Exercise 1

Use the inner product

$$(f, g) = \int_0^1 fg \, \mathrm{d}x$$

to find an orthonormal basis for the span of $\{1, x\}$.

Exercise 1

Use the inner product

$$(f, g) = \int_0^1 fg \,\mathrm{d}x$$

to find an orthonormal basis for the span of $\{1, x\}$.
Solution:

Exercise 1

Use the inner product

$$(f, g) = \int_0^1 fg \, \mathrm{d}x$$

to find an orthonormal basis for the span of $\{1, x\}$.
Solution: Note that 1 is already a unit in this inner product.

Exercise 1

Use the inner product

$$(f, g) = \int_0^1 fg \, \mathrm{d}x$$

to find an orthonormal basis for the span of $\{1, x\}$.
Solution: Note that 1 is already a unit in this inner product. So find $v \perp 1$ and

$$x = c_1(1) + v.$$

Exercise 1

Use the inner product

$$(f, g) = \int_0^1 fg \, \mathrm{d}x$$

to find an orthonormal basis for the span of $\{1, x\}$.
Solution: Note that 1 is already a unit in this inner product. So find $v \perp 1$ and

$$x = c_1(1) + v.$$

Take the inner product with 1:

$$(x, 1) = c_1(1, 1) + (v, 1).$$

Exercise 1

Use the inner product

$$(f, g) = \int_0^1 fg \, \mathrm{d}x$$

to find an orthonormal basis for the span of $\{1, x\}$.
Solution: Note that 1 is already a unit in this inner product. So find $v \perp 1$ and

$$x = c_1(1) + v.$$

Take the inner product with 1:

$$(x, 1) = c_1(1, 1) + (v, 1).$$

Since $(1, 1) = 1$ and $(v, 1) = 0$ we have $c_1 = (x, 1)$.

Exercise 1

Use the inner product

$$(f, g) = \int_0^1 fg \, \mathrm{d}x$$

to find an orthonormal basis for the span of $\{1, x\}$.
Solution: Note that 1 is already a unit in this inner product. So find $v \perp 1$ and

$$x = c_1(1) + v.$$

Take the inner product with 1:

$$(x, 1) = c_1(1, 1) + (v, 1).$$

Since $(1, 1) = 1$ and $(v, 1) = 0$ we have $c_1 = (x, 1)$. Compute:

$$(x, 1) = \int_0^1 x \, \mathrm{d}x = \frac{1}{2}.$$

Solution 1 continued

So $v = x - \frac{1}{2}$. This vector is in the right direction but not normalized.

Solution 1 continued

So $v = x - \frac{1}{2}$. This vector is in the right direction but not normalized. So compute:

$$||x - 1/2|| = \left(\int_0^1 \left(x - \frac{1}{2} \right)^2 \mathrm{d}x \right) = \frac{1}{\sqrt{12}}.$$

So the pair of orthonormal basis is

$$\left\{ 1, \sqrt{12} \left(x - \frac{1}{2} \right) \right\}.$$

Exercise 2

Use the inner product

$$(f, g) = \int_{-1}^{1} fg \,\mathrm{d}x$$

to find all polynomials $p \in P_2(\mathbb{R})$ which are orthogonal to both 1 and x.

Exercise 2

Use the inner product

$$(f, g) = \int_{-1}^{1} fg \, \mathrm{d}x$$

to find all polynomials $p \in P_2(\mathbb{R})$ which are orthogonal to both 1 and x.

Solution:

Let $p = ax^2 + bx + c$.

Exercise 2

Use the inner product

$$(f, g) = \int_{-1}^{1} fg \,\mathrm{d}x$$

to find all polynomials $p \in P_2(\mathbb{R})$ which are orthogonal to both 1 and x.

Solution:

Let $p = ax^2 + bx + c$. We need both $(p, 1)$ and (p, x) to be zero.

Exercise 2

Use the inner product

$$(f, g) = \int_{-1}^{1} fg \,\mathrm{d}x$$

to find all polynomials $p \in P_2(\mathbb{R})$ which are orthogonal to both 1 and x.

Solution:

Let $p = ax^2 + bx + c$. We need both $(p, 1)$ and (p, x) to be zero. Therefore both

$$\int_{-1}^{1} ax^2 + bx + c \,\mathrm{d}x = \int_{-1}^{1} ax^3 + bx^2 + cx \,\mathrm{d}x = 0.$$

Exercise 2

Use the inner product

$$(f, g) = \int_{-1}^{1} fg \, \mathrm{d}x$$

to find all polynomials $p \in P_2(\mathbb{R})$ which are orthogonal to both 1 and x.

Solution:

Let $p = ax^2 + bx + c$. We need both $(p, 1)$ and (p, x) to be zero. Therefore both

$$\int_{-1}^{1} ax^2 + bx + c \, \mathrm{d}x = \int_{-1}^{1} ax^3 + bx^2 + cx \, \mathrm{d}x = 0.$$

Solving these gives: $b = 0$ and $a = -3c$.

Exercise 2

Use the inner product

$$(f, g) = \int_{-1}^{1} fg \,\mathrm{d}x$$

to find all polynomials $p \in P_2(\mathbb{R})$ which are orthogonal to both 1 and x.

Solution:

Let $p = ax^2 + bx + c$. We need both $(p, 1)$ and (p, x) to be zero. Therefore both

$$\int_{-1}^{1} ax^2 + bx + c \,\mathrm{d}x = \int_{-1}^{1} ax^3 + bx^2 + cx \,\mathrm{d}x = 0.$$

Solving these gives: $b = 0$ and $a = -3c$. So the set of polynomials is of the form

$$k(3x^2 - 1)$$

where k is any real number.

Orthogonal complements and "perp space"

The collection of vectors perpendicular to another comes up often enough that it needs a definition:

Definition

Let W be a subset of an inner product space V. $x \in V$ is **orthogonal to** W if

$$(x, y) = 0 \text{ for all } y \in W.$$

The set

$$W^{\perp} := \{y \in V : \text{ for all } x \in W, (x, y) = 0\}$$

is called the **orthogonal complement** of W.

Orthogonal complements and "perp space"

The collection of vectors perpendicular to another comes up often enough that it needs a definition:

Definition

Let W be a subset of an inner product space V. $x \in V$ is **orthogonal to** W if

$$(x, y) = 0 \text{ for all } y \in W.$$

The set

$$W^{\perp} := \{y \in V : \text{ for all } x \in W, (x, y) = 0\}$$

is called the **orthogonal complement** of W.

$$\{1, x\}^{\perp} = \left\{k(3x^2 - 1) : k \in \mathbb{R}\right\}.$$

A Good Example to Keep in Mind

Recall that an equation of the form

$$Ax + By + Cz = 0$$

represents a plane in $\mathbb{R}^3$ through the origin with normal $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$.

A Good Example to Keep in Mind

Recall that an equation of the form

$$Ax + By + Cz = 0$$

represents a plane in $\mathbb{R}^3$ through the origin with normal $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$.

Note that

$$0 = Ax + Bx + Cx = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

A Good Example to Keep in Mind

Recall that an equation of the form

$$Ax + By + Cz = 0$$

represents a plane in $\mathbb{R}^3$ through the origin with normal $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$.

Note that

$$0 = Ax + Bx + Cx = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = n \cdot x.$$

A Good Example to Keep in Mind

Recall that an equation of the form

$$Ax + By + Cz = 0$$

represents a plane in $\mathbb{R}^3$ through the origin with normal $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$.

Note that

$$0 = Ax + Bx + Cx = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = n \cdot x.$$

Therefore the plane $Ax + By + Cz = 0$ is the same as

$$\{n\}^{\perp}.$$

Decomposing Vector Spaces

First some theory:

Theorem

Let W be a subspace of an inner-product space V.

Decomposing Vector Spaces

First some theory:

Theorem

Let W be a subspace of an inner-product space V. Then:

- *$W^{\perp}$ is a subspace of V.*

Decomposing Vector Spaces

First some theory:

Theorem

Let W be a subspace of an inner-product space V. Then:

- $W^{\perp}$ *is a subspace of* V.
- $W \cap W^{\perp} = \{0\}$.

Decomposing Vector Spaces

First some theory:

Theorem

Let W be a subspace of an inner-product space V. Then:

- $W^\perp$ *is a subspace of* V.
- $W \cap W^\perp = \{0\}$.

The proof of the first part is related to an assigned exercise.

Decomposing Vector Spaces

First some theory:

Theorem

Let W be a subspace of an inner-product space V. Then:

- $W^\perp$ *is a subspace of* V.
- $W \cap W^\perp = \{0\}$.

The proof of the first part is related to an assigned exercise. For the second part:

$$x \in W \cap W^\perp$$

Decomposing Vector Spaces

First some theory:

Theorem

Let W be a subspace of an inner-product space V. Then:

- $W^{\perp}$ *is a subspace of* V.
- $W \cap W^{\perp} = \{0\}$.

The proof of the first part is related to an assigned exercise. For the second part:

$$x \in W \cap W^{\perp} \Rightarrow (x, x) = 0$$

Decomposing Vector Spaces

First some theory:

Theorem

Let W be a subspace of an inner-product space V. Then:

- $W^{\perp}$ *is a subspace of* V.
- $W \cap W^{\perp} = \{0\}$.

The proof of the first part is related to an assigned exercise. For the second part:

$$x \in W \cap W^{\perp} \Rightarrow (x, x) = 0 \Rightarrow x = 0.$$

Decomposing Vector Spaces

Now know that a subspace and its orthogonal complement are as "separate" as subspaces can be!

Decomposing Vector Spaces

Now know that a subspace and its orthogonal complement are as "separate" as subspaces can be! Becuase of this it is possible to treat them as "basis spaces"

Decomposing Vector Spaces

Now know that a subspace and its orthogonal complement are as "separate" as subspaces can be! Becuase of this it is possible to treat them as "basis spaces" (Say that five times fast...)

Decomposing Vector Spaces

Now know that a subspace and its orthogonal complement are as "separate" as subspaces can be! Becuase of this it is possible to treat them as "basis spaces" (Say that five times fast...)

Theorem

Let W be a finite dimensional subspace of an inner-prodcut space V. Then every vector $v \in V$ can be written uniquely as

$$v = w + w^*$$

with $w \in W$ and $w^* \in W^\perp$.

Proof

Proof

To prove, let $\{w_1, \ldots, w_n\}$ be an ON basis for W.

Proof

To prove, let $\{w_1, \ldots, w_n\}$ be an ON basis for W. Let v be any vector in V.

Proof

To prove, let $\{w_1, \ldots, w_n\}$ be an ON basis for W. Let v be any vector in V. Then define:

$$w := \sum_{i=1}^{n} (v, w_i) w_i$$

Proof

To prove, let $\{w_1, \ldots, w_n\}$ be an ON basis for W. Let v be any vector in V. Then define:

$$w := \sum_{i=1}^{n} (v, w_i) w_i$$

Let $w^* = v - w$.

Proof

To prove, let $\{w_1, \ldots, w_n\}$ be an ON basis for W. Let v be any vector in V. Then define:

$$w := \sum_{i=1}^{n} (v, w_i) w_i$$

Let $w^* = v - w$. Now

- $w \in W$.

Proof

To prove, let $\{w_1, \ldots, w_n\}$ be an ON basis for W. Let v be any vector in V. Then define:

$$w := \sum_{i=1}^{n} (v, w_i) w_i$$

Let $w^* = v - w$. Now

- $w \in W$.
- For any basis element w_j of W:

$$(w^*, w_j) = (v - w, w_j) = (v, w_j) - (w, w_j)$$

Proof

To prove, let $\{w_1, \ldots, w_n\}$ be an ON basis for W. Let v be any vector in V. Then define:

$$w := \sum_{i=1}^{n} (v, w_i) w_i$$

Let $w^* = v - w$. Now

- $w \in W$.
- For any basis element w_j of W:

$$(w^*, w_j) = (v - w, w_j) = (v, w_j) - (w, w_j) = (v, w_j) - (v, w_j) = 0.$$

Proof

To prove, let $\{w_1, \dots, w_n\}$ be an ON basis for W. Let v be any vector in V. Then define:

$$w := \sum_{i=1}^{n} (v, w_i) w_i$$

Let $w^* = v - w$. Now

- $w \in W$.
- For any basis element w_j of W:

  $$(w^*, w_j) = (v - w, w_j) = (v, w_j) - (w, w_j) = (v, w_j) - (v, w_j) = 0.$$

  So $w^* \in W^\perp$.

Proof part 2

Now suppose that

$$v = w_1 + w_1^* = w_2 + w_2^*$$

with $w_1, w_2 \in W$ and $w_1^*, w_2^* \in W^\perp$.

Proof part 2

Now suppose that

$$v = w_1 + w_1^* = w_2 + w_2^*$$

with $w_1, w_2 \in W$ and $w_1^*, w_2^* \in W^\perp$.
Then

$$w_1 - w_2 = -(w_1^* - w_2^*).$$

Proof part 2

Now suppose that

$$v = w_1 + w_1^* = w_2 + w_2^*$$

with $w_1, w_2 \in W$ and $w_1^*, w_2^* \in W^\perp$.
Then

$$w_1 - w_2 = -(w_1^* - w_2^*).$$

Therefore

$$w_1 - w_2 \in W \cap W^\perp$$

Proof part 2

Now suppose that

$$v = w_1 + w_1^* = w_2 + w_2^*$$

with $w_1, w_2 \in W$ and $w_1^*, w_2^* \in W^\perp$.
Then

$$w_1 - w_2 = -(w_1^* - w_2^*).$$

Therefore

$$w_1 - w_2 \in W \cap W^\perp \Rightarrow w_1 - w_2 = 0$$

Proof part 2

Now suppose that

$$v = w_1 + w_1^* = w_2 + w_2^*$$

with $w_1, w_2 \in W$ and $w_1^*, w_2^* \in W^\perp$.
Then

$$w_1 - w_2 = -(w_1^* - w_2^*).$$

Therefore

$$w_1 - w_2 \in W \cap W^\perp \Rightarrow w_1 - w_2 = 0 \Rightarrow w_1 = w_2.$$

Similarly $w_1^* = w_2^*$.

Projection Computation

The vector:

$$w = \sum_{i=1}^{n} (v, w_i) w_i$$

is called the **Orthogonal Projection of v onto the space W spanned by** $\{w_1, \ldots, w_n\}$.

Projection Computation

The vector:

$$w = \sum_{i=1}^{n} (v, w_i) w_i$$

is called the **Orthogonal Projection of** v **onto the space** W **spanned by** $\{w_1, \ldots, w_n\}$. Or just

$$\mathrm{proj}_W v.$$

Projection Computation

The vector:

$$w = \sum_{i=1}^{n} (v, w_i) w_i$$

is called the **Orthogonal Projection of v onto the space W spanned by** $\{w_1, \ldots, w_n\}$. Or just

$$\operatorname{proj}_W v.$$

Note that it is CRUCIAL in this definition that you have an orthonormal basis...

Example

Let

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } w = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

Example

Let

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } w = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

Write $W = \operatorname{span}\{w\}$ and compute $\operatorname{proj}_W v$:

Example

Let

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } w = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

Write $W = \operatorname{span}\{w\}$ and compute $\operatorname{proj}_W v$:
First we need an ON basis for W:

$$\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \to \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}.$$

Example

Let

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } w = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

Write $W = \operatorname{span}\{w\}$ and compute $\operatorname{proj}_W v$:
First we need an ON basis for W:

$$\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \to \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}.$$

Then we compute the projection:

$$\operatorname{proj}_W v$$

Example

Let

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } w = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

Write $W = \text{span}\{w\}$ and compute $\text{proj}_W v$:
First we need an ON basis for W:

$$\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \to \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}.$$

Then we compute the projection:

$$\text{proj}_W v = \left(\frac{4}{3} - \frac{1}{3} - \frac{6}{3} \right) \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}$$

Example

Let

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } w = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

Write $W = \text{span}\{w\}$ and compute $\text{proj}_W v$:
First we need an ON basis for W:

$$\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \to \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}.$$

Then we compute the projection:

$$\text{proj}_W v = \left(\frac{4}{3} - \frac{1}{3} - \frac{6}{3} \right) \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

Exercise

Find $\operatorname{proj}_W v$ for

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } W = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$$

Exercise

Find $\operatorname{proj}_W v$ for

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } W = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$$

Solution:

Exercise

Find $\mathrm{proj}_W\, v$ for

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } W = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$$

Solution: ON basis for W:

$$W = \mathrm{Span}\left\{ \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

Exercise

Find $\operatorname{proj}_W v$ for

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } W = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$$

Solution: ON basis for W:

$$W = \operatorname{Span}\left\{ \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

So

$$\operatorname{proj}_W v$$

Exercise

Find $\mathrm{proj}_W\, v$ for

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } W = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$$

Solution: ON basis for W:

$$W = \mathrm{Span}\left\{ \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

So

$$\mathrm{proj}_W\, v = - \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} + \frac{5}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Exercise

Find $\text{proj}_W\, v$ for

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } W = \text{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$$

Solution: ON basis for W:

$$W = \text{Span}\left\{ \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

So

$$\text{proj}_W\, v = -\begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} + \frac{5}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 11/6 \\ 1/3 \\ 19/6 \end{bmatrix}$$

Orthogonal Projections are Approximations

Orthogonal Projections are Approximations

Theorem

Let V be an inner product space with subspace W. The vector $w \in W$ which minimizes $||v - w||$ is $w = \operatorname{proj}_W v$.

Orthogonal Projections are Approximations

Theorem

Let V be an inner product space with subspace W. The vector $w \in W$ which minimizes $||v - w||$ is $w = \operatorname{proj}_W v$.

This tells us that $\operatorname{proj}_W v$ is the vector in W which is closest to v!

Proof

We minimize $\|v - w\|^2$ instead.

Proof

We minimize $\|v - w\|^2$ instead.
From the orthogonal decomposition theorem we can write $v = u + u^*$ where $u \in W$ and $u^* \in W^\perp$.

Proof

We minimize $\|v - w\|^2$ instead.
From the orthogonal decomposition theorem we can write $v = u + u^*$ where $u \in W$ and $u^* \in W^\perp$. So:

$$\|v - w\|^2 = \|u + u^* - w\|^2 = \|x + u^*\|^2$$

where $x = u - w \in W$.

Proof

We minimize $\|v - w\|^2$ instead.
From the orthogonal decomposition theorem we can write $v = u + u^*$ where $u \in W$ and $u^* \in W^\perp$. So:

$$\|v - w\|^2 = \|u + u^* - w\|^2 = \|x + u^*\|^2$$

where $x = u - w \in W$.
Then

$$\|x + u^*\|^2 = (x + u^*, x + u^*) = \|x\|^2 + 2(x, u^*) + \|u^*\|^2$$

Proof

We minimize $\|v - w\|^2$ instead.
From the orthogonal decomposition theorem we can write $v = u + u^*$ where $u \in W$ and $u^* \in W^\perp$. So:

$$\|v - w\|^2 = \|u + u^* - w\|^2 = \|x + u^*\|^2$$

where $x = u - w \in W$.
Then

$$\|x + u^*\|^2 = (x + u^*, x + u^*) = \|x\|^2 + 2(x, u^*) + \|u^*\|^2$$

Now $(x, u^*) = 0$, so the above is minimized when $x = 0$.

Proof

We minimize $\|v - w\|^2$ instead.
From the orthogonal decomposition theorem we can write $v = u + u^*$ where $u \in W$ and $u^* \in W^\perp$. So:

$$\|v - w\|^2 = \|u + u^* - w\|^2 = \|x + u^*\|^2$$

where $x = u - w \in W$.
Then

$$\|x + u^*\|^2 = (x + u^*, x + u^*) = \|x\|^2 + 2(x, u^*) + \|u^*\|^2$$

Now $(x, u^*) = 0$, so the above is minimized when $x = 0$. This means $w = u$ is the solution.

Proof

We minimize $\|v - w\|^2$ instead.
From the orthogonal decomposition theorem we can write $v = u + u^*$ where $u \in W$ and $u^* \in W^\perp$. So:

$$\|v - w\|^2 = \|u + u^* - w\|^2 = \|x + u^*\|^2$$

where $x = u - w \in W$.
Then

$$\|x + u^*\|^2 = (x + u^*, x + u^*) = \|x\|^2 + 2(x, u^*) + \|u^*\|^2$$

Now $(x, u^*) = 0$, so the above is minimized when $x = 0$. This means $w = u$ is the solution. Again, by the decomposition theorem $w = u = \text{proj}_W v$.