Exercise

Compute the following determinant in two ways 1) using elimination to reduce to upper (or lower) triangular form and 2) by cofactor expansion.

$$\begin{vmatrix} 2 & 0 & 4 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix}$$

Solutions:

Via elimination:

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$$\det(A) = (0)(-1)\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + (1)(1)\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} + (0)(-1)\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix}$$

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Recall that the ij-th cofactor is

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where M_{ij} is the *minor* matrix created by removing the ith row and jth column from A.

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Exercise: Write A^{-1} above in terms of $|A|$ and the cofactors A_{ij}.

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Question: If A is a general non-singular matrix, can we write A^{-1} in terms of $|A|$ and its cofactors?

Cofactor Representation for Inverse Matrices

Theorem

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Stop.

This is a terrible way to compute inverse matrices.

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Why Inverses via Cofactors is Useful

Let $A = [a_{ij}]$ be a non-singular $n \times n$ matrix and b a n-vector. Then we know that the unique solution to

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Let us write A^{-1} in terms of the cofactors and x and b in terms of their components:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

So

$$x_i = \frac{1}{|A|} \sum_{k=1}^{n} b_k A_{ki}$$

What is x_i?

We have

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$$x_i = \frac{1}{|A|}\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1,i-1} & b_1 & a_{1,i+1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2,i-1} & b_2 & a_{2,i+1} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n,i-1} & b_n & a_{n,i+1} & \dots & a_{nn} \end{vmatrix}.$$

The i-th column of A is replaced by b.

Cramer's Rule

Theorem

Let $Ax = b$ be a linear system as before. If $|A| \neq 0$ then we have

$$x_i = \frac{|B_i|}{|A|}$$

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This tells us an important thing: the solution to a linear system of equations can be expressed *algebraically* in terms of the coefficients of the system matrix and the right hand side using *continuous* operations.

Summary Comments

- Inversion via cofactors/Cramer's rule is important because it tells us that the solution to a linear system can be expressed continuously from the components of the matrix and right hand side vector. This result is used for theoretical results in ODE, PDE, dynamical systems, chaos, etc.

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- What does the determinant have to do with geometry? (a bunch!)