Independence and Expectation

Gregory M. Shinault

Goal

We just want to know the effect independence has on expectation.

This material corresponds to sections 8.2 and 8.3 of the textbook.

Independence and Expectation

Fact

If X and Y are independent then

$$\mathbb{E}XY = \mathbb{E}X\mathbb{E}Y.$$

More generally, if $X_1, \ldots, X_n$ are independent and we have single variable functions $g_1, \ldots, g_n$ then

$$\mathbb{E}[g_1(X_1)g_2(X_2)\cdots g_n(X_n)] = \mathbb{E}[g_1(X_1)]\mathbb{E}[g_2(X_2)]\cdots\mathbb{E}[g_n(X_n)].$$

Independence and Variance

Fact

If $X_1, \ldots, X_n$ are independent then

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n).$$

Example

Find the variance of the negative binomial distribution.

Convolution and MGF

Fact

If X and Y are independent then

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

Example

Prove that if $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Example

Let X have the PMF

k	0	1	2
$\mathbb{P}(X = k)$	0.2	0.5	0.3

and Y have the PMF

k	0	1
$\mathbb{P}(Y = k)$	0.3	0.7

.

Further, assume X and Y are independent. Find the PMF of $X + Y$.

The Wrap Up

Summary

1. If X, Y are independent then

$$\mathbb{E}XY = \mathbb{E}X\mathbb{E}Y.$$

2. If $X_1, X_2, \ldots, X_n$ are independent then

$$\text{Var}(X_1 + X_2 + \cdots + X_n) = \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_n).$$

3. If X, Y are independent then

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$