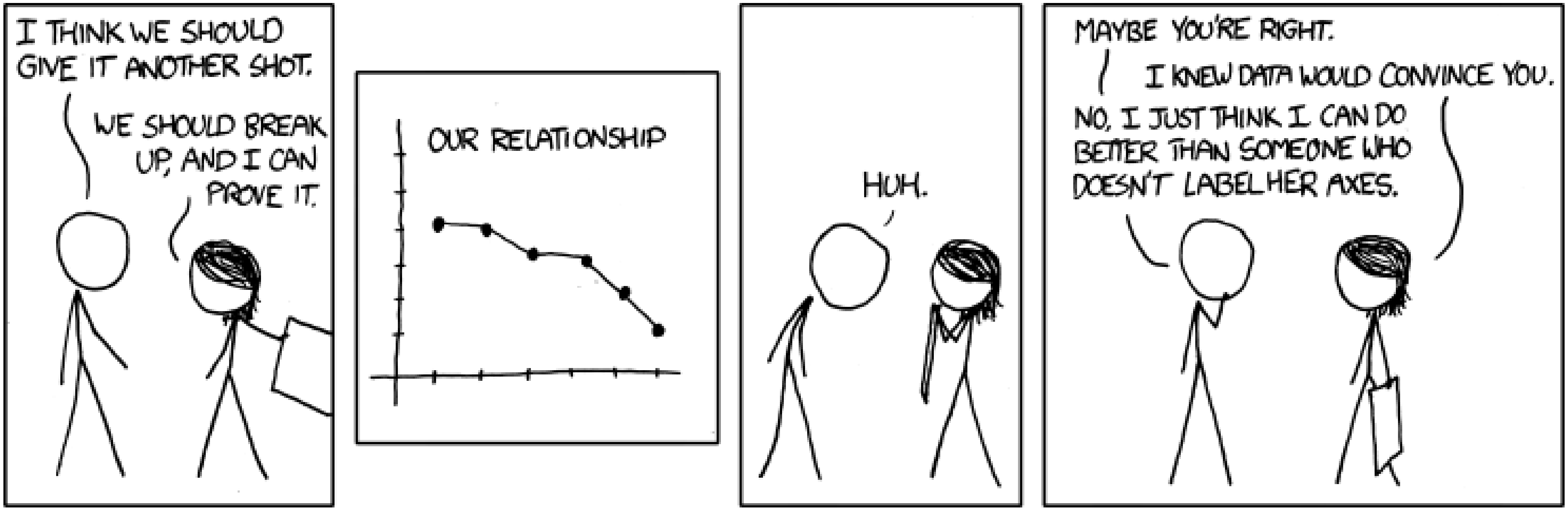
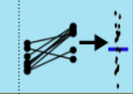
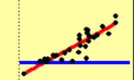
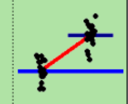
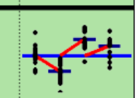
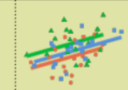
Linear Models - Regression



Common statistical tests are linear models

Last updated: 02 April, 2019

See worked examples and more details at the accompanying notebook: https://lindeloev.github.io/tests-as-linear

	Common name	Built-in function in R	Equivalent linear model in R	Exact?	The linear model in words	Icon
Simple regression: lm(y ~ 1 + x)	**y is independent of x** P: One-sample t-test N: Wilcoxon signed-rank	t.test(y) wilcox.test(y)	lm(y ~ 1) lm(signed_rank(y) ~ 1)	✓ for N >14	One number (intercept, i.e., the mean) predicts **y**. - (Same, but it predicts the *signed rank* of **y**.)	
	P: Paired-sample t-test N: Wilcoxon matched pairs	t.test(y_1, y_2, paired=TRUE) wilcox.test(y_1, y_2, paired=TRUE)	lm(y_2 - y_1 ~ 1) lm(signed_rank(y_2 - y_1) ~ 1)	✓ for N >14	One intercept predicts the pairwise **y_2-y_1** differences. - (Same, but it predicts the *signed rank* of **y_2-y_1**.)	
	y ~ continuous x P: Pearson correlation N: Spearman correlation	cor.test(x, y, method='Pearson') cor.test(x, y, method='Spearman')	lm(y ~ 1 + x) lm(rank(y) ~ 1 + rank(x))	✓ for N >10	One intercept plus **x** multiplied by a number (slope) predicts **y**. - (Same, but with *ranked* **x** and **y**)	
	y ~ discrete x P: Two-sample t-test P: Welch's t-test N: Mann-Whitney U	t.test(y_1, y_2, var.equal=TRUE) t.test(y_1, y_2, var.equal=FALSE) wilcox.test(y_1, y_2)	lm(y ~ 1 + G_2)[A] gls(y ~ 1 + G_2, weights=...[B])[A] lm(signed_rank(y) ~ 1 + G_2)[A]	✓ ✓ for N >11	An intercept for **group 1** (plus a difference if **group 2**) predicts **y**. - (Same, but with one variance *per group* instead of one common.) - (Same, but it predicts the *signed rank* of **y**.)	
Multiple regression: lm(y ~ 1 + x_1 + x_2 +...)	P: One-way ANOVA N: Kruskal-Wallis	aov(y ~ group) kruskal.test(y ~ group)	lm(y ~ 1 + G_2 + G_3 +...+ G_N)[A] lm(rank(y) ~ 1 + G_2 + G_3 +...+ G_N)[A]	✓ for N >11	An intercept for **group 1** (plus a difference if group ≠ 1) predicts **y**. - (Same, but it predicts the *rank* of **y**.)	
	P: One-way ANCOVA	aov(y ~ group + x)	lm(y ~ 1 + G_2 + G_3 +...+ G_N + x)[A]	✓	- (Same, but plus a slope on **x**.) *Note: this is discrete AND continuous. ANCOVAs are ANOVAs with a continuous x.*	
	P: Two-way ANOVA	aov(y ~ group * sex)	lm(y ~ 1 + G_2 + G_3 + ... + G_N + S_2 + S_3+ ... + S_K + G_2*S_2+G_3*S_3+...+G_N*S_K)	✓	Interaction term: changing **sex** changes the **y ~ group** parameters. *Note: **$G_{2 \text{ to } N}$** is an indicator (0 or 1) for each non-intercept levels of the **group** variable. Similarly for **$S_{2 \text{ to } K}$** for sex. The first line (with G_i) is main effect of group, the second (with S_i) for sex and the third is the **group × sex** interaction. For two levels (e.g. male/female), line 2 would just be "S_2" and line 3 would be S_2 multiplied with each G_i.*	*[Coming]*
	Counts ~ discrete x N: Chi-square test	chisq.test(groupXsex_table)	**Equivalent log-linear model** glm(y ~ 1 + G_2 + G_3 + ... + G_N + S_2 + S_3+ ... + S_K + G_2*S_2+G_3*S_3+...+G_N*S_K, family=...)[A]	✓	Interaction term: (Same as Two-way ANOVA.) *Note: Run glm using the following arguments:* `glm(model, family=poisson())` *As linear-model, the Chi-square test is $\log(y_i) = \log(N) + \log(\alpha_i) + \log(\beta_j) + \log(\alpha_i\beta_j)$ where α_i and β_j are proportions. See more info in the accompanying notebook.*	*Same as Two-way ANOVA*
	N: Goodness of fit	chisq.test(y)	glm(y ~ 1 + G_2 + G_3 +...+ G_N, family=...)[A]	✓	(Same as One-way ANOVA and see Chi-Square note.)	*1W-ANOVA*

List of common parametric (P) non-parametric (N) tests and equivalent linear models. The notation y ~ 1 + x is R shorthand for $y = 1 \cdot b + a \cdot x$ which most of us learned in school. Models in similar colors are highly similar, but really, notice how similar they *all* are across colors! For non-parametric models, the linear models are reasonable approximations for non-small sample sizes (see "Exact" column and click links to see simulations). Other less accurate approximations exist, e.g., Wilcoxon for the sign test and Goodness-of-fit for the binomial test. The signed rank function is `signed_rank = function(x) sign(x) * rank(abs(x))`. The variables G_i and S_i are "dummy coded" indicator variables (either 0 or 1) exploiting the fact that when $\Delta x = 1$ between categories the difference equals the slope. Subscripts (e.g., G_2 or y_1) indicate different columns in data. lm requires long-format data for all non-continuous models. All of this is exposed in greater detail and worked examples at https://lindeloev.github.io/tests-as-linear.

[A] See the note to the two-way ANOVA for explanation of the notation.
[B] Same model, but with one variance per group: `gls(value ~ 1 + G2, weights = varIdent(form = ~1|group), method="ML")`.

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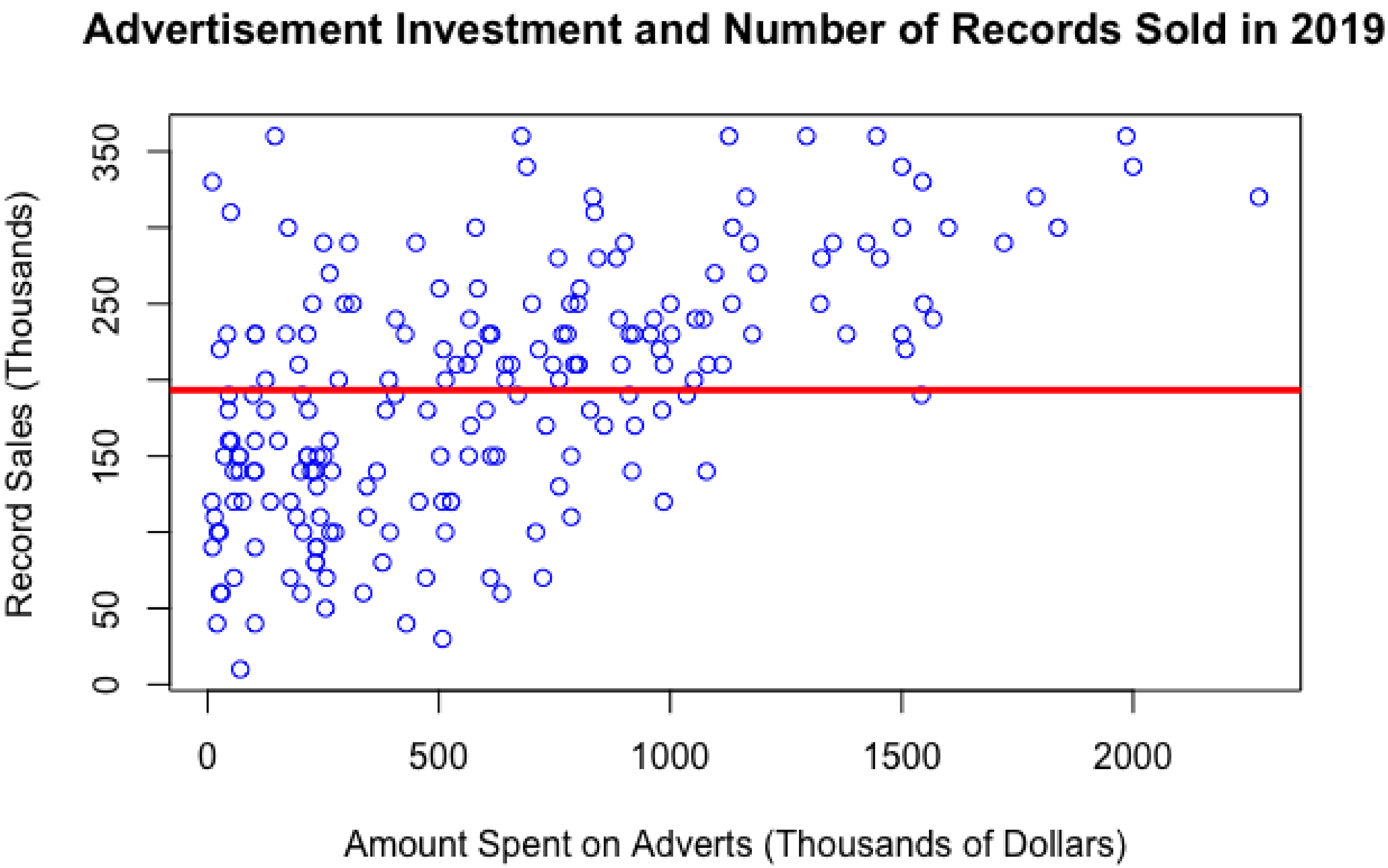
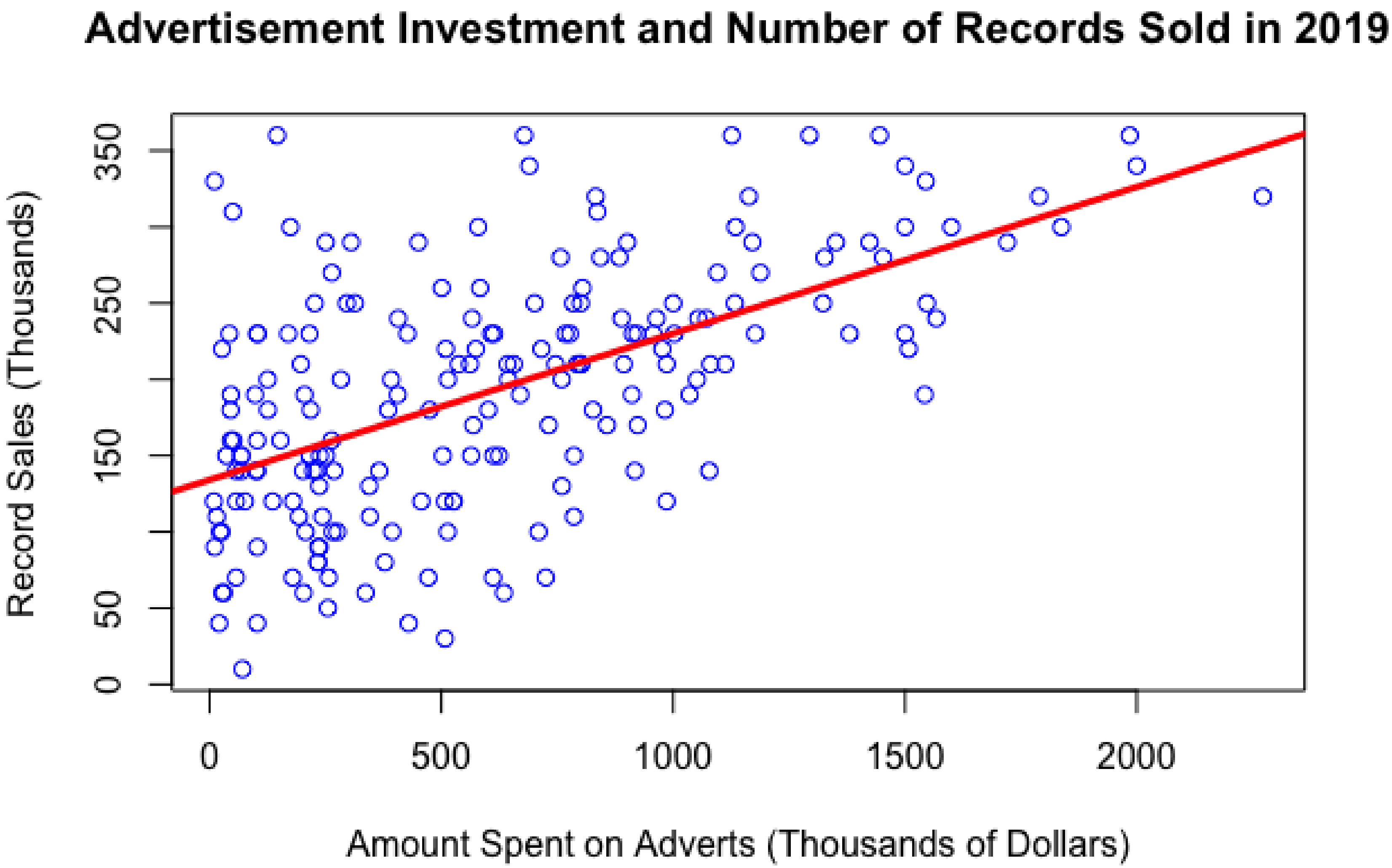
The mean:
A very simple statistical model

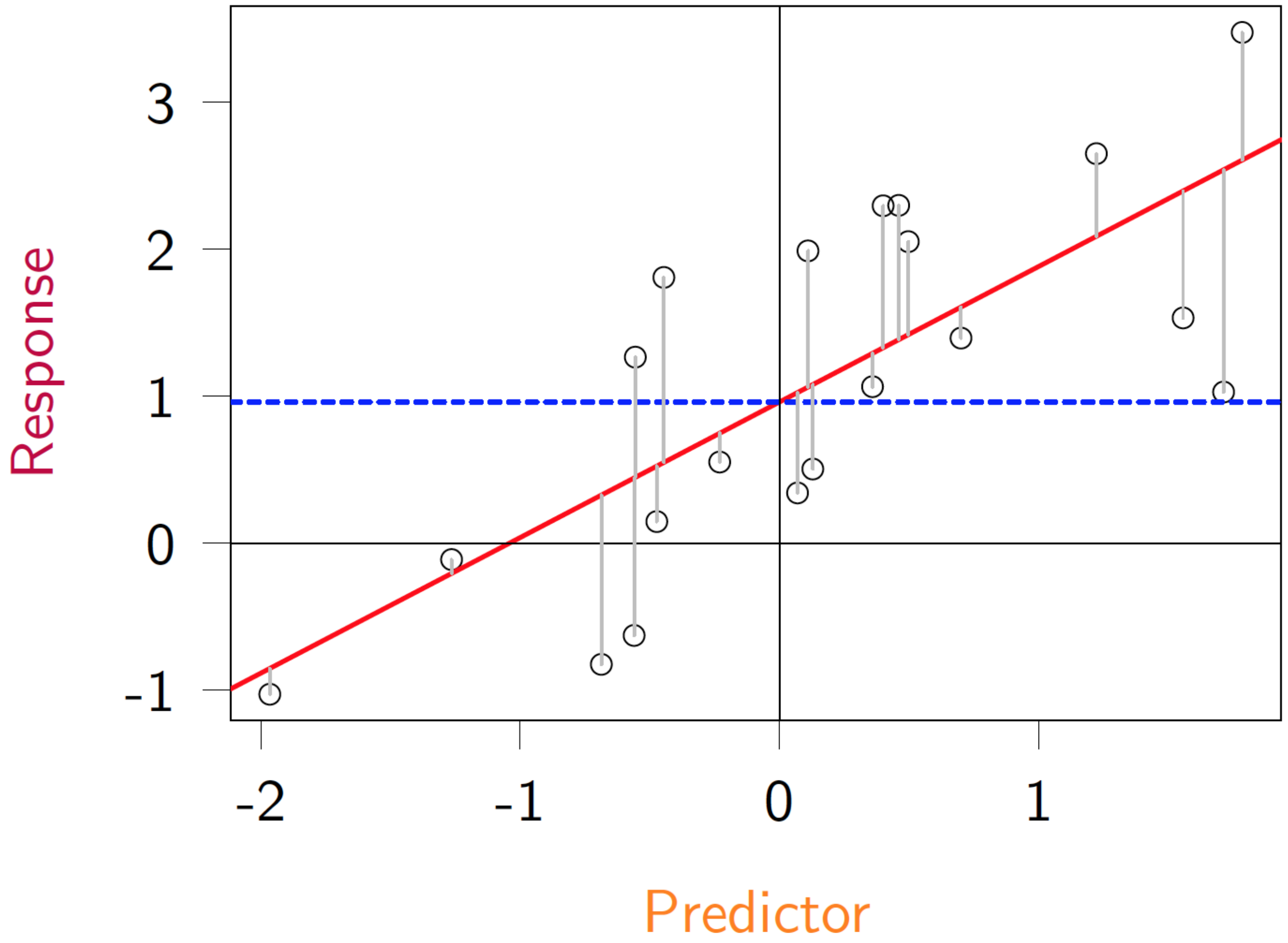


The method of least squares



Linear Regression

Response = **Intercept** + **Slope** × **Predictor** + **Error**

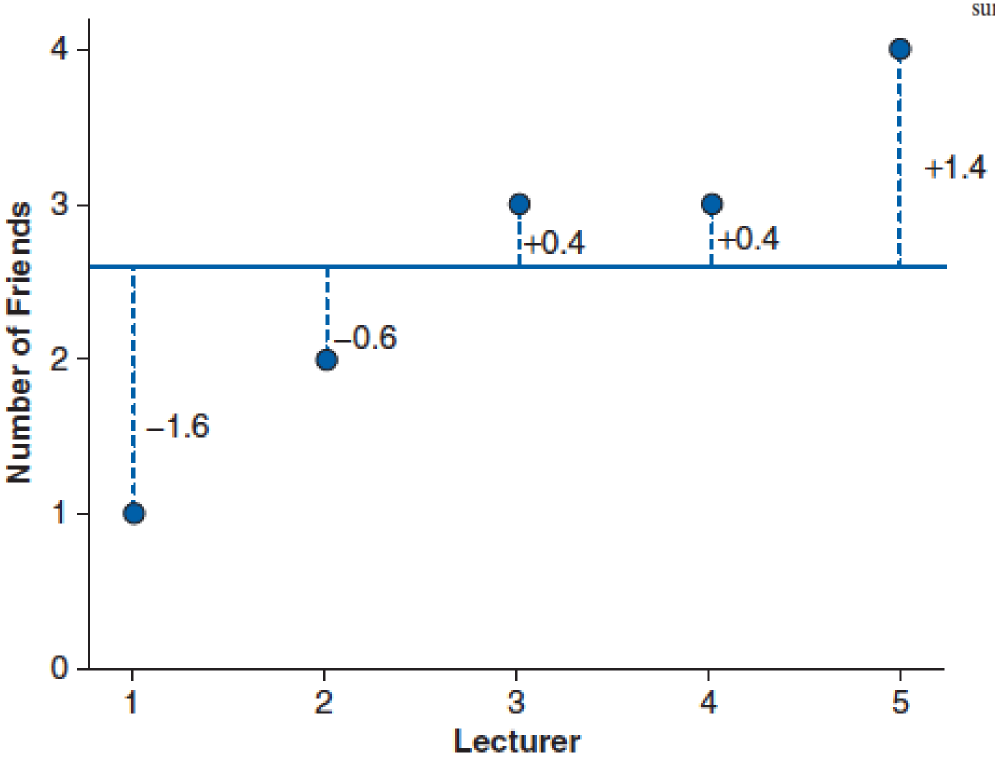


$$Y_i = (b_0 + b_1 X_i) + \varepsilon_i$$

Concept Check - Variance

total error = sum of deviances

$$= \sum (x_i - \bar{x}) = (-1.6) + (-0.6) + (0.4) + (0.4) + (1.4) = 0$$

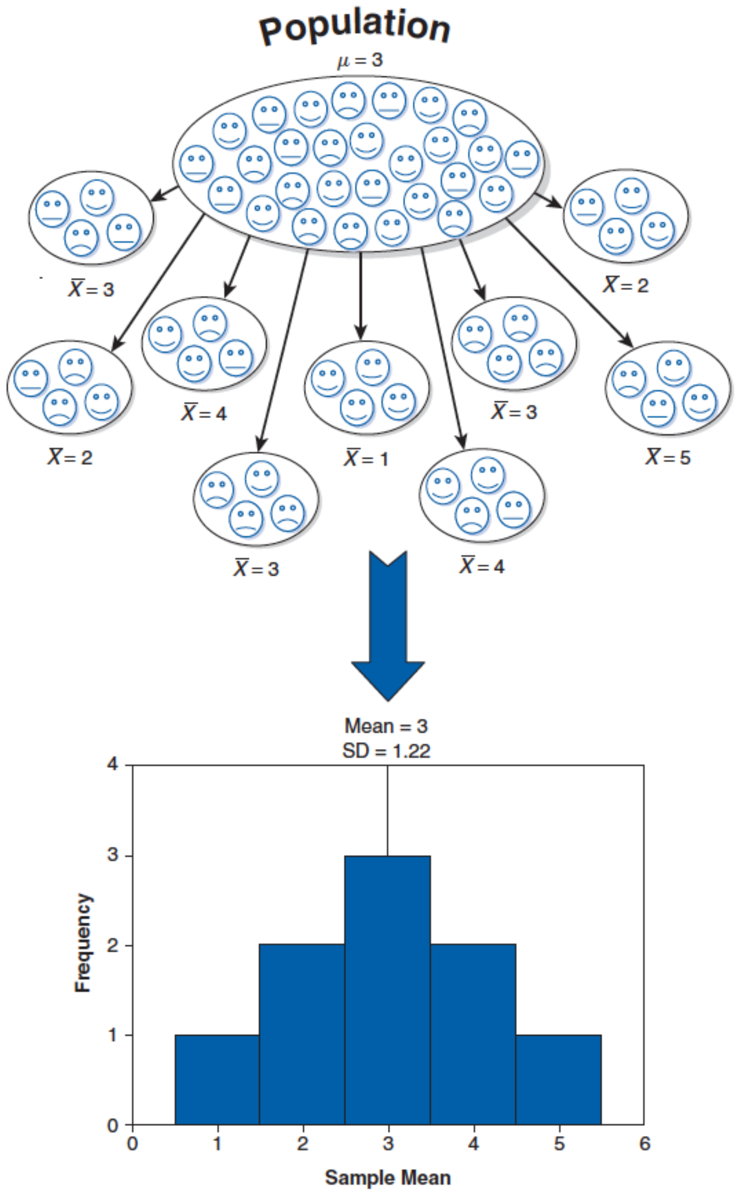


$$\begin{aligned}\text{sum of squared errors (SS)} &= \sum (x_i - \bar{x})(x_i - \bar{x}) \\ &= (-1.6)^2 + (-0.6)^2 + (0.4)^2 + (0.4)^2 + (1.4)^2 \\ &= 2.56 + 0.36 + 0.16 + 0.16 + 1.96 \\ &= 5.20\end{aligned}$$

$$\text{variance } (s^2) = \frac{SS}{N-1} = \frac{\sum (x_i - \bar{x})^2}{N-1} = \frac{5.20}{4} = 1.3$$

$$\begin{aligned} s &= \sqrt{\frac{\sum (x_i - \bar{x})^2}{N-1}} \\ &= \sqrt{1.3} \\ &= 1.14 \end{aligned}$$

Concept Check – Standard Error



$$\sigma_{\bar{X}} = \frac{s}{\sqrt{N}}$$

Constructing Simple Regressions in R

```
album_lm_1 <- lm(sales ~ 1 + adverts, data = album_data)
```

```
summary(album_lm_1)
```

```
Call:
lm(formula = sales ~ 1 + adverts, data = album_data)

Residuals:
     Min       1Q   Median       3Q      Max 
-152.949  -43.796   -0.393   37.040  211.866 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 1.341e+02  7.537e+00  17.799   <2e-16 ***
adverts     9.612e-02  9.632e-03   9.979   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 65.99 on 198 degrees of freedom
Multiple R-squared:  0.3346,    Adjusted R-squared:  0.3313 
F-statistic: 99.59 on 1 and 198 DF,  p-value: < 2.2e-16
```

Exercise 1 – Simple Regression

```
Call:
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The Theory – Regression Output

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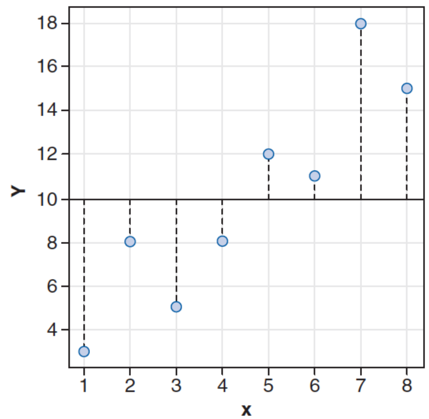
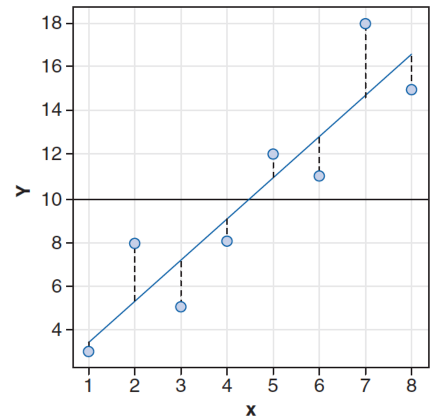
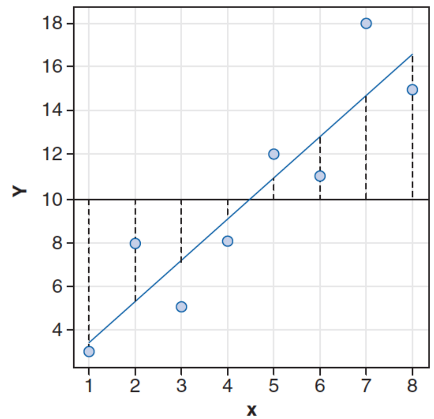
The Model - Assessing Goodness of Fit (R^2)



SS_T uses the differences between the observed data and the mean value of *Y*



SS_R uses the differences between the observed data and the regression line



SS_M uses the differences between the mean value of *Y* and the regression line

$$R^2 = \frac{SS_M}{SS_T}$$

$$1 - (1 - R^2)\frac{n - 1}{n - p - 1}$$

$$r = \sqrt{R^2}$$

$$F = \frac{MS_M}{MS_R}$$

The Theory – Regression Output

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The Predictors – Assessing Individual Predictors

$$t = \frac{b_{\text{observed}} - b_{\text{expected}}}{SE_b}$$

$$= \frac{b_{\text{observed}}}{SE_b}$$

Degrees of freedom (df) = N – p – 1

Where:
N = Total sample size
p = number of predictors

A.2 Critical values of the t-distribution

	Two-Tailed Test		One-Tailed Test	
df	*0.05*	*0.01*	*0.05*	*0.01*
1	12.71	63.66	6.31	31.82
2	4.30	9.92	2.92	6.96
3	3.18	5.84	2.35	4.54
4	2.78	4.60	2.13	3.75
5	2.57	4.03	2.02	3.36
6	2.45	3.71	1.94	3.14
7	2.36	3.50	1.89	3.00
8	2.31	3.36	1.86	2.90
9	2.26	3.25	1.83	2.82
10	2.23	3.17	1.81	2.76
11	2.20	3.11	1.80	2.72
12	2.18	3.05	1.78	2.68
13	2.16	3.01	1.77	2.65
14	2.14	2.98	1.76	2.62
15	2.13	2.95	1.75	2.60
16	2.12	2.92	1.75	2.58
17	2.11	2.90	1.74	2.57
18	2.10	2.88	1.73	2.55
19	2.09	2.86	1.73	2.54
20	2.09	2.85	1.72	2.53
21	2.08	2.83	1.72	2.52
22	2.07	2.82	1.72	2.51
23	2.07	2.81	1.71	2.50
24	2.06	2.80	1.71	2.49
25	2.06	2.79	1.71	2.49
26	2.06	2.78	1.71	2.48
27	2.05	2.77	1.70	2.47
28	2.05	2.76	1.70	2.47
29	2.05	2.76	1.70	2.46

Exercise 2.1 – Deriving the output

```
Call:
lm(formula = sales ~ 1, data = album_data)

Residuals:
   Min     1Q Median     3Q    Max 
-183.2  -55.7    6.8   56.8  166.8 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  193.200      5.706   33.86   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 80.7 on 199 degrees of freedom
```

$$\sigma_{\bar{x}} = \frac{s}{\sqrt{N}}$$

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N-1}}$$

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$$\sigma_{\bar{x}} = \frac{s}{\sqrt{N}}$$

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N-1}}$$

```
estimate <- mean(album_data$sales); estimate
standard_error <- sd(album_data$sales)/sqrt(nrow(album_data)); standard_error
t_value <- estimate/standard_error; t_value
residuals_sales <- album_data$sales - mean(album_data$sales); residuals_sales
quantile_residuals_sales <- quantile(residuals_sales); quantile_residuals_sales
residual_standard_error <- sd(residuals_sales); residual_standard_error
```

Exercise 2.2 – Deriving the output

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Call:
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$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N-1}}$$

$$1 - (1 - R^2)\frac{n-1}{n-p-1}$$

Exercise 2.2 – Deriving the output

```
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```

$$s = \sqrt{\frac{\sum (x_i - \overline{x})^2}{N-1}}$$

$$1 - (1 - R^2)\frac{n-1}{n-p-1}$$

```
RMSE <- sqrt(sum(residuals(album_lm_1)^2)/df.residual(album_lm_1)); RMSE
R2 <- cor(album_data$adverts, album_data$sales)^2; R2
R2_adjusted <- 1 - (1 - R2)*((200-1)/(200-1-1)); R2_adjusted
F_test <- anova(album_lm_0, album_lm_1); F_test

t_adverts <- 0.09612/0.009632; t_adverts
t_intercept <- 134.1/7.537; t_intercept

residuals <- quantile(album_lm_1$residuals); residuals
```

Exercise 3 – Anscombe Quartet

Thank you!