Mathematical Modelling - Discrete Process

Introduction:

Difference equations arise from many sources and the discrete independent variable ‘n’ can signify many different things which can think of difference equations as

Future Value= Current Value + Change

Or

Change = Future Value – Present Value

If change takes place over discrete time periods, it gives a difference equation.

If change takes place continuously with respect to the time, then we get a differential equation.

Difference Equation: If $A = \{a_0, a_1, \ldots a_n\}$ is a sequence of numbers,

$\Delta a_0 = a_1 - a_0, \Delta a_1 = a_2 - a_1, \ldots, \Delta a_{n-1} = a_n - a_{n-1}$

$a_{n+1} = a_n + \Delta a_n$

Future Value = Present Value + Change.

Example:

1. Fibonacci Numbers 1, 1, 2, 3, 5, 8, 13, 21, ….
   The string of numbers referred to as a recurrence relation can be generated from the difference equation
   $x_{n+1} = x_n + x_{n-1}$
2. Newton’s Method given by the difference equation
   $x_{n+1} = x_n - \frac{f'(x_n)}{f(x_n)}$.

Linear difference equations

An equation of the form

$$f(x_{t+n} + x_{t+n-1} + \cdots + x_t, t) = 0$$

is called a difference equation of nth order.

The difference equation of the form

$$a_0 x_{t+n} + a_1 x_{t+n-1} + \cdots + a_n x_t = \phi(t) \text{---(1)}$$

is called a non-homogenous linear difference equation with constant coefficients.

The difference equation of the form

$$a_0 x_{t+n} + a_1 x_{t+n-1} + \cdots + a_n x_t = 0 \text{---(2)}$$

is called a homogenous linear difference equation with constant coefficients.

$$x_t = c_1 g_1(t) + c_2 g_2(t) + \cdots + c_n g_n(t)$$

is called general solution of the homogeneous linear difference equation (2).

$$x_t = \text{Complementary function} + \text{Particular solution}$$

is the general solution of the non-homogenous linear difference equation with constant coefficients (1).

First order difference equation

The equation of the form

$$x_{t+1} = c x_t + f(t+1) \text{ ---(3)}$$

or

$$a_n = c a_{n-1} + f(n)$$

is called a first order non homogeneous difference equation .

$$x_t = c^t x_0 + \sum_{i=1}^{t} c^{t-i} f(i)$$

is the general solution of first order non homogeneous difference equation (3).

If $f(t+1) = 0$, The difference equation (3) becomes homogeneous.

$$x_t = c^t x_0$$

Is the solution of first order homogeneous difference equation.

Problems:

1. Solve $x_{t+1} = 4x_t, x_0 = 3.$

Solution: $x_t = 4^t.3$

2. Solve $x_t = 7x_{t-1}, t \geq 1, x_2 = 98.$

   Solution: from the given equation,

$$x_2 = 7x_1 = 7(7x_0) = 49x_0$$
$$98 = 49x_0 \rightarrow x_0 = 2$$
$$x_t = 7^t.2$$

3. The number of virus affected files in a system is 1000(to start with) and this increases 250% every two hours. Determine the number of virus affected files in the system after one day.

   Let x_t be number of virus affected files in a system after $2t$ hours.

$$x_0 = 1000$$
$$x_1 = 1000 + 2.5 * 1000 = x_0 + 2.5x_0$$
$$x_2 = x_1 + 2.5\, x_1$$
$$x_{t+1} = x_t + 2.5x_t = 3.5\, x_t$$

   It is a first order difference equation. Its solution is

$$x_t = 3.5^t x_0 = 3.5^t * 1000$$

   After one day, ie., 24 hours , $2t = 24, t = 12.$

   $x_{12} = 3.5^{12} * 1000 = 3.37 \times 10^9.$

Second order difference equation:

The equation of the form

$$c_0x_{t+2} + c_1x_{t+1} + c_2x_t = 0 \text{ ---(4)}$$

is called second order linear difference equation.

Auxiliary equation of (4) is $c_0m^2 + c_1m + c_2 = 0$.

Let the roots of Auxiliary equation be $m = r_1, r_2$.

Case (i): If roots are real and distinct

$$x_t = Ar_1^t + Br_2^t$$

Where A, B are arbitrary.

Case(ii): If roots are real and repeated

$$x_t = (A + B)r^t$$

Case(iii): If roots are complex roots $p \pm q$

$$x_t = r^t[A\cos(t\theta) + B\sin(t\theta)]$$

$$\text{Where } r = \sqrt{p^2 + q^2},\ \theta = \tan^{-1}\left(\frac{q}{p}\right)$$

Problems:

Solve the following difference equations.

i) $x_{t+2} + x_{t+1} - 6x_t = 0, x_0 = -1, x_1 = 8.$

Solution: Auxiliary equation $m^2 + m - 6 = 0, m = -3,2$

G.S is $x_t = A(-3)^t + B2^t$

At $t = 0, x_0 = -1 \rightarrow A + B = -1$

At $t = 1, x_1 = 8 \rightarrow -3A + 2B = 8$

Solving above equations, $A = -2, B = 1$

$$\therefore x_t = -2(-3)^t + 2^t$$

ii) $x_{t+2} - 6x_{t+1} + 9x_t = 0$

Solution: Auxiliary equation $m^2 - 6m + 9 = 0, m = 3,3$

$$\therefore x_t = (A + Bt)3^t$$

iii) $x_{t+2} = 2(x_{t+1} - x_t)$

Solution: Auxiliary equation $m^2 - 2m + 2 = 0, m = 1 \pm i$

$$r = \sqrt{1+1} = \sqrt{2}, \theta = \tan^{-1} 1 = \frac{\pi}{4}$$

$$x_t = \sqrt{2}\left[A\cos\frac{\pi}{4}t + B\sin\frac{\pi}{4}t\right] = \sqrt{2}\left[A\cos\frac{t}{\sqrt{2}} + B\sin\frac{t}{\sqrt{2}}\right]$$

Mathematical modeling through difference equations in Economics and Finance

HARROD MODEL

Let $S(t)$ be the savings, $Y(t)$ be the National Income, I(t) be the investment.

We make the following assumptions:

i) Savings made by the people in a country depend on the national income.

$$S(t) = \alpha Y(t) \text{ ----- } (1)$$

ii) The investment depends on the difference between the income of the current year and the last year.

$$I(t) = \beta[Y(t) - Y(t-1)], \beta > 0 \text{ ----- } (2)$$

iii) All the savings made are invested,

$$S(t) = I(t) \text{ -----}(3)$$

$$\text{Ie., } \alpha I(t) = \beta[Y(t) - Y(t-1)]$$

$$Y(t-1) = (\beta - \alpha)Y(t)$$

$$Y(t) = \frac{\beta}{\beta - \alpha} Y(t-1)$$

It is a first order difference equation. Its solution is

$$Y(t) = A\left(\frac{\beta}{\beta-\alpha}\right)^t = Y(0)\left(\frac{\beta}{\beta-\alpha}\right)^t \text{----}(4)$$

Assuming $Y(t)$ is always positive,

$$\beta > 0, \frac{\beta}{\beta - \alpha} > 1$$

So that the national income increases with time, $'t'$.

The national income at different times 0, 1 , 2 , 3 form a geometrical progression.

Thus if all savings made are invested, savings are proportional to the national income and the investment is proportional to the excess of the current years income over the preceding years income, then the national income increases geometrically.

THE COBWEB MODEL

Let p_t be the price of a commodity in the year t and let q_t be the amount of the commodity available in the market in the year t.

The following assumptions are made:

i) Amount of the commodity produced this year and available for sale is a linear function of the price of the commodity in the last year.

$$q_t = \alpha + \beta p_{t-1} \text{ ----(1)}$$

Where, $\beta > 0$, since if the last year's price was high, the amount available this year will also be high.

ii) The price of the commodity this year is a linear function of the amount available this year.

$$p_t = \gamma + \delta q_t \text{ ----(2)}$$

Where $\delta < 0$, since if q_t is large, the price would be low.

From (1) AND (2),

$$p_t = \gamma + \delta(\alpha + \beta p_{t-1})$$

$$= \gamma + \delta\alpha + \beta\delta p_{t-1}$$

$$= (\beta\delta)p_{t-1} + (\gamma + \alpha\delta)$$

Which is a non- homogenous difference equation

Its solution is

$$p_t = (\beta\delta)^t p_0 + \sum_{i=1}^{t} (\beta\delta)^{t-1} (\gamma + \alpha\delta)$$

$$= (\beta\delta)^t p_0 + (\gamma + \alpha\delta)[(\beta\delta)^{t-1} + (\beta\delta)^{t-2} + \cdots + (\beta\delta) + 1]$$

$$= (\beta\delta)^t p_0 + (\gamma + \alpha\delta)\left[\frac{1+(\beta\delta)^t}{1-\beta\delta}\right]$$

$$= (\beta\delta)^t p_0 + \frac{(\gamma+\alpha\delta)}{1-\beta\delta} - \frac{(\gamma+\alpha\delta)}{1-\beta\delta}(\beta\delta)^t$$

$$p_t - \frac{(\gamma+\alpha\delta)}{1-\beta\delta} = \left[p_0 - \frac{(\gamma+\alpha\delta)}{1-\beta\delta}\right](\beta\delta)^t \text{ ------(3)}$$

$$p_{t-1} - \frac{(\gamma+\alpha\delta)}{1-\beta\delta} = \left[p_0 - \frac{(\gamma+\alpha\delta)}{1-\beta\delta}\right](\beta\delta)^{t-1} \text{-----(4)}$$

From (3),

$$p_t - \frac{(\gamma+\alpha\delta)}{1-\beta\delta} = \left[p_0 - \frac{(\gamma+\alpha\delta)}{1-\beta\delta}\right](\beta\delta)^{t-1}(\beta\delta)$$

$$\therefore p_t - \frac{(\gamma+\alpha\delta)}{1-\beta\delta} = \left[p_{t-1} - \frac{(\gamma+\alpha\delta)}{1-\beta\delta}\right](\beta\delta)$$

Since $(\beta\delta)$ is negative, $p_0, p_1, p_2, \ldots$ are alternately greater and less than $\frac{(\gamma+\alpha\delta)}{1-\beta\delta}$.

If $|\beta\delta| > 1$, then the deviation of p_t from $\frac{(\gamma+\alpha\delta)}{1-\beta\delta}$ goes increasingly.

If $|\beta\delta| < 1$, then the deviation of p_t from $\frac{(\gamma+\alpha\delta)}{1-\beta\delta}$ goes on decreasingly and ultimately

. $p_t \to \frac{(\gamma+\alpha\delta)}{1-\beta\delta}$ as $t \to \infty$.

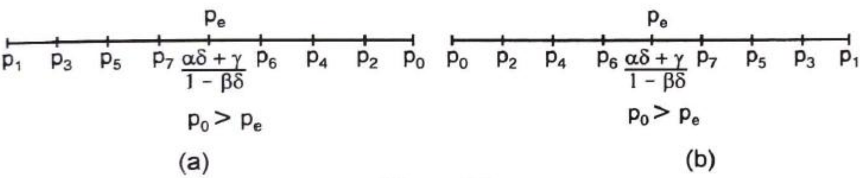


Figure (a) and (b) show how the price approaches equilibrium price $p_e = \frac{(\gamma+\alpha\delta)}{1-\beta\delta}$ as t increases in two cases when $p_0 > p_e$ and $p_0 < p_e$ respectively.

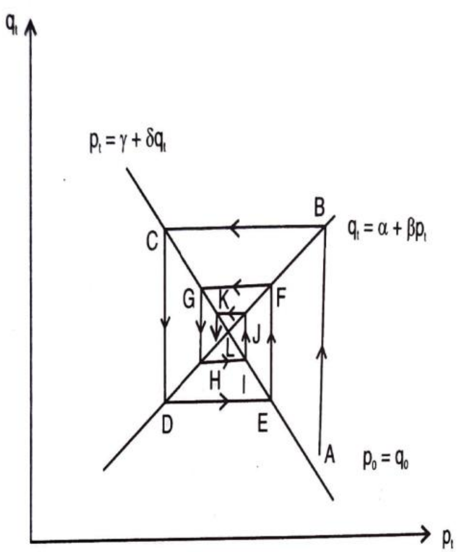
In the same way eliminating p_t from equation (1) and (2), we get

$$q_t = \alpha + \beta\gamma_t + \beta\delta q_{t-1}.$$

which has the solution

$$q_t - \frac{(\alpha+\beta\gamma)}{1-\beta\delta} = \left[q_{t-1} - \frac{(\alpha+\beta\gamma)}{1-\beta\delta}\right](\beta\delta)$$

So that q_t also oscillates about the equilibrium quantityq_e.



Suppose we start in the year zero with price p_0 and quantity q_0 represented by point A. In year one, the quantity q_1 is given by $\alpha + \beta p_0$ and the price by $p_1 = \gamma + \delta q_1$. This brings us to the point C in two steps via B. The path of prices and quantities is thus given by the COBWEB path ABCDEFGHIJKL…. And the equilibrium price and quantity are given by the point of intersection of the two straight lines.

ACTUARIAL SCIENCE

One mathematical aspect of actuarial science is Mathematics of Finance and Investment.

If a sum s_0 is invested at compound interest i per unit amount per unit time and s_t is the amount at the end of time t, then we have the difference equation

$$s_{t+1} = s_t + is_t = (1+i)s_t$$

Which has the solution

$$s_t = s_0(1+i)^t$$

Which is well-known formulae for compound interest. Suppose a person borrows a sum S_0 at compound interest i and wants to 'amortize' his debt ie., pay the amount and the interest back by payment of n equal installments say R.

Let S_t be the amount due at the end of t years, then we have the difference equation

$$s_{t+1} = s_t + is_t - R = (1+i)s_t - R$$

which is a first order non- homogeneous difference equation. Its solution is

$$s_t = (1+i)^t s_0 - \sum_{k=1}^{t} (1+i)^{t-1} R$$

$$s_t = (1+i)^t s_0 - R\{(1+i)^{t-1} + (1+i)^{t-2} + \cdots + (1+i) + 1\}$$

$$= (1+i)^t s_0 - R\left[\frac{(1+i)^t - 1}{1+i-1}\right]$$

At the end of n years $s_n = 0$

Ie., $0 = (1+i)^t s_0 - \frac{R}{i}[(1+i)^t - 1]$

$$\frac{R}{i}[(1+i)^t - 1] = (1+i)^t s_0$$

$$R = \frac{i(1+i)^t s_0}{(1+i)^t - 1} = s_0 \frac{i}{1-(1+i)^{-t}} = s_0 \frac{i}{a_n \rceil\, i}$$

$a_n \rceil\, i$ is called the amortization fraction in the present value of annuity of 1 per unit time for n periods at an interest rate i.

Mathematical modeling through difference equations in GENETICS

Hardy- Weinberg law:

Every characteristic of an individual like height, colour of hair etc., is determined by a pair of genes. One obtained from the father and the other from the mother.

Every gene occurs in two forms, dominant (denoted by 'G') and a recessive ('g'). Thus with respect to a characteristic, an individual may be a dominant (GG), a hybrid (Gg or gG) Or a recessive.

In the 'n'th generation, let the proportions be p_n, q_n and r_n so that

$$p_n + q_n + r_n = 1, p_n, q_n, r_n \geq 0.$$

We assume that individuals in this generation mate at random.
p_{n+1}= The probability that an individual in the $(n+1)th$ generation is dominant (GG)

= (Probability that this individual gets a G from father

$\times$Probability that the individual gets a G from mother)

$$= \left(p_n + \tfrac{1}{2}q_n\right)\left(p_n + \tfrac{1}{2}q_n\right) = \left(p_n + \tfrac{1}{2}q_n\right)^2$$

Similarly,

$$q_{n+1} = \left(p_n + \frac{1}{2}q_n\right)\left(r_n + \frac{1}{2}q_n\right) + \left(r_n + \frac{1}{2}q_n\right)\left(p_n + \frac{1}{2}q_n\right)$$

$$q_{n+1} = 2\left(p_n + \frac{1}{2}q_n\right)\left(r_n + \frac{1}{2}q_n\right)$$

$$r_{n+1} = \left(r_n + \frac{1}{2}q_n\right)^2$$

So that

$$p_{n+1} + q_{n+1} + r_{n+1}$$

$$= \left(p_n + \frac{1}{2}q_n\right)^2 + 2\left(p_n + \frac{1}{2}q_n\right)\left(r_n + \frac{1}{2}q_n\right) + \left(r_n + \frac{1}{2}q_n\right)^2$$

$$= \left(p_n + \frac{1}{2}q_n + r_n + \frac{1}{2}q_n\right)^2 = 1$$

Similarly,

$$p_{n+2} = \left(p_{n+1} + \frac{1}{2}q_{n+1}\right)^2$$

$$= \left[\left(p_n + \tfrac{1}{2}q_n\right)^2 + \left(p_n + \tfrac{1}{2}q_n\right)\left(r_n + \tfrac{1}{2}q_n\right)\right]^2$$

$$= \left[\left(p_n + \tfrac{1}{2}q_n\right)\left(p_n + \tfrac{1}{2}q_n + r_n + \tfrac{1}{2}q_n\right)\right]^2$$

$$= \left[\left(p_n + \frac{1}{2}q_n\right)^2 (p_n + q_n + r_n)^2\right] = \left(p_n + \frac{1}{2}q_n\right)^2 = p_{n+1}$$

and

$q_{n+2} = q_{n+1}, \; r_{n+2} = r_{n+1}$

This means that the proportions of dominants hybrids and recessives in the $(n+2)th$ generation are same as in the $(n+1)th$ generation. Thus in any population in which random mating takes place with respect to a characteristics the proportions of dominants, hybrids and recessive do not change after the first generation. This is known as Hardy-Weinberg law after the mathematician Hardy geneticist Weinberg who jointly discovered it.

Mathematical modeling through difference equations in population dynamics

Let x_t be the population at time $'t'$ and let births and deaths in time interval $(t, t+1)$ be proportional to x_t. Then the population x_{t+1} at time $t+1$ is given

$$x_{t+1} = x_t + bx_t - dx_t = x_t(1 + b - d)$$

$$x_{t+1} = x_t(1 + a) \text{-----} \qquad (1)$$

This is a homogeneous first order difference equation whose solution is

$$x_{t+1} = x_0(1 + a)^t$$

(hint: for homogeneous equations solution $x_{t+1} = c^t x_0$)

The population increases or decreases exponentially as $a > 0$ or $a < 0$ respectively.

Consider the generalization when births and deaths b and d per unit population depend linearly on x_t

$$x_{t+1} = x_t + (b_0 - b_1 x_t)x_t - (d_0 - d_1 x_t)$$

$$= x_t + b_0x_t - b_1x_t^2 - d_0x_t + d_1x_t^2$$
$$= x_t(1 + b_0 - d_0) - x_t^2(b_1 - d_1)$$
$$x_{t+1} = mx_t - rx_t^2$$
$$x_{t+1} = mx_t\left[1 - \frac{r}{m}x_t\right] \text{ ----(2)}$$

This is a non-homogeneous linear generalization of (1) , which gives the discrete version of the logistic law of population growth.

Let $\frac{r}{m}x_t = y_t, x_t = \frac{m}{r}y_t$ then $mx_t = \frac{m^2}{r}y_t$

$(2) \Longrightarrow \frac{m}{r}y_{t+1} = m\frac{m}{r}y_t(1 - y_t)$

$\Longrightarrow y_{t+1} = my_t(1 - y_t)$

One- Period fixed point:

A one-period fixed point of this equation is that value of y_t for which

$$y_{t+1} = y_t$$
$$y_t = my_t(1 - y_t)$$
$$y_t = my_t - my_t^2$$
$$my_t^2 - my_t + y_t = 0$$
$$y_t(my_t - m + 1) = 0$$

$y_t = 0$ or $my_t = m - 1, y_t = \frac{m-1}{m}$

There are two one- period fixed points.

If $y_t = 0$ then $y_1, y_2, \ldots$ are all zero

If $y_t = \frac{m-1}{m}$, then $y_1, y_2, \ldots$ are all $\frac{m-1}{m}$.

If $y_t = 0$, then $y_{t+1}, y_{t+2}, \ldots$ will be zero as well.(y_0 initial time)

Two - Period fixed point:

It is a point which repeats itself after two periods.

$$y_{t+2} = y_t$$

$$y_{t+2} = my_{t+1}(1 - y_{t+1})$$

$$= m(my_t)(1 - y_t)\big[1 - [my_t(1 - y_t)]\big]$$

$$= m^2 y_t(1 - y_t)[1 - my_t + my_t^2] = y_t$$

$$\Rightarrow (m^2 y_t - m^2 y_t^2)[1 - my_t + my_t^2] = y_t$$

$$\Rightarrow y_t\big(my_t(m-1)\big)[m^2 y_t^2 - m(1+m)y_t + (1+m)] = 0$$

This is a fourth degree equation and there can be four fixed points. Two of these are same as one period fixed points(Every one period fixed point is also a two period fixed point). The genuine two period fixed points are obtained by solving the equation

$$m^2 y^2 - m(1+m)y_t + (1+m) = 0.$$

<u>Newton's law of cooling using difference equation</u>

The rate of change temperature of an object is proportional to the difference between the temperature of the object and the surroundings.

Let $T(t)$ be the temperature of the body and let S be the temperature of the surroundings.

By Newton's law of cooling,

$$T(t+1) - T(t) = k\big(S - T(t)\big)$$

where k is the proportional constant.

$$T(t+1) = k - T(t) + T(t) + kS$$

$$T(t+1) = (1-k)T(t) + kS$$

Its solution is

$T(t) = (1-k)^t T(0) + \sum_{i=1}^{t}(1-k)^{t-i} * S$

$$= (1-k)^t T(0)$$
$$+kS[(1-k)^{t-1} + (1-k)^{t-2} + \cdots + (1-k) + 1]$$
$$= (1-k)^t T(0) + kS\left[\frac{1-(1-k)^t}{1-(1-k)}\right]$$
$$= (1-k)^t T(0) + S[1-(1-k)^t]$$

As $t \to \infty$, with $k < 0$, $(1-k) \to 0 \Rightarrow T(t) \to S$.

Problems:

1. Suppose a cup of coffee initially at temperature $190^0 F$, is placed in a room with temperature $70^0 F$. After one minute, the coffee cools down to $180^0 F$. Find the temperature of the coffee after 15 minutes.

   Solution:
   $$T(1) - T(0) = k\big(70 - T(0)\big)$$
   $$180 - 190 = k(70 - 190) \Rightarrow k = 12$$
   At $t = 15$, $T(15) = 102.5^0 F$.

2. Solve the difference equation $x_{t+2}^2 - 5x_{t+1}^2 + 4x_t^2 = 0$, given $x_0 = 4$ and $x_1 = 13$.

   Solution: Let $x_t^2 = y_t \Rightarrow y_{t+2} - 5y_{t+1} + 4y_t = 0$

   A.E is $m^2 - 5m + 4 = 0, m = 4\ \&1$.
   $$y_t = A4^t + B1^t$$
   $$x_0 = 4 \Rightarrow y_0 = x_0^2 = 16$$
   Similarly, $y_1 = x_1^2 = 169$
   $$A = 51, B = -35$$
   $$y_t = 51 * 4^t - 35, x_t = \pm\sqrt{51 * 4^t - 35}$$