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In this note, we look at the correspondence between the Quantum Kicked Rotor at Resonance condition and the Continuous-Time Quantum Random Walk in momentum basis.

1) Quantum Kicked Rotor at Resonance.

The QKR is described by the Hamiltonian:

$$\hat{\mathcal{H}}(\hat{\theta}, \hat{l}, t) = \frac{\hbar^2 \hat{l}^2}{2} + K\cos\hat{\theta} \sum_{j=1}^{T} \delta(t - j\tau)$$

where $\tau \rightarrow$ Kicking period
$K \rightarrow$ Kicking strength
$\hbar \rightarrow$ Planck's constant
$\hat{\theta} \rightarrow$ angle operator
$\hat{l} \rightarrow$ ang. momentum number.

- To completely non-dimensionalize the Hamiltonian we can either do $(\tau = 1)$ or $(\hbar = 1)$.

- At resonance conditions, we set:

  (1) $\tau = 1$, $\hbar = 4\pi$ (For numerical calculations)

  (2) $\hbar = 1$, $\tau = 4\pi$ (Used by experimentalists)

The Floquet operator (Stroboscopic time-evolution operator) then becomes:

$$F = e^{-\frac{i\hbar\hat{l}^2}{2}} e^{-iK\cos\theta}$$

For resonance, we set $\hbar = 4\pi$,

$$F = \underbrace{e^{-(2\pi i)\ell^2}}_{\text{Becomes 1 because } \ell^2 \text{ is a number.}} e^{-ik\cos\theta}$$

So $$\boxed{F = e^{-ik\cos\theta}}$$

At Quantum Resonance,

the momentum distribution expands around its initial state ballistically ie $\sigma(t) \propto t$.

Suppose we start with an initial momentum eigenstate:

$$\psi(0) = \frac{1}{\sqrt{2\pi}} e^{in_0\theta}$$

After 't' kicks we have:

$$\psi(t) = F^t \psi(0)$$

The probability distribution in the momentum basis will be:

$$\langle n | F^t \psi(0) \rangle = \frac{1}{\sqrt{2\pi}} e^{-in\theta} e^{-ikt\cos\theta} \frac{1}{\sqrt{2\pi}} e^{in_0\theta}$$

$$= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-ikt\cos\theta} e^{i(n_0-n)\theta}$$

$$= -i^{n-n_0} J_{n-n_0}(kt)$$

We use the identity: $\frac{1}{2\pi} \int_0^{2\pi} d\theta \, e^{iz\cos\theta} e^{-in\theta} = i^n J_n(z)$

So

$$\boxed{P_K(n,t|n_0) = |J_{n-n_0}(Kt)|^2}$$

* The probability of measuring the momentum state 'n' after time 't' or 't' kicks, given the initial state 'n_0' and kick strength 'K' is given by a Bessel Function.

Continuous time Quantum Walks. (CTQW)

* We obtain CTQW by quantising the classical CT-Markov chains.

For a CTMC,

if the initial distribution is $\vec{p}(0)$ and the transition matrix is M, then $\vec{p}(t)$ is:

$$\vec{p}(t) = M(t)\,\vec{p}(0)$$

where $\boxed{M(t) = e^{-Ht}}$

H is the generating matrix of M.

For the quantum case, we define the unitary operator

$U(t) = e^{-iHt}$ such that:

$$|\Psi(t)\rangle = U(t)\,|\Psi(0)\rangle$$

and

$$P_k = |\langle 4|\Psi(t)\rangle|^2$$

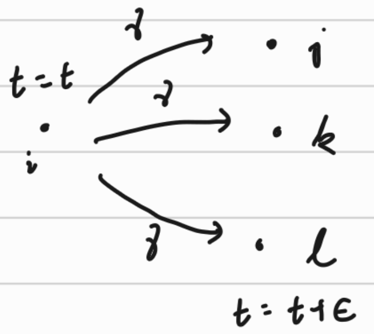
Ex: CTQW on a line:

def $$H_{ij} = \begin{cases} 2\gamma, & \text{if } i = j \\ -\gamma, & \text{if } i \neq j \text{ but } (i,j) \in E(G) \\ 0, & \text{if } i \neq j \text{ and } (i,j) \notin E(G) \end{cases}$$

where $\gamma \rightarrow$ transition rate.

Elucidation

- Consider a random walker on a graph $G = (V, E)$
- When time is a continuous variable, the RW can jump from vertices $x_i \rightarrow x_j$ given $(x_i, x_j) \in E(G)$ at any time.
- We can assume a constant transition probability γ for all vertices to its edges (homogeneity and isotropy)



- Consider a time evolution from $t \rightarrow t + \epsilon$

Recipe to solve the problem:

1. use infinitesimal time interval,
2. setup differential equation
3. Solve the DE.

If initially at 't', the walker is at vertex x_i then at '$t+\epsilon$', the walker is at one of x_i's neighbour with probability '$\epsilon \gamma d_i$' where $d_i \rightarrow$ degree of x_i

then $M_{ij}(t)$ is the probability of the particle which is at vertex x_j going to the vertex x_i in time 't'

So
$$M_{ij}(\epsilon) = \begin{cases} 1 - d_j \epsilon \gamma + O(\epsilon^2) & , i = j \\ \gamma \epsilon & , i \neq j \end{cases}$$

We define an auxillary matrix 'H', called generating matrix

$$H_{ij} = \begin{cases} d_{ij} \gamma & , i = j \\ -\gamma & , i \neq j \text{ and adjacent} \\ 0 & , i \neq j \text{ and non-adjacent} \end{cases}$$

Since the process is Markovian, we can write :

$$M_{ij}(t+\epsilon) = \sum_k M_{ik}(t) M_{kj}(\epsilon)$$

$$= M_{ij}(t) M_{jj}(\epsilon) + \sum_{k \neq j} M_{ik}(t) M_{kj}(\epsilon)$$

$$= M_{ij}(t) \left[1 - \epsilon H_{jj} \right] + \sum_{k \neq j} M_{ik}(t) H_{kj}$$

$$\frac{M_{ij}(t+\epsilon) - M_{ij}(t)}{\epsilon} = -H_{jj} + \frac{1}{\epsilon} \sum_{k \neq j} M_{ik}(t) H_{kj}$$

$\lim_{\epsilon \to 0}$:
$$\frac{dM_{ij}(t)}{dt} = -\sum_k H_{kj} M_{ik}(t)$$

or
$$\boxed{M(t) = e^{-Ht}}$$

Solution of the differential equation

So :
$$\vec{p}(t) = M(t) \vec{p}(0)$$

or
$$\boxed{\vec{p}(t) = e^{-Ht} \vec{p}(0)}$$

Back to CTQW on a line.

$$\hat{\mathcal{H}}_\gamma |m\rangle = -\gamma|m-1\rangle + 2\gamma|m\rangle - \gamma|m+1\rangle$$

In matrix form,

$$(1) \longleftarrow \langle n|\mathcal{H}_\gamma|m\rangle = \begin{cases} 2\gamma, & \text{if } n=m \\ -\gamma, & \text{if } n=m\pm 1 \\ 0, & \text{otherwise} \end{cases}$$

- Consider the Floquet operator of the QKR at resonance:

$$F = e^{-ik\cos\theta} \xrightarrow[\text{generated by}]{\text{looks like}} \mathcal{H}_k = K\cos\theta$$

- In the ang. momentum basis, this becomes:

$$\langle n|\mathcal{H}_K|m\rangle = \langle n| \int_0^{2\pi} \frac{d\theta}{\sqrt{2\pi}} |\theta\rangle\langle\theta| K\cos\theta |m\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \langle n|\theta\rangle K\cos\theta \langle\theta|m\rangle \, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} K \frac{e^{i\theta}+e^{-i\theta}}{2} e^{im\theta} \, d\theta$$

$$= \frac{K}{4\pi}\int_0^{2\pi} d\theta \, e^{i(1-n+m)\theta} \quad e^{-i(1+n-m)}$$

$$(2) \longleftarrow \langle n|\mathcal{H}_K|m\rangle = \begin{cases} \frac{K}{2} & \text{if } m=n\pm 1 \\ 0 & \text{, otherwise} \end{cases}$$

From (1) and (2) we can see that:

$$\boxed{\hat{H}_\gamma = 2\gamma\mathbb{I} - \hat{H}_K} \qquad (K = +2\gamma)$$

Probability of being in the n^{th} momentum eigenstate:

(1) QKR : $$P_k(n,t) = \|\langle n| F^t |\psi_0\rangle\|^2$$
$$= \|\langle n| e^{-iH_k t} |\psi_0\rangle\|^2$$

(2) CTQW :
$$P_\gamma(n,t) = \|\langle n| e^{-iH_\gamma t} |\psi_0\rangle\|^2$$
$$= \|\langle n| e^{-i(2\gamma \mathbb{I} - H_k)t} |\psi_0\rangle\|^2$$
$$= \|\langle n| e^{-i2\gamma t} e^{iH_k t} |\psi_0\rangle\|^2$$

set $\gamma = \frac{k}{2}$

$$= \|\langle n| \underbrace{e^{-ikt}}_{\text{just a phase}} e^{iH_k t} |\psi_0\rangle\|^2$$

$$= \|\langle n| e^{iH_k t} |\psi_0\rangle\|^2 = P_k(n,-t)$$

So we have: $\boxed{P_\gamma(n,t) = P_k(n,-t)}$ $k = 2\gamma$

* If $k = -2\gamma$ is chosen then the probability distributions are equal.

* A simple QKR at resonance conditions directly implements a CTQW in momentum space.