Mathematical Modelling Through Graphs

7.1 SITUATIONS THAT CAN BE MODELLED THROUGH GRAPHS

7.1.1 Qualitative Relations in Applied Mathematics

It has been stated that "Applied Mathematics is nothing but solution of differential equations". This statement is wrong on many counts: (*i*) Applied Mathematics also deals with solutions of difference, differential-difference, integral, integro-differential, functional and algebraic equations (*ii*) Applied Mathematics is equally concerned with inequations of all types (*iii*) Applied Mathematics is also concerned with mathematical modelling; in fact mathematical modelling has to precede solution of equations (*iv*) Applied Mathematics also deals with situations which cannot be modelled in terms of equations or inequations; one such set of situations is concerned with qualitative relations.

Mathematics deals with both quantitative and qualitative relationships. Typical qualitative relations are: y likes x, y hates x, y is superior to x, y is subordinate to x, y belongs to same political party as x, set y has a non-null intersection with set x; point y is joined to point x by a road, state y can be transformed into state x, team y has defeated team x, y is father of x, course y is a prerequisite for course x, operation y has to be done before operation x, species y eats species x, y and x are connected by an airline, y has a healthy influence on x, any increase of y leads to a decrease in x, y belongs to same caste as x, y and x have different nationalities and so on.

Such relationships are very conveniently represented by graphs where a graph consists of a set of vertices and edges joining some or all pairs of these vertices. To motivate the typical problem situations which can be modelled through graphs, we consider the first problem so historically modelled *viz.* the problem of seven bridges of Konigsberg.

7.1.2 The Seven Bridges Problem

There are four land masses *A*, *B*, *C*, *D* which are connected by seven bridges numbered 1 to 7 across a river (Figure 7.1). The problem is to start from any point in one of the land masses, cover each of the seven bridges once and once only and return to the starting point.

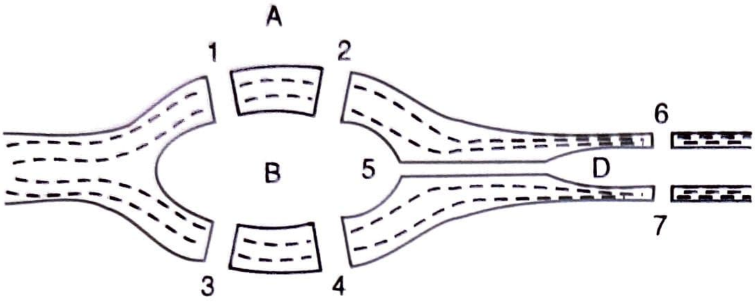

Figure 7.1

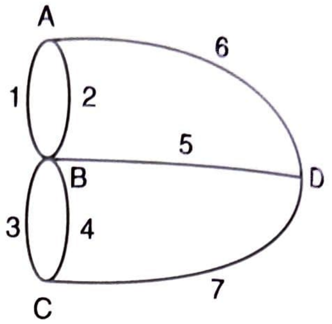

Figure 7.2

There are two ways of attacking this problem. One method is to try to solve the problem by walking over the bridges. Hundreds of people tried to do so in their evening walks and failed to find a path satisfying the conditions of the problem. A second method is to draw a scale map of the bridges on paper and try to find a path by using a pencil.

It is at this stage that concepts of mathematical modelling are useful. It is obvious that the sizes of the land masses are unimportant, the lengths of the bridges or whether these are straight or curved are irrelevant. What is relevant information is that *A* and *B* are connected by two bridges 1 and 2, *B* and *C* are connected by two bridges 3 and 4, *B* and *D* are connected by one bridge number 5, *A* and *D* are connected by bridge number 6 and *C* and *D* are connected by bridge number 7. All these facts are represented by the graph with four vertices and seven edges in Figure 7.2. If we can trace this graph in such a way that we start with any vertex and return to the same vertex and trace every edge once and once only without lifting the pencil from the paper, the problem can be solved. Again trial and error method cannot be satisfactorily used to show that no solution is possible.

The number of edges meeting at a vertex is called the degree of that vertex. We note that the degrees of *A*, *B*, *C*, *D* are 3, 5, 3, 3 respectively and each of these is an odd number. If we have to start from a vertex and return to it, we need an even number of edges at that vertex. Thus it is easily seen that Konigsberg bridges problem cannot be solved.

This example also illustrates the power of mathematical modelling. We have not only disposed of the seven-bridges problem, but we have discovered a technique for solving many problems of the same type.

7.1.3 Some Types of Graphs

A graph is called *complete* if every pair of its vertices is joined by an edge (Figure 7.3 (*a*)).

A graph is called a *directed graph* or a *digraph* if every edge is directed with an arrow. The edge joining A and B may be directed from A to B or from B to A. If an edge is left undirected in a digraph, it will be assumed to be directed both ways (Figure 7.3 (b)).

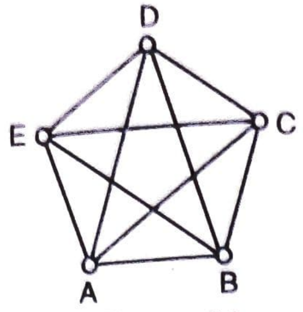

Figure 7.3a

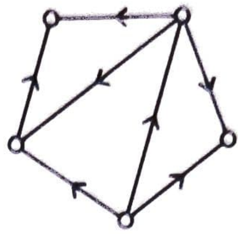

Figure 7.3b

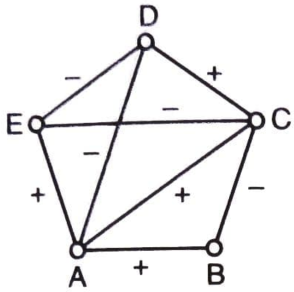

Figure 7.3c

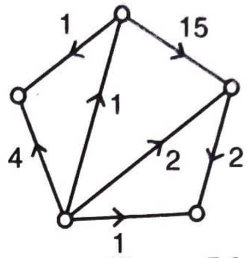

Figure 7.3d

A graph is called a *signed graph* if every edge has either a plus or minus sign associated with it (Figure 7.3(c)).

A digraph is called a *weighted digraph* if every directed edge has a weight (giving the importance of the edge) associated with it (Figure 7.3 (d)). We may also have digraphs with positive and negative numbers associated with edges. These will be called *weighted signed digraphs*.

7.1.4 Nature of Models in Terms of Graphs

In all the applications we shall consider, the length of the edge joining two vertices will not be relevant. It will not also be relevant whether the edge is straight or curved. The relevant facts would be: (a) which edges are joined; (b) which edges are directed and in which direction(s); (c) which edges have positive or negative signs associated with them; (d) which edges have weights associated with them and what these weights are.

EXERCISE 7.1

1. In the Konigsberg problem suggest deletion or addition of minimum number of bridges which may lead to a solution of the problem.
2. Show that in any graph, the sum of local degrees of all the vertices is an even number. Deduce that a graph has an even number of odd vertices.
3. Three houses A, B, C have to be connected with three utilities a, b, c by separate wires lying in the same plane and not crossing one another. Explain why this is not possible.

4. Each of the four neighbours has connected his house with the other three houses by paths which do not cross. A fifth man builds a house nearby. Prove that (*a*) he cannot connect his house with all others by non-intersecting paths (*b*) he can however connect with three of the houses.
5. A graph is called regular if each of its vertices has same degree r. Draw regular graphs with 6 vertices and degree 5, 4 and 3.
6. Show that in Konigsberg, four one-way bridges will be enough to connect the four land masses.

7.2 MATHEMATICAL MODELS IN TERMS OF DIRECTED GRAPHS

7.2.1 Representing Results of Tournaments

The graph (Figure 7.4) shows that:

(*i*) Team A has defeated teams B, C, E.

(*ii*) Team B has defeated teams C, E.

(*iii*) Team E has defeated D.

(*iv*) Matches between A and D, B and D, C and D and C and E have yet to be played.

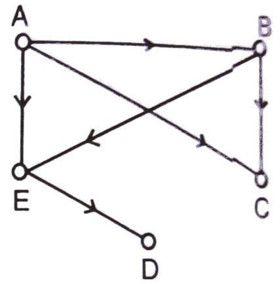

Figure 7.4

7.2.2 One-Way Traffic Problems

The road map of a city can be represented by a directed graph. If only one-way traffic is allowed from point a to point b, we draw an edge directed from a to b. If traffic is allowed both ways, we can either draw two edges, one directed from a to b and the other directed from b to a or simply draw an undirected edge between a and b. The problem is to find whether we can introduce one-way traffic on some or all of the roads without preventing persons from going from any point of the city to any other point. In other words, we have to find when the edges of a graph can be given direction in such a way that there is a directed path from any vertex to every other. It is easily seen that one-way traffic on the road DE cannot be introduced without disconnecting the vertices of the graph (Figure 7.5).

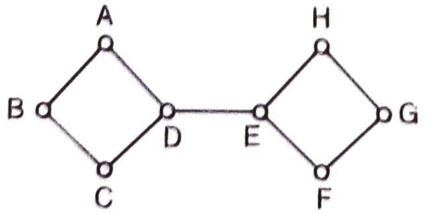

Figure 7.5 (a)

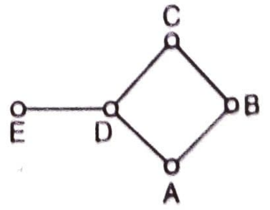

Figure 7.5 (b)

In Figure 7.5(*a*), DE can be regarded as a bridge connecting two regions of the town. In Figure 7.5(*b*) DE can be regarded as a blind street on which a two-way traffic is necessary. Edges like DE are called *separating edges*, while other edges are called *circuit edges*. It is necessary that on separating edges, two-way traffic should be permitted. It can also be shown that this is sufficient. In other words, the following theorem can be established:

If G is an undirected connected graph, then one can always direct the circuit edges of G and leave the separating edges undirected (or both way directed) so that there is a directed path from any given vertex to any other vertex.

7.2.3 Genetic Graphs

In a genetic graph, we draw a directed edge from A to B to indicate that B is the child of A. In general each vertex will have two incoming edges, one from the vertex representing the father and the other from the vertex representing the mother. If the father or mother is unknown, there may be less than two incoming edges. Thus in a genetic graph, the local degree of incoming edges at each vertex must be less than or equal to two. This is a necessary condition for a directed graph to be a genetic graph, but it is not a sufficient condition. Thus Figure 7.6 does not give a genetic graph inspite of the fact that the number of incoming edges at each vertex does not exceed two. Suppose A_1 is male, then A_2 must be female, since A_1, A_2 have a child B_1. Then A_3 must be male, since A_2, A_3 have a child B_2. Now A_1, A_3 being both males cannot have a child B_3.

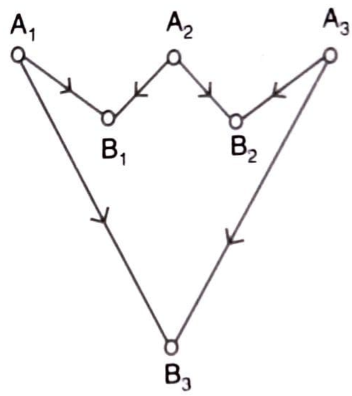

Figure 7.6

7.2.4 Senior-Subordinate Relationship

If a is senior to b, we write aSb and draw a directed edge from a to b. Thus the organisational structure of a group may be represented by a graph like the following [Figure 7.7].

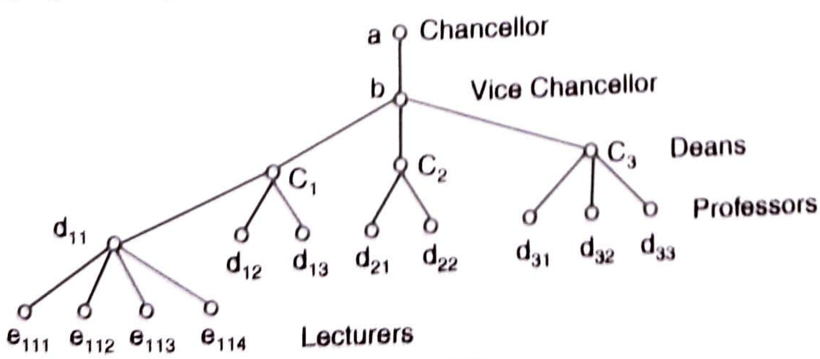

Figure 7.7

The relationship S satisfies the following properties:

(i) $\sim(aSa)$ i.e. no one is his own senior.

(ii) $aSb = \sim(bSa)$ i.e. a is senior to b implies that b is not senior to a.

(iii) $aSb,\ bSc \Rightarrow aSc$ i.e. if a is senior to b and b is senior to c, then a is senior to c.

The following theorem can easily be proved: "The necessary and sufficient condition that the above three requirements hold is that the graph of an organisation should be free of cycles".

We want now to develop a *measure for the status* of each person. The status $m(x)$ of the individual should satisfy the following reasonable requirements:

(*i*) $m(x)$ is always a whole number.

(*ii*) If x has no subordinate, $m(x) = 0$.

(*iii*) If, without otherwise changing the structure, we add a new individual subordinate to x, then $m(x)$ increases.

(*iv*) If, without otherwise changing the structure, we move a subordinate of a to a lower level relative to x, then $m(x)$ increases.

A measure satisfying all these criteria was proposed by Harary. We define the level of seniority of x over y as the length of the shortest path from x to y. To find the measure of status of x, we find n_1, the number of individuals who are one level below x, n_2 the number of individuals who are two levels below x and in general, we find n_k the number of individuals who are k levels below x. Then the Harary measure $h(x)$ is defined by

$$h(x) = \sum_k kn_k \tag{1}$$

It can be shown that among all the measure which satisfy the four requirements given above, Harary measure is the least.

If however, we define the level of seniority of x over y as the length of the longest path from x to y, and then find $H(x) = \sum_k kn_k$, we get another measure which will be the largest among all measures satisfying the four requirements. For Figure 7.8, we get

$h(a) = 1.2 + 4.2 + 2.3 = 16$

$h(b) = 1.3 + 2.4 = 11$

$h(c) = 1.2 + 1.2 = 4$

$H(a) = 1.1 + 3.2 + 2.3 + 2.4 = 21$

$H(b) = 2.1 + 2.2 + 2.3 + 1.4 = 16$

$H(c) = 1.1 + 1.2 + 1.3 = 6$

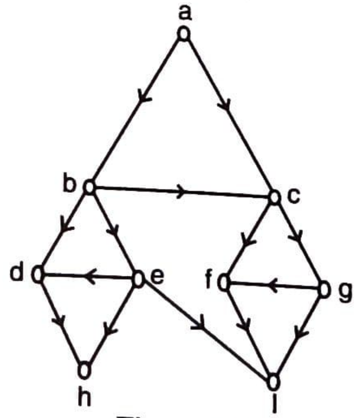

Figure 7.8

$h(d) = 1.1$	$= 1$	$H(d) = 1.1$	$= 1$
$h(e) = 1.3$	$= 3$	$H(e) = 1.2 + 2.1$	$= 4$
$h(f) = 1.1$	$= 1$	$H(f) = 1.1$	$= 1$
$h(g) = 1.2$	$= 2$	$H(g) = 1.2$	$= 2$
$h(k)$	$= 0$	$H(k)$	$= 0$
$h(l)$	$= 0$	$H(l)$	$= 0$

7.2.5 Food Webs

Here aSb if a eats b and we draw a directed edge from a to b. Here also $\sim(aSa)$ and $aSb \Rightarrow \sim(bSa)$. However the transitive law need not hold. Thus consider the food web in Fig. 7.9. Here fox eats bird, bird eats grass, but fox does not eat grass.

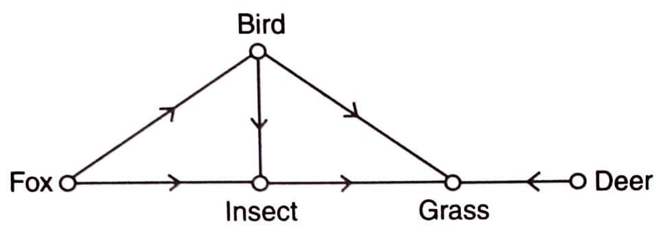

Figure 7.9

We can however calculate measure of the status of each species in this food web by using Eqn. (1) $h(\text{bird}) = 2$, $h(\text{fox}) = 4$, $h(\text{insect}) = 1$, $h(\text{grass}) = 0$, $h(\text{deer}) = 1$.

7.2.6 Communication Networks

A directed graph can serve as a model for a communication network. Thus consider the network given in Figure 7.10. If an edge is directed from a to b, it means that a can communicate with b. In the given network e can communicate directly with b, but b can communicate with e only indirectly through c and d. However every individual can communicate with every other individual.

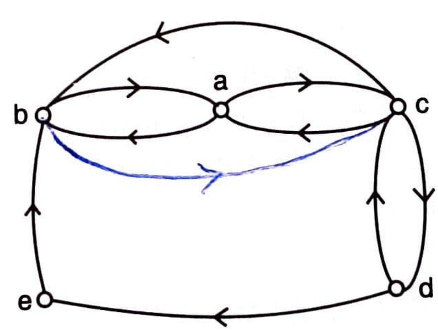

Figure 7.10

Our problem is to determine the importance of each individual in this network. The importance can be measured by the fraction of the messages on an average that pass through him. In the absence of any other knowledge, we can assume that if an individual can send message direct to n individuals, he will send a message to any one of them with probability $1/n$. In the present example, the communication probability matrix is:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad (2)$$

No individual is to send a message to himself and so all diagonal elements are zero. Since all elements of the matrix are non-negative and the sum of elements of every row is unity, the matrix is a stochastic matrix and one of its eigenvalues is unity. The corresponding normalised eigenvector is [11/45, 13/45, 3/10, 1/10, 1/15]. In the long run, these fractions of messages will pass through *a, b, c, d, e* respectively. Thus we can conclude that in this network, *c* is the most important person.

If in a network, an individual cannot communicate with every other individual either directly or indirectly, the Markov chain is not ergodic and the process of finding the importance of each individual breaks down.

7.2.7 Matrices Associated with a Directed Graph

For a directed graph with n vertices, we define the $n \times n$ matrix $A = (a_{ij})$ by $a_{ij} = 1$ if there is an edge directed from i and j and $a_{ij} = 0$ if there is no edge directed from i to j. Thus the matrix associated with the graph of Figure 7.11 is given by

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad (3)$$

We note that (*i*) the diagonal elements of the matrix are all zero (*ii*) the number of non-zero elements is equal to the number of edges (*iii*) the number of non-zero elements in any row is equal to the local outward degree of the vertex corresponding to the row (*iv*) the number of non-zero elements in a column is equal to the local inward degree of the vertex corresponding to the column. Now

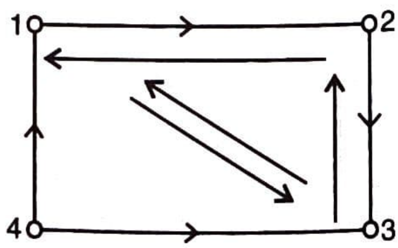

Figure 7.11

$$A^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \end{array} = \left(a_{ij}^{(2)}\right) \tag{4}$$

The element $a_{ij}^{(2)}$ gives the number of 2-chains from i to j. Thus from vertex 2 to vertex 1, there are two 2-chains *viz.* via vertex 3 and vertex 4. We can generalise this result in the form of a theorem *viz.* " The element $a_{ij}^{(2)}$ of A^2 gives the number of 2-chains i.e. the number of paths with two-edges from vertex i to vertex j".

The theorem can be further generalised to "The element $a_{ij}^{(m)}$ of A^m gives the number of m-chains i.e. the number of paths with m edges from vertex i to vertex j". It is also easily seen that "The ith diagonal element of A^2 gives the number of vertices with which i has symmetric relationship".

From the matrix A of a graph, a symmetric matrix S can be generated by taking the elementwise product of A with its transpose so that in our case

$$S = A \times A^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

S obviously is the matrix of the graph from which all unreciprocated connections have been eliminated. In the matrix S (as well as in $S^2, S^3, \ldots$) the elements in the row and column corresponding to a vertex which has no symmetric relation with any other vertex are all zero.

7.2.8 Application of Directed Graphs to Detection of Cliques

A subset of persons in a socio-psychological group will be said to form a clique if (*i*) every member of this subset has a symmetrical relation with every other member of this subset (*ii*) no other group member has a symmetric relation with all the members of the subset (otherwise it will be included in the clique) (*iii*) the subset has at least three members.

If other words a clique can be defined as a maximal completely connected subset of the original group, containing at least three persons. This subset should not be properly contained in any larger completely connected subset.

If the group consists of n persons, we can represent the group by n vertices of a graph. The structure is provided by persons knowing or being connected to other persons. If a person i knows j, we can draw a directed edge from i to j. If i knows j and j knows i, then we have a symmetrical relation between i and j.

With this interpretation, the graph of Figure 7.11 shows that persons 1, 2, 3 form a clique. With very small groups, we can find cliques by carefully observing the corresponding graphs. For larger groups analytical methods based on the following results are useful: (*i*) i is a member of a clique if the ith diagonal element of S^3 is different from zero. (*ii*) If there is only one clique of k members in the group, the corresponding k elements of S^3 will be $(k-1)(k-2)/2$ and the rest of the diagonal elements will be zero. (*iii*) If there are only two cliques with k and m members respectively and there is no element common to these cliques, then k elements of S^3 will be $(k-1)(k-2)/2$, m elements of S^3 will be $(m-1)(m-2)/2$ and the rest of the elements will be zero. (*iv*) If there are m disjoint cliques with $k_1, k_2, \ldots, k_m$ members, then the trace of S^3 is $\frac{1}{2}\sum_{i-1}^{m} k_i(k_i-1)(k_i-2)$. (v) A member is non-cliquical if only if the corresponding row and column of $S^2 \times S$ consists entirely of zeros.

EXERCISE 7.2

1. Show that the graph of Figure 7.12 is a possible genetic graph if and only if n is even.

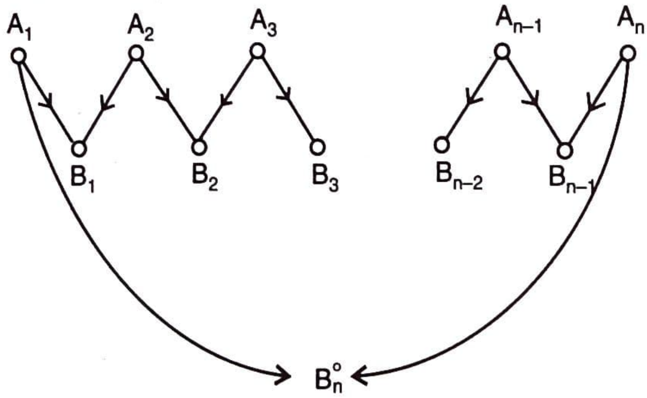

Figure 7.12

2. For each of the following communication networks, set up the corresponding transition probability matrix and find the importance of each member in the network.

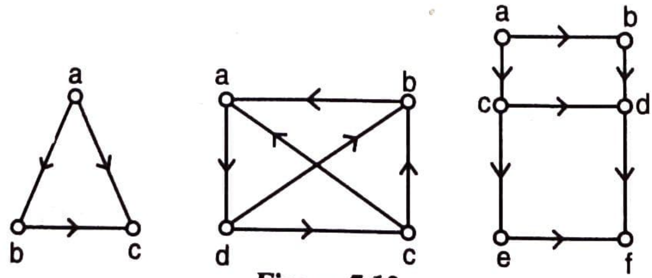

Figure 7.13

3. An intelligence officer can communicate with each of his n subordinates and each subordinate can communicate with him, but the subordinates

cannot communicate among themselves. Draw the graph and find the importance of each subordinate relative to the officer.

4. Find the Harary measure for each individual in the organisational graphs of Figure 7.14.

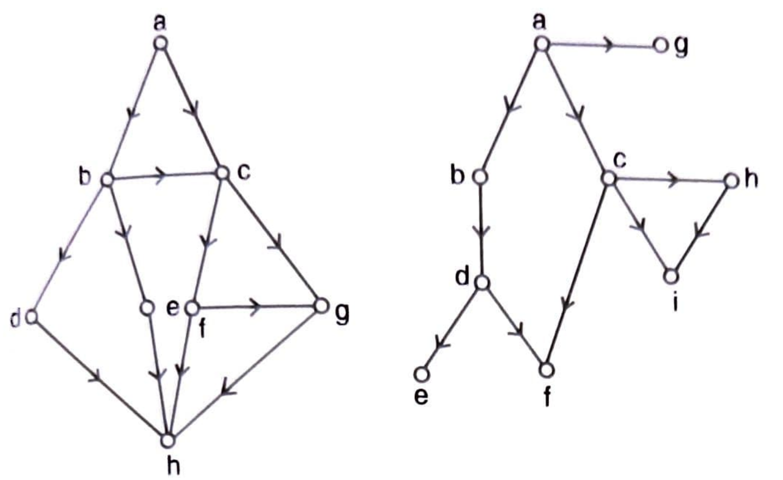

Figure 7.14

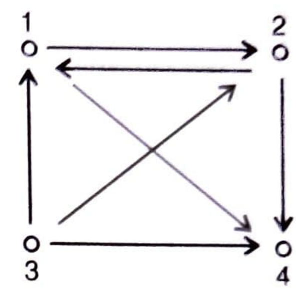

Figure 7.15

5. In Exercise 4, find the measure if the definition of level is based (*i*) on the longest number of steps between two persons (*ii*) on the average of the shortest and longest number of steps between two persons.
6. Find the eigenvector corresponding to the unit eigenvalue of matrix (2).
7. Prove all the theorems stated in Section 7.2.7.
8. Prove all the theorems stated in Section 7.2.8.
9. Write the matrix A associated with the graph of Figure 7.15. Find A^2, A^3, A^4, S, S^2, S^3, and verify the theorems of Sections 7.2.7 and 7.2.8.
10. Enumerate all possible four-cliques.

7.3 MATHEMATICAL MODELS IN TERMS OF SIGNED GRAPHS

7.3.1 Balance of Signed Graphs

A signed (or an algebraic) graph is one in which every edge has a positive or negative sign associated with it. Thus the four graphs of Figure 7.16 are signed graphs. Let positive sign denote friendship and negative sign denote enemity, then in graph (*i*) A is a friend of both B and C and B and C are also friends. In graph (*ii*) A is friend of B and A and B are both jointly enemies of C. In graph (*iii*), A is a friend of both B and C, but B and C are enemies. In graph (*iv*) A is an enemy of both B and C, but B and C are not friends.

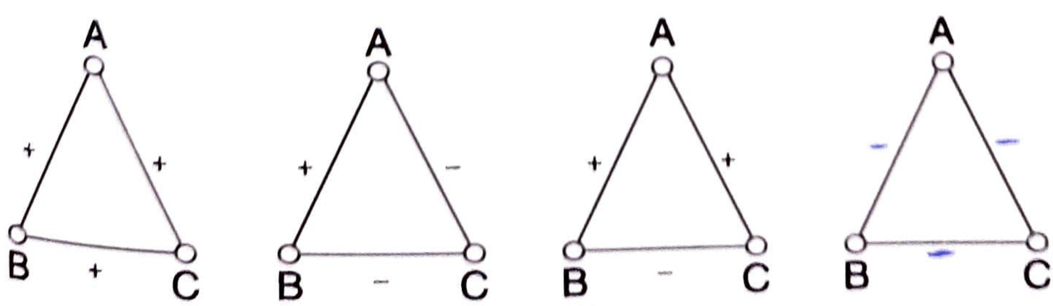

Figure 7.16

The first two graphs represents normal behaviour and are said to be balanced, while the last two graphs represent unbalanced situations since if A is a friend of both B and C and B and C are enemies, this creates a tension in the system and there is a similar tension when B and C have a common enemy A, but are not friends of each other.

We define the sign of a cycle as the product of the signs of component edges. We find that in the two balanced cases, this sign is positive and in the two unbalanced cases, this is negative.

We say that a cycle of length three or a triangle is balanced if and only if its sign is positive. A complete algebraic graph is defined to be a complete graph such that between any two edges of it, there is a positive or negative sign. A complete algebraic graph is said to be balanced if all its triangles are balanced. An alternative definition states that a complete algebraic graph is balanced if all its cycles are positive. It can be shown that the two definitions are equivalent.

A graph is locally balanced at a point a if all the cycles passing through a are balanced. If a graph is locally balanced at all points of the graph, it will obviously be balanced. A graph is defined to be m-balanced if all its cycles of length m are positive. For an incomplete graph, it is preferable to define it to be balanced if all its cycles are positive. The definition in terms of triangle is not satisfactory, as there may be no triangles in the graph.

7.3.4 The Degree of Unbalance of a Graph

For many purposes it is not enough to know that a situation is unbalanced. We may be interested in the degree of unbalance and the possibility of a balancing process which may enable one to pass from an unbalanced to a balanced graph. The possibility is interesting as it can give an approach to group dynamics and demonstrate that methods of graph theory can be applied to dynamic situations also.

Cartwright and Harary define the degree of balance of a group G to be the ratio of the positive cycles of G to the total number of cycles in G. This balance index obviously lies between 0 and 1. G_1 has six negative triangles *viz* (abc), (ade), (bcd), (bce), (bde), (cde) and has four positive triangles. G_2 has four negative triangles *viz* (abc), (abd), (bce) and (bde) and six positive triangles. The degree of balance of G_1 is therefore less than the degree of balance of G_2.

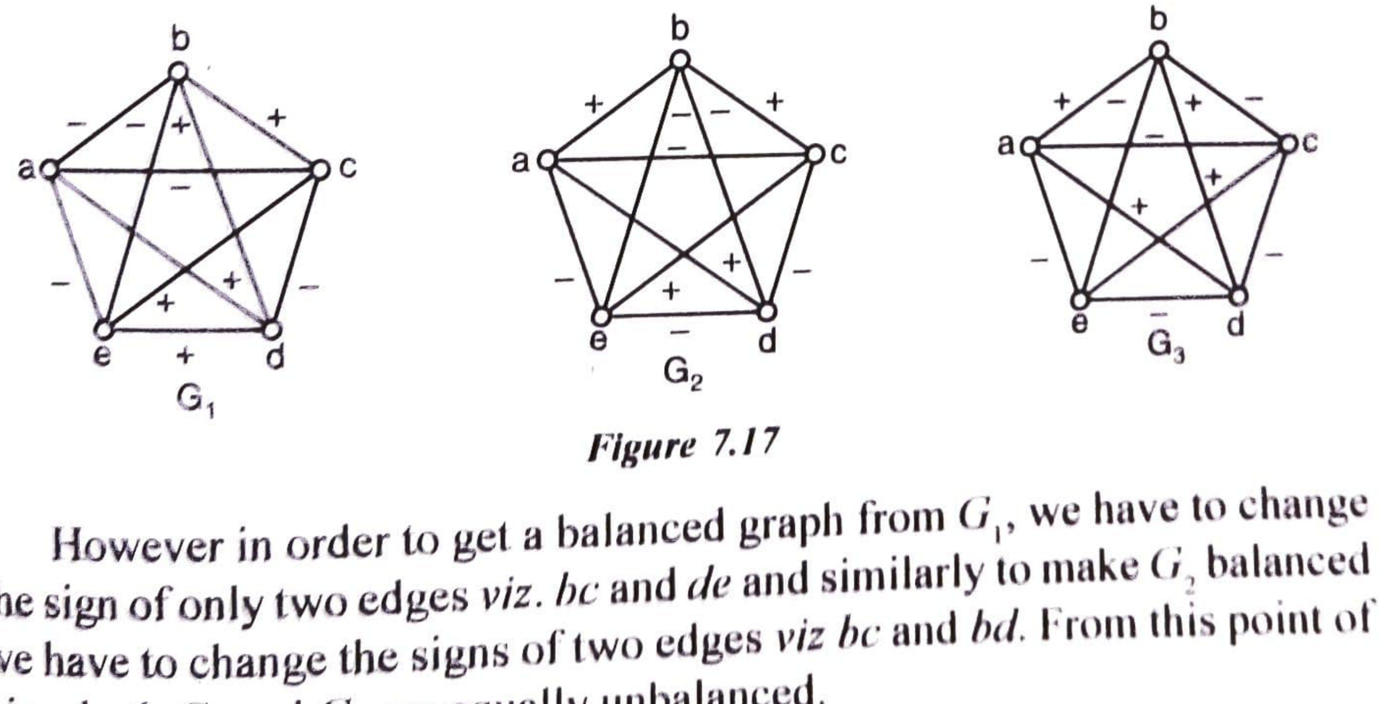

Figure 7.17

However in order to get a balanced graph from G_1, we have to change the sign of only two edges *viz.* bc and de and similarly to make G_2 balanced we have to change the signs of two edges *viz* bc and bd. From this point of view both G_1 and G_2 are equally unbalanced.

Abelson and Rosenberg therefore gave an alternative definition. They defined the degree of unbalance of an algebraic graph as the number of the smallest set of edges of G whose change of sign produces a balanced graph.

The degree of an antibalanced complete algebraic graph (*i.e.*, of a graph all of whose triangles are negative) is given by $[n(n-2)+k]/4$ where $k = 1$

if n is odd and $k = 0$ if n is even. It has been conjectured that the degree of unbalancing of every other complete algebraic graph is less than or equal to this value.

EXERCISE 7.3

1. State which of the following graphs are balanced. If balanced, find the decomposition guaranteed by the structure theorem. If unbalanced, find the degree of unbalance.

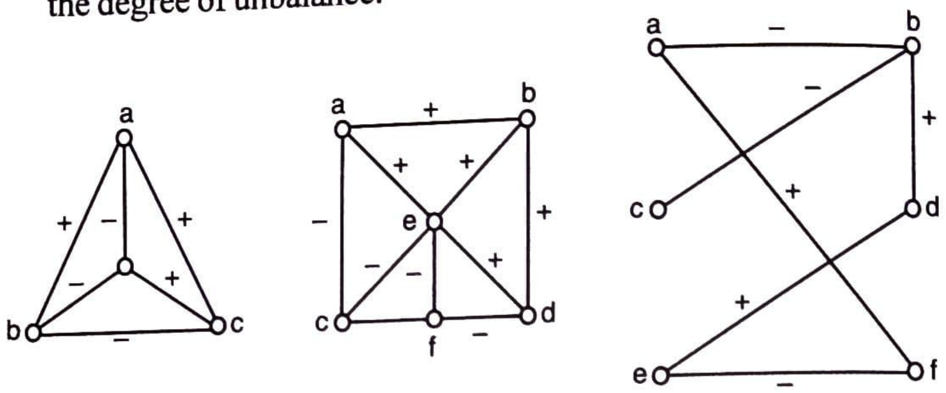

Figure 7.18

2. Draw some antibalanced graphs and verify the structure theorems for them.
3. The adjacency matrix of a signed graph is defined as follows:

   $a_{ij} = 1$ if there is + sign associated with edge i, j

   $= -1$ if there is − sign associated with edge i, j

   $= 0$ if there is no edge i, j.

   Write the adjacency matrices of the four signed graphs is Figure 7.18.
4. A signed graph G is said to have an idealised party structure if the vertices of G can be partitioned into classes so that all edges joining the vertices in the same class have + sign and all edges joining vertices in different sets have negative sign (*a*) Give an example of a signed graph which does not have an idealised party structure (*b*) Give an example of a graph which is not balanced but which has an idealised party structure.
5. Show that a signed graph has an idealised party structure if and only no circuit has exactly one − sign.
6. Show that if all cycles of a signed graph are positive, then all its cycles are also positive. State and prove its converse also.

7.4 MATHEMATICAL MODELLING IN TERMS OF WEIGHTED DIGRAPHS

7.4.1 Communication Networks with Known Probabilities of Communication

In the communication graph of Figure 7.10, we know that a can communicate with both b and c only and in the absence of any other knowledge, we assigned equal probabilities to a's communicating with b or c. However we may have a priori knowledge that a's chances of communicating with b and c are in the ratio 3 : 2, then we assign probability .6 to a's communicating with b and .4 to a's communicating with c. Similarly we can associate a probability with every directed edge and we get the weighted digraph (Figure 7.19) with the associated matrix

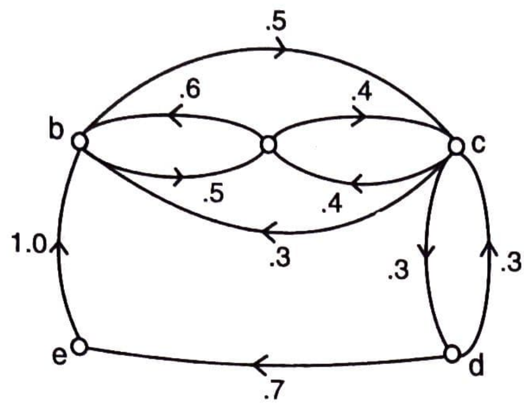

Figure 7.19

$$B = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 0.6 & 0.4 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.4 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0 & .3 & 0 & 0.7 \\ 0 & 1.0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad (6)$$

We note that the elements are all non-negative and the sum of the elements of every row is unity so that B is a stochastic matrix and unity is one of its eigenvalues. The eigenvector corresponding to this eigenvalues will be different from the eigenvector found in Section 7.2.6 and so the relative importance of the individuals depends both on the directed edges as well as on the weights associated with the edges.

7.4.2 Weighted Digraphs and Markov Chains

A Markovian system is characterised by a transition probability matrix. Thus if the states of a system are represented by 1, 2, ..., n and p_{ij} gives the probability of transition from the ith state to jth state, the system is characterised by the transition probability matrix (t.p.m)

$$T = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1j} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots & p_{in} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{n1} & p_{n2} & \cdots & p_{nj} & \cdots & p_{nn} \end{bmatrix} \qquad (7)$$

Since $\sum_{i=1}^{n} p_{ij}$ represents the probability of the system going from ith state to any other state or of remaining in the same state, this sum must be equal to unity. Thus the sum of elements of every row of a t.p.m. is unity.

Consider a set of N such Markov systems where N is large and suppose at any instant NP_1, NP_2,..., NP_n of these ($P_1 + P_2 + ... + P_n = 1$) are in states 1, 2, 3, ..., n respectively. After one step, let the proportions in these states be denoted by $P'_1, P'_2, \dots, P'_n$, then

$$\begin{aligned} P'_1 &= P_1 p_{11} + P_2 p_{21} + P_3 p_{31} + \dots + P_n p_{n1} \\ P'_2 &= P_2 p_{12} + P_2 p_{22} + P_3 p_{32} + \dots + P_n p_{n2} \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ P'_n &= P_1 p_{1n} + P_2 p_{2n} + P_3 p_{3n} + \dots + P_n p_{nn} \end{aligned} \tag{8}$$

or

$$P' = PT \tag{9}$$

where P and P' are row matrices representing the proportions of systems in various states before and after the step and T is the t.p.m.

We assume that the system has been in operation for a long time and the proportions $P_1, P_2, \dots, P_n$ have reached equilibrium values. In this case

$$P = PT \text{ or } P(\mathrm{I} - T) = 0 \tag{10}$$

where I is the unit matrix. This represents a system of n equations for determining the equilibrium values of $P_1, P_2, \dots, P_n$. If the equations are consistent, the determinant of the coefficient must vanish i.e. $|T - I| = 0$. This requires that unity must be an eigenvalue of T. However this, as we have seen already is true. This shows that an equilibrium state is always possible for a Markov chain.

A Markovian system can be represented by a weighted directed graph. Thus consider the Markovian system with the stochastic matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.3 & 0.6 & 0.1 & 0 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{11}$$

Its weighted digraph is given in Figure 7.20.

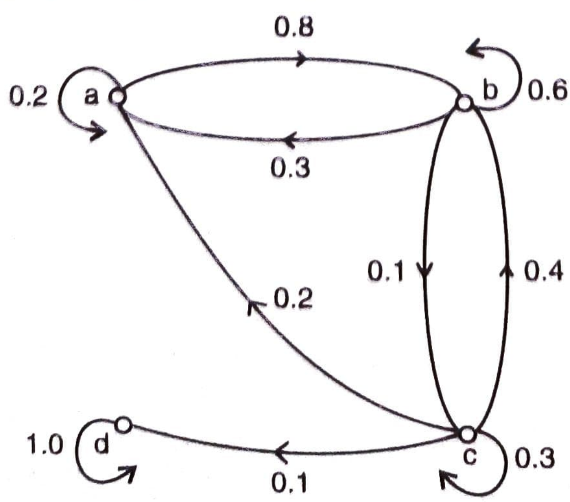

Figure 7.20

7.4.3 General Communication Networks

So for we have considered communication networks in which the weight associated with a directed edge represents the probability of communication along that edge. We can however have more general networks *e.g.*,

(*a*) for communication of messages where the directed edge represents the channel and the weight represents the capacity of the channel say in bits per second.

(*b*) for communication of gas in pipelines where the weights are capacities, say in gallons per hour.

(*c*) communication roads where the weights are the capacities in cars per hour.

An interesting problem is to find the maximum flow rate, of whatever is being communicated, from any vertex of the communication network to any other. Useful graph-theoretic algorithms for this have been developed by Elias. Feinstein ,and Shannon as well as by Ford and Fulkerson.

7.4.4 More General Weighted Digraphs

In the most general case, the weight associated with a directed edge can be positive or negative. Thus Figure 7.21 means that a unit change at vertex 1 at time t causes changes of -2 units at vertex 2, of 2 units at vertex 4 and of 3 units at vertex 5 at time $t + 1$. Similarly a change of 1 unit at vertex 2 causes a change of -3 units at 3 vertex, 4 units at vertex 4 and of 2 units at vertex 5 and so on. Given the values at all vertices at time t, we can find the values at time $t + 1, t + 2, t + 3, \ldots$. The process of doing this systematically is known as the pulse rule.

These general weighted digraphs are useful for representing energy flows, monetary flows and changes in environmental conditions.

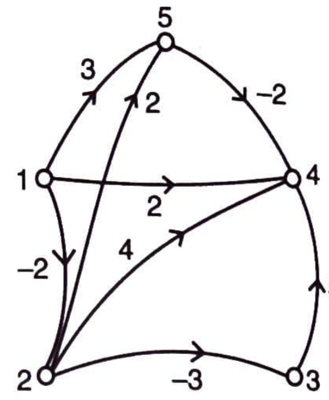

Figure 7.21

7.4.5 Signal Flow Graphs

The system of algebraic equations

$$\begin{aligned} x_1 &= 4y_0 + 6x_2 - 2x_3 \\ x_2 &= 2y_0 - 2x_1 + 2x_3 \\ x_3 &= 2x_1 - 2x_2 \end{aligned} \tag{14}$$

can be represented by the weighted digraph in Figure 7.22. For solving for x_1, we successively eliminate x_3 and x_2 to get the graphs in Figure 7.23 and finally we get

$$x_1 = 4y_0$$

We can similarly represent the solution of any number of linear equations graphically.

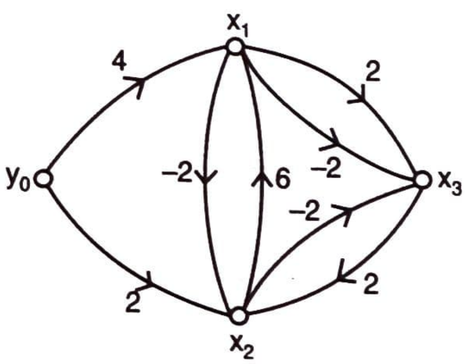

Figure 7.22

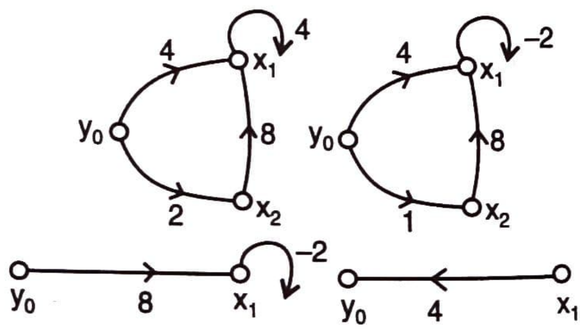

Figure 7.23

7.4.6 Weighted Bipartitic Digraphs and Difference Equations

Consider the system of difference equations

$$\begin{aligned} x_{t+1} &= a_{11}x_t + a_{12}y_t + a_{13}z_t \\ y_{t+1} &= a_{21}x_t + a_{22}y_t + a_{23}z_t \\ z_{t+1} &= a_{31}x_t + a_{32}y_t + a_{33}z_t \end{aligned} \tag{15}$$

This can be represented by a weighted bipartitic digraph (Figure 7.24). The weights can be positive or negative.

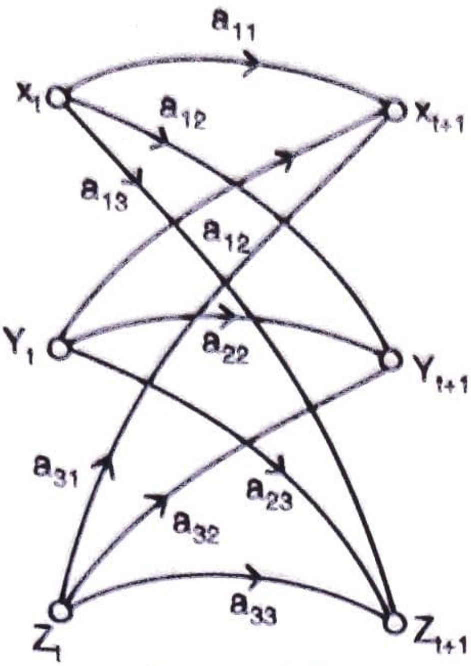

Figure 7.24

EXERCISE 7.4

1. A machine can be in any one of the states a, b, c. The transitions between states are governed by the transition probability matrix

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array} \qquad (16)$$

Draw the weighted digraph and find the limiting probabilities for the machine to be found in each of the three states.

1849 that Kirchoffs' formulation of his laws of electrical currents in graph-theoretic terms led to interest in serious applications of graph theory.

An electrical circuit (Figures 7.25a, b) consists of resistors R_1, R_2, ..., inductances L_1, L_2, ..., capacitors C_1, C_2 and batteries B_1, B_2, etc.

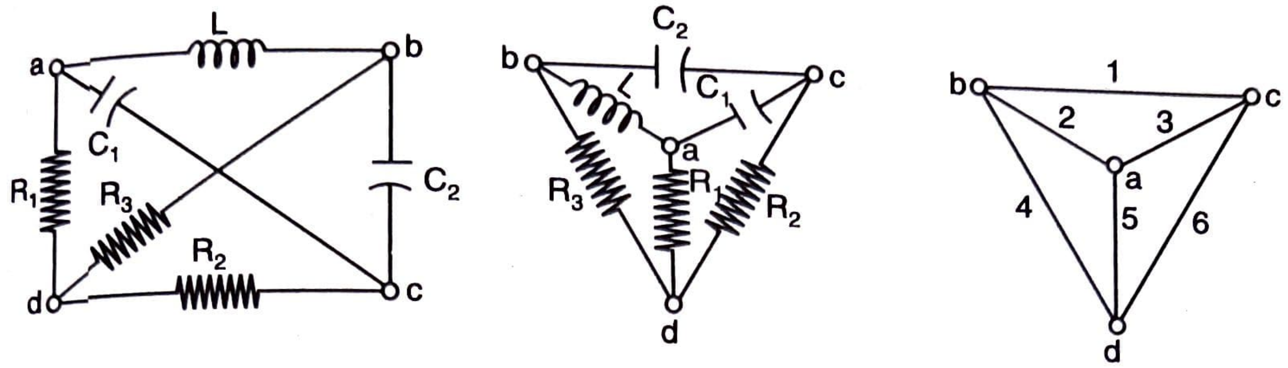

Figure 7.25

The network diagram represents two independent aspects of an electrical network. The first gives the interconnection between components and the second gives voltage-current relationship of each component. The first aspect is called network topology and can be modelled graphically. This aspect is independent of voltages and currents. The second aspect involves voltages and current and is modelled through differential equations.

For topological purposes, lengths and shapes of connections are not important and graphs of Figures 7.25(a), 7.25(b) and 7.25(c) are isomorphic.

For stating Kirchoff's laws, we need two incidence matrices associated with the graph. If v and e denote the number of vertices and edges respectively, we define the vertex or incidence matrix $A = [a_{ij}]$ as follows:

$a_{ij} = 1$; if the edge j is incident at vertex i.

$a_{ij} = 0$; if the edge j is not incident at vertex i.

This consists of v rows and e columns. For graph 7.25, A is given by

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \tag{22}$$

We note that every column has two non-zero elements.

Similarly we define the circuit matrix $B = [b_{kj}]$ as follows:

$b_{kj} = 1$ if element j is in circuit k

$\quad = 0$ if element j is not in circuit k

7.5.3 MAP-COLOURING PROBLEMS

The four colour problem that every plane map, however complex, can be coloured with four colours in such a way that two neighbouring regions get different colours, challenged and fascinated mathematicians for over one hundred years till it was finally solved by Appall and Haken in 1976 by using over 1000 hours of computer time. The problem is essentially graph-theoretic since the sizes and shapes of regions are not important. That four colours are necessary is easily seen by considering the simple graph in Figure 7.26. It was the proof of

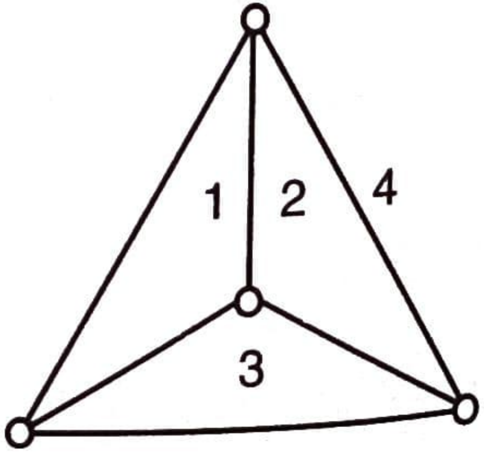

Figure 7.26

the sufficiency that took more than hundred years. However the efforts to solve this problem led to the development of many other graph-theoretic models.

Similar map-colouring problems arise for colouring of maps on surface of a sphere, a torus or other surfaces. However many of these were solved even before the simpler-looking four-colour problem was disposed of.

7.5.4 Planar Graphs

In printing of T.V. and radio circuits; we want that the wires, all lying in a plane, should not intersect. In the graph of Figure 7.27(a) wires appear to intersect, but we can find an isomorphic graph in Figure 7.27(b) in which edges do not intersect. A graph which is such that we can draw a graph isomorphic to it in which edges do not intersect is called a planar graph.

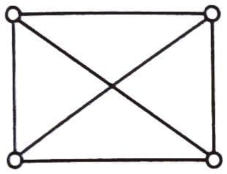

Figure 7.27 (a)

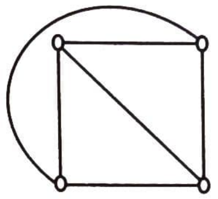

Figure 7.27 (b)

A complete graph with five vertices is not planar (Figure 7.28a). We can draw nine of the edges so that these do not intersect (Figure 7.28b) but however we may draw, we cannot draw all the ten edges without at least two of them intersecting. The proof of this depends on Jordan's theorem that every simple closed curve divides the plane into two regions, one inside the curve and one outside the curve. $ABCDE$ in Figure 7.28(b) is a closed Jordan curve and we cannot draw three edges either inside it or outside it without intersecting.

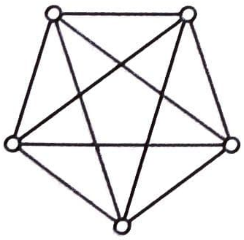

Figure 7.28 (a)

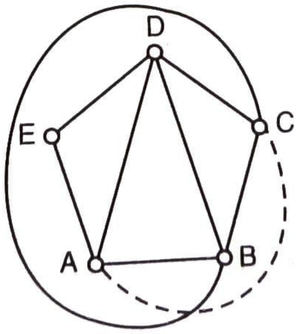

Figure 7.28 (b)

7.5.5 Euler's Formula for Polygonal Graphs

A polygonal graph with n vertices and n straight or curved edges has n vertices, n edges and two faces (one inside and one outside) so that for this graph

$$V - E + F = 2 \tag{26}$$