$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ A : Real & Symm.

$\lambda_1 = 3$, $ev_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓

$\lambda_2 = 1$ $ev_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ✓

$ev_1 \cdot ev_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0.$

$\Rightarrow ev_1 \perp$ to ev_2.

$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

length of eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sqrt{2}$.

$\hat{ev}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $\hat{ev}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$$R = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$R^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = R^T$$

$$R^T R = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(\theta) \end{bmatrix}$$

$$R_\theta^{-1} = R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R_\theta^T$$

A matrix A is Orthogonal matrix if

(i) $A_i \cdot A_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

A_i & A_j are the columns of A.

The eigenvectors corresponding to distinct eigenvalues of a real Symmetric matrix are ORTHOGONAL to each other.

Suppose A is a 2×2 real Symmetric matrix with distinct eigenvalues λ_1 & λ_2.

Let u be the eigenvector associated with λ_1 and v be the eigenvector associated with λ_2.

We know that

$$A u = \lambda_1 u$$
$$A v = \lambda_2 v.$$

Since A is a real symmetric matrix we have

$$(Au)^T = (\lambda_1 u)^T$$
$$\Rightarrow u^T A^T = u^T \lambda_1 = \lambda_1 u^T.$$
$$= u^T A = \lambda_1 u^T \quad (\because A^T = A). \longrightarrow (1)$$

Multiply both sides of (1) by v.

$$u^T A v = \lambda_1 u^T v \rightarrow (2).$$

Similarly we know that

$$A v = \lambda_2 v \quad (*)$$

Pre Multiplying both sides of $(*)$ by u^T

$$u^T A v = u^T \lambda_2 v$$

$$\Rightarrow u^T A v = \lambda_2 u^T v \rightarrow (3)$$

Eqn (2) & Eqn (3) have the same LHS.

$$\Rightarrow \lambda_1 u^T v = \lambda_2 u^T v.$$

$$\Rightarrow (\lambda_1 - \lambda_2)(u^T v) = 0.$$

$\Rightarrow$ either $u^T v = 0$ or $\lambda_1 - \lambda_2 = 0$.

But $\lambda_1 - \lambda_2 \neq 0$ as

$\lambda_1 \neq \lambda_2$ (λ_1, λ_2 distinct)

$\Rightarrow u^T v = 0$

$\Rightarrow u \cdot v = 0$

$\Rightarrow$ The eigenvectors corresponding to distinct eigenvalues of a RSM are orthogonal to each other

Ex: $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$ Projection Matrix that projects every vector onto x_1-axis

$$(A - \lambda I) = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 0-\lambda \end{pmatrix}.$$

$$\det(A - \lambda I) = 0 \Rightarrow (-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \lambda(\lambda - 1) = 0$$

$$\Rightarrow \lambda = \underline{\underline{0}}; \lambda = \underline{\underline{1}}.$$

$\lambda = 0 \Rightarrow \begin{bmatrix} 1-0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x_1 = 0; \quad 0x_2 = 0 \Rightarrow x_2 = k$, real scalar

$\therefore$ eigen vector corresponding to $\lambda = 0$ is $\left\{ k \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\lambda = 1 \Rightarrow \begin{bmatrix} 1-1 & 0 \\ 0 & 0-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad 0x_1 = 0 \Rightarrow x_1 = k$ real scalar

$-x_2 = 0 \Rightarrow x_2 = 0.$

$\therefore$ eigenvector associated with $\lambda = 1$ is $\left\{ k \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$