An Introduction to Conditional Probability

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Goal for this Lecture

Learn the basic concepts surrounding conditional probability. This will include the following topics.

1. The definition and computation of conditional probability
2. The use of the multiplication rule to compute the probabilities of intersections
3. The use of the law of total probability to compute the probability of a single event, using conditional probabilities

The material of this lecture roughly corresponds to Section 2.1 of the textbook.

Conditional Probability

Motivating Example

A family has two children. One of them is a boy. What is the probability the other one is as well?

Definition

Definition: The *conditional probability* of A given B is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)},$$

provided that $\mathbb{P}(B) \neq 0$.

Examples

Two cards are dealt from a 52-card deck. The first is a face card. What is the probability the second is a king? (Do not do this with counting. Use the definition of conditional probability for practice.)

Examples

A family has two children. One of them is a boy that was born on a Tuesday. What is the probability the other one is a boy?

Simulation for Previous Example

```
gender1 <- rbinom(1e+06, p = 0.5, size = 1)
day1 <- sample(1:7, 1e+06, replace = TRUE)
gender2 <- rbinom(1e+06, p = 0.5, size = 1)
day2 <- sample(1:7, 1e+06, replace = TRUE)
bdayDF <- data.frame(cbind(gender1, day1, gender2,
    day2))
cond.DF <- subset(bdayDF, (gender1 == 1 & day1 ==
    2) | (gender2 == 1 & day2 == 2))
twoboy.DF <- subset(cond.DF, gender1 == 1 & gender2 ==
    1)
nrow(twoboy.DF)/nrow(cond.DF)
```

```
## [1] 0.4807663
```

```
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```

```
## [1] 0.4814815
```

Conditional Probability is a Probability Measure

Fact: The conditional probability $\mathbb{P}(\cdot|B)$ satisfies the 3 axioms on the smaller sample space $\Omega_* = B$.

Exercise: Prove this statement.

The Point: Everything we proved in chapter 1 for probability measures is still true for conditional probability.

The Multiplication Rule

Key Idea

Suppose we deal three cards from the deck. What is the probability the first card is a jack, the second is a queen, and the third is another jack?

Multiplication Rule for Two Events

Fact:

$$\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B|A)$$

Multiplication Rule for Three Events

More General Fact:

$$\mathbb{P}(ABC) = \mathbb{P}(A)\mathbb{P}(B|A)\mathbb{P}(C|AB)$$

Multiplication Rule for Many Events

Most General Fact:

$$\mathbb{P}(A_1A_2\cdots A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1A_2)\cdots\mathbb{P}(A_n|A_1A_2\cdots A_{n-1})$$

Examples

Suppose I am going to give a three question quiz (I am not). Questions are randomly drawn from my problem bank, which has 50 problems in it. If you know the answer to 45 of my questions, what is the probability that you answer all of the questions correctly?

The Law of Total Probability

Main Idea

We can use the axiom of countable additivity to decompose a probability for an event into a sum of probabilities for a partition of that event.

If the probabilities are still difficult, we can use the multiplication rule to assist in those computations.

Statement of The Law of Total Probability

Suppose $A_1, A_2, \ldots, A_n$ is a partition of Ω. Then for any event B we have

$$\mathbb{P}(B) = \sum_{j=1}^{n} \mathbb{P}(B|A_j)\mathbb{P}(A_j).$$

Example

We have two urns and a 20-sided die.
Urn 1 has 14 blue marbles and 6 yellow marbles.
Urn 2 has 8 blue marbles and 5 yellow marbles.

We roll the die.

If the outcome is a prime number we choose Urn 1.
Otherwise we choose Urn 2.
Next we randomly select a marble from the chosen urn.

What is the probability the marble is blue?

Comment: All of the "marbles in urn" problems seem contrived, but mathematically they are identical to sampling from a population.

Example

There is a diagnostic test for cancer that has a true positive rate of 0.95 and a false positive rate of 0.02. 1 in 10000 people in the population have this type of cancer. If we select a person from the population at random what is the probability they test positive?

Example

You are conducting a sociology experiment that requires you to determine the percentage of the population that regularly engages in various types of illegal drug use. The plan requires interviewing the experiment participants, but obviously many of the participants will lie. Design an experimental procedure to remedy this problem.

The Wrap Up

Summary

1. The idea behind conditional probability given an event B is to restrict the sample space to B.
2. The multiplication rule is useful if there are sequential events for which conditional probabilities are easy to determine.
3. The law of total probability is a useful way to break the probability of a complicated event into easier pieces.

Next Step

We put these pieces together to form Bayes' formula, which allows us to update beliefs in the face of new information.