Exercises

Which of the following matrices are singular? Why?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Results

Which of the following matrices are singular? Why?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Results

Which of the following matrices are singular? Why?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Results

Which of the following matrices are singular? Why?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So: Singularity is related to the value of $ad - bc$.

Results

Which of the following matrices are singular? Why?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So: Singularity is related to the value of $ad - bc$.

A:

Results

Which of the following matrices are singular? Why?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So: Singularity is related to the value of $ad - bc$.

A: $1 \cdot 4 - 2 \cdot 3 = -2$.

Results

Which of the following matrices are singular? Why?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So: Singularity is related to the value of $ad - bc$.

A: $1 \cdot 4 - 2 \cdot 3 = -2$. Nonsingular.

Results

Which of the following matrices are singular? Why?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So: Singularity is related to the value of $ad - bc$.

A: $1 \cdot 4 - 2 \cdot 3 = -2$. Nonsingular.
B:

Results

Which of the following matrices are singular? Why?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So: Singularity is related to the value of $ad - bc$.

A: $1 \cdot 4 - 2 \cdot 3 = -2$. Nonsingular.
B: $1 \cdot 4 - 2 \cdot 2 = 0$. Singular.

Results

Which of the following matrices are singular? Why?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So: Singularity is related to the value of $ad - bc$.

A: $1 \cdot 4 - 2 \cdot 3 = -2$. Nonsingular.
B: $1 \cdot 4 - 2 \cdot 2 = 0$. Singular.
C:

Results

Which of the following matrices are singular? Why?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So: Singularity is related to the value of $ad - bc$.

A: $1 \cdot 4 - 2 \cdot 3 = -2$. Nonsingular.
B: $1 \cdot 4 - 2 \cdot 2 = 0$. Singular.
C: $1 \cdot 1 - (-1) \cdot (-1) = 0$. Singular.
D:

Results

Which of the following matrices are singular? Why?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So: Singularity is related to the value of $ad - bc$.

A: $1 \cdot 4 - 2 \cdot 3 = -2$. Nonsingular.
B: $1 \cdot 4 - 2 \cdot 2 = 0$. Singular.
C: $1 \cdot 1 - (-1) \cdot (-1) = 0$. Singular.
D: $1 \cdot 1 - 1 \cdot 0 = 1$. Nonsingular.

Determinant of a 2×2 Matrix

For the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the quantity $ad - bc$ is called a **determinant**.

Determinant of a 2×2 Matrix

For the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the quantity $ad - bc$ is called a **determinant**. It has the following nice property:

$$\det(A) = 0 \Leftrightarrow A \text{ is singular.}$$

Determinant of a 2×2 Matrix

For the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the quantity $ad - bc$ is called a **determinant**. It has the following nice property:

$$\det(A) = 0 \Leftrightarrow A \text{ is singular.}$$

Computing a determinant is a simple way to find if A is singular or not.

Determinant of a 2×2 Matrix

For the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the quantity $ad - bc$ is called a **determinant**. It has the following nice property:

$$\det(A) = 0 \Leftrightarrow A \text{ is singular.}$$

Computing a determinant is a simple way to find if A is singular or not.

Question: Are there determinant computations for general $n \times n$ matrices?

Determinant of a 2×2 Matrix

For the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the quantity $ad - bc$ is called a **determinant**. It has the following nice property:

$$\det(A) = 0 \Leftrightarrow A \text{ is singular.}$$

Computing a determinant is a simple way to find if A is singular or not.

Question: Are there determinant computations for general $n \times n$ matrices?

Yes!

Determinant Definition Part 1

A **determinant** is a *function*

$$\det : \mathbb{R}^{n \times n} \to \mathbb{R}$$

with the following properties:

Determinant Definition Part 1

A **determinant** is a *function*

$$\det : \mathbb{R}^{n \times n} \to \mathbb{R}$$

with the following properties:

1. $\det(I_n) = 1$.

Determinant Definition Part 1

A **determinant** is a *function*

$$\det : \mathbb{R}^{n \times n} \to \mathbb{R}$$

with the following properties:

1. $\det(I_n) = 1$.
2. If A and B are row equivalent by *swapping* two rows, then

$$\det(B) = -\det(A).$$

Determinant Definition Part 1

A **determinant** is a *function*

$$\det : \mathbb{R}^{n\times n} \to \mathbb{R}$$

with the following properties:

1. $\det(I_n) = 1$.
2. If A and B are row equivalent by *swapping* two rows, then

   $$\det(B) = -\det(A).$$

   This condition says that the determinant is *antisymmetric* or *alternating* on matrix rows.

Determinant Definition Part 2

3 If A and B are row equivalent by *scaling a row of A by k* then

$$\det(B) = k \det(A).$$

Determinant Definition Part 2

3 If A and B are row equivalent by *scaling a row of A by k* then

$$\det(B) = k\det(A).$$

4 Label the rows of the matrix A by $a_1, a_2, \ldots, a_n$.

Determinant Definition Part 2

3 If A and B are row equivalent by *scaling a row of* A *by* k then

$$\det(B) = k\det(A).$$

4 Label the rows of the matrix A by $a_1, a_2, \ldots, a_n$. Choose one row, say a_j, and a covector $v \in \mathbb{R}^{1\times n}$.

Determinant Definition Part 2

3 If A and B are row equivalent by *scaling a row of A by k* then

$$\det(B) = k\det(A).$$

4 Label the rows of the matrix A by $a_1, a_2, \ldots, a_n$. Choose one row, say a_j, and a covector $v \in \mathbb{R}^{1\times n}$. We have:

$$\det\left(\begin{bmatrix} a_1 \\ \vdots \\ a_{j-1} \\ a_j + v \\ a_{j+1} \\ \vdots \\ a_n \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_1 \\ \vdots \\ a_{j-1} \\ a_j \\ a_{j+1} \\ \vdots \\ a_n \end{bmatrix}\right) + \det\left(\begin{bmatrix} a_1 \\ \vdots \\ a_{j-1} \\ v \\ a_{j+1} \\ \vdots \\ a_n \end{bmatrix}\right).$$

These conditions say that the matrix is *multilinear* on the rows of the matrix.

Exercise

Verify these properties for the 2×2 determinant $ad - bc$.

Exercise

Verify these properties for the 2×2 determinant $ad - bc$.
Here is property 4:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ad - bc$$

Exercise

Verify these properties for the 2×2 determinant $ad - bc$.
Here is property 4:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ad - bc$$

$$B = \begin{bmatrix} a & b \\ e & f \end{bmatrix} \qquad \det(B) = af - be$$

Exercise

Verify these properties for the 2×2 determinant $ad - bc$.
Here is property 4:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ad - bc$$

$$B = \begin{bmatrix} a & b \\ e & f \end{bmatrix} \qquad \det(B) = af - be$$

$$C = \begin{bmatrix} a & b \\ c+e & d+f \end{bmatrix}$$

Exercise

Verify these properties for the 2×2 determinant $ad - bc$.
Here is property 4:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ad - bc$$

$$B = \begin{bmatrix} a & b \\ e & f \end{bmatrix} \qquad \det(B) = af - be$$

$$C = \begin{bmatrix} a & b \\ c+e & d+f \end{bmatrix} \qquad \begin{aligned} \det(C) &= a(d+f) - b(c+e) \\ &= ad + af - bc - be \\ &= \det(A) + \det(B). \end{aligned}$$

Some Comments

- Notation: Sometimes the determinant is written with vertical bars ($|\cdot|$):

$$|A| = \det(A) \quad \text{and} \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$

Some Comments

- Notation: Sometimes the determinant is written with vertical bars ($|\cdot|$):

$$|A| = \det(A) \quad \text{and} \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$

- Suppose A is a matrix with two identical rows. Then $\det(A) = 0$.

Some Comments

- Notation: Sometimes the determinant is written with vertical bars ($|\cdot|$):

$$|A| = \det(A) \quad \text{and} \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$

- Suppose A is a matrix with two identical rows. Then $\det(A) = 0$. Why?

Some Comments

- Notation: Sometimes the determinant is written with vertical bars ($|\cdot|$):

$$|A| = \det(A) \quad \text{and} \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$

- Suppose A is a matrix with two identical rows. Then $\det(A) = 0$. Why? Because if we swap those two rows we get the same matrix and the negative determinant. So

$$|A| = -|A|$$

Some Comments

- Notation: Sometimes the determinant is written with vertical bars ($|\cdot|$):

$$|A| = \det(A) \quad \text{and} \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$

- Suppose A is a matrix with two identical rows. Then $\det(A) = 0$. Why? Because if we swap those two rows we get the same matrix and the negative determinant. So

$$|A| = -|A| \Leftrightarrow |A| = 0.$$

Some Comments

- Notation: Sometimes the determinant is written with vertical bars ($|\cdot|$):

$$|A| = \det(A) \quad \text{and} \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$

- Suppose A is a matrix with two identical rows. Then $\det(A) = 0$. Why? Because if we swap those two rows we get the same matrix and the negative determinant. So

$$|A| = -|A| \Leftrightarrow |A| = 0.$$

- Suppose A and B are row equivalent by *adding two rows of A together and replacing one of them*.

Some Comments

- Notation: Sometimes the determinant is written with vertical bars ($|\cdot|$):

$$|A| = \det(A) \quad \text{and} \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$

- Suppose A is a matrix with two identical rows. Then $\det(A) = 0$. Why? Because if we swap those two rows we get the same matrix and the negative determinant. So

$$|A| = -|A| \Leftrightarrow |A| = 0.$$

- Suppose A and B are row equivalent by *adding two rows of A together and replacing one of them*. Then $\det(A) = \det(B)$.

Some Comments

- Notation: Sometimes the determinant is written with vertical bars ($|\cdot|$):
$$|A| = \det(A) \quad \text{and} \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$
- Suppose A is a matrix with two identical rows. Then $\det(A) = 0$. Why? Because if we swap those two rows we get the same matrix and the negative determinant. So
$$|A| = -|A| \Leftrightarrow |A| = 0.$$
- Suppose A and B are row equivalent by *adding two rows of A together and replacing one of them*. Then $\det(A) = \det(B)$. Why?

Some Comments

- Notation: Sometimes the determinant is written with vertical bars ($|\cdot|$):

$$|A| = \det(A) \quad \text{and} \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$

- Suppose A is a matrix with two identical rows. Then $\det(A) = 0$. Why? Because if we swap those two rows we get the same matrix and the negative determinant. So

$$|A| = -|A| \Leftrightarrow |A| = 0.$$

- Suppose A and B are row equivalent by *adding two rows of A together and replacing one of them*. Then $\det(A) = \det(B)$. Why?

$$|B| = |A| + |C|$$

and $|C| = 0$ because it has two identical rows!

Computing Determinants the Fancy Way

- A determinant of a 1×1 matrix is obvious:

$$\det([a]) = a.$$

Computing Determinants the Fancy Way

- A determinant of a 1×1 matrix is obvious:

$$\det([a]) = a.$$

- We already have a determinant formula for 2×2 matrices. You should memorize it.

Computing Determinants the Fancy Way

- A determinant of a 1×1 matrix is obvious:

$$\det([a]) = a.$$

- We already have a determinant formula for 2×2 matrices. You should memorize it.
- For larger matrices we can determine determinant values directly from the properties!

Computing Determinants the Fancy Way

- A determinant of a 1×1 matrix is obvious:

$$\det([a]) = a.$$

- We already have a determinant formula for 2×2 matrices. You should memorize it.
- For larger matrices we can determine determinant values directly from the properties! For example

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

Computing Determinants the Fancy Way

- A determinant of a 1×1 matrix is obvious:

$$\det([a]) = a.$$

- We already have a determinant formula for 2×2 matrices. You should memorize it.
- For larger matrices we can determine determinant values directly from the properties! For example

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1$$

from condition 2!

Exercises

Argue the following computations:

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -6, \quad \begin{vmatrix} 0 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 6$$

$$\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 6, \quad \begin{vmatrix} 2 & x & y \\ 0 & 3 & z \\ 0 & 0 & 1 \end{vmatrix} = 6$$

Note: in the last case, it does not matter what x, y, z are.