Math 431: Homework 9 Solutions

1. **Exercise 6.2**

   (a) To find the PMF of X we compute the row sums. This gives

k	1	2	3
$p_X(k)$	$\frac{5}{15}$	$\frac{5}{10}$	$\frac{5}{30}$

   To find the PMF of Y we compute the column sums. This gives

k	0	1	2	3
$p_Y(k)$	$\frac{6}{30}$	$\frac{6}{30}$	$\frac{5}{15}$	$\frac{8}{30}$

   (b) We add the appropriate terms that satisfy the inequality.

$$
\begin{aligned}
P(X + Y^2 \leq 2) &= P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 2, Y = 0) \\
&= \frac{1}{15} + \frac{1}{15} + \frac{1}{10} \\
&= \frac{7}{30}.
\end{aligned}
$$

2. **Exercise 6.5**

   (a) First, note that

$$xy + y^2 \geq xy \geq 0$$

   for all $x, y \geq 0$. So the first criterion for a PDF is satisfied. Next,

$$
\begin{aligned}
\int_0^1 \int_0^1 \frac{12}{7}(xy + y^2)\, dx\, dy &= \frac{12}{7} \int_0^1 \frac{y}{2} + y^2\, dy \\
&= \frac{12}{7} \left[\frac{1}{4} + \frac{1}{3}\right] \\
&= \frac{12}{7} \cdot \frac{7}{12} = 1
\end{aligned}
$$

   So the second criterion for a PDF is satisfied.

   (b) To find f_X we integrate the joint PDF over y. For $0 \leq x \leq 1$ we have

$$
\begin{aligned}
f_X(x) &= \int_0^1 \frac{12}{7}(xy + y^2)\, dy \\
&= \frac{12}{7}\left(\frac{x}{2} + \frac{1}{3}\right).
\end{aligned}
$$

For all other values, the PDF is 0. To summarize,

$$f_X(x) = \begin{cases} \frac{12}{7}\left(\frac{x}{2}+\frac{1}{3}\right) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

To find f_Y we integrate the joint PDF over x. For $0 \le y \le 1$ we have

$$\begin{aligned} f_Y(y) &= \int_0^1 \frac{12}{7}(xy+y^2)\,dx \\ &= \frac{12}{7}\left(\frac{y}{2}+y^2\right). \end{aligned}$$

For all other values, the PDF is 0. To summarize,

$$f_Y(y) = \begin{cases} \frac{12}{7}\left(\frac{y}{2}+y^2\right) & : 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c)

$$\begin{aligned} P(X<Y) &= \int_0^1 \int_0^y \frac{12}{7}(xy+y^2)\,dx\,dy \\ &= \frac{12}{7}\int_0^1 \frac{y^3}{2} + y^3\,dy \\ &= \frac{12}{7}\cdot\frac{3}{2}\cdot\frac{1}{4} \\ &= \frac{9}{14} \end{aligned}$$

(d) We compute directly.

$$\begin{aligned} \int_0^1\int_0^1 x^2y\cdot\frac{12}{7}(xy+y^2)\,dx\,dy &= \frac{12}{7}\int_0^1 \frac{y^2}{4}+\frac{y^3}{3}\,dy \\ &= \frac{12}{7}\left(\frac{1}{12}+\frac{1}{12}\right) \\ &= \frac{2}{7}. \end{aligned}$$

3. **Exercise 6.9**

We know that $X \sim \text{Bin}(3, 1/2)$, because it counts the number of heads in three independent fair coin tosses. So we have the PMF

$$p_X(j) = \binom{3}{j}\left(\frac{1}{2}\right)^j\left(\frac{1}{2}\right)^{3-j} \quad \text{for } j = 0,1,2,3$$

$$p_X(j) = \binom{3}{j}\left(\frac{1}{2}\right)^3 \quad \text{for } j = 0,1,2,3$$

For Y we have a uniform distribution on all outcomes for a six-sided die roll. This gives the PMF

$$p_Y(k) = \frac{1}{6} \quad \text{for } k = 1,2,3,4,5,6.$$

X and Y are independent, so their joint PMF is given by

$$p_{X,Y}(j,k) = p_X(j)p_Y(k) = \binom{3}{j}\left(\frac{1}{2}\right)^3 \frac{1}{6} \quad \text{for } 0 \le j \le 3,\ 1 \le k \le 6.$$

For all other values of j and k, the PMF is 0.

4. **Exercise 6.12**

First we find the marginal PDF of X. For $x > 0$ we have

$$\begin{aligned} f_X(x) &= \int_0^\infty 2e^{-(x+2y)}\,dy \\ &= e^{-x}\left[-e^{-2y}\right]_0^\infty \\ &= e^{-x}. \end{aligned}$$

If $x \le 0$ then $f(x,y) = 0$. The the PDF of X is

$$f_X(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now we find the marginal PDF of Y. For $y > 0$ we have

$$\begin{aligned} f_Y(y) &= \int_0^\infty 2e^{-(x+2y)}\,dx \\ &= 2e^{-2y}\left[-e^{-x}\right]_0^\infty \\ &= 2e^{-2y}. \end{aligned}$$

If $y \le 0$ then $f(x,y) = 0$. The the PDF of Y is

$$f_Y(y) = \begin{cases} 2e^{-2y} & \text{if } y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

We see that X and Y are independent, because

$$f_X(x)f_Y(y) = e^{-x}\cdot 2e^{-2y} = 2e^{-(x+2y)} = f(x,y).$$

5. **Exercise 6.22**

It is more straightforward to think in terms of the interpretation of the multinomial distribution. X_1 gives the number of trials that have result 1. X_2 gives the number of trials that have result 2. So $X_1 + X_2$ gives the number of trials out of n that have result 1 or result 2. For any given trial, the probability of result 1 or result 2 is $p_1 + p_2$. As the trials are independent and there are a fixed number of them, we get that $X_1 + X_2 \sim \text{Bin}(n, p_1 + p_2)$.