Math 431 HW04, Part 2: Solutions

1. **Exercise 2.21**

   (a) We start by defining some appropriate random variables.

   $$X = \text{Number of questions on exam Jane gets correct}$$

   and

   $$X_i = \begin{cases} 1 & \text{if Jane gets question } i \text{ correct} \\ 0 & \text{otherwise} \end{cases}$$

   for $i = 1, 2, 3, 4$. We are given that X_1, X_2, X_3, X_4 are independent, because it is assumed the results on different problems are independent. We know that $X_i \sim \text{Bernoulli}(0.8)$. $X \sim \text{Binomial}(4, 0.8)$, because the results on each problem are independent and all have the same success probability.

   With these random variables, we only need to compute $P(X \geq 3)$. This is straightforward using the Binomial$(4, 0.8)$ PMF.

   $$\begin{aligned} P(X \geq 3) &= P(X = 3) + P(X = 4) \\ &= \binom{4}{3} 0.8^3 \cdot 0.2^1 + \binom{4}{4} 0.8^4 \cdot 0.2^0 \\ &\approx 0.8192. \end{aligned}$$

   (b) Now we have assumed that the envent $X_1 = 1$.

   $$\begin{aligned} P(X \geq 3 | X_1 = 1) &= P(X_1 + X_2 + X_3 + X_4 \geq 3 | X_1 = 1) \\ &= \frac{P(X_1 + X_2 + X_3 + X_4 \geq 3, X_1 = 1)}{P(X_1 = 1)} \\ &= \frac{P(X_2 + X_3 + X_4 \geq 2, X_1 = 1)}{P(X_1 = 1)} \\ &= \frac{P(X_2 + X_3 + X_4 \geq 2) P(X_1 = 1)}{P(X_1 = 1)} \\ &= P(X_2 + X_3 + X_4 \geq 2) \end{aligned}$$

   The sum of 3 independent Bernoulli(0.8) random variables has a Binomial(3, 0.8) distribution. So

   $$\begin{aligned} P(X \geq 3 | X_1 = 1) &= P(X_2 + X_3 + X_4 \geq 2) \\ &= P(X_2 + X_3 + X_4 = 2) + P(X_2 + X_3 + X_4 = 3) \\ &= \binom{3}{2} 0.8^2 \cdot 0.2^1 + \binom{3}{3} 0.8^3 \cdot 0.2^0 \\ &\approx 0.896. \end{aligned}$$

2. **Exercise 2.23**

We begin by defining the appropriate random variables.

$$X_i = \begin{cases} 1 & \text{if there is an accident on day } i \\ 0 & \text{if there is not an accident on day } i \end{cases}$$

So $X_i \sim \text{Bernoulli}(0.05)$. We assume accidents occur independently from day to day, so this means that we are assuming $X_1, X_2, X_3, \ldots$ are independent random variables.

(a) In terms of our random variables, this is

$$\begin{aligned} P(X_1 = 0,& X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0, X_6 = 0, X_7 = 0) \\ &= P(X_1 = 0)P(X_2 = 0)P(X_3 = 0)P(X_4 = 0)P(X_5 = 0)P(X_6 = 0)P(X_7 = 0) \\ &= 0.95^7 \end{aligned}$$

(b) If we set $X = X_1 + X_2 + \ldots + X_{30}$ then

$$X = \text{Number of days with accidents over the next 30 days}$$

This is what we want, because September has 30 days. We must compute $P(X = 2)$. X is the sum of independent Bernoulli(0.05) random variables, so $X \sim \text{Binomial}(30, 0.05)$. We can just use the PMF for this distribution to get

$$P(X = 2) = \binom{30}{2} 0.05^2 \cdot 0.95^{28} \approx 0.259.$$

(c) We are given $X_1 = 0$. We must compute

$$\begin{aligned} P(X_2 = 0,& X_3 = 0, X_4 = 0, X_5 = 0, X_6 + \cdots + X_{11} \geq 1 | X_1 = 0) \\ &= P(X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0, X_6 + \cdots + X_{11} \geq 1) \\ &= P(X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0) P(X_6 + \cdots + X_{11} \geq 1) \\ &= 0.95^4 \cdot (1 - P(X_6 + \cdots + X_{11} = 0)) \\ &= 0.95^4 \cdot \left(1 - 0.95^6\right) \\ &\approx 0.2158. \end{aligned}$$

It is also reasonable to have interpreted the problem as meaning the first day had no accidents, and we only consider 9 more days. In that case the solution is the following.

$$\begin{aligned} P(X_2 = 0,& X_3 = 0, X_4 = 0, X_5 = 0, X_6 + \cdots + X_{10} \geq 1 | X_1 = 0) \\ &= P(X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0, X_6 + \cdots + X_{10} \geq 1) \\ &= P(X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0) P(X_6 + \cdots + X_{10} \geq 1) \\ &= 0.95^4 \cdot (1 - P(X_6 + \cdots + X_{10} = 0)) \\ &= 0.95^4 \cdot \left(1 - 0.95^5\right) \\ &\approx 0.184. \end{aligned}$$

Both answers are acceptable.

3. **Exercise 2.25**

We start by defining events.

$$\begin{aligned}
A &= \text{The first roll is 3}\\
B &= \text{The second roll is 4}\\
D_k &= \text{The die used is } k\text{-sided, for } k = 4, 6, 12
\end{aligned}$$

The events A and B are conditionally independent give D_k. Also, D_4, D_6, and D_{12} form a partition of the sample space. In terms of these events, we want to find

$$P(D_6|AB) = \frac{P(AB|D_6)P(AB)}{P(AB)}.$$

We can compute the denominator separately, as it is the largest part of the computation.

So we can use the law of total probability to get

$$\begin{aligned}
P(AB) &= P(AB|D_4)P(D_4) + P(AB|D_6)P(D_6) + P(AB|D_{12})P(D_{12})\\
&= P(A|D_4)P(B|D_4)P(D_4) + P(A|D_4)P(B|D_6)P(D_6) + P(A|D_4)P(B|D_{12})P(D_{12})\\
&= \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{3} + \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{3} + \frac{1}{12}\cdot\frac{1}{12}\cdot\frac{1}{3}\\
&= \left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{12}\right)^2\right)\cdot\frac{1}{3}
\end{aligned}$$

From here, we use the Bayes formula to finish the problem.

$$\begin{aligned}
P(D_6|AB) &= \frac{P(AB|D_6)P(AB)}{P(AB)}\\
&= \frac{\left(\frac{1}{6}\right)^2\cdot\frac{1}{3}}{\left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{12}\right)^2\right)\cdot\frac{1}{3}}\\
&= \frac{2}{7}
\end{aligned}$$

4. **Exercise 2.28**

   (b) We define

   $$X_b = \text{Number of games with at least one ace.}$$

   The number of aces received in each game are independent. So

   $$X_b \sim \text{Binomial}(50, p)$$

   where

   $$p = \text{Probability of receiving at least one ace in a game.}$$

   Using part (a), we have

   $$p = P(X_a \geq 1) = 1 - P(X_a = 0) = 1 - \frac{\binom{48}{13}}{\binom{52}{13}}.$$

(c) We define

$$X_c = \text{Number of games with cards of all the same suit.}$$

The cards received in each game are independent. So

$$X_c \sim \text{Binomial}(50, p)$$

where

$$p = \text{Probability of receiving cards of all the same suit.}$$

This is a more direct probability calculation:

$$p = \frac{4}{\binom{52}{13}}.$$

5. **Exercise 2.47**

We start by defining appropriate random variables.

$$X = \text{Number of patients who have a successful trial}$$

The trials between patients are independent and the all have success probability p, so $X \sim \text{Binomial}(80, p)$.

We also define random variables for the outcomes of individual patients.

$$X_j = \begin{cases} 1 & \text{if the } i\text{-th person has a successful trial} \\ 0 & \text{if the } i\text{-th person does not have a successful trial} \end{cases}$$

This is for $j = 1, 2, \ldots, 80$. Note that $X_j \sim \text{Bernoulli}(p)$, and $X_1, \ldots, X_{80}$ are independent. We also have

$$X = X_1 + X_2 + \cdots + X_{79} + X_{80}.$$

For simplicity, we can assume that our two friends are the 79th and 80th persons in the medical trial. So we want to compute the probability below.

$$\begin{aligned}
P(X_{79} = 1, X_{80} = 1 | X = 55) &= \frac{P(X_{79} = 1, X_{80} = 1, X = 55)}{P(X = 55)} \\
&= \frac{P(X_{79} = 1, X_{80} = 1, X_1 + \cdots + X_{78} = 53)}{P(X = 55)} \\
&= \frac{P(X_{79} = 1, X_{80} = 1)P(X_1 + \cdots + X_{78} = 53)}{P(X = 55)} \\
&= \frac{p^2 \cdot \binom{78}{53} p^{53}(1-p)^{25}}{\binom{80}{55} p^{55}(1-p)^{25}} \\
&= \frac{\binom{78}{53}}{\binom{80}{55}}
\end{aligned}$$

6. **Exercise 2.57**

(a) We start by defining appropriate events and random variables.

$$\begin{aligned}
C_1 &= \text{The first component is working} \\
C_2 &= \text{The second component is working} \\
X_1 &= \text{The number of working elements in the first component} \\
X_2 &= \text{The number of working elements in the second component}
\end{aligned}$$

Because elements work independently of one another and with the same probability, we know that $X_1 \sim \text{Binomial}(8, 0.95)$ and $X_2 \sim \text{Binomial}(4, 0.90)$. This also tells us that C_1 and C_2 are independent.

The connection between these events and random variables is

$$\begin{aligned}
C_1 &= \{X_1 \geq 6\} \\
C_2 &= \{X_2 \geq 3\}
\end{aligned}$$

The system functions if both components are working. So we want that probability $P(C_1C_2) = P(C_1)P(C_2)$, because the elements operate independently.

$$P(C_1) = P(X_1 \geq 6) = \binom{8}{6}0.95^6 \cdot 0.05^2 + \binom{8}{7}0.95^7 \cdot 0.05^1 + \binom{8}{8}0.95^8 \cdot 0.05^0 \approx 0.9942$$

$$P(C_2) = P(X_2 \geq 3) = \binom{4}{3}0.90^3 \cdot 0.10^1 + \binom{4}{4}0.90^4 \cdot 0.10^0 \approx 0.9477$$

Now we just take this product to get our final answer.

$$\begin{aligned}
P(C_1C_2) &= P(C_1)P(C_2) \\
&= P(X_1 \geq 6)P(X_2 \geq 3) \\
&\approx 0.9942 \cdot 0.9477 \\
&\approx 0.9422.
\end{aligned}$$

(b) This is a direct computation in which we can use Bayes' formula.

$$\begin{aligned}
P(C_2^c|(C_1C_2)^c) &= \frac{P((C_1C_2)^c|C_2^c)P(C_2^c)}{P((C_1C_2)^c)} \\
&= \frac{1 \cdot (1 - 0.9477)}{1 - 0.9422} \\
&\approx 0.9048.
\end{aligned}$$

7. **Exercise 2.67**

This is a direct computation.

$$\begin{aligned}
P(X = n + k|X > n) &= \frac{P(X = n + k, X > n)}{P(X > n)} \\
&= \frac{P(X = n + k)}{P(X > n)} \\
&= \frac{(1-p)^{n+k-1}p}{(1-p)^n} \\
&= (1-p)^{k-1}p \\
&= P(X = k).
\end{aligned}$$

8. **Exercise 2.74**

   We start by defining events.

$$\begin{aligned} D &= \text{Steve is a drug user} \\ T_1 &= \text{Steve fails the first test} \\ T_2 &= \text{Steve fails the second test.} \end{aligned}$$

   We are given that

$$\begin{aligned} P(T_k|D) &= 0.99 \\ P(T_k|D^c) &= 0.02 \\ P(D) &= 0.01. \end{aligned}$$

   Note that T_1, T_2 are assumed to be conditionally independent given D (or D^c).

   (a) First we find $P(D|T_1)$ using the Bayes formula.

$$\begin{aligned} P(D|T_1) &= \frac{P(T_1|D)P(D)}{P(T_1|D)P(D) + P(T_1|D^c)P(D^c)} \\ &= \frac{0.99 \cdot 0.01}{0.99 \cdot 0.01 + 0.0.02 \cdot 0.99} \\ &= \frac{1}{3}. \end{aligned}$$

   (b) Now we find $P(T_2|T_1)$.

$$\begin{aligned} P(T_2|T_1) &= \frac{P(T_1T_2)}{P(T_1)} \\ &= \frac{P(T_1T_2|D)P(D) + P(T_1T_2|D^c)P(D^c)}{P(T_1|D)P(D) + P(T_1|D^c)P(D^c)} \\ &= \frac{P(T_1|D)P(T_2|D)P(D) + P(T_1|D^c)P(T_2|D^c)P(D^c)}{P(T_1|D)P(D) + P(T_1|D^c)P(D^c)} \\ &= \frac{0.99^2 \cdot 0.01 + 0.02^2 \cdot 0.99}{0.99 \cdot 0.01 + 0.02 \cdot 0.99} \\ &\approx 0.3433 \end{aligned}$$

   (c) Finally, we find $P(D|T_1T_2)$ using the Bayes formula.

$$\begin{aligned} P(D|T_1T_2) &= \frac{P(T_1T_2|D)P(D)}{P(T_1T_2|D)P(D) + P(T_1T_2|D^c)P(D^c)} \\ &= \frac{P(T_1|D)P(T_2|D)P(D)}{P(T_1|D)P(T_2|D)P(D) + P(T_1|D^c)P(T_2|D^c)P(D^c)} \\ &= \frac{0.99^2 \cdot 0.01}{0.99^2 \cdot 0.01 + 0.0.2^2 \cdot 0.99} \\ &= \frac{99}{103} \approx 0.9612. \end{aligned}$$