Assignment 8

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Download all python codes from

https://github.com/Ananthoju−Pranav−Sai/AI1103/tree/main/Assignment_8/Codes

and latex codes from

https://github.com/Ananthoju−Pranav−Sai/AI1103/blob/main/Assignment_8/main.tex

UGC June 2017 Math set A Q 57

Suppose (X_1, X_2) follows a bivariate normal distribution with

$$E(X_1) = E(X_2) = 0 \tag{0.0.1}$$
$$V(X_1) = V(X_2) = 2 \tag{0.0.2}$$
$$Cov(X_1, X_2) = -1 \tag{0.0.3}$$

If $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy$, then $\Pr(X_1 - X_2 > 6) = ?$

1) $\Phi(-1)$
2) $\Phi(-3)$
3) $\Phi(\sqrt{6})$
4) $\Phi(-\sqrt{6})$

Solution

Definition 0.1. $\mathbf{x} \sim \mathcal{N}(\mu_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ *is a bivariate random vector given by*

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{0.0.4}$$

$$\mu_{\mathbf{x}} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \tag{0.0.5}$$

$$\mathbf{\Sigma_x} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \tag{0.0.6}$$

where $\mathbf{\Sigma_x}$ *is covariance matrix of* $\mathbf{x}$ *and* ρ *is correlation of* X_1 *and* X_2 *which is given by*

$$\rho = \frac{Cov(X_1, X_2)}{\sigma_1\sigma_2} \tag{0.0.7}$$

Theorem 0.1. *Let* $\mathbf{x}$ *be a* $K \times 1$ *multivariate normal random vector with mean* μ_x *and covariance matrix* $\mathbf{\Sigma_x}$. *Let* $\mathbf{a}$ *be an* $L \times 1$ *real vector and* $\mathbf{B}$ *an* $L \times K$ *full-rank real matrix. Then the* $L \times 1$ *random vector* $\mathbf{y}$ *defined by*

$$\mathbf{y} = \mathbf{a} + \mathbf{Bx} \tag{0.0.8}$$

has a multivariate normal distribution with mean

$$\mu_{\mathbf{y}} = \mathbf{a} + \mathbf{B}\mu_{\mathbf{x}} \tag{0.0.9}$$

and covariance matrix

$$\mathbf{\Sigma_y} = \mathbf{B\Sigma_x B}^\top \tag{0.0.10}$$

Proof. The joint moment generating function of $\mathbf{x}$ is

$$M_{\mathbf{x}}(\mathbf{t}) = \mathbb{E}[\exp(\mathbf{t}^\top\mathbf{x})] \tag{0.0.11}$$
$$\implies M_{\mathbf{x}}(\mathbf{t}) = \exp\left(\mathbf{t}^\top\mu_{\mathbf{x}} + \frac{1}{2}\mathbf{t}^\top\mathbf{\Sigma_x t}\right) \tag{0.0.12}$$

Therefore, the joint moment generating function of $\mathbf{y}$ is

$$M_{\mathbf{y}}(\mathbf{t}) = \mathbb{E}[\exp(\mathbf{t}^\top\mathbf{y})] \tag{0.0.13}$$
$$= \mathbb{E}[\exp(\mathbf{t}^\top(\mathbf{a} + \mathbf{Bx}))] \tag{0.0.14}$$
$$= \exp(\mathbf{t}^\top\mathbf{a})\mathbb{E}[(\mathbf{t}^\top\mathbf{Bx})] \tag{0.0.15}$$
$$= \exp(\mathbf{t}^\top\mathbf{a})\mathbb{E}[(\mathbf{B}^\top\mathbf{t})^\top\mathbf{x}] \tag{0.0.16}$$
$$= \exp(\mathbf{t}^\top\mathbf{a})M_{\mathbf{x}}(\mathbf{B}^\top\mathbf{t}) \tag{0.0.17}$$
$$= \exp(\mathbf{t}^\top\mathbf{a})\exp\left((\mathbf{B}^\top\mathbf{t})^\top\mu_{\mathbf{x}} + \frac{1}{2}(\mathbf{B}^\top\mathbf{t})^\top\mathbf{\Sigma_x B}^\top\mathbf{t}\right) \tag{0.0.18}$$
$$= \exp\left(\mathbf{t}^\top(\mathbf{a} + \mathbf{B}\mu_{\mathbf{x}}) + \frac{1}{2}\mathbf{tB\Sigma_x B}^\top\mathbf{t}^\top\right) \tag{0.0.19}$$

which is the moment generating function of a multivariate normal distribution with mean $\mathbf{a} + \mathbf{B}\mu$ and covariance matrix $\mathbf{B\Sigma_x B}^\top$.

$\therefore \mu_{\mathbf{y}} = \mathbf{a} + \mathbf{B}\mu_{\mathbf{x}}$ and $\mathbf{\Sigma_y} = \mathbf{B\Sigma_x B}^\top$ □

Theorem 0.2. *If* (X_1, X_2) *follow bivariate distribution then* $\Pr(X_1 - X_2 > \alpha)$ *is given by* $\Phi\left(\frac{(\mu_1-\mu_2)-\alpha}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}}\right)$.

Proof. Let $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$$X_2 - X_1 = \mathbf{u}^\top \mathbf{x} \tag{0.0.20}$$

Now consider a random variable Y defined as follows

$$Y = X_2 - X_1 \tag{0.0.21}$$
$$\implies y = \mathbf{u}^\top \mathbf{x} \tag{0.0.22}$$

then Y has normal distribution with mean

$$\mu_y = \mathbf{u}^\top \mu_\mathbf{x} \tag{0.0.23}$$

and variance is given by

$$\sigma_y^2 = \mathbf{u}^\top \mathbf{\Sigma_x} \left(\mathbf{u}^\top\right)^\top \tag{0.0.24}$$
$$\implies \sigma_y^2 = \mathbf{u}^\top \mathbf{\Sigma_x} \mathbf{u} \tag{0.0.25}$$

$\therefore\ Y \sim \mathcal{N}(\mu = \mathbf{u}^\top \mu_\mathbf{x}, \sigma^2 = \mathbf{u}^\top \mathbf{\Sigma_x u})$

The Standard Normal, often written Z, is a Normal with $\mu = 0$ and $\sigma^2 = 1$. Thus, $Z \sim \mathcal{N}(\mu = 0, \sigma^2 = 1)$

Now $\Pr(X_1 - X_2 > \alpha)$ can be written as $\Pr(Y < -\alpha)$

$$\Pr(Y < -\alpha) = \Pr\left(\frac{Y - \mu_y}{\sigma_y} < \frac{-\alpha - \mu_y}{\sigma_y}\right) \tag{0.0.26}$$
$$\implies \Pr(Y < -\alpha) = \Pr\left(Z < \frac{-(\alpha + \mathbf{u}^\top \mu_\mathbf{x})}{\sqrt{\mathbf{u}^\top \mathbf{\Sigma_x u}}}\right) \tag{0.0.27}$$
$$\therefore \Pr(X_1 - X_2 > \alpha) = \Phi\left(\frac{-(\alpha + \mathbf{u}^\top \mu_\mathbf{x})}{\sqrt{\mathbf{u}^\top \mathbf{\Sigma_x u}}}\right) \tag{0.0.28}$$

Lemma 0.3.

$$\mu_y = \mathbf{u}^\top \mu_\mathbf{x} \tag{0.0.29}$$
$$\implies \mu_\mathbf{y} = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \tag{0.0.30}$$
$$\implies \mu_\mathbf{y} = \begin{pmatrix} \mu_2 - \mu_1 \end{pmatrix} \tag{0.0.31}$$

Lemma 0.4.

$$\sigma_y^2 = \mathbf{u}^\top \mathbf{\Sigma_x u} \tag{0.0.32}$$
$$\implies \sigma_y^2 = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{0.0.33}$$
$$\implies \sigma_y^2 = \left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right) \tag{0.0.34}$$

$$\therefore \Pr(X_1 - X_2 > \alpha) = \Phi\left(\frac{(\mu_1 - \mu_2) - \alpha}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}\right) \tag{0.0.35}$$

□

Given (X_1, X_2) follow a bivariate normal distribution with

$$\mu_1 = \mu_2 = 0 \tag{0.0.36}$$
$$\sigma_1^2 = \sigma_2^2 = 2 \tag{0.0.37}$$
$$Cov(X_1, X_2) = -1 \tag{0.0.38}$$
$$\rho = \frac{Cov(X_1, X_2)}{\sigma_1\sigma_2} = \frac{-1}{2} \tag{0.0.39}$$

for $\Pr(X_1 - X_2 > 6)$ we can use the above theorem as follows

$$\Pr(X_1 - X_2 > \alpha) = \Phi\left(\frac{(\mu_1 - \mu_2) - \alpha}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}\right) \tag{0.0.40}$$
$$\implies \Pr(X_1 - X_2 > 6) = \Phi\left(\frac{(0 - 0) - 6}{\sqrt{2 + 2 - 2\left(\frac{-1}{2}\right)2}}\right) \tag{0.0.41}$$
$$\implies \Pr(X_1 - X_2 > 6) = \Phi\left(\frac{-6}{\sqrt{6}}\right) \tag{0.0.42}$$
$$\therefore \Pr(X_1 - X_2 > 6) = \Phi(-\sqrt{6}) \tag{0.0.43}$$