Lecture 2: Geometry of Linear Equations in 2 and 3 unknowns

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Recap

- Matrix Representation of system of linear equations

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- Matrix Representation of system of linear equations
- Some interpretations of matrices in different application domains

Today's agenda

- Geometry of linear equations involving 2 and 3 unknowns

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- Solutions to a system of linear equations

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- Solutions to a system of linear equations
- Existence of solution to system of linear equations

Example 1

Consider the system of equations

$$\begin{aligned} l_1: \quad 2x_1 + x_2 &= \quad 3 \\ l_2: \quad x_1 + 2x_2 &= \quad 3 \end{aligned}$$

- l_1 and l_2 represent to two straight lines

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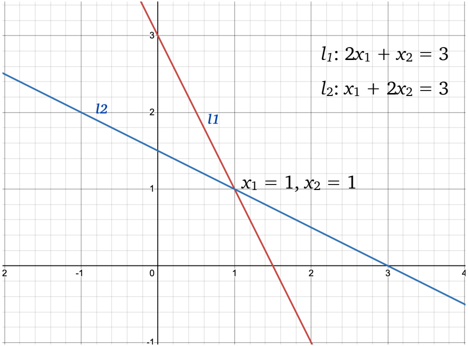

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- Intersection point - A point that lies on both the lines - Solution to a pair of linear equations in 2 unknowns
- Only one point of intersection - **UNIQUE** solution.

Example 2

- $$\begin{aligned} l_1: \quad 2x_1 + x_2 &= \quad 3 \\ l_2: \quad 4x_1 + 2x_2 &= \quad 6 \end{aligned}$$

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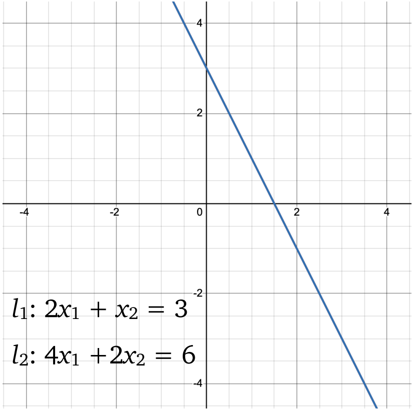

Example 2 - Some Observations

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- Infinitely many solutions

Example 3

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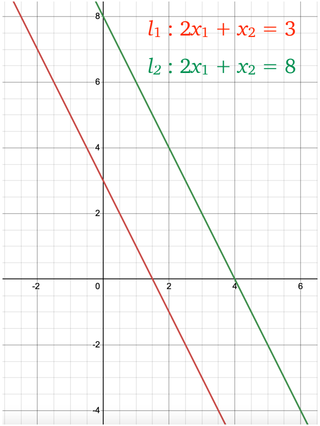

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- **NO SOLUTION**

Some Questions to look answers for!

1. When does a system of linear equations have solution?
2. When does a system of linear equations not have solution?
3. If the system of linear equations has solution, how many solutions does it have?

Consistent & Inconsistent Systems

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Some Examples of Consistent Systems - Unique solution case

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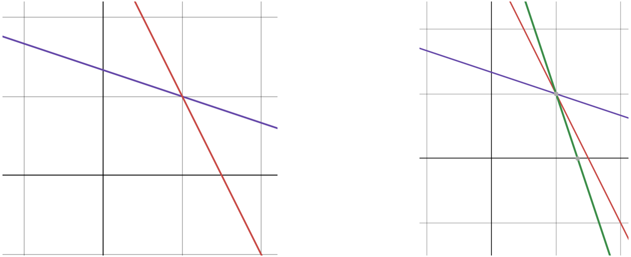

Figure: Unique Solution Scenarios

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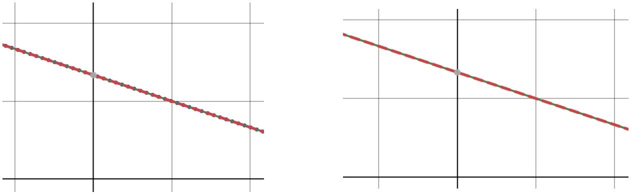

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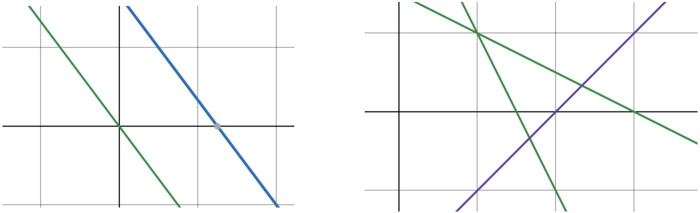

Figure: Inconsistent - No Solution Scenarios

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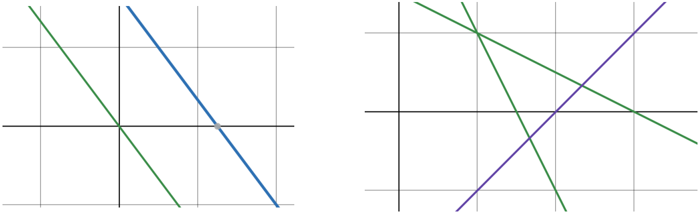

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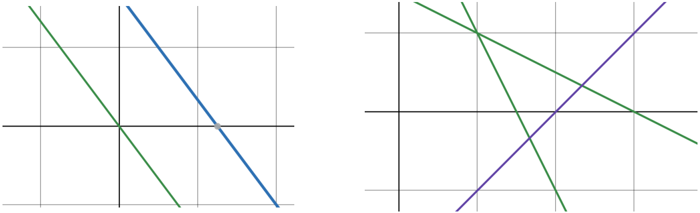

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- Same LHS in both equations, RHS are different - Inconsistency

System of linear equations in 3 unknowns

Example 4

- $$\begin{aligned} 2x + 3y + z &= 6 \\ x - 2y + 3z &= 2 \\ x &= 1 \end{aligned}$$

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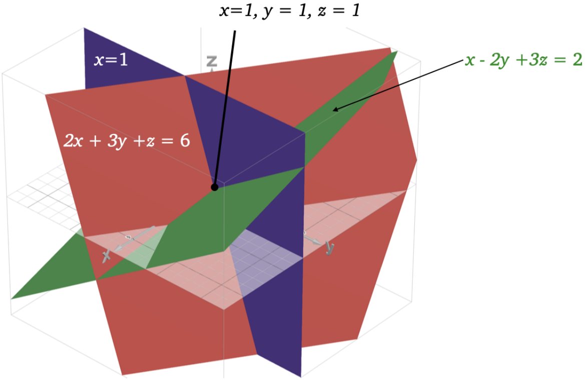

Example 4 - Some Observations

- Point of intersection $(x, y, z) = (1,\ 1,\ 1)$

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- That is **THE** solution

Example 5

- $$\begin{aligned} 2x + 3y + z &= 6 \\ x - 2y + 3z &= 2 \\ 3x + y + 4z &= 8 \end{aligned}$$

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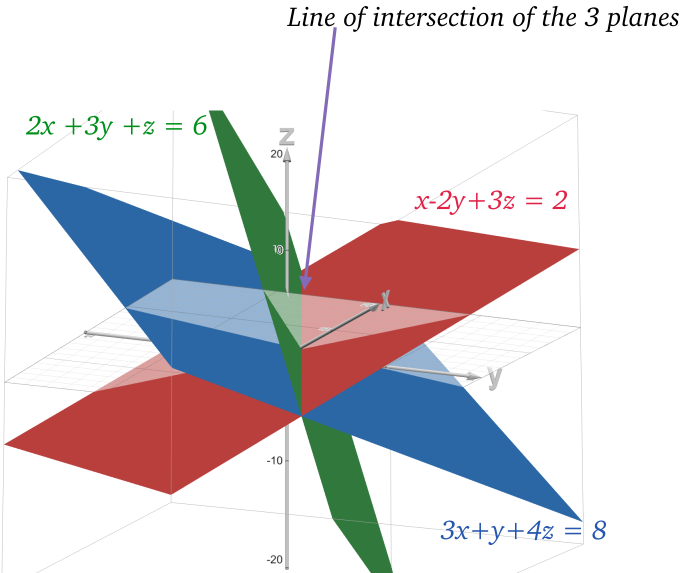
Line of intersection of the 3 planes
2x +3y +z = 6
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x
y
20
10
-10
-20

Example 5 - Some observations

- Intersection of planes in this case - Line

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- The parametric form of the line is given by $(x, y, z) = (-\frac{11}{7}t + \frac{18}{7}, \frac{5t}{7} + \frac{2}{7}, t)$

Example 5 - Some observations

- Intersection of planes in this case - Line
- The parametric form of the line is given by $(x, y, z) = (-\frac{11}{7}t + \frac{18}{7}, \frac{5t}{7} + \frac{2}{7}, t)$
- **Look at where a wall and the adjacent floor meet**

Example 6

- $$\begin{aligned} 2x + y + z &= -15 \\ 2x + y + z &= 20 \\ 2x + y + z &= 70 \end{aligned} \tag{1}$$

Some observations about Example 6

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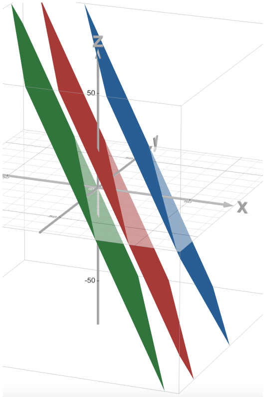

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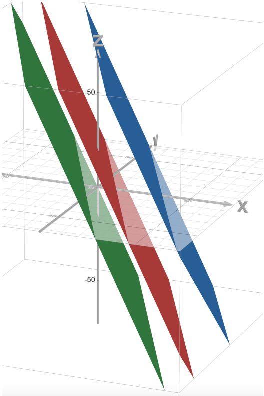
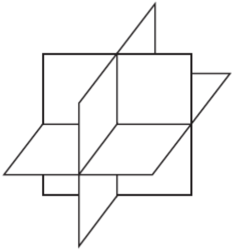


Planes do not intersect at all - **NO SOLUTION**

Visualizing consistent systems in 3D



Consistent System with infinitely many solutions

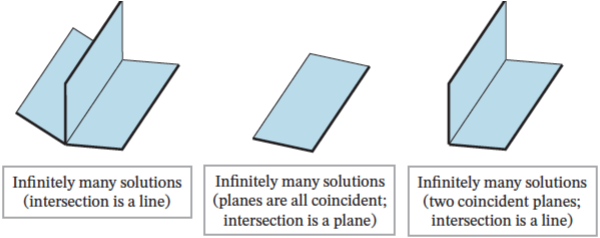


Figure: Infinitely Many solutions - Planes intersect in a line[1]

[1]Image Courtesy: Anton, Rorres, Kaul, *Elementary Linear Algebra*, Wiley, 2019, 12th Ed.

Inconsistent Systems

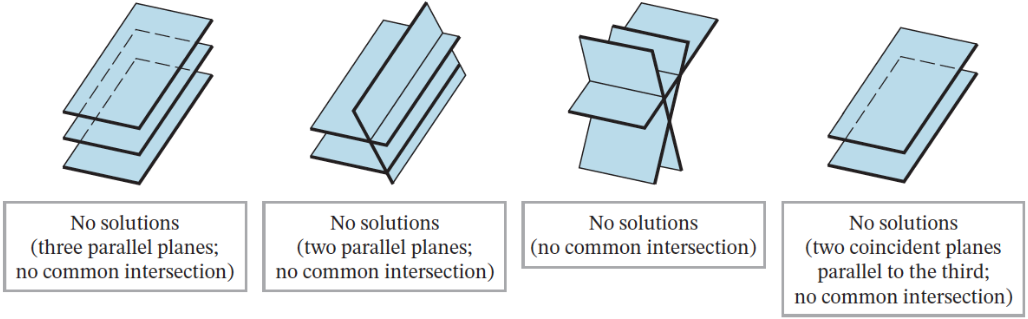


Figure: No solutions[2]

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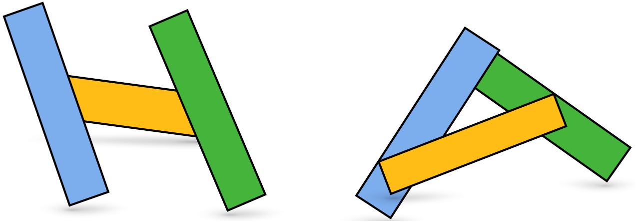

Some Comments

For a system of linear equations, be it in two equations in two unknowns or three equations in three unknowns, **there can only be**

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- An inconsistent system does not have solution
- Next Lecture - General system of linear equations, their solutions

Thank you!, Namaste

Lecture 3: General Linear System of Equations and Matrices

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Recap

- Geometry of system of linear equations in 2 and 3 unknowns
- Focus of this lecture - General system of linear equations followed by some matrix definitions

General System of Linear Equations

- Consider the system of linear equations,

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\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
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- First subscript - equation in which the coefficient occurs,
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- For example: a_{34} is a coefficient in the third equation and multiplies x_4.

Solution to the System of linear equations

- **Particular solution** to this linear system of equations in n unknowns - Sequence of n numbers, say $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ that satisfies each and every equation that constitutes the system of equations $A\mathbf{x} = \mathbf{b}$.

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- The answer is **NEED NOT BE**.

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Consider the following system of equations:

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ x_1 + 2x_2 &= 3 \\ x_1 + 3x_2 &= 4 \\ 4x_1 + 7x_2 &= 11 \end{aligned} \tag{1}$$

Observe the following

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- The number of equations is more than the number of unknowns, i.e., $m > n$
- Tall matrix - **overdetermined** system

Overdetermined System

- More constraints than the unknown
- May or may not have solution to the system of equations

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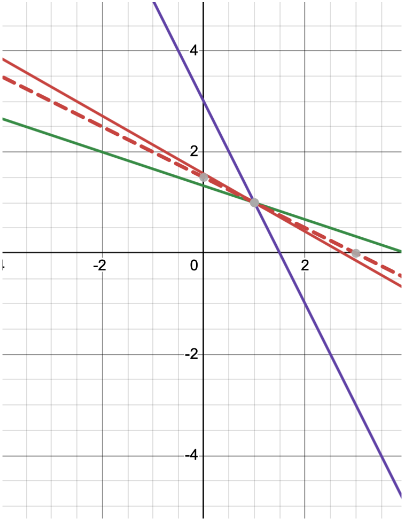

Figure: OverDetermined System with Unique solution. Solution is $x_1 = 1, x_2 = 1$.

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- Example:

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ 4x_1 + 2x_2 &= 6 \\ 3x_1 + 1.5x_2 &= 4.5 \\ 8x_1 + 4x_2 &= 12 \end{aligned}$$

Infinitely Many Solutions

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Example 2

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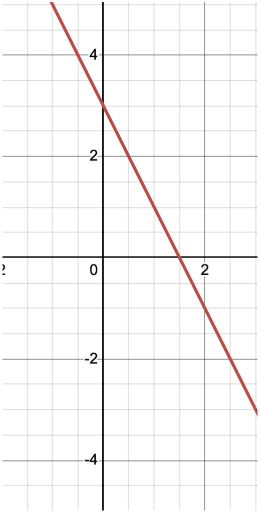

Infinitely Many Solutions

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$$
\begin{aligned}
2x_1 + x_2 &= 3 \\
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$$

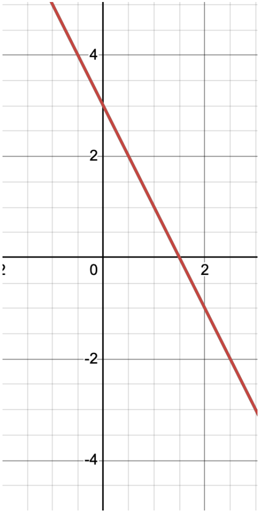

Figure: OverDetermined System with Infinitely many solutions.

No Solutions

- As a next step, we look at system which has constraints that are impossible to be met simultaneously

$$
\begin{aligned}
2x_1 + x_2 &= 3 \\
x_1 + 2x_2 &= 3 \\
x_1 + 3x_2 &= 5 \\
4x_1 + 7x_2 &= 4
\end{aligned}
$$

No Solution case

No Solution case

Overdetermined system with
no solution

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\begin{aligned}
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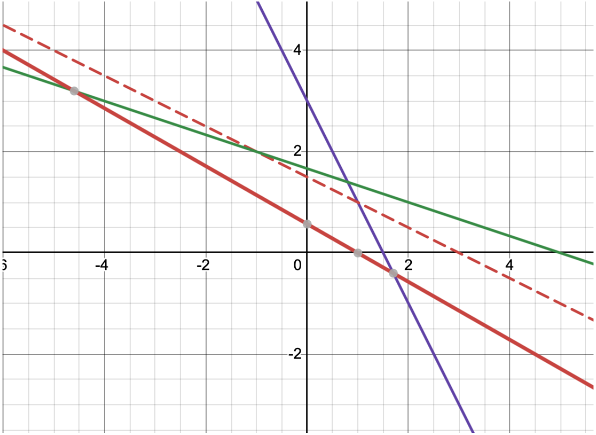

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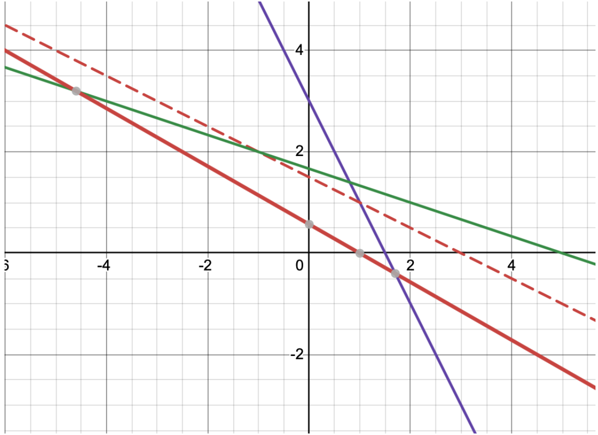

Figure: OverDetermined System with No solutions. Every pair of lines intersect at different points. Together, they do not intersect at any common point.

Overdetermined Systems

An over determined system ($m > n$) can have

- unique solution or

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- Later, from the perspective of vector spaces, what it means to say solution (unique or infinitely many) exists and no solution exists!
- Question: What happens when we have fewer equations than the number of variables?

Underdetermined System

$$
\begin{aligned}
2x_1 + 3x_2 + 4x_3 &= 0 \\
x_1 + 2x_2 + x_3 &= 3
\end{aligned}
$$

- 3 unknowns and only 2 equations

Underdetermined System

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- Suppose n denotes the number of unknowns, and m the number of equations, we see here that $m < n$.
- System of equations with more unknowns and fewer equations - underdetermined system
- What can we say about the solution to the system of equations of an underdetermined system?

Underdetermined System with infinitely many solutions

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 3 \end{aligned}$$

- The equations correspond to two planes

Underdetermined System with infinitely many solutions

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\begin{aligned}
2x_1 + 3x_2 + 4x_3 &= 0 \\
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$$

- The equations correspond to two planes
- Two planes either intersect in a line or do not intersect at all

Underdetermined System with infinitely many solutions

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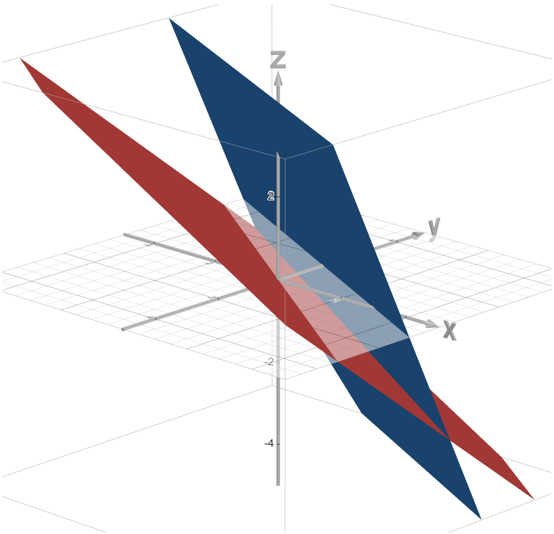
$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 3 \end{aligned}$$

Underdetermined System with infinitely many solutions

$$\begin{aligned} {\color{red}2x_1 + 3x_2 + 4x_3} &= 0 \\ {\color{blue}x_1 + 2x_2 + x_3} &= 3 \end{aligned}$$



Underdetermined System with infinitely many solutions

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 3 \end{aligned}$$

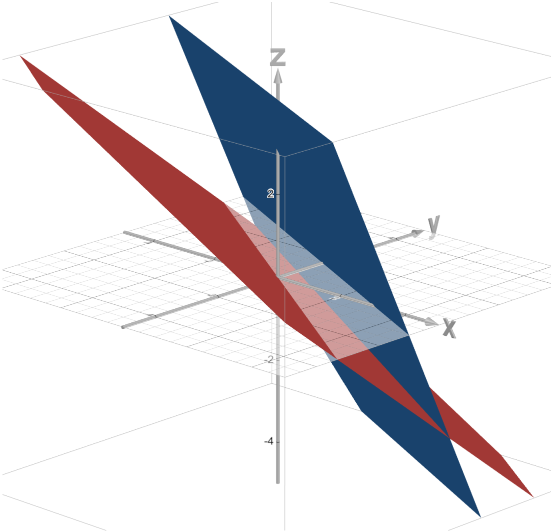


Figure: Underdetermined System with Infinitely many solutions. The two planes intersect in a line

Example 5

Underdetermined system - inconsistent or infinitely many solutions. Never a unique solution

Example 5

Here is an example of an underdetermined system with no solution

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 0 \\ 2x_1 + 3x_2 + 4x_3 &= 3 \end{aligned}$$

Underdetermined system - inconsistent or infinitely many solutions. Never a unique solution

Example 5

Here is an example of an underdetermined system with no solution

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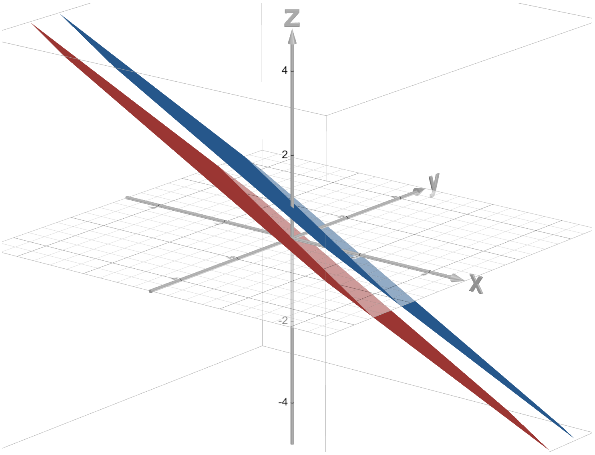

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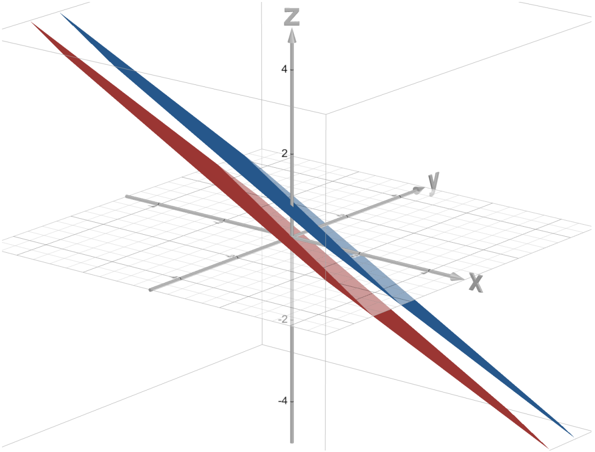

Figure: Underdetermined System with No solutions. The two planes are parallel

Underdetermined system - inconsistent or infinitely many solutions. Never a unique solution

Matrices

Matrices

- A typical matrix is defined as a rectangular array of numbers arranged in m rows and n columns

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$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

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- A : m by n real matrix if all its entries are real numbers

Some examples of matrices - You all know it, still

- $m < n$ - wide matrix. Example:

$$A = \begin{pmatrix} 1 & 0 & 2 & 5.38 \\ 2.1987 & 3 & -1 & 4 \end{pmatrix} \quad 2 \text{ by } 4 \text{ matrix} \ .$$

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- $m > n$ - tall matrix. Example

$$B = \begin{pmatrix} 1 & 0 \\ 2 & 5.38 \\ 2.1987 & \sqrt{2} \\ 3 & -1 \\ 4 & 7 \end{pmatrix} \quad \text{5 by 2 matrix - Tall Matrix.}$$

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- $m = n$, - square matrix. Example:

$$C = \begin{pmatrix} 1 & 0 & 2 \\ 2.1987 & \sqrt{2} & 7 \\ 3 & -17 & 0 \end{pmatrix} \quad \text{3 by 3 matrix.}$$

- A Matrix with only one row is called a row vector or a row matrix. Example:

$$D = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \quad \text{1 by 3 matrix.}$$

- Any matrix - collection of m row vectors
- A matrix with only one column is called a column vector or a column matrix. Example:

$$E = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \text{3 by 1 matrix}$$

- Any matrix - collection of n column vectors
- The 3-by-3 matrix $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is called an 3 by 3 IDENTITY matrixAn n $\times n$ identity matrix is denoted by $\mathcal{I}_{n\times n}$ or simply $\mathcal{I}$

Diagonal matrix

- A diagonal matrix is an n by n matrix with entries above and below the main diagonal are 0, that is a matrix is a diagonal matrix if and only is it is square and $d_{i,j} = 0, fori \neq j$.

  Example: $F = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 7.2 & 0 & 0 \\ 0 & 0 & 0.123 & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix}$
- Two matrices A and B are said to be **EQUAL** is and only if they have the same size and all of their corresponding entries are the same. $A = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, Not equal matrices though their entries are the same

Triangular Matrices

- A matrix is said to be upper triangular U, if it is a square matrix with all entries below the main diagonal equal to 0. An upper triangular matrix has its entries equal to zero at all those positions $a_{i,j}$ where $i > j$. Example: $U = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & \pi \end{pmatrix}$
- A matrix is said to be lower triangular L, if it is a square matrix with all entries aove the main diagonal equal to 0. A lower triangular matrix has its entries equal to zero at all those positions $a_{i,j}$ where $i < j$.

  Example: $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ \pi & -5 & 23 \end{pmatrix}$
- A matrix with all its entries is called a zero matrix. A zero matrix can be square or rectangular.

Summary

- In this lecture, we looked at general system of linear equations.
- We defined what an overdetermined system is. These are systems that have more equations than the unknowns.
- We also looked at underdetermined system as the system with fewer equations than unknowns.
- We looked at the visualizations for such systems
- We defined various matrices.
- In the next lecture, we will look at operations on matrices.

Namaste!!!