Math 431, Homework 8 Solutions

1. **Exercise 4.12**

   Following the hint that $E[X^2] = \frac{2}{\lambda^2}$, we have

$$\begin{aligned} E[X^2] &= \int_0^\infty x^3\, \lambda e^{-\lambda x}\, dx \\ &= \left[x^3 \cdot -\frac{1}{\lambda} e^{-\lambda x} \right]_{x=0}^{\infty} - \int_0^\infty 3x^2 \cdot \lambda e^{-\lambda x}\, dx \\ &= [0-0] + 3\int_0^\infty x^2 \cdots \lambda e^{-\lambda x}\, dx \\ &= 3E[X^2] \\ &= \frac{6}{\lambda^2}. \end{aligned}$$

2. **Exercise 4.14** We start by defining the appropriate random variable.

$$T = \text{Length of lightbulbs lifetime.}$$

   The $T \sim \text{Exp}(\lambda)$ where $\lambda = 1/E[T] = 1/1000$.

   (a) We want to find $P(T > 2000)$. Using the CDF for the exponential distribution we get
$$P(T > 2000) = 1 - F_T(2000) = e^{-\frac{1}{1000}\cdot 2000} = e^{-2}.$$

   (b) We want to find $P(T > 2000 \,|\, T > 500)$. We use the memoryless property in this case.
$$P(T > 2000 \,|\, T > 500) = P(T > 500 + 1500 \,|\, T > 500) = P(T > 1500) = e^{-3/2}.$$

3. **Exercise 4.50**

   We want to compute $P(T > 7+3 \,|\, T > 7)$. This is easiest using the memoryless property of the exponential distribution.

$$P(T > 7+3 \,|\, T > 7) = P(T > 3) = e^{-\frac{1}{3}\cdot 3} = e^{-1}.$$

   More generally for an additional x hours, we have

$$P(T > 7+x \,|\, T > 7) = P(T > x) = e^{-\frac{1}{3}\cdot x} = e^{-x/3}.$$

4. **Exercise 5.2**

(a) We need EX and EX^2 to find the mean and variance of X. So we start by computing the first two derivatives of the MGF.

$$M'_X(t) = \frac{-4}{3} \cdot e^{-4t} + \frac{5}{6} \cdot e^{5t}$$
$$M''_X(t) = \frac{16}{3} \cdot e^{-4t} + \frac{25}{6} \cdot e^{5t}$$

Now we use the general formula $EX^n = M_X^{(n)}(0)$ to get

$$EX = \frac{-4}{3} + \frac{5}{6} = \frac{-3}{6} = -\frac{1}{2}$$
$$EX^2 \frac{16}{3} + \frac{25}{6} = \frac{57}{6} = \frac{19}{2}.$$

Thus,

$$EX = -\frac{1}{2}$$
$$\mathrm{Var}(X) = EX^2 - (EX)^2 = \frac{19}{2} - \left(-\frac{1}{2}\right)^2$$
$$= \frac{19}{2} - \frac{1}{4} = \frac{37}{4}$$

(b) We can express the MGF as

$$M_X(t) = e^{t \cdot 0}\frac{1}{2} + e^{t \cdot -4}\frac{1}{3} + e^{t \cdot 5}\frac{1}{6}.$$

So the PMF is

k	-4	0	5
$p_X(k)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Computing the first two moments using the PMF gives

$$EX = -4 \cdot \frac{1}{3} + 0 \cdot \frac{1}{2} + 5\frac{1}{6} = -\frac{4}{3} + \frac{5}{6} = -\frac{1}{2}$$
$$EX^2 = (-4)^2 \cdot \frac{1}{3} + 0^2 \cdot \frac{1}{2} + 5^2\frac{1}{6} = \frac{16}{3} + \frac{25}{6} = \frac{57}{6} = \frac{19}{2}.$$

The first two moments are the same as our solution to the previous part, so the mean and variance will be equal as well.

5. **Exercise 5.3**

We start with the definition of the MGF.

$$M_X(t) = E[e^{tX}] = \int_0^1 e^{tx} \cdot \frac{1}{1-0}\,dx$$
$$= \left[\frac{1}{t}e^{tx}\right]_{x=0}^1$$
$$= \frac{1}{t}e^t - \frac{1}{t} = \frac{e^t - 1}{t}.$$

This computation is valid for all values of t except $t = 0$. We must find $M_X(0)$ through different methods. Fortunately, this is quite direct:

$$M_X(0) = E[e^{0 \cdot X}] = E[1] = 1$$

To summarize,

$$M_X(t) = \begin{cases} \frac{e^t - 1}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

6. **Exercise 5.6**

   The possible values Y can take are

$$\text{Range}(Y) = \{(-1-1)^2, (0-1)^2, (2-1)^2, (4-1)^2\} = \{4, 1, 1, 9\} = \{1, 4, 9\}.$$

   So we find $p_Y(k) = P(Y = k)$ for $k = 1,\ 4,\ 9$.

$$p_Y(1) = P((X-1)^2 = 1) = P(X = 0) + P(X = 1) = \frac{4}{14} = \frac{2}{7}$$
$$p_Y(4) = P((X-1)^2 = 4) = P(X = -1) + P(X = 3) = \frac{1}{7}$$
$$p_Y(9) = P((X-1)^2 = 9) = P(X = -2) + P(X = 4) = \frac{4}{7}$$

7. **Exercise 5.7**

   We begin by computing the cdf of Y. First note that because the range of X is $[0, +\infty)$, the range of Y is $(-\infty, +\infty)$ (we can ignore the case in which $Y = -\infty$, as this corresponds to $X = 0$ which occurs with probability 0). For any $t \in \mathbb{R}$, we have

$$F_Y(t) = P(Y \leq t) = P(\ln(X) \leq t) = P(X \leq e^t) = 1 - e^{-\lambda e^t}.$$

   By differentiating, we find that the probability density function of Y is given by

$$f_Y(t) = F_Y'(t) = \lambda e^{-\lambda e^t} e^t = \lambda e^{t - \lambda e^t},$$

   for any $t \in \mathbb{R}$.