Random process

A random process is a collection of random variables usually indexed by time.

Markov Process

$$\text{If } P\{X(t_n)=a_n \overset{\text{given that}}{\Big|} X(t_{n-1})=a_{n-1},\ X(t_{n-2})=a_{n-2},\ \ldots\ldots,\ X(t_1)=a_1\}$$

$$= P\{X(t_n)=a_n \mid X(t_{n-1})=a_{n-1}\}\quad \forall\ t_1<t_2\cdots<t_n,$$

then the random process is called a "Markov process"

Markov chain

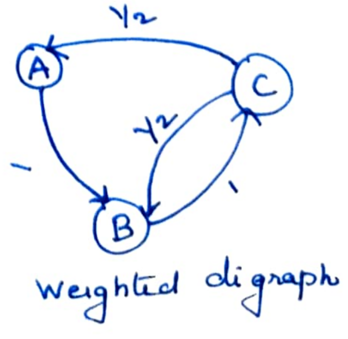
A discrete parameter (time t) markov process is called a Markov chain.

Example

Three children A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C However C is just as likely to throw the ball to B as to A. This is an example of a Markov process (The child throwing the ball is not influenced by those who previously had the ball).

$$S = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

TPM



Weighted digraph

Probability vectors and stochastic Matrices.

A vector $q = [q_1, q_2, \cdots q_n]$ is called a probability vector If its entries are nonnegative and their sum is 1 that is

If ① Each $q_i \geq 0$

(ii) $q_1 + q_2 + \cdots + q_n = 1$

A square matrix $P = [P_{ij}]$ is called a stochastic matrix If each row of P is a probability vector.

S is a stochastic matrix.

Note:- Suppose A and B are stochastic matrices then the product AB is also a stochastic matrix

Regular matrix.

A stochastic matrix P is said to be regular If all the entries of some power P^m of P are positive (all entries >0)

[Note:- P is not regular If '1' occurs in the Principal diagonal]

consider the following matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

Both of them are stochastic Matrices. In particular A is regular since all entries in A^2 are +ve

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$$

B is not regular.

$$B^2 = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 15/16 & 1/16 \end{bmatrix}$$

and every power B^m of B will have 1 and 0 in the first row.

B is not regular.

Accordingly

* A markov chain is said to be irreducible if the associated transition probability matrix is regular

Transition matrix of a Markov Process

A markov process of chain consists of a Sequence of repeated trials of an experiment whose outcomes have the following properties

① Each outcome belongs to a finite set $[a_1, a_2, \dots a_n]$ called the state space of the system. If the outcome on the n^{th} trial is a_i, then the system is in state a_i at time n or at the n^{th} step.

② The outcome of any trial depends, at most, on the outcome of the preceding trial and not on any other previous outcome.

Accordingly with each pair of state (a_i, a_j) there is given the probability p_{ij} that a_j occurs immediately after a_i occurs.

The probabilities p_{ij} form the following n-square matrix

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$$

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Determine which of the following stochastic matrices are regular.

① $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ② $B = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

③ $C = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$ is called

the transition matrix of the Markov process.

With each state a_i there corresponds the i^{th} row $[p_{i1} \; p_{i2} \cdots p_{in}]$ of the transition matrix P.

If the system is in state a_i, this row represents the probabilities of all possible out comes of the next trial and so it is a probability vector.

The transition matrix P of a Markov process is a stochastic matrix.

<u>Example</u> - A man either Takes a bus or drives his car to work each day. Suppose he never Takes the bus 2 days in a row; but If he drives to work, then the next day he is just as likely to drive again as he is to take the bus.

This stochastic process is a Markov chain, since the outcome on any day depends only on what happened the preceding day.

The state space is $\{ b(\text{bus}), d(\text{drive}) \}$

and the transition Matrix is

$$P = \begin{array}{c|cc} & b & d \\ \hline b & 0 & 1 \\ d & 1/2 & 1/2 \end{array}$$

The first row of the matrix P corresponds to the fact that the man never takes the bus 2 days in a row, and so he definitely will drive the day after he takes the bus.

The second row corresponds to the fact that the day after he drives he will drive or take the bus with equal probability.

n step transition probabilities

The probability that a Markov chain will move from state a_i to state a_j in exactly n steps, denoted by $p_{ij}^{(n)} = p_{ij}(n) = P(X_{m+n} = a_j / X_m = a_i)$

ie, $a_i \to a_{k_1} \to a_{k_2} \to a_{k_{n-1}} \to a_j$

Kolmogrov Theorem:

The matrix of n-step transition probability $p^{(n)}$ is obtained by multiplying the matrix of one step transition probability P by itself n times.

ie, $p^{(n)} = p^{(0)} P^n$ [Also $p^{(n)} = p^{(n-1)} P$]

P — Transition matrix.

$p^{(n)}$ — n step transition probability

A students study habit is as follows:
If he studies one night he is 70%
sure not to study the next night.
On the other hand if he does not study
one night he is only 60% sure not
to study the next night as well.
suppose he studied on Monday
what is the probability that he
does not study on the coming
Thursday. Also find how often
in the long run he studies.

$$\text{Let } P = TPM = \begin{array}{c|cc} & S & NS \\ \hline S & 0.3 & 0.7 \\ NS & 0.4 & 0.6 \end{array}$$

suppose he studied on Monday.

$$p^{(0)} = \begin{array}{cc} S & NS \\ \hline 1 & 0 \end{array}$$

$$p^{(3)} = p^{(0)} P^3 \qquad (\text{or } p^{(3)} = p^{(2)} P)$$

$$= [1 \quad 0] \left[\qquad \right]$$

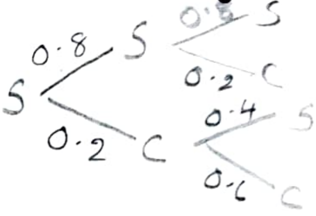
$$[\ \cdot\ ]$$

* A Fixed vector of a tpm P is a probability vector V such that VP = V

It is also called as stationary probability vector

Problems

① In a certain city, If today is sunny, tomorrow will be sunny 80% of the time. If today is cloudy, tomorrow will be cloudy 60% of the time. Supposing today is sunny, what is the probability that it will be cloudy the day after ?



0·8 × 0·2 +
0·8 × 0·6

Solu::

T PM

$$A = \text{current} \begin{array}{c c} & \text{Next} \\ & \begin{array}{cc} S & C \end{array} \\ \begin{array}{c} S \\ C \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

Let the initial vector be (suppose today is sunny)

$$V = \begin{array}{cc} S & C \\ \hline \end{array}$$

$$V = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

For tomorrow

$$VA = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.8 & \overset{v}{0.2} \end{bmatrix}$$

Next day (day after)

$$\begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.72 & 0.28 \end{bmatrix}$$

∴ The probability that It will be cloudy the day after tomorrow = $\underline{\underline{0.28}}$

② Two boys B_1, B_2 and two girls G_1 & G_2 are throwing ball from one to the other. Each boy throws the ball to the other boy with probability $1/2$ and to each girl with probability $1/4$. On the other hand each girl throws the ball to each boy with probability $1/2$ and never to the other girl. In the long run, how often does each receive the ball?

Solu :-

State space $\{B_1, B_2, G_1, G_2\}$ and the associated TPM is

$$P = \begin{array}{c} \\ B_1 \\ B_2 \\ G_1 \\ G_2 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & G_1 & G_2 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} \end{array}$$

To find fix probability vector $V = [a \ b \ c \ d]$ such that $VP = V$

$$[a \quad b \quad c \quad d] \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = [a \quad b \quad c \quad d]$$

$\frac{b+c+d}{2} = a$ — ①, $\frac{a+c+d}{2} = b$ — ②, $\frac{a+b}{4} = c$ — ③, $\frac{a+b}{4} = d$ — ④

also $a+b+c+d = 1$ — ⑤

from ⑤ $b+c+d = 1-a$ — ⑥

substitute ⑥ in ①

$$\frac{1-a}{2} = a$$

$$\Rightarrow 1-a = 2a.$$

$$\boxed{a = 1/3}$$

Also from ⑤ $a+c+d = 1-b$ — ⑦

substitute ⑦ in ② we get

$$\frac{1-b}{2} = b$$

$$\Rightarrow 1-b = 2b$$

$$\boxed{b = 1/3}$$

From ③ $c = \frac{a+b}{4} = \frac{2/3}{4} = 1/6$

ie, $\boxed{c = 1/6}$

and from ④ $d = \frac{a+b}{4} = 1/6$

$$\Rightarrow V = \begin{bmatrix} 1/3 & 1/3 & 1/6 & 1/6 \end{bmatrix}$$

∴ In the long run, every boy receives ball $(1/3 \times 100) = 33.33\%$ of the time and every girl receives $(1/6 \times 100) = 16.6\%$ of the time.

③ P.T the Markov chain whose transition probability matrix is $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducible. Find the corresponding stationary probability vector?

Solu :-

Given $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

All elements of P^2 are non-zero

$\Rightarrow$ P is regular

ie, P is irreducible.

Now to find fixed probability vector $V = [x \;\; y \;\; z]$ such that $VP = V$

ie, $x + y + z = 1$

$\Rightarrow z = 1 - x - y$

$$\therefore \begin{bmatrix} x & y & 1-x-y \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} x & y & 1-x-y \end{bmatrix}$$

$\frac{y}{2} + \frac{1-x-y}{2} = x$ ①, $\frac{2x}{3} + \frac{1-x-y}{2} = y$ ②, $\frac{x}{3} + \frac{y}{2} = 1-x-y$ ③

From ①

$$\cancel{y} + 1 - x - \cancel{y} = 2x$$

$$3x = 1$$

$$\boxed{x = 1/3}$$

From ③

$$\frac{1}{9} + \frac{y}{2} = 1 - \frac{1}{3} - y$$

$$\frac{3}{2}y = 1 - 1/3 - 1/9$$

$$= \frac{9-3-1}{9} = \frac{5}{9}$$

$$\boxed{y = \frac{10}{27}}$$

$$z = 1 - x - y = 1 - \frac{1}{3} - \frac{10}{27}$$

ie, $\boxed{z = \frac{8}{27}}$

$\therefore \quad v = \begin{bmatrix} \frac{1}{3} & \frac{10}{27} & \frac{8}{27} \end{bmatrix}$ is the required Stationary probability vector.

④ Every year a man trades his car for a new car. If he has a Maruti, he trades it for an ambassader, if he has an ambassador, he trades it for a Santro. However if he has a Santro, he is just likely to trade it for a new Santro, or a Maruti or an ambassador. In 2020 he bought his first car which was a Santro

① Find the probability that he has a (a) 2022 Santro b) 2022 Maruti c) 2023 ambassador d) 2023 Santro.

(ii) ~~Find~~ In the long run how often will he have a Santro.

Soln.

Let a_1 : state of having a Maruti car

a_2 : state of having an Ambassador

a_3 : state of having a Santro

The transition matrix is

$$P = \begin{array}{c|ccc} & a_1 & a_2 & a_3 \\ \hline a_1 & 0 & 1 & 0 \\ a_2 & 0 & 0 & 1 \\ a_3 & 1/3 & 1/3 & 1/3 \end{array}$$

In 2020: (Initial state) : $p^{(0)} = [0 \quad 0 \quad 1]$

(i)

a) To find probability that he has 2022 Santro.

To reach 2022, 2-steps

Compute 2-step transition matrix.

$$p^{(2)} = p^{(0)} P^2$$

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}$$

$$\therefore p^{(0)} P^2 = \left[\frac{1}{9} \quad \frac{4}{9} \quad \frac{4}{9}\right]$$

$\therefore$ probability that he has a Santro in 2022 is

$$p_3^{(2)} = \frac{4}{9}$$

b) probability that he has a 2022 Maruti is $p_1^{(2)} = \frac{1}{9}$

c) To reach 2023

3 steps from 2020

or $p^{(3)} = p^{(0)} P^3$

$$p^{(3)} = p^{(0)} P^3 = p^{2} P$$

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} \frac{4}{27} & \frac{7}{27} & \frac{16}{27} \end{bmatrix}$$

∴ probability that he has 2023 ambassador is

$$p_2^{(3)} = \frac{7}{27}$$

d) and ..... he has 2023 Santro is $p_3^{(3)} = \frac{16}{27}$

(ii) To know what happens on the long run, we should find the fixed probability vector v of P

Let $V = [x \quad y \quad z]$ s.t $VP = V$

$$[x \quad y \quad z] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = [x \quad y \quad z]$$

$$\frac{z}{3} = x, \quad x + \frac{z}{3} = y, \quad y + \frac{z}{3} = z$$

$$y = \frac{2z}{3}$$

$x + y + z = 1$

$1 - (x + y) = z$

$$\frac{z}{3} + \frac{2z}{3} + z = 1$$

$$6z = 3$$

$$\boxed{z = 1/2}$$

$x = 1/6$

$y = 1/3$

∴ $V = [1/6, 1/3, 1/2]$

ie, In long run, he has Santro half of the time.

⑤ A gambler's luck follows the pattern, If he wins a game, the probability of winning next game is 0.6. If he losses a game then the probability of winning the next game is 0.3. what is the probability that he wins the 2nd game, If there is an even chance that the gambler wins the first game. In long run, how often he wins the game.

Solu::   Statespace: $\{W, L\}$

TPM

$$P = \begin{array}{c|cc} & \multicolumn{2}{c}{\text{Future}} \\ & W & L \\ \hline W & 0.6 & 0.4 \\ L & 0.3 & 0.7 \\ \text{Present} & & \end{array}$$

Initial state : $p^{(0)} = [0.5 \quad 0.5]$

The probability distribution for the second game is

$$p^{(1)} = p^{(0)} P$$

$$= [0.5 \quad 0.5] \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$$

$$= [\underset{p_1^{(1)}}{0.45} \quad \underset{p_2^{(1)}}{0.55}]$$

∴ The probability that he wins the 2nd game is $p_1^{(1)} = 0.45$

In long run $VP = V$

Let $V = [x \quad y]$

$$\therefore \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$0.6x + 0.3y = x \Rightarrow y = \frac{0.4}{0.3}x \quad — ①$$

$$0.4x + 0.7y = y \quad — ②$$

Also $x + y = 1$ — ③

Substitute ① in ③ $\Rightarrow$

$$x + \frac{0.4}{0.3}x = 1$$

$$\Rightarrow x = \frac{0.3}{0.7} = \frac{3}{7}$$

$$\therefore y = \frac{0.4}{0.7} = \frac{4}{7}$$

$$\therefore V = \begin{bmatrix} 3/7 & 4/7 \end{bmatrix}$$

∴ In the long run he win the game $3/7 \times 100$
$= 42.8\%$ of time