Math 431 HW01 Solutions

1. **Exercise 1.1**

   The sample space is

   $$\Omega = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} = \{(i, j) \mid \text{ where } 1 \leq i, j \leq 6\}.$$

   The event A consists of pairs (i, j) such that $i < j$. There are 15 such integer pairs with $1 \leq i, j \leq 6$. Hence $P(A) = \frac{15}{36} = \frac{5}{12}$.

2. **Exercise 1.2**

   (a) $\Omega = \{\{\text{cereal, eggs}\}, \{\text{cereal, fruit}\}, \{\text{eggs, fruit}\}\}$

   (b) $A = \{\{\text{cereal, eggs}\}, \{\text{cereal, fruit}\}\}$. We can further express A as

   $$A = \Omega \setminus \{\{\text{eggs, fruit}\}\},$$

   or

   $$A = \{\omega \in \Omega : \text{cereal} \in \omega\}.$$

3. **Exercise 1.4**

   (a) This is an ordered sample with replacement, where we are sampling from the population of flags, $S = \{1, \ldots, 50\}$. Here we can say 1 is the Alabama flag, 2 is the Alaska flag, and so on in alphabetical order. So

   $$\Omega = S^3 = \{(\omega_1, \omega_2, \omega_3) : \omega_i \in S\}.$$

   We are using the discrete uniform probability measure, so we have that

   $$P(\omega) = \frac{1}{\#\Omega} = \frac{1}{50^3}$$

   for each $\omega \in \Omega$. Alternately, you could specify that

   $$P(E) = \frac{\#E}{\#\Omega} = \frac{\#E}{50^3}.$$

   (b) This is one specific outcome within the sample space, so

   $$P(\text{WI on Monday, MI on Tuesday, CA on Wednesday}) = \frac{1}{50^3}.$$

(c) There are multiple outcomes in the sample space corresponding to this event, so we must use the more general formula:

$$P(E) = \frac{\#E}{\#\Omega} = \frac{\#E}{50^3}$$

where E is the event that the Wisconsin flag hangs at least two of the days. There are two possibilities.

The first possibility is the Wisconsin flag hangs exactly 3 days. There is only outcome in the sample space corresponding to this event.

The second possibility is the Wisconsin flag hangs exactly 2 days. To count the number of outcomes in this event, we use the multiplication rule. First, note that we can choose 2 out of 3 days for the Wisconsin flag to hang. There are $\binom{3}{2} = 3$ ways to make this decision. Second, we pick a different state's flag to hang on the remaining day. There are 49 other flags to choose from. So in total, there are $3 \cdot 49$ ways for the Wisconsin flag to hang for exactly 2 days.

Therefore,

$$P(E) = \frac{\#E}{\#\Omega} = \frac{1 + 3 \cdot 49}{50^3} = \frac{37}{31250}.$$

4. **Exercise 1.6**

(a) Label the three green balls 1, 2, and 3, and label the yellow balls 4, 5, 6, and 7. We imagine picking the balls in order, and hence take

$$\Omega = \{(i,j) : i, j \in \{1, 2, \ldots, 7\}, i \neq j\}$$

the set of ordered pairs of distinct elements from the set $\{1, 2, \ldots, 7\}$. The event of two different colored balls is,

$$A = \{(i,j) : (i \in \{1, 2, 3\} \text{ and } j \in \{4, \ldots, 7\}) \text{ or } (i \in \{4, \ldots, 7\} \text{ and } j \in \{1, 2, 3\})\}.$$

(b) We have $\#\Omega = 7 \cdot 6 = 42$ and $\#A = 3 \cdot 4 + 4 \cdot 3 = 24$. Thus,

$$P(A) = \frac{24}{42} = \frac{4}{7}.$$

Alternatively, we could have chosen a sample space in which order does not matter. In this case the size of the sample space is $\binom{7}{2}$. There are $\binom{3}{1}$ ways to choose one of the green balls and $\binom{4}{1}$ ways to choose one yellow ball. Hence, the probability is computed as

$$P(A) = \frac{\binom{3}{1}\binom{4}{1}}{\binom{7}{2}} = \frac{4}{7}.$$

5. **Exercise 1.20**

(a) The sample space is

$$\Omega = \{1, 2, 3, 4, 5, 6\}^4 = \{(\omega_1, \omega_2, \omega_3, \omega_4) \mid 1 \leq \omega_1, \omega_2, \omega_3, \omega_4 \leq 6\}.$$

The uniform probability measure is appropriate, so

$$P(\omega) = \frac{1}{\#\Omega} = \frac{1}{6^4} = \frac{1}{1296}.$$

(b) To count the number of outcomes in A we consider the two possible cases: there are 2 rolls that are 5, 3 rolls that are 5, or all 4 rolls are 5. We will count these events separately, and add them together. Let A_k denote the event there are k rolls that are 5. Then

$$\#A_2 = \binom{4}{2} \cdot (5 \cdot 5 \cdot 1 \cdot 1) = 6 \cdot 5^2 = 150,$$

because you first choose one of the four rolls to be 5, then the remaining three rolls can be any of the results 1, 2, 3, 4, or 6. We also have

$$\#A_3 = \binom{4}{3} \cdot (5 \cdot 1 \cdot 1 \cdot 1) = 4 \cdot 5 = 20,$$

because you first choose one of the four rolls to be 5, then the remaining three rolls can be any of the results 1, 2, 3, 4, or 6. We also have

$$\#A_4 = 1$$

because all of the 4 rolls must be 5. So

$$P(A) = \frac{150 + 20 + 1}{1296} = \frac{171}{1296}.$$

To count the number of outcomes in B we consider the two possible cases: there are 0 rolls that are 5 or 1 roll that is 5. We will count these events separately, and add them together. Let B_k denote the event there are k rolls that are 5. Then

$$\#B_0 = 5 \cdot 5 \cdot 5 \cdot 5 = 5^4 = 625,$$

because each of the 4 rolls can be any of the results 1, 2, 3, 4, or 6. We also have

$$\#B_1 = 4 \cdot (5 \cdot 5 \cdot 5 \cdot 1) = 4 \cdot 5^3 = 500,$$

because you first choose one of the four rolls to be 5, then the remaining three rolls can be any of the results 1, 2, 3, 4, or 6. So

$$P(B) = \frac{625 + 500}{1296} = \frac{1125}{1296}.$$

There is a better approach to this problem, which we will explore in part (c).

(c) The set

$$A \cup B = \Omega$$

because $A = B^c$ and $B = A^c$. There are either at most 1 four or at least 2 fours. So we should have

$$1 = P(\Omega) = P(A) + P(B) = \frac{171}{1296} + \frac{1125}{1296} = \frac{1296}{1296}.$$

Our answers from part (b) satisfy this equality. This tells us that a better approach to solving be would be find $P(B)$ or $P(A)$, then use the complement rule to find the other value.

6. **Exercise 1.21**

(a) In this case the sample space Ω is all possible results for an ordered drawing without replacement. The population of chips is size $3 + 2 + 2 = 7$ and the sample is size 3, so $\#\Omega = (7)_3 = 126$. To count the number of outcomes in A, we break this into steps.

    i. Choose an order for the 3 colors to be drawn. There are $3!$ ways to do this.

    ii. Choose which black chip to draw. There are 3 ways to do this.

    iii. Choose which red chip to draw. There are 2 ways to do this.

    iv. Choose which green chip to draw. There are 2 ways to do this.

So $\#A = 3! \cdot 3 \cdot 2 \cdot 2 = 72$. Thus we have

$$P(A) = \frac{\#A}{\#\Omega} = \frac{72}{126} = \frac{12}{35}.$$

(b) In this case the sample space Ω is all possible results for an unordered drawing without replacement. The population of chips is size $3 + 2 + 2 = 7$ and the sample is size 3, so $\#\Omega = \binom{7}{3} = 35$. To count the number of outcomes in A, we break this into steps.

    i. Choose which black chip to draw. There are 3 ways to do this.

    ii. Choose which red chip to draw. There are 2 ways to do this.

    iii. Choose which green chip to draw. There are 2 ways to do this.

So $\#A = 3 \cdot 2 \cdot 2 = 12$. Thus we have

$$P(A) = \frac{\#A}{\#\Omega} = \frac{12}{35}.$$

So the results are the same regardless of method.