Exchangeability

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The Big Idea

Shuffling and sampling problems can be much easier than they seem.

This material corresponds to Section 7.2 of the textbook

Permutation

Definition: A *permutation of the set* $S = \{1, 2, ..., n\}$ is a bijection $\gamma : S \to S$.

Intuition: A permutation of S is just a way to shuffle the elements of S.

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$$

So $\gamma_1 = 2$, $\gamma_2 = 3$, etc.

Exchangeability

Definition: $(X_1, \ldots, X_n)$ are *exchangeable* if

$$\mathbb{P}((X_1, \ldots, X_n) \in B) = \mathbb{P}((X_{\gamma_1}, \ldots, X_{\gamma_n}) \in B)$$

for any permutation γ and subset $B \subset \mathbb{R}$.

Intuition: The joint PMF/PDF of exchangeable RVs is unchanged if you mix up the RVs.

Criteria for Exchangeability

Fact: Suppose $(X_1, \ldots, X_n)$ are discrete RVs. $(X_1, \ldots, X_n)$ are exchangeable if and only if

$$p_{X_1,\ldots,X_n}(k_1, \ldots, k_n) = p_{X_1,\ldots,X_n}(k_{\gamma_1}, \ldots, k_{\gamma_n})$$

for any permutation γ.

Fact: Suppose $(X_1, \ldots, X_n)$ are continuous RVs. $(X_1, \ldots, X_n)$ are exchangeable if and only if

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = f_{X_1,\ldots,X_n}(x_{\gamma_1}, \ldots, x_{\gamma_n})$$

for any permutation γ.

Example | IID Sequences

Fact: Suppose $X_1, \ldots, X_n$ are independent and have the same PMF/PDF (we call this *independent and identically distributed* (IID)). Then $X_1, \ldots, X_n$ are exchangeable.

Special Example | Sample without Replacement

Let $k \leq n$. We take k samples without replacement from the set $\{1, 2, \ldots, n\}$.
X_1 denotes the first sample, X_2 denotes the second sample, and so on to X_k.

Are $X_1, X_2, \ldots, X_k$ exchangeable?

Special Example | Sample without Replacement

Suppose you deal 7 cards from a standard deck. What is the probability the fourth card is a king and the seventh card is a 10?

Nonexample | Multinomial Distribution

Suppose $(X_1, \ldots, X_r) \sim \text{Mult}(n, p_1, \ldots, p_r)$. Are $(X_1, \ldots, X_r)$ exchangeable?

Summary

Key Ideas

1. Exchangeability is the term for a distribution being unchanged by the permutation of random variables.
2. The most important use of exchangeability is for samples without replacement.
3. The multinomial distribution is not exchangeable.