The Exponential Distribution

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Motivation

Remember that a Geo(p) RV is representative of the waiting time for the first success in a sequence of Bernoulli trials. This is a discrete waiting time.

There must be a distribution that is representative of a continuous waiting time. We arrive at this distribution by taking the limit of the geometric distribution.

Goals for this Lecture

1. Define the Exponential(λ) distribution.
2. Learn how to perform some computations for the exponential distribution.
3. See how we can use a limit to pass from a discrete distribution to a continuous distribution.

This material corresponds to section 4.5 of the textbook.

Properties of the Exponential Distribution

Definition

A continuous RV X is said to have the *exponential distribution* with *rate* $\lambda > 0$ if it has the CDF

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We denote this by $X \sim \text{Exp}(\lambda)$.

Alternate Definition

A continuous RV X is said to have the *exponential distribution* with *rate* $\lambda > 0$ if it has the PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the exponential distribution has only one parameter, λ.

Example

Find the mean and variance of the exponential distribution.

Example

Let $X \sim \text{Exp}(\lambda)$. Find a general expression for $\mathbb{P}(a \leq X \leq b)$ where $a, b > 0$.

Memoryless Property

Fact: Let $X \sim \text{Exp}(\lambda)$. Then for $s, t > 0$ we have

$$\mathbb{P}(X > t + s \mid X > t) = \mathbb{P}(X > s).$$

Exercise: Prove this fact.

Radioactive Decay

Setup

Unstable isotopes eventually emit an α-particle (two protons and two neutrons) or a β-particle (one electron) thus making it a different isotope.

This is of interest in a probability course because the waiting time for the emission of the particle is a random variable. Specifically, it is an exponential random variable.

If we are going to model this as an exponential random variable how can we find the rate λ?

Getting the Rate from Half-life

Most decays are described in terms of their half-life. This is defined in terms of a large number of atoms. For example, radiocarbon (carbon-14) has a half-life of 5730 years. It emits a β-particle and becomes nitrogen-14 which is a stable isotope. This means that if we start out with 8 grams of carbon-14, 5730 years later we can expect to be left with 4 grams of carbon-14. For individual atoms this is $\mathbb{P}(T < 5730) = .5$. From this fact, how do we recover the rate λ?

Longer Chains

Many of the more important isotopes do not decay directly into a stable isotope. There is a long chain of decays, all having different decay rates. Often times it is important to have information about the intermediate states of decay. The first four steps in the decay chain for Uranium-235 is as follows:

$$U^{235} \xrightarrow[\alpha]{7E8a} Th^{231} \xrightarrow[\beta]{1.06d} Pa^{231} \xrightarrow[\alpha]{3.3E4a} Ac^{227} \xrightarrow[\beta]{21.8a} Th^{227}$$

Let $T_{U^{235} \to Th^{227}}$ be the time at which an isotope originating as U^{235} decays into Th^{227}.

How do we model this radioactive decay using exponential RVs?

Derivation

Setup

Suppose we need to model a continuous waiting time and we know the average waiting time is $1/\lambda$.

We can approximate this in discrete time.

$$X_n \sim \text{Geo}(\lambda/n)$$
$$T_n = \frac{X_n}{n} \quad \Rightarrow \quad \mathbb{E}T_n = \frac{\mathbb{E}X_n}{n} = \frac{1}{\lambda}$$

So T_n subdivides the intervals into units of $1/n$, while the expected waiting time is always $1/\lambda$.

Limit

Fact: Let $t > 0$. Then

$$\lim_{n\to\infty} \mathbb{P}(T_n > t) = e^{-\lambda t}.$$

Summary

The Key Facts

1. The $\text{Exp}(\lambda)$ distribution is continuous with PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

   We call the parameter $\lambda > 0$ the *rate* of the distribution.

2. The exponential distribution has mean $\frac{1}{\lambda}$ and variance $\frac{1}{\lambda^2}$.
3. The $\text{Exp}(\lambda)$ distribution is generally used to model a waiting time, or survival time.
4. The $\text{Exp}(\lambda)$ distribution is the continuous version of the geometric distribution.