

Assignment 8

Ananthoju Pranav Sai - AI20BTECH11004

Download all python codes from

https://github.com/Ananthoju-Pranav-Sai/AI1103/tree/main/Assignment_8/Codes

and latex codes from

https://github.com/Ananthoju-Pranav-Sai/AI1103/blob/main/Assignment_8/main.tex

UGC JUNE 2017 MATH SET A Q 57

Suppose (X_1, X_2) follows a bivariate normal distribution with

$$E(X_1) = E(X_2) = 0 \quad (0.0.1)$$

$$V(X_1) = V(X_2) = 2 \quad (0.0.2)$$

$$\text{Cov}(X_1, X_2) = -1 \quad (0.0.3)$$

If $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$, then $\Pr(X_1 - X_2 > 6) = ?$

- 1) $\Phi(-1)$
- 2) $\Phi(-3)$
- 3) $\Phi(\sqrt{6})$
- 4) $\Phi(-\sqrt{6})$

SOLUTION

Definition 0.1. $\mathbf{x} \sim \mathcal{N}(\mu_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ is a bivariate random vector given by

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad (0.0.4)$$

$$\mu_{\mathbf{x}} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad (0.0.5)$$

$$\Sigma_{\mathbf{x}} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \quad (0.0.6)$$

where $\Sigma_{\mathbf{x}}$ is covariance matrix of \mathbf{x} and ρ is correlation of X_1 and X_2 which is given by

$$\rho = \frac{\text{Cov}(X_1, X_2)}{\sigma_1\sigma_2} \quad (0.0.7)$$

Theorem 0.1. Let \mathbf{x} be a $K \times 1$ multivariate normal random vector with mean $\mu_{\mathbf{x}}$ and covariance matrix $\Sigma_{\mathbf{x}}$. Let \mathbf{a} be an $L \times 1$ real vector and \mathbf{B} an $L \times K$

full rank real matrix with $K > L$. Then the $L \times 1$ random vector \mathbf{y} defined by

$$\mathbf{y} = \mathbf{a} + \mathbf{B}\mathbf{x} \quad (0.0.8)$$

has a multivariate normal distribution with mean

$$\mu_{\mathbf{y}} = \mathbf{a} + \mathbf{B}\mu_{\mathbf{x}} \quad (0.0.9)$$

and covariance matrix

$$\Sigma_{\mathbf{y}} = \mathbf{B}\Sigma_{\mathbf{x}}\mathbf{B}^T \quad (0.0.10)$$

Proof. The joint moment generating function of \mathbf{x} is

$$M_{\mathbf{x}}(\mathbf{t}) = \mathbb{E}[\exp(\mathbf{t}^T \mathbf{x})] \quad (0.0.11)$$

$$\Rightarrow M_{\mathbf{x}}(\mathbf{t}) = \exp\left(\mathbf{t}^T \mu_{\mathbf{x}} + \frac{1}{2} \mathbf{t}^T \Sigma_{\mathbf{x}} \mathbf{t}\right) \quad (0.0.12)$$

Therefore, the joint moment generating function of \mathbf{y} is

$$M_{\mathbf{y}}(\mathbf{t}) = \mathbb{E}[\exp(\mathbf{t}^T \mathbf{y})] \quad (0.0.13)$$

$$= \mathbb{E}[\exp(\mathbf{t}^T (\mathbf{a} + \mathbf{B}\mathbf{x}))] \quad (0.0.14)$$

$$= \exp(\mathbf{t}^T \mathbf{a}) \mathbb{E}[(\mathbf{t}^T \mathbf{B}\mathbf{x})] \quad (0.0.15)$$

$$= \exp(\mathbf{t}^T \mathbf{a}) \mathbb{E}[(\mathbf{B}^T \mathbf{t})^T \mathbf{x}] \quad (0.0.16)$$

$$= \exp(\mathbf{t}^T \mathbf{a}) M_{\mathbf{x}}(\mathbf{B}^T \mathbf{t}) \quad (0.0.17)$$

$$= \exp(\mathbf{t}^T \mathbf{a}) \exp\left((\mathbf{B}^T \mathbf{t})^T \mu_{\mathbf{x}} + \frac{1}{2} (\mathbf{B}^T \mathbf{t})^T \Sigma_{\mathbf{x}} \mathbf{B}^T \mathbf{t}\right) \quad (0.0.18)$$

$$= \exp\left(\mathbf{t}^T (\mathbf{a} + \mathbf{B}\mu_{\mathbf{x}}) + \frac{1}{2} \mathbf{t}^T \mathbf{B} \Sigma_{\mathbf{x}} \mathbf{B}^T \mathbf{t}\right) \quad (0.0.19)$$

which is the moment generating function of a multivariate normal distribution with mean $\mathbf{a} + \mathbf{B}\mu_{\mathbf{x}}$ and covariance matrix $\mathbf{B}\Sigma_{\mathbf{x}}\mathbf{B}^T$.

$\therefore \mu_{\mathbf{y}} = \mathbf{a} + \mathbf{B}\mu_{\mathbf{x}}$ and $\Sigma_{\mathbf{y}} = \mathbf{B}\Sigma_{\mathbf{x}}\mathbf{B}^T$ \square

Theorem 0.2. If (X_1, X_2) follow bivariate distribution then $\Pr(X_1 - X_2 > \alpha)$ is given by

$$\Phi\left(\frac{(\mu_1 - \mu_2) - \alpha}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}\right).$$

Proof. Let $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$$X_2 - X_1 = \mathbf{u}^\top \mathbf{x} \quad (0.0.20)$$

Now consider a random variable Y defined as follows

$$Y = X_2 - X_1 \quad (0.0.21)$$

$$\implies y = \mathbf{u}^\top \mathbf{x} \quad (0.0.22)$$

then Y has normal distribution with mean

$$\mu_y = \mathbf{u}^\top \mu_{\mathbf{x}} \quad (0.0.23)$$

and variance is given by

$$\sigma_y^2 = \mathbf{u}^\top \Sigma_{\mathbf{x}} (\mathbf{u}^\top)^\top \quad (0.0.24)$$

$$\implies \sigma_y^2 = \mathbf{u}^\top \Sigma_{\mathbf{x}} \mathbf{u} \quad (0.0.25)$$

$$\therefore Y \sim \mathcal{N}(\mu = \mathbf{u}^\top \mu_{\mathbf{x}}, \sigma^2 = \mathbf{u}^\top \Sigma_{\mathbf{x}} \mathbf{u})$$

The Standard Normal, often written Z , is a Normal with $\mu = 0$ and $\sigma^2 = 1$. Thus, $Z \sim \mathcal{N}(\mu = 0, \sigma^2 = 1)$

Now $\Pr(X_1 - X_2 > \alpha)$ can be written as $\Pr(Y < -\alpha)$

$$\Pr(Y < -\alpha) = \Pr\left(\frac{Y - \mu_y}{\sigma_y} < \frac{-\alpha - \mu_y}{\sigma_y}\right) \quad (0.0.26)$$

$$\implies \Pr(Y < -\alpha) = \Pr\left(Z < \frac{-(\alpha + \mathbf{u}^\top \mu_{\mathbf{x}})}{\sqrt{\mathbf{u}^\top \Sigma_{\mathbf{x}} \mathbf{u}}}\right) \quad (0.0.27)$$

$$\therefore \Pr(X_1 - X_2 > \alpha) = \Phi\left(\frac{-(\alpha + \mathbf{u}^\top \mu_{\mathbf{x}})}{\sqrt{\mathbf{u}^\top \Sigma_{\mathbf{x}} \mathbf{u}}}\right) \quad (0.0.28)$$

Lemma 0.3.

$$\mu_y = \mathbf{u}^\top \mu_{\mathbf{x}} \quad (0.0.29)$$

$$\implies \mu_y = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad (0.0.30)$$

$$\implies \mu_y = (\mu_2 - \mu_1) \quad (0.0.31)$$

Lemma 0.4.

$$\sigma_y^2 = \mathbf{u}^\top \Sigma_{\mathbf{x}} \mathbf{u} \quad (0.0.32)$$

$$\implies \sigma_y^2 = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (0.0.33)$$

$$\implies \sigma_y^2 = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) \quad (0.0.34)$$

$$\therefore \Pr(X_1 - X_2 > \alpha) = \Phi\left(\frac{(\mu_1 - \mu_2) - \alpha}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}\right) \quad (0.0.35)$$

□

Given (X_1, X_2) follow a bivariate normal distribution with

$$\mu_1 = \mu_2 = 0 \quad (0.0.36)$$

$$\sigma_1^2 = \sigma_2^2 = 2 \quad (0.0.37)$$

$$\text{Cov}(X_1, X_2) = -1 \quad (0.0.38)$$

$$\rho = \frac{\text{Cov}(X_1, X_2)}{\sigma_1\sigma_2} = \frac{-1}{2} \quad (0.0.39)$$

$$\mu_{\mathbf{x}} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (0.0.40)$$

$$\Sigma_{\mathbf{x}} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad (0.0.41)$$

for $\Pr(X_1 - X_2 > 6)$ we can use (0.0.28)

$$\Pr(X_1 - X_2 > 6) = \Phi\left(\frac{-(6 + \mathbf{u}^\top \mu_{\mathbf{x}})}{\sqrt{\mathbf{u}^\top \Sigma_{\mathbf{x}} \mathbf{u}}}\right) \quad (0.0.42)$$

where $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ using python for vector multiplications we get

$$\implies \Pr(X_1 - X_2 > 6) = \Phi\left(\frac{-6}{\sqrt{6}}\right) \quad (0.0.43)$$

$$\therefore \Pr(X_1 - X_2 > 6) = \Phi(-\sqrt{6}) \quad (0.0.44)$$