Jointly Discrete RVs

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Introduction

We have almost exclusively computed probabilities for a single RV so far (also called *univariate random variables*).

Now we look at the more general case of multiple RVs (*multivariate random variables* or *random vectors*).

Goals for this Lecture

We will learn how to

1. compute probabilities for multiple discrete RVs using their *joint probability mass function*,
2. determine the *marginal probability mass function* for a random variable from the joint PMF,
3. use the law of the unconscious statistician in a multivariate setting, and
4. use the joint PMF to determine if a collection of random variables is independent.

This material corresponds to section 6.1 and part of 6.3 of the textbook.

Joint PMF

Definition

Definition (for 2 RVs): Let X and Y be discrete RVs. Their *joint probability mass function* is defined by

$$p_{X,Y}(j,k) = \mathbb{P}(X = j, Y = k).$$

This is often represented as a table,

$$\begin{array}{cc|cccc}
 & & & Y & & \\
 & & k_1 & k_2 & \cdots & k_m \\
\hline
 & j_1 & p_{X,Y}(j_1,k_1) & p_{X,Y}(j_1,k_2) & \cdots & p_{X,Y}(j_1,k_m) \\
X & j_2 & p_{X,Y}(j_2,k_1) & p_{X,Y}(j_2,k_2) & \cdots & p_{X,Y}(j_2,k_m) \\
 & \vdots & \vdots & & \ddots & \\
 & j_n & p_{X,Y}(j_n,k_1) & p_{X,Y}(j_n,k_2) & \cdots & p_{X,Y}(j_n,k_m)
\end{array}$$

Example

Suppose X and Y have the joint PMF

		Y			
		1	2	3	4
	1	0.05	0.10	0.05	0.20
X	2	0.10	0.05	0	0.15
	3	0.20	0.05	0.05	0

1. Compute $\mathbb{P}(Y = 4)$, $\mathbb{P}(X = 2)$, $\mathbb{P}(X + Y < 4)$ and $\mathbb{P}(X = Y)$.
2. Find the PMF of X.
3. Set $W = \min\{X, Y\}$. Find the PMF of W.

Properties

1. Probabilities are positive:
$$p_{X,Y}(j,k) \geq 0$$
2. Probabilities sum to 1:
$$\sum_{j \in \mathrm{Ran}(X)} \sum_{k \in \mathrm{Ran}(Y)} p_{X,Y}(j,k) = 1$$
3. To find the probabilities for only one of the RVs, you sum over all the values of the other RV:
$$p_X(j) = \sum_{k \in \mathrm{Ran}(Y)} p_{X,Y}(j,k)$$

Joint PMF

Definition (for many RVs): Let $X_1, X_2, \ldots, X_n$ be discrete RVs. Their *joint probability mass function* is defined by

$$p_{X_1,X_2,\ldots,X_n}(k_1, k_2, \ldots, k_n) = \mathbb{P}(X_1 = k_1, X_2 = k_2, \ldots, X_n = k_n).$$

This is cannot be represented as a table.

Special Example (Multinomial Distribution)

Suppose we conduct a medical trial with 3 outcomes: improvement (1), deterioriation (2), no change (3). There are 60 patients. Let X_1, X_2, X_3 be the number of patients with outcome type 1, 2, 3 respectively.

The probabilities of outcomes 1, 2, and 3 are p_1, p_2, and p_3, respectively.

What is the joint PMF of X_1, X_2, X_3?

Multinomial Distribution

Definition: Let $X_1, X_2, \ldots, X_r$ be discrete RVs. They are said to have the *multinomial distribution* with parameters $n, p_1, \ldots, p_r$ if they have the joint PMF

$$p_{X_1,X_2,\ldots,X_r}(k_1, k_2, \ldots, k_r) = \binom{n}{k_1, k_2, \cdots, k_r} p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

for $k_1 + k_2 + \cdots + k_r = n$ and $p_1 + p_2 + \cdots + p_r = 1$.

We denote this by $(X_1, \ldots, X_r) \sim \text{Mult}(n, p_1, \ldots, p_r)$.

Comment: This is probably the only special discrete multivariate distribution we will discuss, compared to the many special discrete univariate distributions we can analyze.

Marginal PMF

Definition

Definition: Let $X_1, X_2, \ldots, X_n$ be discrete RVs with joint PMF $p(k_1, \ldots, k_n)$. The *marginal PMF* of $X_1, X_2, \ldots, X_m$ for $m < n$ is defined by

$$p_{X_1,\ldots,X_m}(k_1, \ldots, k_m) = \mathbb{P}(X_1 = k_1, \ldots, X_m = k_m).$$

This is computed by

$$p_{X_1,\ldots,X_m}(k_1, \ldots, k_m) = \sum_{j_{m+1},\ldots,j_n} p(k_1, \ldots, k_m, j_{m+1}, \ldots, j_n).$$

Comment: You have already found a marginal PMF in the first example.

Example

Let $(X_1, X_2, X_3) \sim \text{Mult}(n, p_1, p_2, p_3)$. Find the marginal pmf of X_2.

Expectation

Law of the Unconscious Statistician

Let $g : \mathbb{R}^n \to \mathbb{R}$. If $X_1, \ldots, X_n$ are discrete random variables with joint PMF p then

$$E[g(X_1, \ldots, X_n)] = \sum_{k_1} \cdots \sum_{k_n} g(k_1, \ldots, k_n) p(k_1, \ldots, k_n)$$

Example (6.6 in textbook)

A fair 4-sided die is rolled twice. Let X_k denote the k-th outcome. Set $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = |X_1 - X_2|$.

- Find the joint PMF of X_1 and X_2.
- Find the joint PMF of Y_1 and Y_2.
- Find the expected value $E[Y_1 Y_2]$.

Independence

Fact

Recall: Discrete RVs are independent if and only if

$$\mathbb{P}(X = j, Y = k) = \mathbb{P}(X = j)\mathbb{P}(Y = k)$$

for all j, k.

Fact: $X_1, \ldots, X_n$ are independent if and only if

$$p_{X_1,\ldots,X_n}(k_1, \ldots, k_n) = p_{X_1}(k_1) p_{X_2}(k_2) \cdots p_{X_n}(k_n).$$

for $k_1, k_2, \ldots, k_n$.

Example

Suppose X and Y have the joint PMF

		Y			
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X	2	0.10	0.05	0	0.15
	3	0.20	0.05	0.05	0

Are X and Y independent?

The Wrap Up

Summary

1. A joint PMF is based on the same idea and has the same properties as an ordinary PMF. The only difference is that it is multivariable.

2. A marginal PMF is obtained by summing over the ranges of the RVs that we are not interested in.
3. Independence is equivalent to splitting the joint PMF into a product of marginal PMFs.
4. The law of the unconscious statistician extends to multiple discrete random variables as you might hope.

Next Step

After completing the discrete version, of course we must cover the case of continuous random variables.