Axioms of Probability and Infinite Sample Spaces

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Goal for this Lecture

- Learn the axioms of probability theory in general.
- Learn how to use the additivity axiom when the sample space contains infinitely many outcomes.

The material of this lecture roughly corresponds to 1.1 and 1.3 of the textbook.

The Axioms of Probability

Recall the axioms of probability:

1. $\mathbb{P}(E) \geq 0$,
2. $\mathbb{P}(\Omega) = 1$, and
3. if $E_1, E_2, E_3, \ldots$ are pairwise disjoint then

$$\mathbb{P}\left(\cup_k E_k\right) = \sum_k \mathbb{P}\left(E_k\right).$$

Today we are going to look at what consequences these simple rules must have on all sample spaces and events.

The First of Many Coin Toss Problems

Prove that a fair coin tossed repeatedly will eventually come up heads.

Follow Up Question

What is the probability that the first heads occurs on an odd flip?

Uncountable Sample Spaces

Equally Likely Outcomes (Uncountable Setting)

Introductory Example: I throw a dart at a dartboard that is 10 inches across. Assuming my aim is terrible and there is no consistency in my throws, what is the probability that the dart is within 2 inches of the bullseye?

Equally Likely Outcomes (Uncountable Setting)

Definition: Suppose Ω is a bounded, uncountable sample space in 1 dimension. The *uniform probability measure* on Ω is given by

$$\mathbb{P}(A) = \frac{\text{Length}(A)}{\text{Length}(\Omega)}.$$

Comment: This generalizes in 2 dimensions to Area, 3 dimensions to Volume, and so on.

Examples

1. You drop a piece of chalk that is 3 inches long, and it breaks into exactly two pieces. The break is equally likely to occur at any position on the chalk. What is the probability that the larger of the two broken pieces is more than 2 inches long?
2. We choose a point uniformly from the triangle with corners at $(0,2)$, $(2,0)$, $(4,2)$. What is the probability that the point is above the line $y = 1$?

The Wrap Up

Summary

1. The additivity axiom is pretty much the best tool we have for dealing with countably many outcomes.
2. The uniform probability measure is the only tool we have (for now) for dealing with uncountably many outcomes.

Next Step

After seeing how the axioms can be useful for dealing with infinitely many outcomes, we are going to look at what consequences they have for sample spaces in general.