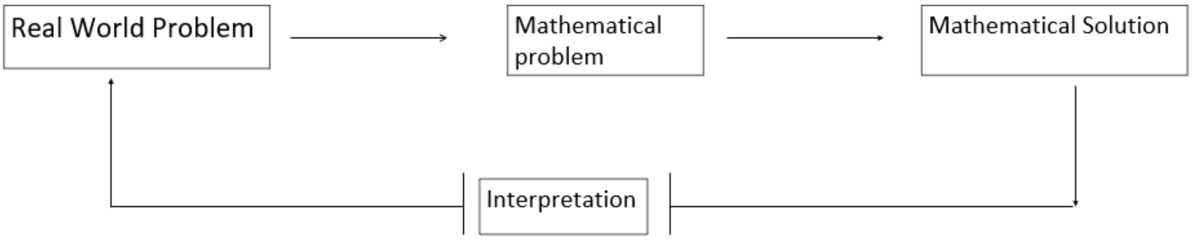
UNIT-1

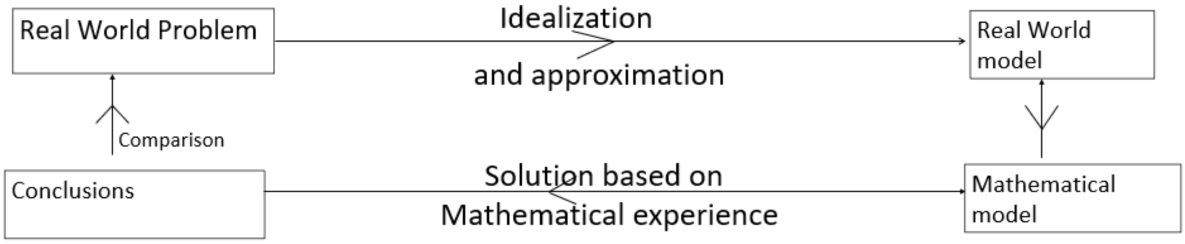
Elementary Mathematical Modelling

Introduction:

Mathematical molding consists of translating real world problems into mathematical problems, solving the mathematical problems and interpreting solutions in the language of the real world.



A real-world problem, in all its generality can seldom be translated into a mathematical problem Even if it can be translated, it may not be able to solve the resulting mathematical problem. As such it becomes necessary 'idealize' or 'simplify' the problems or approximate it by another problem which is quite close to the original problem and can be solved mathematically.
The justification of the idealization assumptions is often found in terms of the closeness of the agreement between observation and predications of the mathematical model. If the comparison Is not satisfactory, we modify either the idealization assumption or search for another structure for the mathematical model.



This leads to the following twelve-point procedure for solving problems through mathematical modeling.

1. Be clear about the real-world situation to be investigated. Find all its essential characteristics relevant to the situation and the aspects that can be ignored. (ie of minimal relevance)
2. Think about all the physical, chemical, biological, social, economic laws that may be relevant to the situation.
3. Formulate the problem into problem language (PL)

4. Think about all the variable $x_1, x_2, \ldots x_n$ and the parameters $a_1, a_2, \ldots. a_m$ involved. Classify them into known and unknown ones.
5. Choose the most appropriate model and translate the problem suitably into Mathematical Language (ML)
6. Think of all possible ways of solving the problem. The methods may be analytical (Preferably), numerical or simulation.
7. If a reasonable change in the assumptions makes analytical solution possible, investigate the possibility.
8. Make an error analysis of the method used. If the error is not within acceptable limits, change the method of solution.
9. Translate the final solution into PL
10. Compare the predictions with available observations or data. If the agreement is good, accept the model else change the assumptions and approximations in the light of the discrepancies observed and proceed as before.
11. Continue the process till a satisfactory model is obtained.
12. Deduce the conclusions from the model and test these conclusions against easier data and additional data that may be collected and see if the agreement still continues to be good.

Classification of Mathematical Models (MM):

1. Mathematical models may be classified based on the subject matter: MM in Physics, MM in Biology, MM in Engineering etc.
2. Based on the mathematical techniques used in solving them:
   Thus, we have MM through linear algebra, MM through ODE, MM through graphs, MM through partial DE's, MM through partial differential equation, MM through integral equations etc.
3. Based on purpose: MM for optimization, MM for prediction, MM for insight, MM for control etc.
4. Based on their nature: MM May be linear or non-linear, discrete or continuous, deterministic or stochastic, static or dynamic.

Classify the following models:

- ❑ $S = a + bp + cp^2$
  $D = \propto + \beta p + \gamma p^2$, where S is the supply, D is the demand and p is price.
  Ans: Nonlinear MM, MM in economics.
- ❑ $\frac{dx}{dt} = ax - bx^2$, where $x(t)$ is population at time t
  Ans:MM through ODE, MM in Biology, continuous MM.

Some characteristics of mathematical models:

1. Realism of model: Trade-off between “mathematically tractable” and “realistic”.
2. Robustness: A MM is said to be robust if small changes in parameter lead to only small changes in the behaviour of the model.
3. Generality and Applicability of models: Some Models are applicable to a wide variety of situations while others are applicable to specific situation only.
4. Dictionary of mathematical models: It is unlikely that we shall ever have a complete dictionary of MM. Familiarity with existing models will always be useful, but new situations will always demand construction of new models.
5. Unity of disciplines: When a Number of different situations are represented by the same mathematical model, it reveals a certain identify of structures of these situation. It can lead to a certain economy of efforts and it can reveal a certain underlying unity between different disciplines.

Linear growth and decay model:

Let $x(t)$ be the population size at time t and let b and d be the birth and death rates i.e, the number of individuals born or died per individual per unit time.
In time interval $(t, t+\Delta t)$, the number of births and deaths would be $bx\Delta t$ and $dx\Delta t$

$x(t+\Delta t) = x(t) + bx\Delta t - dx\Delta t$

$= x(t) + (b-d)x\Delta t$

$x(t+\Delta t) - x(t) = (b-d)x\Delta t$ ----------------(1)

Dividing both sides by Δt and taking limit as $\Delta t \to 0$ (1) becomes

$\frac{dx}{dt} = (b-d)x = ax$ where $a = b - d$

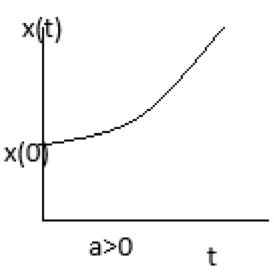
$ie, (D-a)x=0$

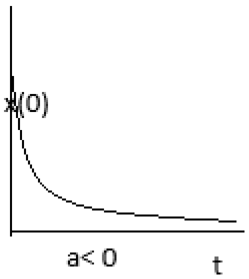
$$x = c_1 e^{at}$$

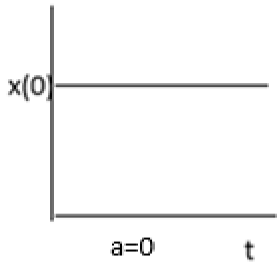
t=0 $x(0) = c_1$

$$\therefore x(t) = x(0)e^{at}$$

This implies population grows exponentially if $a > 0$ and decays exponentially if $a < 0$ and remains constant if $a = 0$.







(i) $a > 0$ then let T be the time in which the population will become double its present size

i.e, $x(t) = 2x(0)$

$\Rightarrow 2x(0) = x(0)e^{aT}$

$$\Rightarrow \log_e 2 = aT$$

$$T = \frac{1}{a}\log_e 2$$

T is called the doubling period of the population and it may be noted that this doubling period is independent of $x(0)$. It depends on only on a and is such that greater the value o a, small is the doubling period.

(ii) If $a < 0$, the population will become half its present size in time , say T'

i.e, $x(t) = \frac{1}{2}|x(t)| \Rightarrow \frac{1}{2}x(0) = x(0)e^{aT'} \Rightarrow \frac{1}{2} = e^{aT'} \Rightarrow \ln\left(\frac{1}{2}\right) = aT'$

$$\therefore T' = \frac{1}{a}\ln(\frac{1}{2})$$

T' is called the half life (period) of the population. It is also independent of $x(0)$ and it decreases as the excess of death rate over birth rate increases.

Examples:

1. Growth of science and scientists:

   If $x(t)$ denotes the number of scientists at time t, $bx\Delta t$ be the number of new scientists trained in time interval $(t, t + \Delta t)$ and $dx\Delta t$ be the number of scientists who retire from science in the same period, then the above model applies to the growth of the number of scientists.

   The same model applies to the growth of Science, Mathematics and technology, where, for example, $x(t)$ represents the amount of mathematics at time t. Then $x(t) = x(0)e^{at}$ would imply that the rate of growth of Mathematics is proportional to the amount of mathematics existing.

2. The model also applies to growth of populations of microorganisms, malignant cells and tree in a forest.

3. Radioactive decay:

   Many substances undergo radioactive decay at a rate proportional to the amount of the radioactive substance present at any time, and each of them have a half life period. For Uranium it is 4.5 billion years , while for carbon 14 (also called radioactive carbon) it is only 5730 years.

   The ratio of radioactive carbon to ordinary carbon (carbon 12) in dead plants and animals enables us to estimate their time of death. This is called radioactive dating (or radiocarbon dating). (in this method, the age of earth estimated to be 45 billion years.

4. Change of Temperature:

   According to Newtons law of cooling the rate of change of temperature of a body is proportional to the difference between the temperature T of the body and temperature T_s of the surrounding medium.

   We have

$$\frac{dT}{dt} = k(T - T_s), k < 0$$

$$\ln(T - t_s) = kt + c_1 \Rightarrow T - T_s = Ce^{kt}$$

$$\therefore T(t) - T_s = (T(0) - T_s)e^{kt}$$

This means the excess of the temperature of the body over that the surrounding medium decays exponentially.

5. Diffusion:

According to Fick's law of diffusion, the rate of movement of a solute across a thin membrane is proportional to the area of the membrane and to the difference in concentrations of the solute on one slide is kept fixed at a and the concentration of the solution on the other side initially $c_0 > a$ then

$$\frac{dc}{dt} = k(a-c)$$

$$a - c = \big(a - c(0)\big)e^{-kt}$$

$\Rightarrow c(t) \to a$ as $t \to \infty$ whatever be the value of $c(0)$.

Nonlinear Growth and decay models (Logistic law of population growth):

As the population increases, due to overcrowding and limitations of resources, the birth rate b decreases and the death rate d increases with the population size x.

A simple assumption is to take

$$B = b_1 - b_2 x, \qquad D = d_1 + d_2 x, \quad b_1, b_2, d_1, d_2 > 0$$

w.k.t, $$\frac{dx}{dt} = (B - D)x$$

$$\Rightarrow \frac{dx}{dt} = \ [b_1 - b_2 x - (d_1 + d_2 x)]x$$

$$\Rightarrow \frac{dx}{dt} = [b_1 - d_1 - (b_2 + d_2)x]x$$

$$\Rightarrow \frac{dx}{dt} = (a - bx)x - - - - - - - - (1)$$

$$\frac{dx}{(a-bx)x} = dt$$

$$\frac{a-bx}{x}\ \frac{(a-bx) - x(-b)}{(a-bx)^2}dx = adt$$

By integrating , $$\ln\left(\frac{x}{(a-bx)}\right) = at + c$$

$$\Rightarrow \frac{x}{a-bx} = Ce^{at}$$

$$\Rightarrow \frac{x(0)}{a-bx(0)} = C$$

$$\therefore \frac{x(t)}{a-bx(t)} = \frac{x(0)}{a-bx(0)}e^{at} - - - - - (2)$$

From (1) and (2) , we have

(i) $a - bx(0) > 0 \ \Rightarrow \frac{a}{b} > x(0) \Rightarrow a - bx(t) > 0 \Rightarrow \frac{a}{b} > x(t) \Rightarrow \frac{dx}{dt} > 0$

$\Rightarrow x(t)$ is a monotonic increasing function of t which approaches $\frac{a}{b}$ as $t \to \infty$

(ii) $a - bx(0) < 0 \Rightarrow \frac{a}{b} < x(0) \Rightarrow a - bx(t) < 0 \Rightarrow \frac{a}{b} < x(t) \Rightarrow \frac{dx}{dt} < 0$

$\Rightarrow x(t)$ is a monotonic decreasing function which approaches $\frac{a}{b}$ as $t \to \infty$

Now (1) $\Rightarrow \frac{d^2x}{dt^2} = a - 2bx \;\Rightarrow \frac{d^2x}{dt^2} \lesseqgtr 0$ according as $x \gtreqless \frac{a}{2b}$

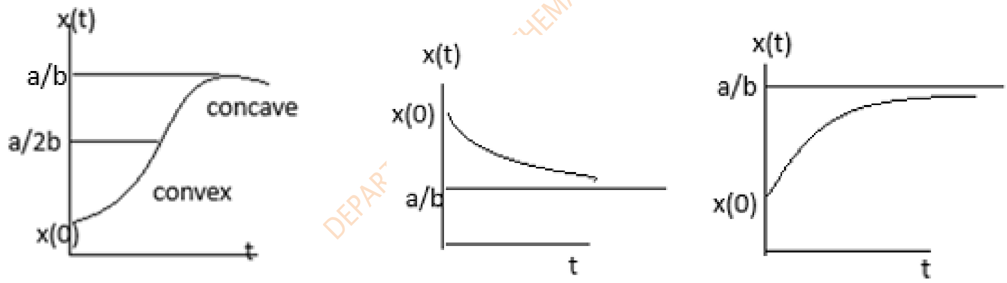
Thus in case (i) the growth curve is convex if $x < \frac{a}{2b}$, concave if $x > \frac{a}{2b}$ and has a point of inflexion at $x = \frac{a}{2b}$.

If $x(0) < \frac{a}{2b}, x(t)$ increases at an increasing rate till $\frac{a}{2b}$, then it increases at a decreasing rate and approaches $\frac{a}{b}$ as $t \to \infty$.

If $\frac{a}{2b} < x(0) < \frac{a}{b}$, then $x(t)$ increases at a decreasing rate and approaches $\frac{a}{b}$ as $t \to \infty$

If $x(0) = \frac{a}{b}$, then $x(t)$ is always equal to $\frac{a}{b}$

If $x(0) > \frac{a}{b}$, then $x(t)$ decreases at a decreasing rate and approaches $\frac{a}{b}$ as $t \to \infty$.



Spread of technological innovations and infectious diseases:

Let $N(t)$ be the number of companies which have adopted a technological innovation till time t, then the rate of change of

N depends on the number of companies which have adopted this innovation and the number of those which have not yet adopted it. If R is the total number of companies in the region, then $\frac{dN}{dt} = kN(R - N) - - - - - - (3)$

Which is the logistic law. This shows that ultimately, all companies would adopt this innovation.

If R is the total number of persons in a system , and if $N(t)$ is the number of infected persons, then equation (3) represents spreading of an infectious disease.

Note: In the above examples(situations) , N(t) is essentially an integer valued variable , but we have treated it as a continuous one . this can be regarded as a n idealisation of the situation or as an approximation to reality.

Law of mass action: Chemical reactions:

Two chemical substances combine in the ratio a:b to form a third substance z
If z(t) is the amount of the third substance at time t, then a proportion $\frac{a}{a+b}z(t)$ of it consists of the first substance and a proportion of $\frac{b}{a+b}z(t)$
of second.
The rate of formation of third substance is proportional to the product of the amount of the two component substances which have not yet combined together.
If A and B are the initial amounts of the two substances, then we get

$$\frac{dz}{dt} = k\left(A - \frac{a}{a+b}z\right)\left(B - \frac{b}{a+b}z\right)$$

This is a nonlinear DE for a second order reaction.
For the n^{th} order reaction we get the non-linear equation as

$$\frac{dz}{dt} = k(A_1 - a_1 z)(A_2 - a_2 z) \ldots \ldots . . (A_n - a_n z)$$

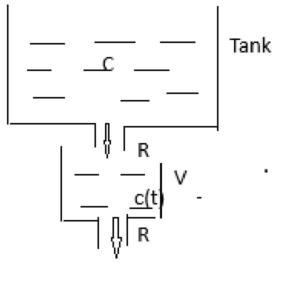
where $a_1 + a_2 + \cdots + a_n = 1$

Compartment models:

Here we use principle of continuity:
"The gain in amount of a substance in a medium in any time is equal to the excess of the amount that has entered the medium in the time over the amount that has left the medium in this time.''

A simple compartment model:

Let a vessel contain a volume V of a solution with concentration c(t) of a substance at time t
Let a solution with constant concentration C in an overhead tank enter the vessel at a constant rate R and after mixing thoroughly with the solution in the vessel, let the mixture with concentration c(t) leave the vessel at the same rate R, so that the volume of the solution in the vessel remains same V



Using the principle of continuity we get

$$V\,[c(t+\Delta t) - c(t)] = RC\Delta t - Rc(t)\,\Delta t$$

Dividing by Δt and taking $\Delta t \to 0$

$$V\frac{dc}{dt} = RC - Rc$$

$$\frac{dc}{dt} = \frac{R}{V}C - \frac{R}{V}c = (C-c)\,\frac{R}{V}$$

$$\frac{1}{C-c}\,dc = \frac{R}{V}\,dt$$

$$-\log(C-c) = \frac{R}{V}t + c_1$$

$$\log(C-c) = -\frac{R}{V}t + c_1$$

$$C-c = Ke^{-\frac{R}{V}t}$$

$$c(t) = C - Ke^{-\frac{R}{V}t}$$

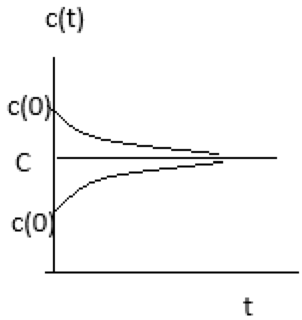
$$c(0) = C - K$$

$$K = C - c(0)$$

$$\therefore c(t) = C - \left(C - c(0)\right)e^{-\frac{R}{V}t}$$

$$= c(0)e^{-\frac{R}{V}t} + C\left(1 - e^{-\frac{R}{V}t}\right)$$

As $t \to \infty$ $c(t) \to C$, so that ultimately the vessel has the same concentration as the overhead tank. Also if $C > c(0)$ the concentration in the vessel increases to C and $C < c(0)$, the concentration in the vessel decreases to C.



Diffusion of glucose or medicine in blood stream:

Let the volume of blood in a human body be V and let the initial concentration of glucose in the blood stream be $c(0)$.

Let glucose be introduced in the blood stream at a constant rate I. Glucose is also removed from the blood stream due to the physiological needs of the human body at a rate proportional to $c(t)$.

By continuity principle we get

$$V\frac{dc}{dt} = I - kc$$

Now let a dose D of a medicine be given to a patient at regular intervals of duration T each.

The medicine also disappears from the system at a rate proportional to $c(t)$, the concentration of the medicine in the blood stream

Then by continuity principle

$$V\frac{dc}{dt} = -kc$$

Integrating we get

$$c(t) = De^{-\frac{k}{V}t} \quad 0 \le t < T \text{ --------- } (4)$$

At time $T(t \ge T)$, the residue of the first dose is $De^{-\frac{k}{V}t}$ and now another dose D is given so that we get

$$c(t) = De^{-\frac{k}{V}t} + De^{-\frac{k}{V}(t-T)} \quad T \le t < 2T$$

The first term gives the residual of the first dose and the second term gives the residual of the second dose. Proceeding in this way after n doses we get

$$c(t) = De^{-\frac{k}{V}t} + De^{-\frac{k}{V}(t-T)} + De^{-\frac{k}{V}(t-2T)} \pm \text{-----} De^{-\frac{k}{V}(t-(n-1)T)}$$

$$= De^{-\frac{k}{V}t}\left[1 + e^{\frac{k}{V}T} + e^{\frac{2k}{V}T} + \text{-----} e^{(n-1)\frac{k}{V}T}\right]$$

$$= De^{-\frac{k}{V}t}\frac{\left(e^{\frac{nk}{V}T}-1\right)}{e^{\frac{k}{V}T}-1}, \ (n-1)T \le t < nT$$

Just before the next dose

$$c(nT-0) \quad = De^{-\frac{nk}{V}T}\left[\frac{\left(e^{n\frac{k}{V}T}-1\right)}{e^{\frac{k}{V}T}-1}\right] = \frac{D\left(1-e^{-n\frac{k}{V}T}\right)}{e^{\frac{k}{V}T}-1}$$

When next dose is given

$$c(nT+0) = De^{-\frac{nk}{V}T}\frac{\left(e^{n\frac{k}{V}T}-1\right)}{e^{\frac{k}{V}T}-1} + D$$

$$= \frac{D\left(e^{\frac{k}{V}T}-e^{-n\frac{k}{V}T}\right)}{e^{\frac{k}{V}T}-1}$$

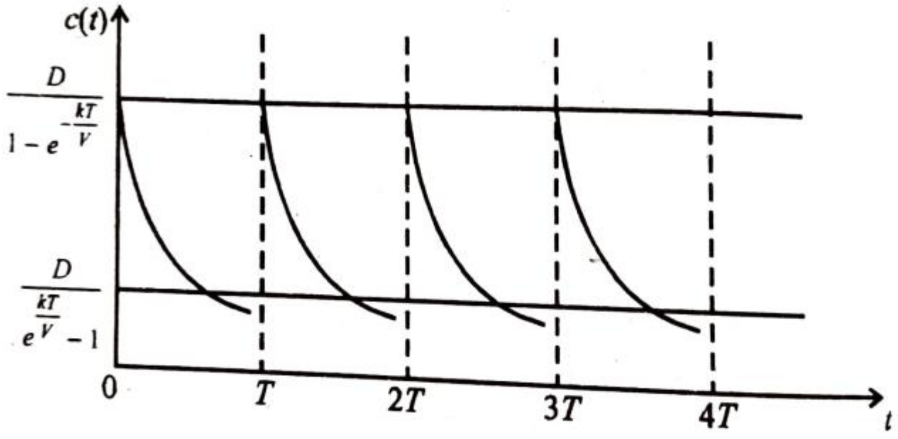
$As\ n \to \infty$ i.e, n gets bigger we have

$$D\,\frac{e^{\frac{k}{V}T}-0}{e^{\frac{k}{V}T}-1}$$

i.e, $$\frac{D}{\frac{\left(e^{\frac{k}{V}T}-1\right)}{e^{\frac{k}{V}T}}} = \frac{D}{1-e^{-\frac{k}{V}T}}$$

i.e, the concentration never exceeds $\frac{D}{1-e^{-\frac{k}{V}T}}$

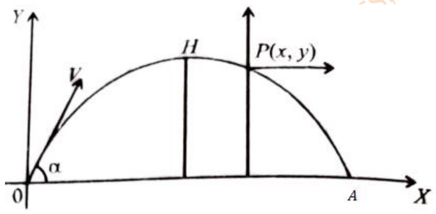
- In each interval concentration decreases
- In any interval, the concentration is maximum at the beginning and that maximum concentration keep increasing interval by interval but will be below $\frac{D}{1-e^{-\frac{k}{V}T}}$
- The minimum value in an interval occurs at the end of each interval. This also keep increasing but lies below $\frac{D}{e^{\frac{k}{V}T}-1}$.
- The concentration curve is piecewise continuous and has points of discontinuity at $T, 2T, 3T, \ldots..$
- By injecting glucose on penicillin in blood and fitting curve (4) to the data, we can estimate the value of k and V. In particular this gives a method for finding the volume of blood in the human body.



Modelling in dynamics

Equations of motion are formed based in the principle:
Mass x acceleration in any direction = Force in that direction which yield second order differential equations.

Motion of a projectile:



A particle of mass m is projected from the origin in vacuum with velocity V inclined at angle α to the horizontal. Suppose at time t, it is at position P $(x(t), y(t))$ and horizontal and vertical velocity components are $u(t), v(t)$ respectively, then the equations of motion are

$m\frac{du}{dt} = 0$ and $m\frac{dv}{dt} = -mg$

Integrating, we get

$u(t) = c_1$ and $v(t) = -gt + c_2$

$u(0) = Vcos(\alpha) = c_1$ and $v(0) = V\sin(\alpha) = -g(0) + c_2$

$\Rightarrow c_1 = Vcos(\alpha)$ and $c_2 = V\sin(\alpha)$

$\therefore u = Vcos(\alpha)$ and $v = V\sin(\alpha) - gt$

i.e, $\frac{dx}{dt} = Vcos(\alpha)$ and $\frac{dy}{dt} = V\sin(\alpha) - gt$

Integrating again,

$x = Vcos(\alpha)t$ and $y = tV\sin(\alpha) - \frac{gt^2}{2}$ ----------(1)

Eliminating t between these two equations,

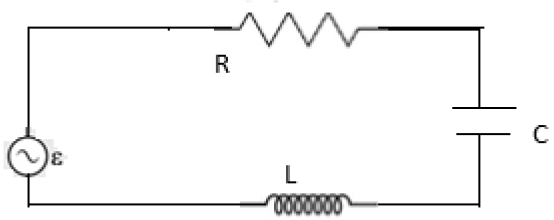
$y = xtan(\alpha) - \frac{1}{2}\frac{gx^2}{v^2\cos^2(\alpha)}$ (equation of trajectory) this is a parabola.

i.e, $y = x\left(tan(\alpha) - \frac{1}{2}\frac{gx}{v^2\cos^2(\alpha)}\right)$

The parabola cuts $y = 0$ when $x = 0$ or $gx = 2V^2 \cos^2 \alpha \tan \alpha,$ i.e, $x = \frac{V^2 \sin 2\alpha}{g}$
Which corresponds to the positions O and A in the figure. Hence the range of the particle is $R = \frac{V^2 \sin 2\alpha}{g}$
Putting $y = 0$ in (1), $t = 0$ or $t = \frac{2V \sin \alpha}{g}$ this gives the time of flight T.

Mathematical modelling through ODE of second order:

Electrical circuits:
Consider an electric circuit as shown in the figure



The current $i(t)$ amperes represent the rate of change of charge q flowing in the circuit, so that

$$i(t) = \frac{dq}{dt} \text{---------------(1)}$$

There is resistance of R ohms in the circuit. This may be provided by a light bulb, an electric heater or any electric device, causing a potential drop of magnitude $E_R = R_i$ volts.
There is an induction of inductance L henry which produces a potential drop $E_L = L\frac{di}{dt}$.
There is a capacitance which produces a potential drop $E_c = \frac{1}{C}q$
All these potential drops are balanced by the battery which produces a voltage Evolts.
According to Kirchoff's voltage law the algebraic sum of the voltage drops round a closed circuit is zero so that

$$R_i + L\frac{di}{dt} + \frac{1}{C}q = E(t) \text{---------(2)}$$

Differentiating and using (1)

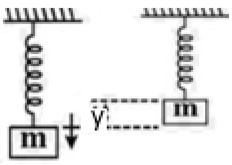
$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dE}{dt} \text{-----------------(3)}$$

Also by substituting (1) in (2)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t) \text{ ------------------(4)}$$

Both equations (3) and (4) represents linear differential equations with constant coefficients and their solutions determine $i(t)$ and $q(t)$.

Model of Mass spring – Dash pot:



Consider an ordinary coil spring suspended from a fixed support. Attach a body at its lower end, for instance, an iron piece at rest. We choose the downward direction as positive.
Let the metal piece be pull down by an amount $y > 0$, then by hook's law, the equation of motion is $my'' = -ky,\ (k > 0)$ where $k\ (> 0)$ is the spring constant. Thus

$$my'' + ky = 0$$

This is a homogeneous linear ODE with constant coefficients.

Solution of this equation is $y(t) = A\cos w_0 t + B\sin w_0 t$ where $w_0 = \sqrt{\frac{k}{m}}$

This is called a harmonic oscillation.

Now add a damping force proportional to the velocity obtaining

$$my'' + cy' + ky = 0$$

This can be done by connecting the ball to a dashpot where c (>0) is the damping constant.

Here $\lambda_1 + \lambda_2 = -\frac{c}{m}$ and $\lambda_1\lambda_2 = \frac{k}{m}$

Case(i): over damping

If the damping constant c is so large that $c^2 > 4mk$, then the roots λ_1 and λ_2 are real, distinct and negative.

$$y(t) = A_1\, e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

Thus $y(t) \to 0$ as $t \to \infty$ and the motion is said to be overdamped.

Case (ii): Critical damping

If $c^2 = 4mk$, then roots $\lambda_1 = \lambda_2 = -\frac{c}{2m}$

$$y(t) = (A_1 + A_2 t)e^{-\frac{c}{2m}t}$$

Again $y(t) \to 0$ as $t \to \infty$ and eventually the body comes to rest, but not as quickly as case (i) because of the first term which is a polynomial. The motion is said to be critically damped.

Case (iii): Under damping

If $c^2 < 4mk$, then the roots are imaginary say $\alpha \pm i\beta$. Then

$$y(t) = e^{\alpha t}(A_1\cos\beta t + A_2\sin\beta t)$$

This means keeps oscillating but since ($\alpha = -\frac{c}{2m}$), the oscillations are damped out and tend to zero as $t \to \infty$. The motion is said to be underdamped.

This model is extended by including an external force $r(t)$ acting on the body. Then we have

$$my'' + cy' + ky = r(t) \text{ ------------(5)}$$

of special interest are periodic external forces of the form $F\cos\omega t$ which gives the non – homogeneous ODE.

$$my'' + cy' + ky = F\cos\omega t$$

Both the equations (4)(in LRC circuit) and (5) (from mass-spring-dashpot) can be compared which gives the following correspondences.

Mass $m \leftrightarrow$ inductance L

Frition coefficient $C \leftrightarrow$ resistance R

Spring constant $K \leftrightarrow$ inverse capacitanc $\frac{1}{C}$

Impressed force $r(t) \leftrightarrow$ impressed voltage $E(t)$

Displacement $y \leftrightarrow$ charge q

Velocity $v \leftrightarrow$ current i

This shows the correspondance between mechanical systems and electrical systems. This forms the basis of analogue computers. A linear ODE of second order can be solved by forming an electrical circuit and measuring the electrical current in it.
Similar analogues exist between hydrodynamical and electrical systems. Mathematical modelling brings out the isomorphisms between mathematical structures of quite different systems and gives a method for solving all these models in terms of the simplest one among them.