Moment Generating Functions

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Motivation

Reminder

$\mathbb{E}X$ tells us the center of X

If we know $\mathbb{E}X^2$ we can find $\mathrm{Var}(X)$ which tells us the spread of X.

A Couple Steps Further

We can go up to EX^3 to find

$$\mathrm{Skew}(X) = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$$

and EX^4 to find

$$\mathrm{Kurt}(X) = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right].$$

These measure the skewness and peakedness of the distribution of X.

If we know all the moments $\mathbb{E}X^n$ of X, does this fully describe the distribution of X? In many cases, yes.

Goals for this Lecture

1. Define the *moment generating function* for any random variable.
2. See exactly how the moment generating function generates moments.
3. Use the MGF theory to easily solve some otherwise difficult problems.

This material corresponds to section 5.1 of the textbook.

Theory

Definition

The *moment generating function* (MGF) of a RV X is defined as

$$M_X(t) = \mathbb{E}e^{tX}.$$

Key Fact I

The moment generating function generates moments. That is,

$$\frac{d^n}{dt^n}\left[M_X(t)\right]_{t=0} = \mathbb{E}X^n.$$

Key Fact II

Suppose there exists some $\delta > 0$ such that $M_X(t) = M_Y(t)$ for all $-\delta < t < \delta$. Then $X \stackrel{d}{=} Y$.

Examples

Example

Find the MGF of $X \sim \text{Poisson}(\lambda)$. Use it to find $\text{Var}(X)$

Example

Suppose X has the MGF

$$M_X(t) = 0.2 + 0.1e^t + 0.2e^{2t} + 0.3e^{4t} + 0.2e^{7t}.$$

Find the PMF of X.

Example

Find the MGF of $X \sim \text{Exp}(\lambda)$. Use it to find $\text{Var}(X)$

The Wrap Up

Summary

1. $M_X(t) = \mathbb{E}e^{tX}$
2. $M_X^{(n)}(0) = \mathbb{E}X^n$
3. If $M_X(t) = M_Y(t)$ then $X \stackrel{d}{=} Y$ (with some restrictions).