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- Idea: We can remove vectors from a finite set until we get to a linearly independent case WITHOUT losing the spanning property!
- The proof hinged on the fact that such a basis is the *LARGEST* linearly independent subset of S.

An example

Find a basis for span(S) where

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

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We need to find the largest LI subset. To find it, consider the homogeneous problem.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = 0$$

and use row reduction.

Via row reduction

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & -2 \end{bmatrix}$$

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Examine the columns with "leading ones". If we *removed* the others, we would have a LI set!

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Examine the columns with "leading ones". If we *removed* the others, we would have a LI set! So

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is the basis for span(S).

Exercise

Find the a basis for the span of the following sets of vectors:

$$A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -6 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$B = \left\{ t^3 + t^2 - 2t + 1, t^2 + 1, t^3 - 2t, 2t^3 + 3t^2 - 4t + 3 \right\}$$

Hint for B: write the coefficients of the polynomials as column vectors.

Solutions

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$$\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & -5 \\ 0 & 2 & 1 & -6 & 1 \\ -1 & 1 & -1 & -3 & 0 \end{bmatrix}$$

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So

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is the largest LI subset and basis for span(A).

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So our basis is made up of the polynomials corresponding to the first and second columns:

$$\left\{ t^3 + t^2 - 2t + 1, t^2 + 1 \right\}.$$

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  We will now work on theory to make the "sweet spot" concrete.

Basis gives the maximum number of LI vectors

Theorem

Let V be a vector space with finite basis $\beta = \{v_1, \ldots, v_n\}$.
Let $W = \{w_1, \ldots, w_r\}$ be a set of linearly independent vectors.
Then $r \leq n$.

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Proof.

This is a simple proof (on the board) assuming that each vector can be represented as a column in $\mathbb{R}^n$. (which we may find later is actually not as huge an assumption as you might think!) Please come chat with me for a proof which does not use this assumption (it uses induction).

$\square$

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Hence the *size* of a basis is unique!

Definition

The **dimension**, $\dim(V)$, of a vector space is the size of a basis for the space.

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Extension to Basis

A final theorem:

Theorem

If L_1 is a linearly independent subset of a vector space V. Suppose $|L_1| = m$ and $\dim(V) = n$. Then there exists $n - m$ vectors L_2 with $L_1 \cup L_2$ a basis for V.

This theorem tells us that we can always assume that we have a basis with some known LI vectors.