Real Symmetric Matrix:

Def: A matrix $A^{n\times n}$ with real entries and the property that $A^T = A$ is called a real symmetric matrix.

For real symmetric matrices A & B

(i) $(A+B)^T = (A+B)$

We know that matrix additions commute.

$\Rightarrow A+B = B+A$.

We also know that

$$(A+B)^T = A^T + B^T$$
$$= A + B, \quad (A \text{ \& } B \text{ Symmetric})$$

$\Rightarrow (A+B)^T = A+B$. $\Rightarrow$ Sum of Symmetric matrices is another symmetric matrix

(ii) Given A, real symmetric matrix, and k any real scalar

kA is also symmetric.

(iii) If A is an invertible real symmetric matrix, A^{-1} is also real symmetric

Proof: We know that
$$A^{-1}A = AA^{-1} = I.$$

We know that $I^T = I$.

$$(AA^{-1}) = I = I^T = (AA^{-1})^T$$
$$= (A^{-1})^T A^T$$

Rearranging we get

$$A^{-1}A = (A^{-1})^T A^T.$$
$$\Rightarrow A^{-1}AA^{-1} = (A^{-1})^T A^T A^{-1}, \quad A^T = A$$
$$\Rightarrow A^{-1}(I) = (A^{-1})^T I. \Rightarrow A^{-1} = (A^{-1})^T.$$

Consider $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow$ SYMM matrix.

$\det(A - \lambda I) = 0 \Rightarrow \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\Rightarrow (\lambda-3)(\lambda-1) = 0$

$\Rightarrow \boxed{\lambda = 3, \quad \lambda = 1.}$

Consider $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, real matrix

Note that A is real & Symmetric.

Let us find out the eigenvalues of A.

$\Rightarrow A - \lambda I = \begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow$

$$(a-\lambda)(c-\lambda) - b^2 = 0$$

$$\Rightarrow \quad \lambda^2 - \lambda(a+c) - b^2 + ac = 0.$$

roots of the above are

$$\lambda = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2}.$$

$$= \frac{(a+c) \pm \sqrt{a^2+2ac+c^2-4ac+4b^2}}{2}$$

$$\lambda = \frac{(a+c) \pm \sqrt{a^2-2ac+c^2+4b^2}}{2}$$

$$= \frac{(a+c) \pm \left(\sqrt{(a-c)^2+4b^2}\right)}{2}.$$

The value $(a-c)^2+4b^2$ is always ≥ 0. Hence the square root of $(a-c)^2+4b^2$ is always real.

Case when $(a-c)^2 + 4b^2 = 0$.

$\Rightarrow a - c = 0 \Rightarrow c = a$.

$4b^2 = 0 \Rightarrow b = 0$.

$\therefore$ Matrix A now becomes

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

If $(a-c)^2 + 4b^2 > 0$

$$\lambda = \frac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2}.$$

The eigenvalues of a real symmetric 2×2 matrix are REAL.

Consider the product A^TA.
where A is a real symmetric matrix.

Let $B = A^TA$.

$B^T = (A^TA)^T = (A^T)(A^T)^T$
$= A^TA = B$.

$B^T = B$.

The product matrix $A^TA = B$ is also a real symmetric matrix.

Similarly the product $\underline{AA^T = C}$ is also symmetric.

$C = AA^T$

$C^T = (AA^T)^T = (A^T)^T A^T = AA^T$.

$C^T = AA^T = C$.