Math 431: Homework 4 Solutions

1. **Exercise 2.2**

   Let T_ℓ denote the event that the ℓ-th flip was tails, H_ℓ denote the event that the ℓ-th flip was heads, and N_k is the event that there were exactly k tails in the three flips. We want to compute $P(T_2|N_0 \cup N_1)$.

$$\begin{aligned} P(T_2|N_0 \cup N_1) &= \frac{P(T_2(N_0 \cup N_1))}{P(N_0 \cup N_1)} \\ &= \frac{P(T_2N_0 \cup T_2N_1)}{P(N_0 \cup N_1)} \\ &= \frac{P(T_2N_1)}{P(N_0) + P(N_1)} \\ &= \frac{P(H_1T_2H_3)}{(1/8) + (3/8)} \\ &= \frac{1/8}{4/8} \\ &= \frac{1}{4}. \end{aligned}$$

2. **Exercise 2.3**

   Our sample space is $\Omega = \{j \in \mathbb{Z} \mid 1 \le j \le 100\}$. We can define the events

$$\begin{aligned} B &= \{j \in \Omega \mid j \text{ is divisible by } 3\} \\ &= \{3, 6, 9, \ldots, 99\} \\ A &= \{j \in \Omega \mid j \text{ contains at least one digit equal to } 5\} \\ &= \{5, 15, 25, 35, 45, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 65, 75, 85, 95\} \\ \Rightarrow BA &= \{15, 45, 51, 54, 57, 75\}. \end{aligned}$$

   We want to compute $P(B|A)$. This can be accomplished using the definition of conditional probability.

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{\#BA/\#\Omega}{\#A/\#\Omega} = \frac{6/100}{19/100} = \frac{6}{19}.$$

3. **Exercise 2.10**

   Define events:

$$A = \{\text{outcome of the roll is } 4\} \quad \text{and} \quad B_k = \{\text{the } k\text{-sided die is picked}\}.$$

Then

$$P(B_6|A) = \frac{P(A \cap B_6)}{P(A)} = \frac{P(A|B_6)P(B_6)}{P(A|B_4)P(B_4) + P(A|B_6)P(B_6) + P(A|B_{12})P(B_{12})}$$
$$= \frac{\frac{1}{6} \cdot \frac{1}{3}}{\frac{1}{4} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{12} \cdot \frac{1}{3}} = \frac{1}{3}.$$

4. **Exercise 2.11**

   Let A be the event that a randomly chosen customer is accident prone. Let B be the event that a randomly chosen person has an accident. We know the following,

   $$P(A) = 0.2, \quad P(A^c) = 0.80, \quad P(B|A) = 0.04, \quad \text{and } P(B|A^c) = 0.01.$$

   We are tasked with finding the probability $P(A^c|B)$:

   $$\begin{aligned} P(A^c|B) &= \frac{P(A^cB)}{P(B)} \\ &= \frac{P(B|A^c)P(A^c)}{P(BA) + P(BA^c)} \\ &= \frac{P(B|A^c)P(A^c)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \\ &= \frac{0.01 \times 0.80}{0.04 \times 0.2 + 0.01 \times 0.80} \\ &= \frac{1}{2}. \end{aligned}$$

   You also could have used the Bayes' formula directly, and skipped what is essentially a re-derivation.

5. **Exercise 2.31**

   (a) The sample space is
   $$\Omega = \{(g, b), (b, g), (b, b), (g, g)\},$$
   and the probability measure is simply

   $$P(g, b) = P(b, g) = P(b, b) = P(g, g) = \frac{1}{4},$$

   since we assume that each outcome is equally likely.

   (b) Let A be the event that there is a girl in the family. Let B be the event that there is a boy in the family. Note that the question is asking for $P(B|A)$. Begin to solve by noting that
   $$A = \{(g, b), (b, g), (g, g)\} \text{ and } P(A) = \frac{3}{4}.$$
   Similarly,
   $$B = \{(g, b), (b, g), (b, b)\} \text{ and } P(B) = \frac{3}{4}.$$
   Finally, we have

   $$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(\{(g, b), (b, g)\})}{3/4} = \frac{2/4}{3/4} = \frac{2}{3}.$$

(c) Let $C = \{(g, b), (g, g)\}$ be the event that the first child is a girl. B is as above. We want $P(B|C)$. Since $P(C) = 1/2$ we have

$$P(B|C) = \frac{P(BC)}{P(C)} = \frac{P\{(g,b)\}}{1/2} = \frac{1/4}{1/2} = \frac{1}{2}.$$

6. **Exercise 2.33**

(a) Let B_k be the event that we choose urn k and let A be the event that we chose a red ball. Then

$$P(B_k) = \tfrac{1}{5}, \qquad P(A|B_k) = \tfrac{k}{10}, \qquad \text{for } 1 \le k \le 5.$$

By conditioning on the urn we chose, we get

$$P(A) = \sum_{k=1}^{5} P(A \mid B_k) P(B_k) = \sum_{k=1}^{5} \tfrac{k}{10} \cdot \tfrac{1}{5} = \tfrac{1+2+3+4+5}{50} = \tfrac{3}{10}.$$

(b)

$$P(B_k \mid A) = \frac{P(A|B_k)P(B_k)}{\sum_{k=1}^{5} P(A \mid B_k)P(B_k)} = \frac{\frac{k}{10} \cdot \frac{1}{5}}{\frac{3}{10}} = \frac{k}{15}.$$

7. **Exercise 2.36**

(a) We start by defining the relevant events.

$$\begin{aligned} D_j &= \text{Event that the } j\text{-sided die is chosen} \\ A &= \text{Event that the outcome of the roll is 6.} \end{aligned}$$

We want to compute $P(A)$. Before we begin, note that D_4, D_6, and D_{12} form a partition of all possible outcomes. With that in mind, we can start computing using the law of total probability.

$$\begin{aligned} P(A) &= P(A|D_4)P(D_4) + P(A|D_6)P(D_6) + P(A|D_{12})P(D_{12}) \\ &= 0 \cdot \frac{7}{12} + \frac{1}{6} \cdot \frac{3}{12} + \frac{1}{12} \cdot \frac{2}{12} \\ &= \frac{8}{144} \\ &= \frac{1}{18}. \end{aligned}$$

(b) Now we want to compute $P(D_6|A)$.

$$\begin{aligned} P(D_6|A) &= \frac{P(D_6 A)}{P(A)} \\ &= \frac{P(A|D_6)P(D_6)}{P(A)} \\ &= \frac{(1/6)(3/12)}{1/18} \\ &= \frac{1/24}{1/18} \\ &= \frac{3}{4} \end{aligned}$$