Exercises

Let $A, B \in \mathbb{R}^{n \times n}$ and $x, y \in \mathbb{R}^n$. For each of the following determine if the statement is true or false (and try to give a reason why):

1. A linear system of three equations can have exactly three different solutions.
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3. $Ax \in \mathbb{R}^n$.
4. $A + A^T$ is symmetric.
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5. Ax is a linear combination of the columns of A.
   TRUE: Write $A = [a_1 \cdots a_n]$ where a_i is the i-th column of A and $x = [x_1 \cdots x_n]^T$. Then
   $$Ax = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i a_i.$$

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OTOH, we know that some systems DO have unique solutions. What types of matrices are they associated with?

Nonsingular and Inverse Matrices

Definition

$A \in \mathbb{R}^{n \times n}$ is called **nonsingular** or **invertible** if there exists another matrix $B \in \mathbb{R}^{n \times n}$ so that $AB = I$.

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It is: $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

A Good Example

Let a, b, c, d be real numbers and $ad - bc \neq 0$. Verify that

$$A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right].$$

is invertible with inverse

$$\frac{1}{ad - bc} \left[\begin{array}{cc} d & -b \\ -c & a \end{array} \right]$$

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Notation: If A is nonsingular then we write A^{-1} for its unique inverse.

Properties of A^{-1}

Let A and B be nonsingular $n \times n$ matrices.

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What is Special About Non-Singular Matrices

Let Suppose $AB = AC$ for matrices A, B, and C. Moreover, assume that A is non-singular with inverse A^{-1}. Then:

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In particular: If A is an invertible matrix then

$$Ax = b \Leftrightarrow x = A^{-1}b$$

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Exercise

Solve the linear system

$$\begin{aligned} 2x + y &= 4 \\ -2x + y &= -4 \end{aligned}$$

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$$\left\{ \begin{array}{c} 2x + y = 4 \\ -2x + y = -4 \end{array} \right\} \Leftrightarrow \left[\begin{array}{cc} 2 & 1 \\ -2 & 1 \end{array} \right] \left[\begin{array}{c} x \\ y \end{array} \right] = \left[\begin{array}{c} 4 \\ -4 \end{array} \right] \Leftrightarrow Ax = b$$

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$$x = A^{-1}b = \left[\begin{array}{cc} 1/4 & -1/4 \\ 1/2 & 1/2 \end{array} \right] \left[\begin{array}{c} 4 \\ -4 \end{array} \right] = \left[\begin{array}{c} 2 \\ 0 \end{array} \right]$$

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$$\left\{ \begin{array}{c} 2x + y = 4 \\ -2x + y = -4 \end{array} \right\} \Leftrightarrow \left[\begin{array}{cc} 2 & 1 \\ -2 & 1 \end{array} \right] \left[\begin{array}{c} x \\ y \end{array} \right] = \left[\begin{array}{c} 4 \\ -4 \end{array} \right] \Leftrightarrow Ax = b$$

So

$$x = A^{-1}b = \left[\begin{array}{cc} 1/4 & -1/4 \\ 1/2 & 1/2 \end{array} \right] \left[\begin{array}{c} 4 \\ -4 \end{array} \right] = \left[\begin{array}{c} 2 \\ 0 \end{array} \right]$$

So the solution set is $\{(x, y) = (2, 0)\}$.

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So we might write:

$$A = \begin{bmatrix} I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$$

Partitioned Matrices Results

In short: Operations on partition matrices work the same as on regular matrices provided the sizes of the partitions make sense. There are some exercises in the HW which explore this.