Recall:

Suppose $A^{n\times n}$ is a real matrix and $\lambda_1, \lambda_2 \ldots \lambda_n$ are n- distinct eigenvalues of $A^{n\times n}$ and $ev_1, ev_2 \ldots ev_n$ be the eigenvectors associated with the eigen values $\lambda_1 \ldots \lambda_n$ respectively

$$P = \begin{bmatrix} ev_1 & ev_2 & \ldots & ev_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

then we have the following.

The algebraic multiplicity of $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$

Geometric Multiplicity $= 1$ corresp to

One l.i. vector each associated with each eigen value.

$\Rightarrow$ We can express

$$\boxed{A = P D P^{-1}}$$

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\Rightarrow$ $A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$

Characteristic eqn is
$\det(A - \lambda I) = 0$.

$\Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1$, twice.

$\Rightarrow$ AM of $\lambda = 1$ is 2.
(Algebraic Multiplicity).

$(A - \lambda I)u = 0$

$\Rightarrow \begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad u_2 = 0,$

$0 u_1 = 0 \Rightarrow u_1 = k$ any scalar.

$\therefore$ The eigenvector associated with $\lambda_1 = 1$ is $\begin{bmatrix} k \\ 0 \end{bmatrix} = k \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$.

Geometric Multiplicity:

Dimension of the invariant subspace associated with a specific eigenvalue λ_i.

For $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\lambda_1 = 1$ has AM $= 1$

GM $= 1$.

Ex:2: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$\therefore$ Char eqn: $(1-\lambda)(2-\lambda)^2 = 0$

$\Rightarrow \lambda_1 = 1; \quad \lambda_2 = 2$, twice.

AM of $\lambda_1 = 1$ is 1

AM of $\lambda_2 = \underline{\underline{2}}$ is 2

$\lambda_1 = 1$

$$\begin{bmatrix} 1-1 & 0 & 0 \\ 0 & 2-1 & 0 \\ 0 & 0 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_2 = 0; \quad x_3 = 0 \quad . \; 0 x_1 = 0 \Rightarrow x_1 = k$

k a real.

Scalar.

The eigenspace associated with $\lambda_1 = 1$ is the subspace $\left\{ k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\lambda_2 = 2$.

$$A - 2I = \begin{bmatrix} 1-2 & 0 & 0 \\ 0 & 2-2 & 0 \\ 0 & 0 & 2-2 \end{bmatrix}$$

$$(A-2I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$.

$x_2 = s$, s real

$x_3 = t$, t real.

$\Rightarrow$ eigenvector: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ s \\ t \end{pmatrix}$

$$\left\{ s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Dimension of the eigenspace associated with $\lambda = 2$ twice is 2.

$\Rightarrow$ GM of $\lambda = 2$ is $\underline{2}$.

Geometric Multiplicity $\leq$ Algebraic Multiplicity.

* If for some eigenvalue, λ, of a matrix A, the GM < AM, then we say that the eigenvalue λ is defective.

* If $\sum gm = n$, then the matrix A can be diagonalized!

A matrix $A^{n \times n}$ is called defective if at least one of the eigen values has GM < AM.

For ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has

$\lambda_1 = 1$ twice AM = 2; GM = 1

$\Rightarrow$ $\lambda_1 = 1$ is defective.

$\Rightarrow$ $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is defective

and therefore cannot be diagonalized.