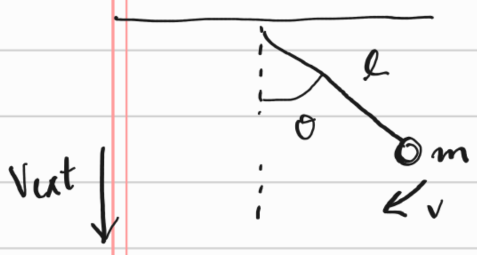
Kicked rotor systems (Revision Jan 2, 2021)

1) Classical simple rotor



$$T = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m l^2 \dot{\theta}^2 \quad (v = l\dot{\theta})$$

$$V = -mgl\cos\theta$$

$$\mathcal{L} = T - V$$
$$= \tfrac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

$$p = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \quad \text{(canonical momentum)}$$

$$\mathcal{H}(p,\theta) = \left(\frac{\partial L}{\partial \dot{\theta}}\right)\dot{\theta} - L$$

$$\boxed{\mathcal{H}(p,\theta) = \frac{p^2}{2ml^2} - mgl\cos\theta}$$

Hamilton's EOM: $(q \equiv \theta)$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{ml^2} \quad ; \quad \dot{p} = -\frac{\partial H}{\partial q} = -mgl\sin q$$

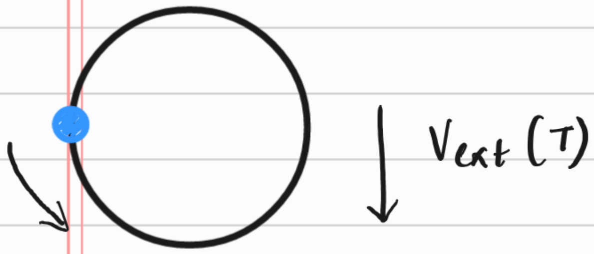
$$\Rightarrow \quad ml^2\ddot{q} = -mgl\sin q$$

$$\boxed{\ddot{q} = -\omega^2 \sin q} \xrightarrow[\text{approx}]{\text{small } q} \ddot{q} = -\omega^2 q$$

$$\omega = \sqrt{\frac{g}{l}}$$

Solution : $\boxed{q(t) = Ae^{i\omega t} + Be^{-i\omega t}}$

2) Classical Kicked rotor : (1D)



(1) The external potential responsible for motion is periodic

$$V_{ext}(t+\tau) = V_{ext}(t)$$

(2) Vext is a delta pulse

periodic kicking $\longrightarrow$ $V_{ext}(t) = \begin{cases} 0, & t \neq n\tau \\ K, & t = n\tau, \; n \in I^+ \end{cases}$

now,

$$\boxed{H(q,p,t) = \frac{p^2}{2ml^2} - mgl\cos\theta\,\delta(t-n\tau)}$$

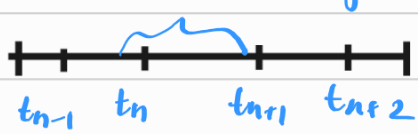
$$\dot{p} = -\frac{\partial H}{\partial q} = -mgl\sin q\,\delta(t-n\tau) \quad \text{—— (1)}$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{ml^2} \quad \text{—— (2)}$$

Now we integrate $\dot{p}$ between two successive kicks: (t_n, t_{n+1})

$$\lim_{\beta \to 0} \int_{t_n-\beta}^{t_{n+1}-\beta} dp = -\lim_{\beta \to 0} mgl\sin q \int_{t_n-\beta}^{t_{n+1}-\beta} \delta(t-n\tau)\,dt$$

limits of integ.



$$\lim_{\beta\to 0}\left[p(t_{n+1}-\beta) - p(t_n-\beta)\right] = -mgl\sin q$$

using $\lim \beta\to 0$, we get :

$$p(t_{n+1}) - p(t_n) = -mgl\sin q \quad — (3)$$

Now we integrate $\dot{q}$ between two successive kick timings.

$$\lim_{\beta\to 0}\int_{t_n-\beta}^{t_{n+1}-\beta} dq = \frac{1}{ml^2}\lim_{\beta\to 0}\int_{t_n-\beta}^{t_{n+1}-\beta} p\,dt$$

$$\boxed{\begin{aligned}\text{put } m&=1\\ l&=1\end{aligned}}$$

$$\lim_{\beta\to 0}\left[q(t_{n+1}-\beta) - q(t_n-\beta)\right] = \lim_{\beta\to 0}\left[p(t_{n+1}-\beta)(t_{n+1}-\beta) - p(t_n-\beta)(t_n-\beta)\right]$$

* between two kicks, 'p' is constant / conserved.

RHS : $\{\, t_{n+1}\,p(t_{n+1}-\beta) - t_n\,p(t_n-\beta) + \beta\left[p(t_n-\beta) - p(t_{n+1}-\beta)\right]$

$\lim_{\beta\to 0}$: $t_{n+1}\,p(t_{n+1}) - t_n\,p(t_n)$

So we have : $q(t_{n+1}) - q(t_n) = \underbrace{t_{n+1}\,p(t_{n+1}) - t_n\,p(t_n)}$

$$= p(t_{n+1}) \quad — (4)$$

(3) and (4) :

$$\boxed{\begin{aligned}p_{n+1} &= p_n + K\sin q_n\\ q_{n+1} &= q_n + p_{n+1}\end{aligned}}$$

CHIRIKOV MAP

$K \to$ stochasticity parameter

Non-dimensionalize the Hamiltonian:

$$\mathcal{H}(q, p, t) = \frac{p^2}{2ml^2} - mgl\cos q\, \delta(t - n\tau)$$

here $(m, g, l) \rightarrow$ constants $(+\tau)$

$(p, q, t) \rightarrow$ physical parameters that vary

let
$$p = p_c \tilde{p}$$
$$q = q_c \tilde{q}$$
$$t = t_c \tilde{t}$$

(p_c, q_c, t_c) all constants.

$(\tilde{p}, \tilde{q}, \tilde{t})$ all new variables.

then

$$\mathcal{H}(\tilde{q}, \tilde{p}, \tilde{t}) = \left(\frac{p_c^2}{ml^2}\right) \frac{\tilde{p}^2}{2} - mgl\cos q_c\tilde{q}\, \delta(t_c\tilde{t} - n\tau)$$

$$\mathcal{H} = \left(\frac{p_c^2}{ml^2}\right) \frac{\tilde{p}^2}{2} - \left(\frac{mgl}{t_c}\right) \cos q_c\tilde{q}\, \delta\left(\tilde{t} - \frac{n\tau}{t_c}\right)$$

choose:

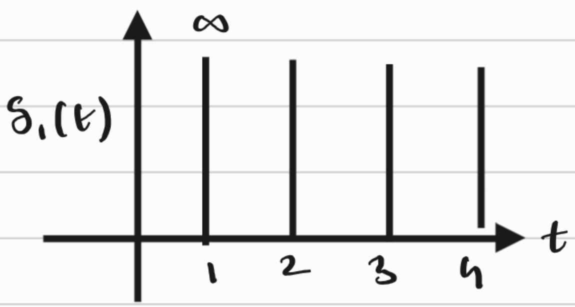
- t_c such that: $\frac{n\tau}{t_c} = 1 \Rightarrow t_c = n_0\tau, \; n_0 \in \mathbb{Z}^+$ (multiple of kick strength)
- p_c such that: $\frac{p_c^2}{ml^2} = 1 \Rightarrow p_c = \sqrt{ml^2}$
- let $\frac{mgl}{t_c} = K$ (kappa)

then

$$H(q,\tilde{p},\tilde{t}) = \frac{\tilde{p}^2}{2} - K\cos q\,\delta(\tilde{t}-1)$$

$$\boxed{H(q,p,t) = \frac{p^2}{2} - K\cos q\,\delta_1(t)}$$

where $\delta_1(t)$:



$K \rightarrow$ kicking strength

- For large 'K', the system shows diffusive behaviour

$K > K_c \approx 0.971 \longrightarrow$ system becomes chaotic and has a positive Maximal Lyapunov Exponent (MLE)

From standard map,

$$p(n) = p(0) + K\sum_{i=0}^{n-1}\sin(x(i))$$

$$\overline{(\Delta p)^2} = \overline{(p(n)-p(0))^2} = K^2\sum_{i=0}^{n-1}\overline{\sin^2(x(i))} \rightsquigarrow 1/2$$

$$+ K^2\sum_{i\neq j}\overline{\sin(x(i))\sin(x(j))} \rightsquigarrow 0$$

So $\boxed{\overline{(\Delta p)^2} = \frac{1}{2}K^2 n}$ Diffusive behaviour