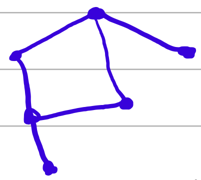
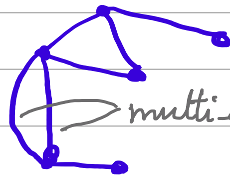
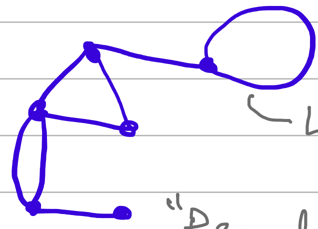
Graph modelling

A graph G consists of a pair (V, E) where V is the set of vertices and E is the set of edges.
nodes points   arcs lines





multiedges



Loop

"Simple graph"
- no multiedges
- no loops.

"Multi-graph"

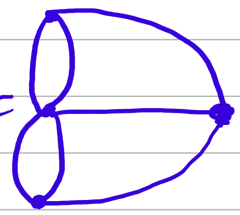
"Pseudo-graph"

- **Konigsberg bridge problem**

· First known problem to be modelled as a "graph"

A

C



D

B

G

"Degree" of a vertex is the number of edges adjacent to it.

$\deg(A) = \deg(B) = \deg(D) = 3, \quad \deg(C) = 5 = \Delta(G)$
↳ max degree

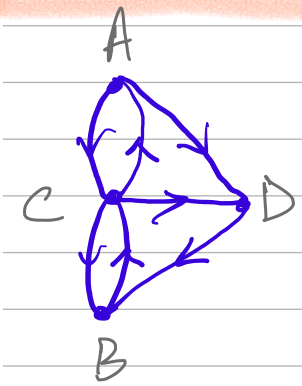
The problem: Start from a vertex (land), pass through every edge (bridge) exactly once and come back to the starting point.

Such a 'walk' is called an 'Euler circuit'.

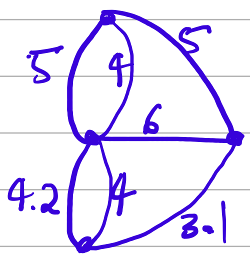
Solution: A graph has Euler circuit iff every vertex is of even degree.

Theorem: $\sum_i \deg(v_i) = 2|E|$

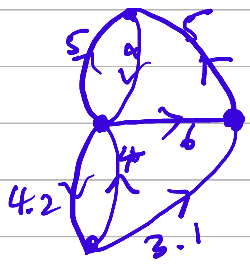
$\because$ every edge is counted twice, once at each end vertex.



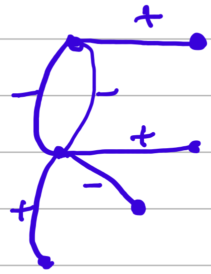
- "directed graph" or "digraph"
- $\deg^+(C) = 3$, $\deg^-(C) = 2$
- $\sum_i \deg^+(v_i) = \sum_i \deg^-(v_i) = |E|$



"weighted graph"

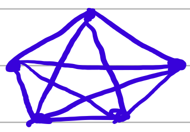


"weighted digraph"



"Signed graph"

- A simple graph in which there is an edge between every pair of vertices is called a "Complete graph" (K_n)
- A graph is said to be "k-regular" if every vertex has the same degree 'k'.

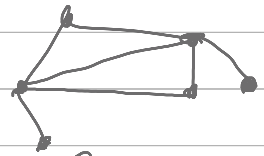


K_5 4-regular

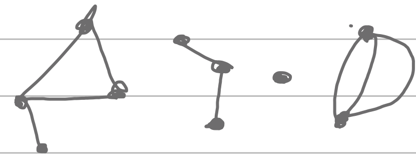
Can we have a 3-regular graph with 5 vertices?

- If a graph G contains a u-v path, then the vertices u & v are said to be connected.

A graph G is "Connected" if every two vertices of G are connected.



G is connected



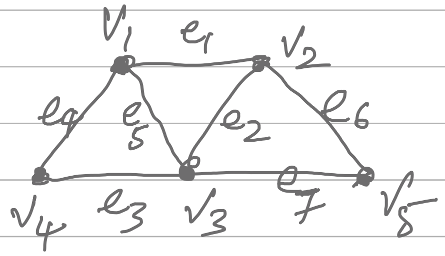
H is disconnected & has 4 "components"

Cut-sets

Let G be a connected graph. A set S of edges of G is said to be a "disconnecting edge set" if the removal of S disconnects G.

A set S of edges of a graph G is called a "cut set" if:

1. S is a disconnecting edge set in G.
2. No proper subset of S is a disconnecting edge set in G.



$S_1 = \{e_1, e_2, e_5\}$ ✗

$S_2 = \{e_1, e_2, e_7, e_5\}$ → disconnecting edge sets

$S_3 = \{e_1, e_2, e_7\}$ → cut-set.

S_3 is said to be a "$v_1 - v_2$ cut" or a "$v_4 - v_5$ cut" etc.

- Suppose G is a weighted graph, then the sum of weights of all the edges in the cut-set S is said to be the "capacity" of S.

- Let the weights of the edges $\{e_1, e_2, \dots, e_7\}$ are respectively $\{2, 1, 1, 3, 3, 4, 2\}$, then the capacity of S_3 is $2+1+2 = 5$.

Network flows

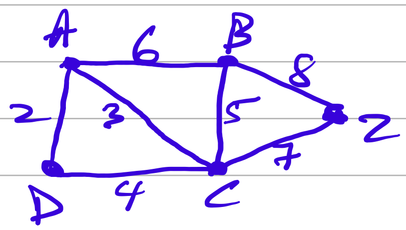
In several networks, like networks of telephone lines, rail roads, oil or water pipelines, a knowledge about the maximum rate of flow that is possible from one station to another is essential. Such networks are modelled using weighted connected graphs where the vertices represent stations and the edges represent the lines of network, and the weight of an edge represent the capacity of a line.

Max-flow min-cut theorem

"The maximum possible flow b/w two vertices A and B in a transport network is equal to the minimum of the capacities of all cut-sets wrt A and B.

Problems:

1. For the network shown below, determine the maximum flow b/w A & Z by identifying a cut-set of minimum capacity.



Sol^n:

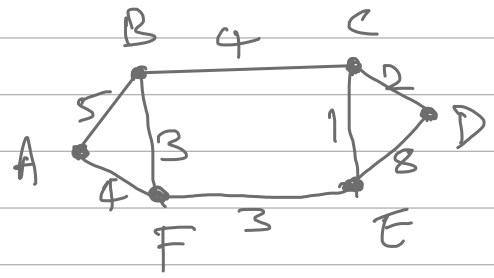
Cut-set	Capacity
{BZ, CZ}	15
{AB, AC, DC}	13
{AB, AC, AD}	**11**
{AB, BC, CZ}	18

The min-cut is {AB, AC, AD} whose capacity is 11.
∴ The maximum-flow from A to Z is 11 units.

2. Enumerate the A-D cuts and determine the maximum flow for the network below.

P.T.O

Solⁿ:

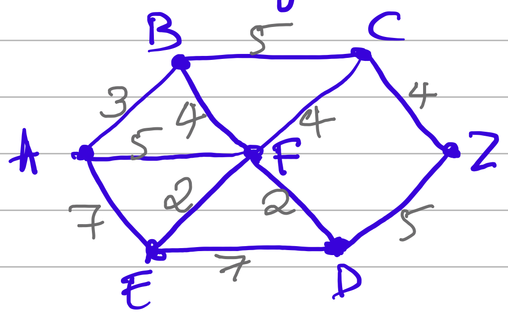


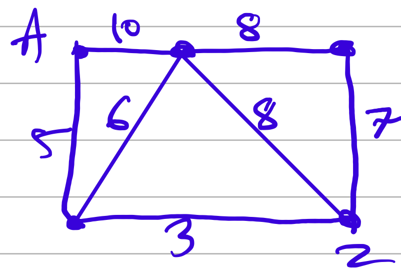
Cut-set	Capacity
{AB, AF}	9
{AB, BF, FE}	11
{AB, BF, CE, ED}	17
{AF, FB, BC}	11
{AF, FB, CE, CD}	10
{BC, FE}	7
{BC, CE, ED}	13
{FE, EC, CD}	6
{CD, ED}	10

Thus, 6 is the minimum capacity & this is possessed by the cut-set {FE, EC, CD}

∴ The maximum flow is 6 units.

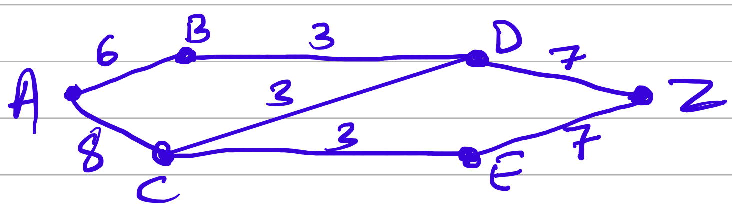
⊗ Find the maximum possible flow in the following networks by enumerating all the cut-sets b/w A & Z.





Ans. 9 units, 15 units

- At the end of the school day, all the students plan on driving from school (vertex A) to the concert (at vertex Z). The edges in the graph below represent one-way roads, and the weight of each edge represents the number of vehicles (in hundreds) that particular road can handle in one hour. What is the greatest number of vehicles that can get from school



Bipartite graph

Suppose a simple graph G is such that its vertex set V is the union of its two mutually disjoint non-empty subsets V_1 & V_2 which are such that every edge in G joins a vertex in V_1 and a vertex in V_2. Then G is called a 'bipartite graph'.

Complete bipartite graph

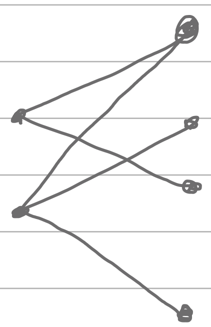
A bipartite graph $G=(V_1, V_2; E)$ is called a complete bipartite graph if there is an edge between every vertex in V_1 and every vertex in V_2.

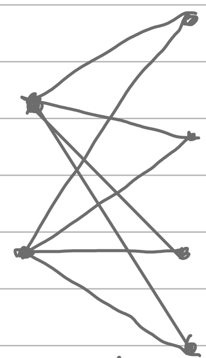
Denoted by $K_{|V_1|,|V_2|}$

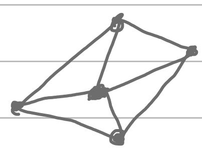
Planar graph

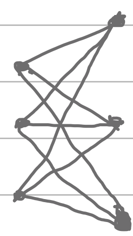
A graph which can be represented by at least one plane drawing (drawing done on a plane surface) in which the edges meet only at the vertices is called a 'planar graph'.

- $|V|$ – "order", $|E|$ – "size" of a graph.


Bipartite graph


Complete bipartite $K_{2,3}$


Planar graph


non-planar graph

Theorem

The graph $K_{3,3}$ ("Kuratowski's second graph) is non-planar.

Graph Coloring

Given a graph G, a 'proper coloring' of G means assigning colours to its vertices such that adjacent vertices have different colors.

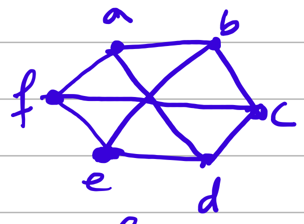
chromatic number

A graph is said to be 'K-colorable' if we can properly color it with K colors.

A graph G which is K-colorable but not K-1 colorable is said to be a 'K-chromatic graph'. Then, K is said to be the chromatic number of G, denoted by $\chi(G)$. (chi)

- $\chi(K_n) = n$
- $\chi(G) \le |V|$

Q Find the chromatic number of the following graphs:

1) 

G

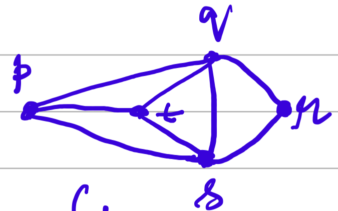
Sol^n: $V_1 = \{a, c, e\}$, $V_2 = \{b, d, f\}$

the vertices of V_1 can have a single color & those in V_2 can have another color.

Thus $\chi(G) = 2$.

2) $K_{m,n}$

Sol^n: let $|V_1| = m$ & $|V_2| = n$ be the partites. all the vertices in each set can have single color as none of them are adjacent with any other vertex of the same partite. $\therefore \chi(K_{m,n}) = 2$.

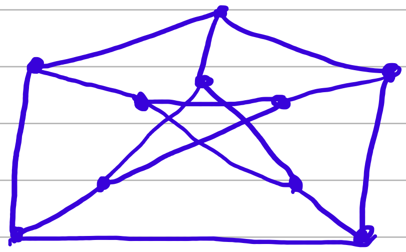
3) 

G_1

Sol^n: $\{p, r\}$, $\{q\}$, $\{s\}$, $\{t\}$

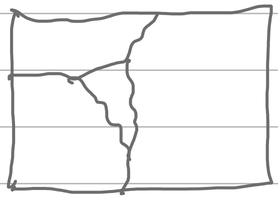
$\therefore \chi(G_1) = 4$.

4) Find the chromatic number of the "Peterson graph":

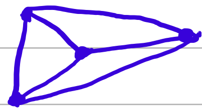


- The four Color theorem
  "No more than four colors are required to color the regions of any map so that no two adjacent regions have the same color"

  i,e. For a loopless planar graph G, $\chi(G) \leq 4$



The map

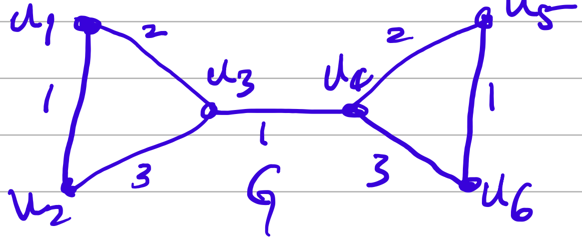


The graph

- Proper Edge coloring: Assigning colors to edges such that no two adjacent edges have same color.

  the 'edge chromatic number' $\chi'(G)$ of a loopless graph G is the least k such that G is k-colorable.

Ⓧ Find the edge chromatic number of the given graph:



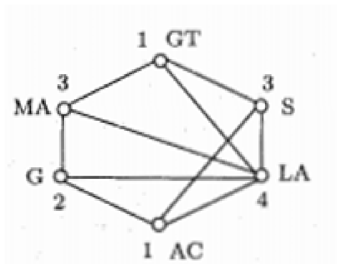
Sol^n: $\chi'(G) = 3$

4 Scheduling problems

Example 5.1: The mathematics department of a certain college plans to schedule the classes Graph Theory (GT), Statistics (s), Liner Algebra (LA), Advanced Calculus (AC), Geometry (G), and Modern Algebra (MA) this summer. Ten students (see below) have indicated the courses they plan to take. With this information, use graph theory to determine the minimum number of time periods needed to offer these courses so that every two classes having student in common are taught at different time periods during the day. Of course, two classes having no students in common can be taught during the same period.

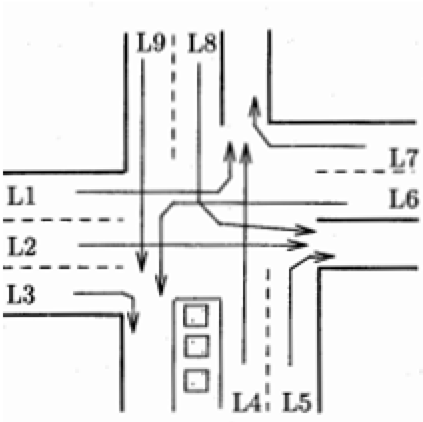
Anden: LA, S	Brynn: MA, LA, G
Chase: MA, G, LA	Denise: G, LA, AC
Everett: AC, LA, S	Francois: G, AC
Greg: GT, MA, LA	Harper: LA, GT, S
Irene: AC, S, LA	Jennie: GT, S

Solution: First, we construct a graph H whose vertices are the six subjects. Two vertices (subjects) are joined by an edge if some student is taking classes in these two subjects(see Figure). The minimum number of time periods is $\chi(H) = 4$.

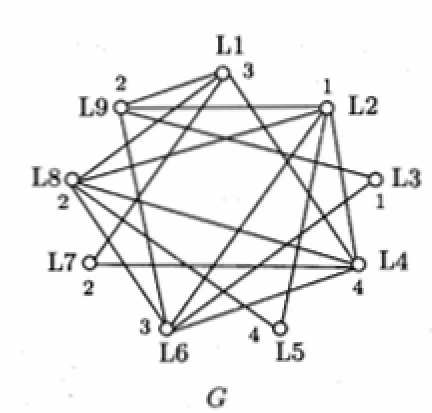


P.T.O

Example 5.2: The following figure shows the traffic lanes $L1, L2, ..., L9$, at the intersection of two busy streets. A traffic light is located at this intersection. During a certain phase of the traffic light, those cars in lanes for which the light is green may proceed safely through the intersection. What is the minimum number of phase needed for the traffic light so that all cars may proceed trough the intersection?



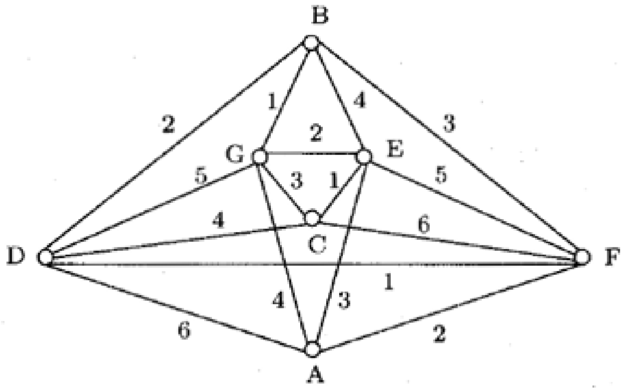
Solution: Construct a graph G to model this situation, where $V(G) = \{L1, L2, ..., L9\}$ and two vertices (lanes) are joined by a edge if vehicles in these two lanes cannot safely enter the intersection at the same time, as there is a possibility of an accident. Answering this question require determining the chromatic number of the graph. First notice that $\langle\{L2, L4, L6, L8\}\rangle \cong K_4$. Since there exists a 4-coloring of G, as indicated in the graph, therefore $\chi(G) = 4$.



P.T.O

Example 5.3: Alvin (A) has invited three married couples to his summer house for a week: Bob(B) and Carrie (C) Hanson, David (D) and Edith (E) Irwin, and Frank (F) and Gena (G) Jackson. Since all six guest enjoy playing tennis match against every other guest except his/her spouse. In addition, Alvin is to play a match against each of David, Edith, Frank, and Gena. If no one is to play two match on the same day, what is a schedule of matches over the smallest number of days.

Solution: First, we construct a graph H whose vertices are the people at Alvin's summer house, so $V(G) = \{A, B, C, D, E, F, G\}$, and two vertices of H are adjacent if the two vertices (people) are to play a tennis match. (The graph H is shown in the below graph). To answer the question, we determine the edge chromatic number of H. The edge chromatic number of the graph H is $\chi^{'}(H) = 6$.

$H:$ 

The above graph gives a 6-edge coloring of H, which provides a schedule of matches.

Day 1:	Bob-Gena, Carrie-Edith, David-Frank
Day 2:	Alvin-Frank, Bob-David, Edith-Gena
Day 3:	Alvin-Edith, Bob-Frank, Carrie-Gena
Day 4:	Alvin-Gena, Bob-Edith, Carrie-David
Day 5:	David-Gena, Edith-Frank
Day 6:	Alvin-David, Carrie-Frank

Mathematical Modelling Through Graphs

7.1 SITUATIONS THAT CAN BE MODELLED THROUGH GRAPHS

7.1.1 Qualitative Relations in Applied Mathematics

It has been stated that "Applied Mathematics is nothing but solution of differential equations". This statement is wrong on many counts: (*i*) Applied Mathematics also deals with solutions of difference, differential-difference, integral, integro-differential, functional and algebraic equations (*ii*) Applied Mathematics is equally concerned with inequations of all types (*iii*) Applied Mathematics is also concerned with mathematical modelling; in fact mathematical modelling has to precede solution of equations (*iv*) Applied Mathematics also deals with situations which cannot be modelled in terms of equations or inequations; one such set of situations is concerned with qualitative relations.

Mathematics deals with both quantitative and qualitative relationships. Typical qualitative relations are: y likes x, y hates x, y is superior to x, y is subordinate to x, y belongs to same political party as x, set y has a non-null intersection with set x; point y is joined to point x by a road, state y can be transformed into state x, team y has defeated team x, y is father of x, course y is a prerequisite for course x, operation y has to be done before operation x, species y eats species x, y and x are connected by an airline, y has a healthy influence on x, any increase of y leads to a decrease in x, y belongs to same caste as x, y and x have different nationalities and so on.

Such relationships are very conveniently represented by graphs where a graph consists of a set of vertices and edges joining some or all pairs of these vertices. To motivate the typical problem situations which can be modelled through graphs, we consider the first problem so historically modelled *viz.* the problem of seven bridges of Konigsberg.

7.1.2 The Seven Bridges Problem

There are four land masses A, B, C, D which are connected by seven bridges numbered 1 to 7 across a river (Figure 7.1). The problem is to start from any point in one of the land masses, cover each of the seven bridges once and once only and return to the starting point.

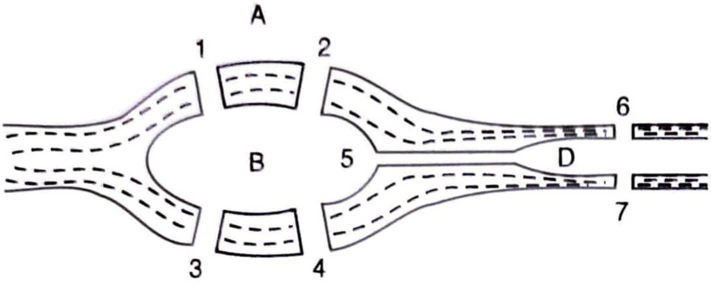


Figure 7.1

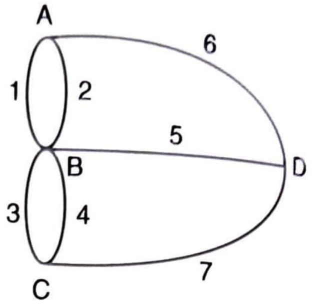


Figure 7.2

There are two ways of attacking this problem. One method is to try to solve the problem by walking over the bridges. Hundreds of people tried to do so in their evening walks and failed to find a path satisfying the conditions of the problem. A second method is to draw a scale map of the bridges on paper and try to find a path by using a pencil.

It is at this stage that concepts of mathematical modelling are useful. It is obvious that the sizes of the land masses are unimportant, the lengths of the bridges or whether these are straight or curved are irrelevant. What is relevant information is that A and B are connected by two bridges 1 and 2, B and C are connected by two bridges 3 and 4, B and D are connected by one bridge number 5, A and D are connected by bridge number 6 and C and D are connected by bridge number 7. All these facts are represented by the graph with four vertices and seven edges in Figure 7.2. If we can trace this graph in such a way that we start with any vertex and return to the same vertex and trace every edge once and once only without lifting the pencil from the paper, the problem can be solved. Again trial and error method cannot be satisfactorily used to show that no solution is possible.

The number of edges meeting at a vertex is called the degree of that vertex. We note that the degrees of A, B, C, D are 3, 5, 3, 3 respectively and each of these is an odd number. If we have to start from a vertex and return to it, we need an even number of edges at that vertex. Thus it is easily seen that Konigsberg bridges problem cannot be solved.

This example also illustrates the power of mathematical modelling. We have not only disposed of the seven-bridges problem, but we have discovered a technique for solving many problems of the same type.

7.1.3 Some Types of Graphs

A graph is called *complete* if every pair of its vertices is joined by an edge (Figure 7.3 (*a*)).

A graph is called a *directed graph* or a *digraph* if every edge is directed with an arrow. The edge joining A and B may be directed from A to B or from B to A. If an edge is left undirected in a digraph, it will be assumed to be directed both ways (Figure 7.3 (b)).

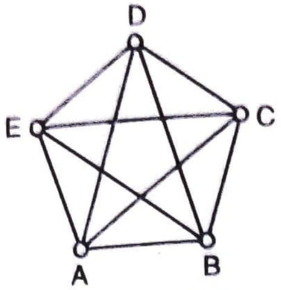


Figure 7.3a

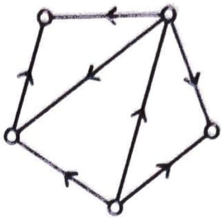


Figure 7.3b

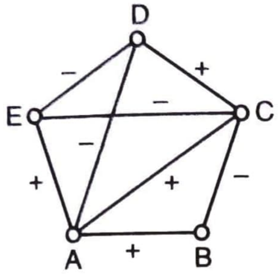


Figure 7.3c

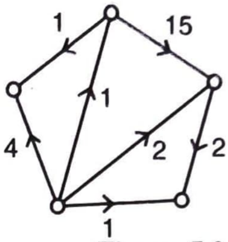


Figure 7.3d

A graph is called a *signed graph* if every edge has either a plus or minus sign associated with it (Figure 7.3(c)).

A digraph is called a *weighted digraph* if every directed edge has a weight (giving the importance of the edge) associated with it (Figure 7.3 (d)). We may also have digraphs with positive and negative numbers associated with edges. These will be called *weighted signed digraphs*.

7.1.4 Nature of Models in Terms of Graphs

In all the applications we shall consider, the length of the edge joining two vertices will not be relevant. It will not also be relevant whether the edge is straight or curved. The relevant facts would be: (a) which edges are joined; (b) which edges are directed and in which direction(s); (c) which edges have positive or negative signs associated with them; (d) which edges have weights associated with them and what these weights are.

EXERCISE 7.1

1. In the Konigsberg problem suggest deletion or addition of minimum number of bridges which may lead to a solution of the problem.
2. Show that in any graph, the sum of local degrees of all the vertices is an even number. Deduce that a graph has an even number of odd vertices.
3. Three houses A, B, C have to be connected with three utilities a, b, c by separate wires lying in the same plane and not crossing one another. Explain why this is not possible.

4. Each of the four neighbours has connected his house with the other three houses by paths which do not cross. A fifth man builds a house nearby. Prove that (*a*) he cannot connect his house with all others by non-intersecting paths (*b*) he can however connect with three of the houses.

5. A graph is called regular if each of its vertices has same degree r. Draw regular graphs with 6 vertices and degree 5, 4 and 3.

6. Show that in Konigsberg, four one-way bridges will be enough to connect the four land masses.

7.2 MATHEMATICAL MODELS IN TERMS OF DIRECTED GRAPHS

7.2.1 Representing Results of Tournaments

The graph (Figure 7.4) shows that:

(*i*) Team *A* has defeated teams *B, C, E*.

(*ii*) Team *B* has defeated teams *C, E*.

(*iii*) Team *E* has defeated D.

(*iv*) Matches between *A* and *D, B* and *D, C* and *D* and *C* and *E* have yet to be played.

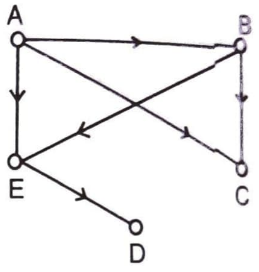


Figure 7.4

7.2.2 One-Way Traffic Problems

The road map of a city can be represented by a directed graph. If only one-way traffic is allowed from point *a* to point *b*, we draw an edge directed from *a* to *b*. If traffic is allowed both ways, we can either draw two edges, one directed from *a* to *b* and the other directed from *b* to *a* or simply draw an undirected edge between *a* and *b*. The problem is to find whether we can introduce one-way traffic on some or all of the roads without preventing persons from going from any point of the city to any other point. In other words, we have to find when the edges of a graph can be given direction in such a way that there is a directed path from any vertex to every other. It is easily seen that one-way traffic on the road *DE* cannot be introduced without disconnecting the vertices of the graph (Figure 7.5).

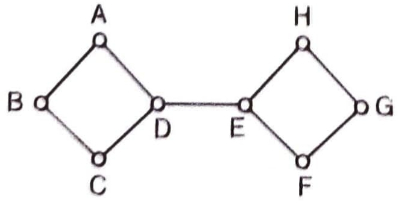


Figure 7.5 (a)

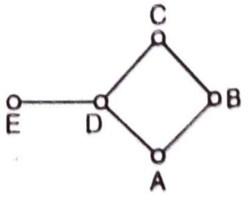


Figure 7.5 (b)

In Figure 7.5(*a*), *DE* can be regarded as a bridge connecting two regions of the town. In Figure 7.5(*b*) *DE* can be regarded as a blind street on which a two-way traffic is necessary. Edges like *DE* are called *separating edges*, while other edges are called *circuit edges*. It is necessary that on separating edges, two-way traffic should be permitted. It can also be shown that this is sufficient. In other words, the following theorem can be established:

If G is an undirected connected graph, then one can always direct the circuit edges of G and leave the separating edges undirected (or both way directed) so that there is a directed path from any given vertex to any other vertex.

7.2.3 Genetic Graphs

In a genetic graph, we draw a directed edge from A to B to indicate that B is the child of A. In general each vertex will have two incoming edges, one from the vertex representing the father and the other from the vertex representing the mother. If the father or mother is unknown, there may be less than two incoming edges. Thus in a genetic graph, the local degree of incoming edges at each vertex must be less than or equal to two. This is a necessary condition for a directed graph to be a genetic graph, but it is not a sufficient condition. Thus Figure 7.6 does not give a genetic graph inspite of the fact that the number of incoming edges at each vertex does not exceed two. Suppose A_1 is male, then A_2 must be female, since A_1, A_2 have a child B_1. Then A_3 must be male, since A_2, A_3 have a child B_2. Now A_1, A_3 being both males cannot have a child B_3.

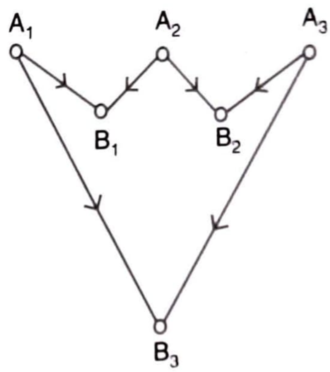


Figure 7.6

7.2.4 Senior-Subordinate Relationship

If a is senior to b, we write aSb and draw a directed edge from a to b. Thus the organisational structure of a group may be represented by a graph like the following [Figure 7.7].

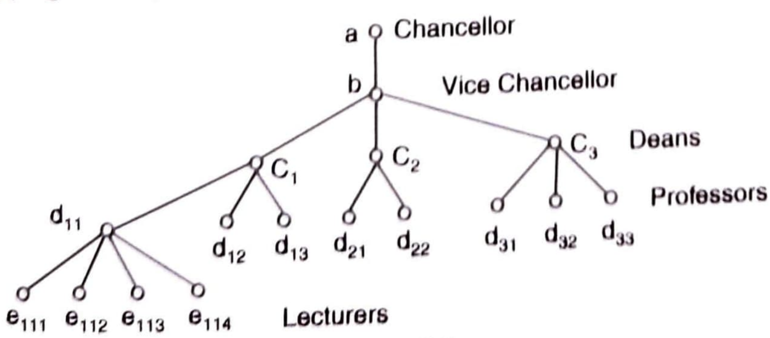


Figure 7.7

The relationship S satisfies the following properties:

(*i*) $\sim(aSa)$ i.e. no one is his own senior.

(*ii*) $aSb = \sim(bSa)$ i.e. a is senior to b implies that b is not senior to a.

(*iii*) aSb, $bSc \Rightarrow aSc$ i.e. if a is senior to b and b is senior to c, then a is senior to c.

The following theorem can easily be proved: "The necessary and sufficient condition that the above three requirements hold is that the graph of an organisation should be free of cycles".

We want now to develop a *measure for the status* of each person. The status $m(x)$ of the individual should satisfy the following reasonable requirements:

(*i*) $m(x)$ is always a whole number.

(*ii*) If x has no subordinate, $m(x) = 0$.

(*iii*) If, without otherwise changing the structure, we add a new individual subordinate to x, then $m(x)$ increases.

(*iv*) If, without otherwise changing the structure, we move a subordinate of a to a lower level relative to x, then $m(x)$ increases.

A measure satisfying all these criteria was proposed by Harary. We define the level of seniority of x over y as the length of the shortest path from x to y. To find the measure of status of x, we find n_1, the number of individuals who are one level below x, n_2 the number of individuals who are two levels below x and in general, we find n_k the number of individuals who are k levels below x. Then the Harary measure $h(x)$ is defined by

$$h(x) = \sum_k k n_k \tag{1}$$

It can be shown that among all the measure which satisfy the four requirements given above, Harary measure is the least.

If however, we define the level of seniority of x over y as the length of the longest path from x to y, and then find $H(x) = \sum_k k n_k$, we get another measure which will be the largest among all measures satisfying the four requirements. For Figure 7.8, we get

$h(a) = 1.2 + 4.2 + 2.3 = 16$

$h(b) = 1.3 + 2.4 = 11$

$h(c) = 1.2 + 1.2 = 4$

$H(a) = 1.1 + 3.2 + 2.3 + 2.4 = 21$

$H(b) = 2.1 + 2.2 + 2.3 + 1.4 = 16$

$H(c) = 1.1 + 1.2 + 1.3 = 6$

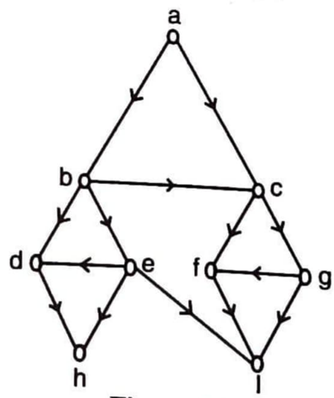


Figure 7.8

$h(d) = 1.1$	$= 1$	$H(d) = 1.1$	$= 1$
$h(e) = 1.3$	$= 3$	$H(e) = 1.2 + 2.1$	$= 4$
$h(f) = 1.1$	$= 1$	$H(f) = 1.1$	$= 1$
$h(g) = 1.2$	$= 2$	$H(g) = 1.2$	$= 2$
$h(k)$	$= 0$	$H(k)$	$= 0$
$h(l)$	$= 0$	$H(l)$	$= 0$

7.2.5 Food Webs

Here aSb if a eats b and we draw a directed edge from a to b. Here also $\sim(aSa)$ and $aSb \Rightarrow \sim(bSa)$. However the transitive law need not hold. Thus consider the food web in Fig. 7.9. Here fox eats bird, bird eats grass, but fox does not eat grass.

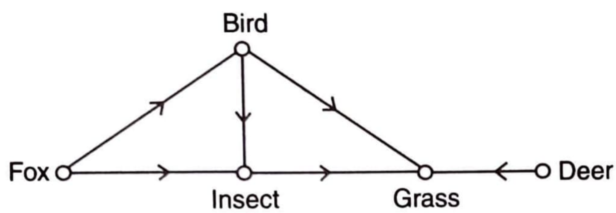


Figure 7.9

We can however calculate measure of the status of each species in this food web by using Eqn. (1) h(bird) = 2, h(fox) = 4, h(insect) = 1, h(grass) = 0, h(deer) = 1.

7.2.6 Communication Networks

A directed graph can serve as a model for a communication network. Thus consider the network given in Figure 7.10. If an edge is directed from a to b, it means that a can communicate with b. In the given network e can communicate directly with b, but b can communicate with e only indirectly through c and d. However every individual can communicate with every other individual.

Our problem is to determine the importance of each individual in this network. The importance can be measured by the fraction of the messages on an average that pass through him. In the absence of any other knowledge, we can assume that if an individual can send message direct to n individuals, he will send a message to any one of them with probability $1/n$. In the present example, the communication probability matrix is:

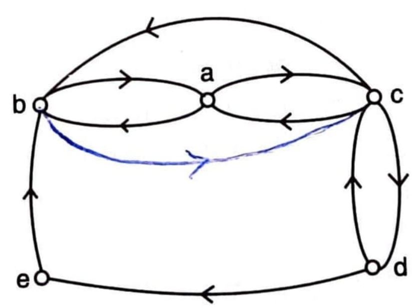


Figure 7.10

edge from i to j. If i knows j and j knows i, then we have a symmetrical relation between i and j.

With this interpretation, the graph of Figure 7.11 shows that persons 1, 2, 3 form a clique. With very small groups, we can find cliques by carefully observing the corresponding graphs. For larger groups analytical methods based on the following results are useful: (*i*) i is a member of a clique if the ith diagonal element of S^3 is different from zero. (*ii*) If there is only one clique of k members in the group, the corresponding k elements of S^3 will be $(k-1)(k-2)/2$ and the rest of the diagonal elements will be zero. (*iii*) If there are only two cliques with k and m members respectively and there is no element common to these cliques, then k elements of S^3 will be $(k-1)(k-2)/2$, m elements of S^3 will be $(m-1)(m-2)/2$ and the rest of the elements will be zero. (*iv*) If there are m disjoint cliques with $k_1, k_2, ..., k_m$ members, then the trace of S^3 is $\frac{1}{2}\sum_{i-1}^{m} k_i(k_i-1)(k_i-2)$. (v) A member is non-cliquical if only if the corresponding row and column of $S^2 \times S$ consists entirely of zeros.

EXERCISE 7.2

1. Show that the graph of Figure 7.12 is a possible genetic graph if and only if n is even.

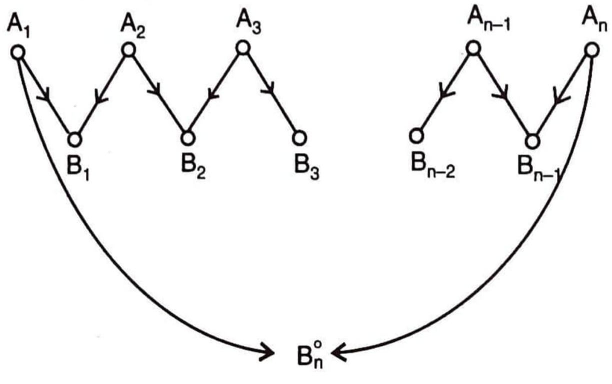


Figure 7.12

2. For each of the following communication networks, set up the corresponding transition probability matrix and find the importance of each member in the network.

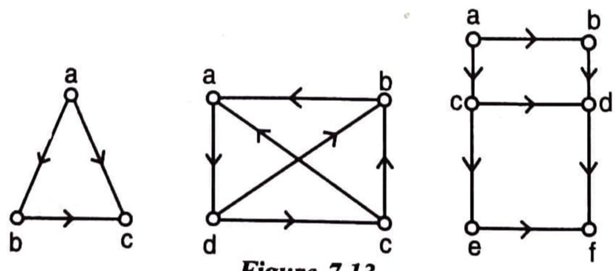


Figure 7.13

3. An intelligence officer can communicate with each of his n subordinates and each subordinate can communicate with him, but the subordinates

cannot communicate among themselves. Draw the graph and find the importance of each subordinate relative to the officer.

4. Find the Harary measure for each individual in the organisational graphs of Figure 7.14.

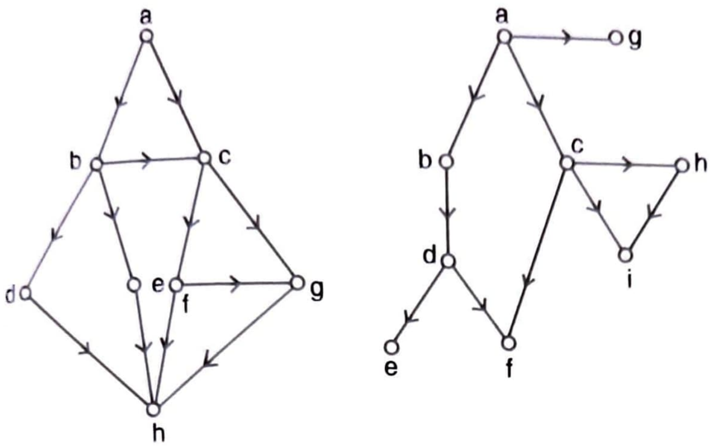


Figure 7.14

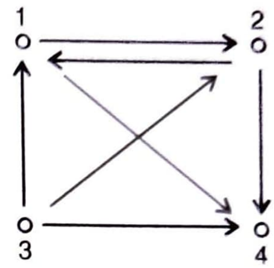


Figure 7.15

5. In Exercise 4, find the measure if the definition of level is based (*i*) on the longest number of steps between two persons (*ii*) on the average of the shortest and longest number of steps between two persons.
6. Find the eigenvector corresponding to the unit eigenvalue of matrix (2).
7. Prove all the theorems stated in Section 7.2.7.
8. Prove all the theorems stated in Section 7.2.8.
9. Write the matrix A associated with the graph of Figure 7.15. Find A^2, A^3, A^4, S, S^2, S^3, and verify the theorems of Sections 7.2.7 and 7.2.8.
10. Enumerate all possible four-cliques.

7.3 MATHEMATICAL MODELS IN TERMS OF SIGNED GRAPHS

7.3.1 Balance of Signed Graphs

A signed (or an algebraic) graph is one in which every edge has a positive or negative sign associated with it. Thus the four graphs of Figure 7.16 are signed graphs. Let positive sign denote friendship and negative sign denote enmity, then in graph (*i*) A is a friend of both B and C and B and C are also friends. In graph (*ii*) A is friend of B and A and B are both jointly enemies of C. In graph (*iii*), A is a friend of both B and C, but B and C are enemies. In graph (*iv*) A is an enemy of both B and C, but B and C are not friends.

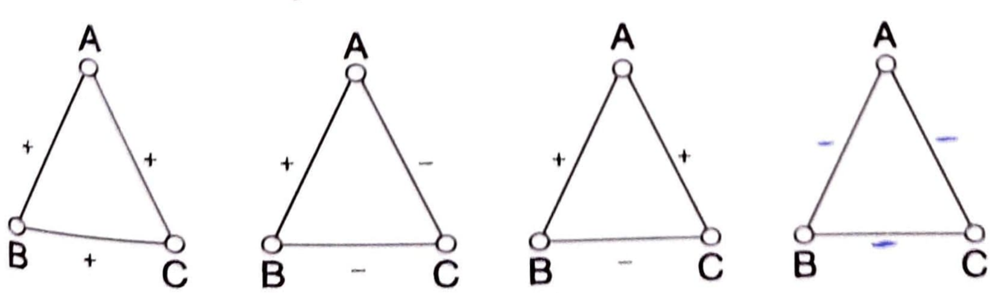


Figure 7.16

The first two graphs represents normal behaviour and are said to be balanced, while the last two graphs represent unbalanced situations since if A is a friend of both B and C and B and C are enemies, this creates a tension in the system and there is a similar tension when B and C have a common enemy A, but are not friends of each other.

We define the sign of a cycle as the product of the signs of component edges. We find that in the two balanced cases, this sign is positive and in the two unbalanced cases, this is negative.

We say that a cycle of length three or a triangle is balanced if and only if its sign is positive. A complete algebraic graph is defined to be a complete graph such that between any two edges of it, there is a positive or negative sign. A complete algebraic graph is said to be balanced if all its triangles are balanced. An alternative definition states that a complete algebraic graph is balanced if all its cycles are positive. It can be shown that the two definitions are equivalent.

A graph is locally balanced at a point a if all the cycles passing through a are balanced. If a graph is locally balanced at all points of the graph, it will obviously be balanced. A graph is defined to be m-balanced if all its cycles of length m are positive. For an incomplete graph, it is preferable to define it to be balanced if all its cycles are positive. The definition in terms of triangle is not satisfactory, as there may be no triangles in the graph.

7.3.4 The Degree of Unbalance of a Graph

For many purposes it is not enough to know that a situation is unbalanced. We may be interested in the degree of unbalance and the possibility of a balancing process which may enable one to pass from an unbalanced to a balanced graph. The possibility is interesting as it can give an approach to group dynamics and demonstrate that methods of graph theory can be applied to dynamic situations also.

Cartwright and Harary define the degree of balance of a group G to be the ratio of the positive cycles of G to the total number of cycles in G. This balance index obviously lies between 0 and 1. G_1 has six negative triangles *viz* (abc), (ade), (bcd), (bce), (bde), (cde) and has four positive triangles. G_2 has four negative triangles *viz* (abc), (abd), (bce) and (bde) and six positive triangles. The degree of balance of G_1 is therefore less than the degree of balance of G_2.

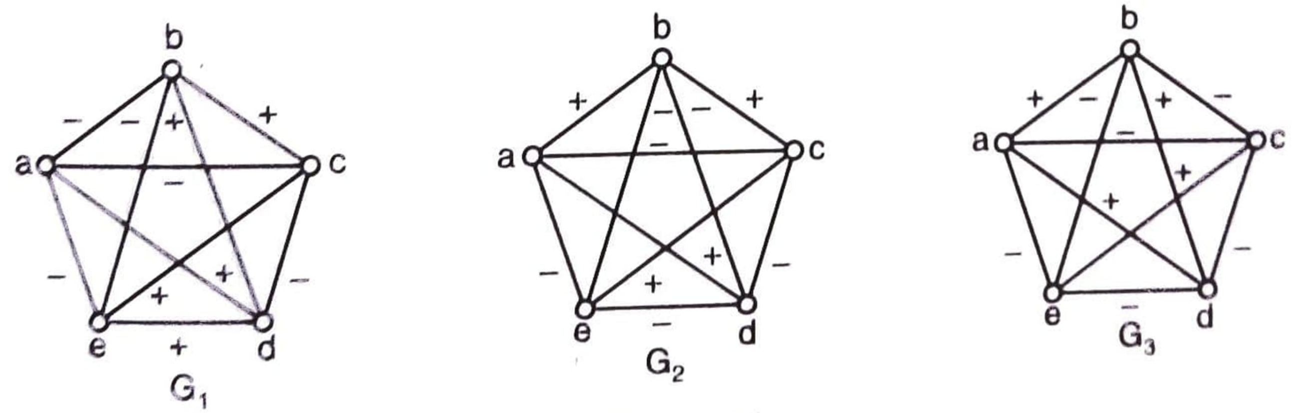


Figure 7.17

However in order to get a balanced graph from G_1, we have to change the sign of only two edges *viz.* bc and de and similarly to make G_2 balanced we have to change the signs of two edges *viz* bc and bd. From this point of view both G_1 and G_2 are equally unbalanced.

Abelson and Rosenberg therefore gave an alternative definition. They defined the degree of unbalance of an algebraic graph as the number of the smallest set of edges of G whose change of sign produces a balanced graph.

The degree of an antibalanced complete algebraic graph (*i.e.*, of a graph all of whose triangles are negative) is given by $[n(n-2)+k]/4$ where $k = 1$

if n is odd and $k = 0$ if n is even. It has been conjectured that the degree of unbalancing of every other complete algebraic graph is less than or equal to this value.

EXERCISE 7.3

1. State which of the following graphs are balanced. If balanced, find the decomposition guaranteed by the structure theorem. If unbalanced, find the degree of unbalance.

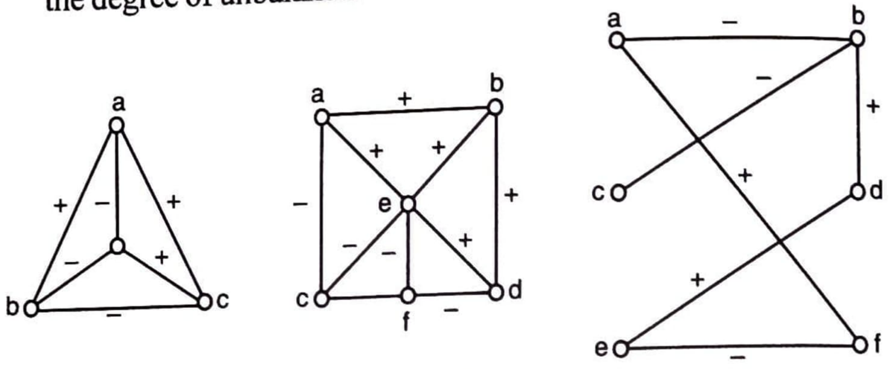


Figure 7.18

2. Draw some antibalanced graphs and verify the structure theorems for them.
3. The adjacency matrix of a signed graph is defined as follows:

   $a_{ij} = 1$ if there is + sign associated with edge i, j

   $= -1$ if there is – sign associated with edge i, j

   $= 0$ if there is no edge i, j.

   Write the adjacency matrices of the four signed graphs is Figure 7.18.
4. A signed graph G is said to have an idealised party structure if the vertices of G can be partitioned into classes so that all edges joining the vertices in the same class have + sign and all edges joining vertices in different sets have negative sign (*a*) Give an example of a signed graph which does not have an idealised party structure (*b*) Give an example of a graph which is not balanced but which has an idealised party structure.
5. Show that a signed graph has an idealised party structure if and only no circuit has exactly one – sign.
6. Show that if all cycles of a signed graph are positive, then all its cycles are also positive. State and prove its converse also.

7.4 MATHEMATICAL MODELLING IN TERMS OF WEIGHTED DIGRAPHS

7.4.1 Communication Networks with Known Probabilities of Communication

In the communication graph of Figure 7.10, we know that a can communicate with both b and c only and in the absence of any other knowledge, we assigned at vertex 5 at time $t+1$. Similarly a change of 1 unit at vertex 2 causes a change of -3 units at 3 vertex, 4 units at vertex 4 and of 2 units at vertex 5 and so on. Given the values at all vertices at time t, we can find the values at time $t+1$, $t+2$, $t+3$, ... The process of doing this systematically is known as the pulse rule.

These general weighted digraphs are useful for representing energy flows, monetary flows and changes in environmental conditions.

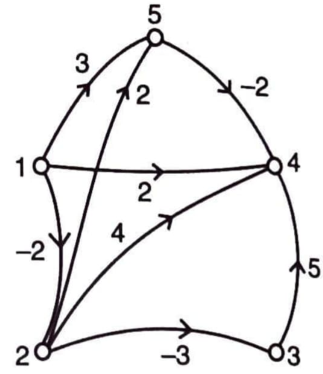


Figure 7.21

7.4.5 Signal Flow Graphs

The system of algebraic equations

$$\begin{aligned} x_1 &= 4y_0 + 6x_2 - 2x_3 \\ x_2 &= 2y_0 - 2x_1 + 2x_3 \\ x_3 &= 2x_1 - 2x_2 \end{aligned} \tag{14}$$

can be represented by the weighted digraph in Figure 7.22. For solving for x_1, we successively eliminate x_3 and x_2 to get the graphs in Figure 7.23 and finally we get

$$x_1 = 4y_0$$

We can similarly represent the solution of any number of linear equations graphically.

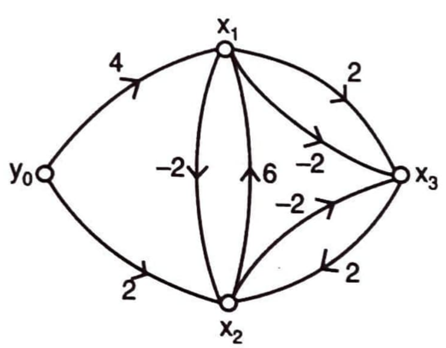


Figure 7.22

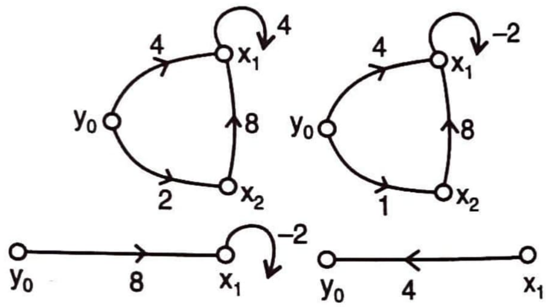


Figure 7.23

7.4.6 Weighted Bipartitic Digraphs and Difference Equations

Consider the system of difference equations

$$\begin{aligned} x_{t+1} &= a_{11}x_t + a_{12}y_t + a_{13}z_t \\ y_{t+1} &= a_{21}x_t + a_{22}y_t + a_{23}z_t \\ z_{t+1} &= a_{31}x_t + a_{32}y_t + a_{33}z_t \end{aligned} \tag{15}$$

This can be represented by a weighted bipartitic digraph (Figure 7.24). The weights can be positive or negative.