Math 431: Homework 5 Solutions

1. **Exercise 3.2**

   (a) We must have

   $$\sum_{k=1}^{6} p(k) = 1 \quad \Rightarrow \quad \sum_{k=1}^{6} ck = 1 \quad \Rightarrow \quad c \cdot \frac{6 \cdot 7}{2} = 1 \quad \Rightarrow \quad c = \frac{1}{21}.$$

   (b) We compute directly:

   $$\begin{aligned} P(X \text{ is odd}) &= P(X=1) + P(X=3) + P(X=5) \\ &= \frac{1}{21} + \frac{3}{21} + \frac{5}{21} \\ &= \frac{9}{21} = \frac{3}{7}. \end{aligned}$$

2. **Exercise 3.3**

   (a) We must verify two things: that $f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)dx = 1$. The first condition is satisfied, because exponential functions are always positive. We compute the second condition, the integral.

   $$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{0}^{\infty} 3e^{-3x}\,dx \\ &= \left[-e^{-3x}\right]_0^{\infty} \\ &= -0 - (-e^0) \\ &= 1. \end{aligned}$$

   So f is a PDF.

   (b) Using the definition of a PDF, this is

   $$P(-1 < X < 1) = \int_{-1}^{1} f(x)\,dx = \int_{0}^{1} 3e^{-3x}\,dx = 1 - e^{-3}.$$

   (c) Similar to part (b), we have

   $$P(X < 5) = \int_{-\infty}^{5} f(x)\,dx = \int_{0}^{5} 3e^{-3x}\,dx = 1 - e^{-15}.$$

   (d) Starting from the definition of conditional probability we have

   $$P(2 < X < 4 \,|\, X < 5) = \frac{P(2 < X < 4, X < 5)}{P(X < 5)} = \frac{P(2 < X < 4)}{P(X < 5)}.$$

So we need to compute these two probabilities.

$$P(2 < X < 4) = \int_2^4 3e^{-3x}\,dx = e^{-6} - e^{-12}.$$

We already have $P(X < 5)$ from part (c). So,

$$P(2 < X < 4 \,|\, X < 5) = \frac{e^{-6} - e^{-12}}{1 - e^{-15}}.$$

3. **Exercise 3.7**

   (a) This is a continuous RV, so

   $$P(a \le X \le b) = F(b) - F(a).$$

   We must choose the smallest value of b such that $F(b) = 1$ and the largest value of a such that $F(a) = 0$ to minimize the interval $[a, b]$. These values are $a = \sqrt{2}$ and $b = \sqrt{3}$.

   (b) X is a continuous RV, so $P(X = 1.6) = 0$.

   (c) We use the CDF. Note that $1 < \sqrt{2} < 3/2 < \sqrt{3}$. So

   $$P\left(1 \le X \le \frac{3}{2}\right) = F(3/2) - F(1) = [(3/2)^2 - 2] - 0 = \frac{1}{4}.$$

   (d) We differentiate the CDF to get the PDF.

   $$f_X(x) = \frac{d}{dx}\left[F_X(x)\right] = \begin{cases} 2x & \text{if } \sqrt{2} < x < \sqrt{3} \\ 0 & \text{otherwise} \end{cases}$$

4. **Exercise 3.9**

   (a)

   $$\begin{aligned} EX &= \int_0^\infty x \cdot 3e^{-3x}\,dx \\ &= \left[-xe^{-3x}\right]_0^\infty - \int_0^\infty -e^{-3x}\,dx \\ &= (0-0) + \left[-\frac{e^{-3x}}{3}\right]_0^\infty \\ &= -0 - \left(-\frac{1}{3}\right) \\ &= \frac{1}{3} \end{aligned}$$

   (b)

   $$\begin{aligned} E[e^{2X}] &= \int_0^\infty e^{2x} \cdot 3e^{-3x}\,dx \\ &= \int_0^\infty 3e^{-x}\,dx \\ &= \left[-3e^{-x}\right]_{x=0}^\infty \\ &= -0 - (-3) \\ &= 3 \end{aligned}$$

5. **Exercise 3.20**

   First we find the CDF of X.

$$\begin{aligned}
F_X(t) &= P(X \leq t) \\
&= \begin{cases} 0 & \text{for } t < 0 \\ \int_0^t \frac{1}{c}\, dx & \text{for } 0 \leq t \leq c \\ 1 & \text{for } t > c \end{cases} \\
&= \begin{cases} 0 & \text{for } t < 0 \\ \frac{t}{c} & \text{for } 0 \leq t \leq c \\ 1 & \text{for } t > c \end{cases}
\end{aligned}$$

   For the CDF of Y, we can proceed from the definition and use the CDF of X.

$$\begin{aligned}
F_Y(t) &= P(Y \leq t) \\
&= P(c - X \leq t) \\
&= P(c - t \leq X) \\
&= 1 - F_X(c - t) \\
&= \begin{cases} 1 - 0 & \text{for } c - t < 0 \\ 1 - \frac{c-t}{c} & \text{for } 0 \leq c - t \leq c \\ 1 - 1 & \text{for } c - t > c \end{cases} \\
&= \begin{cases} 1 & \text{for } c < t \\ \frac{t}{c} & \text{for } c \geq t \geq 0 \\ 0 & \text{for } 0 > t \end{cases} \\
&= F_X(t).
\end{aligned}$$

6. **Exercise 3.30**

   (a) The probability mass function is found by utilizing the multiplication rule. We have

$$\begin{aligned}
P(X = 0) &= P(\text{hit on first shot}) = \frac{1}{2} \\
P(X = 1) &= P(\text{miss on first, then hit}) \\
&= P(\text{hit on second}|\text{miss on first})P(\text{miss on first}) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}.
\end{aligned}$$

   Continuing,

$$\begin{aligned}
P(X = 2) &= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{12} \\
P(X = 3) &= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} = \frac{1}{20} \\
P(X = 4) &= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{1}{5}.
\end{aligned}$$

   (b) The expected value of X, the number of misses, is

$$E[X] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{12} + 3 \cdot \frac{1}{20} + 4 \cdot \frac{1}{5} = \frac{77}{60}.$$

7. **Exercise 3.46**

(a) First we handle the easy cases. For $x < 0$,

$$F_x(x) = P(X \leq x) = 0.$$

The length of the shorter piece cannot be negative. For $x > \ell/2$,

$$F_x(x) = P(X \leq x) = 1.$$

This is because the shorter piece of the thermometer cannot be greater than half of the original thermometer. For $0 \leq x \leq \ell/2$,

$$F_x(x) = P(X \leq x) = \frac{2x}{\ell}.$$

The factor of 2 occurs because the break can occur with x units of the left or right end of the thermometer. Putting the pieces together we have

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{2x}{\ell} & \text{if } 0 \leq x \leq \ell/2 \\ 1 & \text{if } x > \ell \end{cases}$$

(b) To get the PDF we differentiate the CDF. The result is

$$f_X(x) = \begin{cases} \frac{2}{\ell} & \text{if } 0 \leq x \leq \ell/2 \\ 0 & \text{otherwise.} \end{cases}$$