Exercise

Let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$$

Compute the following:

$$A - 2B^T.$$

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Result:

$$A - 2B^T = \begin{bmatrix} -1 & -4 \\ -2 & -1 \\ -3 & -6 \end{bmatrix}.$$

Matrix Operations We Know:

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Q: "Can we multiply matrices?"

A: Yes, but the definition we use is not obvious...

Dot Product

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Definition

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$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

The **dot product** of $\mathbf{x}$ and $\mathbf{y}$ is defined to be

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + x_2 y_2 + \cdots x_n y_n$$

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$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + x_2 y_2 + \cdots x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

Examples/Exercises

Let

$$\mathbf{w} = \begin{bmatrix} .15 \\ .15 \\ .2 \\ .2 \\ .3 \end{bmatrix} \text{ and } \mathbf{g} = \begin{bmatrix} 90 \\ 85 \\ 80 \\ 75 \\ 90 \end{bmatrix}$$

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Answer:

$$.15 * 90 + .15 * 85 + .2 * 80 + .2 * 75 + .3 * 90$$

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Compute $\mathbf{w} \cdot \mathbf{g}$:
Answer:

$$.15 * 90 + .15 * 85 + .2 * 80 + .2 * 75 + .3 * 90 = 84.25.$$

Note: $\mathbf{w}$ can be thought of as a vector of "weights" and $\mathbf{g}$ as a vector of information. Then the dot product of the two is a weighted average.

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In components:

$$\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \sum_{i=1}^{n} v_i w_i.$$

Matrix multiplication

Definition

Let $A = [a_{ik}] \in \mathbb{R}^{m \times \ell}$ and $B = [b_{kj}] \in \mathbb{R}^{\ell \times n}$ Then the **product** of A and B is the matrix

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$$AB := \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right] \in \mathbb{R}^{m \times n}.$$

In words: The i, jth element of AB is the action of the ith row of A on the jth column of B.

Example

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Then

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$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 10 \\ 7 & 8 & 17 \end{bmatrix}.$$

and

$$BA = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix}.$$

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This choice of definition for matrix multiplication is USEFUL:

- This definition agrees with the the behavior of how linear operators are composed (TBD in a month or so).
- The special case of a matrix-vector product gives us useful interpretations.

Special Case: Matrix-Vector Product

Let $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$.

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$$Ax = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

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Interpretation (1): Ax is a linear combination

With A and x as before we have:

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{m3} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

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So Ax is a *linear combination* of the columns of A.

Interpretation (2): $Ax = b$ is a system of linear equations

Now let A and x be as before and consider a vector $b = [b_i] \in \mathbb{R}^m$

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$Ax = b$ **is a system of linear equations!**

Example

Consider the system of linear equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Interpret this system as a matrix-vector equation $Ax = b$. Write out each of A, x, and b explicitly.

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Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 14 \\ -2 \end{bmatrix}$$

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When interpreted as a system of equations, A is called the **coefficient matrix** of the system.

Augmented Matrix for a System of Equations

In the prior example, when interpreted as a system of equations, $Ax = b$ has two "known" parts: A and b. x is considered to be a vector of "unknowns."

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A **augmented matrix** is created from the "known" quantities:

$$\left[\begin{array}{c|c} A & b \end{array}\right] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right]$$

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Write out the system of equations which corresponds to the following augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 3 & 0 & 2 & 5 \end{array}\right]$$

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Solution:

$$\begin{aligned} 2x - y + 3z &= 4 \\ 3x + 2z &= 5 \end{aligned}$$