Quantum Kicked Rotator

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classically,
$$\mathcal{H}(q,p,t) = \frac{p^2}{2} + K\cos q\, \delta_1(t)$$

and EOM: <u>Chirikov map</u>:

$$\begin{cases} p_{n+1} = p_n + K\sin q_n \\ q_{n+1} = q_n + p_n \end{cases}$$

In the quantum version, $q \to \hat{x}$, $p \to \hat{p}$, $\mathcal{H} \to \hat{\mathcal{H}}$

So
$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2} + K\cos\hat{x}\,\delta_1(t)$$

and $[\hat{x}, \hat{p}] = i\hbar$ or

$$\mathcal{H} = \mathcal{H}_0 + V(\theta)\sum_{n=1}^{\infty}\delta(t-n\tau)$$

$$= -\frac{1}{2}\hbar^2\frac{\partial^2}{\partial\theta^2} - K\cos\theta\sum_{n=1}^{\infty}\delta(t-n\tau)$$

particle restricted on a ring (Free Hamiltonian)

Periodic kick

- Homogeneous ring parallel to ring plane.

Given

$$\mathcal{H} = \frac{\hbar^2}{2}\frac{\partial^2}{\partial\theta^2} + K\cos\theta \sum_{n=1}^{\infty} \delta(t-n\tau)$$

$$= \frac{\hat{\ell}^2}{2} + K\cos\hat{\theta} \sum_{n=1}^{\infty} \delta(t-n\tau)$$

$$= \frac{\hbar^2 \ell^2}{2} + K\cos\hat{\theta} \sum_{n=1}^{\infty} \delta(t-n\tau)$$

We know that,

$$i\hbar \frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$

$$\Rightarrow \Psi(t) = \underbrace{e^{\int_0^t \frac{-iH\,dt}{\hbar}}}\,\Psi(0)$$

$$= U(t)\,\Psi(0)$$

We solve $\int_0^\tau \frac{-i\hat{H}(t)dt}{\hbar}$ where τ time period of kicks

$$I = \int_0^\tau -\frac{i\hbar\ell^2 dt}{2} - \int_0^\tau \frac{i K\cos\theta}{\hbar} \sum_{n=1}^{\infty} \delta(t-nt)\,dt$$

$$I = -\frac{i\hbar\ell^2\tau}{2} - \frac{iK\cos\theta}{\hbar}$$

we can set $\tau = 1$

So $$U(t) = \exp\left\{ \frac{-i\hbar\ell^2}{2} + \frac{iK\cos\theta}{\hbar} \right\}$$

and

$$|\Psi(\tau)\rangle = \underbrace{FFT}_{\text{Fourier transform}}\left\{ e^{\frac{-iK\cos\theta}{\hbar}} \left[\underbrace{iFFT}_{\text{Inverse fourier transform.}}\left(e^{-\frac{i\hbar\ell^2}{2}} |\Psi(\tau-1)\rangle \right) \right] \right\}$$

Stroboscopic description

This unitary time evolution operator $U(t)$ is also called the **Floquet operator** $F(\tau)$

For the Quantum kicked rotor,

$$F = \exp\left(-\frac{iK\cos\theta}{\hbar}\right)\exp\left(-\frac{i\tau L^2}{2}\right)$$

In the eigenbasis of L: $\{|n\rangle, |m\rangle\}$ where $\langle\theta|n\rangle = \frac{e^{in\theta}}{\sqrt{2\pi}}$

$$F_{nm} = \langle n|F|m\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\, e^{-\frac{iK\cos\theta}{\hbar}}\, e^{-\frac{i\tau L^2}{\hbar 2}}\, e^{im\theta}\, d\theta$$

$$= e^{-\frac{i\tau \hbar m^2}{2}}\, \frac{1}{2\pi}\int_0^{2\pi} \exp\left(-\frac{iK\cos\theta}{\hbar}\right) e^{+i(m-n)\theta}\, d\theta$$

$$\boxed{F_{nm} = \exp\left(-\frac{i\tau m^2}{2\hbar}\right) i^{m-n}\, J_{m-n}\left(\frac{K}{\hbar}\right)}$$

Bessel functions.

- We can use this F_{nm} matrix in the L-basis to evaluate time-evolution and other properties of the system at various times.

so
$$\langle l|\psi(T)\rangle = F^T\langle l|\psi(0)\rangle$$

and
$$\langle L^2\rangle = \psi^\dagger(T)\, L^2\, \psi(T)$$

$$= \hbar^2 \sum_{l,m,n} l^2 (F^N)_{lm} (F^N)^*_{ln}\, a_m a_n^*$$

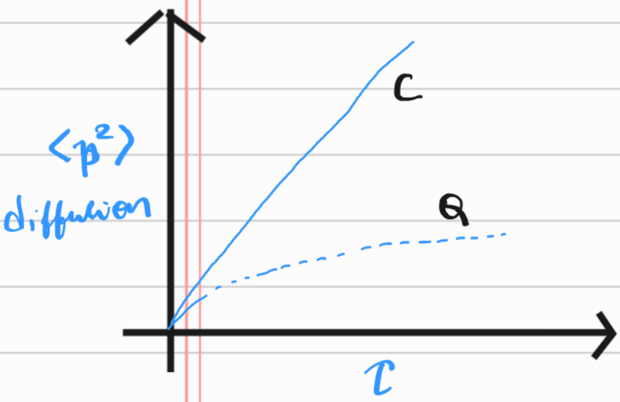
initial state

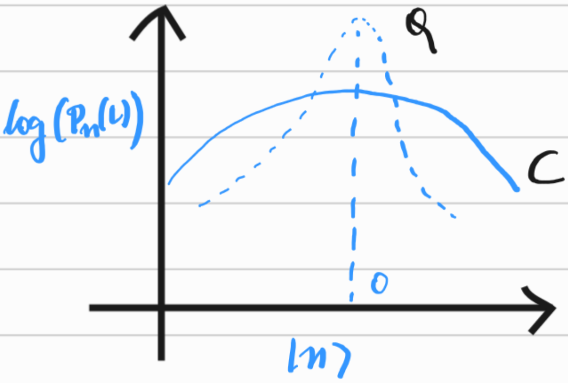
Observation :

for large k values, after say 100 kicks ;

- classical phase space → chaotic (diffusion)

- QM → Saturation after a point (localisation)





Q → exponential localization

So $\quad P_n^q(l) = \frac{1}{l_s} \exp\left\{ -\frac{2|l|}{l_s} \right\}$ localization length