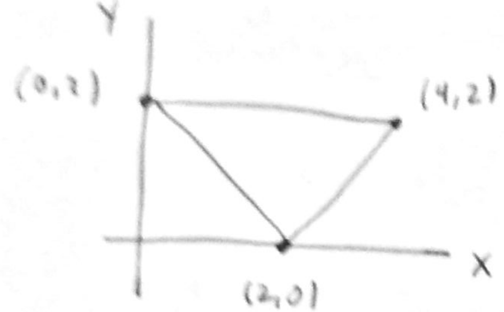
CONTINUOUS RANDOM VARIABLES Comprehensive Problem (Sol)



Total Area: $\frac{1}{2} \cdot 4 \cdot 2 = 4$

(a) $t < 0$: $F_X(t) = 0$

$t > 4$: $F_X(t) = 1$

$0 \le t < 2$: $F_X(t) = P(X \le t) = \dfrac{\frac{1}{2} t \cdot (2 - (2-t))}{4} = \dfrac{t^2}{8}$

$2 \le t \le 4$: $F_X(t) = P(X \le t) = \dfrac{4 - \frac{1}{2}(4-t)^2}{4} = 1 - \frac{1}{8}(4-t)^2$

$$F_X(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{8} t^2 & 0 \le t < 2 \\ 1 - \frac{1}{8}(4-t)^2 & 2 \le t \le 4 \\ 1 & t > 4 \end{cases}$$

(b)

$$f_X(t) = \frac{d}{dt}[F_X(t)] = \begin{cases} t/4 & 0 < t < 2 \\ (4-t)/4 & 2 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

(c) $P(X < 1.5 \mid X \le 2) = \dfrac{P(X < 1.5, X \le 2)}{P(X \le 2)}$

CONTINUED

CONTINUOUS RANDOM VARIABLES Comprehensive Problem (Sol. Cont.)

$$\rightarrow \quad \frac{P(X<1.5,\ X\le 2)}{P(X\le 2)} = \frac{F_X(1.5)}{F_X(2)} = \frac{\frac{1}{8}\cdot 1.5^2}{1-\frac{1}{8}(2)^2}$$

$$= \frac{\frac{1}{8}\cdot\left(\frac{3}{2}\right)^2}{\frac{1}{2}} = \frac{1}{4}\cdot\frac{9}{4} = \frac{9}{16}$$

(d) $$EX = \int_0^2 \frac{1}{4}x^2\,dx + \int_2^4 x\left(1-\frac{1}{4}x\right)dx$$

$$= \left[\frac{1}{4}\cdot\frac{1}{3}x^3\right]_0^2 + \left[\frac{1}{2}x^2 - \frac{1}{4}\cdot\frac{1}{3}x^3\right]_2^4$$

$$= \frac{8}{12} + \left[\left(8-\frac{16}{3}\right)-\left(2-\frac{2}{3}\right)\right]$$

$$= \frac{8}{12} + \frac{8}{3} - \frac{4}{3} = \frac{6}{3} = 2$$

(e) $$EX^2 = \int_0^2 \frac{1}{4}x^3\,dx + \int_2^4 x^2 - \frac{1}{4}x^3\,dx$$

$$= \left[\frac{1}{16}x^4\right]_0^2 + \left[\frac{1}{3}x^3 - \frac{1}{16}x^4\right]_2^4$$

$$= (1-0) + \left[\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-1\right)\right]$$

$$= 1 + \frac{56}{3} + 1 - \frac{48}{3} = \frac{62}{3} - \frac{48}{3} = \frac{14}{3}$$

$$\mathrm{Var}\,X = \frac{14}{3} - 2^2 = \frac{14}{3} - \frac{12}{3} = \frac{2}{3}$$

DISCRETE RANDOM VARIABLES Comprehensive Problem

Solution

(a) $\frac{c}{2\cdot 1} + \frac{c}{3\cdot 2} + \frac{c}{4\cdot 3} + \frac{c}{5\cdot 4} = 1$

$c\cdot\left(\frac{30}{60} + \frac{10}{60} + \frac{5}{60} + \frac{3}{60}\right) = 1$

$c\cdot \frac{48}{60} = 1 \Rightarrow \boxed{c = \frac{5}{4}}$

(b) $EX = \sum_{k=2}^{5} k\cdot \frac{c}{k\cdot(k-1)} = \frac{5}{4}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$

$= \frac{5}{4}\left(\frac{12+6+4+3}{12}\right) = \frac{5}{4}\cdot\frac{25}{12} = \frac{125}{48}$

(c) $E[X(X-1)] = \sum_{k=2}^{5} k(k-1)\cdot\frac{c}{k(k-1)} = 4c = 5$

(d) $E[X(X-1)] = EX^2 - EX = 5$

$\Rightarrow EX^2 = 5 + \frac{125}{48}$

$\Rightarrow \text{Var } X = \left(5 + \frac{125}{48}\right) - \left(\frac{125}{48}\right)^2$

So $SD(X) = \sqrt{\left(5+\frac{125}{48}\right) - \left(\frac{125}{48}\right)^2}$

Binomial Approximation Comprehensive Problem (Solution)

Solution

(a) Let X = # Sentences in paper with a typo.

$\Rightarrow X \sim \text{Binomial}(225, \frac{1}{100})$.

We have $p = \frac{1}{100} \ll 1$, so we use a Poisson approximation with $\lambda = 225 \cdot \frac{1}{100} = 2.25 = \frac{9}{4}$.

We want

$$P(X \geq 3) = 1 - P(X \leq 2)$$

$$= 1 - (P(X=0) + P(X=1) + P(X=2))$$

$$\approx 1 - e^{-\frac{9}{4}}\left(\frac{(9/4)^0}{0!} + \frac{(9/4)^1}{1!} + \frac{(9/4)^2}{2!}\right) \quad //$$

There is no need to simplify further. If you did, the result is

$$P(X \geq 3) \approx 1 - \frac{185}{32} e^{-9/4}.$$

(b) Let X = # Run-on sentences in the paper.

$\Rightarrow X \sim \text{Binomial}(225, 1/5)$.

In this case $p = \frac{1}{5}$ is not tiny. Also,

$$\text{Var}\, X = 225 \cdot \frac{1}{5} \cdot \frac{4}{5} = \frac{225}{25} \cdot 4 = 36 > 10.$$

So we use a $N(\mu, \sigma^2)$ approximation with

$$\mu = 225 \cdot \frac{1}{5} = 45 \quad \text{and} \quad \sigma^2 = 36.$$

Continued...

BINOMIAL APPROXIMATION

Solution (Continued)

We want

$$P(X < 40) = P(X \leq 39)$$

$$= P\left(\frac{X-\mu}{\sigma} \leq \frac{39-\mu}{\sigma}\right)$$

$$= P\left(\frac{X-\mu}{\sigma} \leq -\frac{6}{6}\right)$$

$$\approx \Phi(-1)$$

$$\approx 0.1587.$$

#

MOMENT GENERATING FUNCTIONS Continuous Problem (Solution)

(a) $M_X(t) = E[e^{tX}]$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2-2tx}{2}\right) dx$$

We can complete the square for the exponent:

$$x^2 - 2tx = x^2 - 2tx + t^2 - t^2 = (x-t)^2 - t^2.$$

So

$$M_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-t)^2 - t^2}{2}\right) dx$$

$$= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-t)^2}{2}\right) dx.$$

Using substitution for the integral,

$$u = x - t \Rightarrow du = dx, \quad u(-\infty) = -\infty, \quad u(\infty) = \infty$$

we get

$$M_X(t) = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du = e^{t^2/2} \cdot 1 \quad //$$

(b) Note that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow M_X(t) = \sum_{n=0}^{\infty} \frac{(t^2/2)^n}{n!}$

$$\Rightarrow M_X(t) = \sum_{n=0}^{\infty} \frac{1}{2^n n!} t^{2n} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n n!} \cdot \frac{t^{2n}}{(2n)!}.$$

Thus, $EX^6 = M_X^{(6)}(0) = M_X^{(2\cdot 3)}(0)$

$$= \frac{(2\cdot 3)!}{2^3 \cdot 3!} = \frac{6\cdot 5\cdot 4}{8} = 15 \quad \#$$

Continued...

MGFs Continuous Problem (Solution)

Another acceptable solution is to differentiate six times.

$$M_X'(t) = e^{t^2/2} \cdot \frac{2t}{2} = tM_X(t)$$

$$M_X''(t) = M_X(t) + t \cdot M_X'(t) = (t^2+1)M_X(t)$$

$$M_X'''(t) = (t^2+1)M_X'(t) + (2t) \cdot M_X(t)$$
$$= (t^3+3t)M_X(t)$$

$$M_X^{(4)}(t) = (3t^2+3)M_X(t) + (t^3+3t)M_X'(t)$$
$$= (t^4+6t^2+3)M_X(t)$$

$$M_X^{(5)}(t) = (4t^3+12t)M_X(t) + (t^4+6t^2+3)M_X'(t)$$
$$= (t^5+10t^3+15t)M_X(t)$$

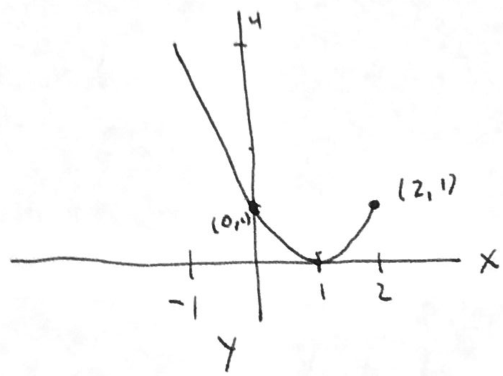
$$M_X^{(6)}(t) = (5t^4+30t^2+15)M_X(t) + (t^5+10t^3+15t)M_X'(t)$$

$$\Rightarrow EX^6 = M_X^{(6)}(0) = 15 \cdot M_X(0) + 0 \cdot M_X'(0)$$
$$= 15 \cdot e^{0^2/2} = 15 \quad \#$$

TRANSFORMATIONS OF RVs Problem

Suppose $X \sim \text{Unif}([-1,2])$ and $Y = (X-1)^2$. Find the PDF of Y.

Solution



For $0 \le y \le 1$, we have

$$F_Y(y) = P(Y \le y) = P((X-1)^2 \le y)$$

$$= P(-\sqrt{y} \le X - 1 \le \sqrt{y})$$

$$= P(1-\sqrt{y} \le X \le 1+\sqrt{y}) = \frac{(1+\sqrt{y})-(1-\sqrt{y})}{3}$$

$$= \frac{2}{3}\sqrt{y}$$

For $1 < y \le 4$, we have

$$F_Y(y) = P(Y \le y) = P((X-1)^2 \le y)$$

$$= P(1-\sqrt{y} \le X \le 1+\sqrt{y}) = P(1-\sqrt{y} \le X \le 2)$$

$$= \frac{1+\sqrt{y}}{3}$$

For $y < 0$, $F_Y(y) = 0$ and $y > 4$, $F_Y(y) = 1$.

Continued...

TRANSFORMATIONS OF RVs Problem

Solution (continued)

So

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{2}{3}\sqrt{y} & \text{if } 0 \le y < 1 \\ \frac{1+\sqrt{y}}{3} & \text{if } 1 \le y \le 4 \\ 1 & \text{if } y > 4 \end{cases}$$

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{3} y^{-1/2} & \text{if } 0 \le y < 1 \\ \frac{1}{6} y^{-1/2} & \text{if } 1 \le y \le 4 \\ 0 & \text{else} \end{cases}$$