Convolution Formulas

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The Problem

Suppose X and Y are independent discrete/continuous RVs. What is the the PMF/PDF of $W = X + Y$?

This lecture corresponds to section 7.1 of the textbook.

Discrete Case

The Convolution Formula

Fact: Suppose X and Y are independent discrete RVs, and $\mathrm{Ran}(X) = \{0, 1, \ldots, n\}$, $\mathrm{Ran}(Y) = \{0, 1, \ldots, \ell\}$. Set $W = X + Y$. Then

$$p_W(k) = \sum_{j=0}^{k} p_X(j) p_Y(k-j) = p_X \star p_Y(k).$$

for $k = 0, 1, 2, \ldots, n + \ell$.

Comment: You can generalize to other ranges for the RVs. It is just necessary to be careful with the indices.

Example

Let X and Y be independent RVs with the distributions $\mathrm{Pois}(\lambda_X)$ and $\mathrm{Pois}(\lambda_Y)$. Find the PMF of $X + Y$.

Special Example

Let $X \sim \mathrm{Geo}(p)$ and $Y \sim \mathrm{Geo}(p)$ be independent RVs. Find the PMF of $X + Y$.

Negative Binomial Distribution

Definition

A RV is said to have the *negative binomial* distribution with parameters k and p if it has PMF

$$p_X(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

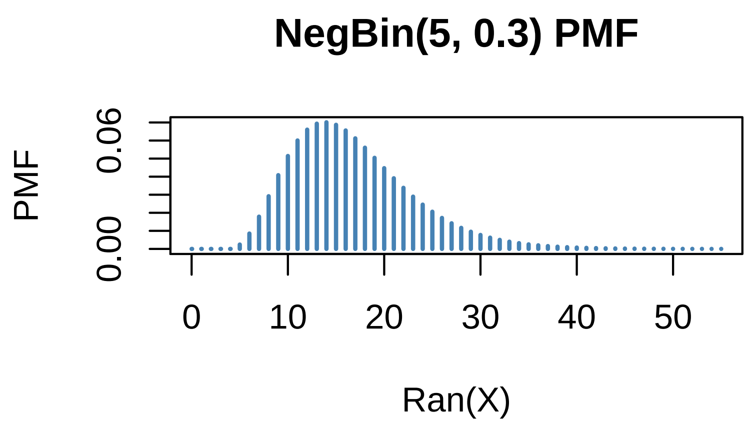
for $n = k, k+1, \ldots$.

Interpretation: $X \sim \text{NegBin}(k, p)$ can be defined as

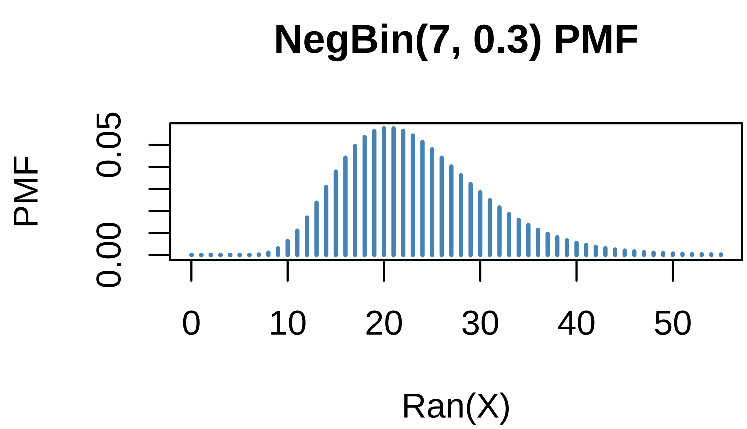
$$X = T_1 + T_2 + \cdots + T_k$$

for independent RVs $T_\ell \sim \text{Geo}(p)$.

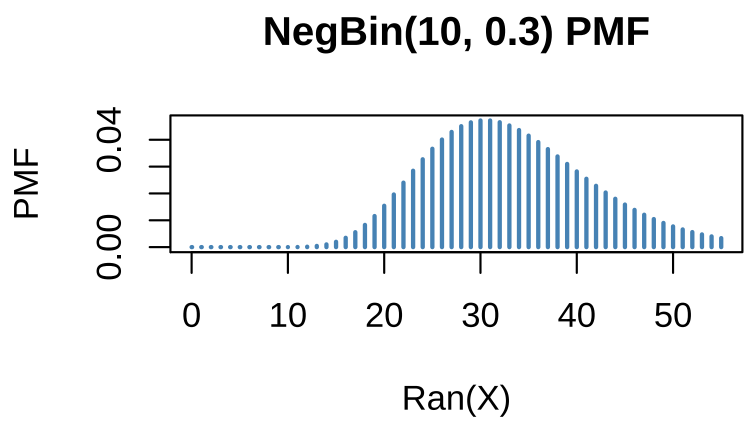
PMF, $k = 5$ and $p = 0.3$



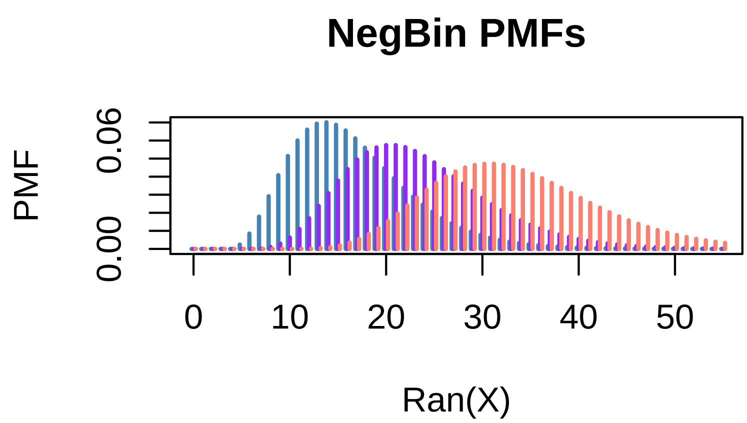
PMF, $k = 7$ and $p = 0.3$



PMF, $k = 10$ and $p = 0.3$



PMFs, Compared



Continuous Setting

Convolution Formula

Fact: Suppose X and Y are independent continuous RVs Set $W = X + Y$. Then

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x)\, dx = f_X \star f_Y(w).$$

for all w.

Example

Let X and Y be independent RVs with the distribution $N(0,1)$. Find the PDF of $X+Y$.

Important Fact

The following fact is a generalization of the previous example.

Fact: Suppose $X_1, \ldots, X_n$ are independent and $X_j \sim N(\mu_j, \sigma_j^2)$. Then

$$X_1 + X_2 + \cdots + X_n \sim N(\mu_1 + \mu_2 + \cdots + \mu_n, \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2).$$

Example

Let $X \sim \mathrm{Exp}(\lambda)$ and $Y \sim \mathrm{Exp}(\lambda)$ be independent RVs. Find the PDF of $X+Y$.

Gamma Distribution

Definition

A RV is said to have the *gamma* distribution with parameters k and λ if it has PDF

$$f_X(x) = \frac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x}$$

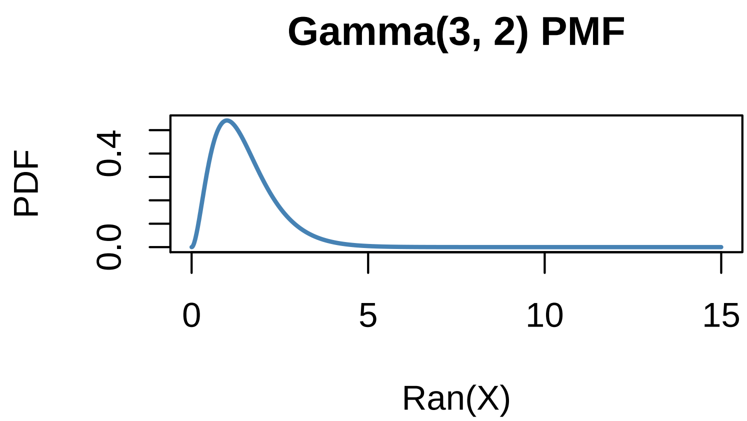
for $x \geq 0$.

Interpretation: $X \sim \mathrm{Gamma}(k, \lambda)$ can be defined as

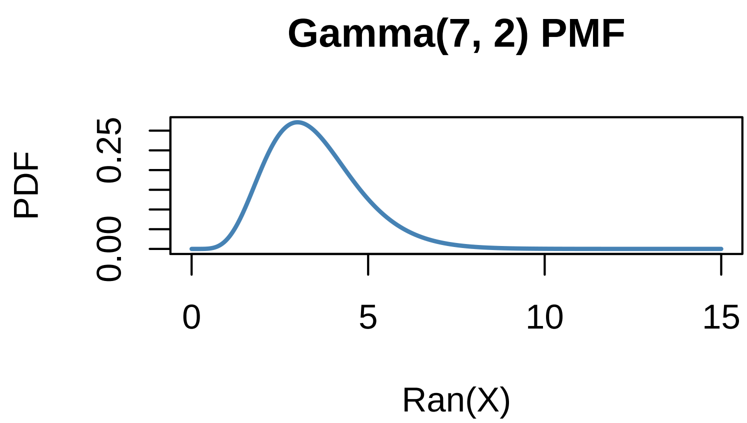
$$X = T_1 + T_2 + \cdots + T_k$$

for independent RVs $T_\ell \sim \mathrm{Exp}(\lambda)$.

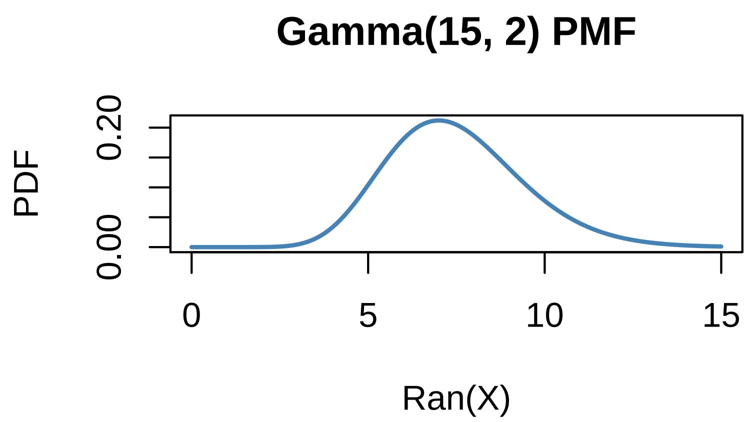
PDF, $n = 3$ *and* $\lambda = 2$



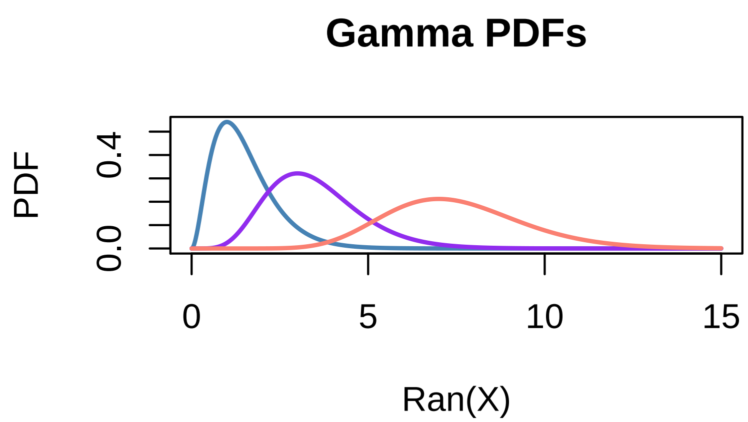
PDF, $n = 7$ *and* $\lambda = 2$



PDF, $n = 17$ and $\lambda = 2$



PDFs, Compared



A Few More Special Uses

1. The sum of independent Poisson RVs is Poisson.
2. The sum of independent Binomial RVs is Binomial.
3. The sum of independent Geometric RVs with the same parameter is NegBin.
4. The sum of independent Normal RVs is Normal.
5. The sum of independent Exponential RVs with the same parameter is Gamma.

6. The $\chi^2(n)$ distribution can be derived as a sum of independent RVs.
7. ... and so on.

The Wrap Up

Summary

1. The convolution formula can be used to find the PMF/PDF of $X + Y$ when X and Y are independent.
2. The sum of independent normal RVs has a normal distribution.
3. We have learned two new special distributions: Negative Binomial and the Gamma.