Variance and Standard Deviation

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Goals for this Lecture

1. Define and interpret the variance and standard deviation of a random variable.
2. Learn some techniques for computation of these quantities.
3. Learn a couple numerical properties.

This material corresponds to section 3.4 of the textbook.

Basic Definitions and Properties

Introduction

Consider the RVs

$$X_1 = \begin{cases} 1000000 & \text{with probability } 0.50 \\ 0 & \text{with probability } 0.50, \end{cases}$$

$$X_2 = \begin{cases} 499999 & \text{with probability } 1/3, \\ 500000 & \text{with probability } 1/3, \\ 500001 & \text{with probability } 1/3. \end{cases}$$

Same expected value, wildly different behavior.

Exercise: Verify X_1 and X_2 have the same expected value.

Variance

Definition

The *variance* of a random variable X with mean $\mu = \mathbb{E}X$ is given by

$$\text{Var } X = \mathbb{E}[(X-\mu)^2].$$

This is often denoted by σ_X^2.

Computation

Fact: The variance of X can be computed by the formula

$$\text{Var } X = \mathbb{E}X^2 - (\mathbb{E}X)^2.$$

Exercise: Prove this fact.

Standard Deviation

Definition

The *standard deviation* of a random variable X is given by

$$\mathrm{SD}(X) = \sqrt{\mathrm{Var}\ X}.$$

This is often denoted by σ_X.

The standard deviation is a more meaningful measure of a random variables fluctuation, but the square root makes it more difficult to work with.

Example

Find the variance and standard deviation of a $\mathrm{Geo}(p)$ RV.

Example

Find the variance and standard deviation of a continuous RV with PDF

$$f_X(x) = \begin{cases} \frac{x^2}{3} & \text{for } -1 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Basic Properties

Scaling and Translation

Fact: For real numbers a, b we have

$$\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X),$$

$$\mathrm{SD}(aX + b) = |a| \mathrm{SD}(X).$$

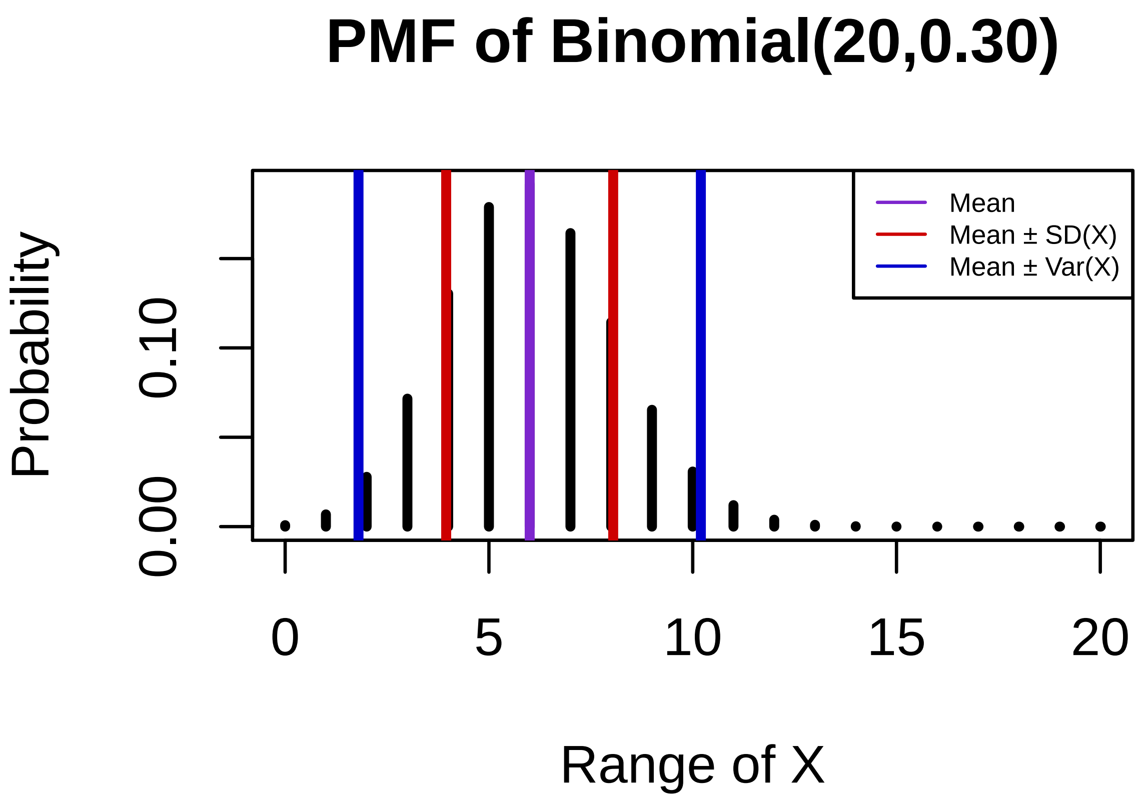
Exercise: Prove this fact.

Comparison of the Two

Binomial(20, 0.30)

Binomial(20, 0.30)

Let's compare $\mathbb{P}(\mu - \sigma_X < X \leq \mu + \sigma_X)$ to $\mathbb{P}(\mu - \sigma_X^2 < X \leq \mu + \sigma_X^2)$.


PMF of Binomial(20,0.30)
Mean
Mean ± SD(X)
Mean ± Var(X)
Probability
0.00
0.10
0
5
10
15
20
Range of X

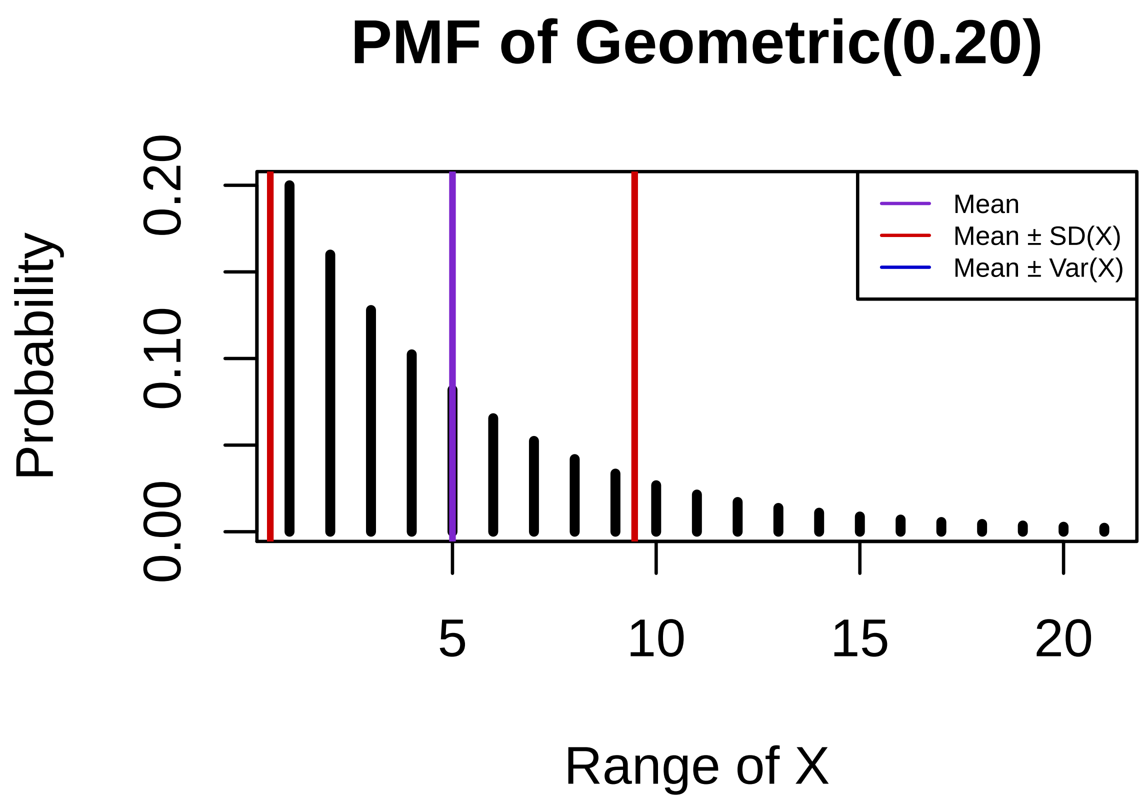
```
pbinom(MeanX+sdX, size = n, prob = p)- pbinom(MeanX-sdX, size = n, prob = p)
```

```
## [1] 0.7795817
```

```
pbinom(MeanX+VarX, size = n, prob = p)- pbinom(MeanX-VarX, size = n, prob = p)
```

```
## [1] 0.9752179
```

Geometric(0.20)



Geometric(0.20)

Let's compare $\mathbb{P}(\mu - \sigma_X < X \leq \mu + \sigma_X)$ to $\mathbb{P}(\mu - \sigma_X^2 < X \leq \mu + \sigma_X^2)$.

```
p=0.20 ; MeanX = 1/p ; VarX <- (1-p)/(p^2) ; sdX <- sqrt(VarX)
print(c(VarX, sdX))
```

```
## [1] 20.000000  4.472136
```

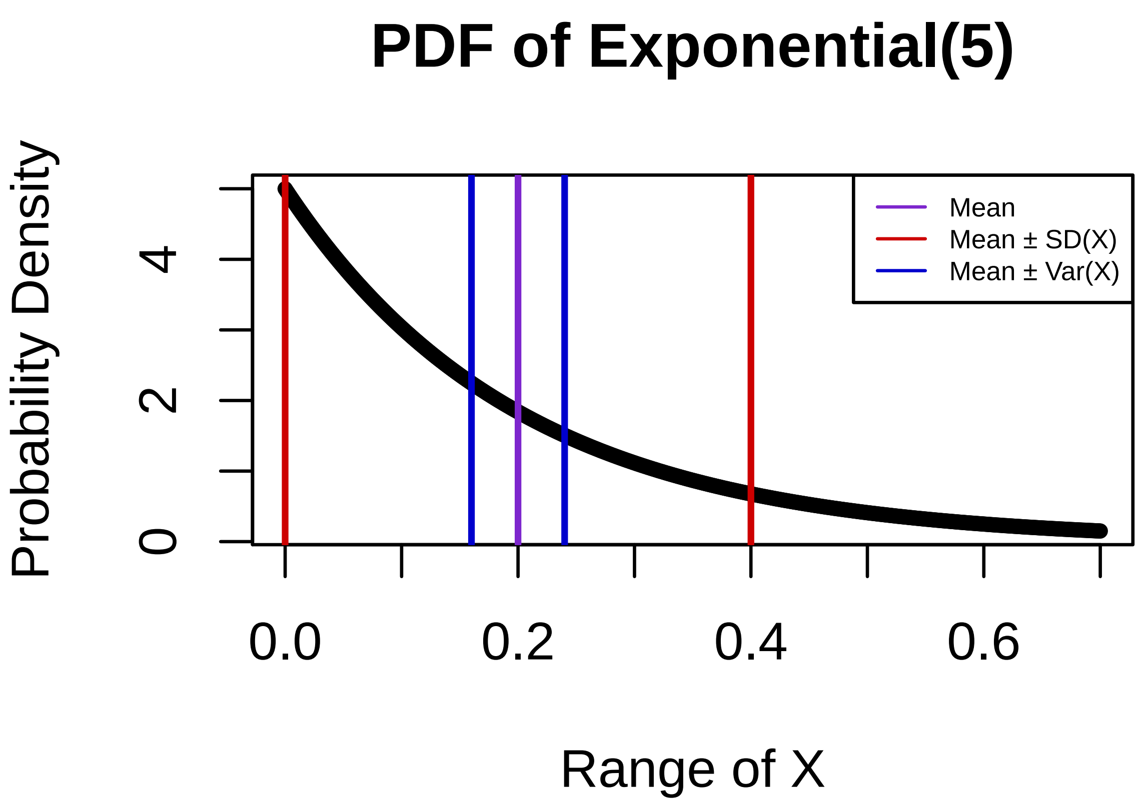
```
pgeom(MeanX+sdX, prob = p) - pgeom(MeanX-sdX, prob = p)
```

```
## [1] 0.6926258
```

```
pgeom(MeanX+VarX, prob = p) - pgeom(MeanX-VarX, prob = p)
```

```
## [1] 0.9969777
```

Exp(5)



Exp(5)

```
Rate=5 ;MeanX = 1/Rate ; VarX <- 1/(Rate^2) ; sdX <- sqrt(VarX)
print(c(VarX, sdX))
```

```
## [1] 0.04 0.20
```

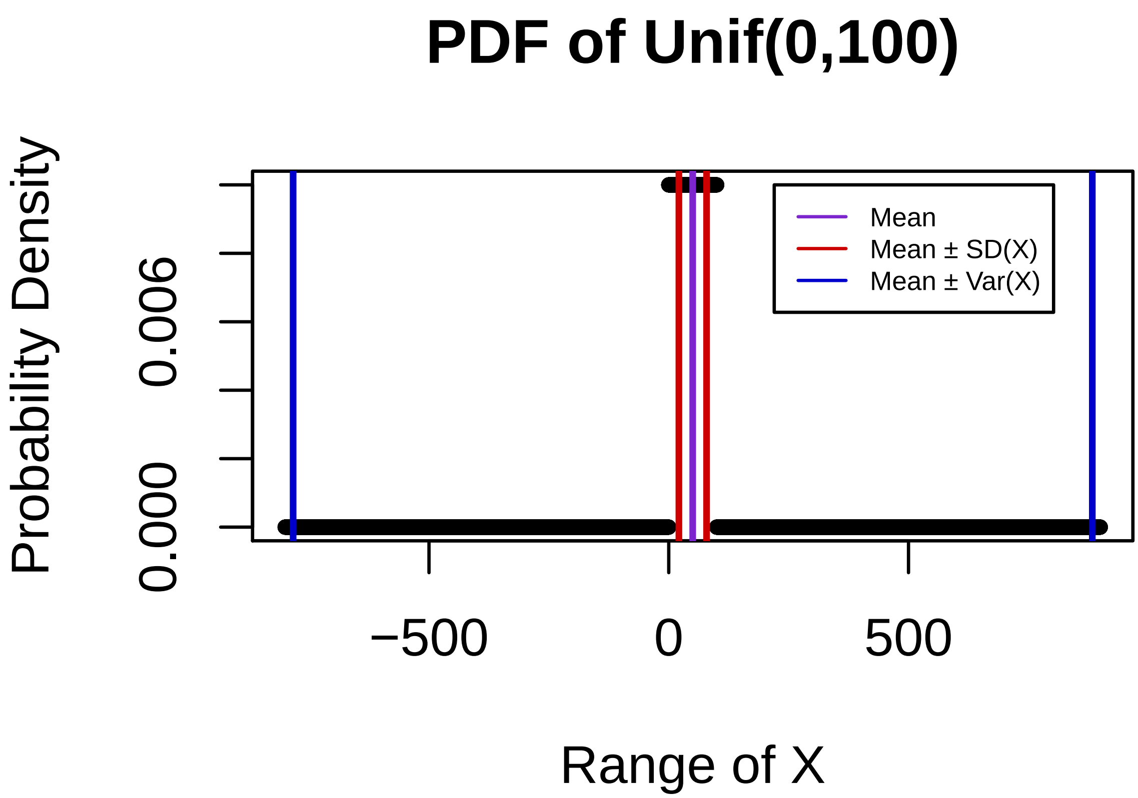
```
pexp(MeanX+sdX, rate = Rate) - pexp(MeanX-sdX, rate = Rate)
```

```
## [1] 0.8646647
```

```
pexp(MeanX+VarX, rate = Rate) - pexp(MeanX-VarX, rate = Rate)
```

```
## [1] 0.1481348
```

Unif(0,100)



The Wrap Up

Summary

1. Variance and standard deviation are used to measure the spread of a RV from its mean.
2. Compute variance using $\mathbb{E}X^2 - \mu^2$.
3. Remember the scaling laws, which tell you standard deviation is the more natural, but frustrating, measure of spread.

Next Step

Now we have all the tools necessary to define the most important continuous distribution in classical probability theory, the Normal distribution.