Exercises

Compute the following (if they exist):

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}^{-1} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}^{-1}$$

Results

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 3/2 & 1/2 & -3/2 \\ -1 & 0 & 1 \end{bmatrix}$$

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$\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$ is singular $(2r_1 + 3r_2 = r_3)$.

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The **elementary column operations** are

1. You can swap any two columns of a matrix.
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3. You can add any two columns together and replace one of them.

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Question: how are these related to row operations?

Review of Row Operation Properties

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Can we reinterpret these properties with column operations? What (matrix) operations makes rows into columns and vice versa?

Column Operations = Row Operations + Transposes

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So: Column operations on an $n \times m$ matrix can be realized by *right* multiplication by an $m \times m$ *elementary* matrix!

Exercise

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 7 & -3 \end{bmatrix}$$

Find matrices E_1, E_2, and E_3 so that:

1. E_1 swaps columns 2 and 3 of A.
2. E_2 multiplies column 2 by 4.
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$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Note: each of these matrices are invertible.

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Definition

Let A and B be matrices. A and B are **equivalent** if there exist two (finite) sets of elementary matrices: $\{E_i\}_{i=1}^{k}$ and $\{F_i\}_{i=1}^{\ell}$ so that:

$$A = \underbrace{E_1 E_2 \cdots E_k}_{\text{Row Ops}} B \underbrace{F_1 F_2 \cdots F_\ell}_{\text{Col Ops}}$$

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Matrices are equivalent if they can be changed from one to the other by a sequence of row and/or column operations.

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Row Operations make A equivalent to a matrix in reduced row echelon form.

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Together we get the following result:

Theorem

Every non-zero $n \times m$ matrix is equivalent to a matrix in the following form:

$$P_r = \begin{bmatrix} I_r & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix}.$$

Where I_r is the $r \times r$ identity matrix and the others are all zeros.

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Corollary

Every nonsingular matrix is equivalent to an identity matrix.

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Such a representation is called a *factorization* of A.
Let's follow a factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & -1 \end{bmatrix}$$

based on elementary row operation steps.

Example

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & -1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -4 \\ 3 & 1 & -1 \end{bmatrix}}_{U_1}$$

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So

$$E_3 E_2 E_1 A = U.$$

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- Variation on this idea is how Gaussian elimination is coded efficiently in computer algorithms.