Math 431: HW08, Part 2 Solutions

1. **Exercise 5.8**

   The range of X is $[-1, 2]$, so the range of Y is $[0, 4]$. Thus, we know that $f_Y(t) = 0$ for $t \notin [0, 4]$.

   For $t \in [0, 4]$ we have

   $$F_Y(t) = P(Y \le t) = P(X^2 \le t) = P(-\sqrt{t} \le X \le \sqrt{t}), \tag{$*$}$$

   and we have to consider multiple cases. For $t \in [0, 1]$ we have $-\sqrt{t} \ge -1$ and so

   $$\begin{aligned} F_Y(t) &= P(Y \le t) = P(-\sqrt{t} \le X \le \sqrt{t}) \\ &= \frac{\sqrt{t} - (-\sqrt{t})}{2 - (-1)} = \frac{2\sqrt{t}}{3} \implies f_Y(t) = F_Y'(t) = \frac{1}{3} t^{-1/2}. \end{aligned}$$

   For $t \in [1, 4]$ we have

   $$F_Y(t) = P(Y \le t) = P(-1 \le X \le \sqrt{t}) = \frac{\sqrt{t} + 1}{3} \implies f_Y(t) = \frac{1}{6} t^{-1/2}.$$

   Putting this together yields:

   $$f_Y(t) = \begin{cases} 0 & t \notin [0, 4] \\ \frac{1}{3} t^{-1/2} & 0 \le t \le 1 \\ \frac{1}{6} t^{-1/2} & 1 < t < 4 \end{cases}.$$

   Note that there is another way to proceed from equation ($*$). We have

   $$F_Y(t) = P(Y \le t) = P(X^2 \le t) = P(-\sqrt{t} \le X \le \sqrt{t}) = F_X(\sqrt{t}) - F_X(-\sqrt{t}).$$

   We can get the pdf of Y by differentiating F_Y, and since $F_X' = f_X$, we get

   $$f_Y(t) = F_Y'(t) = \frac{d}{dt} \Big(F_X(\sqrt{t}) - F_X(-\sqrt{t}) \Big) = \frac{1}{2\sqrt{t}} f_X(\sqrt{t}) + \frac{1}{2\sqrt{t}} f_X(-\sqrt{t}).$$

   The pdf $f_X(x)$ is equal to $\frac{1}{3}$ if $-1 \le x \le 2$ and 0 otherwise. Thus if $0 < t < 1$ then

   $$f_Y(t) = \frac{1}{2\sqrt{t}} f_X(\sqrt{t}) + \frac{1}{2\sqrt{t}} f_X(-\sqrt{t}) = \frac{1}{2\sqrt{t}} \frac{1}{3} + \frac{1}{2\sqrt{t}} \frac{1}{3} = \frac{1}{3\sqrt{t}}$$

   and if $1 < t < 4$ then

   $$f_Y(t) = \frac{1}{2\sqrt{t}} f_X(\sqrt{t}) + \frac{1}{2\sqrt{t}} f_X(-\sqrt{t}) = \frac{1}{2\sqrt{t}} \frac{1}{3} + 0 = \frac{1}{6\sqrt{t}}.$$

   This gives the same case-defined function that we have found before.

2. **Exercise 5.11**

(a) By definition of moment generating function and by applying Fact 3.26, we have

$$\begin{aligned} M_X(t) = E[e^{tX}] &= \int_{-\infty}^{+\infty} e^{tx} f_X(x)dx \\ &= \int_0^{+\infty} e^{tx} x e^{-x} dx \\ &= \int_0^{+\infty} x e^{x(t-1)} dx \end{aligned}$$

Now, we have two cases:

- If $t \geq 1$ then $x(t-1) \geq 0$ for any $x > 0$, which implies that $xe^{x(t-1)} \geq xe^0 = x$ for any $x > 0$. It follows that

$$M_X(t) = \int_0^{+\infty} x e^{x(t-1)} dx \geq \int_0^{+\infty} x dx = \left[\frac{x^2}{2} \right]_0^{+\infty} = +\infty$$

Therefore, if $t \geq 1$ then $M_X(t) = \infty$.

- If $t < 1$ then, by integrating by parts (we can use $u' = e^{x(t-1)}$ and $v = x$),

$$\begin{aligned} M_X(t) &= \int_0^{+\infty} x e^{x(t-1)} dx \\ &= \left[x \frac{e^{x(t-1)}}{t-1} \right]_0^{+\infty} - \int_0^{+\infty} \frac{e^{x(t-1)}}{t-1} dx \\ &= \left[x \frac{e^{x(t-1)}}{t-1} \right]_0^{+\infty} - \left[\frac{e^{x(t-1)}}{(t-1)^2} \right]_0^{+\infty} \end{aligned}$$

To evaluate the last expressions we note that since $t-1 < 0$, both $x\frac{e^{x(t-1)}}{t-1}$ and $\frac{e^{x(t-1)}}{(t-1)^2}$ converge to zero as $x \to \infty$. Thus

$$M_X(t) = -0 \cdot \frac{e^{0(t-1)}}{t-1} + \frac{e^{0(t-1)}}{(t-1)^2} = \frac{1}{(1-t)^2}.$$

Therefore,

$$M_x(t) = \begin{cases} \dfrac{1}{(1-t)^2} & \text{if } t < 1 \\ +\infty & \text{if } t \geq 1 \end{cases}$$

(b) The moment generating function $M_x(t)$ is finite in an interval around 0, therefore we can apply the fact

$$E[X^n] = M_X^{(n)}(0).$$

Now, if we try to compute the nth derivative of $M_X(t)$, we might recognize a familiar pattern. Indeed, if we consider a random variable Y ~Exp(1), we have for any $t < 1$

$$M_X(t) = \frac{1}{(1-t)^2} = \frac{d}{dt}\left(\frac{1}{1-t}\right) = M_Y'(t)$$

Therefore,

$$E[X^n] = M_X^{(n)}(0) = M_Y^{(n+1)}(0) = E[Y^{n+1}] = (n+1)!1^{-n-1} = (n+1)!,$$

where we made use of the fact (proven in class) that the nth moment of an exponential random variable with parameter $\lambda > 0$ is $n!\lambda^{-n}$.
Alternatively, we can explicitly compute the nth derivative of $M_X(t)$. For $t < 1$ we have

$$\begin{aligned} M_X'(t) &= \frac{2(1-t)}{(1-t)^4} = \frac{2}{(1-t)^3} \\ M_X^{(2)}(t) &= \frac{2\cdot 3(1-t)^2}{(1-t)^6} = \frac{2\cdot 3}{(1-t)^4} \\ M_X^{(3)}(t) &= \frac{2\cdot 3\cdot 4(1-t)^3}{(1-t)^8} = \frac{2\cdot 3\cdot 4}{(1-t)^5} \end{aligned}$$

So each time to obtain $M_X^{(j)}(t)$ we multiply $M_X^{(j-1)}(t)$ by $j+1$ and divide it by $(1-t)$. This can be readily seen by the fact that

$$\frac{d}{dt}\left(\frac{1}{(1-t)^{j+1}}\right) = \frac{(j+1)(1-t)^j}{(1-t)^{2j+2}} = \frac{j+1}{(1-t)^{j+2}}.$$

Therefore, for $t < 1$ we have

$$M_X^{(n)}(t) = \frac{(n+1)!}{(1-t)^{n+2}}.$$

Hence,

$$E[X^n] = M_X^{(n)}(0) = (n+1)!$$

3. **Exercise 5.16**

(a) This is a direct computation.

$$E[X^n] = \int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$$

(b) From 5.3 we have

$$M_X(t) = \begin{cases} \frac{e^t-1}{t} & \text{if } t \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Expanding e^t as its Taylor series gives

$$\begin{aligned}
M_X(t) &= \frac{1}{t}(e^t - 1) \\
&= \frac{1}{t}\left(\sum_{n=0}^{\infty} \frac{t^n}{n!} - 1\right) \\
&= \frac{1}{t}\left(\sum_{n=1}^{\infty} \frac{t^n}{n!}\right) \\
&= \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{t^n}{n!}.
\end{aligned}$$

Note that this computation is for $t \neq 0$. However, the formula is also valid for $t = 0$:

$$M_X(0) = 1 = \sum_{n=0}^{\infty} \frac{0^n}{n!}.$$

So the Taylor series found above is valid for all real values of t.
So we have that the n-th coefficient of the MGF's Taylor series is

$$\frac{1}{n+1} = E[X^n],$$

the same as the moments found in (a).

4. **Exercise 5.21**

We approach this directly.

$$\begin{aligned}
M_Y(t) = E[e^{tY}] &= E[e^{t(aX+b)}] \\
&= E[e^{(at)X} e^{tb}] \\
&= e^{tb} M_X(at).
\end{aligned}$$