CHARACTERISTIC FUNCTIONS

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Characteristic function

The Characteristic function of random variable X is defined as

$$C_X(t) = \mathbb{E}[e^{itX}] \tag{1}$$

which can also be written as

$$C_X(t) = \int e^{itx} \, d\mathbb{P}_X \tag{2}$$

If X is a continuous random variable with density function $\mathrm{f}_X(x), then$

$$C_X(t) = \int e^{itx} f_X(x) \, dx \tag{3}$$

Elementary properties of characteristic functions

- If $Y = aX + b$,then $C_Y(t) = e^{ibt} C_X(at)$.
- If X and Y are independent random variables and $Z = X + Y$, then $C_Z(t) = C_X(t) C_Y(t)$

Example Question

Question

Find the characteristic function of $Y = \sum_{r=1}^{n} a_r X_r$, where $a_1, a_2, ..., a_n$ are constants and $X_1, X_2, X_3, .., X_n$ are random variables, each of which takes the values -1 and 1 with probability $\frac{1}{2}$. Taking $a_r = 2^{-r}$ for each r, show that Y converges in distribution to uniform distribution on (-1,1).

Solution

Given,

$$Y = \sum_{r=1}^{n} a_r X_r \tag{4}$$

The characteristic function of a random variable Y is defined as

$$C_Y(t) = E[e^{itY}] \tag{5}$$

$$\implies C_Y(t) = E[e^{it\sum_{r=1}^{n} a_r X_r}] \tag{6}$$

$$\implies C_Y(t) = \prod_{r=1}^{n} E[e^{ita_r X_r}] \tag{7}$$

$$\implies C_Y(t) = \prod_{r=1}^{n} C_{X_r}(a_r t) \tag{8}$$

Solution contd.

By taking $a_r = 2^{-r}$ we get,

$$C_Y(t) = \prod_{r=1}^{n} C_{X_r}\left(\frac{t}{2^r}\right) \tag{9}$$

As random variables $X_r's$ follow discrete uniform distribution with only two possible outcomes ($X_r = -1$ and $X_r = 1$) the characteristic function of X_r is

$$C_{X_r}(t) = \sum_k e^{ikt} \Pr(X_r = k) \tag{10}$$

$$\implies C_{X_r}(t) = \frac{e^{-it}}{2} + \frac{e^{it}}{2} \tag{11}$$

$$\implies C_{X_r}(t) = \frac{1 + e^{2it}}{2e^{it}} \tag{12}$$

$$\implies C_{X_r}\left(\frac{t}{2^r}\right) = \frac{1 + e^{2i\left(\frac{t}{2^r}\right)}}{2e^{i\left(\frac{t}{2^r}\right)}} \tag{13}$$

Solution contd.

using (13) in (9)

$$C_Y(t) = \prod_{r=1}^{n} \left(\frac{1 + e^{2i\left(\frac{t}{2^r}\right)}}{2e^{i\left(\frac{t}{2^r}\right)}} \right) \tag{14}$$

$$\implies C_Y(t) = \frac{\left(1 + e^{it}\right)\left(1 + e^{i\left(\frac{t}{2}\right)}\right)\ldots.\left(1 + e^{i\left(\frac{t}{2^{n-1}}\right)}\right)}{2^n\left(e^{i\sum_{r=1}^{n}\frac{t}{2^r}}\right)} \tag{15}$$

$$\therefore C_Y(t) = \frac{\left(e^{2it} - 1\right)}{2^n e^{it\left(\frac{2^{n+1}-1}{2^{n+1}}\right)}\left(e^{\left(\frac{it}{2^{n-1}}\right)} - 1\right)} \tag{16}$$

Solution contd.

Now consider

$$\lim_{n \to \infty} C_Y(t) \tag{17}$$

using (16) in (17)

$$\implies \lim_{n \to \infty} \frac{\left(e^{2it} - 1\right)}{2^n e^{it\left(\frac{2^{n+1}-1}{2^{n+1}}\right)} \left(e^{\left(\frac{it}{2^{n-1}}\right)} - 1\right)} \tag{18}$$

$$\implies \lim_{n \to \infty} \frac{\left(e^{2it} - 1\right)}{2ite^{it\left(\frac{2^{n+1}-1}{2^{n+1}}\right)} \frac{\left(e^{\left(\frac{it}{2^{n-1}}\right)} - 1\right)}{\left(\frac{it}{2^{n-1}}\right)}} \tag{19}$$

Solution contd.

We know that,

$$\lim_{n\to\infty} e^{it\left(\frac{2^{n+1}-1}{2^{n+1}}\right)} = e^{it} \tag{20}$$

Now, for

$$\lim_{n\to\infty} \frac{\left(e^{\left(\frac{it}{2^{n-1}}\right)} - 1\right)}{\left(\frac{it}{2^{n-1}}\right)} \tag{21}$$

Let $x = \frac{it}{2^{n-1}}$

$$\implies \lim_{x\to 0} \frac{(e^x - 1)}{x} = 1 \tag{22}$$

Since, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$

Solution contd.

$$\therefore \lim_{n\to\infty} \frac{\left(e^{\left(\frac{it}{2^{n-1}}\right)}-1\right)}{\left(\frac{it}{2^{n-1}}\right)} = 1 \tag{23}$$

Using (23) and (20) in (19)

$$\lim_{n\to\infty} C_Y(t) = \frac{\left(e^{2it}-1\right)}{2ite^{it}} \tag{24}$$

Solution contd.

Now, let's assume that Y follows uniform distribution on (-1,1) then it's pdf can be written as

$$f_Y(y) = \begin{cases} \frac{1}{2} & -1 \leq y \leq 1 \\ 0 & otherwise \end{cases} \tag{25}$$

And it's cdf would be

$$F_Y(y) = \begin{cases} 0 & y < -1 \\ \frac{1+y}{2} & -1 \leq y \leq 1 \\ 1 & otherwise \end{cases} \tag{26}$$

Solution contd.

It's characteristic function would be

$$C_Y(t) = \int_{-\infty}^{\infty} e^{ity}.f_Y(y)\,dy \tag{27}$$

$$\implies C_Y(t) = \int_{-1}^{1} e^{ity}.\left(\frac{1}{2}\right)\,dy \tag{28}$$

$$\implies C_Y(t) = \left.\frac{e^{ity}}{2it}\right|_{-1}^{1} \tag{29}$$

$$\implies C_Y(t) = \frac{\left(e^{2it}-1\right)}{2ite^{it}} \tag{30}$$

So from (24) and (30) we conclude that as $(n \to \infty)$ Y converges in distribution to uniform distribution on (-1,1).

CDF plot

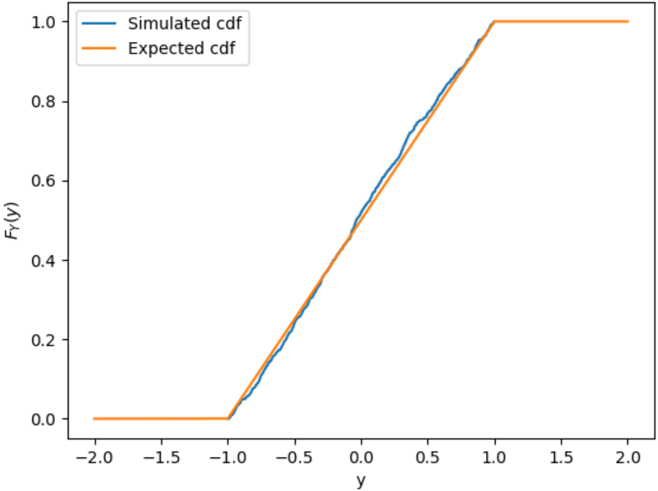

Figure: Simulated vs expected cdf plot of random variable Y