VS Properties

For all vectors u, v, and w in V and scalar values r and s in F we have

1. $u \oplus v = v \oplus u$
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3. There is a vector $e \in V$ so that $u \oplus e = u$
4. There is a vector u^{-1} so that $u \oplus u^{-1} = e$.
5. $r \odot (u \oplus v) = (r \odot u) \oplus (r \odot v)$
6. $(r + s) \odot u = (r \odot u) \oplus (s \odot u)$
7. $r \odot (s \odot u) = (rs) \odot u$
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Examples of vector spaces are matrices, $\mathbb{R}^n$, functions, polynomials, series, and more.

Homogeneous Solution Set

Let $A \in \mathbb{R}^{n \times m}$. Consider the set

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This means that all of the algebraic properties 1-8 hold for H assuming it is closed!

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Definition

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Exercise

Which of the following sets W are/are not subspaces of $\mathbb{R}^3$? Why?

$$W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = 0 \right\}$$

$$W_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : z = x + y \right\}$$

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W_1 and W_3 are. W_2 is not.

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A **(finite) linear combination** of vectors from a vector space V is a sum of the form

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Definition

Let V be a vector space and $W \subset V$. The **span** of W is the set

$$\mathrm{span}(W) = \{v \in V : v \text{ is a linear combination of elements from } W\}.$$

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$$\text{So } v + w = \sum_{i=1}^{n}(a_i + b_i)v_i \in \text{span}(W)$$