Warm Up Exercise

Solve the system:

$$\begin{aligned} x - 2z &= 0 \\ x + 4y - 4z - w &= 0 \\ x + 2y - 2w &= 0 \\ y - z &= 0 \end{aligned}$$

by

- Writing down the appropriate augmented matrix.
- Reducing the augmented matrix by Gaussian elimination to RREF.
- Identifying the solution set.

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- Augmented matrix:

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- RREF:

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 1 & 4 & -4 & -1 & 0 \\ 1 & 2 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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- Sln set:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{rcl} x - w & = & 0 \\ y - w/2 & = & 0 \\ z - w/2w & = & 0 \\ 0 & = & 0 \end{array}$$

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In vector form:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = w \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

Parameterized Form

$$\left\{ \begin{bmatrix} t \\ t/2 \\ t/2 \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

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Note:

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{bmatrix} = t \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

is a line through the origin in the direction of $[1, 1/2, 1/2, 1]^T$ in $\mathbb{R}^4$.

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Question: What relationship is there between singularity (invertiblity) and elementary row operations (elementary matrix multiplication)?

Row equivalence

Recall that matrices A and B are **row equivalent** if there are elementary matrices $E_1, \ldots, E_k$ and

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   and each ${E_i}^{-1}$ is also elementary.
2. Row equivalent augmented matrices represent systems of equations with identical solution sets.

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Lemma

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$$[B|0] \Leftrightarrow \begin{array}{c} x_1 = 0 \\ x_2 = 0 \\ \vdots \\ x_n = 0 \end{array} \Leftrightarrow [I|0]$$

Hence $B = I$ (RREF is unique). $\square$

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$(\Rightarrow)$: If A^{-1} exists, then $Ax = 0$ has unique solution $A^{-1}0 = 0$. So A is row equivalent to I So there exist E_i so that

$$A = E_1 E_2 \cdots E_k I = E_1 E_2 \cdots E_k.$$

$\square$

Summary

All of the following statements are logically equivalent:

- A is nonsingular.
- $Ax = 0$ has only the trivial solution.
- $Ax = b$ has a unique solution for every b.
- A is row equivalent to the identity matrix.
- A is a product of elementary matrices

Application: A Way to Compute Inverses

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Suppose that A is invertible. Then we know that A is row equivalent to I:

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Therefore the partitioned/augmented matrix $[A|I] \sim [I|A^{-1}]$ since

$$E_k \cdots E_2 E_1 [A|I] = [I|A^{-1}].$$

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Therefore the partitioned/augmented matrix $[A|I] \sim [I|A^{-1}]$ since

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Conclusion: the collection of row operations which transform A to I is the same as the row operations which transform I to A^{-1}.

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Compute

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}.$$

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$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

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So

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Exercises

Compute the following (if they exist):

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}^{-1} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}^{-1}$$

Results

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 3/2 & 1/2 & -3/2 \\ -1 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix} \text{ is singular } (2r_1 + 3r_2 = r_3).$$