Exercise

Consider each of the following as subsets of appropriate vector spaces over the real numbers. Identify if the set is linearly dependent or independent.

$$A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}, \quad B = \left\{ \sin^2\theta, \cos^2\theta, 4 \right\}$$

$$C = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} \right\}$$

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So the only solution to

$$a \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} + b \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} -1 & 0 & 5 \\ 1 & 1 & 0 \\ 1 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $a = 5c$ and $b = -5c$. Set $c = 1$ to find

$$5\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So C is linearly dependent.

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In an ideal world, we want unique solutions to things. So we combine the two concepts together!

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$$\beta = \{v_1, \ldots, v_n\}, \quad n \in \mathbb{N}.$$

Examples: Standard basis

1. The "standard basis" for $\mathbb{R}^n$ is the set:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}.$$

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These are usually written as $e_1, e_2, \ldots, e_n$ where e_i is a vector of all zeros except for a 1 in the ith position.

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- For $P_n(\mathbb{R})$ we have

$$\{x^n, x^{n-1}, \ldots, x^2, x, 1\}$$

Examples

Which of the following sets is a basis for $\mathbb{R}^2$?

$$A = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \quad B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \quad C = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$$

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- C is not. Is not linearly independent.

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- There seems to be some sort of "sweet spot" where a set of vectors becomes a basis. Roughly speaking:
    - Not spanning $\leftrightarrow$ Not enough vectors.
    - Not linearly independent $\leftrightarrow$ Too many vectors.

Theoretical Result

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Idea: We can remove vectors from a finite set until we get to a linearly independent case WITHOUT losing the spanning property!

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So $S \subset \text{span}(\beta)$ and therefore $\text{span}(S) = \text{span}(\beta)$.

An example

Find a basis for span(S) where

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

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We need to find the largest LI subset. To find it, consider the homogeneous problem.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = 0$$

Via row reduction

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & -2 \end{bmatrix}$$

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Examine the "leading ones". It appears that the third and fifth columns depend on the first, second, and fourth.

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$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is the basis for span(S).