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- Prove that the cofactor inverse formula **IS** the inverse.
- Geometric interpretation of the determinant.

The Cofactor Expansion Formula is the Determinant

Recall that if $A = [a_{ij}]$ is an $n \times n$ matrix, then the cofactor expansion formula across the i-th row is

$$f(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} f(M_{1j})$$

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We will make one MAJOR assumption: We will assume that the expansion formula is row independent.

Method of Proof: Induction

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So it should work for all sizes!

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So: Cofactor expansion works for 1×1 matrices.

Works for small matrix means it also works for a big one

Do this on the board

Next: The Cofactor Inverse Formula gives the Inverse

Let $A = [a_{ij}]$ be non-singular and define

$$B = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

where

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Lets show $AB = I$.

Compute AB

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because the sum is the determinant of a matrix with two identical rows!

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We can think of an $n \times n$ matrix as a list of $n \times 1$ vectors:

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If we plot the list of vectors in $\mathbb{R}^n$ we can create a *hyperparallelepiped* (HPP)!

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It turns out that $vol_n(A) = |\det(A)|$!

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Let A be an $n \times n$ matrix and consider $f_A : \mathbb{R}^n \to \mathbb{R}^n$ by $f_A(x) = Ax$. Then for a set $S \subset \mathbb{R}^n$ we have:

$$vol_n(f(S)) = |\det(A)| vol_n(S).$$

Example

Let S be the unit circle in $\mathbb{R}^2$. The matrix

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

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Exercise: What is the volume of an ellipsoid with radii a, b, and c?