n/2) + (21/4)]^{1/2},\ n \geq 0$
(b) $a_n^2 - 2a_{n-1} = 0,\ n \geq 1,\ a_0 = 2$
$a_n^2 = 2a_{n-1}$
$\log_2 a_n^2 = \log_2(2a_{n-1}) = \log_2 2 + \log_2 a_{n-1}$
$2\log_2 a_n = 1 + \log_2 a_{n-1}$
Let $b_n = \log_2 a_n$.
The solution of the recurrence relation $2b_n = 1 + b_{n-1}$ is $b_n = A(1/2)^n + 1$.
$b_0 = \log_2 a_0 = \log_2 2 = 1$, so $1 = b_0 = A + 1$ and $A = 0$.
Consequently, $b_n = 1,\ n \geq 0$, and $a_n = 2,\ n \geq 0$.

12. Consider the nth symbol for the strings counted by a_n. For $n \geq 2$, we consider two cases:
(1) If this symbol is 0, 2, or 3, then the preceding $n-1$ symbols provide a string of length $n-1$ counted by a_{n-1}.
(2) If this symbol is 1, then the preceding $n-1$ symbols contain an even number of 1s – there are $4^{n-1} - a_{n-1}$ such strings of length $n-1$.

Since these two cases are exhaustive and have nothing in common we have

$$a_n = 3a_{n-1} + (4^{n-1} - a_{n-1}) = 2a_{n-1} + 4^{n-1}, \quad n \geq 2.$$

Here $a_n = a_n^{(h)} + a_n^{(p)}$, where $a_n^{(p)} = A(4^{n-1})$ and $a_n^{(h)} = c(2^n)$.

Substituting $a_n^{(p)}$ into the above recurrence relation for a_n we find that $A(4^{n-1}) = 2A(4^{n-2}) + 4^{n-1}$, so $4A = 2A + 4$ and $A = 2$.

There is only one string of length 1 where there is an odd number of 1s – namely, the string 1. So

$$a_1 = 1 = c(2) + 2(4^0), \quad \text{and} \quad c = -1/2.$$

Consequently, $a_n = (-1/2)(2^n) + 2(4^{n-1}),\ n \geq 1$.

We can check this result by using an exponential generating function. Here a_n is the coefficient of $x^n/n!$ in $e^x(\frac{e^x - e^{-x}}{2})(e^x)^2 = \frac{1}{2}e^{4x} - \frac{1}{2}e^{2x}$. Hence $a_n = (\frac{1}{2})(4^n) - \frac{1}{2}(2^n),\ n \geq 1$.

13. (a) Consider the 2^n binary strings of length n. Half of these strings (2^{n-1}) end in 0 and the other half (2^{n-1}) in 1. For the 2^{n-1} binary strings of length $(n-1)$, there are t_{n-1} runs. When we append 0 to each of these strings we get $t_{n-1} + (\frac{1}{2})(2^{n-1})$ runs, where the additional $(\frac{1}{2})(2^{n-1})$ runs arise when we append 0 to the $(\frac{1}{2})(2^{n-1})$ strings of length $(n-1)$ that end in 1. Upon appending 1 to each of the 2^{n-1} binary strings of length $n-1$, we get the remaining $t_{n-1} + (\frac{1}{2})(2^{n-1})$ runs. Consequently we find that

$$t_n = 2t_{n-1} + 2^{n-1}, \quad n \geq 2, \quad t_1 = 2.$$

Here $t_n^{(h)} = c(2^n)$, so $t_n^{(p)} = An(2^n)$. Substituting $t_n^{(p)}$ into the recurrence relation we have

$$\begin{aligned} An(2^n) &= 2A(n-1)2^{n-1} + 2^{n-1} \\ &= An(2^n) - A(2^n) + 2^{n-1} \end{aligned}$$

By comparison of coefficients for 2^n and $n2^n$ we learn that $A = \frac{1}{2}$. Consequently, $t_n = t_n^{(h)} + t_n^{(p)} = c(2^n) + n(2^{n-1})$, and $2 = t_1 = c(2) + 1 \Rightarrow c = \frac{1}{2}$, so $t_n = (\frac{1}{2})(2^n) + n(2^{n-1}) = (n+1)(2^{n-1})$, $n \geq 1$.
(b) Here there are 4^n quaternary strings of length n and 4^{n-1} of these end in each of the one symbol suffices 0,1,2, and 3. In this case

$$t_n = 4[t_{n-1} + (\frac{3}{4})4^{n-1}] = 4t_{n-1} + 3(4^{n-1}), \quad n \geq 2, \quad t_1 = 4.$$

Comparable to the solution for part (a), here $t_n^{(h)} = c(4^n)$ and $t_n^{(p)} = An(4^n)$. So $An4^n = 4A(n-1)4^{n-1} + (3)(4^{n-1}) = An4^n - A(4^n) + (\frac{3}{4})4^n$, and $A = \frac{3}{4}$. Consequently, $t_n = c(4^n) + (\frac{3}{4})n4^n$ and $4 = t_1 = 4c + (\frac{3}{4})(4) \Rightarrow c = \frac{1}{4}$, so $t_n = (\frac{1}{4})4^n + (\frac{3}{4})n4^n = 4^{n-1}(1+3n)$, $n \geq 1$.
(c) For an alphabet Σ, where $|\Sigma| = r \geq 1$, there are r^n strings of length n and these r^n strings determine a total of $r^{n-1}[1 + (r-1)n]$ runs. [Note: This formula includes the case where $r = 1$.]

14. (a) $s_{n+1} = s_n + t_{n+1} = s_n + (n+1)(n+2)/2$
$s_{n+1} - s_n = (1/2)(n^2 + 3n + 2)$
$s_n = s_n^{(h)} + s_n^{(p)}$
$s_{n+1}^{(h)} - s_n^{(h)} = 0$, so $s_n^{(h)} = A(1^n) = A$
$s_n^{(p)} = n(Bn^2 + Cn + D) = Bn^3 + Cn^2 + Dn$
$B(n+1)^3 + C(n+1)^2 + D(n+1) - Bn^3 - Cn^2 - Dn = (1/2)(n^2 + 3n + 2) \Longrightarrow$
$B = 1/6$, $C = 1/2$, $D = 1/3$
$s_n = A + (1/6)n^3 + (1/2)n^2 + (1/3)n$
Since $s_1 = t_1 = 1$, $1 = A + (1/6) + (1/2) + (1/3) \Longrightarrow A = 0$, and $s_n = (1/6)(n)(n+1)(n+2)$.

(b) (i) $s_{10,000,000}$ atoms.
(ii) $s_{99,999} - s_{10,000} \doteq 1.665 \times 10^{14}$ atoms.

15.

```
Program Towers_of_Hanoi (input, output);
Var
    number: integer;    {number = number of disks}

Procedure Move_The_Disks (n: integer; start, inter, finish; char);
{This procedure will move n disks from the start peg to the finish peg using inter as
    the intermediary peg.}
Begin
    If n=1 then
        Writeln ('Move disk from  ', start, '  to  ', finish, '.')
    Else
        Begin
            Move_The_Disks (n-1, start, finish, inter);
            Move_The_Disks (1, start, ' ', finish);
            Move_The_Disks (n-1, inter, start, finish)
        End {else}
End; {procedure}
Begin {main program}
    Write ('How many disks are there?  ');
    Readln (number);
    If number < 1 then
        Writeln ('Your input is not appropriate.')
    Else
        Move_The_Disks (number, '1','2','3')
End.
```

Section 10.4

1. (a) $a_{n+1} - a_n = 3^n \quad n \geq 0,\ a_0 = 1$
Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$.
$\sum_{n=0}^{\infty} a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} 3^n x^{n+1}$
$[f(x) - a_0] - xf(x) = x\sum_{n=0}^{\infty}(3x)^n = x/(1-3x)$
$f(x) - 1 - xf(x) = x/(1-3x)$
$f(x) = 1/(1-x) + x/((1-x)(1-3x)) = 1/(1-x) + (-1/2)/(1-x) + (1/2)/(1-3x) = (1/2)/(1-x) + (1/2)(1-3x)$, and $a_n = (1/2)[1+3^n], n \geq 0$.

(b) $a_n = 1 + [n(n-1)(2n-1)/6], n \geq 0$.

(c) $a_{n+2} - 3a_{n+1} + 2a_n = 0,\ n \geq 0,\ a_0 = 1,\ a_1 = 6$

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 3\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + 2\sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 3x\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + 2x^2\sum_{n=0}^{\infty} a_n x^n = 0$$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$(f(x)-1-6x)-3x(f(x)-1)+2x^2f(x) = 0$, and $f(x)(1-3x+2x^2) = 1+6x-3x = 1+3x$. Consequently,

$$f(x) = \frac{1+3x}{(1-2x)(1-x)} = \frac{5}{(1-2x)} + \frac{(-4)}{(1-x)} = 5\sum_{n=0}^{\infty}(2x)^n - 4\sum_{n=0}^{\infty} x^n,$$

and $a_n = 5(2^n) - 4,\ \ n \geq 0$.

(d) $a_{n+2} - 2a_{n+1} + a_n = 2^n,\ n \geq 0,\ a_0 = 1,\ a_1 = 2$
$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 2\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=0}^{\infty} 2^n x^{n+2}$
Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then
$[f(x) - a_0 - a_1x] - 2x[f(x) - a_0] + x^2 f(x) = x^2\sum_{n=0}^{\infty}(2x)^n$
$f(x) - 1 - 2x - 2xf(x) + 2x + x^2f(x) = x^2/(1-2x)$
$(x^2 - 2x + 1)f(x) = 1 + x^2/(1-2x) \Longrightarrow f(x) = 1/(1-x)^2 +$
$x^2/((1-2x)(1-x)^2) = (1-2x+x^2)/((1-x)^2(1-2x)) = 1/(1-2x) = 1 + 2x + (2x)^2 + \ldots,$
so $a_n = 2^n,\ n \geq 0$.

2. $a(n,r) = a(n-1,r-1) + a(n-1,r),\ r \geq 1$.
$\sum_{r=1}^{\infty} a(n,r)x^r = \sum_{r=1}^{\infty} a(n-1,r-1)x^r + \sum_{r=1}^{\infty} a(n-1,r)x^r$
$a(n,0) = 1,\ n \geq 0;\ a(0,r) = 0,\ r > 0$.
Let $f_n = \sum_{r=0}^{\infty} a(n,r)x^r$.
$f_n - a(n,0) = xf_{n-1} + f_{n-1} - a(n-1,0)$
$f_n = (1+x)f_{n-1}$ and $f_n = (1+x)^n f_0$
$f_0 = \sum_{r=0}^{\infty} a(0,r)x^r = a(0,0) + a(0,1)x + a(0,2)x^2 + \ldots = a(0,0) = 1$, so $f_n = (1+x)^n$ generates $a(n,r),\ r \geq 0$.

3. (a) $a_{n+1} = -2a_n - 4b_n$
$b_{n+1} = 4a_n + 6b_n$
$n \geq 0,\ a_0 = 1,\ b_0 = 0$.
Let $f(x) = \sum_{n=0}^{\infty} a_n x^n,\ g(x) = \sum_{n=0}^{\infty} b_n x^n$.
$\sum_{n=0}^{\infty} a_{n+1}x^{n+1} = -2\sum_{n=0}^{\infty} a_n x^{n+1} - 4\sum_{n=0}^{\infty} b_n x^{n+1}$
$\sum_{n=0}^{\infty} b_{n+1}x^{n+1} = 4\sum_{n=0}^{\infty} a_n x^{n+1} + 6\sum_{n=0}^{\infty} b_n x^{n+1}$
$f(x) - a_0 = -2xf(x) - 4xg(x)$
$g(x) - b_0 = 4xf(x) + 6xg(x)$
$f(x)(1+2x) + 4xg(x) = 1$
$f(x)(-4x) + (1-6x)g(x) = 0$

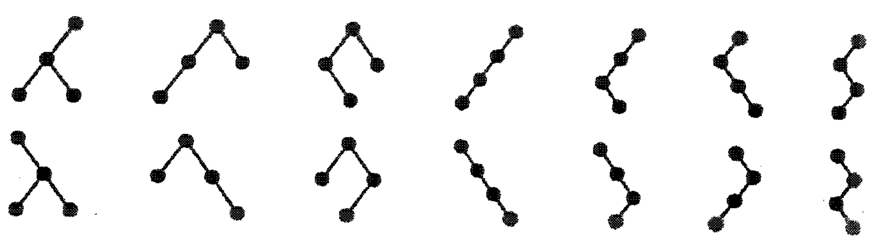
$$f(x) = \frac{\begin{vmatrix} 1 & 4 \\ 0 & (1-6x) \end{vmatrix}}{\begin{vmatrix} (1+2x) & 4x \\ -4x & (1-6x) \end{vmatrix}} = (1-6x)/(1-2x)^2 =$$

$(1-6x)(1-2x)^{-2} = (1-6x)[\binom{-2}{0} + \binom{-2}{1}(-2x) + \binom{-2}{2}(-2x)^2 + \ldots]$
$a_n = \binom{-2}{n}(-2)^n - 6\binom{-2}{n-1}(-2)^{n-1} = 2^n(1-2n),\ n \geq 0$
$f(x)(-4x) + (1-6x)g(x) = 0 \Longrightarrow g(x) = (4x)f(x)(1-6x)^{-1} \Longrightarrow g(x) =$
$4x(1-2x)^{-2}$ and $b_n = 4\binom{-2}{n-1}(-2)^{n-1} = n(2^{n+1}),\ n \geq 0.$

(b) $a_n = (-3/4) + (1/2)(n+1) + (1/4)(3^n),\ n \geq 0$
$b_n = (3/4) + (1/2)(n+1) - (1/4)(3^n),\ n \geq 0$

Section 10.5

1. $b_4 = b_0b_3 + b_1b_2 + b_2b_1 + b_3b_0 = 2(5+2) = 14$
$b_n = [(2n)!/((n+1)!(n!))],\ b_4 = 8!/(5!4!) = 14$



2. $$(1/2)(1/(2n+1))\binom{2n+2}{n+1} = (1/2)(1/(2n+1))\left[\frac{(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!}\right]$$

$= (1/2)[(2n+2)/(n+1)^2][(2n)!/(n!)^2] = (1/(n+1))\binom{2n}{n}$

3. $$\binom{2n-1}{n} - \binom{2n-1}{n-2} = \left[\frac{(2n-1)!}{n!(n-1)!}\right] - \left[\frac{(2n-1)!}{(n-2)!(n+1)!}\right] =$$

$$\left[\frac{(2n-1)!(n+1)}{(n+1)!(n-1)!}\right] - \left[\frac{(2n-1)!(n-1)}{(n-1)!(n+1)!}\right] = \left[\frac{(2n-1)!}{(n+1)!(n-1)!}\right][(n+1)-(n-1)] =$$

$$\frac{(2n-1)!(2)}{(n+1)!(n-1)!} = \frac{(2n-1)!(2n)}{(n+1)!n!} = \frac{(2n)!}{(n+1)(n!)(n!)} = \frac{1}{(n+1)}\binom{2n}{n}$$

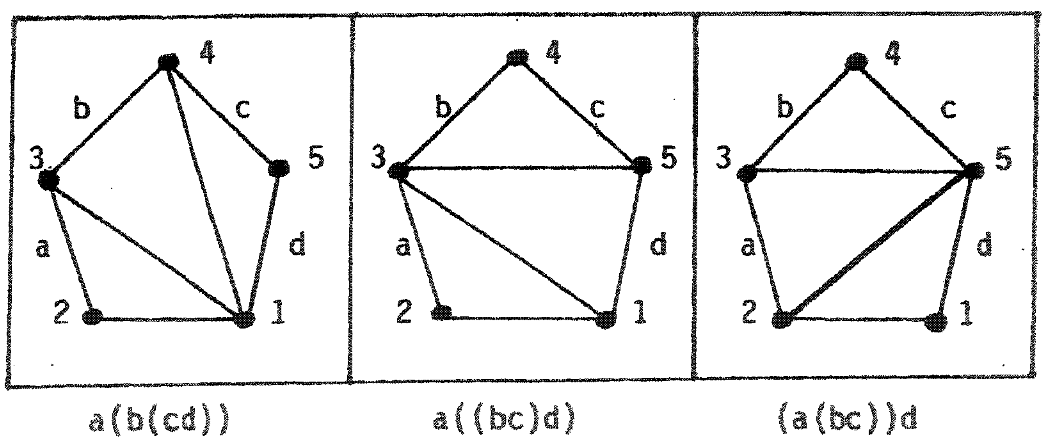
4. (a) No (b) Yes (c) No (d) Yes

5.
(a) $(1/9)\binom{16}{8}$ (b) $[(1/4)\binom{6}{3}]^2$
(c) $[(1/6)\binom{10}{5}][(1/3)\binom{4}{2}]$ (d) $(1/6)\binom{10}{5}$

6. (a) t_{n+1}: For $n \geq 2$, let $v_1, v_2, \ldots, v_{n+1}$ be the vertices of a convex $(n+1)$-gon. In each partition of this polygon into triangles, with no diagonals intersecting, the side v_1v_{n+1} is part of one of these triangles. The triangle is given by $v_1v_iv_{n+1}$, $2 \leq i \leq n$. For each $2 \leq i \leq n$, once the triangle $v_1v_iv_{n+1}$ is drawn, we consider the resulting polygon on $v_1, v_2, \ldots, v_i$ and the other polygon on $v_i, v_{i+1}, \ldots, v_{n+1}$. The former polygon can be partitioned into triangles, with no intersecting diagonals, in t_i ways; the latter polygon in $t_{n+1-i+1} = t_{n+2-i}$ ways. This results in a total of $t_i \cdot t_{n+2-i}$ triangular partitions with no overlapping diagonals. As i varies from 2 to n we have $t_{n+1} = t_2t_n + t_3t_{n-1} + \ldots + t_{n-1}t_3 + t_nt_2 = \sum_{i=2}^{n} t_it_{n+2-i}$.

(b) From Example 10.36, $t_n = b_{n-2}$, $n \geq 2$. With $b_n = (2n)!/[(n+1)!n!]$ we have $t_n = (2n-4)!/[(n-1)!(n-2)!]$, $n \geq 2$.

7. (a)



(b) (iii) $(((ab)c)d)e$ (iv) $(ab)(c(de))$

8. (a) $b_{n+1} = \left(\frac{1}{n+2}\right)\binom{2n+2}{n+1} = \left(\frac{1}{n+2}\right)\left[\frac{(2n+2)!}{(n+1)!(n+1)!}\right] =$

$$\frac{(2n+2)(2n+1)(2n)!}{(n+2)(n+1)^2(n!)^2} = \frac{2(2n+1)}{(n+2)} \cdot \frac{(2n)!}{(n+1)(n!)^2} = \frac{2(2n+1)}{(n+2)}\left[\frac{1}{n+1}\binom{2n}{n}\right]$$

$$= \frac{2(2n+1)}{(n+2)} b_n.$$

9. In Fig. 10.23 note how vertex 1 is always paired with an even numbered vertex. This must be the case for each $n \geq 0$, otherwise we end up with intersecting chords.
For each $n \geq 1$, let $1 \leq k \leq n$, so that $2 \leq 2k \leq 2n$. Drawing the chord connecting vertex 1 with vertex $2k$, we divide the circumference of the circle into two segments – one containing the vertices $2, 3, \ldots, 2k-1$, and the other containing the vertices $2k+1, 2k+2, \ldots, 2n$. These vertices can be connected by nonintersecting chords in $a_{k-1}a_{n-k}$ ways, so

$a_n = a_0 a_{n-1} + a_1 a_{n-2} + a_2 a_{n-3} + \ldots + a_{n-2} a_1 + a_{n-1} a_0.$

Since $a_0 = 1, a_1 = 1, a_2 = 2$, and $a_3 = 5$, we find that $a_n = b_n$, the nth Catalan number.

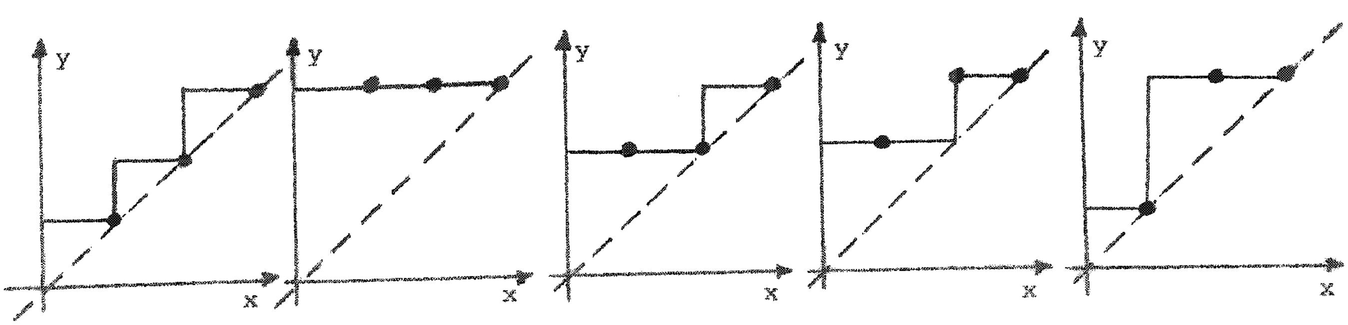
10. Consider, for example, the second mountain range in Fig. 10.24. This path is made up from the moves N S N N S S. Replace each 'N' by a '1' and each 'S' by a '0' to get 1 0 1 1 0 0 – a sequence of three 1's and three 0's, where the number of 0's never exceeds the number of 1's as the sequence is read from left to right. We know that the number of such sequences is $5 (= b_3)$. In general, for $n \in \mathbf{N}$, there are b_n such sequences and, consequently, b_n such mountain ranges. [Note: the above argument could also be established by replacing 'N' by 'push' and 'S' by 'pop', setting up a one-to-one correspondence between the mountain ranges and the permutations obtained with the stack.]

11.

(a)

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$
1	1	3	2	2	1
2	2	3	2	3	3
3	3	3	3	3	3

(b) The functions in part (a) correspond with the following paths from $(0,0)$ to $(3,3)$.



(c) The mountain ranges in Fig. 10.24 of the text.
(d) For $n \in \mathbf{Z}^+$, the number of monotone increasing functions $f : \{1, 2, 3, \ldots, n\} \rightarrow \{1, 2, 3, \ldots, n\}$ where $f(i) \geq i$ for all $1 \leq i \leq n$, is $b_n = (1/(n+1))\binom{2n}{n}$, the n-th Catalan number. This follows from Exercise 3 in Section 1.5. There is a one-to-one correspondence between the paths described in that exercise and the functions being dealt with here.

12.

(a)

x	$g_1(x)$	$g_2(x)$	$g_3(x)$	$g_4(x)$	$g_5(x)$
1	1	1	1	1	1
2	2	1	2	1	1
3	3	1	2	2	3

(b) For $1 \leq i \leq 5$, f_i [in part (a) of the previous exercise] corresponds with g_i. We demonstrate the correspondence for $i = 1, 2,$ and 4.

($i = 1$)

x	$f_1(x)$	$g_1(x)$
1	1	1
2	2	2
3	3	3

($i = 2$)

x	$f_2(x)$	$g_2(x)$
1	3	1
2	3	1
3	3	1

($i = 4$)

x	$f_4(x)$	$g_4(x)$
1	2	1
2	3	1
3	3	2

Consider the column for any f_i. In that column replace each entry k by $3 - (k - 1)$: so 1's and 3's are interchanged while 2's remain as 2's. Then reverse the order of this new column. The result is the column for g_i. [In order to generalize this to the case where the domain and codomain are $\{1, 2, 3, \ldots, n\}$, $n \in \mathbf{Z}^+$, we write down two columns — one for $1, 2, 3, \ldots, n$ and another listing $f_i(1), f_i(2), f_i(3), \ldots, f_i(n)$. Each entry k [in the column for f_i] is replaced by $n - (k - 1)$. Then the order of the column is reversed, giving us the image under the corresponding function g_i.

(c) For $n \in \mathbf{Z}^+$, the number of monotone increasing functions $g : \{1, 2, 3, \ldots, n\} \rightarrow \{1, 2, 3, \ldots, n\}$, where $g(i) \leq i$ for all $1 \leq i \leq n$, is $(1/(n+1))\binom{2n}{n} = b_n$, the n-th Catalan number.

13. For $n \in \mathbf{N}$, let a_n count the number of these arrangements for a row of n contiguous pennies. Here $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, and $a_3 = 5$. For the general situation, let $n \in \mathbf{N}$ and consider a contiguous row of $n + 1$ pennies. These $n + 1$ pennies provide n possible locations for placing a penny on the second level. There are two cases to consider:
(1) The first location (as the second level is scanned from left to right) that is empty is at position i, where $1 \leq i \leq n$. So there are $i - 1$ pennies (above the first i pennies in the bottom contiguous row) in the positions to the left of position i. These $i - 1$ contiguous pennies provide a_{i-1} possible arrangements. The $n - [(i - 1) + 1] = n - i$ positions (on the second level) to the right of position i are determined by a row of $n - i + 1$ contiguous pennies at the bottom level and these $n - i + 1$ contiguous pennies provide a_{n-i+1} arrangements. As i goes from 1 to n we get a total of
$\sum_{i=1}^{n} a_{i-1}a_{n-i+1} = a_0a_n + a_1a_{n-1} + a_2a_{n-2} + \cdots + a_{n-1}a_1$ arrangements.
(2) The only situation not covered in case 1 occurs when there is no empty position on the second level. So we have a row of $n + 1$ contiguous pennies on the bottom level and n contiguous pennies on the second level — and above these $2n + 1$ pennies there are $a_n (= a_na_0)$ possible arrangements.

From cases (1) and (2) we have $a_{n+1} = a_0a_n + a_1a_{n-1} + a_2a_{n-2} + \cdots + a_{n-1}a_1 + a_na_0$, $a_0 = 1$, so $a_n = b_n = (1/(n+1))\binom{2n}{n}$, the nth Catalan numbers.

14. (a) $s_2 = 6$
(b) $s_2 = \binom{4}{0}b_2 + \binom{3}{1}b_1 + \binom{2}{2}b_0$
(c) $s_3 = \binom{6}{0}b_3 + \binom{5}{1}b_2 + \binom{4}{2}b_1 + \binom{3}{3}b_0 = 22.$
(d) Consider those paths from $(0,0)$ to (n,n) where there are r diagonal moves, for $0 \le r \le n$. How can one generate such a path? It must contain $(n-r)$ R's and $(n-r)$ U's and these $2(n-r)$ letters provide 1 (location at the start) $+2(n-r)-1$ [locations between the $(n-r)$ R's and $(n-r)$ U's] $+1$ (location at the end) $= 2(n-r)+1$ locations in total, for inserting the r D's. Further, these $2(n-r)+1$ locations are selected with repetitions allowed. So there are $\binom{2(n-r)+1+r-1}{r}b_{n-r} = \binom{2n-r}{r}b_{n-r}$ paths with r D's, $n-r$ R's, and $n-r$ U's (with the path never crossing the line $y = x$). Summing over r we have $s_n = \sum_{r=0}^{n}\binom{2n-r}{r}b_{n-r}$.

15.

(a) 1 3 2, 2 3 1: $E_3 = 2$

(b)

1 3 2 5 4	3 4 1 5 2
1 4 2 5 3	3 4 2 5 1
1 4 3 5 2	3 5 1 4 2
1 5 2 4 3	3 5 2 4 1
1 5 3 4 2	
2 3 1 5 4	4 5 1 3 2
2 4 1 5 3	4 5 2 3 1
2 4 3 5 1	
2 5 1 4 3	
2 5 3 4 1	$E_5 = 16$

(c) For each rise/fall permutation, n cannot be in the first position (unless $n = 1$); n is the second component of a rise in such a permutation. Consequently, n must be at position 2 or 4 ... or $2\lfloor n/2 \rfloor$.

(d) Consider the location of n in a rise/fall permutation $x_1x_2x_3\ldots x_{n-1}x_n$ of $1, 2, 3, \ldots, n$. The number n is in position $2i$ for some $1 \le i \le \lfloor n/2 \rfloor$. Here there are $2i-1$ numbers that precede n. These can be selected in $\binom{n-1}{2i-1}$ ways and give rise to E_{2i-1} rise/fall permutations. The $(n-1)-(2i-1) = n-2i$ numbers that follow n give rise to E_{n-2i} rise/fall permutations. Consequently, $E_n = \sum_{i=1}^{\lfloor n/2 \rfloor}\binom{n-1}{2i-1}E_{2i-1}E_{n-2i}$, $n \ge 2$.

(e) Comparable to part (c), here we realize that for $n \ge 2$, 1 is at the end of the permutation or is the first component of a rise in such a permutation. Therefore, 1 must be at position 1 or 3 or ... or $2\lfloor (n-1)/2 \rfloor + 1$.

(f) As in part (d) look now for 1 in a rise/fall permutation of $1, 2, 3, \ldots, n$. We find 1 is position $2i+1$ for some $0 \le i \le \lfloor (n-1)/2 \rfloor$. Here there are $2i$ numbers that precede 1. These can be selected in $\binom{n-1}{2i}$ ways and give rise to E_{2i} rise/fall permutations. The remaining $(n-1)-2i = n-2i-1$ numbers that follow 1 give rise to E_{n-2i-1} rise/fall permutations. Therefore, $E_n = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1}{2i} E_{2i}E_{n-2i-1},\ n \geq 1$.

(g) From parts (d) and (f) we have:

(d) $E_n = \binom{n-1}{1}E_1E_{n-2} + \binom{n-1}{3}E_3E_{n-4} + \cdots + \binom{n-1}{2\lfloor n/2 \rfloor - 1}E_{2\lfloor n/2 \rfloor -1}E_{n-2\lfloor n/2 \rfloor}$

(f) $E_n = \binom{n-1}{0}E_0E_{n-1} + \binom{n-1}{2}E_2E_{n-3} + \cdots + \binom{n-1}{2\lfloor (n-1)/2 \rfloor}E_{2\lfloor (n-1)/2 \rfloor}E_{n-2\lfloor (n-1)/2 \rfloor -1}$

Adding these equations we find that $2E_n = \sum_{i=0}^{n-1} \binom{n-1}{i} E_iE_{n-i-1}$ or $E_n =$ $(1/2)\sum_{i=0}^{n-1} \binom{n-1}{i} E_iE_{n-i-1}$.

(h)

$$\begin{aligned}
E_6 &= (1/2)\textstyle\sum_{i=0}^{5} \binom{5}{i} E_iE_{5-i} \\
&= (1/2)[\binom{5}{0}E_0E_5 + \binom{5}{1}E_1E_4 + \binom{5}{2}E_2E_3 + \binom{5}{3}E_3E_2 + \binom{5}{4}E_4E_1 + \binom{5}{5}E_5E_0] \\
&= (1/2)[1\cdot 1\cdot 16 + 5\cdot 1\cdot 5 + 10\cdot 1\cdot 2 + 10\cdot 2\cdot 1 + 5\cdot 5\cdot 1 + 1\cdot 16\cdot 1] \\
&= (1/2)[16+25+20+20+25+16] = 61
\end{aligned}$$

$$\begin{aligned}
E_7 &= (1/2)\textstyle\sum_{i=0}^{6} \binom{6}{i} E_iE_{6-i} \\
&= (1/2)[1\cdot 1\cdot 61 + 6\cdot 1\cdot 16 + 15\cdot 1\cdot 5 + 20\cdot 2\cdot 2 + 15\cdot 5\cdot 1 + 6\cdot 16\cdot 1 + 1\cdot 61\cdot 1] \\
&= 272
\end{aligned}$$

(i) Consider the Maclaurin series expansions
$\sec x = 1 + x^2/2! + 5x^4/4! + 61x^6/6! + \cdots$ and
$\tan x = x + 2x^3/3! + 16x^5/5! + 272x^7/7! + \cdots$
One finds that $\sec x + \tan x$ is the exponential generating function of the sequence 1,1,1,2,5,16,61,272,... – namely, the sequence of Euler numbers.

Section 10.6

1. (a) $f(n) = (5/3)(4n^{\log_3 4} - 1)$ and $f \in O(n^{\log_3 4})$ for $n \in \{3^i | i \in \mathbf{N}\}$
(b) $f(n) = 7(\log_5 n + 1)$ and $f \in O(\log_5 n)$ for $n \in \{5^i | i \in \mathbf{N}\}$

2. As in the proof of Theorem 10.1 we find that $f(n) = a^k f(1) + c[1 + a + a^2 + \ldots + a^{k-1}] =$ $a^k d + c[1 + a + a^2 + \ldots + a^{k-1}]$.
(a) For $a = 1$, $f(n) = d + ck = d + c\log_b n$, since $n = b^k$.
(b) For $a > 1$, $f(n) = a^k d + c[(a^k - 1)/(a-1)]$
$n = b^k \Longrightarrow k = \log_b n$
$a^k = a^{\log_b n} = n^x \Longrightarrow \log_b(a^{\log_b n}) = \log_b n^x \Longrightarrow (\log_b n)(\log_b a) = x(\log_b n) \Longrightarrow x = \log_b a$.
So for $a > 1$, $f(n) = dn^{\log_b a} + (c/(a-1))[n^{\log_b a} - 1]$.

3. (a) $f \in O(\log_b n)$ on $\{b^k | k \in \mathbf{N}\}$
(b) $f \in O(n^{\log_b a})$ on $\{b^k | k \in \mathbf{N}\}$

4. (a) $d = 0,\ a = 2,\ b = 5,\ c = 3$

$f(n) = 3[n^{\log_5 2} - 1]$
$f \in O(n^{\log_5 2})$
(b) $d = 1,\ a = 1,\ b = 2,\ c = 2$
$f(n) = 1 + 2\log_5 n$
$f \in O(\log_5 n)$

5. (a) $f(1) = 0$
$f(n) = 2f(n/2) + 1$
From Exercise 2(b), $f(n) = n - 1$.

(b) The equation $f(n) = f(n/2) + (n/2)$ arises as follows: There are $(n/2)$ matches played in the first round. Then there are $(n/2)$ players remaining, so we need $f(n/2)$ additional matches to determine the winner.

6. (i) Corollary 10.1: From Theorem 10.1

(1) $f(n) = c\,(\log_b n + 1)$ for $n = 1, b, b^2, \ldots,$ when $a = 1$. Hence $f \in O(\log_b n)$ on $S = \{b^k | k \in \mathbf{N}\}$.

(2) $f(n) = [c/(a-1)][an^{\log_b a} - 1]$ for $n = 1, b, b^2, \ldots,$ when $a \geq 2$. Therefore $f \in O(n^{\log_b a})$ on $S = \{b^k | k \in \mathbf{N}\}$.

(ii) Theorem 10.2(b): Since $f \in O(g)$ on S, and $g \in O(n \log n)$, it follows that $f \in O(n \log n)$ on S. So by Definition 10.1 we know that there exist constants $m \in \mathbf{R}^+$ and $s \in \mathbf{Z}^+$ such that $f(n) = |f(n)| \leq m|n \log n| = mn \log n$ for all $n \in S$ where $n \geq s$. We need to find constants $M \in \mathbf{R}^+$ and $s_1 \in \mathbf{Z}^+$ so that $f(n) \leq Mn \log n$ for all $n \geq s_1$ – not just those $n \in S$.

Choose $t \in \mathbf{Z}^+$ so that $s < b^k < t < b^{k+1}$ (and $\log s \geq 1$). Since f is monotone increasing and positive,

$$\begin{aligned}
f(t) \leq f(b^{k+1}) &\leq m\, b^{k+1} \log(b^{k+1}) \\
&= m\, b^{k+1}[\log b^k + \log b] \\
&= m\, b^{k+1} \log b^k + m\, b^{k+1} \log b \\
&= m\, b[b^k(\log b^k + \log b)] \\
&< m\, b[b^k \log b^k(1 + \log b)] \\
&= m\, b(1 + \log b)(b^k \log b^k) \\
&< m\, b(1 + \log b)t \log t
\end{aligned}$$

So with $M = m\, b(1 + \log b)$, and $s_1 = b^k + 1$, we find that for all $t \in \mathbf{Z}^+$, if $t \geq s_1$ then $f(t) < M(t \log t)$ (so $f(t) \leq M(t \log t)$, and $f \in O(n \log n)$).

7. $O(1)$

8. (a) Here $f(1) = 0$, $f(2) = 1$, $f(3) = 3$, $f(4) = 4$, so $f(1) \leq f(2) \leq f(3) \leq f(4)$. To show that f is monotone increasing we shall use the Alternative Form of the Principle of Mathematical Induction. We assume that for all $i, j \in \{1, 2, 3, \ldots, n\}$, $j > i \Longrightarrow f(j) \geq f(i)$. Now we consider the case for $n + 1$, where $n \geq 4$.

(Case 1: $n+1$ is odd) Here we write $n+1 = 2k+1$ and have $f(n+1) = f(k+1) + f(k) + 2 \geq$

$f(k)+f(k)+2 = f(2k) = f(n)$, since $k, k+1 < n$, and by the induction hypothesis $f(k+1) \geq f(k)$.

(Case 2: $n+1$ is even) Now we write $n+1 = 2r$, where $r \in \mathbf{Z}^+$ (and $r \geq 3$). Then $f(n+1) = f(2r) = f(r)+f(r)+2f(r)+f(r-1)+2 = f(2r-1) = f(n)$, because $f(r) \geq f(r-1)$ by the induction hypothesis.

Therefore f is a monotone increasing function.

(b) From part (a), Example 10.48, and Theorem 10.2 (c) it follows that $f \in O(n)$ for *all* $n \in \mathbf{Z}^+$.

9. (a)

$$\begin{array}{rcll}
f(n) & \leq & af(n/b) & + \; cn \\
af(n/b) & \leq & a^2f(n/b^2) & + \; ac(n/b) \\
a^2f(n/b^2) & \leq & a^3f(n/b^3) & + \; a^2c(n/b^2) \\
a^3f(n/b^3) & \leq & a^4f(n/b^4) & + \; a^3c(n/b^3) \\
\vdots & & & \quad \vdots \\
a^{k-1}f(n/b^{k-1}) & \leq & a^kf(n/b^k) & + \; a^{k-1}c(n/b^{k-1})
\end{array}$$

Hence $f(n) \leq a^kf(n/b)^k + cn[1+(a/b)+(a/b)^2+\ldots+(a/b)^{k-1}] = a^kf(1)+cn[1+(a/b)+(a/b)^2+\ldots+(a/b)^{k-1}]$, since $n = b^k$. Since $f(1) \leq c$ and $(n/b^k) = 1$, we have $f(n) \leq cn[1+(a/b)+(a/b)^2+\ldots+(a/b)^{k-1}+(a/b)^k] = (cn)\sum_{i=0}^{k}(a/b)^i$.

(b) When $a = b$, $f(n) \leq (cn)\sum_{i=0}^{k} 1^i = (cn)(k+1)$, where $n = b^k$, or $k = \log_b n$. Hence $f(n) \leq (cn)(\log_b n + 1)$ so $f \in O(n \log_b n) = O(n \log n)$, for any base greater than 1.

(c) For $a \neq b$, $cn\sum_{i=0}^{k}(a/b)^i = cn\left[\dfrac{1-(a/b)^{k+1}}{1-(a/b)}\right]$

$$= (c)(b^k)\left[\frac{1-(a/b)^{k+1}}{1-(a/b)}\right] = c\left[\frac{b^k-(a^{k+1}/b)}{1-(a/b)}\right] = c\left[\frac{b^{k+1}-a^{k+1}}{b-a}\right] =$$

$$= c\left[\frac{a^{k+1}-b^{k+1}}{a-b}\right].$$

(d) From part (c), $f(n) \leq (c/(a-b))[a^{k+1}-b^{k+1}]$
$= (ca/(a-b))a^k - (cb/(a-b))b^k$. But $a^k = a^{\log_b n} = n^{\log_b a}$ and $b^k = n$, so $f(n) \leq (ca/(a-b))n^{\log_b a} - (cb/(a-b))n$.

(i) When $a < b$, then $\log_b a < 1$, and $f \in O(n)$ on $\mathbf{Z}^+$.
(ii) When $a > b$, then $\log_b a > 1$, and $f \in O(n^{\log_b a})$ on $\mathbf{Z}^+$.

10. (a) $a = 9$, $b = 3$, $n^{\log_b a} = n^{\log_3 9} = n^2$

$h(n) = n \in O(n^{\log_3 9-\epsilon})$ for $\epsilon = 1$.
So by case (i) for the Master Theorem we have $f \in \Theta(n^2)$.
(b) $a = 2$, $b = 2$, $n^{\log_b a} = n^{\log_2 2} = n^1 = n$
$h(n) = 1 \in O(n^{\log_2 2} - \epsilon)$ for $\epsilon = 1$.
By case (i) for the Master Theorem it follows that $f \in \Theta(n)$.
(c) $a = 1$, $b = 3/2$, $n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$.
$h(n) = 1 \in \Theta(n^{\log_{3/2} 1})$
Here case (ii) for the Master Theorem applies and we find that $f \in \Theta(n^{\log_b a} \log_2 n) = \Theta(\log_2 n)$.
(d) $a = 2$, $b = 3$, $n^{\log_b a} = n^{\log_3 2} \doteq n^{0.631}$
$h(n) = n \in \Omega(n^{\log_3 2+\epsilon})$ where $\epsilon \doteq 0.369$.
Further, for all sufficiently large n, $a\, h(n/b) = 2h(n/3) = 2(n/3) = (2/3)n \leq (3/4)n = c\, h(n)$, for $0 < c = 3/4 < 1$. Thus, case (iii) of the Master Theorem tells us that $f \in \Theta(n)$.
(e) $a = 4$, $b = 2$, $n^{\log_b a} = n^{\log_2 4} = n^2$
$h(n) = n^2 \in \Theta(n^{\log_2 4})$
From case (ii) of the Master Theorem we have $f \in \Theta(n^{\log_2 4} \log_2 n) = \Theta(n^2 \log_2 n)$.

Supplementary Exercises

1. $\binom{n}{k+1} = \dfrac{n!}{(k+1)!(n-k-1)!} = \dfrac{(n-k)}{(k+1)} \dfrac{n!}{k!(n-k)!} = \left(\dfrac{n-k}{k+1}\right)\binom{n}{k}$

2. (a) Consider the element $n+1$ in $S = \{1, 2, 3, \ldots, n, n+1\}$. For each partition of S we consider the size of the subset containing $n+1$. If the size is 1, then $n+1$ is by itself and there are B_n partitions where this happens. If the size is 2, there are $\binom{n}{1} = n$ ways this can occur, and B_{n-1} ways to partition the other $n-1$ integers. This results in $\binom{n}{1}B_{n-1}$ partitions of S. In general, if $n+1$ is in a subset of size $i+1$, $0 \leq i \leq n$, there are $\binom{n}{i}$ ways this can occur with $\binom{n}{1}B_{n-1}$ resulting partitions of S. By the rule of sum $B_{n+1} = \sum_{i=0}^{n} \binom{n}{i}B_{n-i} = \sum_{i=0}^{n} \binom{n}{n-i}B_{n-i} = \sum_{i=0}^{n} \binom{n}{i}B_i$.
(b) For $n \geq 0$, $B_n = \sum_{i=0}^{n} S(n,i)$. $[S(0,0) = 1]$.

3. There are two cases to consider. Case 1: (1 is a summand) – Here there are $p(n-1, k-1)$ ways to partition $n-1$ into exactly $k-1$ summands. Case 2: (1 is not a summand) – Here each summand $s_1, s_2, \ldots, s_k > 1$. For $1 \leq i \leq k$, let $t_i = s_i - 1 \geq 1$. Then $t_1, t_2, \ldots, t_k$ provide a partition of $n-k$ into exactly k summands. These cases are exhaustive and disjoint, so by the rule of sum $p(n,k) = p(n-1, k-1) + p(n-k, k)$.

4. Here $a_1 = 1$ and $a_2 = 1$.
For $n \geq 3$ write $n = x_1 + x_2 + \ldots + x_t$, where each x_i, for $1 \leq i \leq t$, is an odd positive integer (and $1 \leq t \leq n$, for n odd; $2 \leq t \leq n$, for n even). If $x_1 = 1$, then $n-1 = x_2 + \ldots + x_t$ and this summation is counted in a_{n-1}. If $x_1 \neq 1$, then

$x_1 \geq 3$ and $n-2 = (x_1 - 2) + x_2 + \ldots + x_t$, a summation counted in a_{n-2}. Consequently, $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$, and $a_n = F_n$, the nth Fibonacci number, for $n \geq 1$.

5. (a)

$$A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_3 & F_2 \\ F_2 & F_1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} F_4 & F_3 \\ F_3 & F_2 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} F_5 & F_4 \\ F_4 & F_3 \end{bmatrix}.$$

(b) Conjecture: For $n \in \mathbf{Z}^+$, $A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$,

where F_n denotes the nth Fibonacci number.

Proof: For $n = 1$, $A = A^1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}$, so the result is true in this first case. Assume the result true for $n = k \geq 1$, i.e.,

$A^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}$. For $n = k+1$, $A^n = A^{k+1} = A^k \cdot A = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{bmatrix} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}.$$

Consequently, the result is true for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.

6. (a) $M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $M^2 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, $M^3 = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}$, $M^4 = \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix}$.

(b) $M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_2 & F_3 \end{bmatrix}$ $\qquad M^2 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} F_3 & F_4 \\ F_4 & F_5 \end{bmatrix}$

$M^3 = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} = \begin{bmatrix} F_5 & F_6 \\ F_6 & F_7 \end{bmatrix}$ $\qquad M^4 = \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix} = \begin{bmatrix} F_7 & F_8 \\ F_8 & F_9 \end{bmatrix}$

We claim that for $n \in \mathbf{Z}^+$, $M^n = \begin{bmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{bmatrix}$.

Proof: We see that the claim is true for $n = 1$ (as well as, 2, 3, and 4). Assume the result true for $k (\geq 1)$ and consider what happens when $n = k+1$.

$$M^n = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} F_{2k-1} & F_{2k} \\ F_{2k} & F_{2k+1} \end{bmatrix}$$

$$= \begin{bmatrix} F_{2k-1} + F_{2k} & F_{2k} + F_{2k+1} \\ F_{2k-1} + 2F_{2k} & F_{2k} + 2F_{2k+1} \end{bmatrix}$$

$$= \begin{bmatrix} F_{2k+1} & F_{2k+2} \\ (F_{2k-1} + F_{2k}) + F_{2k} & (F_{2k} + F_{2k+1}) + F_{2k+1} \end{bmatrix}$$

$$= \begin{bmatrix} F_{2k+1} & F_{2k+2} \\ F_{2k+1} + F_{2k} & F_{2k+2} + F_{2k+1} \end{bmatrix} = \begin{bmatrix} F_{2k+1} & F_{2k+2} \\ F_{2k+2} & F_{2k+3} \end{bmatrix}$$

$$= \begin{bmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{bmatrix}.$$

It follows from the Principle of Mathematical Induction that $M^n = \begin{bmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{bmatrix}$ for all $n \geq 1$.

7. From $x^2-1 = 1+\frac{1}{x}$ we find that $x^3-x = x+1$, or $x^3-2x-1 = 0$. Since $(-1)^3-2(-1)-1 = -1+2-1 = 0$, it follows that -1 is a root of x^3-2x-1. Consequently, $x-(-1) = x+1$ is a factor and we have $x^3-2x-1 = (x+1)(x^2-x-1)$. So the roots of x^3-2x-1 are $-1, (1+\sqrt{5})/2$, and $(1-\sqrt{5})/2$.
For $x = -1$, $y = (-1)^2 - 1 = 0$.
For $x = (1+\sqrt{5})/2$, $y = [(1+\sqrt{5})/2]^2 - 1 = (1/4)(6+2\sqrt{5}) - 1 = [(3+\sqrt{5})/2] - 1 = (1+\sqrt{5})/2$.
For $x = (1-\sqrt{5})/2$, $y = [(1-\sqrt{5})/2]^2 - 1 = (1/4)(6-2\sqrt{5}) - 1 = [(3-\sqrt{5})/2] - 1 = (1-\sqrt{5})/2$.

So the points of intersection are $(-1,0)$, $((1+\sqrt{5})/2, (1+\sqrt{5})/2) = (\alpha,\alpha)$, and $((1-\sqrt{5})/2, (1-\sqrt{5})/2) = (\beta,\beta)$.

8. (a) $\alpha^2 = (1+\sqrt{5})^2/4 = (6+2\sqrt{5})/4 = (3+\sqrt{5})/2$
$\alpha + 1 = (1+\sqrt{5})/2 + 1 = (3+\sqrt{5})/2$
$\beta^2 = (1-\sqrt{5})^2/4 = (6-2\sqrt{5})/4 = (3-\sqrt{5})/2$
$\beta + 1 = (1-\sqrt{5})/2 + 1 = (3-\sqrt{5})/2$
(b) $\sum_{k=0}^{n} \binom{n}{k} F_k = \sum_{k=0}^{n} \binom{n}{k} (\alpha^k - \beta^k)/(\alpha - \beta)$
$= [1/(\alpha-\beta)][\sum_{k=0}^{n} \binom{n}{k} \alpha^k - \sum_{k=0}^{n} \binom{n}{k} \beta^k)]$
$= [1/(\alpha-\beta)][(1+\alpha)^n - (1+\beta)^n] = [1/(\alpha-\beta)][(\alpha^2)^n - (\beta^2)^n]$
$= (\alpha^{2n} - \beta^{2n})/(\alpha-\beta) = F_{2n}$
(c) $\alpha^3 = \alpha(\alpha^2) = [(1+\sqrt{5})/2][(3+\sqrt{5})/2] = (8+4\sqrt{5})/4 = 2+\sqrt{5}$
$1 + 2\alpha = 1 + 2[(1+\sqrt{5})/2] = 2+\sqrt{5}$
$\beta^3 = \beta(\beta^2) = [(1-\sqrt{5})/2][(3-\sqrt{5})/2] = (8-4\sqrt{5})/4 = 2-\sqrt{5}$
$1 + 2\beta = 1 + 2[(1-\sqrt{5})/2] = 2-\sqrt{5}$
(d) $\sum_{k=0}^{n} \binom{n}{k} 2^k F_k = \sum_{k=0}^{n} \binom{n}{k} 2^k (\alpha^k - \beta^k)/(\alpha-\beta)$

$= [1/(\alpha-\beta)][\sum_{k=0}^{n}\binom{n}{k}2^k\alpha^k - \sum_{k=0}^{n}\binom{n}{k}2^k\beta^k]$
$= [1/(\alpha-\beta)][\sum_{k=0}^{n}\binom{n}{k}(2\alpha)^k - \sum_{k=0}^{n}\binom{n}{k}(2\beta)^k]$
$= [1/(\alpha-\beta)][(1+2\alpha)^n - (1+2\beta)^n] = [1/(\alpha-\beta)][\alpha^{3n}-\beta^{3n}] = (\alpha^{3n}-\beta^{3n})/(\alpha-\beta) = F_{3n}$.

9. (a) Since $\alpha^2 = \alpha+1$, it follows that $\alpha^2+1 = 2+\alpha$ and $(2+\alpha)^2 = 4+4\alpha+\alpha^2 = 4(1+\alpha)+\alpha^2 = 5\alpha^2$.
(b) Since $\beta^2 = \beta+1$ we find that $\beta^2+1 = \beta+2$ and $(2+\beta)^2 = 4+4\beta+\beta^2 = 4(1+\beta)+\beta^2 = 5\beta^2$.

(c) $\displaystyle\sum_{k=0}^{2n}\binom{2n}{k}F_{2k+m} = \sum_{k=0}^{2n}\binom{2n}{k}\left[\frac{\alpha^{2k+m}-\beta^{2k+m}}{\alpha-\beta}\right]$

$$= (1/(\alpha-\beta))\left[\sum_{k=0}^{2n}\binom{2n}{k}(\alpha^2)^k\alpha^m - \sum_{k=0}^{2n}\binom{2n}{k}(\beta^2)^k\beta^m\right]$$

$$= (1/(\alpha-\beta))[\alpha^m(1+\alpha^2)^{2n} - \beta^m(1+\beta^2)^{2n}]$$

$$= (1/(\alpha-\beta))[\alpha^m(2+\alpha)^{2n} - \beta^m(2+\beta)^{2n}]$$

$$= (1/(\alpha-\beta))[\alpha^m((2+\alpha)^2)^n - \beta^m((2+\beta)^2)^n]$$

$$= (1/(\alpha-\beta))[\alpha^m(5\alpha^2)^n - \beta^m(5\beta^2)^n]$$

$$= 5^n(1/(\alpha-\beta))[\alpha^{2n+m} - \beta^{2n+m}] = 5^n F_{2n+m}.$$

10. (a) Let $p_0 = \$4000$, the price first set by Renu, and let $p_1 = \$3000$, the first offer made by Narmada. For $n \geq 0$, we have

$$p_{n+2} = (1/2)(p_{n+1}+p_n).$$

This gives us the characteristic equation $2x^2 - x - 1 = 0$; the characteristic roots are 1 and $-1/2$. So

$$p_n = A(1)^n + B(-1/2)^n,\ n \geq 0.$$

From $p_0 = 4000$, $p_1 = 3000$ it follows that $A = 10,000/3$, $B = 2000/3$.

Narmada's fifth offer occurs for $n = 9 (= 2\cdot 5 - 1)$ and $p_9 = \$3332.03$. Her 10th offer occurs for $n = 19$ and $p_{19} = \$3333.33$. For $k \geq 1$, her kth offer occurs when $n = 2k-1$ and $p_n = (10,000/3) + (2000/3)(-1/2)^{2k-1}$.

(b) As n increases the term $(-1/2)^n$ decreases to 0, so $p(n)$ approaches $\$10,000/3 = \$3333.33$.

(c) Here $p_n = A(1)^n + B(-1/2)^n$, $n \geq 0$, with $p_0 = \$4000$. As n increases p_n approaches $A = \$3200$. So $4000 = p_0 = 3200 + B$, and $B = 800$.
With $p_n = 3200 + 800(-1/2)^n$ we find the solution $p_1 = 3200 + 800(-1/2) = \$2800$.

11. Consider the case where n is even. (The argument for n odd is similar.) For the fence $\mathcal{F}_n = \{a_1, a_2, \ldots, a_n\}$, there are c_{n-1} order-preserving functions $f : \mathcal{F}_n \to \{1,2\}$ where $f(a_n) = 2$. [Note that $(\{1,2\}, \leq)$ is the same partial order as $\mathcal{F}_2$.] When such a function satisfies $f(a_n) = 1$, then we must have $f(a_{n-1}) = 1$, and there are c_{n-2} of these order-preserving functions. Consequently, since these two cases have nothing in common and cover all possibilities, we find that

$$c_n = c_{n-1} + c_{n-2}, \quad c_1 = 2,\ c_2 = 3.$$

So $c_n = F_{n+2}$, the $(n+2)$nd Fibonacci number.

12. This combinatorial identity follows by observing that F_{n+2} and $\sum_{k=0}^{m} \binom{n-k+1}{k}$, for $m = \lfloor (n+1)/2 \rfloor$, each count the number of subsets of $\{1, 2, 3, \ldots, n\}$ that contain no consecutive integers.

13. (a) For $n \geq 1$, let a_n count the number of ways one can tile a $1 \times n$ chessboard using the 1×1 white tiles and 1×2 blue tiles. Then $a_1 = 1$ and $a_2 = 2$.

For $n \geq 3$, consider the nth square (at the right end) of the $1 \times n$ chessboard. Two situations are possible here:
(1) This square is covered by a 1×1 white tile, so the preceding $n-1$ squares (of the $1 \times n$ chessboard) can be covered in a_{n-1} ways;
(2) This square and the preceding ($(n-1)$st) square are both covered by a 1×2 blue tile, so the preceding $n-2$ squares (of the $1 \times n$ chessboard) can be covered in a_{n-2} ways. These two situations cover all possibilities and are disjoint, so we have

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 3, \quad a_1 = 1, \quad a_2 = 2.$$

Consequently, $a_n = F_{n+1}$, the $(n+1)$st Fibonacci number.
(b) (i) There is only $1 = \binom{n}{0} = \binom{n-0}{n-2\cdot 0}$ way to tile the $1 \times n$ chessboard using all white squares.
(ii) Consider the equation $x_1 + x_2 + \cdots + x_{n-1} = n-1$, where $x_i = 1$ for $1 \leq i \leq n-1$. We can select one of the x_i, where $1 \leq i \leq n-1$, in $\binom{n-1}{1} = \binom{n-1}{n-2} = \binom{n-1}{n-2\cdot 1}$ ways. Increase the value of this x_i to 2 and we have

$$x_1 + x_2 + \cdots + x_{n-1} = n.$$

In terms of our tilings we have $i-1$ white tiles, then the one blue tile, and then $n-i-1$ white tiles on the right of the blue tile – for a total of $(i-1)+1+(n-i-1) = n-1$ tiles.
(iii) There are $\binom{n-2}{2} = \binom{n-2}{n-4} = \binom{n-2}{n-2\cdot 2}$ tilings where we have exactly two blue tiles and $n-4$ white ones.
(iv) Likewise we have $\binom{n-3}{3} = \binom{n-3}{n-6} = \binom{n-3}{n-2\cdot 3}$ tilings that use 3 blue tiles and $n-6$ white ones.
(v) For $0 \leq k \leq \lfloor n/2 \rfloor$, there are $\binom{n-k}{k} = \binom{n-k}{n-2k}$ tilings with k blue tiles and $n-2k$ white ones.

(c) $F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{n-2k}$. [Compare this result with the formula presented in the previous exercise.]

14. $c^2 = 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}} = 1 + c$. So $c^2 - c - 1 = 0$ and $c = \alpha$ or $c = \beta$. Since $c > 0$ it follows that $c = \alpha = (1 + \sqrt{5})/2$.

15. (a) For each derangement, 1 is placed in position i, $2 \leq i \leq n$. Two things then occur.

Case 1: (i is in position 1) – Here the other $n-2$ integers are deranged in d_{n-2} ways. With $n-1$ choices for i this results in $(n-1)d_{n-2}$ such derangements.

Case 2: (i is not in position 1 (or position i)). Here we consider 1 as the new natural position for i, so there are $n-1$ elements to derange. With $n-1$ choices for i we have $(n-1)d_{n-1}$ derangements. Since the two cases are exhaustive and disjoint, the result follows from the rule of sum.

(b) $d_0 = 1$ (c) $d_n - nd_{n-1} = d_{n-2} - (n-2)d_{n-3}$

(d) $d_n - nd_{n-1} = (-1)^i[d_{n-i} - (n-i)d_{n-i-1}]$
Let $i = n-2$.
$d_n - nd_{n-1} = (-1)^{n-2}[d_2 - 2d_1] = (-1)^{n-2} = (-1)^n$

(e) $d_n - nd_{n-1} = (-1)^n$
$(d_n - nd_{n-1})(x^n/n!) = (-1)^n(x^n/n!)$
$\sum_{n=2}^{\infty}(d_n - nd_{n-1})(x^n/n!) = \sum_{n=2}^{\infty}(-x)^n/n! = e^{-x} - 1 + x$
$\sum_{n=2}^{\infty} d_n x^n/n! - x\sum_{n=2}^{\infty} d_{n-1}x^{n-1}/(n-1)! = e^{-x} - 1 + x$
$[f(x) - d_1 x - d_0] - x[f(x) - d_0] = e^{-x} - 1 + x$
$f(x) - 1 - xf(x) + x = e^{-x} - 1 + x$ and $f(x) = e^{-x}/(1-x)$

16. Drawing the $(n+1)$st oval, $n \geq 0$, we get $2n$ new points of intersection which split the perimeter of this oval into $2n$ segments. Each segment takes an existing region and divides it into two regions. So

$a_{n+1} = a_n + 2n$, $n \geq 1$, $a_1 = 2$.
$a_n^{(h)} = A$, $a_n^{(p)} = n(Bn + C)$
$(n+1)[B(n+1)+C] = n(Bn+C)+2n \Longrightarrow B(n^2+2n+1)+Cn+C = Bn^2+Cn+2n \Longrightarrow$ $2B + C = C + 2$, $B + C = 0 \Longrightarrow B = 1$, $C = -1$, so $a_n = A + n^2 - n$. $2 = a_1 = A \Longrightarrow$ $a_n = n^2 - n + 2 = 2[n(n-1)/2] + 2$.

17. (a) $a_n = \binom{2n}{n}$

(b) $(r + sx)^t = r^t(1 + (s/r)x)^t = r^t[\binom{t}{0} + \binom{t}{1}(s/r)x + \binom{t}{2}(s/r)^2x^2 + \ldots] = a_0 + a_1x + a_2x^2 + \ldots = 1 + 2x + 6x^2 + \ldots$
$r^t = 1 \Longrightarrow r = 1$
$\binom{t}{1}s = 2 = ts$, $\binom{t}{2}s^2 = 6 = s^2[t(t-1)/2] = s(t-1) = 2 - s$, so $s = -4$, $t = -1/2$, and $(1-4x)^{-1/2}$ generates $\binom{2n}{n}$, $n \geq 0$.

(c) Let a coin be tossed $2n$ times with the sequence of H's and T's counted in a_n. For $1 \le i \le n$, there is a smallest i where the number of H's equals the number of T's for the first time after $2i$ tosses. This sequence of $2i$ tosses is counted in b_i; the given sequence of $2n$ tosses is counted in $a_{n-i}b_i$. Since $b_0 = 0$, as i varies from 0 to n, $a_n = \sum_{i=0}^{n} a_i b_{n-i}$.

(d) Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$, $f(x) = \sum_{n=0}^{\infty} a_n x^n = (1-4x)^{-1/2}$.
$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty}(a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0)x^n \Longrightarrow f(x) - a_0 = f(x)g(x)$ or $g(x)$
$= 1 - [1/f(x)] = 1 - (1-4x)^{1/2}$.
$(1-4x)^{1/2} = [\binom{1/2}{0} + \binom{1/2}{1}(-4x) + \binom{1/2}{2}(-4x)^2 + \ldots]$

The coefficient of x^n in $(1-4x)^{1/2}$ is $\binom{1/2}{n}(-4)^n =$

$$\frac{(1/2)((1/2)-1)((1/2)-2)\cdots((1/2)-n+1)}{n!}(-4)^n = \frac{(-1)(1)(3)(5)\cdots(2n-3)}{n!}(2^n) =$$

$$\frac{(-1)(1)(3)\cdots(2n-3)(2)(4)\cdots(2n-2)(2n)}{n!n!} = \frac{(-1)}{(2n-1)}\frac{(2n)!}{n!n!} = [-1/(2n-1)]\binom{2n}{n}.$$

Consequently, the coefficient of x^n in $g(x)$ is $b_n = [1/(2n-1)]\binom{2n}{n}$, $n \ge 1$, $b_0 = 0$.

18. $|\beta| = |\frac{1-\sqrt{5}}{2}| = \frac{\sqrt{5}-1}{2} < 1$, so $\sum_{k=0}^{\infty}\beta^k = \frac{1}{1-\beta} = \frac{1}{1-(\frac{1-\sqrt{5}}{2})} = \frac{1}{\frac{1+\sqrt{5}}{2}} = \frac{2}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{2-2\sqrt{5}}{1-5} = \frac{-1+\sqrt{5}}{2} = -(\frac{1-\sqrt{5}}{2}) = -\beta$.

Since $\alpha + \beta = (\frac{1+\sqrt{5}}{2}) + (\frac{1-\sqrt{5}}{2}) = 1$, it follows that $\alpha - 1 = -\beta$.

$\sum_{k=0}^{\infty}|\beta|^k = \sum_{k=0}^{\infty}(\frac{\sqrt{5}-1}{2})^k = \frac{1}{1-(\frac{\sqrt{5}-1}{2})} = \frac{1}{(\frac{3-\sqrt{5}}{2})} = \frac{2}{3-\sqrt{5}}\cdot\frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{6+2\sqrt{5}}{9-5} = \frac{6+2\sqrt{5}}{4} = (\frac{1}{2})(3+\sqrt{5})$,
and $\alpha^2 = (\frac{1+\sqrt{5}}{2})^2 = (\frac{6+2\sqrt{5}}{4}) = (\frac{1}{2})(3+\sqrt{5})$.

19. For $x, y, z \in \mathbf{R}$,

$f(f(x,y),z) = f(a+bxy+c(x+y),z) = a+b[(a+bxy+c(x+y))z]+c[(a+bxy+c(x+y))+z)]$
$= a + ac + c^2x + bcxy + b^2xyz + bcxz + c^2y + bcyz + abz + cz$, and

$f(x, f(y,z)) = f(x, a + byz + c(x+y))$
$= a + b[x(a + byz + c(y+z)) + c[x + (a + byz + c(y+z))]$
$= a + ac + abx + cx + c^2y + c^2z + b^2xyz + bcxy + bcxz + bcyz$.

f associative $\Rightarrow f(f(x,y),z) = f(x,f(y,z)) \Rightarrow c^2x + (ab+c)z = abx + cx + c^2z$. With $ab = 1$ it follows that

$$c^2x + z + cz = x + c^2z + cx, \quad \text{or} \quad (c^2 - c - 1)x = (c^2 - c - 1)z.$$

Since x, z are arbitrary, we have $c^2 - c - 1 = 0$. Consequently, $c = \alpha$ or $c = \beta$.

20. (a) $\alpha - \beta = (\frac{1}{2})(1+\sqrt{5}) - (\frac{1}{2})(1-\sqrt{5}) = \sqrt{5}$

$\alpha^2 - \alpha^{-2} = (\frac{1+\sqrt{5}}{2})^2 - (\frac{2}{1+\sqrt{5}})^2 = \frac{6+2\sqrt{5}}{4} - \frac{4}{6+2\sqrt{5}} = \frac{3+\sqrt{5}}{2} - \frac{2}{3+\sqrt{5}} \cdot \frac{3-\sqrt{5}}{3-\sqrt{5}} = \frac{3+\sqrt{5}}{2} - \frac{6-2\sqrt{5}}{4} = (\frac{1}{2})(3+\sqrt{5}-3+\sqrt{5}) = \sqrt{5}$.

$\beta^{-2} - \beta^2 = (\frac{2}{1-\sqrt{5}})^2 - (\frac{1-\sqrt{5}}{2})^2 = \frac{4}{6-2\sqrt{5}} - \frac{6-2\sqrt{5}}{4} = \frac{2}{3-\sqrt{5}} \cdot \frac{3+\sqrt{5}}{3+\sqrt{5}} - \frac{3-\sqrt{5}}{2} = \frac{6+2\sqrt{5}}{4} - \frac{3-\sqrt{5}}{2} = (\frac{1}{2})(3+\sqrt{5}-3+\sqrt{5}) = \sqrt{5}$.

(b) Using the Binet form we have

$F_{n+1}^2 - F_{n-1}^2 = \left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)^2 - \left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right)^2$

$= \frac{\alpha^{2n+2}+\beta^{2n+2}-2(\alpha\beta)^{n+1}-\alpha^{2n-2}-\beta^{2n-2}+2(\alpha\beta)^{n-1}}{(\alpha-\beta)^2}$

$= \frac{\alpha^{2n}(\alpha^2-\alpha^{-2})-\beta^{2n}(\beta^{-2}-\beta^2)}{(\alpha-\beta)^2}$ (since $\alpha\beta = -1$)

$= (\alpha^{2n} - \beta^{2n})/(\alpha-\beta)$ [from the results in part (a)] $= F_{2n}$.

(c) Here the base angles are 60° and the altitude is $(1/2)(\sqrt{3})F_n$. Consequently, the area of T is $(1/2)(\sqrt{3}/2)F_n[F_{n-1} + F_{n+1}] = (\sqrt{3}/4)F_n[F_{n-1} + F_{n+1}]$.

Returning to part (b) we find that $F_{2n} = F_{n+1}^2 - F_{n-1}^2 = (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1}) = F_nF_{n+1} + F_nF_{n-1}$. Consequently, the area of $T = (\sqrt{3}/4)F_{2n}$.

21. Since $A \cap B = \emptyset$, $Pr(S) = Pr(A \cup B) = Pr(A) + Pr(B)$. Consequently, we have $1 = p + p^2$, so $p^2 + p - 1 = 0$ and $p = (-1 \pm \sqrt{5})/2$. Since $(-1-\sqrt{5})/2 < 0$ it follows that $p = (-1+\sqrt{5})/2 = -\beta$.

22. The probability that Sandra wins is $p + (1-p)(1-p)^2p + (1-p)(1-p)^2(1-p)(1-p)^2p + \cdots = p[1 + (1-p)^3 + (1-p)^6 + (1-p)^9 + \cdots] = p[1/[1-(1-p)^3]]$.

For the game to be fair we must have $1/2 = p[1/[1-(1-p)^3]]$, so

$$\begin{aligned} p &= (1/2)[1-(1-p)^3] \\ 2p &= [1-(1-p)^3] = 1 - (1 - 3p + 3p^2 - p^3) \\ 2p &= 3p - 3p^2 + p^3, \text{ and} \\ 0 &= p^3 - 3p^2 + p = p(p^2 - 3p + 1). \end{aligned}$$

Since $p > 0$, it follows that $p^2 - 3p + 1 = 0$, or $p = (3 \pm \sqrt{5})/2$. Since $p < 1$, we find that

$$p = (3-\sqrt{5})/2 = [(1-\sqrt{5})/2]^2 = \beta^2.$$

23. Here $a_1 = 1$ (for the string 0) and $a_2 = 2$ (for the strings 00, 11). For $n \geq 3$, consider the nth bit of a binary string (of length n) where there is no run of 1's of odd length.
(i) If this bit is 0 then the preceding $n-1$ bits can arise in a_{n-1} ways; and
(ii) If this bit is 1, then the $(n-1)$st bit must also be 1 and the preceding $n-2$ bits can arise in a_{n-2} ways.

Since the situations in (i) and (ii) have nothing in common and cover all cases we have

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 3, a_1 = 1, a_2 = 2.$$

Here $a_n = F_{n+1}, n \geq 1$, and so we have another instance where the Fibonacci numbers arise.

24. Here $x_0 = a$, $x_1 = b$, $x_2 = x_1 x_0 = ba$, $x_3 = x_2 x_1 = b^2 a$, $x_4 = x_3 x_2 = b^3 a^2$, and $x_5 = x_4 x_3 = b^5 a^3$. These results suggest that $x_0 = a$ and, for $n \geq 1$, $x_n = b^{F_n} a^{F_{n-1}}$, where F_n denotes the nth Fibonacci number (for $n \geq 1$). To establish this in general we proceed by mathematical induction. The result is true for $n = 0$, as well as for $n = 1, 2, 3, 4, 5$.

Assume the result true for $n = 0, 1, 2, \ldots, k-1, k$, where k is a fixed (but arbitrary) positive integer. Hence $x_{k-1} = b^{F_{k-1}} a^{F_{k-2}}$ and $x_k = b^{F_k} a^{F_{k-1}}$, so $x_{k+1} = x_k x_{k-1} = (b^{F_k} a^{F_{k-1}})(b^{F_{k-1}} a^{F_{k-2}}) = b^{F_k + F_{k-1}} a^{F_{k-1} + F_{k-2}} = b^{F_{k+1}} a^{F_k}$, by the recursuve definition of the Fibonacci numbers. Consequently, by the alternative form of the Principle of Mathematical Induction the result is true for ($n = 0$ and) all $n \geq 1$.

(Second Solution). For $n \geq 0$ let $y_n = \log x_n$. Then $y_0 = \log a$, $y_1 = \log b$, and $y_n = y_{n+1} + y_{n-2}$, $n \geq 2$. So $y_n = c_1 \alpha^n + c_2 \beta^n$, where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

$\log a = c_1 + c_2, \log b = c_1 \alpha + c_2 \beta \Rightarrow$
$c_2 = (-1/\sqrt{5}) \log b + [(1+\sqrt{5})/2\sqrt{5}] \log a,$
$c_1 = (1/\sqrt{5}) \log b + [(-1+\sqrt{5})/2\sqrt{5}] \log a,$
where the base for the log function is 10 (although any positive real number, other than 1, may be used here for the base).

Consequently,

$$\begin{aligned}
y_n &= c_1 \alpha^n + c_2 \beta^n \\
&= [(1/\sqrt{5}) \log b + [(-1+\sqrt{5})/2\sqrt{5}] \log a] \alpha^n \\
&\quad + [(-1/\sqrt{5}) \log b + [(1+\sqrt{5})/2\sqrt{5}] \log a] \beta^n, \quad \text{so} \\
x_n &= 10^{c_1 \alpha^n + c_2 \beta^n} \\
&= 10^{[[(-1+\sqrt{5}/2\sqrt{5}] \log a + (1/\sqrt{5}) \log b] \alpha^n} \cdot \\
&\quad 10^{[(1+\sqrt{5})/2\sqrt{5}] \log a + (-1/\sqrt{5}) \log b] \beta^n} \\
&= a^{[[(-1+\sqrt{5})/2\sqrt{5}] \alpha^n + [(1+\sqrt{5})/2\sqrt{5}] \beta^n]} b^{(\alpha^n - \beta^n)/\sqrt{5}} \\
&= a^{(\alpha^{n-1} - \beta^{n-1})/\sqrt{5}} \; b^{(\alpha^n - \beta^n)/\sqrt{5}} \\
&= a^{F_{n-1}} b^{F_n},
\end{aligned}$$

since $F_n = (\alpha^n - \beta^n)/(\alpha - \beta) = (\alpha^n - \beta^n)/\sqrt{5}$.

(a) ($n = 0$) $F_1^2 - F_0 F_1 - F_0^2 = 1^2 - 0 \cdot 1 - 0^2 = 1$
($n = 1$) $F_2^2 - F_1 F_2 - F_1^2 = 1^2 - 1 \cdot 1 - 1^2 = -1$
25. ($n = 2$) $F_3^2 - F_2 F_3 - F_2^2 = 2^2 - 1 \cdot 2 - 1^2 = 1$
($n = 3$) $F_4^2 - F_3 F_4 - F_3^2 = 3^2 - 2 \cdot 3 - 2^2 = -1$
(b) Conjecture: For $n \geq 0$,

$$F_{n+1}^2 - F_n F_{n+1} - F_n^2 = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

(c) Proof: The result is true for $n = 0, 1, 2, 3$, by the calculations in part (a). Assume the result true for $n = k (\geq 3)$. There are two cases to consider – namely, k even and

k odd. We shall establish the result for k even, the proof for k odd being similar. Our induction hypothesis tells us that $F_{k+1}^2 - F_kF_{k+1} - F_k^2 = 1$. When $n = k+1 (\geq 4)$ we find that $F_{k+2}^2 - F_{k+1}F_{k+2} - F_{k+1}^2 = (F_{k+1}+F_k)^2 - F_{k+1}(F_{k+1}+F_k) - F_{k+1}^2 = F_{k+1}^2 + 2F_{k+1}F_k + F_k^2 - F_{k+1}^2 - F_{k+1}F_k - F_{k+1}^2 = F_{k+1}F_k + F_k^2 - F_{k+1}^2 = -[F_{k+1}^2 - F_kF_{k+1} - F_k^2] = -1$. The result follows for all $n \in \mathbf{N}$, by the Principle of Mathematical Induction.

26. The answer is the number of subsets of $\{1, 2, 3, \ldots, n\}$ which contain no consecutive entries. We learned in Section 10.2 that this is F_{n+2}, the $(n+2)$nd Fibonacci number.

27. (a) $r(C_1, x) = 1 + x$ $\quad$ $r(C_4, x) = 1 + 4x + 3x^2$
$r(C_2, x) = 1 + 2x$ $\quad$ $r(C_5, x) = 1 + 5x + 6x^2 + x^3$
$r(C_3, x) = 1 + 3x + x^2$ $\quad$ $r(C_6, x) = 1 + 6x + 10x^2 + 4x^3$

In general, for $n \geq 3$, $r(C_n, x) = r(C_{n-1}, x) + xr(C_{n-2}, x)$.

(b) $r(C_1, 1) = 2$ $\quad$ $r(C_3, 1) = 5$ $\quad$ $r(C_5, 1) = 13$
$r(C_2, 1) = 3$ $\quad$ $r(C_4, 1) = 8$ $\quad$ $r(C_6, 1) = 21$

[Note: For $1 \leq i \leq n$, if one "straightens out" the chessboard C_i in Fig. 10.28, the result is a $1 \times i$ chessboard – like those studied in the previous exercise.]

28. For $0 \leq n \leq 18$, let p_n be the probability that Jill bankrupts Cathy when Jill has n quarters. Then $p_0 = 0$ and $p_{18} = 1$ and the answer to this problem is p_{10}. For $0 < n < 18$, if Jill has n quarters, then after playing another game of checkers,

$$p_n = \underbrace{(1/2)p_{n-1}}_{\substack{\text{Jill has lost} \\ \text{the game}}} + \underbrace{(1/2)p_{n+1}}_{\substack{\text{Jill wins} \\ \text{the game}}}$$

$p_{n+1} - 2p_n + p_{n-1} = 0$ has characteristic roots $r = 1, 1$, so $p_n = A + Bn$. $p_0 = 0 \Longrightarrow A = 0$, $1 = p_{18} \Longrightarrow B = 1/18$, so $p_n = n/18$. Hence Jill has probability $10/18 = 5/9$ of bankrupting Cathy.

29. (a) The partitions counted in $f(n, m)$ fall into two categories:

(1) Partitions where m is a summand. These are counted in $f(n-m, m)$, for m may occur more than once.

(2) Partitions where m is not a summand – so that $m-1$ is the largest possible summand. These partitions are counted in $f(n, m-1)$.

Since these two categories are exhaustive and mutually disjoint it follows that $f(n, m) = f(n-m, m) + f(n, m-1)$.

(b)

```
Program Summands(input,output);
Var
    n: integer;
```

```
Function  f(n,m: integer): integer;
Begin
     If n=0 then
          f := 1
     Else if (n < 0) or (m < 1) then
               f := 0
          Else f := f(n,m-1) + f(n-m,m)
End; {of function f}

Begin
     Writeln ('What is the value of n?');
     Readln (n);
     Writeln ('What is the value of m?');
     Readln (m);
     Write ('There are  ', f(n,m):0, '  partitions of  ');
     Write (n:0, '  where  ', m:0, '  is the largest  ');
     Writeln ('summand possible.')
End.
```

(c)

```
Program  Partitions(input,output);
Var
     n: integer;

Function  f(n,m: integer): integer;
Begin
     If n=0 then
          f := 1
     Else if (n < 0) or (m < 1) then
               f := 0
          Else f := f(n,m-1) + f(n-m,m)
End; {of function f}

Begin
     Writeln ('What is the value of n?');
     Readln (n);
     Write ('For n = ', n :0, '  the number of  ');
     Write ('partitions p(', n:0, ') is  ', f(n,n):0, '.')
End.
```

30. Let $|B| = n = 1$ and $|A| = m$. Then $f : A \to B$ where $f(a) = b$ for all $a \in A$ and $\{b\} = B$, is the only onto function from A to B. Hence $a(m, 1) = 1$.

For $m \geq n > 1$, $n^m =$ the total number of functions $f : A \to B$. If $1 \leq i \leq n-1$, there are $\binom{n}{i} a(m,i)$ onto functions g with domain A and range a subset of B of size i. Furthermore, any function $h : A \to B$ that is not onto is found among these functions g. Consequently, $a(m,n) = n^m - \sum_{i=1}^{n-1} \binom{n}{i} a(m,i)$.

31. The following program will print out the units digit of the first 130 Fibonacci numbers: $F_0 - F_{129}$.

```
Program Units(input, output);
Var
    FibUnit: array[0..129] of integer;
        i,j: integer;

Begin
    FibUnit[0] := 0;
    FibUnit[1] := 1;
    For i := 2 to 129 do
        FibUnit[i] := (FibUnit[i-1] + FibUnit[i-2]) Mod 10;
    For i :=0 to 12 do
        For j := 0 to 9 do
            If j < 9 then
                Write (FibUnit[10 * i + j]: 4)
            Else {j = 9}
                Writeln (FibUnit[10 * i + 9]: 4)
End.
```