SVD

We know the following:

$A_{m \times m}$; Real Symmetric then EV-EV

$A x_i = \lambda_i x_i \quad i = 1 \ldots n$; $\lambda_i \rightarrow$ Distinct.

$M \triangleq [x_1 \ldots x_m]$ Prove that orthogonal $\{x_i\}_{i=1}^{m}$

$M^{-1} A M = D$; $\equiv$ orthonormal ;

$\boxed{M^T A M = D} \rightarrow$ Orthogonally Diagonizable

A: Real Sym. Matrix

Distinct Eigen Values

Note: = Gram Schmidt Orthonormalization Process (Projection)

We are interested in the following.

$A_{m\times n} \overset{?}{\equiv}$ "Diagonal Form ?"

$A_{m\times n} \Rightarrow (A)^{-1}_{m\times n}$

$Ax = b$, $m \times m$, $x = A^{-1}b$

$Ax = b$, $m \times n$, $m \neq n$ $\Rightarrow$ "A^{-1}" $\Rightarrow$ $\boxed{A^{+}b = x}$

$A^{+} = (A^{T}A)^{-1}A^{T}$

$\hat{A}\hat{b} = \hat{x}$

$\hat{x} \neq x$

$\Rightarrow A_{m\times m} = \{\lambda_i x_i\}$ $\hookrightarrow$ Eigen like Decomposition of

$A_{m\times n}$; $m \neq n$

→ Step: $A_{m\times n} \rightarrow \begin{cases} A^TA \\ AA^T \end{cases}$ Square & Real Symmetric

$A_{m\times n}$ is Real.

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① "Close" cousin of $A_{m\times n} \rightarrow A^TA_{n\times n}$.

$A_{m\times n} \rightarrow AA^T_{m\times m}$

② $(I_{n\times n}\lambda - A^TA)$ or $(I_{m\times m}\lambda - AA^T)$

Recall:

A Square $\begin{cases} \lambda \text{ is eigenvalue of } A \\ \lambda \text{ is also an Eigenvalue of } A^T \end{cases}$

$\begin{cases} \lambda \text{ is Eigenvalue of } A \text{ or } A^T \\ \lambda^2 \text{ is an Eigenvalue of } A^2 \text{ or } A^TA. \end{cases}$

Conceptually:

A ($m \times n$, $m \neq n$, $m > n$) → AA^T ($m \times m$), A^TA ($n \times n$)

Rank of A ($m \times n$) $\leq \min(m, n)$ $\not> n$

Define Singular value as $\sqrt{\lambda_i^2}$

λ_i is an Eigen value of A^TA.

$$(I\lambda - A^TA) = (\lambda_1^2 \; \lambda_2^2 \cdots \lambda_n^2)$$

$A_{m \times m} = \lambda_1, -\lambda_2$ $\quad (A^TA) \rightarrow \lambda_1^2, \lambda_2^2$ +ve Sign remains

Singular Value $\sigma_i = \sqrt{\lambda_i^2}$

+ve. -ve eigen value → σ_i

Recall: $A_{m\times m} \equiv D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$M: M^TM = I$

$A_{m\times n}$ $m > n$ $\equiv D$

$$\begin{matrix} & 1 & & & n \\ 1 & & & & \\ \vdots & & & & \\ n & & \ddots & & \\ m & & & \sigma_n & \\ & & 0 & & \end{matrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \\ & & 0 & \end{bmatrix}$$

Diagonal Equivalent of $A_{m\times m}$

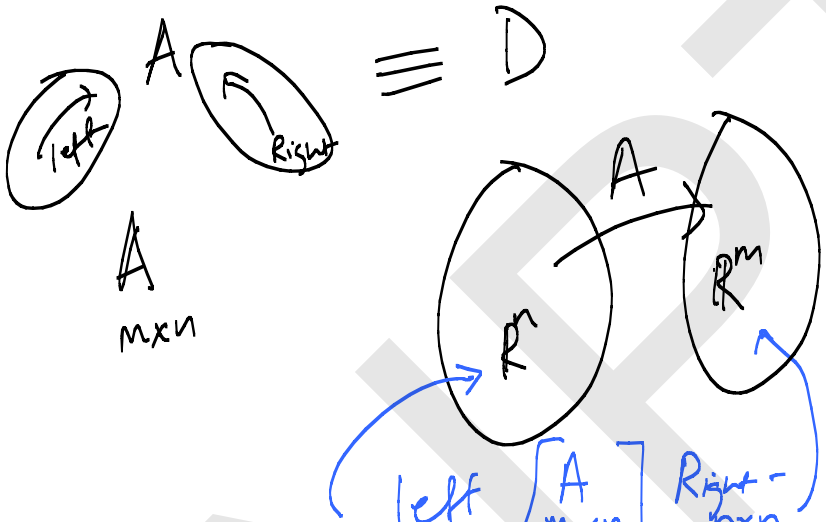
A $m < n$ $\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_m & \end{bmatrix}$ $\begin{matrix} \\ \\ \\ \\ 0 \end{matrix}$

How do I get these Structures?

Recall A: Real Sym $\Big\}$ M S.T

$M^T A M = D$ (mxm mxn mxm = mxn)

$A : \mathbb{R}^n \to \mathbb{R}^m$



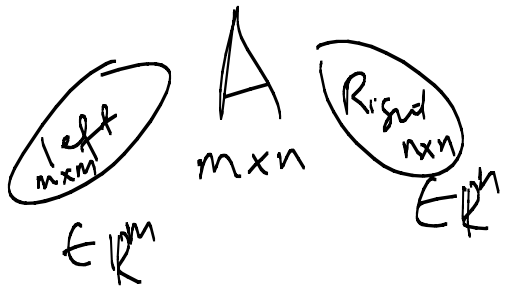
Further: $A \equiv M^T A M = D$

A: $m \times m$

M: orthonormal matrix

EV - Ev's of A

Very Similarly



$\in \mathbb{R}^m$ $\in \mathbb{R}^n$

$\rightarrow \underset{(n \times n)}{A^T A} = (V \lambda - A^T A)$; $\lambda_1^2 \; \lambda_2^2 \cdots \lambda_n^2$

$\{\sigma_i\}_{i=1}^{n} = \sigma_1 \; \sigma_2 \cdots \sigma_n$

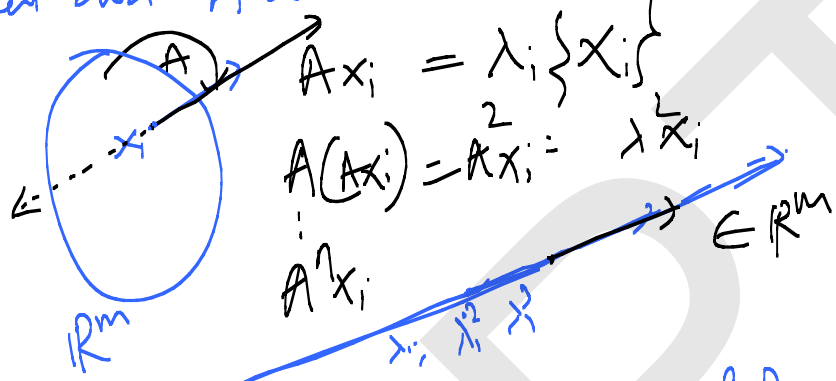
We can do ordering S.T. $\sigma_1 \geq \sigma_2 \geq \sigma_3 \cdots \geq \sigma_n$ } Singular Values!

Singular Vectors!

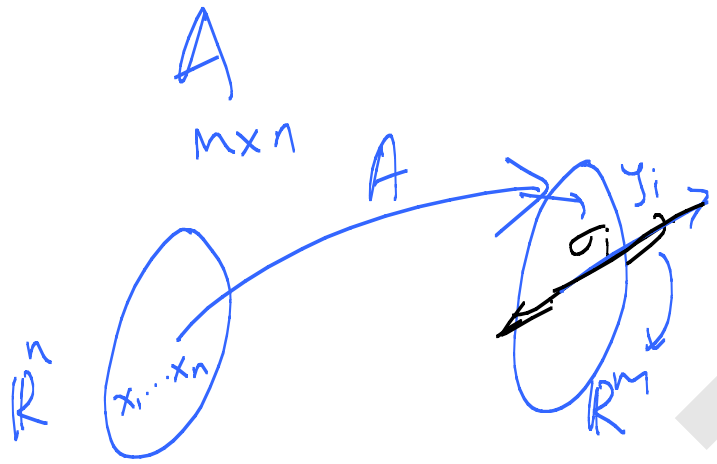
$A_{m \times m}$: $Ax_i = \lambda_i x_i$

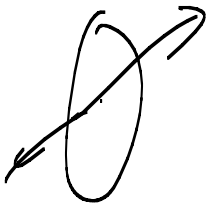
$\{x_i\} \in \mathbb{R}^m$, $\lambda_i \in \mathbb{R}$

What did A do on x_i



What do Singular vectors do?





$A x_i \longrightarrow \in \mathbb{R}^m$

$x_i \in \mathbb{R}^n$

$A x_i = \sigma_i y_i$

$m \times n \quad (n \times n)$

$\exists$ EiEv

$(A^T A) x_i = \sigma_i y_i$

EV-EV Pair

$A x_i = \lambda_i x_i$

$A x_i = \sigma_i y_i$

$x_i \in \mathbb{R}^n$

$y_i \in \mathbb{R}^m$

σ_i = Sing Valu

$n < m$: In simple

Possible to find $x_1 \ldots x_n ; \sigma_1 \ldots \sigma_n$

S.T. $A x_i = \sigma_i y_i \Big\}_{i=1}^{n}$

$x_i \equiv$ Eigen Decomposing $\underset{n \times n}{(A^T A)}$

Eigen Decomposition Problem Based on a Covariance Matrix (Real Symmetric) $\triangleq A^T A \, (AA^T)$

Invariant Singular Vectors are in the Transformed Space.

So the final Structure (Geometric)

Recall: $M^T M = I$; $M M^T = I$

M : $n \times n$

Diagonalization:

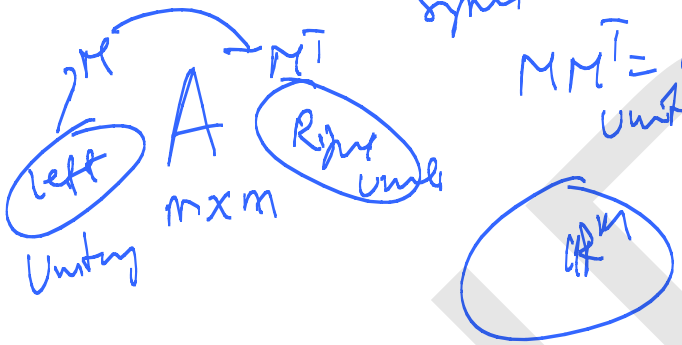
$A =$ Real Symm

$M^T A M = I$

$M A M^T = I$

$M M^T = M M^T = I$

Unitary Matrices.



SVD

$A_{m \times n}$ $m \neq n$ $\equiv$ left $\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & \cdots & 0 \end{bmatrix}_{m \times n}$ Right

$U_{m \times m} \in \mathbb{R}^{m}$ $\quad V_{n \times n} \in \mathbb{R}^{n}$

$U U^T = I_m$

$V V^T = I_n$

$$A_{m \times n} = U \Sigma V^T$$

$$A_{m \times m} = M D M^T$$

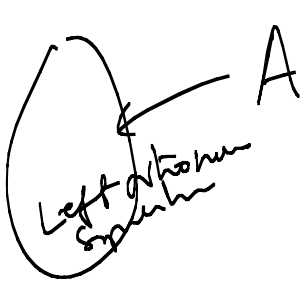
$$= M^T A M = D$$

Working this

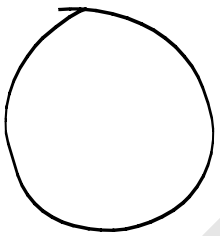
$A_{m \times n}$: $(I\lambda_{n \times n} - A^T A)v_i = 0_{n \times n}$

$m \neq n$

$\{v_1 \cdots v_n\}$ O.N $\triangleq V = [v_1 \cdots v_n]$ Singular Vectors



$= [V]$



U

mxm

$Av_i = \sigma_i u_i$

$\{u_1 \cdots u_n\}$ O.N

Av_i --

$\{u_{n+1} \ldots u_m\}$ $(m-n)$; ? ~~$\exists$~~ $V_{1 \ldots n}$

Basis Extension Problem

Null Space of V

$\{u_1 \ldots u_m\} \leftarrow \{u_{n+1} \ldots u_n\}$

$\rightarrow$ O.N. Extension of Set $(u_1 \ldots u_m)$: $u_{n+1} \ldots u_n$

$U = [u_1 \ldots u_m]$

$Av_i = \sigma_i u_i$

$(m-n)$ Nonzero vecs. L.I

$\leftarrow m > n \Rightarrow \{u_1 \ldots u_m\}$

$$A = U \Sigma V^T \text{ or } U^T A V = \Sigma$$

$A_{m \times n}$; $m < n$ $[\sigma_1 \ldots \sigma_n]$

This is SVD of A

Connections

σ_i } $Av_i = \sigma_i u_i$

$(u_1 \ldots u_m) \equiv$ Principal Components of a Real Symmetric Matrix

$A x_i = \lambda_i x_i$ ⟷ Eigen Configuration

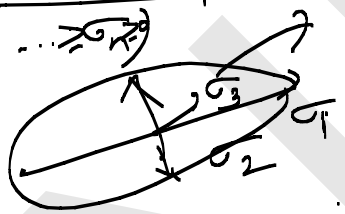
A — $m \times m$ Real Sym

2 — Sym — ... — A

R_m σ_i u_i R_M

DATA Space

$(\sigma_1 \geq \sigma_2 \geq \sigma_3 \ldots \geq \sigma_n)$



Generate a Ellipsoid

Ellipsoid : $\sigma_1 \sigma_2 \sigma_3$

: n Dim Ellipsoid

$A_{m \times n}$ $m \geq n$

$AA^T_{m \times m} = [u_1 \ldots u_{min} \ldots u_m]$ (non zero)

$Min \leq min(m,n)$ $u_{min+1} \ldots u_m$ non zero in Null A^T.

Later we will do PCA

→ Applications Data Compression Using L.A. ≡ SVD And SVD = more commonly used [A] $_{m \times n}$ = in More Common → PCA ≡ Step ≡ $A = U \Sigma V^T$ → Decompo. SVD.