First Midterm Exam: Practice

Name

Please circle the section number of your lecture:

SECTION 1	SECTION 2	SECTION 3
MWF 9:55am	MWF 2:25pm	TuTh 9:30am

- Budget your time wisely! Read the problems carefully!
- You are not allowed to use a calculator or other handheld devices.
- Please present your solutions in a clear manner. Justify your steps.
- Only use the back of your page for scratchwork.
- When there are 10 minutes left, you must remain seated until I dismiss you. This is to prevent disruptions for other students still working on their exams, and to make exam collection easier.

1. Suppose you are stranded on a desert island with a probability-savvy friend. You only have one remaining coconut to eat, but two fair six-sided dice for games. Your friend suggests a game to decide who gets to eat the last coconut. If the maximum of the two dice is 4 or less, you get the coconut. If the maximum of the two dice is 5 or 6, your friend get the coconut.

   (a) Give an appropriate sample space for the outcomes of the two dice rolls.

   (b) Describe the event that you win the coconut as a subset of the sample space.

   (c) What is the probability that you get the coconut?

   Solution.

   (a)
   $$\Omega = \{1, 2, 3, 4, 5, 6\}^2 = \{(\omega_1, \omega_2) \,|\, 1 \leq \omega_1, \omega_2 \leq 6\}$$

   (b)
   $$E = \{(\omega_1, \omega_2) \,|\, 1 \leq \omega_1, \omega_2 \leq 4\}$$

   (c)
   $$P(E) = \frac{\#E}{\#\Omega} = \frac{16}{36} = \frac{4}{9}$$

2. You must prove the following statements using axioms and known formulas of probability theory.

   (a) Suppose $P(A) = 1$ and $P(B) = 1$. Prove $P(AB) = 1$.
   Solution.

$$\begin{aligned} P(A \cup B) &\leq 1 \\ \Rightarrow P(A) + P(B) - P(AB) &\leq 1 \\ \Rightarrow 1 + 1 - 1 &\leq P(AB) \end{aligned}$$

   The probability of all events must be less than or equal to 1, so we have

$$1 \leq P(AB) \leq 1 \quad \Rightarrow \quad P(AB) = 1$$

   (b) Prove that $P(A \cup B \cup C) = 1 - P(A^c)P(B^c|A^c)P(C^c|A^cB^c)$.
   Solution.

$$\begin{aligned} P(A \cup B \cup C) &= 1 - P((A \cup B \cup C)^c) \\ &= 1 - P(A^cB^cC^c) \\ &= 1 - P(A^c)P(B^c|A^c)P(C^c|A^cB^c) \end{aligned}$$

3. An ordinary deck of cards contains 52 cards. Of those 52 cards, 13 of them are hearts. You remove one card from the deck at at time, without replacement, until you remove a heart. Let X be the number of cards removed from the deck. As an example, if the first card is a non-heart and the second card is a heart, then $X = 2$.

   (a) What are the possible values this random variable can take? In other words, find the range of the random variable.

   Solution.

$$\text{Range}(X) = \{1, 2, 3, \ldots, 40\}$$

   (b) Find the probability mass function of the random variable. You will want to give an explicit formula for $p_X(k)$ in this case.

   Solution. Start by looking at the first few terms to see if you can infer a pattern.

$$\begin{aligned}
P(X = 1) &= P(\text{First card is a } \heartsuit) = \frac{13}{52} \\
P(X = 2) &= P(\text{First card is not a } \heartsuit, \text{ Second card is a } \heartsuit) = \frac{39 \cdot 13}{52 \cdot 51} \\
P(X = 3) &= P(\text{First two cards are not a } \heartsuit, \text{ Third card is a } \heartsuit) = \frac{39 \cdot 38 \cdot 13}{52 \cdot 51 \cdot 50}
\end{aligned}$$

   The pattern should become clear (possibly after a few more examples), that

$$\begin{aligned}
P(X = k) &= P(\text{First } k \text{ cards are not a } \heartsuit, \ k\text{-th card is a } \heartsuit) \\
&= \frac{\#\{\text{Ways to deal } k - 1 \text{ non-}\heartsuit\} \cdot \#\{\heartsuit \text{ in deck}\}}{\#\text{Ways to deal } k \text{ cards}} \\
&= \frac{(39)_{k-1} \cdot 13}{(52)_k}
\end{aligned}$$

   for $k = 1, 2, \ldots, 40$.

   (c) Find the probability that the first heart comes on the third card or later.

   Solution.

$$\begin{aligned}
P(X \geq 3) &= 1 - P(X < 3) \\
&= 1 - P(X = 2) - P(X = 1) \\
&= 1 - \frac{13}{52} - \frac{39 \cdot 13}{52 \cdot 51}
\end{aligned}$$

4. Suppose we have two urns. Urn 1 has 2 blue marbles and 1 yellow marble in it. Urn 2 has 1 blue marble and 2 yellow marbles in it.

   You take one marble from Urn 1 and one marble from Urn 2. You place the marble from Urn 1 into Urn 2 and the marble from Urn 2 into Urn 1.

   This process is repeated many times.

   We define the events

$$\begin{aligned} B_{k\ell} &= \text{The } \ell\text{th draw from the } k\text{th urn is blue} \\ Y_{k\ell} &= B_{k\ell}^c = \text{The } \ell\text{th draw from the } k\text{th urn is yellow} \end{aligned}$$

   (a) What is the probability that the second draw from Urn 1 is blue? Be certain to show supporting work using the established formulas of probability theory.
   Solution.

$$\begin{aligned} P(B_{12}) &= P(B_{12}|B_{11}B_{21})P(B_{11}B_{21}) + P(B_{12}|B_{11}Y_{21})P(B_{11}Y_{21}) \\ &\quad + P(B_{12}|Y_{11}B_{21})P(Y_{11}B_{21}) + P(B_{12}|Y_{11}Y_{21})P(Y_{11}Y_{21}) \\ &= \frac{2}{3}\cdot\frac{2\cdot 1}{3\cdot 3} + \frac{1}{3}\cdot\frac{2\cdot 2}{3\cdot 3} + \frac{3}{3}\cdot\frac{1\cdot 1}{3\cdot 3} + \frac{2}{3}\cdot\frac{1\cdot 2}{3\cdot 3} \\ &= \frac{4+4+3+4}{27} = \frac{15}{27} = \frac{5}{9}. \end{aligned}$$

(b) Suppose the second draw from Urn 1 is blue. What is the probability that the first draw from both urns was yellow?

Solution. We can compute directly.

$$\begin{aligned}
P(Y_{11}Y_{21}|B_{12}) &= \frac{P(Y_{11}Y_{21}B_{12})}{P(B_{12})} \\
&= \frac{P(Y_{11}Y_{21})P(B_{12}|P(Y_{11}Y_{21})}{15/27} \\
&= \frac{4/27}{15/27} = \frac{4}{15}
\end{aligned}$$

(c) Let X be the number of blue marbles in Urn 1 after only 1 round of this process. Find the probability mass function of X.

Solution. We start by noting that $\text{Range}(X) = \{1, 2, 3\}$.

$$\begin{aligned}
&P(X = 1) = P(B_{11}Y_{21}) = \frac{4}{9} \\
&P(X = 2) = P(B_{11}B_{21}) + P(Y_{11}Y_{21}) = \frac{4}{9} \\
&P(X = 3) = P(Y_{11}B_{21}) = \frac{1}{9}
\end{aligned}$$

5. Give a short justification for your response. If you find yourself taking more than a few lines to provide a response, you must consider a simpler approach.

   (a) Suppose the events A and B are mutually exclusive and independent. Further, we know that $P(A) > 0$. What is the value of $P(B)$?
   Solution. From the mutual exclusivity we have

   $$AB = \varnothing, \quad \text{so} \quad P(AB) = 0.$$

   Independence also gives

   $$P(AB) = P(A)P(B), \quad \text{so} \quad P(A)P(B) = 0.$$

   Thus $P(A) = 0$ or $P(B) = 0$. We know $P(A) > 0$. That means that $P(B) = 0$.

   (b) List the equations that must be satisfied for the events A, B, and C to be independent.
   Comment: There are two acceptable solutions. One of them requires you to list 8 equations, the other requires you to list 4 equations. Either is acceptable, but a mixture of the two solutions is not.
   Solution. Option 1:

   $$P(AB) = P(A)P(B), \quad P(AC) = P(A)P(C), \quad P(BC) = P(B)P(C),$$
   $$P(ABC) = P(A)P(B)P(C).$$

   Option 2:

   $$\begin{aligned} P(ABC) &= P(A)P(B)P(C) & P(A^cBC) &= P(A^c)P(B)P(C) \\ P(AB^cC) &= P(A)P(B^c)P(C) & P(ABC^c) &= P(A)P(B)P(C^c) \\ P(AB^cC^c) &= P(A)P(B^c)P(C^c) & P(A^cBC^c) &= P(A^c)P(B)P(C^c) \\ P(A^cB^cC) &= P(A^c)P(B^c)P(C) & P(A^cB^cC^c) &= P(A^c)P(B^c)P(C^c) \end{aligned}$$

   (c) Suppose you flip a fair coin three times. A is the event that the first flip is heads and B is the event that at least one of flips is heads. Are A and B independent? Prove your answer.

Solution. A and B are dependent. It is equivalent to prove that A and B^c are dependent.

$$\begin{aligned} P(A) &= \frac{1}{2} \\ P(B^c) &= P(\text{All flips are tails}) = \frac{1}{8} \\ P(AB^c) &= P(\text{No heads and first flip is heads}) = 0 \end{aligned}$$

So $P(AB^c) \neq P(A)P(B^c)$. A and B^c are dependent, so A and B are also dependent.

You can also approach the problem more directly.

$$\begin{aligned} P(A) &= \frac{1}{2} \\ P(B) &= 1 - P(B^c) = 1 - \frac{1}{8} = \frac{7}{8} \\ P(AB) &= P(A) = \frac{1}{2} \end{aligned}$$

So $P(AB) \neq P(A)P(B)$. A and B are dependent.

6. Suppose you are doctor giving flu vaccines one afternoon. You plan to stay at the clinic administering vaccines until a patient has an adverse reaction. You will then leave with the patient to assist with their situation. The probability a patient has an adverse reaction is 0.05. Let X denote the number of patients you administer the vaccine to (including the one who has an adverse reaction).

   (a) What is the name of the probability distribution which X must follow, and the values of its parameters? What assumptions do you make to come to this conclusion?

   Solution. X must follow the *geometric* distribution with parameter $p = 0.05$.

   We assume that patients have adverse reactions independent of one another, so that these are independent trials. If we do not have independent trials, we cannot conclude the geometric distribution is applicable.

   (b) What is the probability that you see at least twenty patients? Your solution be a simple expression, not a long sum or series.

   Hint. You can solve this problem without evaluating a sum or series.

   Solution.

$$\begin{aligned} P(X >= 20) &= P(X > 19) = P(\text{First 19 patients have no adverse reaction}) \\ &= 0.95^{19} \end{aligned}$$

   (c) Assume the first four patients you see do not have an adverse reaction. What is the probability that you see exactly twenty patients before leaving?

   Solution.

$$P(X = 20 \,|\, X > 4) = P(X = 16) = 0.95^{15} \cdot 0.05$$

   or

$$\begin{aligned} P(X = 20 \,|\, X > 4) &= \frac{P(X = 20, X > 4)}{P(X > 4)} = \frac{P(X = 20)}{P(X > 4)} \\ &= \frac{0.95^{19} \cdot 0.05}{0.95^4} = 0.95^{15} \cdot 0.05 \end{aligned}$$

   (d) Now another doctor shows up. She will administer the vaccine to twenty patients over the course of the afternoon. Let Y denote the number of patient this doctor administers the vaccine to who have an adverse reaction. Find the probability that less than three patients have an adverse reaction. It is not necessary to simplify your answer.

   Solution. $Y \sim \text{Binomial}(20, 0.05)$, so

$$\begin{aligned} P(Y < 3) &= P(Y = 0) + P(Y = 1) + P(Y = 2) \\ &= \binom{20}{0} 0.05^0 \cdot 0.95^{20} + \binom{20}{1} 0.05^1 \cdot 0.95^{19} + \binom{20}{2} 0.05^2 \cdot 0.95^{18}. \end{aligned}$$