Assignment 8

Ananthoju Pranav Sai - AI20BTECH11004

Download the latex code from

https://github.com/Ananthoju−Pranav−Sai/AI1103/blob/main/Assignment_8/main.tex

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Suppose (X_1, X_2) follows a bivariate normal distribution with

$$E(X_1) = E(X_2) = 0 \tag{0.0.1}$$
$$V(X_1) = V(X_2) = 2 \tag{0.0.2}$$
$$Cov(X_1, X_2) = -1 \tag{0.0.3}$$

If $\Phi(x)=\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy$, then $\Pr(X_1 - X_2 > 6) = ?$

1) $\Phi(-1)$
2) $\Phi(-3)$
3) $\Phi(\sqrt{6})$
4) $\Phi(-\sqrt{6})$

Solution

Given, (X_1, X_2) follows a bivariate normal distribution with

$$\mu_1 = \mu_2 = 0 \tag{0.0.4}$$
$$\sigma_1^2 = \sigma_2^2 = 2 \tag{0.0.5}$$
$$Cov(X_1, X_2) = -1 \tag{0.0.6}$$
$$\rho = \frac{Cov(X_1, X_2)}{\sigma_1\sigma_2} = \frac{-1}{2} \tag{0.0.7}$$

where ρ is correlation of x_1 and x_2
We define mean matrix μ

$$\mu_{\mathbf{x}} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.0.8}$$

and covariance matrix Σ as follows

$$\mathbf{\Sigma_x} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{0.0.9}$$

Let $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ then,

$$X_2 - X_1 = \mathbf{u}^\top\mathbf{x} \tag{0.0.10}$$

Now consider a random variable Z defined as follows

$$Y = X_2 - X_1 \tag{0.0.11}$$
$$\implies \mathbf{z} = \mathbf{u}^\top\mathbf{x} \tag{0.0.12}$$

then $\mathbf{y}$ has normal distribution with mean

$$\mu_{\mathbf{y}} = \mathbf{u}^\top\mu_{\mathbf{x}} \tag{0.0.13}$$
$$\implies \mu_{\mathbf{y}} = \begin{pmatrix} 0 \end{pmatrix} \tag{0.0.14}$$

and covariance matrix is given by

$$\Sigma_{\mathbf{y}} = \mathbf{u}^\top\mathbf{\Sigma_x}\left(\mathbf{u}^\top\right)^\top \tag{0.0.15}$$
$$\implies \Sigma_{\mathbf{y}} = \begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{0.0.16}$$
$$\implies \Sigma_{\mathbf{y}} = \begin{pmatrix} 6 \end{pmatrix} \tag{0.0.17}$$

Therefore, $Y \sim \mathcal{N}(\mu = 0, \sigma^2 = 6)$
The Standard Normal, often written Z, is a Normal with $\mu = 0$ and $\sigma^2 = 1$. Thus, $Z \sim \mathcal{N}(\mu = 0, \sigma^2 = 1)$

Now $\Pr(X_1 - X_2 > 6)$ can be written as $\Pr(Y < -6)$

$$\Pr(Y < -6) = \Pr\left(\frac{Y - \mu_y}{\sigma_y} < \frac{-6 - \mu_y}{\sigma_y}\right) \tag{0.0.18}$$
$$\implies \Pr(Y < -6) = \Pr\left(Z < \frac{-6}{\sqrt{6}}\right) \tag{0.0.19}$$
$$\implies \Pr(Y < -6) = \Pr\left(Z < -\sqrt{6}\right) \tag{0.0.20}$$
$$\implies \Pr(Y < -6) = \Phi(-\sqrt{6}) \tag{0.0.21}$$
$$\therefore \Pr(X_1 - X_2 > 6) = \Phi(-\sqrt{6}) \tag{0.0.22}$$