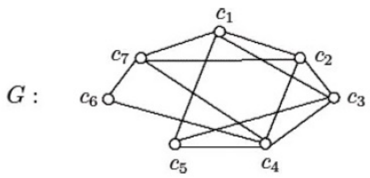
GRAPH THEORY I

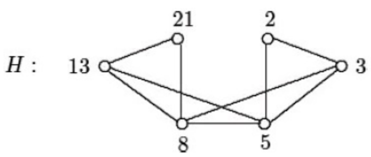
A major publishing company has ten editors(referred to by 1, 2, ...,10) in the scientific, technical and computing areas. These ten editors have a standard meeting time during the first Friday of every month and have divided themselves into seven committees to meet later in the day to discuss specific topics of interest to the company, namely, (i) advertising, (ii) securing reviewers, (iii) contacting new potential authors, (iv) finances, (v) used copies and new editions, (vi) competing textbooks and (vii) textbook representatives.

The ten editors have decided on the seven committees: $c_1 = \{1, 2, 3\}, c_2 = \{1, 3, 4, 5\}, c_3 = \{2, 5, 6, 7\}, c_4 = \{4, 7, 8, 9\}, c_5 = \{2, 6, 7\}, c_6 = \{8, 9, 10\}, c_7 = \{1, 3, 9, 10\}$. They have set aside three time periods for the seven committees to meet on those Fridays when all ten editors are present. Some pairs of committees cannot meet during the same period because one or two of the editors are on both committees. This situation can be modeled visually as shown in the below figure.
In this figure there are several seven small circles, representing the seven committees and a straight line segment is drawn between two circles if the committees they represent have at least one committee member in common. In other words, a straight line segment between two committees tells us that these two committees should be scheduled to meet at the same time. This gives us a picture or a model of the committees and the overlapping nature of their membership.



The above figure is called a **graph**. A graph **G** consists of a finite nonempty set **V** of objects called **vertices** and a set **E** of 2-element subsets of **V** called **edges**. The set **V** and **E** are the **vertex set** and **edge set** of **G**, respectively. So a graph **G** is a pair **G = { V, E }**.

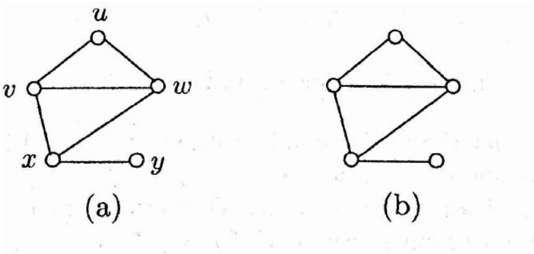
Consider the sequence of integers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ..., known as the **Fibonacci numbers**.
Consider the set S = { 2, 3, 5, 8, 13, 21 } of six specific Fibonacci numbers. There are some pairs of distinct integers belonging to S whose sum or difference(in absolute value) also belongs to S, namely, { 2, 3 }, { 2, 5 }, { 3, 5 }, { 3, 8 }, { 5, 8}, { 5, 13 }, { 8, 13 }, { 8, 21 } and { 13, 21 }.
We can visualize these pairs, by the graph H. In this case V(H) = { 2, 3, 5, 8, 13, 21 } and E(H) = { 2, 3 }, { 2, 5 }, { 3, 5 }, { 3, 8 }, { 5, 8}, { 5, 13 }, { 8, 13 }, { 8, 21 }, { 13, 21 }.



The number of vertices in G is often called the **order** of G, while the number of edges is its **size**. Since the vertex set of every graph is nonempty, the order of every graph is at least 1. A graph with exactly one vertex is called a **trivial graph**, implying that the order of a **nontrivial graph** is at least 2.

The above graph G has order-7 and size-13, graph H has order-6 and size-9. We often use n and m for order and size, respectively, of a graph.

A graph G with $V(G) = \{u, v, w, x, y\}$ and $E(G) = \{uv, uw, vw, vx, wx, xy\}$ is shown below.

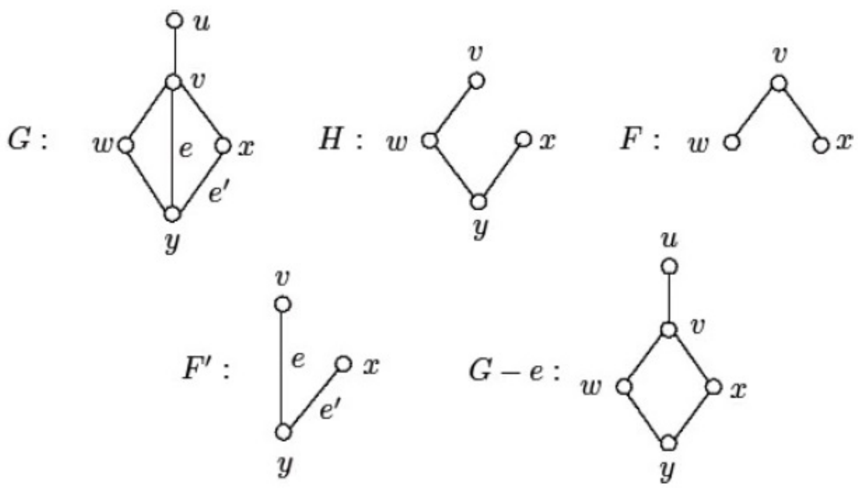


: **A labeled graph and an unlabeled graph**

In this case, graph (a) is drawn without labeling its vertices. For this reason, the graph (a) is a **labeled graph** and (b) is an **unlabeled graph**.

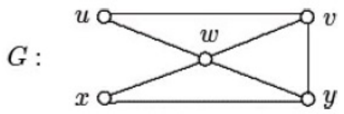
A graph G consists of a finite nonempty set V of vertices and a set E of 2-element subsets of V called edges. If $e = uv$ is an edge of G, then u and v are **adjacent vertices**. We also say u and v are joined by the edge e. The vertices u and v are referred to as **neighbors** of each other. In this case, the vertex u and the edge e(as well as v and e) with each other. Distinct edges incident with a common vertex are **adjacent edges**.

A graph H is called a **subgraph** of a graph G, written $H \subset G$, if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. We also say that G contains H as a subgraph. If $H \subset G$ and either $V(H)$ is proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a **proper subgraph** of G. If a subgraph of a graph G has the same vertex set as G, then it is a **spanning subgraph** of G.



A subgraph F of a graph G is called an **induced subgraph** of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well. If S is a nonempty set of vertices of a graph G, then the **subgraph of G induced by S** is the induced subgraph with vertex set S. The induced subgraph is denoted by $\langle S \rangle$. To emphasize that this is an induced subgraph of G, we sometimes denote this subgraph by $\langle S \rangle_G$. For a nonempty set X of edges, the subgraph $\langle X \rangle$ induced by X has edge set X and consists of all vertices that are incident with at least one edge in X. This subgraph is called an **edge-induced subgraph** of G. Sometimes $G[S]$ and $G[X]$ are used for $\langle S \rangle$ and $\langle X \rangle$ respectively.
Any proper subgraph of a graph G can be obtained by removing vertices and edges from G. For an edge e of G, we write $G - e$ for the spanning subgraph of G whose edge set consists of all edges of G except e. More generally, if X is a set of edges of G, then $G - X$ is the spanning subgraph of G with $E(G - X) = E(G) - X$.

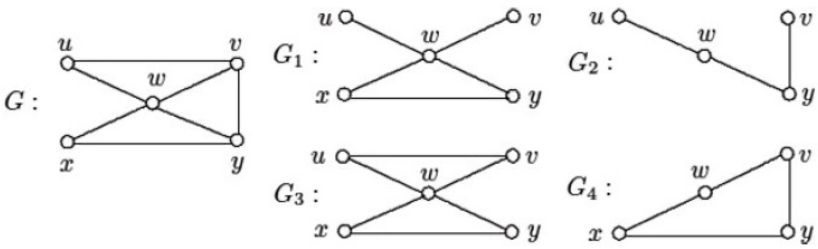
Let's start at some vertex u of a graph G. If we proceed from u to a neighbour of u and then to a neighbour of that vertex, and so on, until we finally come to a stop at a vertex, v, then we have just described a walk from u to v in G. More formally, a $u-v$ **walk** W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent, that is, we can express W as $W: u = v_0, v_1, ..., v_k = v$, where $k \geq 0$ and v_i and v_{i+1} are adjacent for $i = 0, 1, 2, ..., k-1$. Each vertex v_i, $(0 \leq i \leq k)$ and each edge $v_iv_{i+1}(0 \leq i \leq k-1)$ is said to lie on or belong to W. If $u = v$, then the walk W is **closed**; while if $u \neq v$, then W is **open**. As we move from one vertex of W to the next, we are actually encountering or traversing edges of G, possibly traversing some edges of G more than once. The number of edges encountered in a walk(including multiple occurrences of an edge) is called the **length** of the walk.



In the above graph $W: x, y, w, y, v, w$ is therefore a walk, indeed an $x-w$ walk of length 5.
A walk of length 0 is a **trivial walk**. So $W: v$ is a trivial walk.
A $u-v$ **trail** in a graph G is a $u-v$ walk in a graph in which no edge is traversed more than once. $T: u, w, y, x, w, v$ is a $u-v$ trail in the graph G.
A $u-v$ walk in a graph in which no vertices are repeated is a $u-v$ **path**. $P: u, w, y, v$ is a $u-v$ path. If no vertex in a walk is repeated(thereby producing a path), then no edge is repeated either. Hence every path is a trail.

Theorem: If a graph G contains a $u-v$ walk of length l, then G contains a $u-v$ path of length at most l.

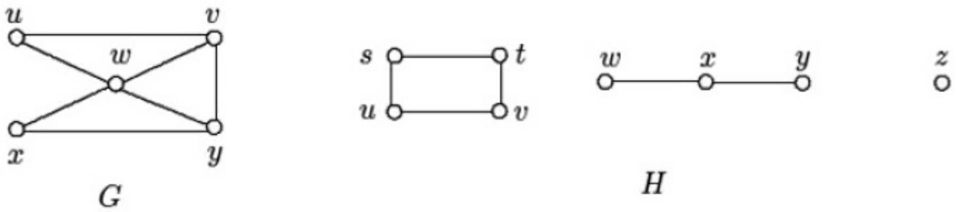
A **circuit** in a graph G is a closed trail of length 3 or more. Hence a circuit begins and ends at the same vertex but repeats no edges. A circuit can be described by choosing any of its vertices as the beginning(and ending) vertex provided the vertices are listed in the same cyclic order. In a circuit, vertices can be repeated, in addition to the first and last.
In the above graph: $C: y, w, u, v, w, x, y$ or $C: x, y, w, u, v, w, x$ or $C: w, x, y, w, u, v, w$ is a circuit.
A circuit that repeats no vertex, except for the first and last, is a **cycle**. A **k-cycle** is a cycle of length k. A 3-cycle is also referred to as a **triangle**. A cycle of odd length is called an **odd cycle**; while, a cycle of even length is called an **even cycle**.
In the above graph: $C': x, y, v, w, x$ is a cycle, namely a 4-cycle.
If a vertex of a cycle is deleted, then a path is obtained. This is not true for circuits, however.



The subgraphs G_1, G_2, G_3, G_4 of the graph G are a trail, path, circuit and cycle respectively.

If G contains a $u-v$ path, then u and v are said to be **connected** and u is connected to v(and v is connected to u). So, saying that u and v are connected only means that there is some $u-v$ path in G. A graph G is **connected** if every two vertices of G are connected, that is, if G contains a $u-v$ path for every pair u, v of distinct vertices of G.
Note: A graph G is connected if and only if G contains a $u-v$ walk for every pair u, v of vertices of G. Since every vertex is connected to itself, the trivial graph is connected.

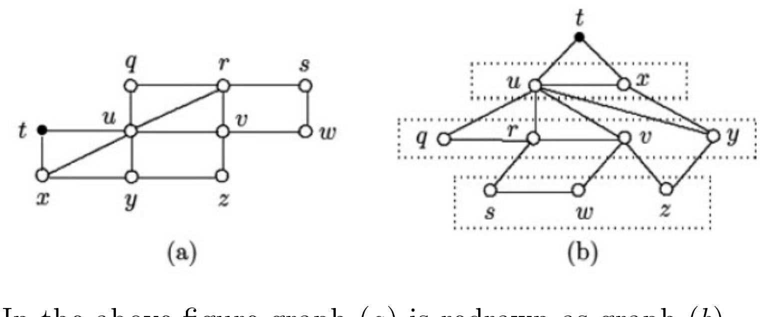
A graph G that is not connected is called **disconnected**. A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a **component** of G. A graph G is then connected if and only if it has exactly one component.



The above graph G is connected, where as the graph H is disconnected.
The graph H has three components, namely H_1, H_2 and H_3.

In general for subgraphs $G_1, G_2, ..., G_k, k \geq 2$, of a graph G, with mutually disjoint vertex sets, we write $G = G_1 \cup G_2 \cup ...G_k$ if every vertex and every edge of G belong to exactly one of these subgraphs. In particular, we write $G = G_1 \cup G_2 \cup ...G_k$ if $G_1, G_2, ..., G_k$ are components of G.
Therefore, we can write $H = H_1 \cup H_2 \cup H_3$ for the graph in the above figure.

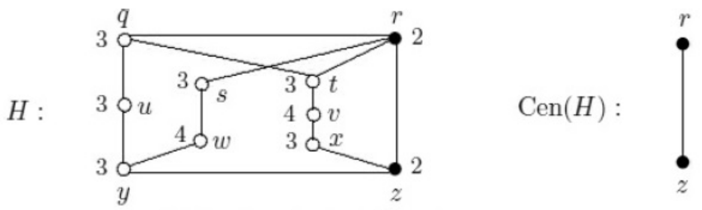
Let G be a connected graph of order n and let u and v be two vertices of G . The **distance** between u and v is the smallest length of any $u-v$ path in G and is denoted by $d_G(u, v)$ or simply $d(u, v)$ if the graph G under consideration is clear. Hence if $d(u, v) = k$, then there exists a $u-v$ path $P : u = v_0, v_1, ..., v_k = v$ of length k in G, but no $u-v$ path of smaller length exists in G. A $u-v$ path of length $d(u, v)$ is called a $u-v$ **geodesic**. In fact, since the path P is a $u-v$ geodesic, not only is $d(u, v) = d(u, v_k) = k$ but $d(u, v_i) = i$ for every i with $0 \leq i \leq k$.



In the above figure graph (a) is redrawn as graph (b).

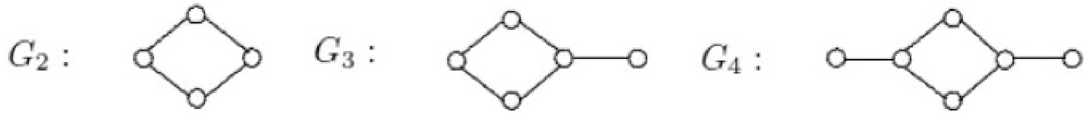
The greatest distance between any two vertices of a connected graph G are called the **diameter** of G and is denoted by $diam(G)$. The diameter of the above graph is 3. The path $P' : y, u, r, s$ is a $y-s$ geodesic whose length is $diam(H)$.
If G is a connected graph such that $d(u, v) = diam(G)$ and $w \neq u, v$, then no $u-w$ geodesic can contain v, for otherwise $d(u, w) > d(u, v) = diam(G)$, which is impossible.

For a vertex v in a connected graph G, the eccentricity $e(v)$ of v is the distance between v and a vertex farthest from v in G. The minimum eccentricity among the vertices of G is its radius and the maximum eccentricity is its diameter, which are denoted by $rad(G)$ and $diam(G)$, respectively. A vertex v in G is a central vertex if $e(v) = rad(G)$ and the subgraph induced by the central vertices of G is the center $Cen(G)$ of G. If every vertex of G is a central vertex, then $Cen(G) = G$ and G is called self-centered. For example, if $G = C_n$ where $n \geq 3$, then G is self-centered.
Consider the graph H of the below figure, where each vertex is labeled by its eccentricity. Since the smallest eccentricity is 2, $rad(H) = 2$. Because the largest eccentricity is 4, $diam(H) = 4$. The center of H is also shown in the figure.



rad$(H) = 2$ and diam$(H) = 4$

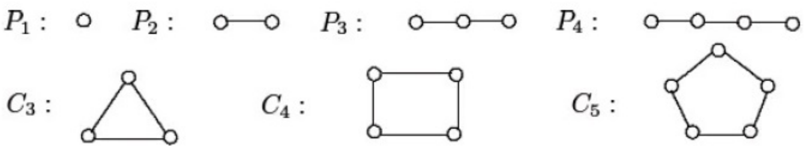
The below figure shows three graphs G_2, G_3 and G_4, each of which has radius 2, where $diam(G_k) = k$ for $k = 2, 3, 4$.



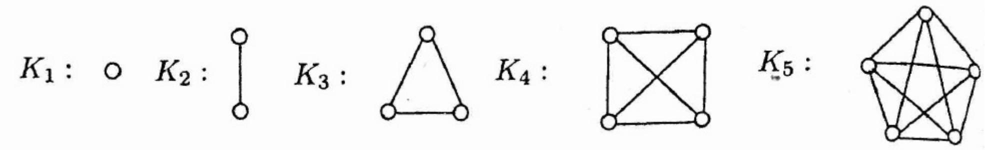
Theorem: For every nontrivial connected graph G, $rad(G) \leq diam(G) \leq 2\, rad(G)$.

Common Classes of Graphs

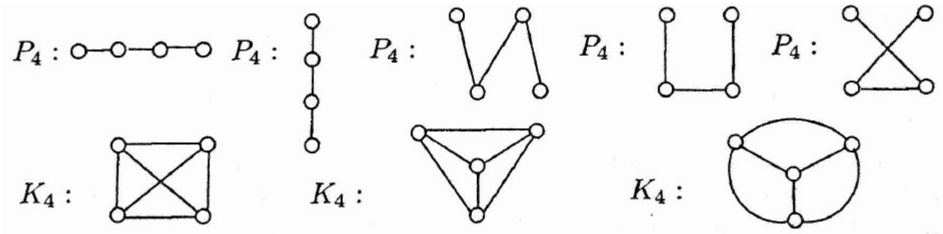
If a graph G of order n can be labeled(or relabeled) $v_1, v_2, ..., v_n$ so that its edges are $v_1v_2, v_2v_3, ..., v_{n-1}v_n$, the G is called a **path**; while if the vertices of a graph G of order $n \geq 3$ can be labeled(or relabeled) $v_1, v_2, ..., v_n$, so that its edges are $v_1v_2, v_2v_3, ..., v_{n-1}v_n$ and v_1v_n, then G is called a **cycle**. A graph that is a path of order n is denoted by P_n, while a graph that is a cycle of order $n \geq 3$ is denoted by C_n. Several paths and cycles are shown in the below figure.



A graph G is **complete** if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n. Therefore, K_n has the maximum possible size for a graph with n vertices. Since every two distinct vertices of K_n are joined by an edge, the number of pairs of vertices in K_n is $\binom{n}{2}$ and so the size of K_n is $\binom{n}{2} = \frac{n(n-1)}{2}$.



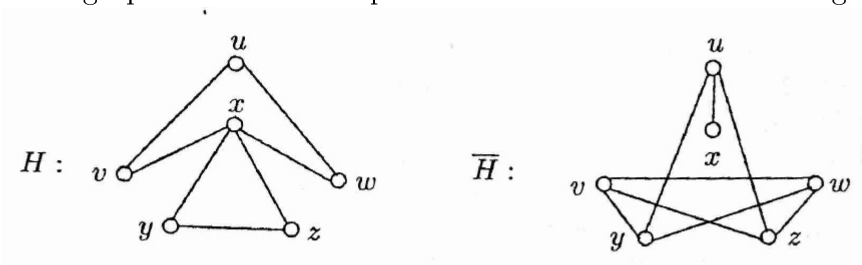
The above graphs are complete graphs.



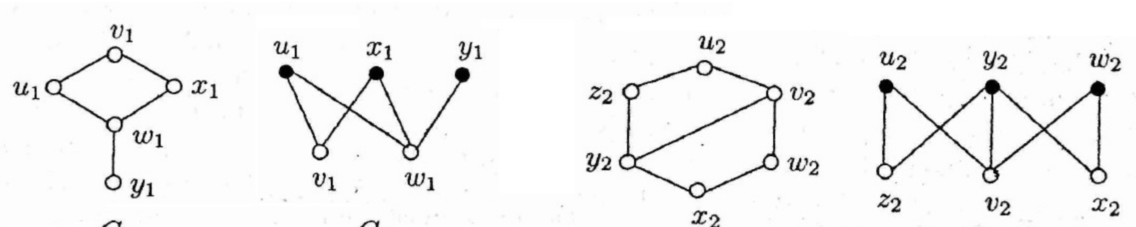
Given any graph it can be drawn in different ways.

The **complement** $\overline{G}$ of a graph G is that graph whose vertex set is $V(G)$ such that for each pair u, v of vertices of G, uv is an edge of $\overline{G}$ if and only if uv is not an edge of G. Observe that if G is a graph of order n and size m, then $\overline{G}$ is a graph of order n and size $\binom{n}{2} - m$. The graph $\overline{K_n}$ then has n vertices and no edges; it is called the **empty graph** of order n. Therefore, empty graph have empty edge sets. In fact, if G is any graph of order n, then $G - E(G)$ is the empty graph $\overline{K_n}$. By definition, no graph can have an empty vertex set.

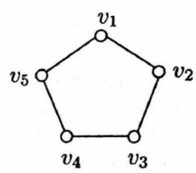
A graph H and its complement are shown in the below figure.



A graph G is a **bipartite graph** if $V(G)$ can be partioned into two subsets U and W, called **partite sets**, such that every edge of G joins a vertex of U and a vertex of W. The connected graphs G_1 and G_2 of the below figure are bipartite, as every edge of G_1 joins a vertex of $U_1 = \{u_1, x_1, y_1\}$and a vertex of $W_1 = \{v_1, w_1\}$, while every edge of G_2 joins a vertex of $U_2 = \{u_2, w_2, y_2\}$ and a vertex of $W_2 = \{v_2, x_2, z_2\}$. By letting $U = U_1 \cup U_2$ and $W = W_1 \cup W_2$, we see that every edge of $G = G_1 \cup G_2$ joins a vertex of U and a vertex of W. This illustrates the observation that a graph is bipartite if and only if each of its components is bipartite.

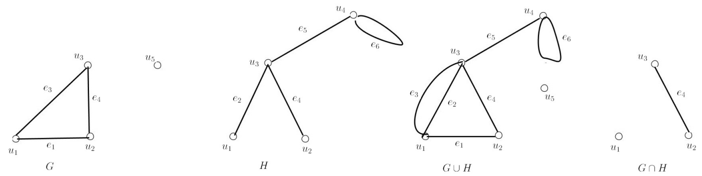


G_1 G_1 G_2 G_2

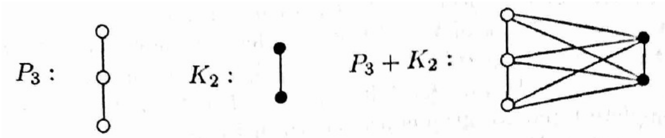


The following graph C_5 is not bipartite.
In fact, no odd cycle is bipartite. Indeed, any graph that contains an odd cycle is not bipartite. The converse is true as well.

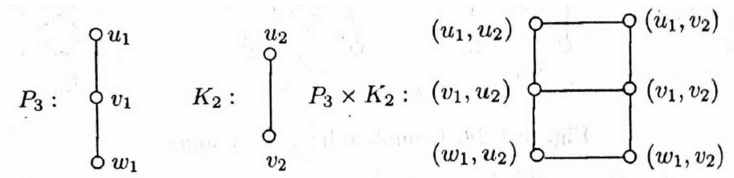
There are several different ways to produce a new graph from a given pair of graphs. For two vertex-disjoint graphs G and H, the **union** $G \cup H$ is that(disconnected) graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The **intersection** of G and H with at least one vertex in common, is $G \cap H$ with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$. The union and intersection of the graphs G and H is shown below.



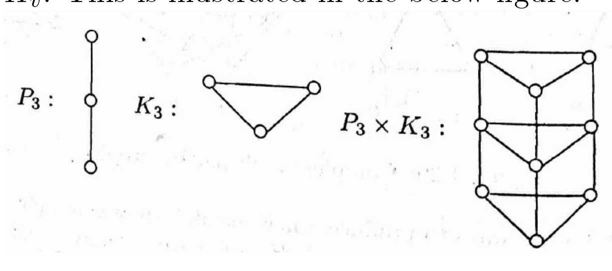
The **join** $G + H$ consists of $G \cup H$ and all edges joining a vertex of G and a vertex of H. The join of P_3 and K_2 is shown in the below figure.



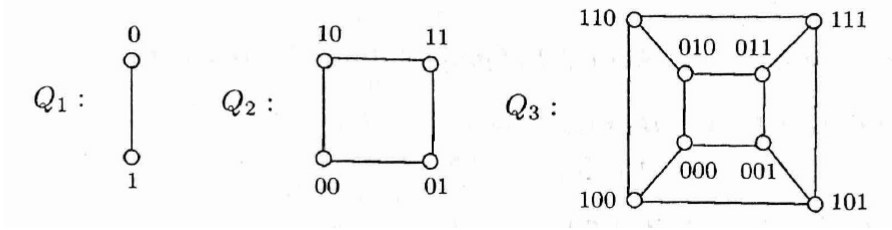
For two(not necessarily vertex-disjoint) graphs G and H, the **Cartesian product** $G \times H$ has vertex set $V(G \times H) = V(G) \times V(H)$, that is , every vertex of $G \times H$ is an ordered pair (u, v), where $u \in V(G)$ and $v \in V(H)$.Two distinct vertices (u, v) and (x, y) are adjacent in $G \times H$ if either (i) $u = x$ and $vy \in E(H)$ or (ii) $v = y$ and $ux \in E(G)$. The below figure shows the Cartesian product of P_3 and K_2.



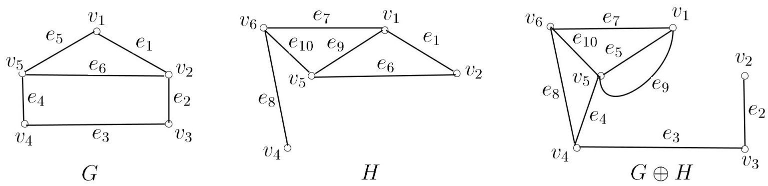
Note: The order in which the graphs G and H are written is structurally irrelevant, that is $G \times H$ and $H \times G$ are the same graph, that is, they are isomorphic graphs.
There is an informal way of drawing the graph $G \times H$(or $H \times G$) that doesn't require us to label the vertices. Replace each vertex x of G by a copy $H_x$of the graph H. Let u and v be two vertices of G. If u and v are adjacent in G, then we join corresponding vertices of H_u and H_v by an edge. If u and v are not adjacent in G, then we add no edges between H_u and H_v. This is illustrated in the below figure.



Notice that $K_2 \times K_2$ is the 4-cycle. The graph $C_4 \times K_2$ is often denoted by Q_3 and is called the **3-cube**. More generally, we define Q_1 to be K_2 and for $n \geq 2$, define Q_n to be $Q_{n-1} \times K_2$. The graphs Q_n are then called **n-cubes** or **hypercubes**. The n-cube can also be defined as that graph whose vertex set is the set of order n-tuples of 0s and 1s(commonly called the **n-bit strings**) and where two vertices are adjacent if their ordered n-tuples differ in exactly one position(coordinate). The n-cubes for $n = 1, 2, 3$ are shown in the below figure.



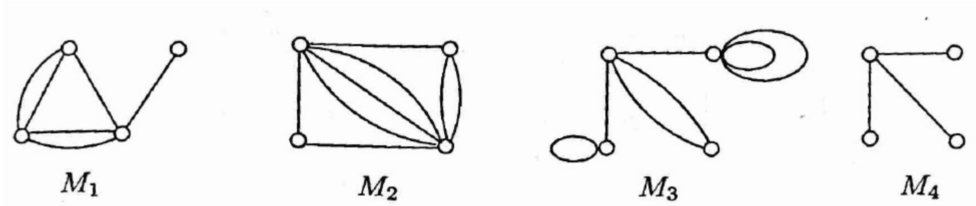
For two graphs G and H the ringsum $G \oplus H$ is that graph with vertex set $V(G) \cup V(H)$ and the edge set is $E(G \oplus H) = E(G \cup H) - E(G \cap H)$. The below graph shows the ringsum of the two graphs G and H.



Multigraphs

In a graph, two vertices are either adjacent or they are not, that is, two vertices are joined by one edge or no edge. A **multigraph** M consists of a finite nonempty set V of vertices and a set E of edges, where every two vertices of M are joined by a finite number of edges(possibly zero). If two or more edges join the same pair of(distinct) vertices, then these edges are called **parallel edges**. In a **pseudograph**, not only are parallel edges permitted but an edge is also permitted to join a vertex to itself. Such an edge is called a **loop**. If a loop e joins a vertex v to itself, then e is said to be a loop at v. There can be any finite number of loops at the same vertex in a pseudograph.

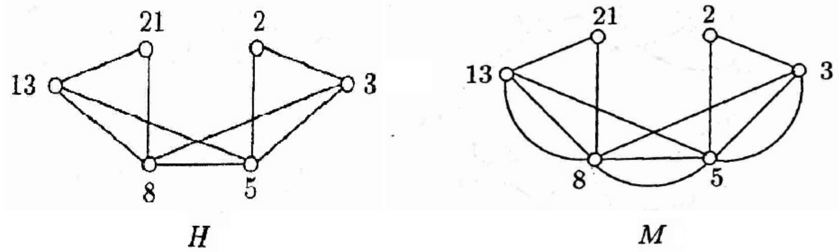
Some multigraphs and pseudograph are shown in the below figure.



In the above figure, M_1 and M_2 are multigraphs, M_3 is a pseudograph and M_4 is a graph. In fact, M_4 is a multigraph and all four are pseudographs.

If M is a multigraph with vertex set V, then it is no longer appropriate to regard an edge of M as a 2-element subset of V as we must somehow indicate the multiplicity of the edge and make allowance for the existence of loops.

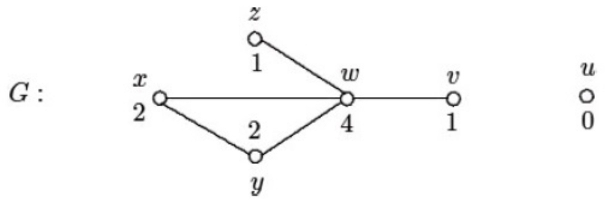
Let us consider the set $S = \{2, 3, 5, 8, 13, 21\}$. The graph H indicates the situation, where the sum or the difference(in absolute value) belongs to S. The graph M indicates the situation, where the sum as well as the difference(in absolute value) belongs to S.



Degrees

The **degree of a vertex** v in a graph G is the number of edges incident with v and is denoted by $deg_G\ v$ or simply by $deg\ v$ if the graph G is clear from the context. Also, $deg\ v$ is the number of vertices adjacent to v. Two adjacent vertices are referred to as **neighbors** of each other. The set $N(v)$ of neighbors of a vertex v is called the **neighborhood** of v. Thus $deg\ v = |N(v)|$.

A vertex of degree 0 is referred to as an **isolated vertex** and a vertex of degree 1 is an **end-vertex**(or a **leaf**). The **minimum degree** of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$; the **maximum degree** of G is denoted by $\Delta(G)$. So if G is a graph of order n and v is any vertex of G, then $0 \leq \delta(G) \leq deg\ v \leq \Delta(G) \leq n-1$.



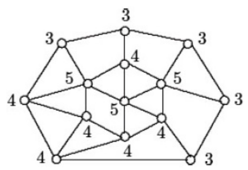
The above graph G has order 6 and size 5. u is an isolated vertex. Hence $\delta(G) = 0$. Both v and z are end-vertices. w has the largest degree i.e., $deg\ w = 4$. Hence $\Delta(G) = 4$. Adding the degrees of all the vertices, we obtain $0+1+1+2+2+4 = 10$, which is twice the size of G.

Problem: A certain graph G has order 14 and size 27. The degree of each vertex of G is 3, 4 or 5. There are six vertices of degree 4. How many vertices of G have degree 3 and how many have degree 5?

Ans: Let x be the number of vertices of G having degree 3. Then,

$3 \cdot x + 4 \cdot 6 + 5 \cdot (8 - x) = 2 \cdot 27 \implies x = 5$. Hence G has five vertices of degree 3 and three vertices of degree 5.

Another way of soling the given problem is by drawing the graph as below.

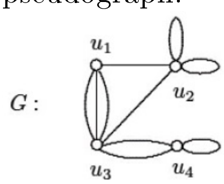


OR $x + y = 8$ and $3x + 4 \cdot 6 + 5y = 2 \cdot 27 = 54 \implies x = 5$ and $y = 3$ satisfies the equations.

Suppose that G is a bipartite graph of size m with partite sets $U = \{u_1, u_2, ..., u_s\}$ and $W = \{w_1, w_2, ..., w_t\}$. Since every edge of G joins a vertex of U and a vertex of W, it follows that adding the degrees of the vertices in U (or in W) gives the number of edges in G, that is, $\sum_{i=1}^{s} deg\ u_i = \sum_{j=1}^{t} deg\ w_j = m$

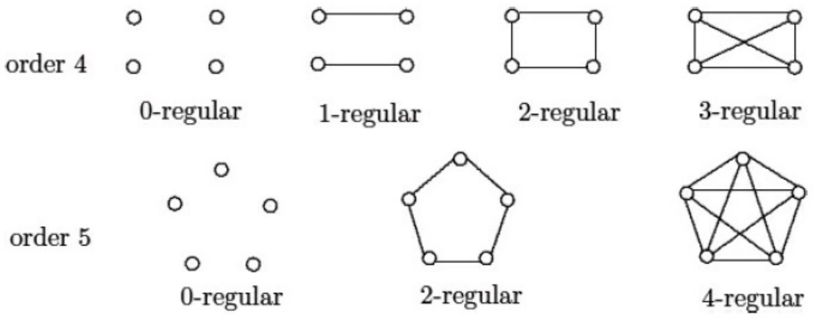
A vertex of even degree is called an **even vertex**, while a vertex of odd degree is an **odd vertex**.

For a vertex v in a multigraph or pseudograph G, the **degree** $deg\ v$ of v in G is the number of edges of G incident with v, where there is a contribution of 2 for each loop at v in a pseudograph.

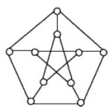


For the pseudograph G above $deg\ u_1 = 4, deg\ u_2 = 6, deg\ u_3 = 6, deg\ u_4 = 4$

Regular Graphs
We have already mentioned that $0 \leq \delta(G) \leq \Delta(G) \leq n-1$ for every graph G of order n. If $\delta(G) = \Delta(G)$ then the vertices of G have the same degree and G is called **regular**.
If $deg\ v = r$ for every vertex v of G, where $0 \leq r \leq n-1$, then G is **r-regular** or **regular of degree r**. The only regular graphs of order 4 or 5 are shown in the below figure. There is no 1-regular or 3-regular graph of order 5, as no graph contains an odd number of odd vertices by a Corollary.



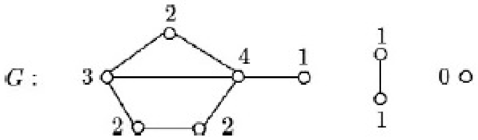
A 3-regular graph is also referred to as a **cubic graph**. The graphs $K_4, K_{3,3}$ and Q_3 are cubic graphs. However the best known cubic graph may very well be the **Petersen graph**, shown in the below figure.



By a Corollary, there are no r-regular graphs of order n if r and n are both odd.

Theorem: Let r and n be integers with $0 \leq r \leq n-1$. There exists an r-regular graph of order n if and only if at least one of r and n is even.

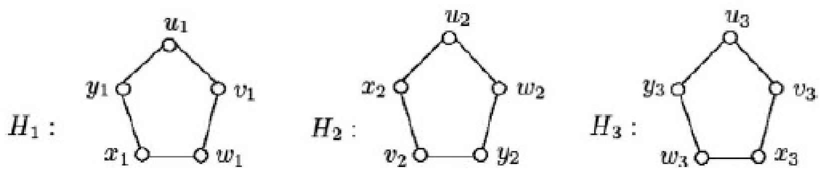
Degree Sequences
It is typical for the vertices of a graph to have a variety of degrees. If the degrees of the vertices of a graph G are listed in a sequence s, then s is called a degree sequence of G. For example,
$s : 4,3,2,2,2,1,1,1,0$; $s' : 0,1,1,1,2,2,2,3,4$; $s'' : 4,3,2,1,2,2,1,1,0$
all of the sequences are degree sequences of the graph G of the below figure, each of whose vertices is labeled by its degree.



The sequence s is non-increasing, s' is non-decreasing and s'' is neither.
Suppose that we are given a finite sequence s of nonnegative integers. This finite sequence of nonnegative integers is called **graphical** if it is a degree sequence of some graph.

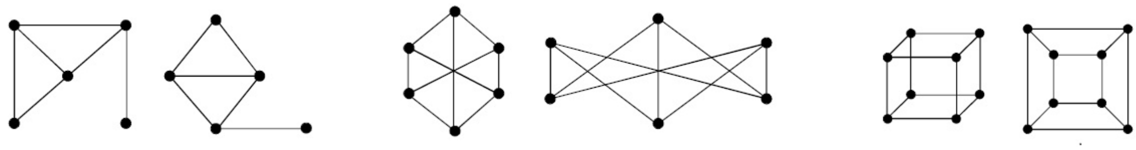
Which of the following sequences are graphical?
(i) $s_1 : 3,3,2,2,1,1$ (ii) $s_2 : 6,5,5,4,3,3,3,2,2$ (iii) $s_3 : 7,6,4,4,3,3,3$ (iv) $s_4 : 3,3,3,1$
Ans: (i) Yes (ii) No (iii) No (iv) No.

Two(labeled) graphs G and H are **isomorphic**(have the same structure) if there exists a one-to-one correspondence from $V(G)$ to $V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. In this case, f is called an **isomorphism** from G to H. Thus, if G and H are isomorphic graphs, then we say that G is isomorphic to H and we write $G \cong H$. If G and H are unlabeled, then they are isomorphic if, under any labeling of their vertices, they are isomorphic as labeled graphs. If two graphs G and H are not isomorphic, then they are called **non-isomorphic** graphs and we write $G \ncong H$.
The below graphs are isomorphic



The necessary conditions for two graphs to be isomorphic are
1. Both must have the same number of vertices.
2. Both must have the same number of edges.
3. Both must have equal number of vertices with the same degree.
4. They must have the same degree sequence and same cycle vector $(c_1, c_2, ..., c_n)$, where c_i is the number of cycles of length i.

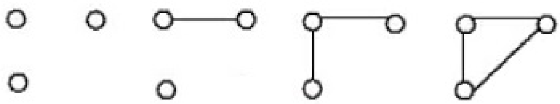
Below are some pairs of non-isomorphic graphs.



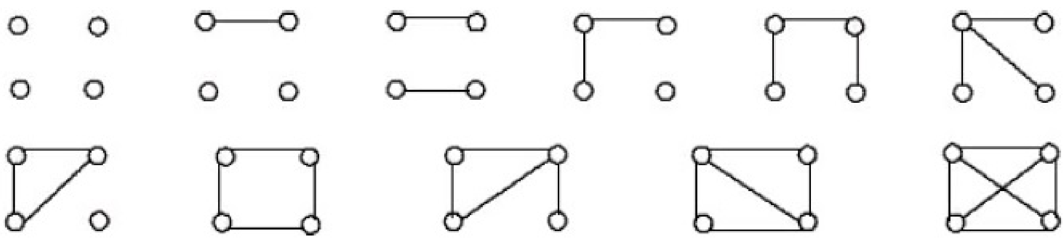
There is only one non-isomorphic graph of order 1,

two non-isomorphic graphs of order 2

and four non-isomorphic graphs of order 3.

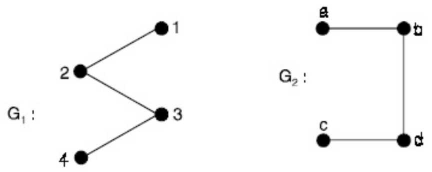


There are eleven non-isomorphic graphs of order 4 and these are shown in the below figure.

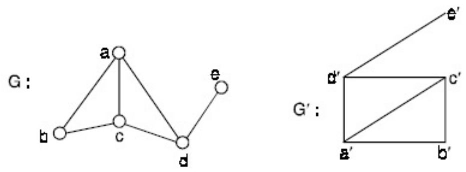


Problem:
Show that the below two graphs are isomorphic.

(i)

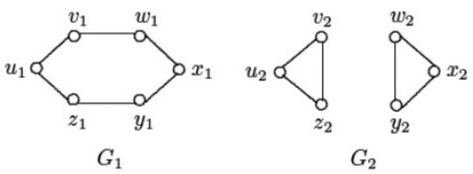


(ii)

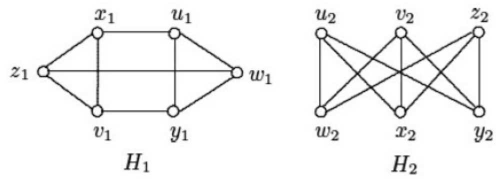


Problem:
Are the two graphs given below isomorphic?

(i)

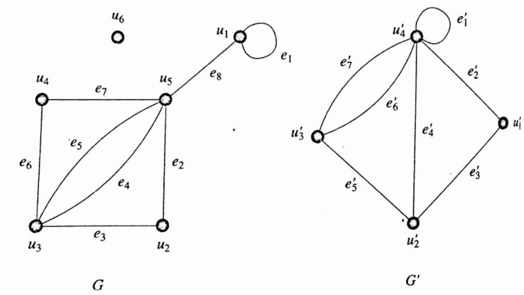


(ii)

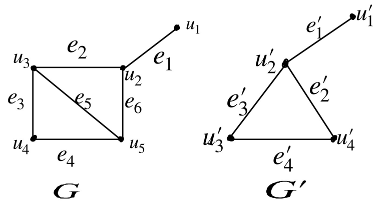


In the above figures G_1 and G_2 are non-isomorphic.
H_1 and H_2 are complements of G_1 and G_2 respectively.
H_1 and H_2 are non-isomorphic as well.
Note: Two graphs G and H are isomorphic if and only if their complements and are isomorphic.

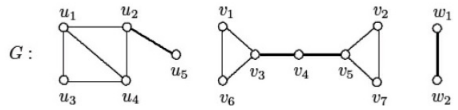
Suppose $G = V, E$ and $G' = V', E'$ be two graphs. A homomorphism $f : G \longrightarrow G'$ from G to G' is an ordered pair $f = (f_1, f_2)$ of maps $f_1 : V \longrightarrow V'$ and $f_2 : E \longrightarrow E'$ satisfying the following condition: $\phi(e) = \{u, v\} \implies \phi'(f_2(e)) = \{f_1(u), f_1(v)\}$. That is, u and v are the end vertices of e in G, then $f_1(u)$ and $f_1(v)$ are the end vertices of $f_2(e)$ in G'.



Consider the two graphs G and G' shown above. Define the maps f_1 and f_2 as follows, first f_1 by $f_1(u_1) = f_1(u_4) = f_1(u_5) = f_1(u_6) = u_4', f_1(u_2) = u_2', f_1(u_3) = u_3'$, and then f_2 for the edges by $f_2(e_1) = f_2(e_7) = f_2(e_8) = e_1', f_2(e_2) = e_4', f_2(e_3) = e_5', f_2(e_4) = f_2(e_5) = e_6', f_2(e_6) = e_7'$.
The two graphs are homomorphic as $e_1 = (u_1, u_1), e_2 = (u_2, u_5), e_3 = (u_2, u_3), e_4 = (u_3, u_5), e_5 = (u_3, u_5), e_6 = (u_3, u_4), e_7 = (u_4, u_5), e_8 = (u_1, u_5)$ are edges of G, whereas the corresponding edges given by $e_1' = (u_4', u_4'), e_4' = (u_2', u_4'), e_5' = (u_2', u_3'), e_6' = (u_3', u_4'), e_6' = (u_3', u_4'), e_7' = (u_3', u_4'), e_1' = (u_4', u_4'), e_1' = (u_4', u_4')$ are the edges of G'.
Problem: Verify whether the below graphs are homomorphic.



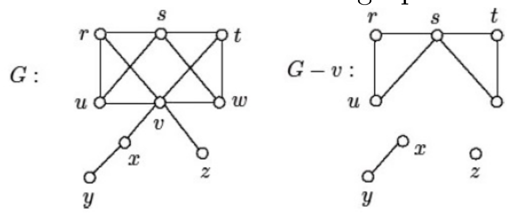
An edge $e = uv$ of a connected graph G is called a **bridge** of G if $G - e$ is disconnected.



In the above figure the bridges are represented in bold.

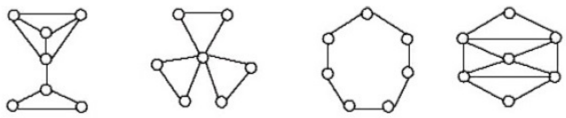
Theorem: An edge e of a graph G is a bridge if and only if e lies on no cycle of G.

A vertex v in a connected graph G is a **cut-vertex** of G if $G - v$ is disconnected.



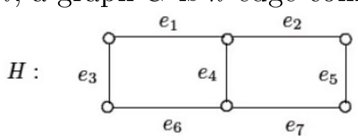
Corollary: Let G be a connected graph of order 3 or more. If G contains a bridge, then G contains a cut-vertex.

By a **vertex-cut** in a graph G, we mean a set U of vertices of G such that $G - U$ is disconnected. A vertex-cut of minimum cardinality in G is called a **minimum vertex-cut**. For a graph G that is not complete, the **vertex-connectivity**(or simply the **connectivity**) $\kappa(G)$ of G is defined as the cardinality of a minimum vertex-cut of G. If $G = K_n$ for some positive integer n, then $\kappa(G)$ is defined to be $n - 1$. For a nonnegative integer k, a graph G is said to be k-connected if $\kappa(G) \geq k$.



In the above figure, the first two graphs are 1-connected($\kappa(G) = 1$) and the last two graphs are 2-connected($\kappa(G) = 2$).

An **edge-cut** in a nontrivial graph G is a set X of edges of G such that $G - X$ is disconnected. An edge-cut X of a connected graph G is minimal if no proper subset of X is an edge-cut of G. If X is a minimal edge-cut of a connected graph G, then $G - X$ contains exactly two components G_1 and G_2. If X is an edge-cut of a connected graph G that is not minimal, then there is a proper subset Y of X that is a minimal edge-cut. An edge-cut of minimum cardinality is called a **minimum edge-cut**. The **edge-connectivity** $\lambda(G)$ of a nontrivial graph G is the cardinality of a minimum edge-cut of G. For a nonnegative integer k, a graph G is k-edge-connected if $\lambda(G) \geq k$. For every graph of order n $0 \leq \lambda(G) \leq n - 1$.



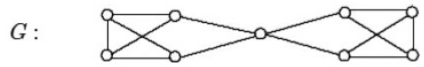
Here $X_1 = \{e_3, e_4, e_5\}, X_2 = \{e_1, e_2, e_6\}$ and $X_3 = \{e_1, e_6\}$ are the edge-cut sets. Hence $\lambda(G) = 2$.

Theorem: For every positive integer n, $\lambda(K_n) = n - 1$.

Theorem: For every graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

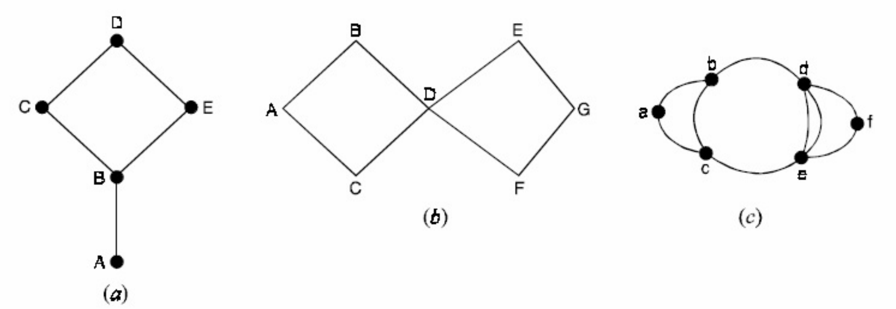
Theorem: If G is a cubic graph, then $\kappa(G) = \lambda(G)$.

Theorem: If G is a graph of order n and size $m \geq n - 1$, then $\kappa(G) \leq \lambda(G) \leq \delta(G) \leq 2m/n$



In the above graph $\kappa(G) = 1, \lambda(G) = 2$ and $\delta(G) = 3$.

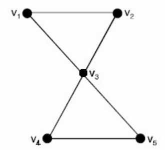
A circuit C in a graph G is called an **Eulerian circuit** if C contains every edge of G. Since no edge is repeated in a circuit, every edge appears exactly once in an Eulerian circuit. A connected graph that contains an Eulerian circuit is called an **Eulerian graph**. In a connected graph G, an open trail that contains every edge of G as an **Eulerian trail**.
Examples:



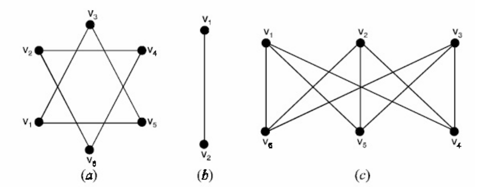
(a) has an Euler trail but no Euler circuit
(b) has both Euler circuit and Euler trail
(c) has an Euler trail but no Euler circuit.

Corollary: A connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at the other.

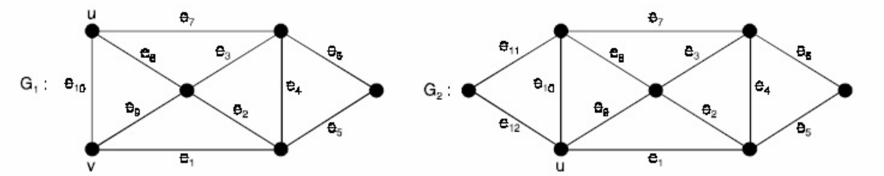
Problem:
Verify if the below graph G has an Eulerian circuit.



Ans: Yes, the graph has an Eulerian circuit.
Show that the graphs in the figure below contain no Eulerian circuit.

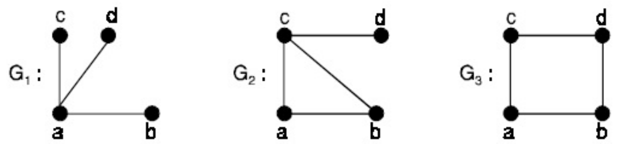


Ans: (a) is a disconnected graph, (b) and (c) have vertices with odd degree.
Which of the following graphs have Eulerian trail and Eulerian circuit.



Ans: G_1 has an Eulerian trail, as it has two vertices with odd degree.
G_2 has an Eulerian circuit, as all the vertices are of even degree.

A cycle in a graph G that contains every vertex of G is called a **Hamiltonian cycle** of G. Thus a Hamiltonian cycle of G is a spanning cycle of G. A **Hamiltonian graph** is a graph that contains a Hamiltonian cycle. The graph $C_n (n \geq 3)$ is Hamiltonian. Also, for $n \geq 3$, the complete graph K_n is a Hamiltonian graph.
A path in a graph G that contains every vertex of G is called a **Hamiltonian path** in G. If a graph contains a Hamiltonian cycle, then it contains a Hamiltonian path. In fact, removing any edge from a Hamiltonian cycle produces a Hamiltonian path. If a graph contains a Hamiltonian path, however, it need not contain a Hamiltonian cycle.

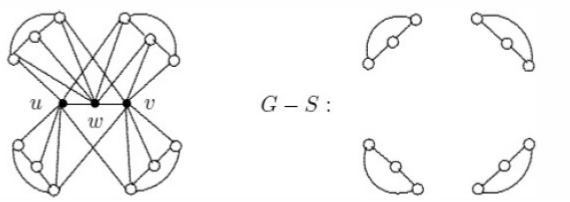


The graph G_1 has no hamiltonian path(and so hamiltonian cycle), G_2 has hamiltonian path but no hamiltonian cycle, while G_3 has both hamiltonian path and hamiltonian cycle.
A simple graph G is called **maximal non-hamiltonian** if it is not hamiltonian but the addition to it any edge connecting two non-adjacent vertices forms a hamiltonian graph. The graph G_2 is a maximal non-hamiltonian since the addition of an edge bd gives hamiltonian graph.

Theorem: If G is a Hamiltonian graph, then for every nonempty proper set S of vertices of G, $\kappa(G - S) \leq |S|$.

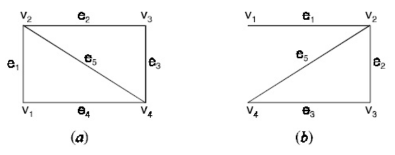
Theorem: Let G be a graph. If $\kappa(G - S) > |S|$ for some nonempty proper subset S of $V(G)$, then G is not Hamiltonian.

Note: If a graph G contains a cut-vertex, then G cannot be Hamiltonian.

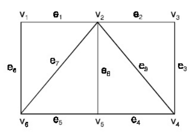
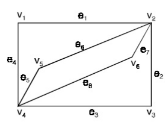


Theorem: Let G be a graph of order $n \geq 3$. If $deg\, u + deg\, v \geq n$ for each pair u, v of non adjacent vertices of G, then G is Hamiltonian.(Converse need not be true)
Corollary: Let G be a graph of order $n \geq 3$. If $deg\, v \geq \frac{n}{2}$ for every vertex v of G, then G is Hamiltonian.(Converse need not be true)

Problem:
Which of the graphs given in the below figure is Hamiltonian?



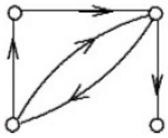
Ans: (a) is hamiltonian, (b) is non-hamiltonian.

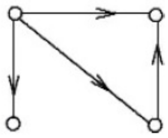
Hamiltonian but not Eulerian. Eulerian but not Hamiltonian.

A **digraph**(or **directed graph**) D is a finite nonempty set V of objects called **vertices** together with a set E of $ordered pairs$ of distinct vertices. The elements of E are called **directed edges** or **arcs**. If (u, v) is a directed edge, then we indicate this in a diagram representing D by drawing a directed line segment or curve from u to v. Then u is said to be **adjacent to** v and v is **adjacent from** u. The vertices u and v are also said to be **incident with** the directed edge (u, v). Arcs (u, v) and (v, u) may both be present in some directed graph. If, in the definition of digraphs, for each pair u, v of distinct vertices, at most one of (u, v) and (v, u) is a directed edge, then the resulting digraph is an **oriented graph**. Thus an oriented graph D is obtained by assigning a direction to each edge of some $graph G$. The digraph D is also called an **orientation** of G. The below figure shows two digraphs D_1 and D_2, where D_2 is an oriented graph but D_1 is not.

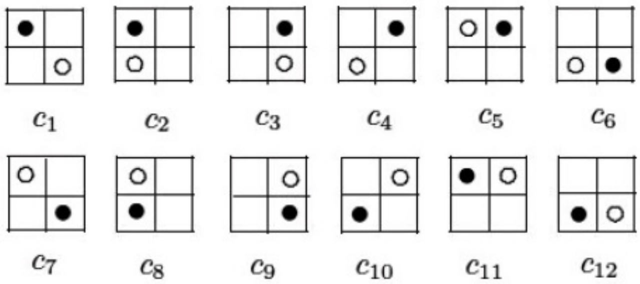
D_1 :



D_2 :

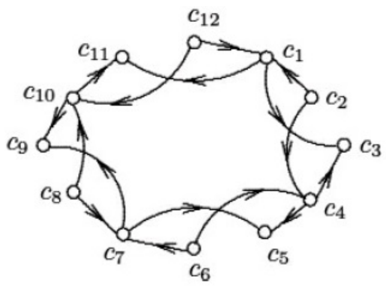


Considered twelve configurations of two coins (one silver, one gold), which were denoted by $c_1, c_2, ..., c_12$.



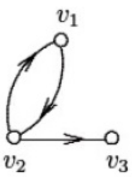
Now, we say that c_i can be transformed into c_j if c_j can be obtained by moving one of the coins in c_i to the right or up. Modeling this situation requires a digraph, namely, the digraph D shown in the below figure, which is an oriented graph.

D :



For a vertex v in a digraph D, the **outdegree** od v of v is the number of vertices of D to which v is adjacent, while the **indegree** id v of v is the number of vertices of D from which v is adjacent.

D :



For the digraph D, $od\, v_1 = 1, id\, v_1 = 1, od\, v_2 = 2, id\, v_2 = 1, od\, v_3 = 0, id\, v_3 = 1$.

The First Theorem of Graph Theory:
If G is a graph of size m, then $\sum\limits_{v\in V(G)} deg\ v = 2m$

Proof: When summing the degrees of the vertices of G, each edge of G is counted twice, once for each of its two incident vertices.

Corollary: Every graph has an even number of odd vertices.

Proof: Let G be a graph of size m.
Divide $V(G)$ into two subsets V_1 and V_2, where V_1 consists of the odd vertices of G and V_2 consists of the even vertices of G.
By the First Theorem of Graph Theory,

$$\sum_{v\in V(G)} deg\ v = 2m$$

$$\implies \sum_{v\in V_1} deg\ v + \sum_{v\in V_2} deg\ v = 2m$$

The number $\sum\limits_{v\in V_2} deg\ v$ is even since it is a sum of even integers.
Thus

$$\sum_{v\in V_1} deg\ v = 2m - \sum_{v\in V_2} deg\ v$$

which implies $\sum\limits_{v\in V_1} deg\ v$ is even.
Since each of the numbers $deg\ v, v \in V_1$ is odd, the number of odd vertices of G is even.

Result: If $x_1, x_2, ..., x_k \geq 1$ are real numbers and $x = x_1 + x_2 + ... + x_k$, then $\sum\limits_{i=1}^{k} x_i^2 \leq x^2 - (k-1)(2x-k)$

Proof: For each i let $x_i = y_i + 1$, where $y_i \geq 0$. Let $y = y_1 + y_2 + ... + y_k$
Then $x^2 - (k-1)(2x-k)$
$= (x_1 + x_2 + ... + x_k)^2 - (k-1)(2x-k)$
$= (y_1 + 1 + y_2 + 1 + ... + y_k + 1)^2 - (k-1)(2x-k)$
$= (y+k)^2 - (k-1)(2x-k)$
$= y^2 + 2y + k$
$= (\sum\limits_{i=1}^{k} y_i)^2 + 2(\sum\limits_{i=1}^{k} y_i) + k$
$\geq \sum\limits_{i=1}^{k} y_i^2 + 2\sum\limits_{i=1}^{k} y_i + k$
$= \sum\limits_{i=1}^{k}(y_i^2 + 2y_i + 1)$
$= \sum\limits_{i=1}^{k} x_i^2$
Hence $\sum\limits_{i=1}^{k} x_i^2 \leq x^2 - (k-1)(2x-k)$

Theorem: For a simple graph G with n vertices and k components $m \leq \dfrac{(n-k)(n-k+1)}{2}$.

Proof: Let $H_1, ..., H_k$ be the components of G.
For $1 \leq i \leq k$, denote by n_i the number of vertices of H_i.
Note that $n = n_1 + n_2 + ... + n_k$. Observe that each H_i is a simple graph. H_i can have at most $n_i(n_i - 1)/2$ edges.
Hence, the maximum number of edges G has is

$$= \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$
$$= \frac{1}{2}(\sum_{i=1}^{k} n_i^2) - \frac{1}{2}(\sum_{i=1}^{k} n_i)$$
$$\leq \frac{1}{2}(n^2 - (k-1)(2n-k)) - \frac{n}{2}$$
$$= \frac{n^2 - 2nk + k^2 + 2n - k - n}{2}$$
$$= \frac{n^2 - 2nk + k^2 + n - k}{2}$$
$$= \frac{n^2 - nk + n - nk + k^2 - k}{2}$$
$$= \frac{n(n-k+1) - k(n-k+1)}{2}$$
$$= \frac{(n-k)(n-k+1)}{2}.$$

Theorem: A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.

Proof: Assume first that G is Eulerian. Then G contains an Eulerian circuit C. Suppose that C begins at the vertex u (and therefore ends at u). Let v be a vertex of G different from u. Since C neither begins nor ends at v, each time that v is encountered on C, two edges are accounted for(one to enter v and another to exit v). Thus v has even degree. Since C begins at u, this accounts for one edge. Another edge is accounted for because C ends at u. If u is encountered at other times, two edges are accounted for. So u is even as well.
For the converse, assume that G is a nontrivial connected graph in which every vertex is even. Among all trails in G, let T be one of maximum length. Suppose that T is a $u - v$ trail. We claim that $u = v$. If not, then T ends at v. It is possible that v may have been encountered earlier in T. Each such encounter involves two edges of G, one to enter v and another to exit v. Since T ends at v, an odd number of edges at v has been encountered. But v has even degree. This means that there is at least one edge at v, say vw, that does not appear on T. But then T can be extended to w, contradicting the assumption that T has maximum length. Thus T is a $u-u$ trail, that is, $C = T$ is a $u-u$ circuit. If C contains all edges of G, then C is an Eulerian circuit and the proof is complete.

Theorem: A nontrivial graph G is a bipartite graph if and only if G contains no odd cycles.

Proof: Let the graph G have a cycle C_k, where k is odd. If C_k were bipartite, then its vertex set could be partitioned into two sets U and W such that every edge of C_k joins a vertex of U and a vertex of W. The vertex v_1 must belong to either U or W, say $v_1 \in U$. Since v_1v_2 is an edge of C_k, it follows that $v_2 \in W$. Since v_2v_3 is an edge of C_k, it follows that $v_3 \in U$. Similarly, $v_{k-1} \in W$ and $v_k \in U$. However, $v_1, v_k \in U$ and v_1v_k is an edge of C_k. This is a contradiction. Therefore, C_k is not bipartite for k odd.
To prove the converse, let G be a nontrivial graph having no odd cycles and assume that G is connected. Let u be any vertex of G, let U consist of all vertices of G whose distance from u is even and let W consist of all vertices whose distance from u is odd. Thus $\{U, W\}$ is a partition of $V(G)$. Since $d(u, u) = 0$, it follows that $u \in U$. We claim that every edge of G joins a vertex of U and a vertex of W. Assume, to the contrary, that there exist two adjacent vertices in U or two adjacent vertices in W. Since these two situations are similar, we will assume that there are vertices v and w in W such that $vw \in E(G)$. Since $d(u, v)$ and $d(u, w)$ are both odd, $d(u, v) = 2s + 1$ and $d(u, w) = 2t + 1$ for nonnegative integers s and t. Let $P' = (u = v_0, v_1, ..., v_{2s+1} = v)$ be a $u - v$ geodesic and let $P'' = (u = w_0, w_1, ..., w_{2t+1} = w)$ be a $u - w$ geodesic in G. Certainly, P' and P'' have their initial vertex u in common but they may have other vertices in common as well. Among the vertices P' and P'' have in common, let x be the last vertex. Perhaps $x = u$. In any case, $x = v_i$ for some integer $i \geq 0$. Thus $d(u, vi) = i$. Since x is on P'' and w_i is the only vertex of P'' whose distance from u is i, it follows that $x = w_i$. So $x = v_i = w_i$. However then, $C = (v_i, v_{i+1}, ..., v_{2s+1}, w_{2t+1}, w_{2t}, ..., w_i = v_i)$ is a cycle of length $(2s + 1) - i + (2t + 1) - i + 1 = 2s + 2t - 2i + 3 = 2(s + t + i) + 1$and so C is an odd cycle, which is a contradiction.