Spectrum:

Set of eigenvalues of A

Spectral Theorem:

A real symmetric matrix $A \in \mathbb{R}^{n \times n}$ has the following properties:

(i) The number of real eigenvalues $= n$ (Counting the AM).

(ii) Dimension of the eigenspace for each eigenvalue λ = Algebraic multiplicity of λ.

(iii) the eigenvectors corresponding to distinct eigenvalues are mutually orthogonal.

(iv) A is ORTHOGONALLY DIAGONALIZABLE

For RSM A, with distinct eigenval,

$$Ax = \begin{pmatrix} \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \dots \\ + \lambda_n u_n u_n^T \end{pmatrix} x$$

where $u_i \in \mathbb{R}^n$ = $u_i = \begin{pmatrix} u_{1i} \\ u_{2i} \\ u_{3i} \\ \vdots \\ u_{ni} \end{pmatrix}$

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T$$

→ Spectral decomposition of A.

Quadratic forms:

For a pair of real numbers (x_1, x_2)

we define

$$H(x_1, x_2) = a x_1^2 + 2b x_1 x_2 + c x_2^2.$$

$H(x_1, x_2)$: Quadratic form.

$$H(x_1, x_2) = a x_1^2 + 2b x_1 x_2 + c x_2^2$$

$$= \overset{u}{\begin{bmatrix} a x_1 + b x_2 \\ b x_1 + c x_2 \end{bmatrix}} \cdot \overset{v}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \qquad = v^T u$$

$$= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a x_1 + b x_2 \\ b x_1 + c x_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The quadratic form

$$H(x_1, x_2) = x^T A x$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

A: The matrix of the quadratic form.

Suppose we have

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \overset{\text{Invertible.}}{P} y = P \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

$$y = P^{-1} x.$$

$$Q(\underset{\sim}{x_1}, x_2) = x^T A x$$

$$= (Py)^T A Py$$

$$= \underline{\underline{y^T \underbrace{P^T A P}_{} y}}.$$

The new matrix of the quadratic form is $P^T A P$.

Recall:

A : Symmetric

$\Rightarrow$

We have an orthogonal matrix P s.t

$$P^T A P = D.$$

Recall For RSMA,

$A = PDP^T$

$D = P^T A P.$

$\Rightarrow \quad y^T P^T A P y = \underline{y^T D y}$

with no cross-term.

Principal Axes Theorem:

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix,

Then, we have an orthogonal change of variable $x = Py$, that transforms the quadratic form $x^T A x$ to $y^T D y$;

$y^T D y$

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} y_1 & y_2 \end{pmatrix}\begin{pmatrix} d_1 y_1 \\ d_2 y_2 \end{pmatrix}$$

$\Rightarrow \underline{d_1 y_1^2 + d_2 y_2^2}$

$x^T A x$

$= a x_1^2 + \underline{2b x_1 x_2} + c x_2^2.$

A quadratic form Q is called

(i) Positive definite if $Q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$.

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

(ii) Negative Definite if $Q(\vec{x}) < 0$ for all $\vec{x} \neq \vec{0}$

(iii) Indefinite: $Q(\vec{x})$ assumes both positive and negative values.

(iv) Positive Semidefinite: $Q(\vec{x}) \geq 0$ for all $\vec{x}$

(v) Negative Semidefinite: $Q(\vec{x}) \leq 0$ for all $\vec{x}$.

Let A be any real Symmetric matrix. Then the quadratic form $x^T A x$ is

(i) Positive definite if the eigenval of A are all positive

(ii) Negative definite if the eigenval of A are all negative

(iii) indefinite if A has both positive and negative eigen val.

Let $\vec{x} \in \mathbb{R}^n$

$\|x\| = x^T x = x_1^2 + x_2^2 + \dots + x_n^2$

$= 0$ if $\vec{x} = \vec{0}$

> 0 if $\vec{x} \neq \vec{0}$

$x^T I x$.

Proof:

For a real symmetric matrix A, by the principal axes theorem, we have an orthogonal change of variable

$$\vec{x} = P\vec{y} \quad s.t$$

$$Q(\vec{x}) = x^T A x = y^T D y$$
$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

where $\lambda_1 \dots \lambda_n$ are the eigenval of the RSM A.