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Definition

The **coordinate representation of x in the basis β** is

$$[x]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Note: the *order* of the basis is important in this definition. From now on, we always assume the basis is ordered unless otherwise specified.

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Verify this.

Exercise

Give the coordinate representation for the vector $4t^2 - 2t + 3$ in the basis

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So

$$\left[4t^2 - 2t + 3\right]_\beta = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

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Therefore $a_i = b_i$! (Why?) And the two representations are the same.

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Essentially an isomorphism tells us 1) that two vector spaces have the "same" elements (bijection) and 2) that they have the same algebraic structure (linearity).

If both of these are true, then the only difference between V and W is their names!

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For the following arguments assume that the basis is

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Therefore $[v+w]_\beta = [v]_\beta + [w]_\beta$.

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Once we finish this we will do the same for maps between vector spaces!