PCA Example Problem.

$$X = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 5 & 4 \\ 1 & 0 \end{bmatrix} \Rightarrow$$

There are 2 Variables corresponding to the 2 columns & each variable has 4 measurements, corresponding to each row.

Let the 2 variables be u & v

$u = [4 \quad 2 \quad 5 \quad 1]$ $\qquad$ $v = [1 \quad 3 \quad 4 \quad 0]$

$\bar{u} = \frac{4+2+5+1}{4} = 3$ $\qquad$ $\bar{v} = \frac{1+3+4+0}{4} = 2.$

$\therefore$ Mean vector is: $[\mu_1, \mu_2] = [\bar{u}, \bar{v}] = [3, 2]$.

Data: $\underset{d_1}{(4,1)}, \underset{d_2}{(2,3)}, \underset{d_3}{(5,4)}, \underset{d_4}{(1,0)}$.

every d_i has a pair of values (u_i, v_i).

$$(u_1-\bar{u},\ v_1-\bar{v}) = (4-3,\ 1-2) = (1,\ -1)$$

$$(u_2-\bar{u},\ v_2-\bar{v}) = (2-3,\ 3-2) = (-1,\ 1)$$

$$(u_3-\bar{u},\ v_3-\bar{v}) = (5-3,\ 4-2) = (2,2)$$

$$(u_4-\bar{u},\ v_4-\bar{v}) = (1-3,\ 0-2) = (-2,-2).$$

Finding the variance covariance matrix:

$$C_x = \frac{1}{N}\left(\sum_{i=1}^{4}\begin{pmatrix} u_i-\bar{u} \\ v_i-\bar{v}\end{pmatrix}\begin{pmatrix} u_i-\bar{u} & v_i-\bar{v}\end{pmatrix}\right)$$

$$= \frac{1}{4}\left[\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}-1 & 1\end{pmatrix} + \begin{pmatrix}2\\2\end{pmatrix}\begin{pmatrix}2 & 2\end{pmatrix} + \begin{pmatrix}-2\\-2\end{pmatrix}\begin{pmatrix}-2 & -2\end{pmatrix}\right.$$

$$= \frac{1}{4}\left[\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \begin{bmatrix}4 & 4\\4 & 4\end{bmatrix} + \begin{bmatrix}4 & 4\\4 & 4\end{bmatrix}\right]$$

$$C_x = \frac{1}{4}\begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

Now find the eigenval & eigenvectors of C_x

Computing Char. polynomial

$$(10-\lambda)^2 - 36 = 0. \qquad \lambda^2 - 20\lambda + 100 - 36 = 0$$

$$= \lambda^2 - 20\lambda + 64 = 0 \quad \Rightarrow \quad \lambda = 16 \text{ or } \lambda = 4.$$

The eigenvectors are for $\lambda = 16$ it is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ &

$\lambda = 4$ it is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore$ The principal component corresponds to the eigen vector associated with the dominant eigen value, which here is 16; which is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Unit vector is $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.