Exercises

Recall that a “linear combination” of objects means to take those things, multiply them by (real) numbers, and add them together.

Let

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad w = u - 2v.$$

1. Graph the three vectors u, v, and w on the same axes.
2. Write a sentence in your notebook using the phrase “linear combination” and the letters u, v, w.
3. Express v as a linear combination of u and w.

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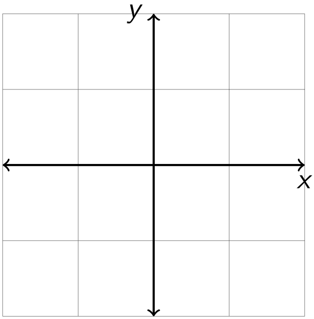

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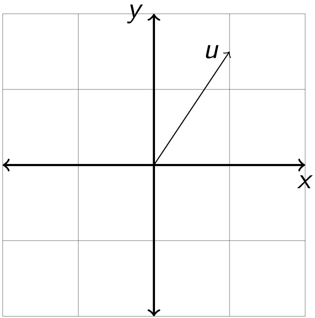

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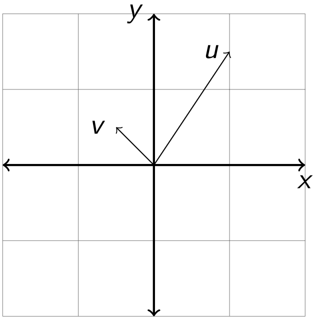

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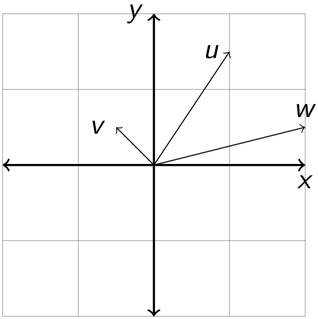

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$$\begin{aligned} w &= u - 2v \\ 2v &= u - w \\ v &= \frac{1}{2}u - \frac{1}{2}w. \end{aligned}$$

Matrix Multiplication Defines a Function

Let $A \in \mathbb{R}^{n \times m}$ and $v \in \mathbb{R}^m$.

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- $f_A(v)$ is the **image** of v.
- The set

  $$\mathrm{Im}(f_A) = \{f_A(v) \in \mathbb{R}^n : v \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

  is the **image** of f_A. It is a subset of the range.

Example One

Let

$$A = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

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- Image of $\left[\begin{array}{c} x \\ y \\ z \end{array} \right]$:

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- Image of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$: $f_A\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$.

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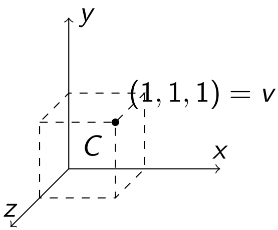
- Image of f_A: $\mathbb{R}^2$.

If image of f is the same as the range of f then we say that f is **onto**.

Geometry of f_A



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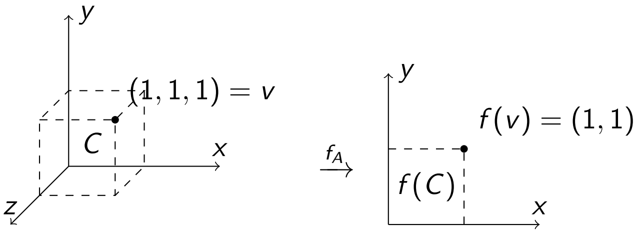

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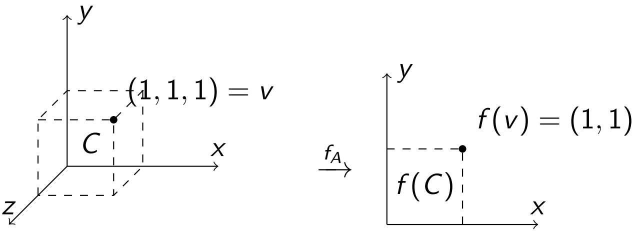

f "collapses" $\mathbb{R}^3$ onto $\mathbb{R}^2$.

Example Two

Let

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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- Domain of f_B:

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- Image of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$:

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- Domain of f_B: $\mathbb{R}^3$.
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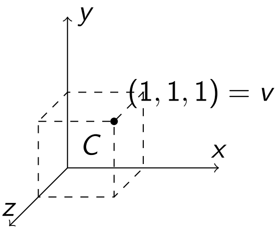
f is NOT onto, since the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is NOT in the image set.

Geometry of f_B



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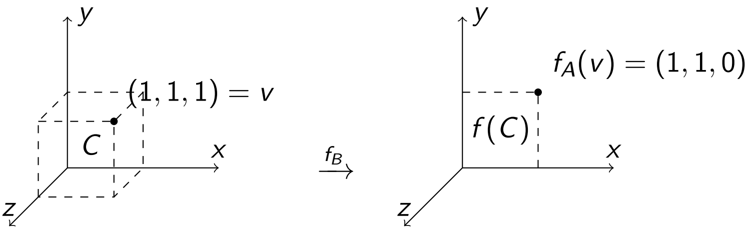

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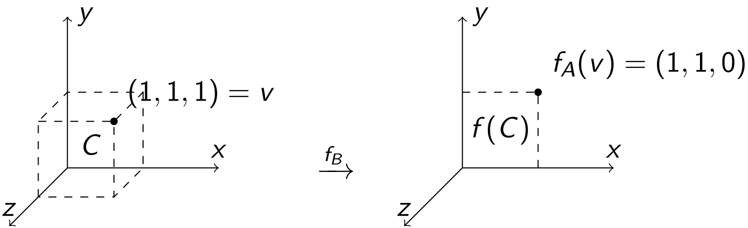

f "collapses" $\mathbb{R}^3$ onto the x, y-plane of $\mathbb{R}^3$.

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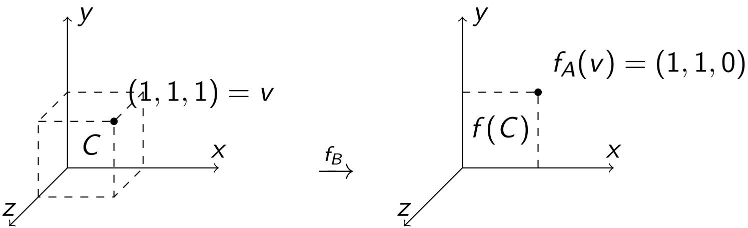


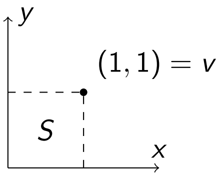
f "collapses" $\mathbb{R}^3$ onto the x, y-plane of $\mathbb{R}^3$.
In this case the range and domain are the same. We call such a collapse a **projection**.

Exercises

Interpret each of the following matrices as linear transformations. What is the image of $\begin{bmatrix} x \\ y \end{bmatrix}$? What is the image of the unit square? Try to identify how the transform manipulates geometry.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

f_A



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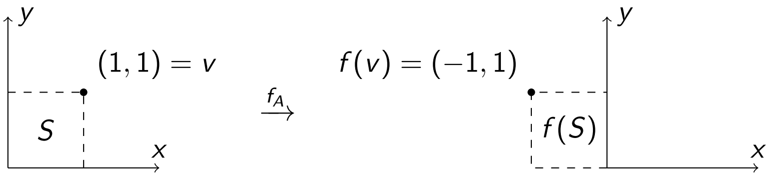
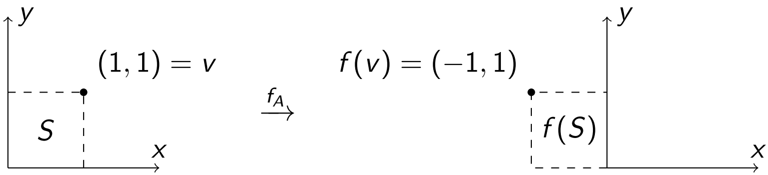
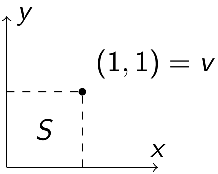
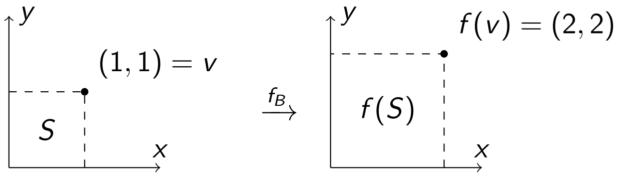




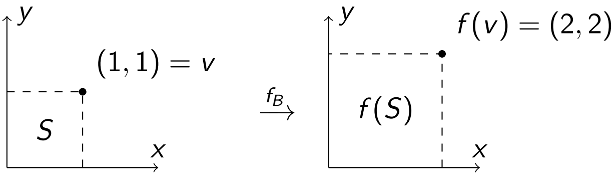
f_A is a **reflection** across the y-axis.


y
$(1,1) = v$
S
x

f_B

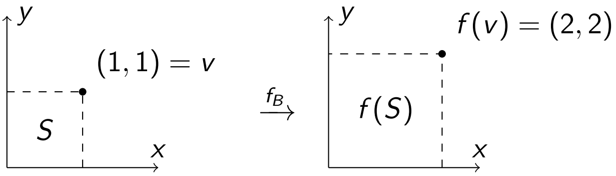


f_B



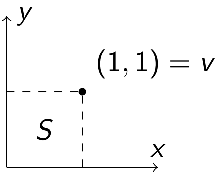
f_B is a **dilation**.

f_B

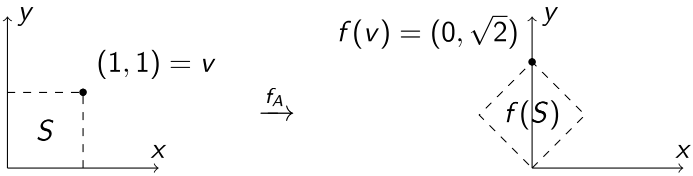


f_B is a **dilation**. If the diagonal elements were less than one (but positive) it would be a **contraction**.

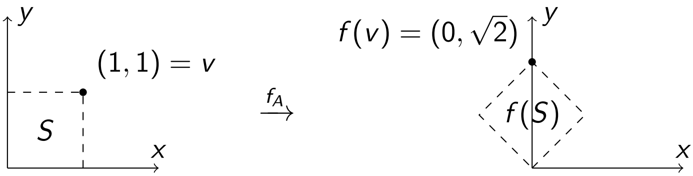
f_C



f_C



f_C



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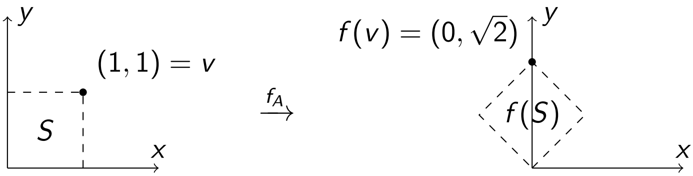

f_C is a **rotation** by $\pi/2$. Note: in general, counter-clockwise is the "positive" direction for rotations.

Rotations

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So $R_\theta^{-1} = R_{-\theta}$ is a rotation by $-\theta$.