Properties of Addition and Scalar Multiplication in $\mathbb{R}^n$:

Let u, v, and w be vectors in $\mathbb{R}^n$ with scalar numbers r and s in $\mathbb{R}$. Then

1. $u + v = v + u$
2. $u + (v + w) = (u + v) + w$
3. $u + 0 = 0 + u = u$
4. $u + (-u) = 0$
5. $r(u + v) = ru + rv$
6. $(r + s)u = ru + su$
7. $r(su) = (rs)u$
8. $1u = u$

Question: What is more important? The operations which produce the properties? Or just the properties themselves?

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This is the same collection of properties with different elements (matrices) and operations (matrix addition).

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For functions f, g, and h; and real numbers r and s we have:

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2. $f + (g + h) = (f + g) + h$
3. $f + e = e + f = f$ where e is the function $f(x) = 0$.
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This is the same collection of properties with different elements (functions) and operations (function addition).

Moral: These are all examples of the same thing

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Definition

Let F be a field (i.e., the real numbers). A **vector space** over F is a set V (the vectors) and operations

$\oplus : V \times V \to V$ (vector addition) $\quad \odot : F \times V \to V$ scalar multiplication

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with the following properties:

VS Properties

For all vectors u, v, and w in V and scalar values r and s in F we have

1. $u \oplus v = v \oplus u$
2. $u \oplus (v \oplus w) = (u \oplus v) \oplus w$
3. There is a vector $e \in V$ so that $u \oplus e = u$
4. There is a vector u^{-1} so that $u \oplus u^{-1} = e$.
5. $r \odot (u \oplus v) = (r \odot u) \oplus (r \odot v)$
6. $(r + s) \odot u = (r \odot u) \oplus (s \odot u)$
7. $r \odot (s \odot u) = (rs) \odot u$
8. $1 \odot u = u$

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Second example: Consider polynomials of even degree with function addition and scalar multiplication. Then:

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But $2x + 3$ is of odd degree and so the set is not *closed* under addition.

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In all of our examples of vector spaces, there are other things which can be shown to be true. For example, in $\mathbb{R}^n$ we have

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This is true even if we replace u with a function, or a matrix, etc. Why not include this property in our list?

The collection of properties 1-8 form some sort of *minimal* collection of assumptions. All others can (should) be derived from them!

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   So $e_1 = e_2$ and the identity is unique. Now that we know it is unique, we often write 0 for the identity.

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This means that all of the algebraic properties 1-8 hold for H assuming it is closed!

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In conclusion: H is a vector space. It is also a subset of another vector space $\mathbb{R}^m$. Such an object is called a **subspace**.