Math 431: Homework 3 Solutions

1. **Exercise 1.13**

   (a) We define our events.

   $$\begin{aligned} B &= \text{The student wears a bracelet} \\ W &= \text{The student wears a watch.} \end{aligned}$$

   We are give

   $$\begin{aligned} P(B) &= 0.30 \\ P(W) &= 0.25 \\ P(B^cW^c) &= 0.60 \end{aligned}$$

   We want to compute $P(B \cup W)$. The natural instinct is to attempt the inclusion-exclusion-rule, because this is a union. It turns out to be easier to use the complement rule.

   $$\begin{aligned} P(B \cup W) &= 1 - P((B \cup W)^c) \\ &= 1 - P(B^cW^c) \\ &= 1 - 0.60 \\ &= 0.40. \end{aligned}$$

   (b) Now we want $P(BW)$. In this case we use inclusion-exclusion.

   $$\begin{aligned} P(B \cup W) &= P(B) + P(W) - P(BW) \\ \Rightarrow 0.40 &= 0.30 + 0.25 - P(BW) \\ \Rightarrow P(BW) &= 0.15. \end{aligned}$$

2. **Exercise 1.14**

   From the inclusion-exclusion principle we get

   $$P(A \cup B) = P(A) + P(B) - P(AB) = 0.4 + 0.7 - P(AB) = 1.1 - P(AB).$$

   Rearranging this we get $P(AB) = 1.1 - P(A \cup B)$.

   Since $P(A \cup B)$ is a probability, it is at most 1, so

   $$P(AB) = 1.1 - P(A \cup B) \geq 1.1 - 1 = 0.1.$$

   On the other hand, $AB \subset A$ so $P(AB) \leq P(A) = 0.4$. Putting these together we get $0.1 \leq P(AB) \leq 0.4$.

3. **Exercise 1.16**

   Note that we are flipping a coin 5 times, so the total number of outcomes is $2^5 = 32$. The coin is fair so these outcomes are equally likely.

   Out net winnings, X, is determined by the number of heads in the 5 flips. For each flip that is heads we earn a dollar, but for each tails we lose a dollar. Let H_k denote the event of getting k heads in the 5 flips (thus $5-k$ tails in those flips). Then

$$\begin{aligned} H_5 &= \{X = 5 - 0 = 5\} \\ H_4 &= \{X = 4 - 1 = 3\} \\ H_3 &= \{X = 3 - 2 = 1\} \\ H_2 &= \{X = 2 - 3 = -1\} \\ H_1 &= \{X = 1 - 4 = -3\} \\ H_0 &= \{X = 0 - 5 = -5\} \end{aligned}$$

   So the range of X is $\{-5, -3, -1, 1, 3, 5\}$.

   The probability of H_k is the number of ways to get k heads in 5 flips, divided by the size of the sample space. This is $\binom{5}{k}$, because you only need to choose k of the flips to be heads. So we have

$$P(H_k) = \frac{\binom{5}{k}}{32}.$$

   This is all we need to compute the PMF of X, because $P(X = 5) = P(H_5)$, $P(X = 3) = P(H_4)$, and so on.

   The PMF of X is

$$\begin{array}{c|cccccc} j & -5 & -3 & -1 & 1 & 3 & 5 \\ \hline p_X(j) & \frac{1}{32} & \frac{5}{32} & \frac{10}{32} & \frac{10}{32} & \frac{5}{32} & \frac{1}{32} \end{array}.$$

   This could be written as the formula

$$p_X(k) = \frac{\binom{5}{(k+5)/2}}{32} \qquad \text{for } k = -5, -3, -1, 1, 3, 5.$$

   Either form is acceptable.

4. **Exercise 1.18**

   The range of X is $\{3, 4, 5\}$ as these are the possible lengths of the words. The probability mass function is

$$\begin{aligned} &P(X = 3) = P(\text{we chose one of the letters of ARE}) = \frac{3}{16} \\ &P(X = 4) = P(\text{we chose one of the letters of SOME or DOGS}) = \frac{8}{16} = \frac{1}{2} \\ &P(X = 5) = P(\text{we chose one of the letters of BROWN}) = \frac{5}{16}. \end{aligned}$$

5. **Exercise 1.36**

   In both parts we have to identify the region corresponding to the appropriate event, and then we can compute the probability by taking ratios of areas.

(a) The points (X, Y) in the unit square with $a < X < b$ will form a rectangle with vertices $(a, 0), (b, 0), (b, 1), (a, 1)$. Thus

$$\begin{aligned} P(a < X < b) &= P(\text{point lies in rectangle with coordinates } (a,0),(b,0),(b,1),(a,1)) \\ &= \frac{\text{area of rectangle with coordinates } (a,0),(b,0),(b,1),(a,1)}{\text{area of square with coordinates } (0,0),(1,0),(1,1),(0,1)} \\ &= b - a. \end{aligned}$$

Thus, X has a uniform distribution on $[0, 1]$.

(b) The region of the x-y plane for which $|x - y| \leq 1/4$ consists of the region between the lines $y = x - 1/4$ and $y = x + 1/4$. Intersecting this region with the unit square gives a region with an area of $7/16$. (The easiest way to check this is by computing the area of the complement within the unit square.) Thus, the desired probability is also $7/16$ since the unit square has an area of one.

6. **Exercise 1.41**

Using the events

$$A_i = \{\text{Person } i \text{ wins no games}\},$$

the probability we want to compute is

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) \\ &\quad - P(A_1A_2) - P(A_1A_3) - P(A_2A_3) \\ &\quad + P(A_1A_2A_3) \\ &= 3 \cdot \frac{2^4}{3^4} - 3 \cdot \frac{1^4}{3^4} + 0 \\ &= \frac{45}{81} = \frac{5}{9}. \end{aligned}$$

7. **Exercise 1.44**

(a) In both cases we have

$$\text{Ran}(X) = \text{Ran}(Y) = \{1, 2, 3, 4, 5, 6\}.$$

(b) We can compute this directly. For any k in $\{1, 2, 3, 4, 5, 6\}$, we have

$$\begin{aligned} P(X \leq k) &= P(\text{Both dice are} \leq k) \\ &= \frac{k^2}{36}. \end{aligned}$$

For $k < 1$ we have

$$P(X \leq k) = 0$$

and for $k > 6$ we have

$$P(X \leq k) = 1.$$

As for the probability mass function of X, we have for any k in $\{1, 2, 3, 4, 5, 6\}$ the value

$$P(X = k) = P(X \leq k) - P(X \leq k - 1) = \frac{k^2}{36} - \frac{(k-1)^2}{36} = \frac{2k-1}{36}.$$

It would also be acceptable to carry this procedure out for all six possible values of k explicitly, but this is more work than is necessary.

(c) We can compute this using a similar technique from (a). For any k in $\{1, 2, 3, 4, 5, 6\}$, we have

$$\begin{aligned} P(Y \leq k) &= 1 - P(Y > k) \\ &= 1 - P(\text{Both dice are} > k) \\ &= 1 - \frac{(6-k)^2}{36}. \end{aligned}$$

For the probability mass function of Y, we have for any k in $\{1, 2, 3, 4, 5, 6\}$ the value

$$\begin{aligned} P(Y = k) &= P(Y \leq k) - P(Y \leq k - 1) \\ &= \left(1 - \frac{(6-k)^2}{36}\right) - \left(1 - \frac{(6-(k-1))^2}{36}\right) = \frac{13-2k}{36}. \end{aligned}$$

It would also be acceptable to carry this procedure out for all six possible values of k explicitly, but this is more work than is necessary.