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$$\mathrm{proj}_x(\sin x) = \underbrace{\left(\int_{-\pi}^{\pi} \sqrt{\frac{3}{2\pi^3}} x \sin x \, \mathrm{d}x\right)}_{(u, \sin x)} \underbrace{\sqrt{\frac{3}{2\pi^3}} x}_{u}$$

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&= \frac{3x}{2\pi^3} \int_{-\pi}^{\pi} x \sin x \, \mathrm{d}x
\end{aligned}$$

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&= \frac{3x}{2\pi^3} \int_{-\pi}^{\pi} x \sin x \, \mathrm{d}x \\
&= \frac{3x}{2\pi^3} 2\pi = \frac{3}{\pi^2} x.
\end{aligned}$$

Linear Transformations

Definition

Let V and W be vector spaces under the same field F. A function $L : V \to W$ is called **linear** if

1. $L(u + v) = L(u) + L(v)$ for all vectors $u, v \in V$.
2. $L(av) = aL(v)$ for all vectors $v \in V$ and scalars $a \in F$.

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3. $I : C^0 \to \mathbb{R}$ by $f \mapsto \int_0^1 f \, \mathrm{d}x$ is linear.
4. $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^2 \\ 2y \end{bmatrix}$ is not linear.

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4. $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^2 \\ 2y \end{bmatrix}$ is not linear. Since

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = L \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq 2L \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Exercise

For each of the following maps, decide if they are or are not linear.

1. $L : P(\mathbb{R}) \to P(\mathbb{R})$ by $p(x) \mapsto xp(x)$.
2. $L : P(\mathbb{R}) \to \mathbb{R}$ by $f \mapsto f(0)$.
3. $L : \mathbb{R} \to \mathbb{R}$ by $x \mapsto |x|$.
4. $L : \mathbb{R}^3 \to \mathbb{R}^3$ by $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}$.

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1. $L(0) = 0$ *where* 0 *is the identity in the appropriate vector space.*
2. $L(u - v) = L(u) - L(v)$ *for all* $u, v \in V$.

The first result is nice since it gives insight into the *kernel* of the operator!

Big Theorem: Basis vectors characterize linear maps

Suppose V has basis $\beta = \{v_1, \ldots, v_n\}$.

Theorem

For any n vectors w_i in W there exists a unique linear map $T : V \to W$ where

$$T(v_i) = w_i$$

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For any any $x \in V$ write $x = \sum_{i=1}^{n} \alpha_i v_i$. Then define

$$T(v) := \sum_{i=1}^{n} \alpha_i w_i.$$

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That this map is well defined and unique come from the fact that β is a basis! $\square$

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$$A = [A_1\, A_2\, \ldots\, A_n] = [L(e_1)\, L(e_2)\, \ldots\, L(e_n)]\,.$$

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Theorem

With notation as above, the matrix A is the unique matrix with the property that

$$L(x) = Ax.$$

Exercise

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} a + 2b \\ 3b - 2c \end{bmatrix}$$

Find the standard matrix representing L.

Example 2: Change of Basis

Let V be a finite dimensional vector space with basis

$$S = \{v_1, v_2, \ldots, v_n\} \text{ and } T = \{w_1, w_2, \ldots, w_n\}.$$

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Recall the coordinate map:

$$v = \sum_{i=1}^{n} c_i v_i \Rightarrow [v]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

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Can we construct the linear operator $P_{T\to S} : \mathbb{R}^n \to \mathbb{R}^n$ so that $P_{T\to S}[v]_T = [v]_S$?

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Compute:

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So (as in the prior case) the linear operator is matrix multiplication by a matrix with i-th column $[w_i]_S$.

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(on board)

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- BUT ALSO: we know that if V is a finite dimensional vector space, then $V \simeq \mathbb{R}^n$!
- So what, if any, relation is there between a map of finite dimensional vector spaces and a matrix operation on the isomorphic real spaces?