Review: Matrix Transformation Maps

Let $A \in \mathbb{R}^{m \times n}$. Then we can construct a map:

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  - This will help us learn if the map is onto.

Null Space/Kernel

As before, let A be an $m \times n$ matrix and T_A its associated transformation.

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Kernel is the more robust definition and we will need it later. For now, we will work only with the null space.

Null Space = Homogeneous slns!

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The null space of A is exactly the set of solutions to $Ax = 0$. Therefore the null space is the set of solutions to the homogeneous linear problem! In particular: it is a subspace of $\mathbb{R}^n$ and therefore has a basis!

Exercise

Find the null space of the following matrix. In particular, express it as the span of appropriate basis vectors.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & -1 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}$$

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It has dimension 2.

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The **range** of T_A is the set of elements in $\mathbb{R}^m$ which are mapped to:

$$\mathrm{range}(T_A) = \{y \in \mathbb{R}^m : \text{there is an } x \in \mathbb{R}^m \text{ so that } T_A(x) = y\}$$

Column Space of A is Range of T_A

To see this, just recall that:

$$\sum_{i=1}^{n} x_i a_i = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = Ax$$

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In fact, we already know how to find the basis for col(A)! It is the exact same problem as finding a basis for the span of a collection of vectors!

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$$\operatorname{col}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix} \right\}$$

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Note that the dimension of the column space is 3.

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Theorem (Dimension Theorem)

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$$\operatorname{rank} A + \operatorname{nullity} A = n.$$

Proof.

Reduce A to RREF.

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Proof.

Reduce A to RREF. Then the n columns are split into two groups: Those which describe the basis for the column space and those which give the null space. $\square$

Example

The 5×5 matrix

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$

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So rank $A = 3$ and nullity $A = 2$.

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So $\operatorname{rank} A = 3$ and $\operatorname{nullity} A = 2$. And of course, $3 + 2 = 5$.

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6. *$|A| \neq 0$.*
7. *The rank of A is n.*
8. *The nullity of A is 0.*
9. *The columns of A are linearly independent.*

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10. The rows of A are linearly independent.