Exercise

Which of the following sets span all of $\mathbb{R}^3$?

$$A = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}, \quad B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Solutions

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A does not, since

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \notin \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

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B does, since the matrix

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C does, since the set contains a spanning set already!

Focus on sets B and C

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$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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We note that each element $b \in \mathbb{R}^3$ must have a UNIQUE representation as a linear combination of elements in B. For example:

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Not true for C:

$$\begin{aligned}\begin{bmatrix}1\\1\\1\end{bmatrix} &= -1\begin{bmatrix}-1\\1\\1\end{bmatrix} + 2\begin{bmatrix}0\\1\\2\end{bmatrix} - 2\begin{bmatrix}0\\0\\1\end{bmatrix} + 0\begin{bmatrix}1\\0\\1\end{bmatrix}\\ &= -2\begin{bmatrix}-1\\1\\1\end{bmatrix} + 3\begin{bmatrix}0\\1\\2\end{bmatrix} - 2\begin{bmatrix}0\\0\\1\end{bmatrix} + -1\begin{bmatrix}1\\0\\1\end{bmatrix}\end{aligned}$$

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- No.
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We have done some work on the yes/no question already (invertibility, determinants). We will do more in the future. But for now we concentrate on uniqueness.

Linear Dependence and Independence

Definition

Let V be a vector space and $W = \{v_1, \ldots v_n\}$ a (finite and nonempty) set of vectors in V. W is **linearly dependent** if there exist scalars a_i so that

$$\sum_{i=1}^{n} a_i v_i = a_1 v_1 + \cdots a_n v_n = 0$$

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and *at least one of* $a_i \neq 0$. Otherwise, the set W is **linearly independent**.

Note: *linearly independence* means that if

$$\sum_{i=1}^{n} a_i v_i = 0$$

then $a_i = 0$.

Exercise

Show that

$$B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly independent whereas

$$C = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is dependent.

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Note the theorem is true if we replace 'columns' with 'rows'.

Solution for Set C

For C:

$$\begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

So the solutions to $Ax = 0$ are NOT unique and we can find at least one non-trivial solution:

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So the vectors here are *linearly dependent*.

Linear Independence/Dependence is a "tipping point"

Theorem

Let $S_1 \subseteq S_2 \subseteq V$ be two subsets of a vector space V.

- If S_2 is LI then so is S_1.
- If S_1 is LD then so is S_2.

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Proof.

Write $S_1 = \{v_1, v_2, \ldots, v_n\}$ and $S_2 = S_1 \sqcup \{w_1, \ldots, w_m\}$.
The result follows from

$$\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} a_i v_i + \sum_{j=1}^{m} 0 w_i.$$

$\square$

Some Results

- The set $\{0\}$ is Linearly dependent.
- The set consisting of a single non-zero vector is linearly independent.
- Any set which contains the zero vector is linearly dependent.
- If a set has more than two vectors and two of them are the same, then it is linearly dependent.
- If one vector in a set is a linear combination of the others, then it must be a linearly dependent set.

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Now suppose that $w = \sum a_i v_i = \sum b_i v_i$ are two different representations of the vector w.

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$$0 = \sum (a_i - b_i) v_i.$$

If V is LI then $a_i - b_i = 0$ and $a_i = b_i$. So they are the same representation. $\square$