First Midterm Exam Solutions

1. Suppose we have a class of 24 children. Exactly one of the children is named Mary. Consider the following scenarios.

   (a) A team of three children is chosen at random. What is the probability that Mary is on the team?

   Solution. Let E denote the event that Mary is on the team. Assuming all possible choices for a team are equally likely, we have

   $$P(E) = \frac{\#\text{Possible teams with Mary}}{\#\text{Possible teams}} = \frac{\binom{23}{2}}{\binom{24}{3}}.$$

   You must choose 3 of the 24 for a team. For a team with Mary, you choose 2 of the remaining 23 students.

   (b) A team of three children is chosen at random, and one of the three is chosen as the team leader. What is the probability that Mary is on the team, but is **not** the team leader?

   Solution. Let E denote the event that Mary is on the team, but not the leader. Assuming all possible choices for a team are equally likely, we have

   $$P(E) = \frac{\#\text{Possible teams with Mary, but not as leader}}{\#\text{Possible teams}} = \frac{23 \cdot 22}{24 \cdot \binom{23}{2}}.$$

   You must choose one of the 24 for a leader, then 2 of the remaining 23 for other team members.. For a team with Mary, you choose one of the remaining 23 to be a leader, and one of the remaining 22 as another member.

2. The symmetric difference of two events is defined as follows:

$$A \triangle B = AB^c \cup A^c B.$$

(a) If A and B are independent and $\Pr(A) = 0.3$ and $\Pr(B) = 0.5$, compute $\Pr(A \triangle B)$.
Solution. This is a direct computation.

$$\begin{aligned} P(A \triangle B) &= P(AB^c \cup A^c B) = P(AB^c) + P(A^c B) \\ &= P(A)P(B^c) + P(A^c)P(B) = 0.3 \cdot 0.5 + 0.7 \cdot 0.5 \\ &= 0.5. \end{aligned}$$

(b) Regardless of whether A and B are independent, prove that

$$\mathbb{P}(A \triangle B) = \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(AB).$$

Solution. The symmetric difference can be expressed as

$$A \triangle B = (A \cup B) \backslash AB,$$

so

$$\begin{aligned} P(A \triangle B) &= P(A \cup B) - P(AB) \\ &= P(A) + P(B) - P(AB) - P(AB) \\ &= P(A) + P(B) - 2P(AB). \end{aligned}$$

3. There is a bag with 3 fair dice. One is 4-sided (with numbers $1, 2, 3, 4$), one is 6-sided (with numbers $1, 2, 3, 4, 5, 6$), and one is 12-sided (with numbers 1 through 12). I reach into the bag, pick one at random and roll it. The outcome of the roll is 4. What is the probability that I pulled out the 6-sided die?

Solution. We define the events

$$D_k = \{\text{The die chosen is } k\text{-sided}\}.$$

We must compute $P(D_6|4)$. We use the Bayes formula.

$$\begin{aligned} P(D_6|4) &= \frac{P(4|D_6)P(D_6)}{P(4|D_4)P(D_4) + P(4|D_6)P(D_6) + P(4|D_{12})P(D_{12})} \\ &= \frac{\frac{1}{6}\cdot\frac{1}{3}}{\frac{1}{4}\cdot\frac{1}{3}+\frac{1}{6}\cdot\frac{1}{3}+\frac{1}{12}\cdot\frac{1}{3}} \\ &= \frac{2}{3+2+1} \\ &= \frac{1}{3}. \end{aligned}$$

4. Suppose you roll a fair six-sided die three times. We define the following events:

$$
\begin{aligned}
A &= \text{The first and second roll are equal} \\
B &= \text{The second and third roll are equal} \\
C &= \text{The first and third roll are equal}
\end{aligned}
$$

(a) Prove that the events are *pairwise independent*.

Solution. We must check if each pair of events is independent.

$$
\begin{aligned}
P(AB) &= P(\text{All three are equal}) = \frac{6}{6^3} = \frac{1}{6^2}, \\
P(A) &= \frac{6}{6^2} = \frac{1}{6}, \qquad P(B) = \frac{6}{6^2} = \frac{1}{6} \\
\Rightarrow P(AB) &= P(A)P(B)
\end{aligned}
$$

$$
\begin{aligned}
P(AC) &= P(\text{All three are equal}) = \frac{6}{6^3} = \frac{1}{6^2}, \qquad P(C) = \frac{6}{6^2} = \frac{1}{6} \\
\Rightarrow P(AC) &= P(A)P(C)
\end{aligned}
$$

$$
\begin{aligned}
P(BC) &= P(\text{All three are equal}) = \frac{6}{6^3} = \frac{1}{6^2}, \\
\Rightarrow P(BC) &= P(B)P(C)
\end{aligned}
$$

All three pairs of events are independent, so A, B, and C are pairwise independent.

(b) Prove that the events are NOT *mutually independent*.

Solution. All that remains to check is the equation for all three events.

$$P(ABC) = P(\text{All three are equal}) = \frac{6}{6^3} = \frac{1}{6^2}.$$

However,

$$P(A)P(B)P(C) = \frac{1}{6^3} \neq P(ABC).$$

So the events are not mutually independent.

5. A teacher is hosting an engineering contest for her students. They must drop an egg off the roof of their classroom in a container of their design. Their design is considered a success if the egg does not break. Suppose the probability that a student is successful is 0.20, and each student's success is independent of the other students. There are 40 students in the class. Let X denote the number of students with a successful design.

   (a) What is the distribution of X? Give the name, parameters, and probability mass function.

   Solution. X follows the Binomial distribution for $n = 40$ trials and success probability $p = 0.20$. This can be abbreviated as $X \sim \text{Binomial}(40, 0.20)$.

   (b) Suppose that for the first 10 students, there are 4 successes. Given this information, what is the probability that there are exactly 8 successes for all 40 students? Be certain to explain your answer.

   Solution. This is equivalent to asking the probability that there are exactly 4 successes for the last 30 students. If we define Y to be the number of successes for the last 30 students then $Y \sim \text{Binomial}(30, 0.20)$. So

$$P(Y = 4) = \binom{30}{4} 0.20^4 \cdot 0.8^{26}.$$

   (c) Now we change the experiment so that the teacher drops the students' eggs off the roof until one of the eggs breaks (so the design fails). What is the probability that more than 8 eggs are dropped off the roof?

   Solution. This is equivalent to asking the probability that the first eight trials are all successful, which is 0.20^8.