Discrete Mathematics
BCSC 0010
Module 2
Predicate logic

(Predicates and Quantifiers)

Predicates and Quantifiers

- Propositional logic cannot adequately express the meaning of statements in mathematics and in natural language.

Example

- "Every computer connected to the university network is functioning properly."

Introduction

- **Predicate logic** is used to express the meaning of a wide range of statements in mathematics and computer science

- Predicates

- Quantifiers

Predicates

- "$x > 3$"
- "$x = y + 3$"
- "$x + y = z$"
- "computer x is under attack by an intruder,"
- "computer x is functioning properly,“

- These statements are **neither true nor false** when the values of the variables are not specified.

Predicates

- $x > 3$
- x is the subject of the statement.
- “is greater than 3” is predicate of the statement
- Predicate refers to a property that the subject of the statement can have.
- We can denote the statement "x is greater than 3" by $P(x)$,
- where P denotes the predicate "is greater than 3" and x is the variable.
- The statement $P(x)$ is also said to be the value of the propositional function P at x.
- Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

Example

- Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

- ***Solution:***
- We obtain the statement $P(4)$ by setting $x = 4$ in the statement "$x > 3$."
- Hence, $P(4)$, which is the statement "4 > 3," is true.
- However, $P(2)$, which is the statement "2 > 3," is false.

Example

Let *A(x)* denote the statement "Computer *x* is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH 1 are currently under attack by intruders. What are truth values of *A(CS1), A(CS2),* and *A* (MATH 1)?

•*Solution:*

- We obtain the statement *A(CS1)* by setting *x* = CS1 in the statement
- "Computer *x* is under attack by an intruder."
- Because CS1 is not on the list of computers currently under attack, we conclude that *A(CS1)* is false.
- Similarly, because CS2 and MATH I 1 are on the list of computers under attack, we know that *A(CS2)* and A(MATH1) are true.

Statements with more than one variable

- Consider the statement "$x = y + 3$."
- We can denote this statement by $Q(x, y)$, where x and y are variables and Q is the predicate.
- When values are assigned to the variables x and y, the statement $Q(x, y)$ has a truth value.

Example

Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

- ***Solution:***
- To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$.
- Hence, $Q(1,2)$ is the statement "$1 = 2 + 3$" which is false.
- The statement $Q(3,0)$ is the proposition "$3 = 0 + 3$" which is true.

Statement with n variables

- In general, a statement involving the *n* variables *x1, x2, ….. , xn* can be denoted by

$$P(x_1, x_2, \ldots, x_n)$$

- A statement of the form *P(x1,x2,……..xn)* is the value of the propositional function *P* at the n-tuple *(x1,x2,……..xn)* and *P* is also called a n-place predicate or a n-ary predicate.

Quantifiers

- When the variables in a **propositional function** are assigned values, the resulting statement becomes a **proposition** with a certain truth value.

- However, there is another important way, called quantification, to create a proposition from a propositional function.

- Quantification expresses the **extent** to which a **predicate is true** over a **range of elements**.

Quantifiers

- In English, the words *all, some, many, none,* and *few* are used in quantifications.

- **Two types of quantification**
- universal quantification: tells us that a predicate is true for **every** element under consideration,
- existential quantification: which tells us that there is **one or more element** under consideration for which the predicate is true.

- The area of logic that deals with **predicates and quantifiers** is called the predicate calculus.

THE UNIVERSAL QUANTIFIER

- The domain specifies the possible values of the variable *x*.

- When property is true for all values of a variable in a particular domain, **s**uch a statement is expressed using universal quantification.

- The meaning of the universal quantification of ***P(x)*** **changes** when we **change the domain.**

- The domain must always be **specified** when a universal quantifier is used; without it, the universal quantification of a statement is not defined.

- Domain is also referred as domain of discourse or the universe of discourse

The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier.** We read $\forall x P(x)$ as "for all $x P(x)$" or "for every $x P(x)$." An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$.

Example

Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Solution: Because $P(x)$ is true for all real numbers x, the quantification

$$\forall x P(x)$$

is true. ◀

Example

Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus

$$\forall x Q(x)$$

is false. ◀

Example

Suppose that $P(x)$ is "$x^2 > 0$." To show that the statement $\forall x P(x)$ is false where the universe of discourse consists of all integers, we give a counterexample. We see that $x = 0$ is a counterexample because $x^2 = 0$ when $x = 0$, so that x^2 is not greater than 0 when $x = 0$.

Note

When all the elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$—it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction

$$P(x_1) \land P(x_2) \land \cdots \land P(x_n),$$

because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

Example

What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution: The statement $\forall x P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

because the domain consists of the integers 1, 2, 3, and 4. Because $P(4)$, which is the statement "$4^2 < 10$," is false, it follows that $\forall x P(x)$ is false.

Example

What does the statement $\forall x N(x)$ mean if $N(x)$ is "Computer x is connected to the network" and the domain consists of all computers on campus?

Solution: The statement $\forall x N(x)$ means that for every computer x on campus, that computer x is connected to the network. This statement can be expressed in English as "Every computer on campus is connected to the network." ◀

THE EXISTENTIAL QUANTIFIER

- Many mathematical statements assert that there is an element with a certain property.

- Such statements are expressed using **existential quantification.**

- With existential quantification, we form a proposition that is true if and only if $P(x)$ is **true** for **at least** one value of x in the domain.

THE EXISTENTIAL QUANTIFIER

The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the **existential quantifier.**

THE EXISTENTIAL QUANTIFIER

A domain must always be specified when a statement $\exists x P(x)$ is used. Furthermore, the meaning of $\exists x P(x)$ changes when the domain changes. Without specifying the domain, the statement $\exists x P(x)$ has no meaning. The existential quantification $\exists x P(x)$ is read as

"There is an x such that $P(x)$,"
"There is at least one x such that $P(x)$,"

or

"For some $x P(x)$."

Besides the words "there exists," we can also express existential quantification in many other ways, such as by using the words "for some," "for at least one," or "there is."

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Example

Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

Solution: Because "$x > 3$" is sometimes true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists x P(x)$, is true. ◀

Example

Let $Q(x)$ denote the statement "$x = x + 1$." What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?

Solution: Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x\, Q(x)$, is false. ◀

Note

When all elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$— the existential quantification $\exists x P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n),$$

because this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

Example

What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution: Because the domain is $\{1, 2, 3, 4\}$, the proposition $\exists x P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Because $P(4)$, which is the statement "$4^2 > 10$," is true, it follows that $\exists x P(x)$ is true. ◀

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence then all logical operators from propositional calculus. For example, $\forall x\, P(x) \vee Q(x)$ is the disjunction of $\forall x\, P(x)$ and $Q(x)$. In other words, it means $(\forall x\, P(x)) \vee Q(x)$ rather than $\forall x (P(x) \vee Q(x))$.

Binding Variables

- When a quantifier is used on the variable *x*, we say that this occurrence of the variable is **bound.**
- An occurrence of a variable that is not bound by a quantifier is said to be **free.**

- **Example:**

In the statement $\exists x(x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable. This illustrates that in the statement $\exists x(x + y = 1)$, x is bound, but y is free.

Example

In the statement $\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$, all variables are bound. The scope of the first quantifier, $\exists x$, is the expression $P(x) \wedge Q(x)$ because $\exists x$ is applied only to $P(x) \wedge Q(x)$, and not to the rest of the statement. Similarly, the scope of the second quantifier, $\forall x$, is the expression $R(x)$. That is, the existential quantifier binds the variable x in $P(x) \wedge Q(x)$ and the universal quantifier $\forall x$ binds the variable x in $R(x)$.