exercises

Use ONLY ELIMINATION to solve the following

1.

$$\begin{aligned} x_1 - 3x_2 &= -7 \\ 2x_1 + x_2 &= 7 \end{aligned}$$

2.

$$\begin{aligned} x - 3y &= -7 \\ 2x - 6y &= 7 \end{aligned}$$

3.

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Answers

1. $(2, 3)$
2. System is not consistent.
3. $(1, -2, 3)$.

Exercise (from text)

A manufacturer makes three types of chemicals: A, B, C.

- A ton of A needs 2 hours to refine and 2 hours to package.
- A ton of B needs 3 hours to refine and 2 hours to package.
- A ton of C needs 4 hours to refine and 3 hours to package.

The refining machine is available 80 hours per week, meanwhile the packing machine is available 60 hours per week. How much of A, B, and C can be manufactured in a week?

1. Give an appropriate system of equations which represents the scenario given in the problem.
2. In your opinion (and without doing any computations) what types of solution sets are possible?

Solutions

1. Let a be the amount (in tons) of chemical A, b for B, and c for C. Then we have the system:

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2a + 3b + 4c &= 80 \\
2a + 2b + 3c &= 60
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Solutions

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2. Interpret the linear equations as planes in $\mathbb{R}^3$. Then we can see that we have a consistent system with an infinite number of solutions since they are not parallel.

Let's do some elimination on the board over here $\longleftarrow$.

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One thing we notice as we go through the elimination steps is that (so long as we are careful) location determines variables and ONLY THE COEFFICIENTS AND WHERE THEY ARE MATTER!

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$$\Updownarrow$$

$$\textbf{Matrix} \longrightarrow \begin{bmatrix}
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a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
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\end{bmatrix} \Leftrightarrow [a_{ij}] \, .$$

Example 1 of Matrices

A matrix of air travel times between several airports:

	To: JFK	To: LAX
From: JFK	0	6:20
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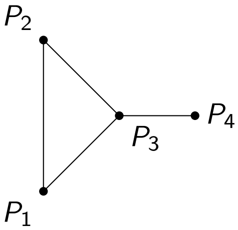
$$\begin{bmatrix} 0 & 6.333 \\ 5.25 & 0 \end{bmatrix}$$

Example 2: **Incidence Matrix**

An incidence matrix for a **graph** tells you if two nodes are adjacent:



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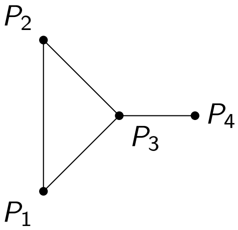

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$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1/10 & 1/10 \\ 9/10 & 9/10 \end{bmatrix}$$

Definition and Standard Terminology

Definition (Matrix)

Let m and n be positive integers. An $m \times n$ **Matrix** is a rectangular array of elements (for us: real or complex numbers) arranged in m (horizontal) rows and n (vertical) columns:

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Remark

Technically we are using the term *vector* rather colloquially. For example, we will see later in the course that vectors, covectors, and even matrices themselves can be thought of as vectors in a more abstract way.

Operations on Matrices

Matrices have natural algebraic operations.

- *Scalar Multiplication*: Let $r \in \mathbb{R}$ and $A = [a_{ij}] \in \mathbb{R}^{m \times n}$. Then the matrix $rA := [ra_{ij}] \in \mathbb{R}^{m \times n}$.

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Answer:

$$\begin{bmatrix} -1 & -1 & 2 \\ -1 & -3 & 4 \\ -2 & 5 & -1 \end{bmatrix}$$

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Let $A = [a_{ij}] \in \mathbb{R}^{n \times m}$. Then the **transpose** of A is the matrix $A^T := [a_{ij}^T] \in \mathbb{R}^{m \times n}$ where

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Let $A \in \mathbb{R}^{2 \times 7}$. How many rows and columns does A^T have?
$A^T \in \mathbb{R}^{7 \times 2}$. So 7 rows and 2 columns.