Assignment - 1

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Subject - Discrete Mathematics

Course - B.Tech CSE (B)

Date
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Main Numerical Problem

A System is designed to automatically verify logical statements used in access control. Each access rule is represented as a propositional formula. Consider the formula:

$[(p \rightarrow q) \wedge (\neg r \vee q)] \rightarrow (p \rightarrow r)$

Where
- p : User is authenticated
- q : User has a valid token
- r : User can access confidential files

Tasks:
1. Construct the truth table for this expression.
2. Determine whether the statement is a tautology, contradiction, or ~~contingency~~ contingency.
3. Show how this logic can prevent unauthorized access when q is false.

Solution We are given the formula:

$((p \rightarrow q) \wedge (\neg r \vee q)) \rightarrow (p \rightarrow r)$

Where
- p = user is authenticated
- q = user has a valid token
- r = user can access confidential files

1) Truth Table

p	q	r	$p \to q$	$\neg r$	$\neg r \vee q$	$(p \to q) \wedge (\neg r \vee q)$	$p \to r$
F	F	F	T	T	T	T	T
F	F	T	T	F	F	F	T
F	T	F	T	T	T	T	T
F	T	T	T	F	T	T	T
T	F	F	F	T	T	F	F
T	F	T	F	F	F	F	T
T	T	F	T	T	T	T	F
T	T	T	T	F	T	T	T

The formula is false only in row 7 (when $p = T$, $q = T$, $r = F$).

2) Classification

Since the formula is true in some rows and false in atleast one, it is a Contingency.

3) How it prevents unauthorized access when q = False.

- If q = False (no valid token), then the expression $(\neg r \vee q)$ becomes just $\neg r$
- In that case, the antecedent can never force $(p \to r)$ to hold unless all conditions match.
- This means: just being authenticated (p = true) is not enough. Without a token (q = False), access (r = true) is not guaranteed.

So the logic ensures that no access is given without a Valid token, which helps block unauthorized users.

Sub-Problem 1 : Propositional Equivalences

Prove that the following propositions are logically equivalent

$\neg(p \to q) \equiv p \wedge \neg q$

Tasks :

1) Prove the above using truth tables,
2) Use logical Identities (e.g. Implication, De Morgan's law) to transform the left side to Right side.

Solution Truth Table

p	q	$p \to q$	$\neg(p \to q)$	$p \wedge \neg q$
f	f	T	f	f
f	T	T	f	f
T	f	f	T	T
T	T	T	f	f

Both Columns are the Same, So they are equivalent.

(B) logical Identities proof
Start with left side

1. $\neg(p \to q)$
2. Replace Implication $p \to q \equiv \neg p \vee q$, So $\neg(p \to q) \equiv \neg(\neg p \vee q)$.
3. Apply De Morgan: $\neg(\neg p \vee q) \equiv \neg(\neg p) \wedge \neg q$.
4. Simplify: $\neg(\neg p) \equiv p$, So result is $p \wedge \neg q$

Hence Proved.

Sub Problem 2: Predicate logic and Quantifiers

Let the domain be all students in a University. Let:

- Smart (x) : "x is Smart"
- Studies (x) : "x Studies regularly".

Given : "All Smart Students Study regularly."

Tasks :

1. Write the above in Symbolic form using quantifiers.
2. Negate the statement.
3. Test the truth table of both original and negated statements for a sample set of 3 students with known attributes.

Solution 1. Symbolic form

All. ~~Students~~ Smart Students Study regularly.

$\forall x(\text{Smart}(x) \rightarrow \text{Studies}(x))$.

2) Negation

~~∃x(Smart(x) ∧ ¬Studies~~

$\exists x(\text{Smart}(x) \land \neg \text{Studies}(x))$

Meaning: There exists a Smart Student who does not study regularly.

3) Example with 3 Students

Let Students = {Alice, Bob, Cara} with attributes

- Alice : Smart = T, Studies = T
- Bob : Smart = T, Studies = F
- Cara : Smart = F, Studies = T

Evaluate original Statement $\forall x(\text{Smart}(x) \rightarrow \text{Studies}(x))$.

- Alice : Smart $\rightarrow$ Studies = $T \rightarrow T = T$
- Bob : $T \rightarrow F = F$
- Cara : $F \rightarrow T = T$

Because Bob gives False, the Universal Statement is False.

Negated Statement $\exists x(\text{Smart}(x) \wedge \neg \text{Studies}(x))$:

- Bob Satisfies $\text{Smart}(\text{Bob}) \wedge \neg \text{Studies}(\text{Bob}) = T \wedge T = T$

So the negation is True for this Sample Set, Consistent with logic.

Sub Problem 3 : Nested Quantifiers & Rules of Inference

Let:

- $M(x, y)$: "x is a Mentor of y"
- $G(x)$: "x is a good mentor".

Date / DELTA Pg No

Given:

1. $\forall x \forall y (M(x,y) \rightarrow G(x))$
2. $M(Amit, Reena)$

Tasks:

1. Using rules of Inference, prove that Amit is a good Mentor.
2. Express the above Inference chain in Steps (Universal Instantiation, Modus Ponens, etc).

Solution- Given:

1. $\forall x \forall y (M(x,y) \rightarrow G(x))$ → "If x is Mentor of y, then x is a good Mentor" (for all x, y),"
2. $M(Amit, Reena)$ "Amit is Mentor of Reena"

Prove Amit is a good Mentor ($G(Amit)$):

1. From (1), by universal Instantiation (put x = Amit, y = Reena) we get:
   $M(Amit, Reena) \rightarrow G(Amit)$.
2. We have $M(Amit, Reena)$. given.
3. By Modus Ponens on Step 1 & 2, infer $G(Amit)$.

Proved - Amit is a good Mentor.

Sub-Problem 4: Set Theory operations

Let:

- $A = \{1, 2, 3, 4\}$
- $B = \{2, 4, 6, 8\}$
- $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (Universal Set)

Tasks:

1. Find:
   - $A \cup B$
   - $A \cap B$
   - $A - B$
   - A', B' (Complement w.r.t. U)
2. Verify: $(A \cup B)' = A' \cap B'$

Solution Given:

- $A = \{1, 2, 3, 4\}$
- $B = \{2, 4, 6, 8\}$
- $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

1. Compute Sets
- $A \cup B = \{1, 2, 3, 4, 6, 8\}$
- $A \cap B : \{2, 4\}$
- $A - B$: elements in A not in B $= \{1, 3\}$
- A' (Complement of A wrt U) $= U - A = \{5, 6, 7, 8, 9\}$
- $B' = U - B = \{1, 3, 5, 7, 9\}$

Date
DELTA Pg No

2. Verify De Morgan: $(A \cup B)' = A' \cap B'$.

- $A \cup B = \{1, 2, 3, 4, 6, 8\}$
- $(A \cup B)' = U - (A \cup B) = \{5, 7, 9\}$
- $A' \cap B' = \{5, 6, 7, 8, 9\} \cap \{1, 3, 5, 7, 9\} = \{5, 7, 9\}$

Since $(A \cup B)' = \{5, 7, 9\} = A' \cap B'$, De Morgan's law is verified for these sets.

Sub Problem 5: Relations - Properties and Matrix Representation

Let $A = \{1, 2, 3\}$ and define a relation R on A as: R
$\{(1,1), (2,2), (3,3), (1,2), (2,3)\}$

Tasks:

1. Represent relation R as a 0-1 Matrix
2. Determine whether R is:
   - Reflexive
   - Symmetric
   - Transitive
3. Draw the digraph of R.

Solutions Let $A = \{1, 2, 3\}$ and $R = \{(1,1) (2,2) (3,3), (1,2) (2,3)\}$

0-1 Matrix representation (rows = domain elements 1, 2, 3; column = Codomain (1, 2, 3)

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 \end{array}$$

(Entry at row i column j = 1 if $(i,j) \in R$, else 0).

2. Properties

- Reflexive. Yes - all $(1,1)$ $(2,2)$ $(3,3)$ are present
- Symmetric : No - $(1,2) \in R$ but $(2,1) \notin R$, so not symmetric.
- Transitive : No - $(1,2)$ and $(2,3)$ are in R but $(1,3)$ is not in R, violating transitivity.

3. Digraph

The digraph has nodes 1, 2, 3 Edges

- loop at 1, loop at 2, loop at 3
- directed edge $1 \to 2$
- directed edge $2 \to 3$.

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(1) - - -> (2) - - -> (3)
 ^                     |
 |                     |
 |                     |
 + - - - - - - - - - - +
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(loops at each node : 1, 2, 3)

- The digraph visually shows why R is not symmetric (no arrow 2→ and not transitive (there is no direct edge $1 \to 3$ in R).