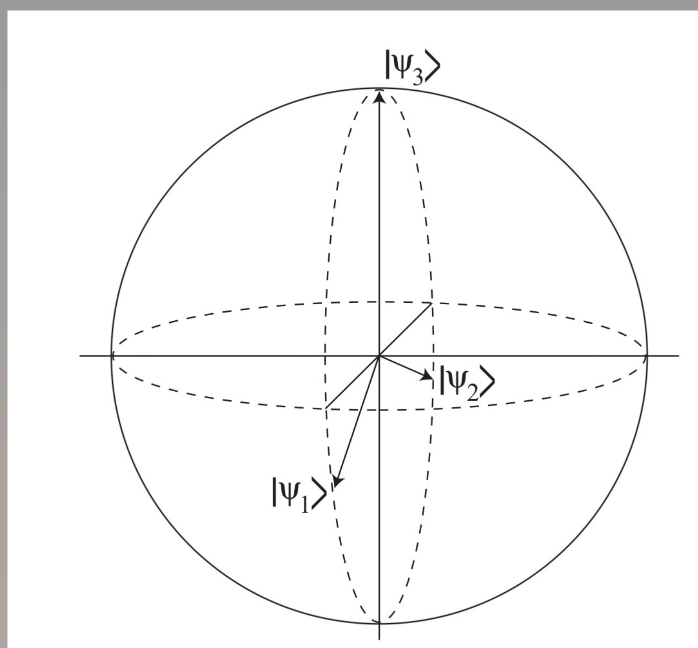
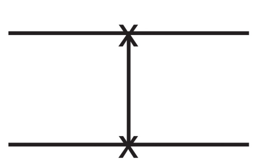
Qubits, measurements & gates

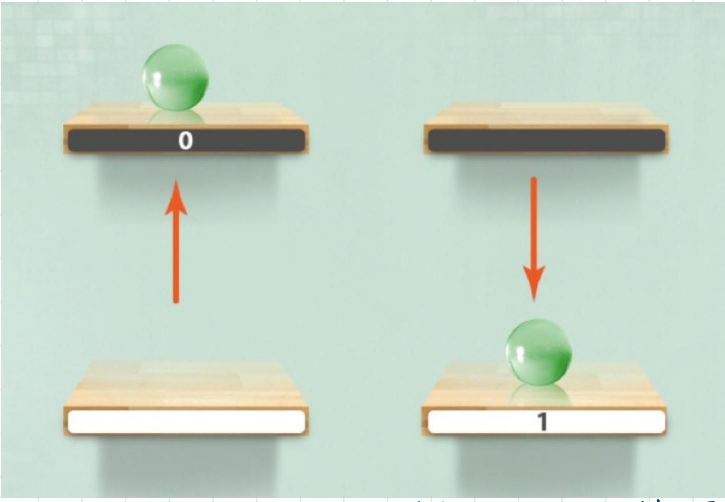
15/12/2020





$$\hat{u}_{\text{swap}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What we can learn about quantum physics from a single qubit ?



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The classical bit

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The qubit = the simplest Q.M. system & generalizes the classical bit.

qubit $\Leftrightarrow$ 2D vector with complex coefficients which is an element of the vector space $\mathbb{C}^2$.

$\hookrightarrow$ Superposition ($\approx$ an interference of the 2 possibilities)

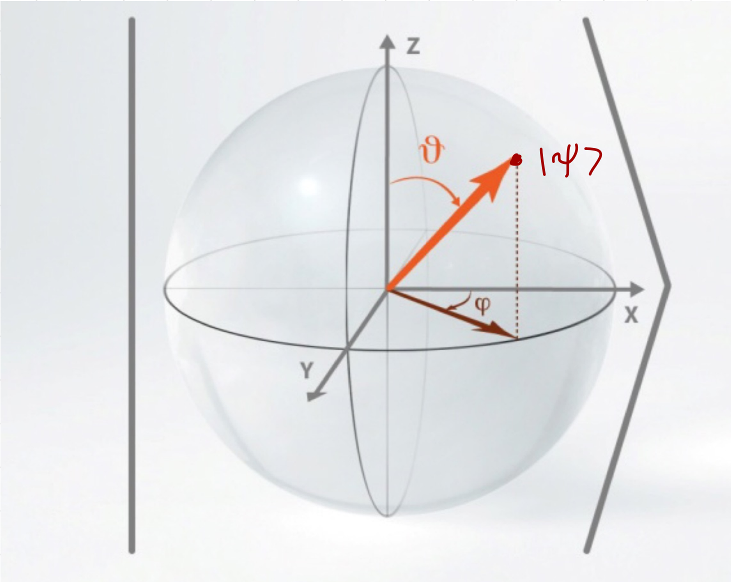
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$$

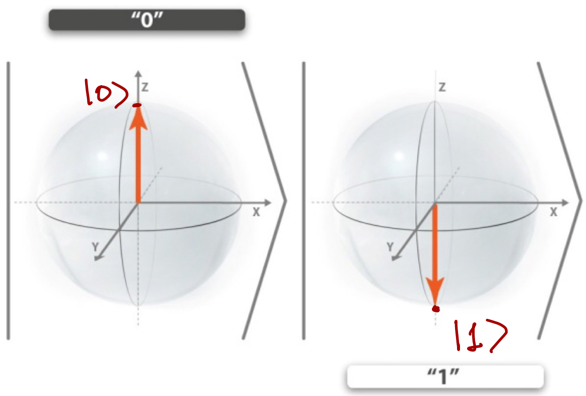
$$\langle\psi|\phi\rangle = \alpha^*\gamma + \beta^*\delta$$

$$|\langle\psi|\psi\rangle|^2 = \alpha^2 + \beta^2 \equiv 1$$

$\therefore$ There is a useful way of visualizing single qubit operations / states in terms of the Bloch Sphere. $\Rightarrow$



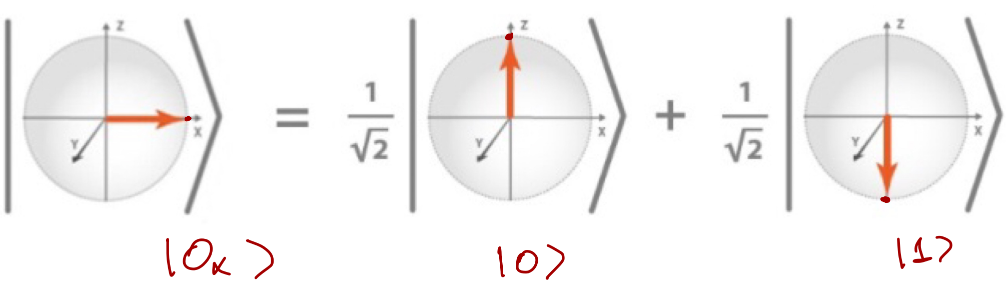
$$|\psi\rangle = \cos\frac{\vartheta}{2}|0\rangle + \sin\frac{\vartheta}{2}e^{i\varphi}|1\rangle$$



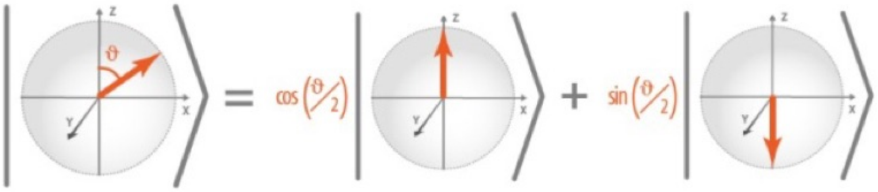
Q.M. Superposition states :

$$|0_x\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|1_x\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

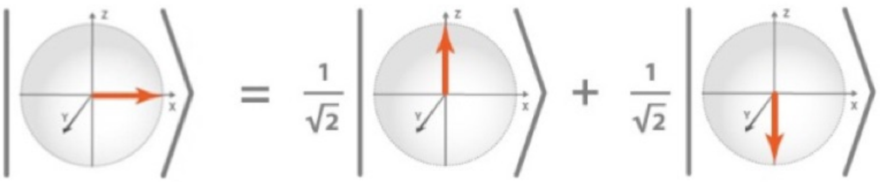


Bloch sphere representation of different superposition states



$|\psi\rangle$

$\Rightarrow |\psi\rangle = \cos\frac{\vartheta}{2}|0\rangle + \sin\frac{\vartheta}{2}e^{i\varphi}|1\rangle.$



$|0_x\rangle$

$\Rightarrow |0_x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

The quantum state of a qubit can be manipulated, i.e. evolved in time $\Rightarrow$ by unitary operation $\hat{U}$

$\hat{U}^+\hat{U} = UU^+ = \mathbb{1} \rightarrow 2\times 2$ matrix $= \hat{U} \rightarrow$ from $SU(2)$ space group.

a qubit manipulation corresponds to a <u>rotation</u> of the state vector on the <u>Bloch sphere</u>

$\therefore \hat{U}$ - must be unitory ; $\hat{U}$ - 2×2 - $SU(2)$ matrix

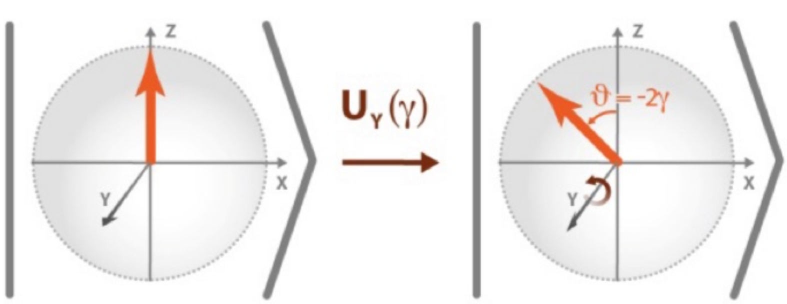
$$U_y(\gamma)|0\rangle = \cos\gamma|0\rangle - \sin\gamma|1\rangle,$$

TIME EVOLUTION

$\Downarrow$

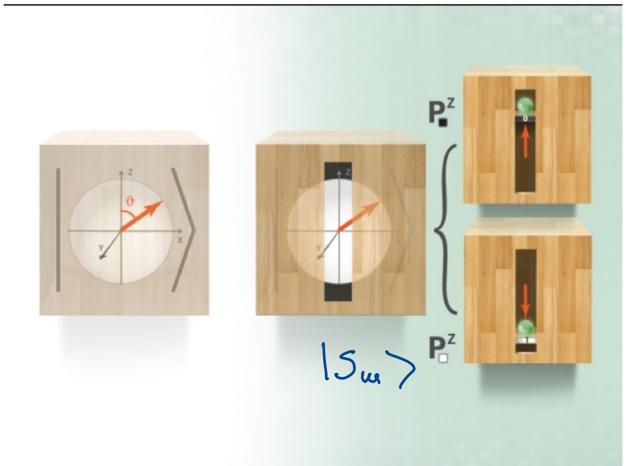
UNITARY Operator

$\hat{U}$



A unitary operation acting on a qubit corresponds to a rotation of the state vector with a certain angle among a fixed axes. Here, the rotation $U_y(\gamma)$ acting on the initial state $|0\rangle$ is depicted, corresponding to a rotation with angle $\vartheta = -2\gamma$ among the y-axes.

Measurement:



$|S_m\rangle$

Illustration of a z-measurement (observable σ_z) performed on a qubit in state $|\psi\rangle = \cos\frac{\vartheta}{2}|0\rangle + \sin\frac{\vartheta}{2}|1\rangle$. The state vector is not oriented in slit direction, and hence the measurement process enforces a rotation of the vector in positive or negative z-direction. This leads to a random measurement result $|0\rangle$ or $|1\rangle$, with probability $p_0 = \cos^2\frac{\vartheta}{2}$ and $p_1 = \sin^2\frac{\vartheta}{2}$, and a change of the state vector after the measurement.

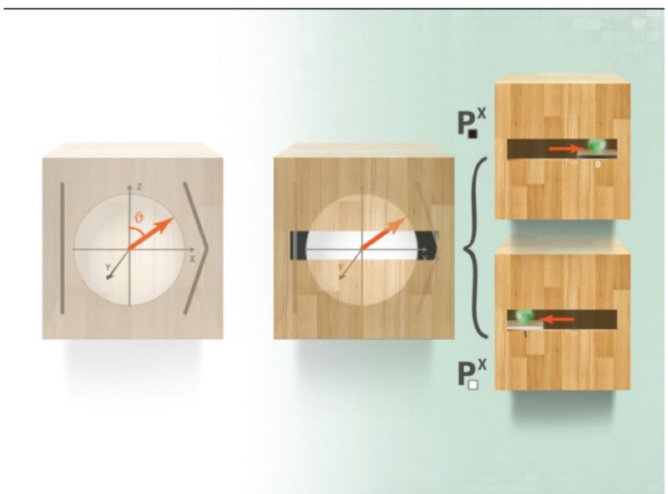


Illustration of a x-measurement (observable σ_x) performed on a qubit in state $|\psi\rangle = \cos\frac{\vartheta}{2}|0\rangle + \sin\frac{\vartheta}{2}|1\rangle$. The slit is oriented in x-direction. The measurement process enforces a rotation of the vector in positive or negative x-direction. This leads to a random measurement result $|0_x\rangle$ or $|1_x\rangle$, with probability $p_0 = 1/2 + \cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2}$ and $p_0 = 1/2 - \cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2}$, and a change of the state vector after the measurement.

MATH <=> Measurement

↓

Projection operator

$$\hat{\mathbb{P}} = |S_m\rangle\langle S_m|$$

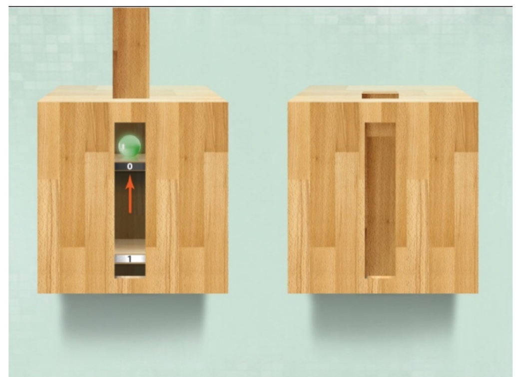
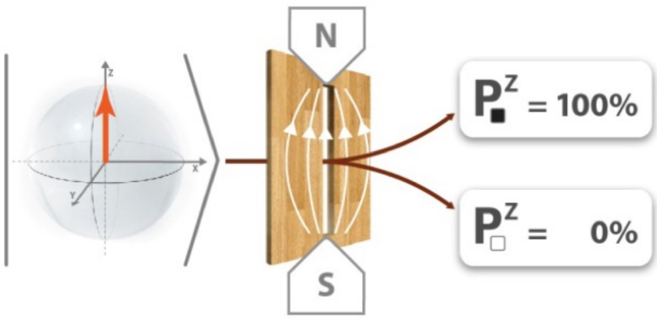
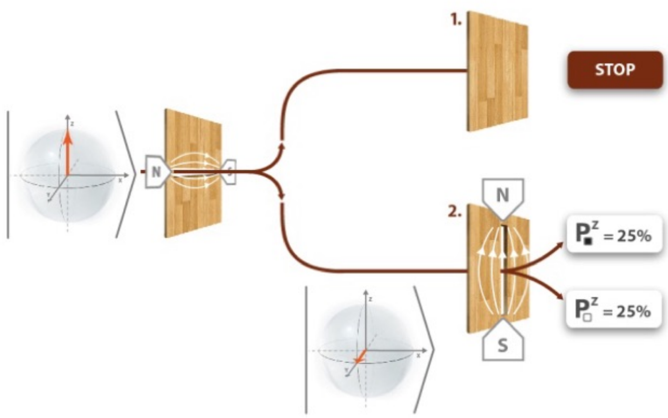


Illustration of a z-measurement (observable σ_z), corresponding to orientation of the slit in z-direction. The measurement determines whether the state vector is oriented in $+z$ or $-z$ direction (measurement result $|0\rangle$ or $|1\rangle$).



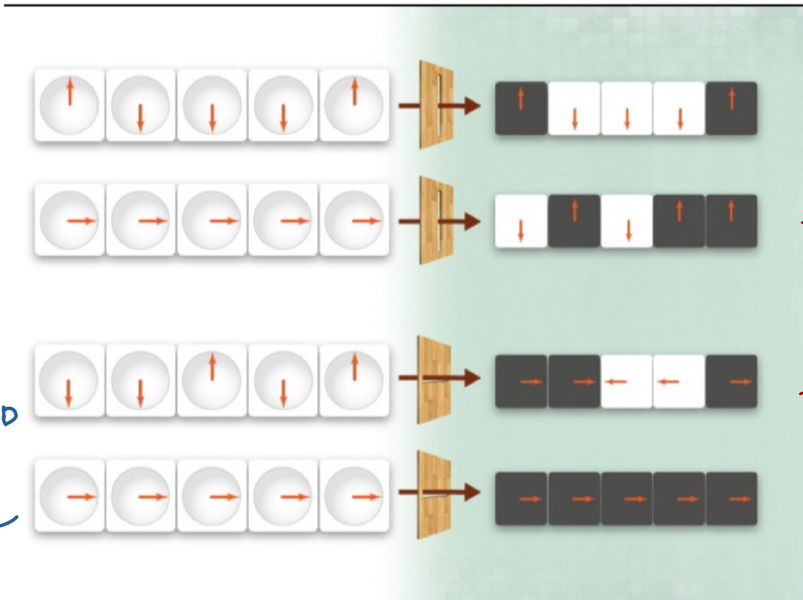
A filter can be realized by blocking one branch of a Stern-Gerlach measurement device. Blocking e.g. the upper branch, only particles in state $|1\rangle$ after the measurements can pass. A qubit prepared in $|0\rangle$ will never pass the filter.



Sequence of measurements in different measurement bases. Adding an additional filter oriented in x-direction (i.e. only states $|1_x\rangle$ can pass) before the z-filter a qubit can pass the two filters with probability $1/4$, even though it can not pass without the additional filter.

Mixed

Pure



pure state

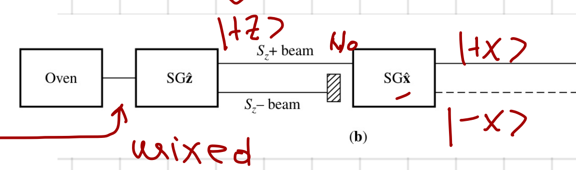


Illustration of the difference between a mixed state (ensemble of random bits in state $|0\rangle$ or $|1\rangle$) and a pure state (all qubits in state $|0_x\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$). The two situations are indistinguishable by z-measurements. In both cases one obtains random outcomes (see line 1 and 3). A x-measurement, however, leads to random measurement results for the mixed state (line 3), while for the pure states, always the same measurement result $|0_x\rangle$ is found.

- The superposition principle
- Stochastic behavior under measurements
- State change due to measurement

PHYSICAL REALIZATIONS OF A QUBIT

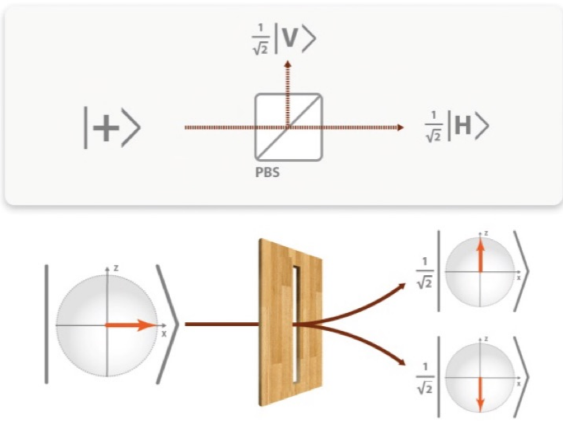


Illustration of the measurement of the polarization degree of freedom of a single photon. The polarizing beam splitter leads to a transmission of horizontally polarized photons, while vertically polarized photons are reflected by the beam splitter. Photons are detected at both branches with help of single-photon detectors, where only one of the two detectors will register a photon. This corresponds to a z-measurement. i.e. measurement of the observable σ_z.

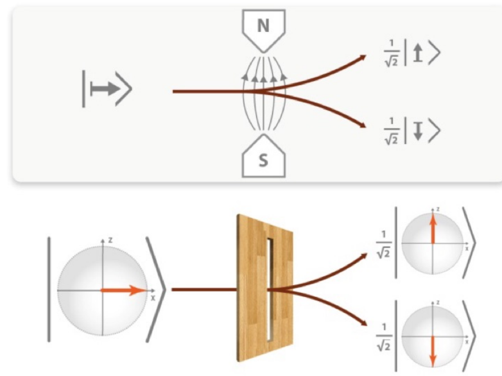
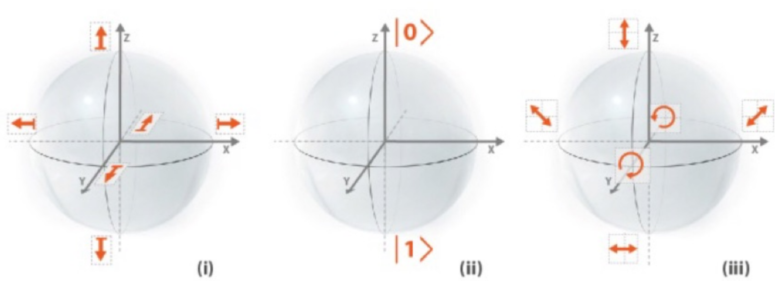
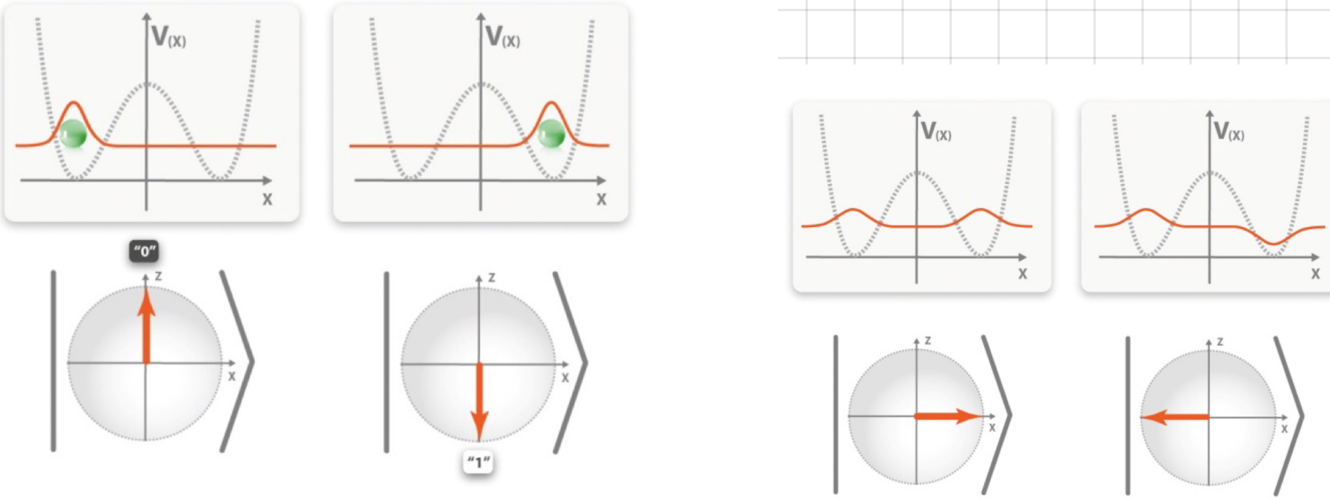


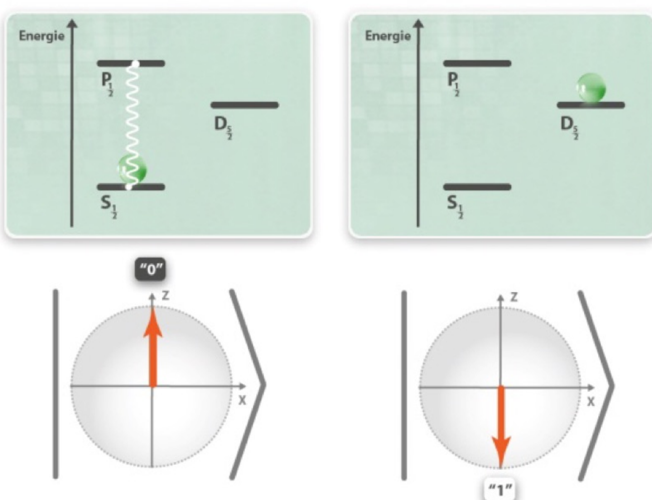
Illustration of the measurement process for a spin with a Stern-Gerlach apparatus. The orientation of the inhomogeneous magnetic field determines the measurement direction (slot). Depicted is a measurement in z-direction. Coupling of the spin to the inhomogeneous magnetic filed leads to a discrete displacement of the particle, corresponding to the two measurement results + (spin up) or -1 (spin down).



Comparison of Bloch sphere representation for spin and photon. While the orientation of the spin is given by the orientation of the Bloch vector, orthogonal polarization states such as $|H\rangle$ and $|V\rangle$ are antiparallel on the Bloch sphere representation. A 45° polarized photons corresponds to a Bloch vector in $\pm x$ direction, while circular polarized photons are described by a Bloch vector in $\pm y$ direction.



Realization of a qubit using spatial degrees of freedom of a single atom. The state $|0\rangle$ corresponds to the localization of the atom in the left well of the double well potential, while the state $|1\rangle$ corresponds to the localization in the right well.



Realization of a qubit using electronic states of an ion or atom (atomic level scheme). The state $|0\rangle$ corresponds to the occupation of the $S_{1/2}$ level, while the state $|1\rangle$ corresponds to the occupation of the $D_{5/2}$ level. A measurement is realized by scattering laser light on the $S_{1/2} - P_{1/2}$ transition.

STATE TOMOGRAPHY :

$|\psi\rangle = a|0\rangle + b|1\rangle$ –

where $|a|^2 + |b|^2 = 1$ with $a, b \in \mathbb{C}$.

$\Rightarrow$ we can write : $\left.\begin{aligned} |a| &= \cos\frac{\theta}{2} \\ |b| &= \sin\frac{\theta}{2} \end{aligned}\right\}$ $\begin{aligned} a &= e^{i\alpha}\cos\theta/2 \\ b &= e^{i\beta}\sin\theta/2 \end{aligned}$

& $\left.\begin{aligned} &0 \leq \alpha, \beta < 2\pi \\ &0 \leq \theta \leq \pi \end{aligned}\right\}$ since $|a|, |b| > 0$

$\Rightarrow$ $|\psi\rangle = e^{i\alpha}\left[\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right]$

where $\varphi \equiv \beta - \alpha$

$\hookrightarrow$ since overall phase cannot be measured – so set to "0".

$\therefore$ $\boxed{|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle}$ $\begin{aligned} &0 \leq \theta \leq \pi \\ &0 \leq \varphi < 2\pi \end{aligned}$

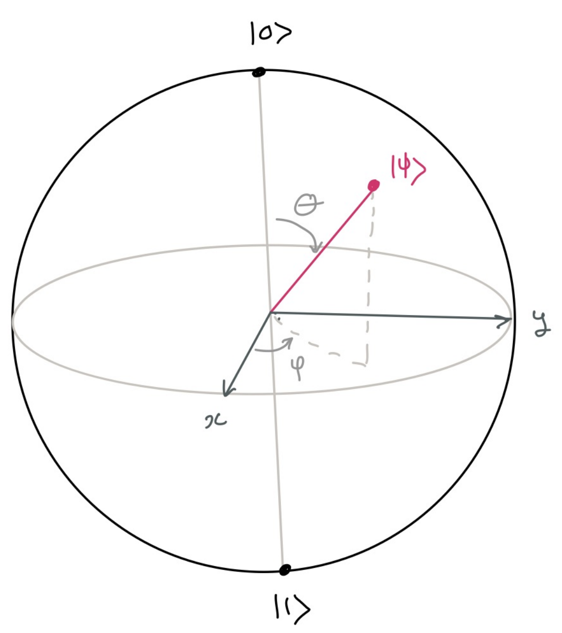
$\downarrow$ 2 continuous dof

We can visualize this state on the surface of in 3D where:

$x = \cos\varphi \sin\theta$

$y = \sin\varphi \sin\theta$

$z = \cos\theta$



Measurement of $|\psi\rangle$ - recall 3 Pauli matrices.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So for the general state $|\psi\rangle \Rightarrow$

$$\langle\psi|\sigma_x|\psi\rangle = e^{i\varphi} \cos\frac{\theta}{2} \sin\frac{\theta}{2} + e^{-i\varphi} \sin\frac{\theta}{2} \cos\frac{\theta}{2}$$

$$= \cos\theta \cos\varphi = \underline{X = \langle\sigma_x\rangle}$$

$$\langle\psi|\sigma_y|\psi\rangle = \sin\theta \sin\varphi = \underline{Y}$$

$$\langle\psi|\sigma_z|\psi\rangle = \cos\theta = \underline{Z}$$

However, we can do even better & only make measurements in standard computational basis!

Consider:

$$P_0 \equiv |\langle 0_z|\psi\rangle|^2 = \cos^2\frac{\theta}{2}$$

$$P_1 \equiv |\langle 1_z|\psi\rangle|^2 = \sin^2\frac{\theta}{2}$$

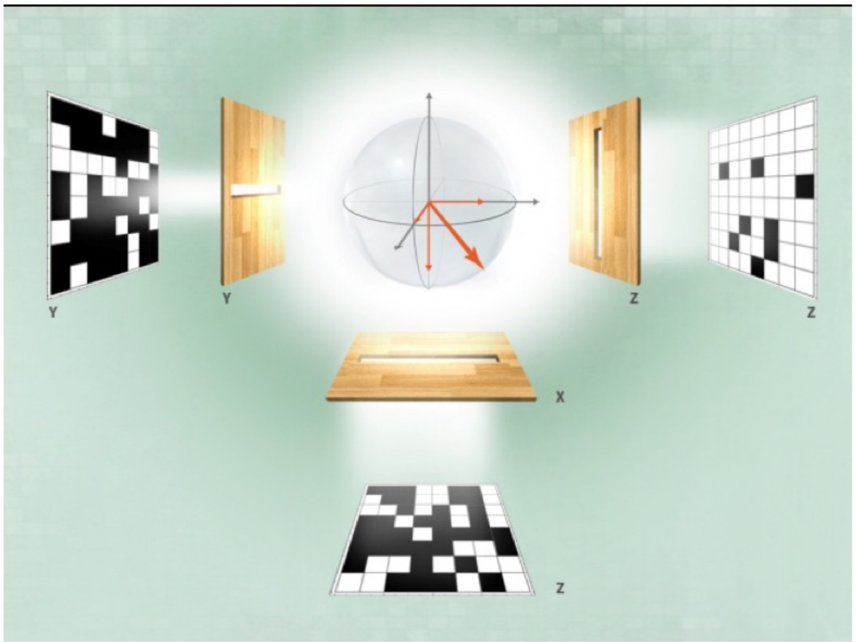
$$\Rightarrow \boxed{P_0 - P_1 = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta = Z}$$

$\therefore$ Coordinate Z is probability difference to obtain outcomes 0 & 1 from a measurement of Z (σ_z), i.e. suppose we can prepare many identically prepared systems, then if N is # of times we measure σ_z - we have $\Rightarrow$

$$Z \xrightarrow[N_0 \gg 1]{} \langle\sigma_z\rangle = \frac{N_0 - N_1}{N_0 + N_1}$$

Similarly: $P_{0x} - P_{1x} = X$

$P_{0y} - P_{1y} = Y$



State tomography: determining an unknown quantum mechanical state requires a large number of measurements performed on an ensemble of identically prepared qubits. From measurements in $x-$, $y-$ and $z-$ direction, the expectation vaules of the observables σ_x, σ_y and σ_z can be determined, which corresponds to the projections of the Bloch vector onto the x-,$y-$ and $z-$ axes, respectively.

No - cloning theorem

We can't copy an unknown quantum state.

1 measurement $\rightarrow$ @ most 1 bit of information

$\therefore$ to copy $|\psi\rangle$ - multiple copies are required

but can't copy superposition

$$|\psi\rangle|0\rangle \rightarrow (|00\rangle + |11\rangle)\frac{1}{\sqrt{2}} \neq |\psi\rangle|\psi\rangle$$

$|0\rangle$ & $|1\rangle$ $\Rightarrow$ $|00\rangle \rightarrow |00\rangle$

$|10\rangle \rightarrow |11\rangle$

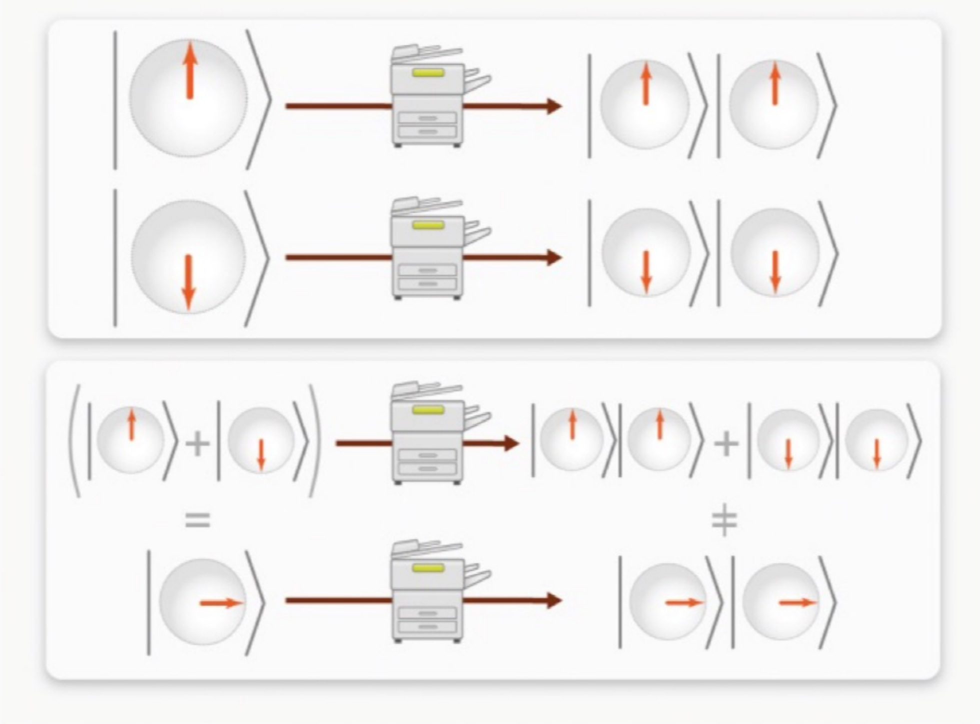
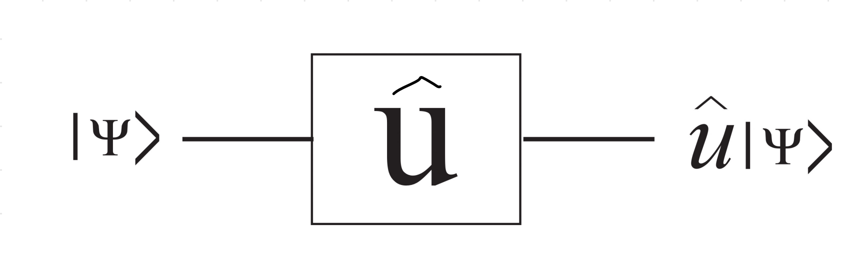


Illustration of the No-cloning theorem. Any copy machine capable of successfully cloning states $|0\rangle$ and $|1\rangle$ would provide an incorrect result for the superposition state $|0\rangle+|1\rangle$. Hence no perfect quantum cloning machine can exist.

QUANTUM GATES



a general 1-qubit gate

6 - common 1 - qubit gates

the associated $\hat{U}$.

Hadamard	H	$\hat{H}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	
Pauli-X	X	$\hat{X}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\hat{\sigma}_x$
Pauli-Y	Y	$\hat{Y}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\hat{\sigma}_y$
Pauli-Z	Z	$\hat{Z}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\hat{\sigma}_z$
Phase	S	$\hat{S}$	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	
$\pi/8$	T	$\hat{T}$	$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$	

$\therefore$ The " most important 1 qubit gate - Hadamard

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

Note: $H^2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$ – Acting twice is the same as doing nothing.

$H \Longleftrightarrow$ a rotation by $\frac{\pi}{2}$ about y & a rotation of π about x.

All possible 2×2 unitary matrices can be written in terms of 4 real numbers: $\alpha, \beta, \gamma, \delta \in \mathbb{R}$

2-qubit gates:

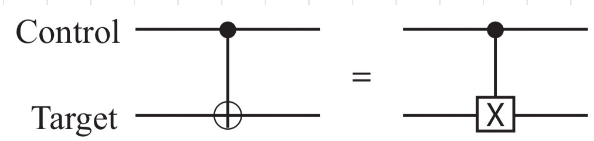
the controlled gates: $|0\rangle$ – control qubit
$|1\rangle$ – target qubit

$$\text{controlled-}\hat{U} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \hat{U}$$

CNOT - gate

$$\boxed{\hat{U}_{\text{CNOT}} = |0\rangle\langle 0| \otimes \hat{\mathrm{I}} + |1\rangle\langle 1| \otimes \hat{\sigma}_x,}$$

$$\hat{U}_{\text{CNOT}} = \begin{array}{c} \text{C} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ \text{T} \end{array}.$$



C	T	C	T
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

The CNOT gate and its effect on the computational basis states.

initial state = eigenstates of $\hat{\sigma}_x$

$$|0_x\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|1_x\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

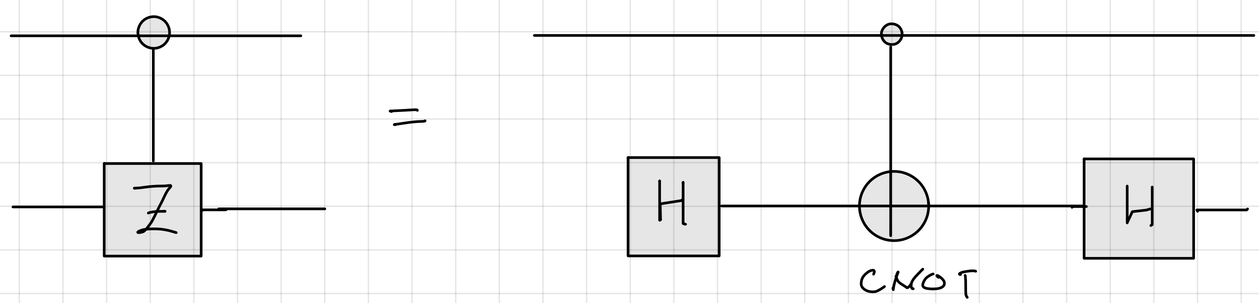
$$\hat{U}_{CNOT}\,|A'\rangle_C\,|B'\rangle_T = |A' \oplus B'\rangle_C\,|B'\rangle_T$$

↳ CNOT ⟹ interchange of the role of C & T.

$|0_x\rangle_C\,|0_x\rangle_T$ – unentangled initial state

$$\boxed{\begin{aligned}\hat{U}_{CNOT}\,|0_x\rangle_C\,|0_x\rangle_T &= \\ &= \frac{1}{\sqrt{2}}\left(|0_x\rangle_C\,|0_x\rangle_T + |1\rangle_C\,|1\rangle_T\right)\end{aligned}}$$

one of the Bell entangled states.



* Every unitary operator on n qubits can be constructed from one & two qubit gates.

* Every 2-qubit quantum gate can be constructed from one-qubit gates & CNOT gates