Assignment 8

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Download the latex code from

https://github.com/Ananthoju−Pranav−Sai/AI1103/blob/main/Assignment_8/main.tex

UGC June 2017 Math set A Q 57

Suppose (X_1, X_2) follows a bivariate normal distribution with

$$E(X_1) = E(X_2) = 0 \tag{0.0.1}$$
$$V(X_1) = V(X_2) = 2 \tag{0.0.2}$$
$$Cov(X_1, X_2) = -1 \tag{0.0.3}$$

If $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$, then $\Pr(X_1 - X_2 > 6) = ?$

1) $\Phi(-1)$
2) $\Phi(-3)$
3) $\Phi(\sqrt{6})$
4) $\Phi(-\sqrt{6})$

Solution

Given, (X_1, X_2) follows a bivariate normal distribution with

$$\mu_1 = \mu_2 = 0 \tag{0.0.4}$$
$$\sigma_1^2 = \sigma_2^2 = 2 \tag{0.0.5}$$
$$V_{12} = -1 \tag{0.0.6}$$
$$\rho = \frac{V_{12}}{\sigma_1\sigma_2} = \frac{-1}{2} \tag{0.0.7}$$

where ρ is correlation of x_1 and x_2
We define mean matrix μ

$$\mu_{\mathbf{x}} = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \tag{0.0.8}$$

and covariance matrix Σ as follows

$$\mathbf{\Sigma_x} = \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix} = \begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix} \tag{0.0.9}$$

Let $\mathbf{u} = \begin{pmatrix}-1 & 1\end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix}X_1\\ X_2\end{pmatrix}$ then,

$$X_2 - X_1 = \mathbf{ux} \tag{0.0.10}$$

Now consider a random variable Z defined as follows

$$Z = X_2 - X_1 \tag{0.0.11}$$
$$\implies \mathbf{z} = \mathbf{ux} \tag{0.0.12}$$

then $\mathbf{z}$ has normal distribution with mean

$$\mu_{\mathbf{z}} = \mathbf{u}\mu_{\mathbf{x}} \tag{0.0.13}$$
$$\implies \mu_{\mathbf{z}} = \begin{pmatrix}0\end{pmatrix} \tag{0.0.14}$$

and covariance matrix is given by

$$\Sigma_{\mathbf{z}} = \mathbf{u\Sigma_x u}^\top \tag{0.0.15}$$
$$\implies \Sigma_{\mathbf{z}} = \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} \tag{0.0.16}$$
$$\implies \Sigma_{\mathbf{z}} = \begin{pmatrix}6\end{pmatrix} \tag{0.0.17}$$

pdf of z is given by

$$f_{\mathbf{z}}(z) = \frac{1}{\sqrt{2\pi\left|\Sigma_{\mathbf{z}}\right|}}\exp\left(-\frac{1}{2}(\mathbf{z}-\mu_{\mathbf{z}})\Sigma_{\mathbf{z}}^{-1}(\mathbf{z}-\mu_{\mathbf{z}})^\top\right) \tag{0.0.18}$$
$$\implies f_{\mathbf{z}}(z) = \frac{1}{\sqrt{2\pi(6)}}\exp\left(-\frac{1}{2}(z)\left(\frac{1}{6}\right)(z)\right) \tag{0.0.19}$$
$$\implies f_{\mathbf{z}}(z) = \frac{1}{\sqrt{12\pi}}e^{-\frac{z^2}{12}} \tag{0.0.20}$$

Then $\Pr(X_1 - X_2 > 6)$ can be written as $\Pr(Z < -6)$

Now for $\Pr(Z < -6)$

$$\Pr(Z < -6) = \int_{-\infty}^{-6} f_Z(z)\,dz \tag{0.0.21}$$
$$= \int_{-\infty}^{-6}\frac{1}{\sqrt{12\pi}}e^{-\frac{z^2}{12}}\,dz \tag{0.0.22}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-6}\frac{1}{\sqrt{6}}e^{-\frac{z^2}{12}}\,dz \tag{0.0.23}$$

let $y = \frac{z}{\sqrt{6}}$ then (0.0.23) can be written as

$$\Pr(Z < -6) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{6}} e^{-\frac{y^2}{2}}\, dy \quad (0.0.24)$$

$$\implies \Pr(Z < -6) = \Phi(-\sqrt{6}) \quad (0.0.25)$$

$$\therefore \Pr(X_1 - X_2 > 6) = \Phi(-\sqrt{6}) \quad (0.0.26)$$