Math 431: Homework 6 Solutions

1. **Exercise 3.15**

   (b) $E[X^2] = \mathrm{Var}(X) + (EX)^2 = 4 + 9 = 13$

   (d) $\mathrm{Var}(4X - 2) = 4^2\mathrm{Var}(X) = 64$

2. **Exercise 3.18**

   (a) This is a direct computation.

$$\begin{aligned} P(2 < X < 6) &= P\left(\frac{2-3}{\sqrt{4}} < \frac{X-\mu}{\sigma} < \frac{6-3}{\sqrt{4}}\right) \\ &= P\left(-\frac{1}{2} < Z < \frac{3}{2}\right) \\ &= \Phi(1.5) - \Phi(-0.5) \\ &\approx 0.6247. \end{aligned}$$

   (b) We proceed by computation.

$$P(X > c) = P\left(\frac{X-\mu}{\sigma} > \frac{c-3}{2}\right) = P\left(Z > \frac{c-3}{2}\right) = 1 - \Phi\left(\frac{c-3}{2}\right) = 0.33$$

   So

$$\frac{c-3}{2} = \Phi^{-1}(0.67) = 0.44 \quad \Rightarrow \quad c = 3.88.$$

   (c) $EX^2 = \sigma^2 + \mu^2 = 4 + 3^2 = 13.$

3. **Exercise 3.25**

   For both portions of the problem, it is necessary to ensure two conditions are satisfied. First, the function must be non-negative. Second, it must integrate over the real numbers to 1.

   (a) We start be integrating to determine a possible value for b.

$$\begin{aligned} & \int_1^3 f(x)\,dx = 1 \\ \Rightarrow\quad & \left[\frac{x^3}{3} - bx\right]_1^3 = 1 \\ \Rightarrow\quad & (9 - 3b) - \left(\frac{1}{3} - b\right) = 1 \\ \Rightarrow\quad & \frac{23}{3} = 2b \\ \Rightarrow\quad & \frac{23}{6} = b \end{aligned}$$

So if f is a PDF, then we must have $b = \frac{23}{6}$. However,

$$1 \leq x < \sqrt{\frac{23}{6}} \quad \Rightarrow \quad x^2 - \frac{23}{6} < 0.$$

So f cannot be a PDF.

(b) In this case, we know immediately the we cannot have $b > \pi/2$. We integrate to find a potential value for b.

$$\begin{aligned}
&\int_{-b}^{b} f(x)\,dx = 1 \\
\Rightarrow\quad & \sin x|_{-b}^{b} = 1 \\
\Rightarrow\quad & \sin(b) - \sin(-b) = 1 \\
\Rightarrow\quad & \sin(b) + \sin(b) = 1 \\
\Rightarrow\quad & \sin(b) = \frac{1}{2} \\
\Rightarrow\quad & b = \frac{\pi}{6}
\end{aligned}$$

So for $b = \frac{\pi}{6}$, the function g is a PDF. This is the only possible value, because we already established that $b \leq \pi/2$ to ensure g is non-negative.

4. **Exercise 3.31**

(a) The total probability must be 1. So we compute

$$\begin{aligned}
1 &= \int_1^{\infty} \frac{c}{x^4}\,dx \\
&= \left[\frac{c}{-3x^3}\right]_1^{\infty} \\
&= \frac{c}{3}
\end{aligned}$$

So $c = 3$. Note that $f(x) \geq 0$ for all x with this choice of c, so it is a valid PDF.

(d) We integrate the PDF.

$$P(2 < X < 4) = \int_2^4 3x^{-4}\,dx = \left[-x^{-3}\right]_2^4 = \frac{1}{8} - \frac{1}{64} = \frac{7}{64}.$$

(e) First we handle the easy case. For $x < 1$ we have $F_X(x) = P(X \leq x) = 0$. The non-trivial case is $x \geq 1$. Here we have

$$\begin{aligned}
F_X(x) = P(X \leq x) &= \int_1^x 3t^{-4}\,dt \\
&= \left[-t^{-3}\right]_1^x \\
&= 1 - \frac{1}{x^3}.
\end{aligned}$$

(f) First we find EX.

$$EX = \int_1^{\infty} x \cdot 3x^{-4}\,dx = \int_1^{\infty} 3x^{-3}\,dx = \left[\frac{3}{-2}x^{-2}\right]_1^{\infty} = \frac{3}{2}$$

Now we find $\text{Var}(X)$. As an intermediate step, we find EX^2.

$$EX^2 = \int_1^\infty x^2 \cdot 3x^{-4}\ dx = \int_1^\infty 3x^{-2}\ dx = \left[\frac{3}{-1}x^{-1}\right]_1^\infty = 3$$

Assembling the pieces, we get

$$\text{Var}(X) = EX^2 - (EX)^2 = 3 - (3/2)^2 = \frac{3}{4}.$$

(g) Using our solutions from part (f), we have

$$E[5X^2 + 3X] = 5E[X^2] + 3E[X] = 5 \cdot 3 + 3 \cdot \frac{3}{2} = 19.5.$$

(h) We can compute with caution.

$$\begin{aligned} EX^n &= \int_1^\infty x^n \cdot 3x^{-4}\ dx = \int_1^\infty 3x^{n-4}\ dx \\ &= \begin{cases} \left[\frac{3}{n-3}x^{n-3}\right]_1^\infty & \text{for } n < 3 \\ \infty & \text{for } n \geq 3 \end{cases} \\ &= \begin{cases} -\frac{3}{n-3} & \text{for } n < 3 \\ \infty & \text{for } n \geq 3 \end{cases} \\ &= \begin{cases} \frac{3}{3-n} & \text{for } n < 3 \\ \infty & \text{for } n \geq 3 \end{cases} \end{aligned}$$

5. **Exercise 3.52**

Following the hint,

$$\begin{aligned} \sum_{k=1}^\infty P(X \geq k) &= \sum_{k=1}^\infty \sum_{i=k}^\infty P(X = i) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i P(X = i) \\ &= \sum_{i=1}^\infty iP(X = i) \\ &= \sum_{i=0}^\infty iP(X = i) \\ &= E[X]. \end{aligned}$$

The difficult part is switching the order of summation correctly.

6. **Exercise 3.54**

(a) Evaluating this as a series we have

$$\begin{aligned}
P(X \geq k) &= \sum_{i=k}^{\infty} P(X = i) = \sum_{i=k}^{\infty} (1-p)^{i-1} p \\
&= p \sum_{i=k-1}^{\infty} (1-p)^{i} \\
&= p \cdot \frac{(1-p)^{k-1}}{1-(1-p)} \\
&= p \cdot \frac{(1-p)^{k-1}}{p} \\
&= (1-p)^{k-1}.
\end{aligned}$$

Alternately, the event $X \geq k$ is equivalent to the event that the first $k-1$ trials were all failures. So

$$P(X \geq k) = P(k-1 \text{ consecutive failures}) = (1-p)^{k-1}.$$

(b) Using the formula from 3.52 and part (a) of the current problem,

$$\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} P(X \geq k) = \sum_{k=1}^{\infty} (1-p)^{k-1} \\
&= \sum_{k=0}^{\infty} (1-p)^{k} = \frac{1}{1-(1-p)} = \frac{1}{p}
\end{aligned}$$

7. **Exercise 3.67**

(a) The most important technique is integration by parts.

$$\begin{aligned}
E[Z^3] &= \int_{-\infty}^{\infty} z^3 \varphi(z)\, dz \\
&= \int_{-\infty}^{\infty} z^2 \cdot (z\varphi(z))\, dz \\
&= \left[z^2 \cdot (-\varphi(z))\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -2z\varphi(z)\, dz \\
&= (0-0) + 2\int_{-\infty}^{\infty} z\varphi(z)\, dz \\
&= 2EZ = 0.
\end{aligned}$$

(b) We can use our knowledge that $EZ = 0$, $EZ^2 = 1$, and $EZ^3 = 0$ to solve this without integration.

$$\begin{aligned}
EX^3 &= E[(\mu + \sigma Z)^3] \\
&= E[\sigma^3 Z^3 + 3\mu\sigma^2 Z^2 + 3\mu^2 \sigma Z + \mu^3] \\
&= \sigma^3 EZ^3 + 3\mu\sigma^2 EZ^2 + 3\mu^2\sigma EZ + \mu^3 \\
&= 3\mu\sigma^2 + \mu^3
\end{aligned}$$

8. **Exercise 3.71** Let X denote the number of minutes past noon that the bus arrives. We want to find $P(X > 5)$.

$$
\begin{aligned}
P(X > 5) &= 1 - P(X \leq 5) \\
&= 1 - P\left(\frac{X - 0}{6} \geq \frac{5 - 0}{6}\right) \\
&= 1 - \Phi\left(\frac{5}{6}\right) \\
&\approx 1 - 0.7967 = 0.2033.
\end{aligned}
$$