Jointly Continuous RVs

Gregory M. Shinault

Introduction

After working with jointly discrete RVs, the natural step to take is into jointly continuous RVs.

Goals for This Lecture

We will learn how to

1. compute probabilities for multiple continuous RVs using their *joint probability density function*,
2. determine the *marginal probability density function* for a random variable from the joint PDF,
3. use the law of the unconscious statistician in a multivariate setting, and
4. use the joint PDF to determine if a collection of random variables is independent.

This material corresponds to section 6.2 and part of 6.3 of the textbook.

Joint PDF

Bivariate Case

Definition (for 2 RVs): Let X and Y be continuous RVs. Their *joint probability density function* is defined as the function which determines joint probabilities for X and Y:

$$\mathbb{P}(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y)\, dy\, dx$$

or for any region $R \subseteq \mathbb{R}^2$,

$$\mathbb{P}((X,Y) \in R) = \iint_R f_{X,Y}(x,y)\, dy\, dx$$

Example

Suppose X and Y have the joint PDF

$$f(x,y) = \begin{cases} \frac{1}{2} & \text{for } 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\mathbb{P}(Y = 1)$, $\mathbb{P}(X + Y < 1)$, $\mathbb{P}(X < Y)$, and $\mathbb{P}(X = Y)$.

Properties

1. Probabilities are positive:

$$f_{X,Y}(x,y) \ge 0$$

2. Total probability is 1:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx = 1$$

3. To find the PDF for only one of the RVs, you integrate over all the values of the other RV:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

Many Variables

Definition (for many RVs): Let $X_1, X_2, \ldots, X_n$ be continuous RVs. Their *joint probability density function* is defined as the function which determines joint probabilities for $X_1, X_2, \ldots, X_n$:

$$\begin{aligned} \mathbb{P}(a_1 \le \quad & X_1 \le b_1, a_2 \le X_2 \le b_2, \ldots, a_1 \le X_n \le b_n) \\ &= \textstyle\int_{a_n}^{b_n} \cdots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n, \end{aligned}$$

or more generally for region $R \subseteq \mathbb{R}^n$.

$$\begin{aligned} \mathbb{P}((X_1, \quad & X_2, \ldots, X_n) \in R) \\ &= \textstyle\int \cdots \int_R f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n. \end{aligned}$$

Example

Suppose X, Y, and Z have the joint PDF

$$f_{X,Y,Z}(x,y,z) = \begin{cases} 8xyz & \text{for } 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\mathbb{P}(Z \ge XY)$, and $\mathbb{P}(X < Y)$.

Marginal PDF

Definition: Let $X_1, X_2, \ldots, X_n$ be continuous RVs with joint PDF $f(x_1, \ldots, x_n)$. The *marginal PDF* of $X_1, X_2, \ldots, X_m$ for $m < n$ is defined by

$$f_{X_1,\ldots,X_m}(x_1, \ldots, x_m)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)\, dx_{m+1}\, dx_{m+2} \cdots dx_n.$$

Uniform Distribution

Definition: Let $X_1, X_2, \ldots, X_n$ be continuous RVs. They are said to have the *uniform distribution* on the region A if they have the joint PDF

$$f(x_1, x_2, \ldots, x_n) = \begin{cases} \frac{1}{\text{Volume}(A)} & \text{for } (x_1, x_2, \ldots, x_n) \in A \\ 0 & \text{otherwise.} \end{cases}$$

Problem: Suppose X and Y are uniformly distributed on the unit circle, $x^2 + y^2 \leq 1$. Find the marginal PDF of Y.

Expectation

Law of the Unconscious Statistician

Let $g : \mathbb{R}^n \to \mathbb{R}$. If $X_1, \ldots, X_n$ are continuous random variables with joint PDF f then

$$E[g(X_1, \ldots, X_n)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_n) f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

Example (6.14 in textbook)

Suppose X, Y have joint PDF

$$f(x, y) = \begin{cases} \frac{3}{2}(xy^2 + y) & \text{if } 0 \leq x, y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the value of $P(X < Y)$ and $E[X^2 Y]$.

Independence

Fact

Fact: Continuous RVs $X_1, \ldots, X_n$ are independent if and only if

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n).$$

for all $x_1, x_2, \ldots, x_n$.

Example

Suppose X and Y are independent RVs with distributions $\mathrm{Exp}(\lambda)$ and $\mathrm{Exp}(\gamma)$, respectively. Compute $\mathbb{P}(X < Y)$.

The Wrap Up

Summary

1. A joint PDF is based on the same idea and has the same properties as an ordinary PDF. The only difference is that it is multivariable.
2. A marginal PDF is obtained by integrating over the ranges of the RVs that we are not interested in.
3. Independence is equivalent to splitting the joint PDF into a product of marginal PDFs.
4. Independence is equivalent to splitting the joint PDF into a product of marginal PDFs.