Exercise

Find all solutions to $Ax = 0$ for

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$

Solution:

Use row reduction to show:

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Therefore

$$\begin{aligned} x_1 &= -2x_3 - x_5 \\ x_2 &= -2x_3 + x_5 \\ x_3 &= x_3 \\ x_4 &= -2x_5 \\ x_5 &= x_5 \end{aligned}$$

Parametric Form of Solution

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Null Space

Definition

The **Null Space** of a matrix A is the set

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Based on our exercise, we can write

$$N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

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The nullity of A is 2.

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Find a basis for the null space (and therefore also the nullity) for the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

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For B: The null space is *trivial*! It has no basis and the nullity is 0.

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Summary:

- Solutions to a linear system differ by a vector in the null space.
- Solutions to linear systems are the null space shifted by a solution vector.

Example

Consider the linear system

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ 3 & 6 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

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$$N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Note also that $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is a solution.

Example Continued

Therefore the set of all solutions to the problem is

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

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This is exactly the parametric form for a plane through the point $(2, 0, 0)$! That is, it is the plane "shifted" to $(2, 0, 0)$.