The Poisson Approximation

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Introduction

Reminder

If $Var(X) = np(1-p) > 10$, the normal approximation to the binomial distribution should be fairly accurate.

What can we do if n is large, but p makes this condition fail?

Goals for this Lecture

1. Derive the correct probability mass function for approximating the binomial distribution for small success probabilities.
2. Define the Poisson(λ) distribution.
3. Learn how to use the Poisson approximation, and when it is appropriate to use.

This material corresponds to section 4.4 of the textbook.

Poisson Distribution

Definition: We say that X is a Poisson(λ) RV if it has PMF

$$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

for $k = 0, 1, 2, \ldots$. We denote this by $X \sim \text{Pois}(\lambda)$.

Fact

If $X \sim \text{Pois}(\lambda)$ then $\mathbb{E}X = \lambda$ and $\text{Var}(X) = \lambda$.

Exercise: Prove this fact.

Poisson Approximation

Big Idea

Suppose $X \sim \text{Bin}(n, p)$. If $p \ll 1$ then $X \overset{d}{\approx} Y \sim \text{Pois}(np)$.

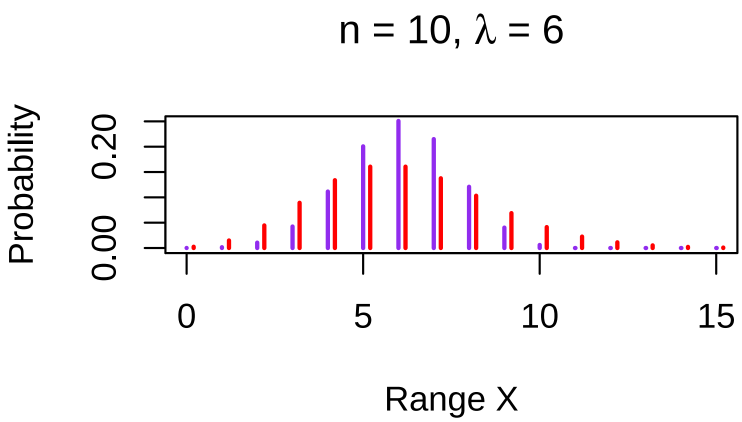
Formal Statement

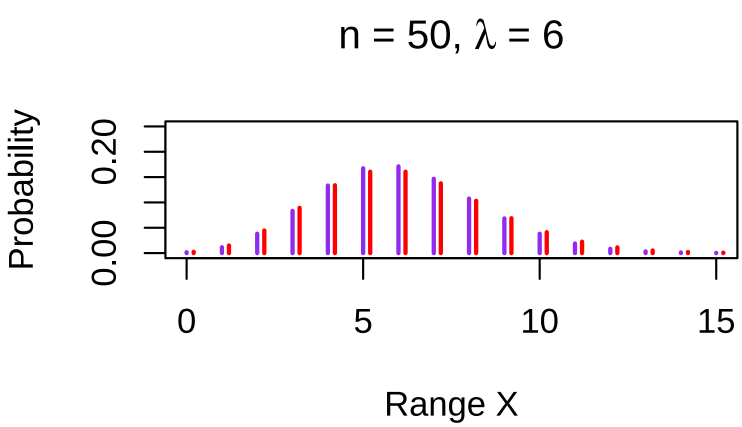
Fact: Let $\lambda > 0$, set $p_n = \lambda/n$. Suppose $S_n \sim \mathrm{Bin}(n, p_n)$ (so that $\mathbb{E}S_n = \lambda$). Then

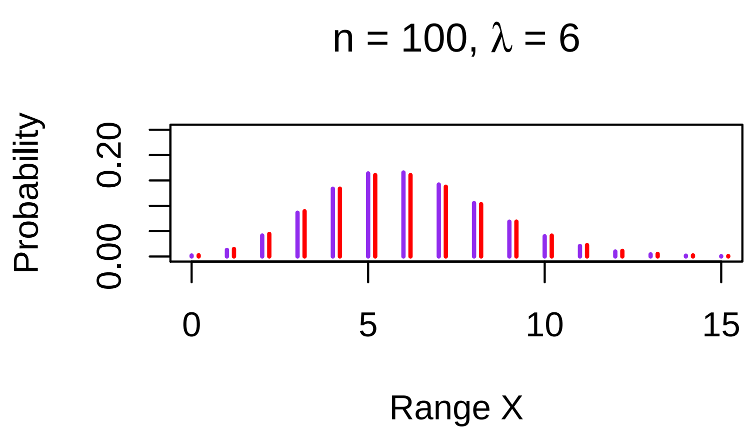
$$\lim_{n\to\infty} \mathbb{P}(S_n = k) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

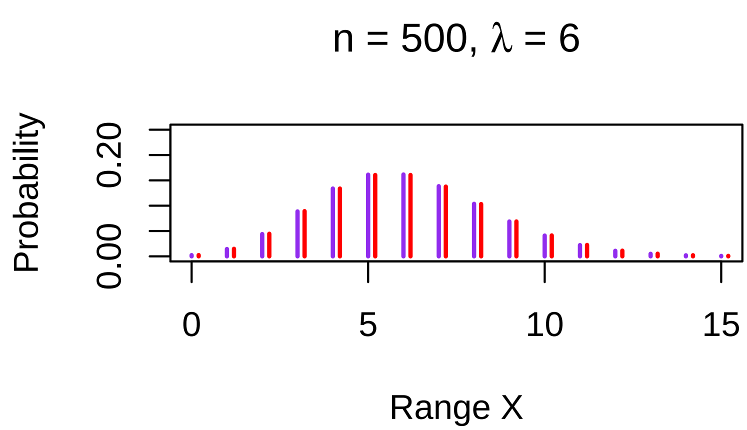
Proof Picture

We can look at some graphs comparing the Binomial and Poisson PMFs to get a feeling for when the approximation is useful.









Practical Considerations and Examples

Example

The probability of dying by horse kick in the Prussian cavalry in the late 19th century is $\frac{1}{10000}$. There are 150,000 soldiers in the cavalry.

What is the probability that more than 20 people die by horse kick in 1893? (Is normal approximation appropriate? Why not?)

Solution in R

```
SampleSize <- 150000
Prob <- 1e-04
MeanX <- SampleSize * Prob
Actual <- 1 - pbinom(20, size = SampleSize, prob = Prob)
Approx <- 1 - ppois(20, lambda = MeanX)
ApproxNorm <- 1 - pnorm(20, mean = MeanX, sd = sqrt(MeanX *
    (1 - Prob)))
c(Actual, Approx, ApproxNorm)

## [1] 0.08296046 0.08297091 0.09834161
```

Example

We have a weekly card game with a friend, in which we play 50 hands of poker. Our friend is a cheater and deals the cards so that the probability of getting a 2 of a kind more than 5 times in a week is 0.01. We have this card game every week for 5 years. What is the approximate probability that in at least 2 weeks we get 2 of a kind more than 5 times?

Solution in R

```
SampleSize <- 5 * 52
Prob <- 0.01
MeanX <- SampleSize * Prob
Actual <- 1 - pbinom(1, size = SampleSize, prob = Prob)
Approx <- 1 - ppois(1, lambda = MeanX)
ApproxNorm <- 1 - pnorm(1, mean = MeanX, sd = sqrt(MeanX *
    (1 - Prob)))
c(Actual, Approx, ApproxNorm)

## [1] 0.7341664 0.7326151 0.8406849
```

Error Bound

Fact: Let $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Pois}(np)$. Then for any a, b

$$|\mathbb{P}(a \leq X \leq b) - \mathbb{P}(a \leq Y \leq b)| \leq np^2.$$

Note no dependence on a, b!

Proof is a bit advanced for this course.

Applying the Error Bound

Note this explains why the previous example is well approximated by the Poisson distribution:

$$\mathrm{Var}(X) = 150000 \cdot 0.0001 \cdot 0.9999 = 14.9985 > 10,$$

so normal approximation is probably pretty good. However,

$$|\mathbb{P}(X \geq 21) - \mathbb{P}(Y \geq 21)| \leq 150000 \cdot (0.0001)^2 = 0.0015.$$

So Poisson approximation will be great.

Lesson: If p is really tiny, Poisson approximation will be excellent.

Note on Applications

The Poisson distribution is used for many applications largely due to the approximation theorem.

However, it has the fairly strong condition that

$$\mathbb{E}X = \mathrm{Var}X.$$

The Wrap Up

Summary

1. The Poisson(λ) distribution has PMF
$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$
for $k = 0, 1, 2, \ldots$ with mean λ and variance λ.
2. If p is tiny, then $\mathrm{Bin}(n, p)$ is approximately the same distribution as $\mathrm{Pois}(np)$.
3. Sometimes Poisson approximation is better than normal approximation even when $\mathrm{Var}(X) > 10$.

Next Step

We have seen how the binomial distribution can be approximated by the normal distribution, which is continuous. Is there something similar we can do for the geometric distribution?