Warm Up Exercises

Let

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. Compute the following: E_1M, E_2M, and E_3M.
2. What effect does each of E_i have on M.
3. Can you generalize? That is, for a given matrix $M \in \mathbb{R}^{m\times n}$ can you produce a matrix E so that the product EA replicates one of the effects you describe in 2?

Elementary Row Operation I: Swapping Rows

The product $E_1 M$ is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

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The effect is that E_1 **swapped rows** 1 and 2 of matrix M.

General Row Swap Matrix

In general, if M is an $m \times n$ matrix, then we can create a row swapping matrix $E_1 \in \mathbb{R}^{m \times m}$ by swapping the corresponding rows of I_m.

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Example: what will EM look like for

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}?$$

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The result will swap the 2nd and 5th rows of M.

Elementary Row Operation 2: Scaling a Row

The product $E_2 M$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 12 & 15 & 18 \\ 7 & 8 & 9 \end{bmatrix}$$

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The effect is that E_2 **multiplied row** 2 of matrix M by 3.

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In general, if M is an $m \times n$ matrix, then we can create a row scaling matrix $E_2 \in \mathbb{R}^{m \times m}$ by multiplying the corresponding row of I_m.

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$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}?$$

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In general, if M is an $m \times n$ matrix, then we can create a row scaling matrix $E_2 \in \mathbb{R}^{m \times m}$ by multiplying the corresponding row of I_m.

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The result will multiply the 4th row of M by k.

Elementary Row Operation 3: Adding Rows Together

The product E_3M is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \\ 7 & 8 & 9 \end{bmatrix}$$

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The effect is that E_3 **summed** rows 1 and 2 of Matrix M **and replaced** row 2 of matrix M with the sum.

General Row Addition Matrix

In general, if M is an $m \times n$ matrix, then we can create a row addition matrix $E_3 \in \mathbb{R}^{m \times m}$ by placing an additional 1 in an entry of I_m.

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$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} ?$$

The result will add the 2nd row to the 4th row of M.

Exercise

What do you think the following will do to $3 \times n$ matrix M with rows m_1, m_2, m_3?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} - & m_1 & - \\ - & m_2 & - \\ - & m_3 & - \end{bmatrix}$$

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$$= \begin{bmatrix} - & m_1 - 2m_3 & - \\ - & m_2 & - \\ - & m_3 & - \end{bmatrix}$$

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   - What is its inverse? Replace the scaling factor k with $\frac{1}{k}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/k & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

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Note: the "subtract row" matrix can be factored as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Summary for Elementary Row Operations

There are three elementary row operations for matrices:

I Swapping two rows of a matrix.

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III Replacing a row of a matrix with the component sum of it and another row.

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- Each inverse is a product of elementary row operations.

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- $\hat{E}$ is a (finite) product of elementary row operations of various types:

$$\hat{E} = \prod_{i=1}^{n} E_i, \text{ where each } E_i \text{ are different EROMs.}$$

Quick Note

In practice, applying elementary row operations is done computationally without referring to the EROM. That is: we do not create the matrices and then multiply. However, it is important is that we know they exist and have these nice properties!

Example: The following are row equivalent

$$\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 0 & 2 & 3 \\ 2 & 2 & -5 & 2 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 0 & 2 & 3 \\ 2 & 2 & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 1 & 11/2 & -5 \\ 0 & 0 & 1 & 3/2 \end{bmatrix}$$

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The matrix on the right is said to be in **row echelon form**

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If in addition:

- Each column with "leading one" has only zero entries above and below the one

then the matrix is said to be in **reduced row echelon form** (RREF).

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$$\text{REF} \leftrightarrow \text{closest approximation to } \textit{an} \text{ upper triangular matrix}$$
$$\text{RREF} \leftrightarrow \text{closest approximation to } \textit{the} \text{ identity matrix}$$

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