Exercise

For each of the following maps, decide if they are or are not linear.

1. $L : P(\mathbb{R}) \to P(\mathbb{R})$ by $p(x) \mapsto xp(x)$.
2. $L : P(\mathbb{R}) \to \mathbb{R}$ by $f \mapsto f(0)$.
3. $L : \mathbb{R} \to \mathbb{R}$ by $x \mapsto |x|$.
4. $L : \mathbb{R}^3 \to \mathbb{R}^3$ by $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}$.

Theory

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1. $L(0) = 0$. *0 is the identity in the appropriate vector space.*
2. $L(u - v) = L(u) - L(v)$ *for all* $u, v \in V$.

The first result is nice since it gives insight into the *kernel* of the operator!

Big Theorem: Basis vectors characterize linear maps

Suppose V has basis $\beta = \{v_1, \ldots, v_n\}$.

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For any n vectors w_i in W there exists a unique linear map $T : V \to W$ where

$$T(v_i) = w_i$$

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Proof.

For any any $x \in V$ write $x = \sum_{i=1}^{n} \alpha_i v_i$. Then define

$$T(v) := \sum_{i=1}^{n} \alpha_i w_i.$$

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That this map is well defined and unique come from the fact that β is a basis! $\square$

Application

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Theorem

With notation as above, the matrix A is the unique matrix with the property that

$$L(x) = Ax.$$

Exercise

Let $L : \mathbb{R}^3 \to IR^2$ by

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} a + 2b \\ 3b - 2c \end{bmatrix}$$

Find the standard matrix representing L.

Moral(s) and a Question

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But before that, some definitions....

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$\ker(L)$ *is a subspace of* V *and* $\text{range}(L)$ *is a subspace of* W*.*

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So

$$\ker(L_A) = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

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So let's look at non-matrix examples too:

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So a basis will be those column vectors which are linearly independent:

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$$\text{range}(L_A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

It is two dimensional.

Range Characterization

If L_A is a linear operator defined by matrix multiplication by A then

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Again, this is a problem that we have already solved!

Range, Kernel, Invertiblity

A linear map $T : V \to W$ is invertible if and only if it is

Definition

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- *T is onto* $\Leftrightarrow \text{range}(T) = W$.
- *T is one to one* $\Leftrightarrow \ker(T) = \{0\}$.

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$$T(v_1 - v_2) = T(v_1) - T(v_2) = 0.$$

  So $v_1 - v_2 \in \ker(T) \Rightarrow v_1 - v_2 = 0$. Therefore $v_1 = v_2$ and T is one to one.

The Dimension Theorem Again

Theorem

Let V be a finite dimensional vector space and $T : V \to W$ a linear map. Then

$$\dim(V) = \operatorname{rank}(T) + \operatorname{nullity}(T)$$

where

$$\operatorname{rank}(T) = \dim \operatorname{range} T \quad \textit{and} \quad \operatorname{nullity}(T) = \dim \ker T$$