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Properties:

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2. $u + (v + w) = (u + v) + w$
3. $u + 0 = 0 + u = u$
4. $u + (-u) = 0$
5. $r(u + v) = ru + rv$
6. $(r + s)u = ru + su$
7. $r(su) = (rs)u$
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From this example, we move to abstract vector spaces by *using the properties as the definition*!

Definition of a Vector Space

Definition

Let F be a field (i.e., the real numbers). A **vector space** over F is a set V (the vectors) and operations

$$\oplus : V \times V \to V \text{ (vector addition)} \quad \odot : F \times V \to V \text{ scalar multiplication}$$

with the following properties:

For all vectors u, v, and w in V and scalar values r and s in F we have

1. $u \oplus v = v \oplus u$
2. $u \oplus (v \oplus w) = (u \oplus v) \oplus w$
3. There is a vector $e \in V$ so that $u \oplus e = u$
4. There is a vector u^{-1} so that $u \oplus u^{-1} = e$.
5. $r \odot (u \oplus v) = (r \odot u) \oplus (r \odot v)$
6. $(r + s) \odot u = (r \odot u) \oplus (s \odot u)$
7. $r \odot (s \odot u) = (rs) \odot u$
8. $1 \odot u = u$

Today:

We will do the same thing with the dot product.

Properties of the Dot Product

Let u, v, and w be vectors in $\mathbb{R}^n$ and c a scalar

1. $u \cdot u \geq 0$ and $u \cdot u = 0 \Leftrightarrow u = 0$.
2. Commutativity: $u \cdot v = v \cdot u$.
3. Distributivity: $u \cdot (v + w) = u \cdot v + u \cdot w$.
4. Scalar associativity: $c(u \cdot v) = (cu) \cdot v$.

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So let's use these properties to DEFINE geometry in general!

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2. $(u, v) = (v, u)$.
3. $(u, v + w) = (u, v) + (u, w)$.
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Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for a finite dimensional vector space. Let $[\cdot]_\beta$ be its associated coordinate map. Then consider

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That this has the properties of an inner product is evident from its definition involving the dot product.

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- *C is symmetric.*
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Note that symmetry is clear.

Proof

Write

$$v = \sum_{i=1}^{n} a_i v_i \text{ and } w = \sum_{j=1}^{n} b_j v_j.$$

Then

$$(v, w) = \left(\sum_{i=1}^{n} a_i v_i, \sum_{j=1}^{n} b_j v_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j (v_i, v_j).$$

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Use the definition of C:

$$(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i C_{ij} b_j = \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{n} C_{ij} b_j \right).$$

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So

$$(v, w) = [v]_\beta \cdot C[w]_\beta.$$

The matrix C is called the *matrix of the inner product with respect to the basis* β.

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The question "Does f have a local minimum" can be answered by asking if H_f is the matrix of an inner product!

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Sln:

$$C = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$$

Lengths, Distances, and Angles

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Definition

- The **norm** of v is
$$\|v\| := \sqrt{(v, v)}.$$
- The **distance** between vectors v and w is
$$d(v, w) := \|v - w\|.$$
- The **angle** between vectors v and w is
$$\cos^{-1} \frac{(v, w)}{\|v\| \|w\|}.$$

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Sln:

$$\int_0^{2\pi} \cos x \sin x \,\mathrm{d}x = \frac{1}{2}\sin^2(x)\Big]_0^{2\pi} = 0.$$

Orthogonal Implies Linear Independence

An *orthogonal set* of vectors is a set

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with the property that

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Theorem

A finite orthogonal set of non-zero vectors is linearly independent.

Proof.

Suppose $\sum_{i=1}^{n} a_i v_i = 0$.

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So we have $0 = a_j(v_j, v_j)$. Now, $(v_j, v_j) \neq 0$ therefore $a_j = 0$. $\square$

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These results rely ONLY on the properties of the inner products and not HOW you compute them or WHAT the vectors represent! They are important/useful for doing analytic geometry in inner product spaces.