Event Algebra and Axioms

Gregory M. Shinault

Sample Spaces and Events

Recall the following facts for a random experiment:

1. The *sample space* Ω is the set of all outcomes.
2. An *event* is a subset of the the sample space. These are what we compute probabilities of.
3. A *probability measure* is needed to assign probabilities to events.

Goal for this Lesson

Learn how to combine events mathematically.

The corresponding textbook section is Appendix B and Section 1.1.

Operations on Sets

Working Example

$$\begin{aligned}\Omega &= \text{Die roll outcomes} = \{1,2,3,4,5,6\}\\ A &= \text{Roll is even} = \{2,4,6\}\\ B &= \text{Roll is prime} = \{2,3,5\}\end{aligned}$$

The "AND" Operation / Set Intersection

In English: Event A AND Event B occur

Mathematical Notation: $A \cap B = AB$

Computation: $\{2,4,6\} \cap \{2,3,5\} = \{2\}$

The "OR" Operation / Set Union

In English: Event A OR Event B occurs (not XOR)

Mathematical Notation: $A \cup B$

Computation: $\{2,4,6\} \cup \{2,3,5\} = \{2,3,4,5,6\}$

The "NOT" Operation / Set Complement

In English: Event A does NOT occur

Mathematical Notation: $A^c = \Omega \backslash A$

Computation: $\{2,4,6\}^c = \{1,2,3,4,5,6\} \backslash \{2,4,6\} = \{1,3,5\}$

Algebraic Properties of Set Operations

Distributive Laws

$$A(B \cup C) = AB \cup AC$$
$$A \cup (BC) = (A \cup B)(A \cup C)$$

De Morgan's Law

$$(AB)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c B^c$$

Examples

A simple lottery is being held for a large sum of money. The lottery tickets are labeled 1 through 500. I buy tickets labeled 37, 311, 104, 425, 117. My arch nemesis Matthew McConaughey buys tickets labeled 37, 117, 87, 75. Let G be the event that I win the lottery and M be the event that McConaughey wins the lottery.

Describe the following events in terms of G and M, then explicitly write out the set corresponding to the event (if it is reasonably small).

(a) I win the lottery.
(b) McConaughey wins the lottery.
(c) We both win the lottery.
(d) I win and McConaughey does not.
(e) We both lose.

Partitions

Big Idea

A common problem-solving technique is to decompose your main problem into smaller, easier problems.

Within probability theory, this often means breaking your event into smaller events.

With appropriate restrictions, we call this decomposition a **partition**.

Formalism

In English: Event A can be broken down into the distinct cases $A_1, A_2, \ldots$

Mathematical Notation: $A = A_1 \cup A_2 \cup \cdots$ where $A_i A_j = \varnothing$ for $i \neq j$. Some people write $A = A_1 \sqcup A_2 \sqcup \cdots$

Definition: We say that A and B are *disjoint* if $AB = \varnothing$.

Definition: We say that $A_1, A_2, \ldots$ form a *partition* of A if 1. $A_1, A_2, \ldots$ are pairwise disjoint, and 2. $A = A_1 \cup A_2 \cup \cdots$.

Partitioning Countably Infinite Sample Spaces

Give three different ways to partition the sample space $\Omega = \mathbb{N} = \{1, 2, 3, \ldots\}$ into a finite or countable number of events.

Partitioning Uncountably Infinite Sample Spaces

Give three different ways to partition the sample space

- $\Omega = [0, 10]$
- $\Omega = \mathbb{R}^+ = [0, \infty)$
- $\Omega = [0, 3] \times [2, 4]$

into a finite or countable number of events.

Comment: The difference between bounded and unbounded sample spaces is significant. We will address this mysterious comment in detail a bit later.

The Axioms of Probability

Definition

A *probability measure* and a sample space Ω is a function $\mathbb{P} : \text{Events} \to \mathbb{R}$ that satisfies

1. $\mathbb{P}(E) \geq 0$,

2. $\mathbb{P}(\Omega) = 1$, and
3. if $E_1, E_2, E_3, \ldots$ are pairwise disjoint then

$$\mathbb{P}\left(\cup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathbb{P}\left(E_k\right).$$

Comment: These are the only assumptions needed for probability theory to work.

Wrap Up

Summary

The event algebra is a mathematically formal way to describe events.

1. The fundamental operations we will use are set intersection, union, and complement.
2. The fundamental properties we will use in conjunction with these operations are the distributive and De Morgan's laws.

Next Step

Okay, we have enough set theory to do some serious probability. Now we need to learn how to count.