Exercises

Compute the following determinants directly from the definition:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix}.$$

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Result: They are both 1. Note that the matrices are transposes.

Determinant Definition

Recall that the determinant is a function:

$$\det : \mathbb{R}^{n\times n} \to \mathbb{R}$$

so that

1. $\det(I) = 1$.
2. det is *antisymmetric* on the rows of A.
3. det is *linear* on each row of A.

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These properties can be applied to the elementary row operations:

Old Matrix	Operation	New Matrix	Det Prop
A	Switch two rows of A	B	$\lvert B\rvert = -\lvert A\rvert$
A	Scale one row of A by k	B	$\lvert B\rvert = k\lvert A\rvert$
A	Add/replace rows of A	B	$\lvert B\rvert = \lvert A\rvert$

Immediate Results:

Theorem

Let A be an $n \times n$ matrix then

1. *If A is diagonal, upper triangular, or lower triangular then*

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One goal today is to strengthen (2.2) into an 'if and only if' statement.

Determinants and Elementary Matrices

Recall that each elementary row operation can be represented by *left* multiplication by some matrix E:

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Lemma

Let A be a matrix and E be an elementary matrix. Then

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Result from Lemma

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A is non-singular $\Rightarrow \det(A) \neq 0$.

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Suppose A is non-singular. Then there exist elementary row operation matrices E_i so that

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Therefore $\det(A) \neq 0$. $\square$

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Let A and B be $n \times n$ matrices. Then

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  - Exercise: Prove this using $AA^{-1} = I$.

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How do we prove this?

Suppose there are two functions $(\det_1)$ and $(\det_2)$ which satisfy all the properties of the determinant. We need to show that $\det_1(A) = \det_2(A)$ for every matrix A.

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Case 2 A is non-singular. Then $A = E_1 \cdots E_n$ for some elementary matrices E_i. We have $\det_1(E_i) = \det_2(E_i)$. Therefore:

$$\begin{aligned}
\det_1(A) &= \det_1(E_1 \cdots E_n) \\
&= \det_1(E_1) \cdots \det_1(E_n) \\
&= \det_2(E_1) \cdots \det_2(E_n) \\
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Now we know: if a computation satisfies all of the properties of the determinant, then it must be the determinant!

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Next: We will look at a computational interpretation for the determinant!