Consequences of the Probability Axioms

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Goal for this Lesson

Learn how to use the basic properties of probability measures for computation. This requires three pieces of knowledge.

1. The statements of the three axioms of probability
2. The formulas that are consequences of the axioms
3. How to use the formulas to solve problems

The material of this lecture roughly corresponds to Section 1.4 of the textbook.

The Axioms of Probability

Recall the axioms of probability:

1. $\mathbb{P}(E) \geq 0$,
2. $\mathbb{P}(\Omega) = 1$, and
3. if $E_1, E_2, E_3, \ldots$ are pairwise disjoint then
$$\mathbb{P}\left(\cup_k E_k\right) = \sum_k \mathbb{P}\left(E_k\right).$$

Today we are going to look at what consequences these simple rules must have on all sample spaces and events.

Complement Rule

Statement

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c)$$

Exercise: Prove this.

Examples

1. Compute the probability that two people in this room share a birthday.
2. Approximately 20% of cats are calico or have 6 toes. A cat walks by the window. Find the probability that the kitty is neither calico nor has 6 toes.

Difference Rule

Statement

If $A \subset B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$ and $\mathbb{P}(BA^c) = \mathbb{P}(B) - \mathbb{P}(A)$.

Exercise: Prove this.

Personal Comment: I feel like I rarely actually use this one, but it is important to keep in mind for probability intuition.

Conceptual Example / Conjunction Fallacy

Survey given to study participants

> **Rate the following statements according to their likelihood:**
> Linda is active in the feminist movement
> Linda is a psychiatric social worker
> Linda works in a bookstore and takes yoga classes
> Linda is a bank teller and is active in the feminist movement
> Linda is a teacher
> Linda is a member of the League of Women Voters
> Linda is a bank teller
> Linda sells insurance

Tversky, A. and Kahneman, D. (1982) "Judgments of and by representativeness". In D. Kahneman, P. Slovic & A. Tversky (Eds.), Judgment under uncertainty: Heuristics and biases. Cambridge, UK: Cambridge University Press.

Inclusion-Exclusion Principle

Simple Version

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)$$

Exercise: Prove this.

More General Version

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = \quad & \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ & - \mathbb{P}(AB) - \mathbb{P}(AC) - \mathbb{P}(BC) \\ & + \mathbb{P}(ABC)\end{aligned}$$

Fully General Version

$$\mathbb{P}\left(\cup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \left((-1)^{j+1} \sum_{\substack{I \subset \{1,\ldots,n\} \\ |I|=j}} \mathbb{P}\left(\cap_{\ell \in I} A_\ell\right) \right)$$

Comment: This formula is not that helpful to memorize. It is better to understand how to use it.

Example

There are currently courses being offered in analysis, botany, and continuum mechanics. Of 100 students chosen, we have the following statistics:

Courses	Enrollment	Courses	Enrollment
Analysis	35	Analysis and CM	25
Botany	50	Botany and CM	30
Cotinuum Mechanics	40	All three	15
Analysis and Botany	20		

If we select one of the 100 students at random, what is the probability that they are taking at least one of the three courses?

Classic Envelope Problem

You are an important executive, sending n letters to n different people. Unfortunately you hired a fool to stuff the letters into the appropriate envelope. He randomly grabs a letter and stuffs it in an envelope that is already addressed, repeating the procedure until all the envelopes are stuffed.

What is the probability that at least one person gets the correct letter? What is this probability as we consider $n \to \infty$?

The Wrap Up

Summary

1. All probability facts follow from the axioms.
2. Computing a probability is usually some mixture of the additive axiom, complement rule, inclusion-exclusion principle, and differ-

ence rule.

Next Step

We have all the basics of the probability space, which is the fundamental object of probability theory. Now we learn how functions behave in probability.