Linearity of Expectation

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The Big Idea

Sometimes expectation and variance are difficult to compute. We can get at them indirectly by splitting a random variable into a sum of more simple RVs.

This material corresponds to section 8.1 of the textook.

Multivariate Expectation

Law of the Unconscious Statistician

Fact: For jointly discrete RVs $X_1, \ldots, X_n$ and a function $g : \mathbb{R}^n \to \mathbb{R}$ we have

$$\mathbb{E}[g(X_1, \ldots, X_n)] \\ = \sum_{k_1 \in Ran(X_1)} \cdots \sum_{k_1 \in Ran(X_1)} g(k_1, \ldots, k_n) p_{X_1,\ldots,X_n}(k_1, \ldots, k_n).$$

For jointly continuous RVs $X_1, \ldots, X_n$ and a function $g : \mathbb{R}^n \to R$ we have

$$\mathbb{E}[g(X_1, \ldots, X_n)] \\ = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_n) f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

Linearity

Fact: For jointly distributed RVs $X_1, \ldots, X_n$ we have

$$\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

Method of Indicators

Indicators

Definition: Suppose A is some event. The *indicator* of A is the random variable

$$I_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$$

Note that I_A is a Bernoulli(p) RV with $p = \mathbb{P}(A)$.

Facts

1. $\mathbb{E}I_A = \mathbb{P}(A)$.
2. $\mathrm{Var}(I_A) = \mathbb{P}(A)(1 - \mathbb{P}(A))$.
3. $I_A I_B = I_{AB}$.

Big Idea

If X is a RV that counts the number of events that occur, we can represent it as a sum of indicators. This makes the expected value incredibly easy to compute. In some cases we can even compute the variance.

Example

As I stroll down Williamson Street on a lovely Sunday morning, there are 4 bakeries I can duck into for a nice baked good: Lazy Jane's, Batch Bakehouse, the Co-Op, and Madison Sourdough. As a lover of baking, I might stop into any number of the 4. The probabilities I stop into each bakery are 0.7, 0.6, 0.05, and 0.3. What is the expected number of bakeries I stop into on Sunday?

Example

Incredible Lesson: We computed an expected value of a random variable without knowing its PMF. Now we will see an example where we can also find the variance.

Example

Suppose we deal 20 cards from a 52 card deck. Let X be the number of face cards dealt. Find $\mathbb{E}X$ and $\mathrm{Var}(X)$.

Example

Suppose we flip a coin 1000 times. How many runs of 5 heads should we expect to occur?

A run of 5 heads is exactly 5 heads in a row. No more, no less: $THHHHHT$.

Summary

1. The natural extension of the Law of the Unconscious Statistician holds in the multivariate setting.
2. We can use this to prove the expectation of a sum is the sum of expectations.
3. Representing counting RVs as a sum of indicators allows us to compute expectation easily. If the indicators are exchangeable, this even works for variance.