Assignment 8

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Download the latex code from

https://github.com/Ananthoju−Pranav−Sai/AI1103/blob/main/Assignment_8/main.tex

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Suppose (X_1, X_2) follows a bivariate normal distribution with

$$E(X_1) = E(X_2) = 0 \tag{0.0.1}$$
$$V(X_1) = V(X_2) = 2 \tag{0.0.2}$$
$$Cov(X_1, X_2) = -1 \tag{0.0.3}$$

If $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$, then $\Pr(X_1 - X_2 > 6) = ?$

1) $\Phi(-1)$
2) $\Phi(-3)$
3) $\Phi(\sqrt{6})$
4) $\Phi(-\sqrt{6})$

Solution

Given, (X_1, X_2) follows a bivariate normal distribution with

$$\mu_1 = \mu_2 = 0 \tag{0.0.4}$$
$$\sigma_1^2 = \sigma_2^2 = 2 \tag{0.0.5}$$
$$V_{12} = -1 \tag{0.0.6}$$
$$\rho = \frac{V_{12}}{\sigma_1\sigma_2} = \frac{-1}{2} \tag{0.0.7}$$

where ρ is correlation of x_1 and x_2
We define mean matrix μ

$$\mu = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{0.0.8}$$

and covariance matrix Σ as follows

$$\mathbf{\Sigma} = \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix} = \begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix} \tag{0.0.9}$$

The characteristic function of bivariate is given by

$$\phi_{X_1X_2}(t_1, t_2) = \mathbb{E}[e^{i(t_1X_1+t_2X_2)}] \tag{0.0.10}$$

which can also be written as

$$\phi_{X_1X_2}(t_1, t_2) = \exp(i\mathbf{u}^\top\mu - \frac{1}{2}\mathbf{u}^\top\mathbf{\Sigma}\mathbf{u}) \tag{0.0.11}$$

where

$$\mathbf{u} = \begin{pmatrix}t_1\\t_2\end{pmatrix} \tag{0.0.12}$$

by using (0.0.8) and (0.0.9) in (0.0.10) we get

$$\phi_{X_1X_2}(t_1, t_2) = \exp\left(0 - \frac{1}{2}\begin{pmatrix}t_1 & t_2\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}t_1\\t_2\end{pmatrix}\right) \tag{0.0.13}$$

$$\implies \phi_{X_1X_2}(t_1, t_2) = \exp\left(-\frac{1}{2}\begin{pmatrix}t_1 & t_2\end{pmatrix}\begin{pmatrix}2t_1 - t_2\\ 2t_2 - t_1\end{pmatrix}\right) \tag{0.0.14}$$

$$\implies \phi_{X_1X_2}(t_1, t_2) = e^{-\left(t_1^2 - t_1t_2 + t_2^2\right)} \tag{0.0.15}$$

Now consider a random variable Z defined as follows

$$Z = X_2 - X_1 \tag{0.0.16}$$

Then $\Pr(X_1 - X_2 > 6)$ can be written as $\Pr(Z < -6)$ characteristic function of Z can be written as

$$\phi_Z(t) = \mathbb{E}[e^{itZ}] \tag{0.0.17}$$
$$\implies \phi_Z(t) = \mathbb{E}[e^{it(X_2-X_1)}] \tag{0.0.18}$$

clearly form (0.0.10), (0.0.18) can be written as

$$\phi_Z(t) = \phi_{X_1X_2}(-t, t) \tag{0.0.19}$$

from (0.0.15)

$$\phi_Z(t) = e^{-\left(t^2+t^2+t^2\right)} \tag{0.0.20}$$
$$\implies \phi_Z(t) = e^{-3t^2} \tag{0.0.21}$$

Now we can find pdf of Z by using a variation of Gil-Peleaz inversion formula

$$f_Z(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itz}\phi_Z(t)\,dt \tag{0.0.22}$$
$$\implies f_Z(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\left(-3t^2 - itz\right)}\,dt \tag{0.0.23}$$

Using the result of Gaussian integral

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}\,dx = e^{b^2/4a}\sqrt{\frac{\pi}{a}} \tag{0.0.24}$$

in (0.0.23)

$$\implies f_Z(z) = \frac{1}{2\pi}e^{\frac{(-iz)^2}{12}}\sqrt{\frac{\pi}{3}} \tag{0.0.25}$$

$$\implies f_Z(z) = \frac{1}{\sqrt{12\pi}}e^{-\frac{z^2}{12}} \tag{0.0.26}$$

Now for $\Pr(Z < -6)$

$$\Pr(Z < -6) = \int_{-\infty}^{-6} f_Z(z)\,dz \tag{0.0.27}$$

$$= \int_{-\infty}^{-6} \frac{1}{\sqrt{12\pi}}e^{-\frac{z^2}{12}}\,dz \tag{0.0.28}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-6} \frac{1}{\sqrt{6}}e^{-\frac{z^2}{12}}\,dz \tag{0.0.29}$$

let $y = \frac{z}{\sqrt{6}}$ then (0.0.29) can be written as

$$\Pr(Z < -6) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{6}} e^{-\frac{y^2}{2}}\,dy \tag{0.0.30}$$

$$\implies \Pr(Z < -6) = \Phi(-\sqrt{6}) \tag{0.0.31}$$

$$\therefore \Pr(X_1 - X_2 > 6) = \Phi(-\sqrt{6}) \tag{0.0.32}$$