eigenvalues of real symmetric matrices are real.

Let A be a RSM. Let λ be the eigenvalue of A and let u be the eigenvector associated with λ.

Let λ be complex

$\Rightarrow \lambda = a + ib$.

Complex conjugate of $\lambda = a - ib = \lambda^*$

$\lambda^* = a - ib$.

The components of u may be complex numbers.

u^*: Complex conjugate of u.

$\lambda^* u^*$ is the complex conjugate of λu.

$Au = \lambda u \Rightarrow Au^* = \lambda^* u^*$

$\therefore (\underline{u^*})^T A = (u^*)^T \lambda^*$

Let us look at the dot product of Au with u^*

$$\Rightarrow (u^*)^T A u = (u^*)^T \lambda u . \rightarrow (1)$$

Similarly we can look at

$$A u^* = \lambda^* u^*$$

$$(u^*)^T A = (u^*)^T \lambda^*$$

$$(u^*)^T A u = (u^*)^T \lambda^* u . \quad - (2)$$

LHS of (1) and (2) are the same.

$\therefore$ we have

$$(u^*)^T \lambda u = (u^*)^T \lambda^* u$$

$$\lambda (u^*)^T u = \lambda^* (u^*)^T u .$$

$$\Rightarrow (u^*)^T u = |u_1|^2 + |u_2|^2 + \dots \Rightarrow \text{Squared length} \neq 0 .$$

$\Rightarrow \lambda = \lambda^*$

$a + ib = a - ib$

$\Rightarrow b = 0$

$\Rightarrow \lambda$ is real

The eigenvalues of a real Symmetric matrix are real.

Consider $A^{2\times 2}$, a real Symmetric matrix with distinct eigenvalues λ_1 & λ_2.

Let u be the eigenvector associated with λ_1 and v be that associated with λ_2.

$u \cdot v = 0$

$\Rightarrow u^T v = 0$

(eigenvectors corresp. to distinct eigenvalues of a RSM are orthogonal to each other).

$P = \begin{bmatrix} u & v \end{bmatrix}$

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$AP = PD.$

Consider $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

Scaling by it's length we have a unitvector u.

Similarly we can get a unit vector v by dividing v by its length.

$u \cdot u = 1, \Rightarrow u^T u = 1$

$v \cdot v = 1 \Rightarrow v^T v = 1.$

$u \cdot v = u^T v = v^T u = v \cdot u = 0$

$\Rightarrow P = \begin{pmatrix} u & v \end{pmatrix}$ u, v: Unit vectors

$\Rightarrow$ Matrix P is an orthogonal matrix

For an orthogonal matrix

P, $P^{-1} = P^T$.

$$P = \begin{bmatrix} u & v \end{bmatrix} \qquad P^T = \begin{bmatrix} u^T \\ v^T \end{bmatrix}$$

$$P^TP = \begin{bmatrix} u^T \\ v^T \end{bmatrix} \begin{bmatrix} u & v \end{bmatrix}$$

$$= \begin{bmatrix} u^Tu & u^Tv \\ v^Tu & v^Tv \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Recall that if A is a real Symmetric matrix

$$A = PDP^T$$

(For any diagonalizable matrix A,
$A = PDP^{-1}$)

Let x be any vector in $\mathbb{R}^2$.

$Ax = ?$ What does this geometrically mean?

$$Ax = PDP^Tx$$

$$= \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \end{bmatrix} x.$$

$$= \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u^Tx \\ v^Tx \end{bmatrix}$$

$$= \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \lambda_1 u^Tx \\ \lambda_2 v^Tx \end{bmatrix}$$

$$= u\lambda_1 u^Tx + v\lambda_2 v^Tx.$$

$$Ax = \lambda_1 uu^Tx + \lambda_2 vv^Tx$$

$$= \left(\lambda_1 \frac{uu^T}{1} + \lambda_2 \frac{vv^T}{1}\right)x.$$

$$= \left(\lambda_1 \frac{uu^T}{u^Tu} + \lambda_2 \frac{vv^T}{v^Tv}\right)x.$$

$$= \lambda_1 \frac{uu^T}{u^Tu}x + \lambda_2 \frac{vv^T}{v^Tv}x$$

Note that $\frac{uu^T}{u^Tu}$ is a rank 1 projection matrix that projects any vector onto the direction of u.

$\frac{vv^T}{v^Tv}$: Rank 1 matrix that projects any vector onto the direction of v.

$$Ax = \lambda_1 \frac{uu^T}{u^Tu} x + \lambda_2 \frac{vv^T}{v^Tv} x.$$

The action of a real symmetric matrix $A \rightarrow$ Interpreted as a linear combination of projections onto the orthogonal eigenvectors.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\lambda_1 = 3, \quad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\lambda_2 = 1, \quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$Ax = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}.$$

$$\underline{Ax} = \lambda_1 u u^T x + \lambda_2 v v^T x.$$

$$\Rightarrow 3 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= 3 \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= 3 \begin{pmatrix} 7/2 \\ 7/2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 21/2 - 1/2 \\ 21/2 + 1/2 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$$