Singular Value Decomposition (SVD)

- Motivation for SVD.

$Ax = \lambda x$: Eigen Value Eigen Vector Problem.

- $Ax = b$; Classical Systems of Linear Equation.

If $A_{m \times m}$; Full rank,

then solution is $\underline{x = A^{-1} b}$

Algebraic

A ; find A^{-1} ; Matrix Inversion

$Ax = b$; $\Rightarrow \underline{A^{-1}}$

→ $Ax = b$; $A_{m \times m}$ matrix Full rank.

Realistic Situation is

Solve $Ax = b$; $A_{m \times n}$ - real matrix

$m \neq n$.

$m > n$; more rows than Columns

→ TALL matrices

$m < n$; FAT matrices

more Columns than rows.

→ Tall: → $x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$ → Linear systems

$Y = HX = \begin{bmatrix} Y_1 \\ \vdots \\ Y_k \end{bmatrix} = \begin{bmatrix} h_{11} & & \\ & \ddots & \\ & & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_m \end{bmatrix}$

Sources

~~Diff~~ Eqs

$k \times m \quad k > m$

$$X = H^{-1} Y$$

FAT: → Process Control / Metallurgy

$[\; H \;]^{-1}$ TO Solve

REALISTIC.

Very Important.

→ Inverting Square matrices, Full rank, is a bad problem

→ Size is Large (> 100 x 100) much worse.

→ Tall: large Size (1000 x 100)

→ Inverting it → Nightmare

→ SVD comes in.

What is our problem NOW!

Solve $Ax = b$ — ① $A_{m \times n}$, $m > n$,

A real matrix.

Algebra problem: <u>Symbol manipulation.</u>

Square $_{m=n}$ $\left(x = A^{-1} b\right)$ keep it in mind.

Premultiply both sides by A^T

$$A^T A x = A^T b \quad \text{—} \; ②$$

$A^T A \longrightarrow \underbrace{A^T_{n \times m} A_{m \times n}}$: Real Symmetric Matrix

$n \times n$: Square.

$$(A^T A)x = A^T b \quad -(3)$$

$$\boxed{\hat{A}x = \hat{b}} \quad -(3a)$$

$$\therefore \hat{x} = (\hat{A})^{-1}\hat{b}$$

$$\hat{x} = (A^T A)^{-1} A^T b \qquad A^{+} = (A^T A)^{-1} A^T$$

$$A^{+} b$$

$$A x = b \rightarrow \qquad \hat{x} \neq x, \quad \hat{b} \neq b$$

A^{+} : Generalized Inverse of A

Imp: Structured a "Easy"
Solution to Invert an NON Square matrix.

Solve for x; You got an answer $\hat{x}$

$\|\hat{x} - x\| = \epsilon > 0$; Not Exact

Solution; but "Approximate" to $\underline{x}$

Least Squared; (Not exact)

$Ax = b$: Inconsistent, Overdetermined or Underdetermined.

$\longrightarrow \infty$ solutions, you pick the "best" = LS Sense.

$$\underset{m\times n}{A}x = b \qquad m \neq n$$

$$\hat{A}\hat{x} = \hat{b} \quad ; \quad \left.\begin{array}{l}\hat{x} \simeq x \\ \hat{b} \simeq b\end{array}\right\} \text{Not Exact.}$$

No Solution for x, for given A & b.

$\exists$ $\hat{A}$, $\hat{b}$ for which $\hat{x}$ Exists.

$\left.\begin{array}{l}\hat{b} \text{ "Close" to } b \\ \hat{x} \text{ "Close" to } x\end{array}\right\}$ Close - 2 norm (min Length).

(AA^T) A^{-1} ; $\boxed{A^{+} \triangleq (A^TA)^{-1}A^T}$

$\underset{m \times n}{A} \to A^T A_{n \times n}$ } Square Real Symmetric

$\to A A^T_{m \times m}$ }

$\to$ Square ; Real Symmetric Matrices

$\underset{m \times m}{A}$ real Sym. matrices ; $\{\lambda_i\}_{i=1}^{m}$ are distinct;

$\{A x_i = \lambda_i x_i\}_{i=1}^{m}$

We can find $U = [x_1\, x_2 \cdots x_m]$

Eigen vectors

Normalized

ortho

S.T. $U^T A U = D$

$\Rightarrow U$; S.T. $U^T U \quad U U^T = I$

Orthonormal ~~matrix~~ Orthogonal

→ To do SVD we use many things

→ Orthogonal - orthonormal vectors

→ Eigenspace of Real Sym. Matrix

$U^T A U = D$: $A \to D$

$A \to U \xrightarrow[\text{orthonormal}]{} U^T\ U$

$U^T A U = D \equiv \boxed{U^T D U = A}$

Reverse Sense: A - Factorized

A = 3 Components; U, U^T, D

SVD → Matrix Factorization → LU → LUD → QR.

→ Recap:

$Ax = b$; $\underset{m \times n}{A}$ $m \neq n$

→ $A^{+} = (A^T A)^{-1} A^T$

$\hat{A}\hat{x} = \hat{b}$ – $\left.\begin{matrix} \hat{b} \neq b \\ \hat{x} \neq x \end{matrix}\right\}$ "Close" LS Sense.

→ Algebraic

→ Geometric way to Solve the Same.