



12080CH07

INTEGRALS

❖ *Just as a mountaineer climbs a mountain – because it is there, so a good mathematics student studies new material because it is there. — JAMES B. BRISTOL* ❖

7.1 Introduction

Differential Calculus is centred on the concept of the derivative. The original motivation for the derivative was the problem of defining tangent lines to the graphs of functions and calculating the slope of such lines. Integral Calculus is motivated by the problem of defining and calculating the area of the region bounded by the graph of the functions.

If a function f is differentiable in an interval I , i.e., its derivative f' exists at each point of I , then a natural question arises that given f' at each point of I , can we determine the function? The functions that could possibly have given function as a derivative are called anti derivatives (or primitive) of the function. Further, the formula that gives all these anti derivatives is called the *indefinite integral* of the function and such process of finding anti derivatives is called integration. Such type of problems arise in many practical situations. For instance, if we know the instantaneous velocity of an object at any instant, then there arises a natural question, i.e., can we determine the position of the object at any instant? There are several such practical and theoretical situations where the process of integration is involved. The development of integral calculus arises out of the efforts of solving the problems of the following types:

- (a) the problem of finding a function whenever its derivative is given,
- (b) the problem of finding the area bounded by the graph of a function under certain conditions.

These two problems lead to the two forms of the integrals, e.g., indefinite and definite integrals, which together constitute the *Integral Calculus*.



G .W. Leibnitz
(1646 -1716)

There is a connection, known as the **Fundamental Theorem of Calculus**, between indefinite integral and definite integral which makes the definite integral as a practical tool for science and engineering. The definite integral is also used to solve many interesting problems from various disciplines like economics, finance and probability.

In this Chapter, we shall confine ourselves to the study of indefinite and definite integrals and their elementary properties including some techniques of integration.

7.2 Integration as an Inverse Process of Differentiation

Integration is the inverse process of differentiation. Instead of differentiating a function, we are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called *integration* or *anti differentiation*.

Let us consider the following examples:

We know that $\frac{d}{dx}(\sin x) = \cos x$... (1)

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2 \quad \dots (2)$$

and $\frac{d}{dx}(e^x) = e^x$... (3)

We observe that in (1), the function $\cos x$ is the derived function of $\sin x$. We say that $\sin x$ is an anti derivative (or an integral) of $\cos x$. Similarly, in (2) and (3), $\frac{x^3}{3}$ and e^x are the anti derivatives (or integrals) of x^2 and e^x , respectively. Again, we note that for any real number C , treated as constant function, its derivative is zero and hence, we can write (1), (2) and (3) as follows :

$$\frac{d}{dx}(\sin x + C) = \cos x, \quad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2 \quad \text{and} \quad \frac{d}{dx}(e^x + C) = e^x$$

Thus, anti derivatives (or integrals) of the above cited functions are not unique. Actually, there exist infinitely many anti derivatives of each of these functions which can be obtained by choosing C arbitrarily from the set of real numbers. For this reason C is customarily referred to as **arbitrary constant**. In fact, C is the **parameter** by varying which one gets different anti derivatives (or integrals) of the given function.

More generally, if there is a function F such that $\frac{d}{dx} F(x) = f(x)$, $\forall x \in I$ (interval), then for any arbitrary real number C , (also called *constant of integration*)

$$\frac{d}{dx}[F(x) + C] = f(x), \quad x \in I$$