Exercise

Compute the following determinant by using elimination to reduce to upper (or lower) triangular form:

$$\begin{vmatrix} 2 & 1 & 4 \\ 1 & 3 & 2 \\ 1 & -7 & 1 \end{vmatrix}$$

One solution:

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Therefore from the last case, we have:

$$(-1)\left(\frac{-1}{5}\right)\det(A) = (1)(1)(-1)$$

so $\det(A) = -5$.

Today: Direct Computational Methods

Recall from before: *The determinant is unique!*. Therefore, so long as we have a computational method which satisfies the properties:

1. Results in 1 when applied to the identity matrix.
2. Is antisymmetric and multilinear.

Then it is a way to compute the determinant!

Combinatorial Method

A **permutation** on n elements is a one-to-one and onto map:

$$\sigma : \{1, 2, 3, \ldots, n\} \to \{1, 2, 3, \ldots, n\}$$

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It turns out that you can create a *signature* function:

$$\text{sgn} : S_n \to \{-1, 1\} \text{ by } \text{sgn}(\sigma) = \begin{cases} 1 & \sigma \text{ uses even flips} \\ -1 & \sigma \text{ uses odd flips} \end{cases}$$

Combinatorial Method:

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(A) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(j)} \right)$$

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Do not worry so much about this definition, though if you are interested in (combinatorics, graph theory, decision theory, data mining, abstract/modern algebra) you might start thinking about how to use it and how to prove it is a determinant.

Cofactor Expansion Part 1: Minor Matrices

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

Definition

The M_{ij} **minor** of A is an $(n-1) \times (n-1)$ matrix created by eliminating the ith row and jth column.

Example:

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$$

Write out M_{11} and M_{23}.

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The Cofactor Expansion Method

Theorem

Let $A = [a_{ij}]$*. Then for any fixed* $1 \leq \mathbf{i} \leq n$

$$|A| = \sum_{j=1}^{n} a_{\mathbf{i}j}(-1)^{\mathbf{i}+j} |M_{\mathbf{i}j}| .$$

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- This definition is via *row expansion*. It turns out that you can create a similar definition for column expansions (just transpose the i and j.

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- This definition is via *row expansion*. It turns out that you can create a similar definition for column expansions (just transpose the i and j.
- The computation does not depend on which row (or column) you choose!
- This definition is *recursive* in that we need to compute 'smaller' cofactor expansions ($|M_{ij}|$) to get 'larger' ones ($|A|$).

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&= 3(-1)^{1+1}\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} + (-1)(-1)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} + 2(-1)^{1+3}\begin{vmatrix} 4 & 5 \\ 7 & 1 \end{vmatrix}
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&= 3(10-6) + (8-42) + 2(4-35) \\
&= 12 - 34 - 62 = -84 \text{ You should check this.}
\end{aligned}$$

Exercises

Compute

$$\begin{vmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{vmatrix}$$

using

1) cofactor expansion across the second row.

2) cofactor expansion across the third column.

Compute

$$\begin{vmatrix} 3 & -1 & 2 & 300 \\ 4 & 5 & 6 & -10 \\ 0 & 0 & 0 & 1 \\ 7 & 1 & 2 & \pi \end{vmatrix}$$

Hint: What row/column do you think is best to use for the expansion?