Assignment 8

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Download the latex code from

https://github.com/Ananthoju−Pranav−Sai/AI1103/blob/main/Assignment_8/main.tex

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Suppose (X_1, X_2) follows a bivariate normal distribution with

$$E(X_1) = E(X_2) = 0 \tag{0.0.1}$$
$$V(X_1) = V(X_2) = 2 \tag{0.0.2}$$
$$Cov(X_1, X_2) = -1 \tag{0.0.3}$$

If $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$, then $\Pr(X_1 - X_2 > 6) = ?$

1) $\Phi(-1)$
2) $\Phi(-3)$
3) $\Phi(\sqrt{6})$
4) $\Phi(-\sqrt{6})$

Solution

Given, (X_1, X_2) follows a bivariate normal distribution. So any random variable $Z = aX_1 + bX_2$ is normal.

So consider a random variable Z defined as follows

$$Z = X_2 - X_1 \tag{0.0.4}$$

Then $\Pr(X_1 - X_2 > 6)$ can be written as $\Pr(Z < -6)$

As $Z = X_2 - X_1$ mean of Z is,

$$\mu_Z = E(Z) \tag{0.0.5}$$
$$\implies \mu_Z = E(X_2) + E(X_1) \tag{0.0.6}$$
$$\implies \mu_Z = 0 \tag{0.0.7}$$

and variance of Z,

$$\sigma_Z^2 = Var(Z) \tag{0.0.8}$$
$$\implies \sigma_Z^2 = Var(X_2 - X_1) \tag{0.0.9}$$
$$\implies \sigma_Z^2 = Var(X_2) + Var(X_1) - 2Cov(X_1, X_2) \tag{0.0.10}$$
$$\implies \sigma_Z^2 = 6 \tag{0.0.11}$$

As Z follows normal distribution its charactersitic function can be written as

$$\phi_Z(t) = e^{\left(i\mu_Z t - \sigma_Z^2 t^2/2\right)} \tag{0.0.12}$$
$$\implies \phi_Z(t) = e^{\left(0 - 6t^2/2\right)} \tag{0.0.13}$$
$$\implies \phi_Z(t) = e^{-3t^2} \tag{0.0.14}$$

Now we can find pdf of Z by using a variation of Gil-Peleaz inversion formula

$$f_Z(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itz}\phi_Z(t)\,dt \tag{0.0.15}$$
$$\implies f_Z(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\left(-3t^2 - itz\right)}\,dt \tag{0.0.16}$$

Using the result of Gaussian integral

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}\,dx = e^{b^2/4a}\sqrt{\frac{\pi}{a}} \tag{0.0.17}$$

in (0.0.16)

$$\implies f_Z(z) = \frac{1}{2\pi}e^{\frac{(-iz)^2}{12}}\sqrt{\frac{\pi}{3}} \tag{0.0.18}$$
$$\implies f_Z(z) = \frac{1}{\sqrt{12\pi}}e^{-\frac{z^2}{12}} \tag{0.0.19}$$

Now for $\Pr(Z < -6)$

$$\Pr(Z < -6) = \int_{-\infty}^{-6} f_Z(z)\,dz \tag{0.0.20}$$
$$= \int_{-\infty}^{-6} \frac{1}{\sqrt{12\pi}}e^{-\frac{z^2}{12}}\,dz \tag{0.0.21}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-6} \frac{1}{\sqrt{6}}e^{-\frac{z^2}{12}}\,dz \tag{0.0.22}$$

let $y = \frac{z}{\sqrt{6}}$ then (0.0.22) can be written as

$$\Pr(Z < -6) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{6}} e^{-\frac{y^2}{2}}\,dy \tag{0.0.23}$$
$$\implies \Pr(Z < -6) = \Phi(-\sqrt{6}) \tag{0.0.24}$$
$$\therefore \Pr(X_1 - X_2 > 6) = \Phi(-\sqrt{6}) \tag{0.0.25}$$