Sampling and Counting

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Goal for This Lesson

We have seen the uniform probability measure for finite sets. It is defined as

$$P(E) = \frac{\#E}{\#\Omega} = \frac{\text{Number of outcomes in } E}{\text{Number of outcomes in } \Omega}.$$

To use this for events and samples spaces that are not tiny, we need a systematic method to count the number of elements in a set. This branch of mathematics is called combinatorics.

The material in this section will correspond to some parts of Appendix C in the textbook, and Section 1.2.

The Multiplication Rule

Introductory Example

Suppose you are buying a new robot. There are three ways to customize your robot.

1. It can be a dog robot or kitty robot.
2. The robot can be black, brown, or magenta.
3. The robot can either speak English or make animal sounds.

How many different robots can be made?

Formal Statement

If we have k successive decisions to make with $n_1, n_2, \ldots, n_k$ choices respectively, then there are $n_1 \cdot n_2 \cdots n_k$ ways to make the k decisions.

Examples

Apply the multiplication rule to determine the number of elements in the following sets.

1. Glass Nickel Pizza offers 37 different toppings. How many different pizzas can Glass Nickel make?

2. 205 countries participated in the 2012 Summer Olympics. We must select the order for the countries to march in for the opening ceremonies. How many ways can we do that?
3. The bottom bookshelf in my office has 37 books. I have decided to group them by subject: Analysis, probability, algebra, and physics. There are 9 analysis books, 11 probability books, 6 algebra books, and 8 physics books. How many different ways can my books be arranged, given I group them by subject?

Sample with Replacement, Ordered

Suppose you have an urn filled with 12 marbles, numbered 1 through 12. You take one marble from the urn, record its number, then replace it in the urn. You repeat this process 4 times, keeping track of the order the marbles are sampled from the urn.

1. What is an appropriate sample space for this experiment?
2. How many outcomes are in this sample space?
3. Suppose we generalize this experiment to n marbles and k samples. Answer questions 1 and 2 with these more general assumptions.

Permutations

Introductory Example

There are 43 instructors in the the math department. Of those instructors, there must be a Department Chair, Vice Chair, and Advising Director. How many ways are there for the department to fill these roles?

Definition

Suppose there is a set of n elements. Any arrangement of k elements of the set is call a *permutation*.

The number of size k permutations from a set with n elements is denoted by $(n)_k$, and spoken as "n permute k"

Fact

The number of size k permutations from a set with n elements is

$$(n)_k = n(n-1)(n-2)\cdots(n-k+2)(n-k+1) = \frac{n!}{(n-k)!}.$$

Note this is just a special case of the multiplication rule.

Examples

1. I have 200 books, and 41 of them will fit on the top shelf of my bookcase. How many different arrangements of books are possible on the top shelf?
2. A man forgets his 5-digit PIN, but remembers there is a 37 and 85 in there. What is the maximum number of guesses will he need to get into his account?
3. There are five couples at a fancy dinner party. How many different seating arrangements are there for each couple to be seated together at a round table with 10 chairs? Now the couples are going to the a midnight screening of the cult classic *Billy Jack*. If the party secures 10 seats in a row, how many seating arrangements are there for each couple to be seated together?

Sample Without Replacement, Ordered

Suppose you have an urn filled with 12 marbles, numbered 1 through 12. You take one marble from the urn, record its number, then throw the marble away. You repeat this process 4 times, keeping track of the order the marbles are sampled from the urn.

1. What is an appropriate sample space for this experiment?
2. How many outcomes are in this sample space?
3. Suppose we generalize this experiment to n marbles and k samples. Answer questions 1 and 2 with these more general assumptions.

Combinations

Introductory Example

There are 43 instructors in the the math department. Of those instructors, a 3 person committee must be formed to determine graduate school admission. How many ways are there for the department to form the committee?

Idea

A *combination* is like a permutation, but the ordering does not matter.

Definition: Let Ω be a set of size n. A subset of size k is called a *combination* of k elements of Ω.

The number of combinations of size k is denoted by $\binom{n}{k}$, which is pronounced "n choose k."

Fact

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}$$

Examples

1. A senate subcommittee is being formed to investigate the actions of **INSERT TIMELY CULTURAL REFERENCE**. There will be 6 members on this committee. 3 will be Democrats, 2 will be Republicans, 1 will be independent. There are currently 53 Democrats, 45 Republicans, and 2 independents in the senate. How many ways are there to form the subcommittee?
2. We have a box of 50 screws. 47 of them have a Phillips head, but 3 of them have a slotted head. We needed 10 screws for our current job, so we pull 10 out of the box at random. What is the probability we pulled out all 3 slotted head screws?
3. Which of the following is larger/largest:
$$\binom{20}{9}, \binom{20}{10}, \text{ or } \binom{20}{11}?$$
$$\binom{20}{1} \text{ or } \binom{20}{19}?$$
Use this (and a few more examples if necessary) to make a conjecture for binomial coefficients in general.
Bonus: If possible, prove your conjectures.

Sample Without Replacement, Unordered

Suppose you have an urn filled with 12 marbles, numbered 1 through 12. You take one marble from the urn, record its number, then throw the marble away. You repeat this process 4 times, but you do not keep track of the order the marbles are sampled from the urn.

1. What is an appropriate sample space for this experiment?

2. How many outcomes are in this sample space?
3. Suppose we generalize this experiment to n marbles and k samples. Answer questions 1 and 2 with these more general assumptions.

Additional Problems

Statements

1. Poker dice is played by rolling 5 dice. What is the probability of rolling a pair? Two pairs?

The Wrap Up

Summary

We learned a few things about counting.

1. The *multiplication rule* is fundamental to solving most counting problems.
2. *Permutations* and *combinations* are just simple counting problems that are usually part of a larger problem.
3. The key relationship of counting to probability and statistics is through sampling.

Next Step

We know enough to handle counting based probability. That is nice, but incredibly limited. The first situation we will look at where counting based techniques are insufficient is infinite sample spaces.