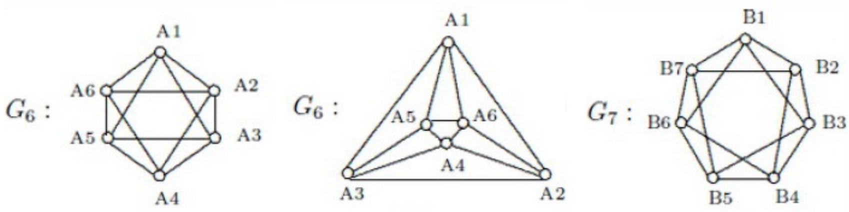
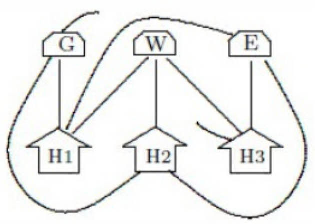
GRAPH THEORY III

A graph G is called a **planar graph** if G can be drawn in the plane so that no two of its edges cross each other. A graph that is not planar is called **nonplanar**. A graph G is called a **plane graph** if it is drawn in the plane so that no two edges of G cross.

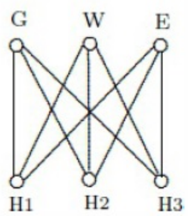


Puzzle:

There are three utilities (gas, water and electricity) that need to be connected to three houses by gas lines, water mains and electrical lines. Can this be done without any of the lines or mains crossing each other? This situation is shown in the below figure.



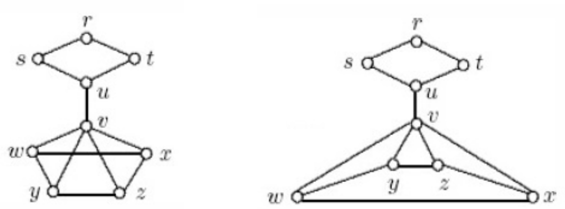
This problem is referred to as the **Three Houses and Three Utilities Problem**. The situation described in this problem can be modeled by the graph of the below figure, which, in fact, is the graph $K_{3,3}$.



In graph theory terms then, the Three Houses and Three Utilities Problem asks whether the graph $K_{3,3}$ is planar.

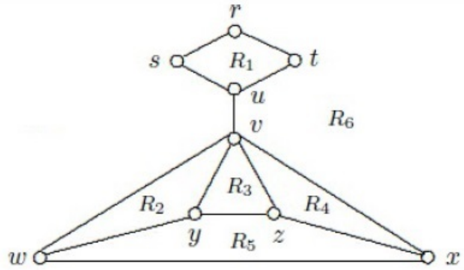
There are some well-known classes of planar graphs. Every cycle is planar. Every path and every star are planar. Indeed, every tree is planar. Of course, every graph that can be drawn in the plane without any two of its edges crossing is planar.

Consider the first graph shown in the below figure.

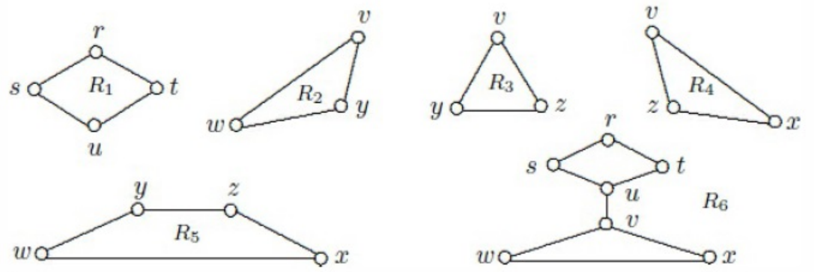


The graph is connected. But it is also planar, as we can see from the second graph, where it is drawn as a plane graph.

A plane graph divides the plane into connected pieces called **regions**. For example, in the case of the plane graph of the above figure, there are six regions. This graph is redrawn below, where the six regions are denoted by $R_1, R_2, ..., R_6$.



In every plane graph, there is always one region that is unbounded. This is the **exterior region**. For the graph of the below figure, R_6 is the exterior region. The subgraph of a plane graph whose vertices and edges are incident with a given region R is the **boundary** of R. The boundaries of the six regions of the above graph are also shown in the below figure.

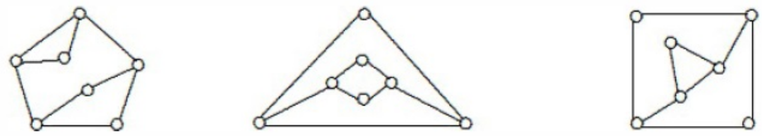


Notice that uv is a bridge in the above graph and is on the boundary of one region only, namely the exterior region. In fact, a bridge is always on the boundary of exactly one region. An edge that is not a bridge lies on the boundary of two regions. For example, vy lies on the the boundary of both R_2 and R_3. If we were to remove the edge vy, then the resulting graph is a plane graph as well but has one less region as R_2 and R_3 become part of a single region. On the other hand, the graph $G - uv$ is disconnected but there is no change in the number of regions.

Note: If G is a connected plane graph with at least three edges, then the boundary of every region of G has at least three edges.

Theorem: (The Euler Identity) If G is a connected plane graph of order n, size m and having r regions, then $n - m + r = 2$.
Proof: First, if G is a tree of order n, then $m = n - 1$ and $r = 1$; so $n - m + r = 2$. Therefore, we need only be concerned with connected graphs that are not trees.
Assume, to the contrary, that the theorem does not hold. Then there exists a connected plane graph G of smallest size for which the Euler Identity does not hold. Suppose that G has order n, size m and r regions. So $n - m + r \neq 2$. Since G is not a tree, there is an edge e that is not a bridge. Thus $G - e$ is a connected plane graph of order n and size $m - 1$ having $r - 1$ regions. Because the size of $G - e$ is less than m, the Euler Identity holds for $G - e$. So $n - (m - 1) + (r - 1) = 2$ but then $n - m + r = 2$, which is a contradiction.

The below figure shows a planar graph G and several ways of drawing G as a plane graph.



However, since G has a fixed order $n = 7$ and fixed size $m = 9$ and the Euler Identity holds ($n - m + r = 7 - 9 + r = 2$), each drawing of G as a plane graph always produces the same number of regions, namely $r = 4$.

Theorem: If G is a planar graph of order $n \geq 3$ and size m, then $m \leq 3n - 6$.
Proof: First, suppose that G is connected. If $G = P_3$, then the inequality holds. So we can assume that G has at least three edges. Draw G as a plane graph, where G has r regions denoted by $R_1, R_2, ..., R_r$. The boundary of each region contains at least three edges. So if m_i is the number of edges on the boundary of $R_i (1 \leq i \leq r)$, then $m_i \geq 3$. Let $M = \sum_{i=1}^{r} m_i \geq 3r$. The number M counts an edge once if the edge is a bridge and counts it twice if the edge is not a bridge. So $M \leq 2m$. Therefore, $3r \leq M \leq 2m$ and so $3r \leq 2m$. Applying the Euler Identity to G, we have $6 = 3n - 3m + 3r \leq 3n - 3m + 2m = 3n - m$. Solving the above inequality for m, we get $m \leq 3n - 6$.
If G is disconnected, then edges can be added to G to produce a connected plane graph of order n and size m', where $m' > m$. From what we have just shown, $m' \leq 3n - 6$ and so $m < 3n - 6$.

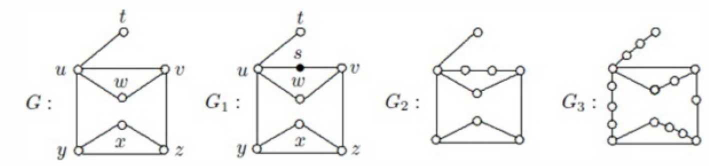
The above theorem provides a necessary condition for a graph to be planar and so provides a sufficient condition for a graph to be nonplanar. In particular, the contrapositive of the above Theorem gives us the following:
If G is a graph of order $n \geq 3$ and size m such that $m > 3n - 6$, then G is nonplanar.

Corollary: Every planar graph contains a vertex of degree 5 or less.
Proof: Suppose that G is a graph, every vertex of which has degree 6 or more. Let G have order n and size m. Certainly, $n \geq 7$. Then $2m = \sum_{v \in V(G)} deg\ v \geq 6n$. Thus $m \geq 3n > 3n - 6$. Hence G is nonplanar.

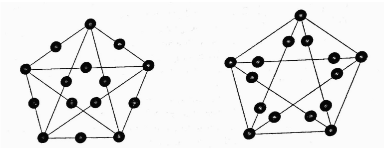
Corollary: The complete graph K_5 is nonplanar.
Proof: The graph K_5 has order $n = 5$ and size $m = 10$. Since $m = 10 > 9 = 3n - 6$, it follows that K_5 is nonplanar by the Theorem.
Theorem: The graph $K_{3,3}$ is nonplanar.
Proof: Assume, to the contrary, that $K_{3,3}$ is planar and draw $K_{3,3}$ as a plane graph. Since $n = 6$ and $m = 9$, it follows by the Euler Identity that $n - m + r = 6 - 9 + r = 2$ and so $r = 5$. Let $R_1, R_2, ..., R_5$ be the five regions and let m_i be the number of edges on the boundary of $R_i (1 \leq i \leq 5)$. Since $K_{3,3}$ has no triangles, $m_i \geq 4$ for $1 \leq i \leq 5$ and because $K_{3,3}$ contains no bridges, it follows that $2m = \sum_{i=1}^{5} m_i \geq 20$, and so $m \geq 10$. This is a contradiction. Hence the graph $K_{3,3}$ is nonplanar.

Note: There exists a graph of order $n \geq 3$ and size $m > 3n - 6$ that contains neither K_5 nor $K_{3,3}$ as a subgraph.

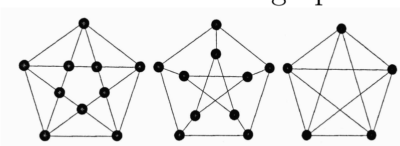
Let G be a graph and $e = uv$ an edge of G. A **subdivision** of e is the replacement of the edge e by a simple path $u_0, u_1, ..., u_k$, where $u_0 = u$ and $u_k = v$ are the only vertices of the path in $V(G)$. We say that G' is a **subdivision** of G, if G' is obtained from G by a sequence of subdivisions of edges in G.



A graphs G_1 is said to be **homeomorphic** to G_2, if G_2 can be obtained from G_1 by insertion or deletion of vertices of degree 2 between the edges of G_1.
The below graphs are homeomorphic to each other.



Whereas the below graphs are nonhomeomorphic to each other.

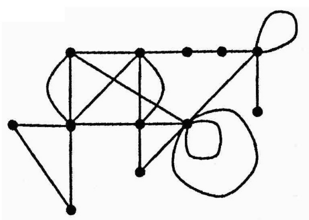


Theorem: (Kuratowski's Theorem) A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.
OR
Theorem: (Kuratowski's Theorem) A graph G is planar if and only if G does not contain any subgraph homeomorphic to K_5 or $K_{3,3}$.

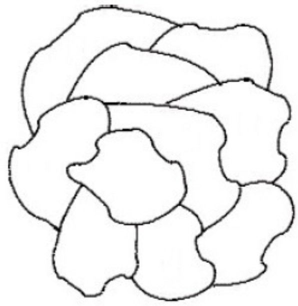
Detection of planarity:
1. Eliminate edges in parallel by removing all but one edge between every pair of vertices.
2. Remove all self loops.
3. Eliminate all edges in series.

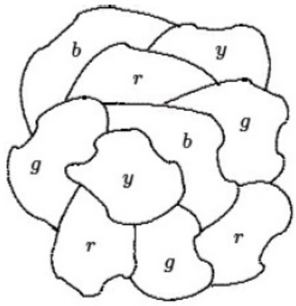
Repeating the above steps yields one of the following graphs:
(i) A single edge,
(ii) A complete graph on four vertices.
(iii) $m > 3n - 6$
(iv) A K_5 or $K_{3,3}$ subgraph.
If the graph reduces to (i) or (ii), the given graphs is planar.
If the graph reduces to (iI) or (iv), the given graphs is nonplanar.

Problem:
Check the planarity of the following graph by the method of elementary deduction.

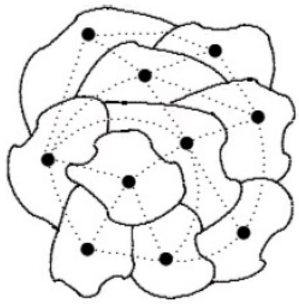


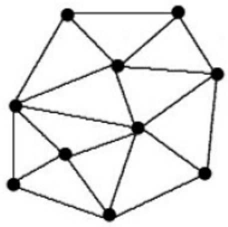
Consider the problem of coloring the regions on a map with different colors, such that no two neighboring regions have the same color.



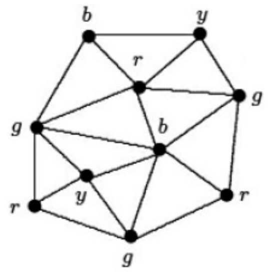


It can be seen that each region of the map can be assigned one of four given colors such that neighboring regions are colored differently. Indeed, one such coloring is shown in the figure, where r, b, g and y denote red, blue, green and yellow, respectively.

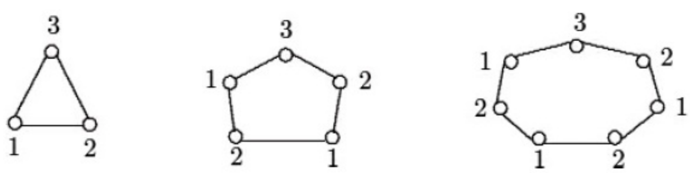




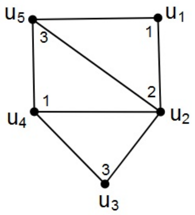
With each map, there is associated a graph G, called the **dual** of the map, whose vertices are the regions of the map and such that two vertices of G are adjacent if the corresponding regions are neighboring regions.
Coloring the regions of a map suggests coloring the vertices of its dual. Indeed, it suggests coloring the vertices of any graph. By a **proper coloring**(or, more simply, a **coloring**) of a graph G, we mean an assignment of colors(elements of some set) to the vertices of G, one color to each vertex, such that adjacent vertices are colored differently.



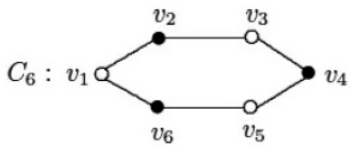
The smallest number of colors in any coloring of a graph G is called the **chromatic number** of G and is denoted by $\chi(G)$. If it is possible to color (the vertices of) G from a set of k colors, then G is said to be k-**colorable**. A coloring that uses k colors is called a k-**coloring**. If $\chi(G) = k$, then G is said to be k-**chromatic** and every k-coloring of G is a **minimum coloring** of G.
A graph G containing only one edge, requires at least two colors to color. $\chi(G) = 1$ if and only if $G = \overline{K_n}$ for some positive integer n. Below are cycles of order 3, 5 and 7 colored with 3 colors.



A proper coloring of a graph naturally induces a partition of the vertices into different subsets. For example the coloring in the following graph produces the partitioning, called the **chromatic partitioning**, $=V_1 = \{u_1, u_4\}, V_2 = \{u_2\}, V_3 = \{u_3, u_5\}$.

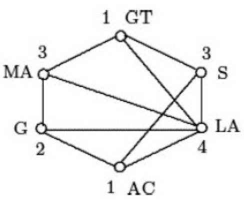


In any coloring of a graph G, no two vertices that are colored the same can be adjacent. Sets of vertices, no two of which are adjacent, are called **independent sets**. Ordinarily, a graph has many independent sets of vertices. A **maximum independent set** is an independent set of maximum cardinality. The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$ and is called the **vertex independence number**(or, more simply, the **independence number**) of G.



For the graph $G = C_6$ in the above figure $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_2, v_4, v_6\}$ are both independent sets. Since no independent set of G contains more than three vertices, $\alpha(G) = 3$. If G is a k-chromatic graph, then it is possible to partition $V(G)$ into k independent sets $V_1, V_2, ..., V_k$, called **color classes**.

Problem: The mathematics department of a certain college plans to schedule the classes Graph Theory(GT), Statistics(S), Linear Algebra(LA), Advanced Calculus(AC), Geometry(G) and Modern Algebra(MA). Ten students have indicated the course they plan to take. With this information, use graph theory to determine the minimum number of time periods needed to offer these courses so that every two classes having a student in common are taught at different time periods during the day. Of course, two classes having no students in common can be taught during the same period. Below is mentioned the student preferences. A: LA,S; B: MA,LA,G; C: MA,G,LA; D: G,LA,AC; E: AC,LA,S; F: G,AC; G: GT,MA,LA; H: LA,GT,S; I: AC,S,LA; J: GT,S.
Solution: The above situation can be represented by the below graph, where the vertices are the six subjects. Two vertices(subjects) are joined by an edge if some student is taking both classes. The graph can be colored by 4 colors as shown below. Hence $\chi(H) = 4$. This also tells us one way to schedule these six classes during four time periods, namely, Period 1: Graph Theory, Advanced Calculus; Period 2: Geometry; Period 3: Statistics, Modern Algebra; Period 4: Linear Algebra.



A graph G of order n has chromatic number n if and only if $G = K_n$.
A **clique** in a graph G is a complete subgraph of G. The order of the largest clique in a graph G is its **clique number**, which is denoted by $\omega(G)$.
Note: $\alpha(G) = k$ iff $\omega(\overline{G}) = k$.
Theorem: For every graph G of order n, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \dfrac{n}{\alpha(G)}$.

A coloring of a graph G can also be thought of as a function c from $V(G)$ to the set N of positive integers (or natural numbers) such that adjacent vertices have distinct functional values, that is, a **coloring** of G is a function $c : V(G) \longrightarrow N$ such that $uv \in E(G)$ implies that $c(u) \neq c(v)$.

Theorem: (Greedy Algorithm) For every graph G, $\chi(G) \leq 1 + \Delta(G)$.
Proof: Let $V(G) = \{v_1, v_2, ..., v_n\}$. Define a coloring $c : V(G) \longrightarrow N$ recursively as follows: $c(v_1) = 1$. Once $c(v_i)$ has been defined,$1 \leq i \leq n$, define $c(v_{i+1})$ as the smallest positive integer not already used to color any of the neighbors of $v_i + 1$. Since v_{i+1} has $degv_{i+1}$ neighbors, at least one of the integers $1, 2, ..., 1 + degv_{i+1}$ is available for $c(v_{i+1})$. Therefore, $c(v_{i+1}) \leq 1 + degv_{i+1}$. If the maximum color assigned to the vertices of G is $c(v_j)$, say, then $\chi(G) \leq c(v_j) \leq 1 + degv_j \leq 1 + \Delta(G)$ as desired.

Theorem:(Brooks' Theorem) For every connected graph G that is not an odd cycle or a complete graph, $\chi(G) \leq \Delta(G)$.

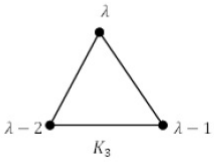
Theorem:(Five Color Theorem) For every planar graph G, we have $\chi(G) \leq 5$.
Proof: We can assume that G is simple. We will use induction on n, the number of vertices of G.
Since the theorem is clearly true for any simple graph on five vertices or fewer vertices, we can assume $n \geq 6$ and that the theorem is true for all planar graphs on $n - 1$ and fewer vertices.
We know that planar graph contains a vertex of degree five or less. Therefore there is a vertex u of G with $deg(u) \leq 5$. If $deg(u) \leq 4$, then by induction hypothesis the graph $G - u$ has a proper 5-coloring. Since we have five colors to choose from, at least one color is distinct from all of the colors of neighbors of u in G. Therefore, this coloring can be extended to a proper 5-coloring of G. If $deg(u) = 5$, then u_1, u_2, u_3, u_4, u_5 be the neighbors of u. Name the color so that $C(u_i) = i$.
Let $G_{i,j}$ denote the subgraph of $G - u$ induced by the verices by the vertices of colors i and j. Switching the two colors on any components of $G_{i,j}$ yields another proper 5-coloring of $G - u$. If the component of $G_{i,j}$ containing u_i does not contain u_j then we can switch the colors on it to remove color i from $N(u)$. Now giving color i to u produces a proper 5-coloring of G. Thus G is 5-colorable unless, for each choice of i and j, the components of $G_{i,j}$ containing u_i also contains u_j . Let $P_{i,j}$ be a path in $G_{i,j}$ from u_i to u_j , illustrated below for $(i, j) = (1, 3)$.
Consider the cycle C computed with$P_{1,3}$ by u; this separates v_2 from v_4. By the Jordan Curve Theorem, the path $P_{2,4}$ must cross C. Since G is planar, paths can cross only at shared vertices, The vertices of $P_{1,3}$ all have color 1 or 3, and the vertices of $P_{2,4}$ all have color 2 or 4, so they have no common vertex. By this contradiction, G is 5- Colorable.

Theorem:(The Four Color Theorem) The chromatic number of every planar graph is at most 4.

Coloring Enumeration:

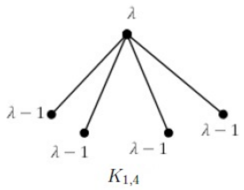
Let G be a graph and $\lambda \in \mathbb{N}$. Define be the number of $P(G : \lambda)$ to be the number of proper λ-vertex colorings $c : V(G) \to \{1, 2, 3, ..., \lambda\}$. This property of a graph expressed by means of a polynomial. This polynomial is called the **chromatic polynomial** of G.
i.e Let G be a labeled graph. A coloring of G from λ colors is a coloring of G which uses λ or fewer colors. Two colorings of G from λ colors will be considered different is at least one of the labeled vertex is assigned different colors.
Note:(i) For each $\lambda < \chi(G)$ we have $P(G : \lambda) = 0$
(ii) For each $\lambda \geq \chi(G)$ we have $P(G : \lambda) > 0$
(iii) Indeed the smallest λ for which $P(G : \lambda) > 0$ is the chromatic number of G.

There are λ ways of coloring any given vertex of K_3. For a second vertex, any of $\lambda - 1$ colors may be used, while there are $\lambda - 2$ ways of coloring the remaining vertex. Thus $P(K_3 : \lambda) = \lambda(\lambda - 1)(\lambda - 2)$.



This can be generalized to any complete graph $P(K_n : \lambda) = \lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1)$. The corresponding polynomial of the totally disconnected graph (Null graph) $\overline{K_n}$ is particularly easy to find since each of its n vertices may be colored independently in any of λ ways. Thus $P(\overline{K_n} : \lambda) = \lambda^n$.

The Central vertex v_0 of $K_{1,4}$ may be colored in any λ ways while each end vertex may be colored in any $\lambda - 1$ ways . Therefore $P(K_{1,4} : \lambda) = \lambda(\lambda - 1)^4$



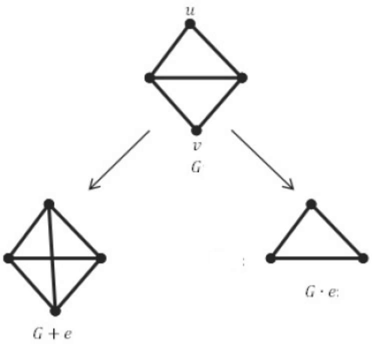
Certainly, every two isomorphic graphs have the same chromatic polynomial. However, there are often several nonisomorphic graphs with the same chromatic polynomial; in fact, all trees with n vertices have equal chromatic polynomials.

A graph G with n vertices is a tree if and only if $P(G, \lambda) = \lambda(\lambda - 1)^{n-1}$

Theorem: Let u and v be two non adjacent vertices in a graph G. Let $G + e$ be a graph obtained by adding an edge between u and v. Let $G \cdot e$ be a simple graph obtained from G by fusing the vertices u and v together and replacing sets of parallel edges with single edge. Then $P(G, \lambda) = P(G + e, \lambda) + P(G \cdot e, \lambda)$.

Problem: Find the chromatic polynomial of the following graph.

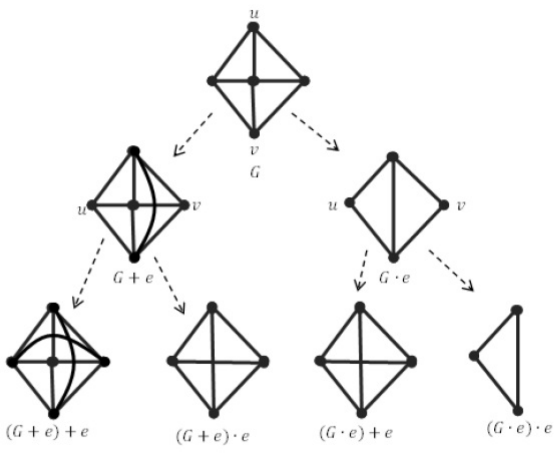
Solution:



$P(G, \lambda) = P(G + e, \lambda) + P(G \cdot e, \lambda)$
$= P(K_4, \lambda) + P(K_3, \lambda)$
$= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2)$
$= \lambda(\lambda - 1)(\lambda - 2)[(\lambda - 3) + 1]$
$= \lambda(\lambda - 1)(\lambda - 2)^2$

Problem: Find the chromatic polynomial of the following graph.

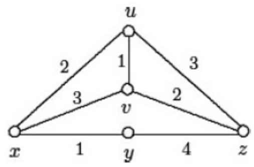
Solution:



$p(G, \lambda) = P(G + e, \lambda) + P(G \cdot e, \lambda)$
$= P((G + e) + e, \lambda) + P((G + e) \cdot e, \lambda) + P((G \cdot e) + e, \lambda) + P((G \cdot e) \cdot e, \lambda)$
$= P(K_5, \lambda) + P(K_4, \lambda) + P(K_4, \lambda) + P(K_3, \lambda)$
$= P(K_5, \lambda) + 2P(K_4, \lambda) + P(K_3, \lambda)$
$= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4) + 2\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2)$
$= \lambda(\lambda - 1)(\lambda - 2)[(\lambda - 3)(\lambda - 4) + 2(\lambda - 3) + 1]$
$= \lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 5\lambda + 7)$

Edge Coloring

A **k-edge-coloring** of G is a labeling $f : E(G) \to S$, where $|S| = k$. The labels are colors; the edges of one color from a color class. A k-edge-coloring is proper if incident edges have different labels; that is, if each color class is a matching. A graph is **k-edge-colorable** if it has a proper k-edge-coloring. The **edge-chromatic number** (**Chromatic index**) $\chi^{'}(G)$ of a loopless graph G is the least k such that G is k-edge-colorable.
The edge chromatic number of the following graph is four.



For a graph G and any vertex $u \in V(G)$, all edges with u as an end vertex are adjacent and hence must receive different colors in a proper edge coloring of G. Hence, we note the obvious lower bound for the edge chromatic number of G $\chi^{'}(G) \geq \Delta(G)$, the maximum degree in G.

Edge chromatic number of some basic graphs:

$$\chi^{'}(K_n) = \begin{cases} n & if\, n\, is\, odd, \\ n-1 & if\, n\, is\, even,. \end{cases} \qquad \chi^{'}(C_n) = \begin{cases} 2 & if\, n\, is\, even, \\ 3 & if\, n\, is\, odd. \end{cases} \qquad \chi^{'}(p_n) = 2$$

Theorem: For a bipartite graph G, we have $\chi^{'}(G) = \Delta(G)$.

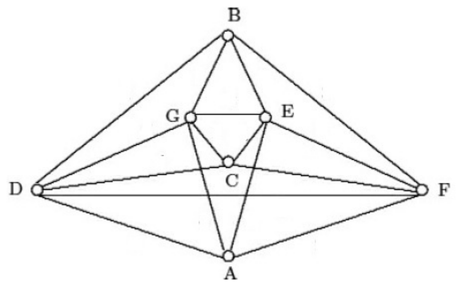
Theorem: For the complete bipartite graph $K_{m,n}$, $\chi^{'}(K_{m,n}) = max(\{m, n\})$.

Theorem: If G is a simple graph, $\Delta(G) \leq \chi^{'}(G) \leq \Delta(G) + 1$.

Theorem:(Vizing's theorem) For a non empty graph G, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = 1 + \Delta(G)$.

Theorem: Let G be a graph of order n and size m. If $m > \dfrac{(n-1)\Delta(G)}{2}$, then $\chi'(G) = 1 + \Delta(G)$.

Problem: Alvin(A) has invited three married couples to his summer house for a week: Bob(B) and Carrie(C)Hanson, David(D) and Edith(E)Irwin and Frank(F) and Gena(G)Jackson. Since all six guests enjoy playing tennis, he decides to set up some tennis matches. Each of his six guests will play a tennis match against every other guest except his/her spouse. In addition, Alvin will play a match against each of David, Edith, Frank and Gena. If no one is to play two matches on the same day, what is a schedule of matches over the smallest number of days?
Solution: First, we construct a graph H whose vertices are the people at Alvin's summer house, so $V(H) = A, B, C, D, E, F, G$, where two vertices of H are adjacent if the two vertices (people) are to play a tennis match. To answer the question, we determine the chromatic index of H.



First, observe that $\Delta(H) = 5$. Hence $\chi'(H) = 5$ or $\chi'(H) = 6$. Also, the order of H is $n = 7$ and its size is $m = 16$. Since $m = 16 > 15 = \dfrac{(7-1)\cdot 5}{2} = \dfrac{(n-1)\Delta(H)}{2}$ it follows that $\chi'(H) = 6$. The below figure gives a 6-edge coloring of H, which provides a schedule of matches that takes place over the smallest number of days (namely six).