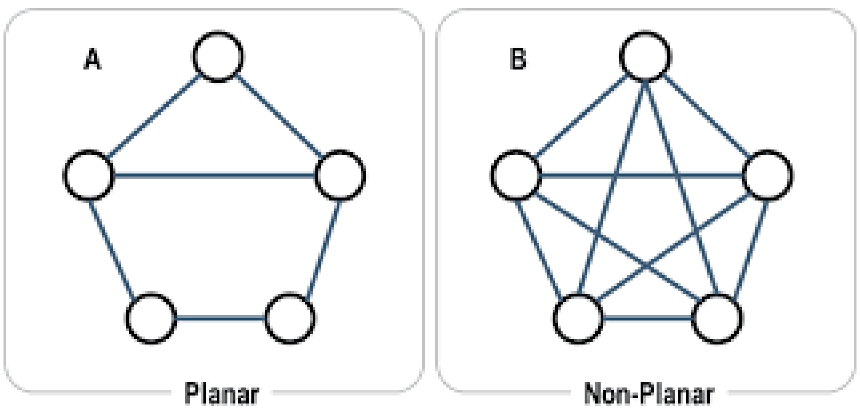
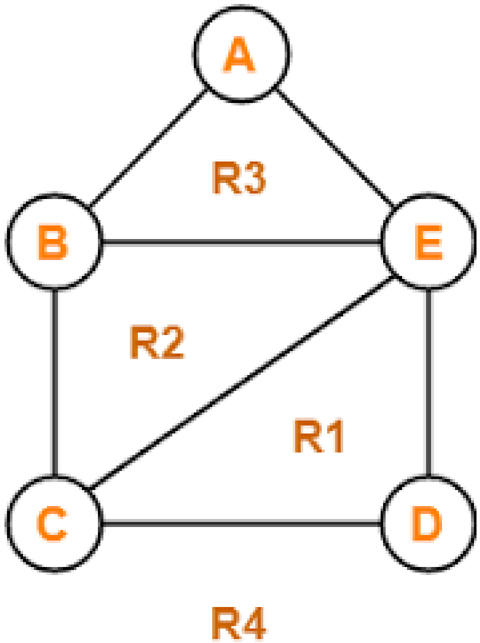
Chapter-3.2
Planar Graph

1 Introduction

Definition 1.1. *Planar Graph A Graph G is called* ***Planar graph*** *if G can be drawn in the plane so that no two of its edges cross each other, expect at an vertex. A graph that is not planer is called* ***Non-planar graph***.



A planar graph divides the plane into connected pieces called **Regions**.



Regions of Plane

Theorem 1.2. *[The Euler Identity]*

If G is a connected planar graph of order n and size m and having r regions then $n - m + r = 2$.

Proof. [Induction; on r]
Step 1: If $r = 1$ then G cannot contain any cycle, since G is connected G is a tree by definition of tree $m = n - 1$

$$\begin{aligned} \therefore n - m + f &= 2 \\ n - (n-1) + 1 &= 2 \\ 2 &= 2 \end{aligned}$$

Step 2: Assume that the result holds for all the graph with $r \geq 2$.
Step 3: If $r \geq 2$, then G is not a tree and hence it has a cycle, let e be an edge on the cycle, e is on the boundary of two distinct regions S_1 and S_2, by removing the edge e the two regions S_1 and S_2 merge and form a new region S', Since $G - e$ now have $m' = m - 1$ edges and $r' = r - 1$ regions. Applying

inductions hypothesis to $G' = G - e$ we get

$$\begin{aligned} 2 &= n - m' + r' \\ &= n - (m-1) + (r-1) \\ &= n - m + r \end{aligned}$$

Which completes the proof of the theorem. $\square$

Corollary 1.3. *If G is a planar graph of order $n \geq 3$ and size m then*

1. $m \leq 3n - 6$
2. $m \leq 2n - 4$ *if G has no 3-cycles*

Proof. First, suppose that G is connected . If $G \cong P_3$ then the inequality holds so we can assume that G has at least three edges. Draw G as a planar graph, where G has $r - regions$ denoted by $R_1, R_2, ...R_r$ the boundary of each region contains at least three edges so if m_i is the number of edges on the boundary of R_i $(1 \leq i \leq r)$ then $m_i \geq 3$.
Let

$$M = \sum_{i=1}^{r} m_i \geq 3r$$

The number M counts an edge once if the edge is a bridge and counts it twice if the edge is not a bridge.

$$M \leq 2m$$

$$\therefore 3r \leq M \leq 2m$$

$$\implies 3r \leq 2m$$

By Euler Identity

$$\begin{aligned} 2 &= n - m + r \\ 6 &= 3n - 3m + 3r \\ 6 &\leq 3n - 3m + 2m \\ 6 &\leq 3n - m \\ m &\leq 3n - 6 \end{aligned}$$

If G has no 3-cycle then each region r has at least four edges then

$$M = \sum_{i=1}^{r} m_i \geq 4r$$

$$4r \leq M \leq 2m$$

$$4r \leq 2m$$

By Euler Identity

$$\begin{aligned} 2 &= n - m + r \\ 8 &= 4n - 4m + 4r \\ 8 &\leq 4n - 4m + 2m \\ m &\leq 2n - 4 \end{aligned}$$

□

Remark:

1. If G is a graph with $n \geq 3$ and size $m > 3n - 6$ then G is a non planar graph graph with no triangles (cycle of length three).

2. If G is a graph with no triangles and $m > 2n - 4$ then G is a non planar graph.

Corollary 1.4. *Every planar graph contains a vertex of degree five or less.*

Proof. Suppose that G is a graph every vertex of which has degree or more, this implies that $n \geq 7$.

$$2m = \sum_{u \in V(G)}^{max} deg(u) \geq 6n$$

thus

$$m \geq 3n$$

$$\Longrightarrow m > 3n - 6$$

Hence G is non planar graph.
therefore if G is a planar graph then contains a vertex of degree five or less □

Corollary 1.5. *The complete graph K_5 is non planar.*

Proof. In K_5, $n = 5$ and $m = 10$

$$m = 10 > 9 = 3n - 6$$
$$\implies m > 3n - 6$$

From Remark 2, K_5 is non planar. □

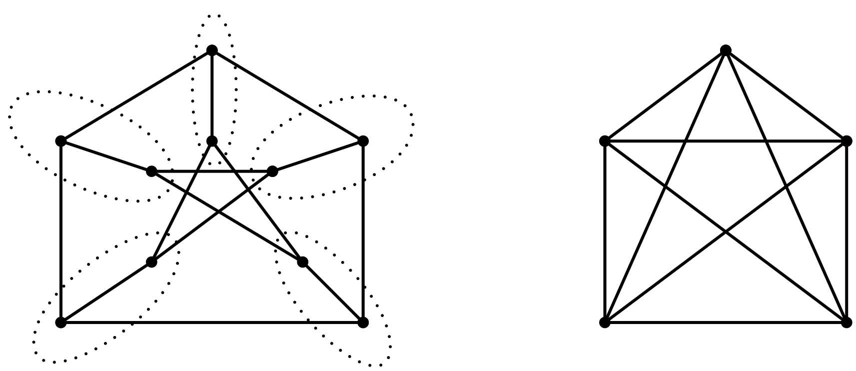
Corollary 1.6. *The complete bipartite graph $K_{3,3}$ is non planar.*

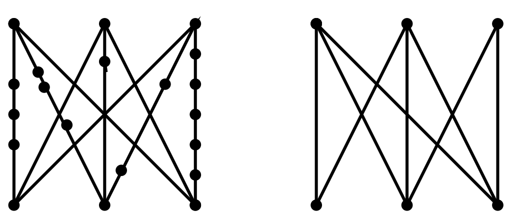
Proof. In $K_{3,3}$,m $n = 6$, and $m = 9$

$$m = 9 > 8 = 2n - 4$$
$$\implies m > 2n - 4$$

From Remark 2, $K_{3,3}$ is non planar. □

Theorem 1.7. *[Kuratowski]*
A graph G is planar if and only if it has no subgraphs homeomorphic to either K_5 or $K_{3,3}$.





Detection of Planarity Of A Graph: If a given graph G is planar or non planar is an important problem. We must have some simple and efficient criterion. We take the following simplifying steps:

Elementary Reduction:

Step 1: Since a disconnected graph is planar if and only if each of its components is planar, we need consider only one component at a time. Also, a separable graph is planar if and only if each of its blocks is planar. Therefore, for the given arbitrary graph G, determine the set.

$$G = G1, G2, ...Gk$$

where each Gi is a non separable block of G.

Then we have to test each Gi for planarity.

Step 2: Since addition or removal of self-loops does not affect planarity, remove all self-loops.

Step 3: Since parallel edges also do not affect planarity, eliminate edges in parallel by removing all but one edge between every pair of vertices.

Step 4: Elimination of a vertex of degree two by merging two edges in series does not affect planarity. Therefore, eliminate all edges in series. Repeated application of step 3 and 4 will usually reduce a graph drastically.