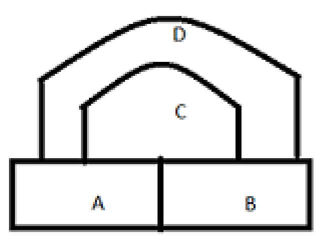
Chapter-3.2
Graph Coloring

1 Vertex Coloring

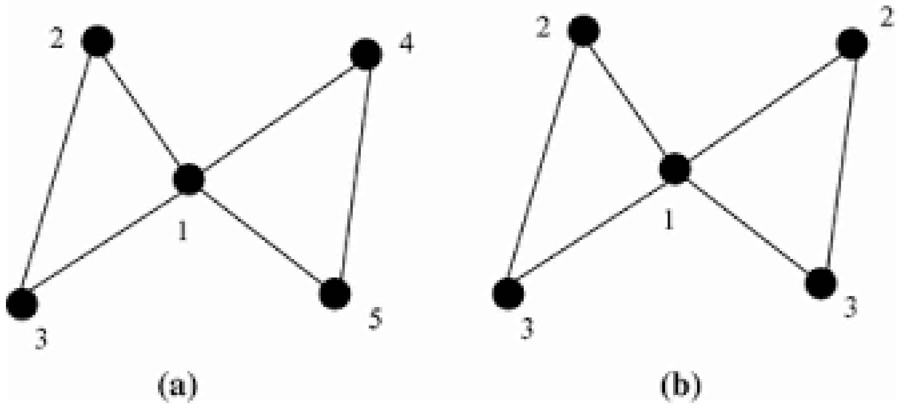
Coloring is one of the important branches of graph theory and has attracted the attention of almost all graph theorists, mainly because of the four color theorem. In 1852 Farncis Guthrie (1831-1899), a recent graduate of University College London, observed that the counties of England could be colored with four colors so that neighboring counties were colored differently. Farncis found maps where three color weren't enough but he felt that four color were enough for all maps and he attempted to prove this. He showed his proof to his younger brother Frederick, who was taking class at the time from the well-known Augustus De Morgan.Francis was not completely happy with the proof he had given, however. With Francis's permission, Frederick showed what Francis had written to De Morgan on October 23, 1852. De Morgan was pleased with his and felt it was new. The very same day, De Morgan wrote the following letter to the celebrated mathematician William Rowan Hamilton:

> A student of mine asked me today to give him a reason for a fact which I did not know was a fact-an do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figurers with any portion of common boundary line are differently coloured- four colours may be wanted but not more-the following is his case in which four are wanted.



Definition 1.1. *A $k-$Coloring of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are* ***colors****; the vertices have of one color form a*

color class. *A* $k-$*coloring is a* ***proper*** *if adjacent vertices have different labels. A graph is* ***k-colorable*** *if it has a proper k-coloring. The chromatic number* $\chi(G)$ *is the least* k *such that* G *is* $k-$*colorable*



We observe that colouring any one of the components in a disconnected graph does not affect the colouring of its other components. Also, parallel edges can be replaced by single edges, since it does not affect the adjacencies of the vertices. Thus, for colouring considerations, we opt only for simple connected graphs.
The following observations are the immediate consequences of the definitions introduced above.

1. A graph is 1-chromatic if and only if it is totally disconnected.
2. A graph having at least one edge is at least 2-chromatic (bichromatic).
3. A graph G having n vertices has $\chi(G) \leq n$.
4. If H is subgraph of a graph G, then $\chi(H) \leq \chi(G)$.
5. $\chi(K_n) = n$ and $\chi(\overline{K_n}) = 1$.
6. $\chi(C_{2n}) = 2$ and $\chi(C_{2n+1}) = 3$.
7. If $G_1, G_2, ..., G_r$ are the components of a disconnected graph G, then $\chi(G) = max\{\chi(G_1), \chi(G_2), ..., \chi(G_r)\}$
8. $\chi(K_{m,n}) = 2$

A Characterization of bicolorable (2-colorable) graph was given by Köning

Theorem 1.2. *A graph is bicolorable if and only if it has no odd cycles(bipartite graph).*

Theorem 1.3. *For every graph G* $\chi(G) \leq 1 + \Delta(G)$

Proof. [Greedy Algorithm]

Let $V(G) = \{v_1, v_2, ..., v_n\}$. Define a coloring $c : V(G) \to N$ recursively as follows: $c(v_1) = 1$. Once $c(v_i)$ has been defined, $1 \leq i \leq n$, define $c(v_{i+1})$ as the smallest positive integer not already used to color any of the neighbors of v_{i+1}, since v_{i+1} has $degv_{i+1}$ neighbors, at least one of the integer $1, 2, ..., 1 + degv_{i+1}$ is available for $c(v_{i+1})$. Therefore $c(v_{i+1}) \leq 1 + degv_{i+1}$. If the maximum color assigned to the vertices of G is $c(v_j)$, then

$$\chi(G) \leq c(v_j) \leq 1 + degv_j \leq 1 + \Delta(G)$$

□

Theorem 1.4. *[Brooks Theorem]*

For a connected simple graph G, which is neither complete nor a cycle of odd length we have $\chi(G) \leq \Delta(G)$.

Theorem 1.5. *[Six Color Theorem]*

For every planar graph G, we have $\chi(G) \leq 6$

Theorem 1.6. *[Five Color Theorem]*

For every planar graph G, we have $\chi(G) \leq 5$

Theorem 1.7. *[Four Color Theorem]*

For every planar graph G, we have $\chi(g) \leq 4$.

2 Coloring Enumeration

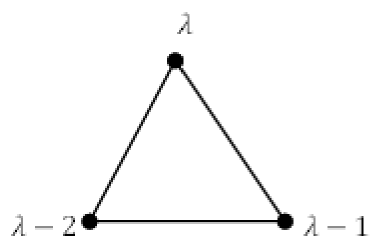
Definition 2.1. *Let G be a graph and* $\lambda \in \mathbb{N}$. *Define be the number of* $P(G : \lambda)$ *to be the number of proper* λ*-vertex colorings* $c : V(G) \to \{1, 2, 3, ..., \lambda\}$. *This property of a graph expressed by means of a polynomial. This polynomial is called the chromatic polynomial of G.*

i.e Let G be a labeled graph. A coloring of G from λ colors is a coloring of G which uses λ or fewer colors. Two colorings of G from λ colors will be considered different is at least one of the labeled vertex is assigned different colors.

1. For each $\lambda < \chi(G)$ we have $P(G : \lambda) = 0$
2. For each $\lambda \geq \chi(G)$ we have $P(G : \lambda) > 0$
3. Indeed the smallest λ for which $P(G : \lambda) > 0$ is the chromatic number of G.

Example 1: There are λ ways of coloring any given vertex of K_3. For a second vertex, any of $\lambda - 1$ colors may be used, while there are $\lambda - 2$ ways of coloring the remaining vertex. Thus

$$P(K_3 : \lambda) = \lambda(\lambda - 1)(\lambda - 2)$$



K_3

This can be generalized to any complete graph

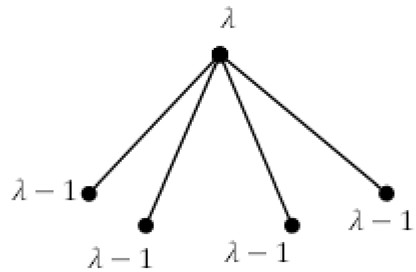
$$P(K_n : \lambda) = \lambda(\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1)$$

The corresponding polynomial of the totally disconnected graph (Null graph) $\overline{K_n}$ is particularly easy to find since each of its n vertices may be colored independently in any of λ ways

$$P(\overline{K_n} : \lambda) = \lambda^n$$

Example 2: The Central vertex v_0 of $K_{1,4}$ may be colored in any λ ways while each end vertex may be colored in any $\lambda - 1$ ways . Therefore

$$P(K_{1,4} : \lambda) = \lambda(\lambda - 1)^4$$



$K_{1,4}$

Certainly, every two isomorphic graphs have the same chromatic polynomial. However, there are often several nonisomorphic graphs with the same chromatic polynomial; in fact, all trees with n vertices have equal chromatic polynomials

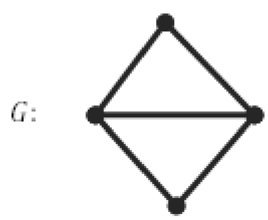
Theorem 2.2. *A graph G with n vertices is a tree if and only if*

$$P(G, \lambda) = \lambda(\lambda-1)^{n-1}$$

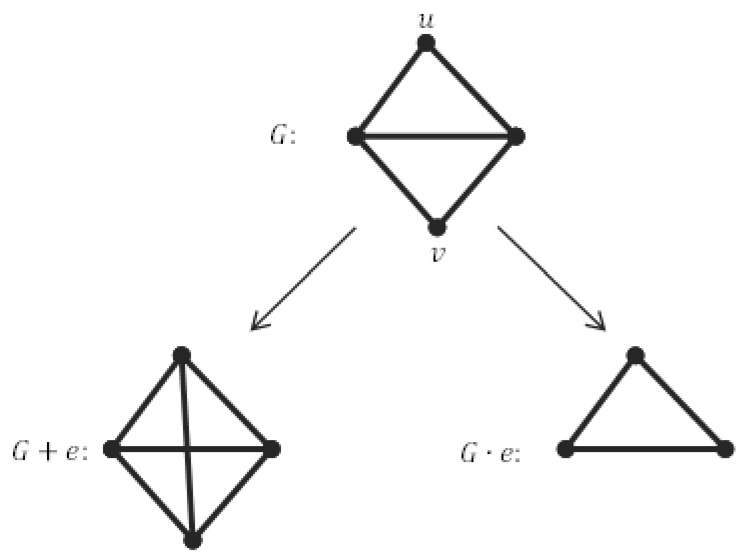
Theorem 2.3. *Let u and v be two non adjacent vertices in a graph G. Let $G+e$ be a graph obtained by adding an edge between u and v. Let $G \cdot e$ be a simple graph obtained from G by fusing the vertices u and v together and replacing sets of parallel edges with single edge. Then*

$$P(G, \lambda) = P(G+e, \lambda) + P(G \cdot e, \lambda)$$

Example 3: Find the chromatic polynomial of the following graph.



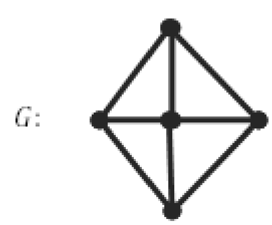
Solution: We obtain



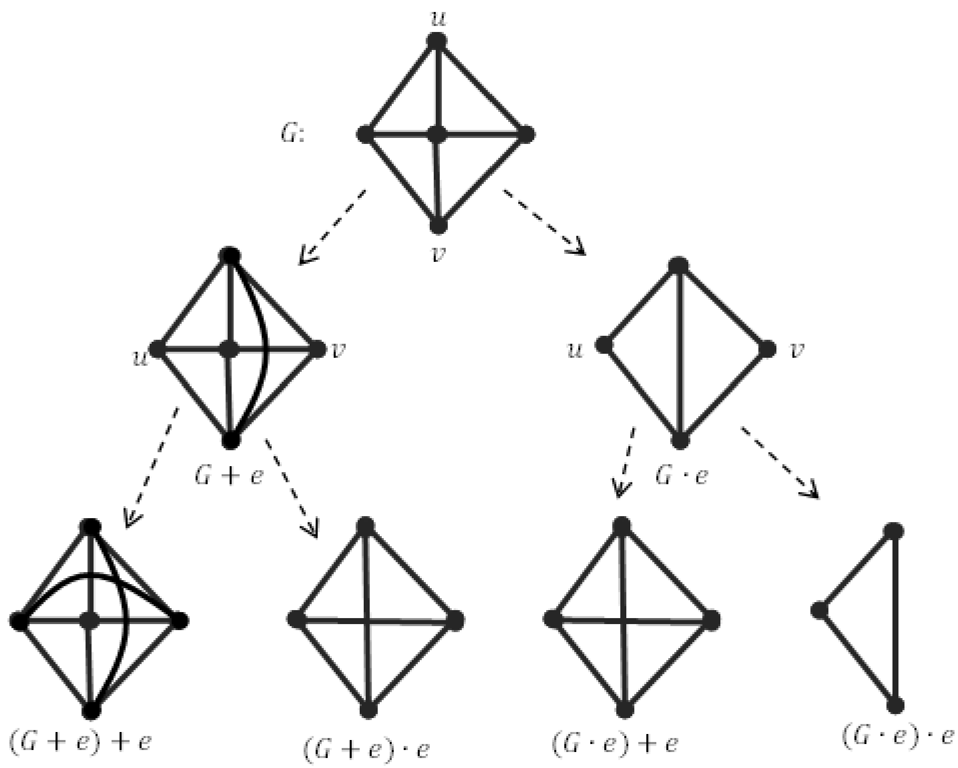
Hence

$$\begin{aligned}
p(G,\lambda) &= P(G+e,\lambda) + P(G\cdot e,\lambda)\\
&= P(K_4,\lambda) + P(K_3,\lambda)\\
&= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)\\
&= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)+1]\\
&= \lambda(\lambda-1)(\lambda-2)^2
\end{aligned}$$

Example 4: Find the chromatic polynomial of the following graph.



Solution: We obtain



Hence

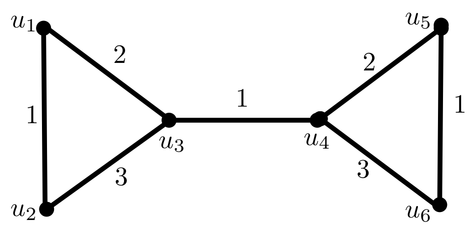
$$\begin{aligned}
p(G,\lambda) &= P(G+e,\lambda) + P(G\cdot e,\lambda) \\
&= P((G+e)+e,\lambda) + P((G+e)\cdot e,\lambda) + P((G\cdot e)+e,\lambda) + P((G\cdot e)\cdot e,\lambda) \\
&= P(K_5,\lambda) + P(K_4,\lambda) + P(K_4,\lambda) + P(K_3,\lambda) \\
&= P(K_5,\lambda) + 2P(K_4,\lambda) + P(K_3,\lambda) \\
&= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) \\
&= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)(\lambda-4) + 2(\lambda-3) + 1] \\
&= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)
\end{aligned}$$

3 Edge Coloring

Definition 3.1. *A $k-$ edge-coloring of G is a labeling $f : E(G) \to S$, where $|S| = k$. The labels are colors; the edges of one color from a color class.*

A k-edge-coloring is proper if incident edges have different labels; that is, if each color class is a matching. A graph is k-edge-colorable if it has a proper k-edge-coloring. The edge-chromatic number (Chromatic index) $\chi^{'}(G)$ of a loopless graph G is the least k such that G is $k-$edge-colorable.

Example 7: The edge chromatic number of the following graph is three.



For a graph G and any vertex $u \in V(G)$, all edges with u as an end vertex are adjacent and hence must receive different colors in a proper edge coloring of G. Hence, we note the obvious lower bound for the edge chromatic number of G

$$\chi^{'}(G) \geq \Delta(G),$$

the maximum degree in G.

Edge chromatic number of some basic graphs:

1. $$\chi^{'}(K_n) = \begin{cases} n & if\, n\, is\, odd, \\ n-1 & if\, n\, is\, even, . \end{cases}$$

2. $$\chi^{'}(C_n) = \begin{cases} 2 & if\, n\, is\, even, \\ 3 & if\, n\, is\, odd. \end{cases}$$

3. $$\chi^{'}(p_n) = 2$$

Theorem 3.2. *For a bipartite graph G, we have*

$$\chi^{'}(G) = \Delta(G).$$

From this theorem we obtain the next corollary.

Corollary 3.3. *For the complete bipartite graph $K_{m,n}$, we have*

$$\chi^{'}(K_{m,n}) = max(\{m, n\}).$$

The following Theorem gives us as tight bound for $\chi^{'}(G)$ as we can hope for when G is a simple graph. It was proved by Vizing in 1964 and independently by Gupta in 1966, although the latter proof never was published excepted as an abstract. It is usually referred to as Vizing's Theorem.

Theorem 3.4. *If G is a simple graph, then*

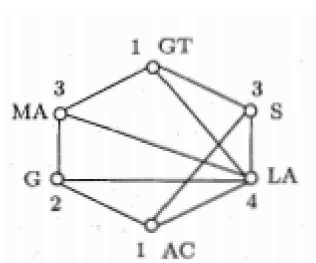
$$\Delta(G) \leq \chi^{'}(G) \leq \Delta(G) + 1$$

4 Scheduling problems

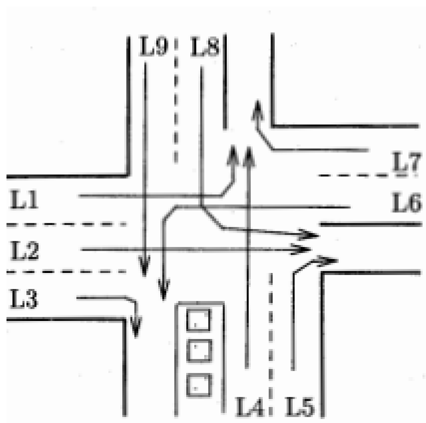
Example 5.1: The mathematics department of a certain college plans to schedule the classes Graph Theory (GT), Statistics (s), Liner Algebra (LA), Advanced Calculus (AC), Geometry (G), and Modern Algebra (MA) this summer. Ten students (see below) have indicated the courses they plan to take. With this information, use graph theory to determine the minimum number of time periods needed to offer these courses so that every two classes having student in common are taught at different time periods during the day. Of course, two classes having no students in common can be taught during the same period.

Anden: LA, S	Brynn: MA, LA, G
Chase: MA, G, LA	Denise: G, LA, AC
Everett: AC, LA, S	Francois: G, AC
Greg: GT, MA, LA	Harper: LA, GT, S
Irene: AC, S, LA	Jennie: GT, S

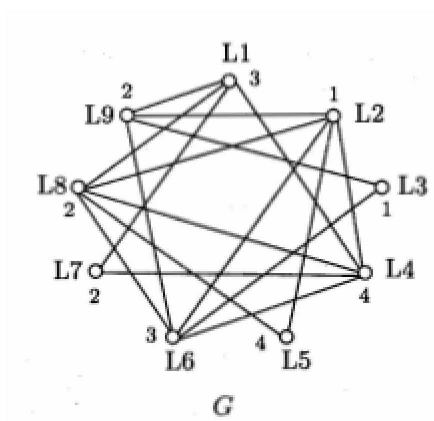
Solution: First, we construct a graph H whose vertices are the six subjects. Two vertices (subjects) are joined by an edge if some student is taking classes in these two subjects(see Figure). The minimum number of time periods is $\chi(H) = 4$.



Example 5.2: The following figure shows the traffic lanes $L1, L2, ..., L9$, at the intersection of two busy streets. A traffic light is located at this intersection. During a certain phase of the traffic light, those cars in lanes for which the light is green may proceed safely through the intersection. What is the minimum number of phase needed for the traffic light so that all cars may proceed trough the intersection?



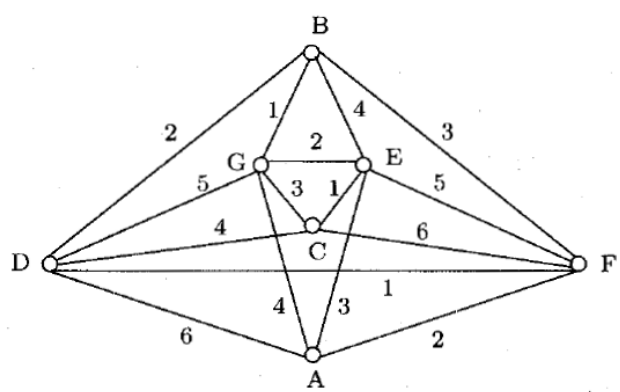
Solution: Construct a graph G to model this situation, where $V(G) = \{L1, L2, ..., L9\}$ and two vertices (lanes) are joined by a edge if vehicles in these two lanes cannot safely enter the intersection at the same time, as there is a possibility of an accident. Answering this question require determining the chromatic number of the graph. First notice that $\langle\{L2, L4, L6, L8\}\rangle \cong K_4$. Since there exists a 4-coloring of G, as indicated in the graph, therefore $\chi(G) = 4$.



Example 5.3: Alvin (A) has invited three married couples to his summer house for a week: Bob(B) and Carrie (C) Hanson, David (D) and Edith (E) Irwin, and Frank (F) and Gena (G) Jackson. Since all six guest enjoy playing tennis match against every other guest except his/her spouse. In addition, Alvin is to play a match against each of David, Edith, Frank, and Gena. If no one is to play two match on the same day, what is a schedule of matches over the smallest number of days.

Solution: First, we construct a graph H whose vertices are the people at Alvin's summer house, so $V(G) = \{A, B, C, D, E, F, G\}$, and two vertices of H are adjacent if the two vertices (people) are to play a tennis match. (The graph H is shown in the below graph). To answer the question, we determine the edge chromatic number of H. The edge chromatic number of the graph H is $\chi^{'}(H) = 6$.

$H :$



The above graph gives a 6-edge coloring of H, which provides a schedule of matches.

Day 1:	Bob-Gena, Carrie-Edith, David-Frank
Day 2:	Alvin-Frank, Bob-David, Edith-Gena
Day 3:	Alvin-Edith, Bob-Frank, Carrie-Gena
Day 4:	Alvin-Gena, Bob-Edith, Carrie-David
Day 5:	David-Gena, Edith-Frank
Day 6:	Alvin-David, Carrie-Frank