Inner Products

Let V be a vector space. An *inner product* on V is a "pairing" of vectors:

$$(\cdot, \cdot) : V \times V \to F$$

which maps a pair of vectors to an element of the field. It must satisfy the properties

1. $(u, u) \geq 0$ and $(u, u) = 0 \Leftrightarrow u = 0$.
2. $(u, v) = (v, u)$.
3. $(u, v + w) = (u, v) + (u, w)$.
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- Dot product in $\mathbb{R}^n$.
- $\int_a^b fg \, \mathrm{d}x$ on integrable functions.

Geometry

Inner products allow us to do geometry:

- The **norm** of v is
$$\|v\| := \sqrt{(v, v)}.$$
- The **distance** between vectors v and w is
$$d(v, w) := \|v - w\|.$$
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$$\cos^{-1} \frac{(v, w)}{\|v\| \|w\|}.$$

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Exercise:
Show that $\frac{1}{\sqrt{\pi}} \sin(x)$ is a unit vector using the inner product $\int_0^{2\pi} fg \, \mathrm{d}x$.

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The point is that these results rely ONLY on the properties of the inner products and not HOW you compute them or WHAT the vectors represent!

Orthogonal and Orthonormal Sets

Definition

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then the set is called **orthonormal**. That is, it is an orthogonal set of unit vectors.

Orthogonal Implies Linear Independence

Theorem

A finite orthogonal set of non-zero vectors is linearly independent.

Proof.

Suppose $\sum_{i=1}^{n} a_i v_i = 0$.

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$$\left(v_j, \sum_{i=1}^{n} a_i v_i \right) = \sum_{i=1}^{n} a_i (v_j, v_i)$$

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So we have $0 = a_j (v_j, v_j)$. Now, $(v_j, v_j) \neq 0$ therefore $a_j = 0$. $\square$

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This means that *vector coordinates* can be expressed in terms of inner products!

Orthonormal Vectors

If our set of vectors is orthonormal, then we an do better:

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Theorem

Let $V = \mathrm{span}(S)$ *where* $S = \{v_1, \ldots, v_n\}$ *is an orthonormal set.*
Let $x, y \in V$ *and write them as*

$$x = a_1 v_1 + \cdots a_n v_n = \sum_{i=1}^{n} a_i v_i, \quad y = b_1 v_1 + \cdots b_n v_n = \sum_{i=1}^{n} b_i v_i.$$

Then

$$(x, y) = \sum_{i=1}^{n} a_i b_i.$$

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To prove, let's compute (x, y):

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So far, we have many nice properties IF we have an orthonormal basis.

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Question: given an vector space, can we always find an orthonormal basis? YES!

The Gram-Schmidt Process

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Let V be a (nontrivial) vector space of dimension m with an inner product. Then V has an orthonormal basis.

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Let V be a (nontrivial) vector space of dimension m with an inner product. Then V has an orthonormal basis.

The proof of this is constructive and the method is known as the *Gram-Schmidt* process. The method is worth knowing over the actual result of the theorem!

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Clearly, v_1 is a unit vector and $\operatorname{span}\{v_1\} = \operatorname{span}\{w_1\}$.

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Next, we want to find a vector v_2 so that $\{v_1, v_2\}$ is orthonormal and $\text{span}\{v_1, v_2\} = \text{span}\{w_1, w_2\}$.

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Therefore $bv_2 = w_2 - (v_1, w_2)v_1$.

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$$\|bv_2\| = \|w_2 - (v_1, w_2)v_1\| \Rightarrow b = \|w_2 - (v_1, w_2)v_1\|.$$

The Gram-Schmidt Process

In general: Let $\{v_1, \ldots, v_n\}$ be an ON set so that

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The Gram-Schmidt Process

In general: Let $\{v_1, \ldots, v_n\}$ be an ON set so that

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In order to find a vector v_{n+1} so that $\{v_1, \ldots, v_{n+1}\}$ is ON and

$$\mathrm{span}\{v_1, \ldots, v_{n+1}\} = \mathrm{span}\{w_1, \ldots, w_{n+1}\}$$

Compute a unit vector in the direction of

$$w_{n+1} - \sum_{i=1}^{n} (v_i, w_{n+1}) v_i.$$

Exercise

Compute an ON basis for the subset of $\mathbb{R}^3$ spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$