Expectation

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Goals for this Lecture

1. Define, use, and interpret the expectation of a random variable.
2. Learn the key properties that mathematical expectation possesses.

This material corresponds to section 3.3 of the textbook.

Basics of Expectation

Motivation

The definition of the expected value is motivated by the definition of the average.

Definition

For a discrete RV X the *expected value* is given by

$$\mathbb{E}X = \sum_{k \in \mathrm{Ran}(X)} k \cdot \mathbb{P}(X = k).$$

For a continuous RV X the *expected value* is given by

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

Example

Find the expected value of a Binomial(n, p) random variable.

Example

Let X be a continuous random variable with pdf

$$f_X(x) = \begin{cases} \frac{x^2}{3} & \text{for } -1 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected value of X.

Example

Your dumb friend forgot his 8-digit PIN and decides to start guessing at random. What is the expected number of guesses he will make until he is correct?

This is really a computer security problem. If someone were to attack your system, you should be more interested in how long it would typically take to gain access than maximum possible.

History of the Problem of Points

1494: Proposed by Pacioli (an accountant)
~1550: Tartaglia realized his solution was flawed
1654: Resolved by Pascal and Fermat in a series of letters
1657: Huygens ran with their solution to write the first treatise on probability, *On Reasoning in Games of Chance*

Subsequently: Casinos use expectation to determine the fair price of games of chance, then set the price unfairly in their favor. Insurance companies pay actuaries to do essentially the same thing.

Example of the Problem of Points

You are playing a game with a friend in which the winner takes home the prize pot of $20 dollars. The game consists of independent rounds in which each player is equally likely to win. The winner of the game is the first to win 5 rounds. I have won 2 rounds, my opponent has won 3 rounds. The game is interrupted and we must divide the pot. What is a fair quantity to give to me?

Properties of Expectation

Law of the Unconscious Statistician

Fact: For a real function $g(x)$ and discrete RV X we have

$$\mathbb{E}[g(X)] = \sum_{k \in \mathrm{Ran}(X)} g(k) \cdot \mathbb{P}(X = k).$$

For a real function $g(x)$ and continuous RV X we have

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

Scaling of Expectation

The following fact is a corollary to the Law of the Unconscious Statistician.

Fact: For real constants a, b we have

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b.$$

Example

Let X be a continuous RV with PDF

$$f_X(x) = \begin{cases} \frac{x^2}{3} & \text{for } -1 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Define $Y = X^3$. Find $\mathbb{E}Y$.

St. Petersburg Paradox

Problem Statement

There is a boring casino game based on coin tossing. You flip a fair coin until you get heads. If you do this in k flips you win $\$2^k$. What is a fair price to play this game?

A Few Ideas

There are many proposed solutions to this paradox. The most practical (in my opinion) requires you to consider the number of games to be played and the bankrolls of the player and casino (which are finite).

Other good solutions discard the notion that expectation is the best determination of a fair price. See Feller's classic book on introductory probability for a good example.

Simulation of 100 Games

```
NumFlips <- rgeom(100, 0.5) + 1
Winnings <- 2^NumFlips
mean(Winnings)
```

```
## [1] 7.88
```

Simulation of 1000 Games

```
NumFlips <- rgeom(1000, 0.5) + 1
Winnings <- 2^NumFlips
mean(Winnings)

## [1] 12.104
```

Simulation of 10000 Games

```
NumFlips <- rgeom(10000, 0.5) + 1
Winnings <- 2^NumFlips
mean(Winnings)

## [1] 41.6062
```

Simulation of 100000 Games

```
NumFlips <- rgeom(1e+05, 0.5) + 1
Winnings <- 2^NumFlips
mean(Winnings)

## [1] 17.9782
```

Simulation of 1000000 Games

```
NumFlips <- rgeom(1e+06, 0.5) + 1
Winnings <- 2^NumFlips
mean(Winnings)

## [1] 23.11274
```

The Wrap Up

Summary

1. Mathematical expectation is the theoretical foundation for the average of a randomly generated dataset.
2. The Law of the Unconscious Statistician significantly simplifies expected value calculations.
3. Expectation is used to determine fairness in situations involving randomness (gambling, insurance, etc.).

4. This is only helpful for a large number of repetitions of games of chance. If the game will only be played once, the most likely outcome is best to determine your appropriate course of action.