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Department of Mathematics

CALCULUS OF VARIATION

Introduction

- Minimization principles form one of the most wide-ranging means of formulating mathematical models governing the equilibrium configurations of physical systems. Moreover, many popular numerical integration schemes such as the powerful finite element method are also founded upon a minimization paradigm. In these notes, we will develop the basic mathematical analysis of nonlinear minimization principles on infinite-dimensional function spaces — a subject known as the “calculus of variations”.
- The calculus of variations is a field of mathematics about solving optimization problems. The methods of calculus of variations to solve optimization problems are very useful in mathematics, physics and engineering. Therefore, it is an important field in contemporary research. However, the calculus of variations has a very long history, which is interwoven with the history of mathematics.
- In 1696, Johan Bernoulli came up with one of the most famous optimization problems: the brachistochrone problem. His brother Jakob Bernoulli and the Marquis de l'Hôpital immediately were interested in solving this problem, but the first major developments in the calculus of variations appeared in the work of Leonhard Euler. He started in 1733 with some important contributions in his Elementa Calculi Variationum. Joseph-Louis Lagrange and Adrien-Marie Legendre came up with some important contributions. These big names were not the only contributors to the calculus of variations. Isaac Newton, Gottfried Leibniz, Vincenzo Brunacci, Carl Friedrich Gauss, Siméon Poisson, Mikhail Ostrogradsky and Carl Jacobi also worked on the subject. Not forget to mention Karl Weierstrass: he was the first to place the subject on an unquestionable foundation.

Introduction

- The underlying physical principle, first formulated by the seventeenth century French mathematician Pierre de Fermat, is that, when a light ray moves through an optical medium, it travels along a path that minimizes the travel time.
- In the 20th century, David Hilbert, Emmy Noether, Leonida Tonelli, Henri Lebesgue and Jacques Hadamard studied the subject. The calculus of variations is thus a subject with a long history, a huge importance in classical and contemporary research and a subject where many big names in mathematics and physics have worked on. Therefore, it is a very interesting subject to study. The basic ideas that are needed will be explained. When the theory behind the calculus of variations is understood, some basic problems will be solved.
- Minimization problems that can be analyzed by the calculus of variations serve to characterize the equilibrium configurations of almost all continuous physical systems, ranging through elasticity, solid and fluid mechanics, electro-magnetism, gravitation, quantum mechanics, string theory, and many, many others. Many geometrical configurations, such as minimal surfaces, can be conveniently formulated as optimization problems. Moreover, numerical approximations to the equilibrium solutions of such boundary value problems are based on a nonlinear finite element approach that reduces the infinite-dimensional minimization problem to a finite-dimensional problem.
- The best way to appreciate the calculus of variations is by introducing a few concrete examples of both mathematical and practical importance. Some of these minimization problems played a key role in the historical development of the subject. And they still serve as an excellent means of learning its basic constructions such as Minimal Curves, Optics, and Geodesics. The minimal curve problem is to find the shortest path between two specified locations. A closely related problem arises in geometrical optics, and to construct the geodesics on a curved surface, meaning the curves of minimal length.

Functionals:

Calculus of variations is a subject that deals with functionals. So in order to understand the method of calculus of variations, first need to know what functional are. In a very short way, a functional is a function of a function.

To make it more clear a functional is a quantity whose values are determined by one or several functions. Thus the domain of a functional is a set of admissible functions, rather than a region of a coordinate space.

Defn:

Consider the function $f(x, y, y')$ in the interval $[x_1, x_2]$.

The definite integral $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$ is called a functional associated with the function f.

Functional is a mapping from a set of functions to set of real numbers.

Extremal of a functional:

Consider the function $f(x, y, y')$ in the interval $[x_1, x_2]$. Let $Y = y(x) + \epsilon\eta(x)$, where ϵ is independent of x and $Y' = y'(x) + \epsilon\eta'(x)$.

Consider the functional associated with $f(x, y, y')$.

Let I denote this functional, $I = \int_{x_1}^{x_2} f(x, Y, Y')\, dx$. (1)

I depends on value of ϵ. Therefore, the necessary condition for I to be extremal is $\frac{dI}{d\epsilon} = 0$. (The condition in ordinary calculus)

In particular if y(x) has to be the extremal curve,

then $\frac{dI}{d\epsilon} = 0$, when $\epsilon = 0$. (2)

Euler Lagrange equation:

Euler – Lagrange equation is a differential equation which gives the condition for extremization of the functional.

Statement: A necessary condition for the functional $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$ to be an extremum is that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

Particular cases:

1. When f is independent of y, i.e, $\frac{\partial f}{\partial y} = 0.$

Therefore Euler equation becomes $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0,$

$$\Rightarrow \frac{\partial f}{\partial y'} = \text{constant.}$$

2. When f is independent of x, i.e., $\frac{\partial f}{\partial X} = 0.$

From equation (5) $\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0,$

$$\Rightarrow f - y'\frac{\partial f}{\partial y'} = \text{constant.}$$

3. When f is independent of both x and y, i.e. $\frac{\partial f}{\partial X} = 0,\ \frac{\partial f}{\partial y} = 0.$

From equation (6) $\frac{\partial^2 f}{\partial (y')^2} y'' = 0,$

If $y'' \neq 0$ $\frac{\partial^2 f}{\partial (y')^2} = 0,\ \Rightarrow \frac{\partial f}{\partial y'} = \text{constant.}$

If $\frac{\partial^2 f}{\partial (y')^2} \neq 0$ then $y'' = 0 \Rightarrow y = ax + b$ is the extremal curve.

Examples:

1. Find the extremal of the functional $\int_{x_1}^{x_2}[(y')^2 + 2y]dx$.

Solution: $f(x, y, y') = (y')^2 + 2y$.

Euler's equation to extremize the functional is

$$f - y'\frac{\partial f}{\partial y'} = \text{constant},$$

$$(y')^2 + 2y - y'(2y') = \text{c},$$

$$2y - (y')^2 = \text{c},$$

$$(y')^2 = 2y - \text{c}$$

$$\frac{dy}{\sqrt{2y - \text{c}}} = \pm dx$$

$$\sqrt{2y - \text{c}} = \pm x$$

$$2y - \text{c} = x^2$$

$$y = \frac{1}{2}(x^2 + \text{c})$$

Example

2. Obtain the extremal of the functional $I = \int_0^1 [(y')^2 + 12xy]dx,\ y(0) = 0, y(1) = 1.$

Solution: $f(x, y, y') = (y')^2 + 12xy.$

Euler's equation to extremize the functional is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0,$$

$$12x - \frac{d}{dx}(2y') = 0,$$

$$\Rightarrow y'' = 6x.$$

Integrating,

$$y' = 3x^2 + c_1.$$

Again integrating ,

$$y = x^3 + c_1 x + c_2.$$

When $x = 0, y = 0 \Rightarrow c_2 = 0.$

When $x = 1, y = 1 \Rightarrow 1 + c_1 = 1 \Rightarrow c_1 = 0.$

$\therefore$ The curve $y = x^3$ extrimizes the functional .

Application of calculus of variation

Geodesics:

Geodesics is a curve of shortest length joining two points on any surface.

Method of finding geodesics on any surface:

Let A and B be two points on the surface and ds represents the element of arc length of the curve joining A and B. As there are infinite number of paths joining the two points, to find the geodesic it is required to extremise the functional for arc length given by $s = \int_A^B ds$.

Geodesics on a plane:

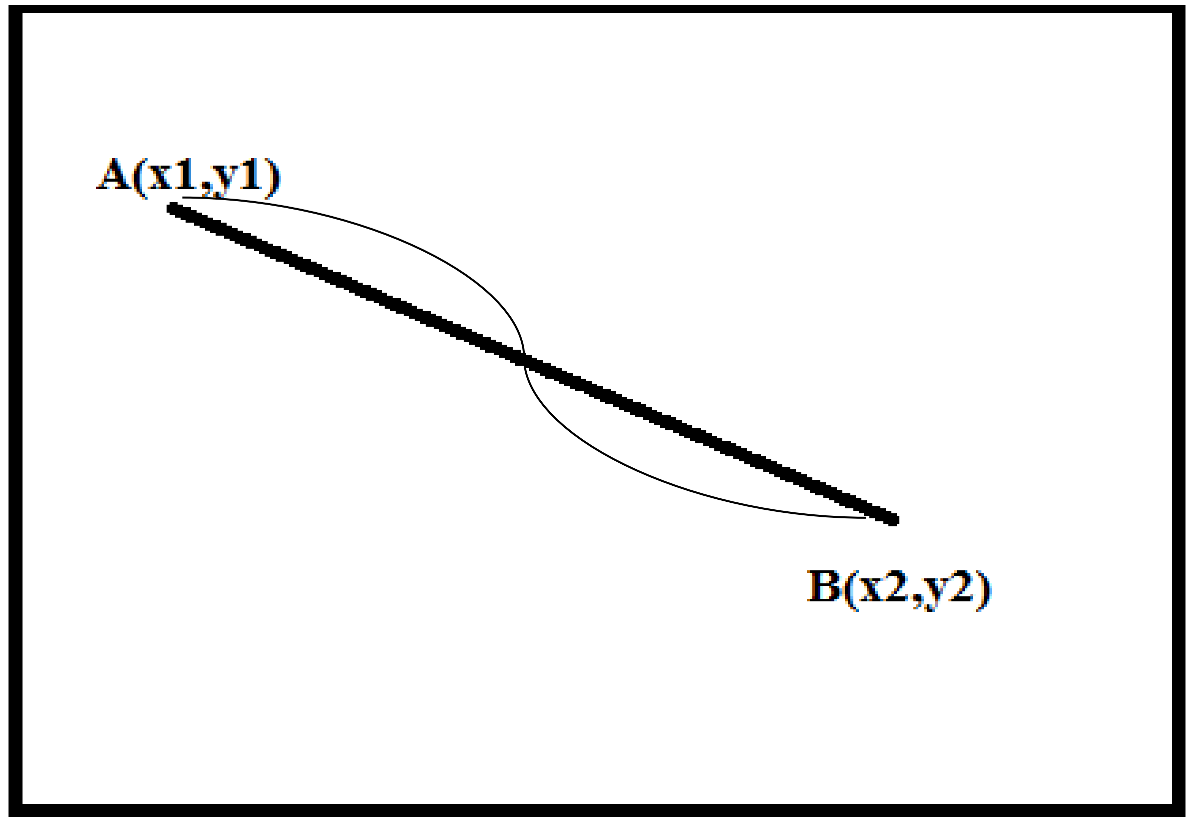
Show that the straight line is the shortest distance curve joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in a plane.

OR

Show that geodesic on the plane is a straight line.

Proof:

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points on the plane. Let ds be the element of arc length of the curve joining these two points



Geodesics on a plane cont.…

For a plane element of arc length is

$$ds = \sqrt{(dx)^2 + (dy)^2},$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

$$ds = \sqrt{1 + (y')^2}\, dx.$$

∴ The total length of the arc joining A and B is $s = \int_A^B ds$.

$$s = \int_A^B \sqrt{1 + (y')^2}\, dx = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx \,. \qquad (1)$$

To find the geodesic, it is required to extremise the functional given by (1),

Consider $\int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$.

Here $f(x, y, y') = \sqrt{1 + (y')^2}$.

Geodesics on a plane cont.…

As f is independent of both x and y, the modified form of Euler equation is $y''\frac{\partial^2 f}{\partial (y')^2} = 0$, (2)

$$\frac{\partial f}{\partial y'} = \frac{2y'}{2\sqrt{1+(y')^2}},$$

$$\frac{\partial^2 f}{\partial (y')^2} = \frac{[1+(y')^2]-(y')^2}{(1+(y')^2)\sqrt{1+(y')^2}},$$

$$\frac{\partial^2 f}{\partial (y')^2} = \frac{1}{(1+(y')^2)\sqrt{1+(y')^2}} \neq 0$$

$$\therefore y'' = 0.$$

Integrating, $y' = constant = m.$

Integrating again, $y = mx + c.$ (3)

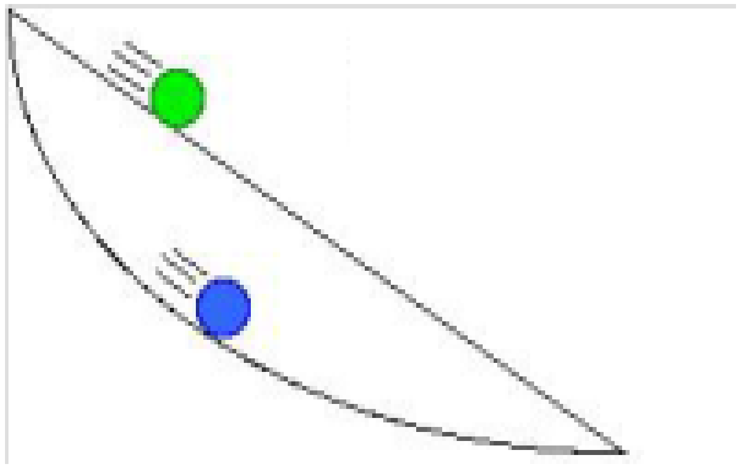
The curve given by (3) is the extremal curve of the functional (1) which can be minimizing or maximizing. knowing the second variation, it is proved that $y = mx + c$ is a curve of minimum length. Therefore straight line is the geodesics for a plane. The constant m and c can be found using boundary conditions at (x_1, y_1) and (x_2, y_2).

Brachistochrone Problem (Greek words, Brachistos means shortest and Chronos means time)

Johann Bernoulli's Challenge

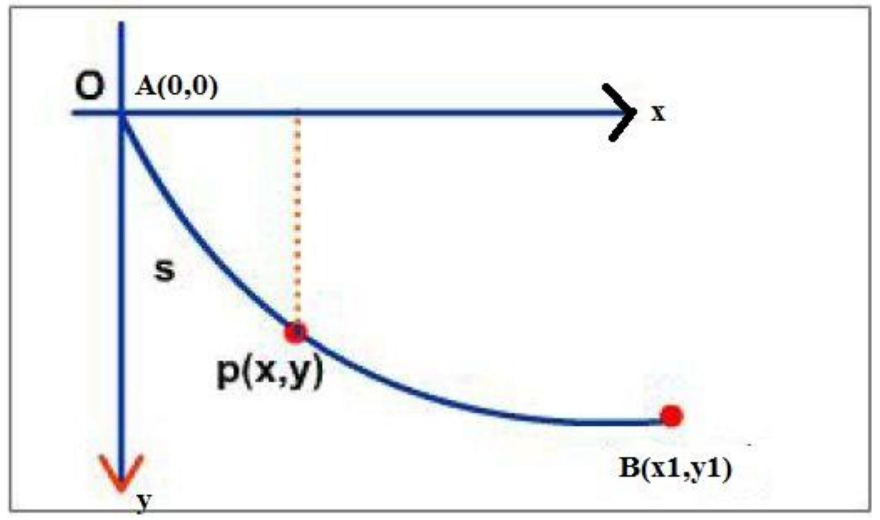
The calculus of variations as a recognizable as a part of mathematics had its origins in Johann Bernoulli's challenge in 1696 to the mathematicians of Europe to find the curve of quickest descent, or brachistochrone. The brachistochrone means the "shortest time" in Greek .

Start the two balls at the top at the same time .The one rolling along the curved path travels further , but reaches the bottom first.



Brachistochrone Problem (Greek words, Brachistos means shortest and Chronos means time)

Find the path in which a particle, in the absence of friction, will slide from one point to another in the shortest time under the action of gravity.



Proof:

Consider the particle initially at point A. for convenience; consider the point A to coincide with origin. Let the horizontal line through A be the x −axis and let the y −axis be vertically downwards as the particle is sliding down due to gravity. Let $P(x, y)$ be any point on the curve joining $A(0,0)$ and $B(x_1, y_1)$, where A and B are on different vertical planes but not along the same vertical line. Let ds represent the element of arc length. If dt represents the time taken by the particle to travel a distance ds, then velocity of the particle $v = \frac{ds}{dt}$.

From the law of conservation of energy,

Brachistochrone Problem cont.…

Kinetic Energy(KE) + (Potential energy)$PE = constant$, i.e., at any point on the curve, gain in $KE =$ loss in PE.

i.e., $\frac{1}{2}mv^2 = mgy$

where $m \rightarrow$ mass of the particle

$g \rightarrow$ acceleration due to gravity

$y \rightarrow$ vertical displacement of the particle

$$\therefore \frac{1}{2}\left(\frac{ds}{dt}\right)^2 = gy, \qquad (1)$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{2gy},$$

$$\Rightarrow dt = \frac{ds}{\sqrt{2gy}}.$$

Brachistochrone Problem cont.…

If $'T'$ represents the total time taken by the particle in sliding down to point B, then

$$T = \int_A^B dt,$$

$$\Rightarrow T = \int_0^{x_1} \frac{ds}{\sqrt{2gy}} = \int_0^{x_1} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\, dx \quad . \qquad (2)$$

To find the path of minimum time, it is required to extremize the functional (2).

Consider $= \frac{1}{\sqrt{2g}} \sqrt{\frac{1+(y')^2}{y}}$.

As f is independent of x, the Euler's equation is $f - y' \frac{df}{dy'} = c$.

$$\Rightarrow \frac{1}{\sqrt{2gy}} \left[\sqrt{1} + (y')^2 - \; y' \frac{y'}{\sqrt{1+(y')^2}}\right] = c,$$

$$\Rightarrow \frac{1}{\sqrt{2g}} \frac{1}{\sqrt{y(1+(y')^2)}} = c,$$

Brachistochrone Problem cont.…

$$\Rightarrow \sqrt{y(1+(y')^2)} = \frac{1}{c\sqrt{2g}},$$

$$\Rightarrow y[1+(y')^2] = \frac{1}{2c^2 g},$$

$$\Rightarrow y[1+(y')^2] = k^2 \quad \text{where } k^2 = \frac{1}{2c^2 g},$$

$$\Rightarrow 1+(y')^2 = \frac{k^2}{y} \; or \; (y')^2 = \frac{k^2-y}{y},$$

$$\Rightarrow y' = \frac{dy}{dx} = \sqrt{\frac{k^2-y}{y}},$$

Separating the variables,

$$\Rightarrow \int \sqrt{\frac{y}{k^2-y}}\, dy = \int dx,$$

$$\Rightarrow x = \int \sqrt{\frac{y}{k^2-y}}\, dy + k_1.$$

Brachistochrone Problem cont….

Put $y = k^2 \sin^2\left(\frac{\theta}{2}\right) = \frac{k^2}{2}(1 - cos\theta)$, (3)

$$\Rightarrow dy = k^2 sin\left(\frac{\theta}{2}\right) cos\left(\frac{\theta}{2}\right) d\theta,$$

$$\therefore x = \int \sqrt{\frac{k^2 \sin^2\left(\frac{\theta}{2}\right)}{k^2\left(1-\sin^2\left(\frac{\theta}{2}\right)\right)}}\, k^2 sin\left(\frac{\theta}{2}\right) cos\left(\frac{\theta}{2}\right) d\theta + k_1,$$

$$\Rightarrow x = \int \sqrt{\frac{\sin^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right)}}\, k^2 sin\left(\frac{\theta}{2}\right) cos\left(\frac{\theta}{2}\right) d\theta + k_1,$$

$$\Rightarrow x = k^2 \int \sin^2\left(\frac{\theta}{2}\right) d\theta + k_1,$$

$$\Rightarrow x = \frac{k^2}{2} \int (1 - cos\theta) d\theta + k_1,$$

$$\Rightarrow x = \frac{k^2}{2}[\theta - sin\theta] + k_1 \;, \qquad (4)$$

Brachistochrone Problem cont.…

From (3) and (4)

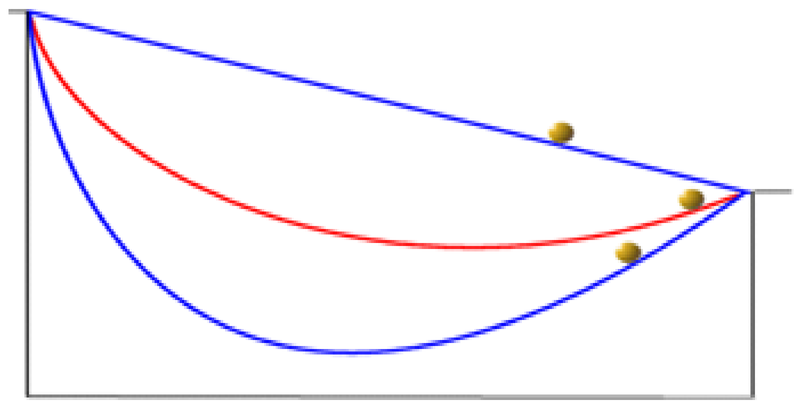
$x = \frac{k^2}{2}[\theta - sin\theta] + k_1, y = \frac{k^2}{2}(1 - cos\theta)$, which are in the parametric form.

When $\theta = 0, y = 0$ $k_1 = x$.

As the particle is sliding from the point A, when $y = 0, x = 0 \Rightarrow k_1 = 0$.

$\therefore x = a(\theta - sin\theta), y = a(1 - cos\theta)$, where $a = \frac{k^2}{2}$, which represents a cycloid.

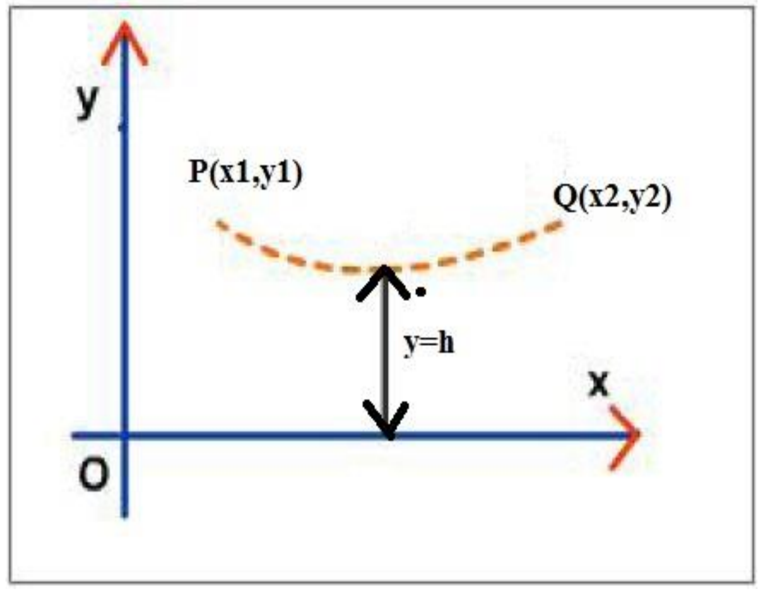
Therefore particle slides from point A to B, along a frictionless curve which is in the form of a cycloid in a minimum time or the distance covered in shortest time is along a cycloid.



The red brachistochrone (inverted cycloid) curve is the curve of fastest descent between two points.

Hanging Cable problem:

Show that a heavy cable hangs freely under gravity between two fixed points in the shape of a catenary.



Proof:

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two fixed point of the hanging cable. Let x- axis be the line of reference. Let ds be the element of arc length of the cable. Let ρ be the density of the cable so that ρds is the mass of the element. If g is the acceleration due to gravity then the potential energy is given by

$$PE = mgh = (\rho ds)gy.$$

Therefore total PE of the cable is given by,

$$I = \int_P^Q \rho dsgy = \int_{x_1}^{x_2} \rho gy \frac{ds}{dx} dx.$$

Hanging Cable problem cont….

But $\frac{ds}{dx} = \sqrt{1+(y')^2}$.

Therefore

$$I = \int_{x_1}^{x_2} \rho g y \sqrt{1+(y')^2}\, dx \qquad (1)$$

To find the shape of the curve, it is required to extremize the functional given by (1).

Consider $f(x, y, y') = \rho g y \sqrt{1+(y')^2}$.

As the functional is independent of x, the Euler equation is given by

$$f - y' \frac{\partial f}{\partial y'} = c,$$

$$\Rightarrow \rho g y \left[\sqrt{1+(y')^2} - y' \rho g y \frac{2y'}{2\sqrt{1+(y')^2}} \right] = c,$$

$$\Rightarrow \rho g [y(1+(y')^2) - y(y')^2] = c\sqrt{1+(y')^2},$$

Hanging Cable problem cont….

$$\Rightarrow \rho g y = c\sqrt{1+(y')^2},$$

$$\Rightarrow \left(\frac{\rho g}{c}\right)^2 y^2 = 1+(y')^2,$$

$$\Rightarrow k^2y^2 = 1+(y')^2 \Rightarrow (y')^2 = k^2y^2-1,$$

$$\Rightarrow y' = \frac{dy}{dx} - \sqrt{k^2y^2-1},$$

$$\Rightarrow \frac{dy}{\sqrt{k^2y^2-1}} = dx,$$

$$\frac{dy}{k\sqrt{y^2-\frac{1}{k^2}}} = dx.$$

Integrating,

$$\cosh^{-1}\left(\frac{y}{1/k}\right) = kx + k_1,$$

$$y = \frac{1}{k}\cosh(kx+k_1),$$

which represents a catenary.

Therefore a cable between two fixed points hangs in the form of a catenary.

Examples:

1.

A chain hanging from points forms a catenary.

3.

The silk on a spider's web forming multiple elastic catenaries.

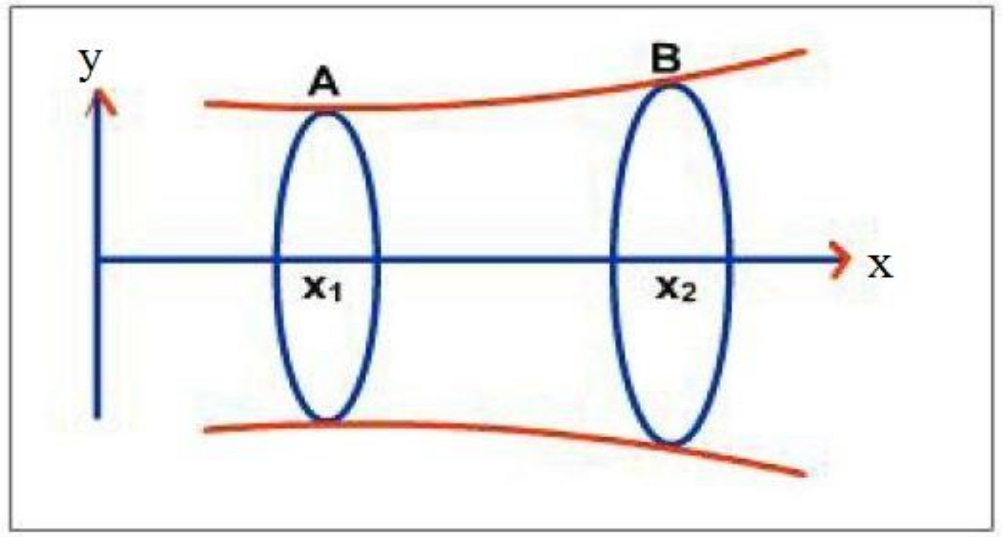
2.

Freely-hanging electric power cables (for example, those used on electrified railways) can also form a catenary.

Minimal surface of revolution

Find the curve passing through the points (x_1, y_1) and (x_2, y_2) which when rotated about the x-axis gives a minimum surface area.

Proof:



Let ds be the element of arc length on the curve joining A and B

when the curve is rotated about x-axis it covers a distance of $2\pi y$ units.

Therefore, area of the small strip=$2\pi y ds$.

To find the complete area of the surface generated by the revolving curve, it is given by $A = \int_A^B 2\pi y ds$,

$$A = \int_{x_1}^{x_2} 2\pi y\sqrt{1 + (y')^2}\, dx, \qquad (1)$$

To find the shape of the curve, which generates surface of minimum area, it is required to extremise the functional given by (1).

Consider $f(x, y, y') = 2\pi y\sqrt{1 + (y')^2}$.

As the functional is independent of x, the Euler's equation is given by

$$f - y'\frac{\partial f}{\partial y'} = c,$$

$$\Rightarrow 2\pi y\left[\sqrt{1 + (y')^2} - y'\frac{2y'}{2\sqrt{1+(y')^2}}\right] = c,$$

Minimal surface of revolution cont.…

$$\Rightarrow 2\pi y\left[\sqrt{1+(y')^2} - y'\frac{2y'}{2\sqrt{1+(y')^2}}\right] = c,$$

$$\Rightarrow 2\pi y\left[\frac{1}{\sqrt{1+(y')^2}}\right] = c,$$

$$\Rightarrow \sqrt{1+(y')^2} = ky \text{ where } k = \frac{c}{2\pi},$$

$$\Rightarrow 1+(y')^2 = k^2y^2, \qquad \Rightarrow y' = k\sqrt{y^2 - \frac{1}{k^2}},$$

$$\Rightarrow \frac{dy}{dx} = k\sqrt{y^2 - \frac{1}{k^2}}, \qquad \Rightarrow \frac{dy}{\sqrt{y^2-\frac{1}{k^2}}} = kdx.$$

Integrating,

$$\int \frac{dy}{\sqrt{y^2-\frac{1}{k^2}}} = \int kdx,$$

$$\Rightarrow \cosh^{-1}\left(\frac{y}{1/k}\right) = kx + k_1,$$

$$y = \frac{1}{k}\cosh(kx+k_1). \qquad (2)$$

Minimal surface of revolution cont…

$$\int \frac{dy}{\sqrt{y^2-\frac{1}{k^2}}} = \int k\,dx,$$

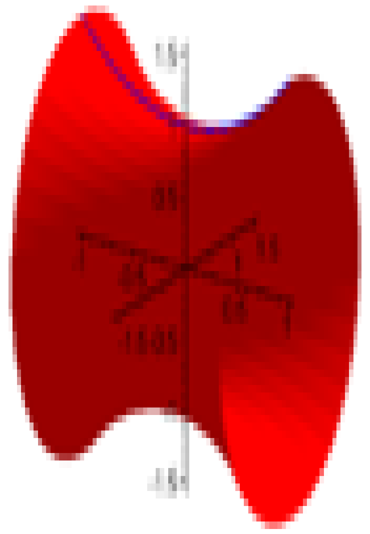
$$\Rightarrow \cosh^{-1}\left(\frac{y}{1/k}\right) = kx + k_1,$$

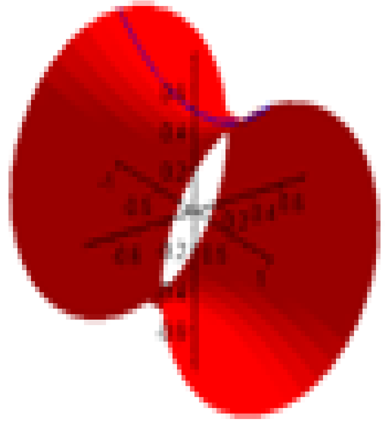
$$y = \frac{1}{k}\cosh(kx + k_1). \qquad (2)$$

The expression given by (2) is the extremal curve of the functional (1) which minimizes or maximizes the area. The curve given by (2) represents a catenary and the surface generated is called a catenoid. Knowing the boundary conditions at $x_1 and\ x_2$, it is possible to find the constants k and k_1.

Examples

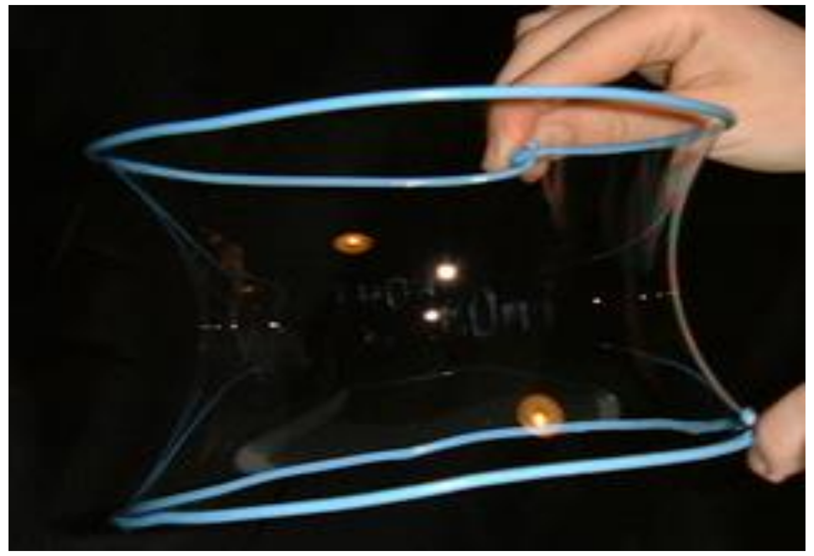
1.Surface of revolution generated by catenaries(graphs of hyperbolic cosines)are called catenoids.





Minimal surface of revolution cont.…

2. Stretching a soap film between two parallel circular wire loops generates a catenoidal minimal surface of revolution.



Video links:

1. Right circular cylinder - https://www.youtube.com/watch?v=C8-rU9XoGxs
2. Sphere - https://www.youtube.com/watch?v=48Mal2asfEY
3. Right circular cone -https://www.youtube.com/watch?v=oN0q2Q5fh68
4. Brachistochrone problem - https://www.youtube.com/watch?v=skvnj67YGmw , https://www.youtube.com/watch?v=Cld0p3a43fU
5. Application problem - https://www.youtube.com/watch?v=l_ffdarcJiQ

All the best