# Math 431, Homework 10 Solutions

1. **Exercise 7.2**

   The possible values for both $X$ and $Y$ are 0 and 1, hence $X+Y$ can take the values $0, 1$ and 2. If $X+Y=0$ then we must have $X=0$ and $Y=0$ and by independence we get

   $$P(X+Y=0) = P(X=0, Y=0) = P(X=0)P(Y=0) = (1-p)(1-r).$$

   Similarly, if $X+Y=2$ then we must have $X=1$ and $Y=1$:

   $$P(X+Y=2) = P(X=1, Y=1) = P(X+1)P(Y=1) = pr.$$

   We can now compute $P(X+Y)=1$ by considering the complement:

   $$P(X+Y=1) = 1-P(X+Y=0)-P(X+Y=2) = 1-(1-p)(1-r)-pr = p+r-2pr.$$

   We have computed the probability mass function of $X+Y$ which identifies its distribution.

2. **Exercise 7.4** We have

   $$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise,} \end{cases} \qquad f_Y(y) = \begin{cases} \mu e^{-\mu y}, & \text{if } y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

   Since $X$ and $Y$ are both positive, $X+Y>0$ with probability one, and $f_{X+Y}(z)=0$ for $z \le 0$. For $z > 0$, using the convolution formula

   $$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx = \int_0^z \lambda e^{-\lambda x} \mu e^{-\mu(z-x)} dx.$$

   In the second step we used that $f_X(x)f_Y(z-x) \neq 0$ if and only if $x > 0$ and $z - x > 0$ which means that $0 < x < z$.

   Returning to the integral

   $$\begin{aligned} f_{X+Y}(z) &= \int_0^z \lambda e^{-\lambda x} \mu e^{-\mu(z-x)} dx = \lambda \mu e^{-\mu z} \int_0^z e^{(\mu-\lambda)x} dx \\ &= \lambda \mu e^{-\mu z} \frac{e^{(\mu-\lambda)x}}{\mu-\lambda} \bigg|_{x=0}^{x=z} = \lambda \mu e^{-\mu z} \frac{e^{(\mu-\lambda)z}-1}{\mu-\lambda} = \lambda\mu \frac{e^{-\lambda z}-e^{-\mu z}}{\mu-\lambda}. \end{aligned}$$

   Note that we used $\lambda \neq \mu$ when we integrated $e^{(\mu-\lambda)x}$.

   Hence the probability density function of $X+Y$ is

   $$f_{X+Y}(z) = \begin{cases} \lambda\mu \frac{e^{-\lambda z}-e^{-\mu z}}{\mu-\lambda}, & \text{if } z > 0 \\ 0, & \text{otherwise.} \end{cases}$$

3. **Exercise 7.5**

   (a) By Fact 7.9 the distribution of $W$ is normal, with

$$\mu_W = 2\mu_x - 4\mu_Y + \mu_Z = -7, \qquad \sigma_W^2 = \sigma_X^2 + 16\sigma_Y^2 + \sigma_Z^2 = 25.$$

   Thus $W \sim N(-7, 25)$.

   (b) Using part (a) we know that $\frac{W+7}{\sqrt{25}}$ is a standard normal. Thus

$$P(W > -2) = P\left(\frac{W+7}{5} > \frac{-2+7}{5}\right) = 1 - \Phi(1) \approx 1 - 0.8413 = 0.1587.$$

4. **Exercise 8.2**

   Let $X_k$ be the number showing on the $k$-sided die. We need $E[X_4 + X_6 + X_{12}]$. By linearity of expectation

$$E[X_4 + X_6 + X_{12}] = E[X_4] + E[X_6] + E[X_{12}].$$

   We can compute the expectation of $X_k$ by taking the average of the numbers $1, 2, \ldots, k$:

$$E[X_k] = \sum_{j=1}^{k} j \cdot \frac{1}{k} = \frac{k(k+1)}{2k} = \frac{k+1}{2}.$$

   This gives

$$E[X_4 + X_6 + X_{12}] = \frac{4+1}{2} + \frac{6+1}{2} + \frac{12+1}{2} = \frac{25}{2}.$$

5. **Exercise 8.20** By linearity, $E[X_3 + X_{10} + X_{22}] = E[X_3] + E[X_{10}] + E[X_{22}]$. The random variables $X_1, \ldots, X_{30}$ are exchangeable, thus $E[X_k] = E[X_1]$ for all $1 \le k \le 30$. This gives

$$E[X_3 + X_{10} + X_{22}] = 3E[X_1].$$

   The value of the first pick is equally likely to be any of the first 30 positive integers, hence

$$E[X_1] = \sum_{k=1}^{30} k \frac{1}{30} = \frac{30 \cdot 31}{2 \cdot 30} = \frac{31}{2},$$

   and

$$E[X_3 + X_{10} + X_{22}] = 3E[X_1] = \frac{93}{2}.$$