Review: Geometric Realization of Vectors in $\mathbb{R}^n$

Recall that vectors in $\mathbb{R}^n$ can be thought of as "arrows."

Exercise: Graph the vectors u, v, and $u - v$ where:

$$u = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \quad v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

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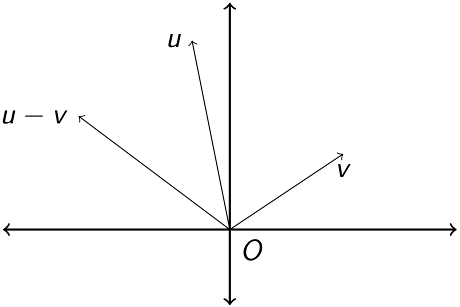
Sln:



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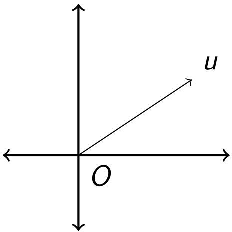
It turns out that there is a *single* computation which can help us to answer both questions!

But we will tackle one-at-a-time.

Length of a Vector

Consider the generic vector

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



The "length" of u should be the distance from O to the head of u. Pythagoras gives:

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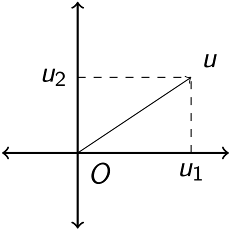

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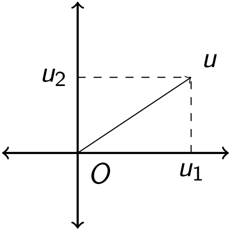

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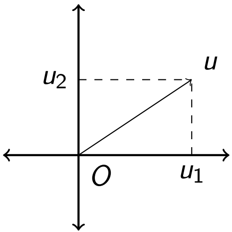

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Length/Magnitude/Norm

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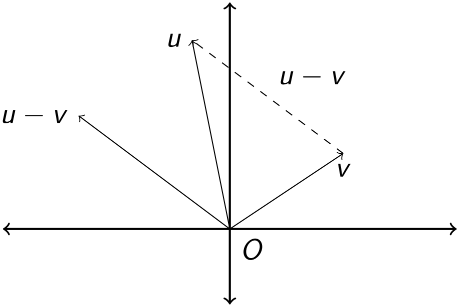
Solutions:

$$\|u\| = \sqrt{(-1)^2 + 5^2} = \sqrt{26}, \quad \|v\| = \sqrt{13}, \quad \|u - v\| = \sqrt{25}.$$

Distance Between Vectors

Let's return to our vectors u, v, and $u - v$:



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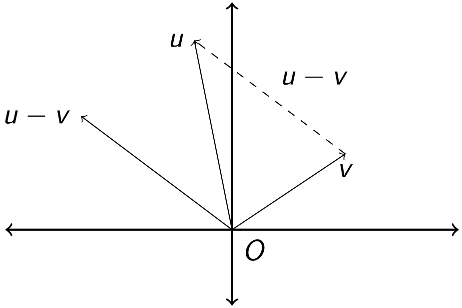


Definition

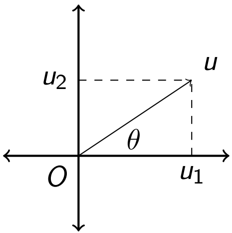
The **distance** between the vectors u and v in $\mathbb{R}^n$ is

$$d(u, v) := \|u - v\|.$$

Note: Distance measurements are CRUCIAL to answer questions like "How good is our approximation?"

Direction of a Vector

Return to our generic vector in $\mathbb{R}^2$:



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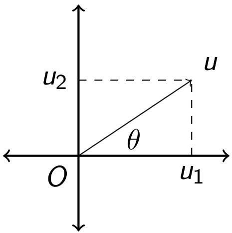

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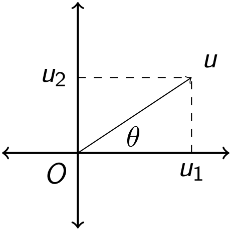

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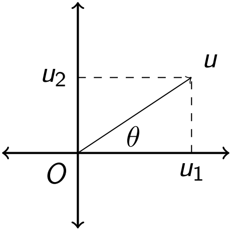

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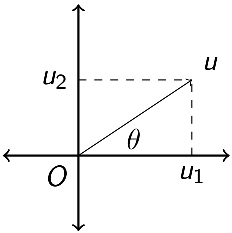

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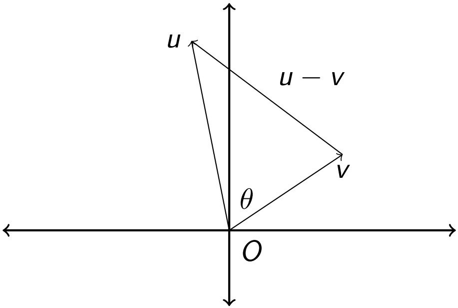
Note: the norm of u in the denominator. Somehow the angle is related to sizes.

Angle Between Two Vectors

A slightly longer computation:



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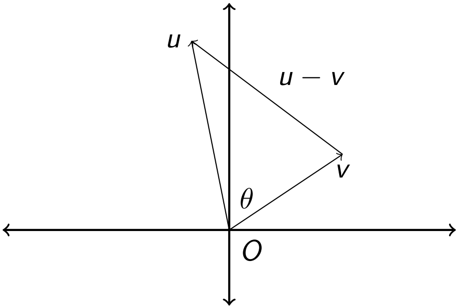

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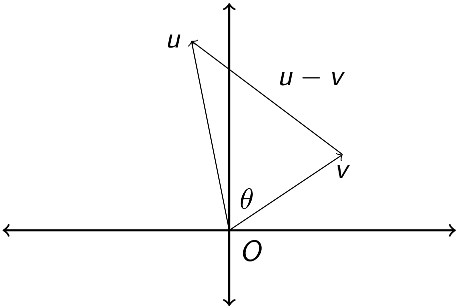

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After some expansion and algebra you can show

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Multiplying individual vector coordinates and then summing comes up in many contexts.

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Let u and v be vectors in $\mathbb{R}^n$. The **dot product** or **standard inner product** of u and v is

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- Angle: $\cos(\theta) = \frac{u \cdot v}{\|u\|\|v\|}$.

We can also say that the element in the (i, j)-th position in the product matrix AB is the dot product of the i-th row of A and the j-th column of B.

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6. *Orthogonality property:* $u \perp v \Leftrightarrow u \cdot v = 0.$

Simple Application: unit vectors

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Exercises:

1. Show that $u = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ is a unit vector.
2. Show that u above and the vector $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$ are parallel. (What is the angle between them?)
3. Find a unit vector in the same direction as $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. (What value can you use to *scale* v?)