CSL105: Discrete Mathematical Structures

Ragesh Jaiswal, CSE, IIT Delhi

Resolution Principle

- *Resolution Principle* is another way of showing that an argument is correct.
- Definitions:
  - Literal: A variable or a negation of a variable is called a literal.
  - Sum and Product: A disjunction of literals is called a sum and a conjunction of literals is called a product.
  - Clause: A disjunction of literals is called a clause.
  - Resolvent: For any two clauses C_1 and C_2, if there is a literal L_1 in C_1 that is complementary to literal L_2 in C_2, then delete L_1 and L_2 from C_1 and C_2 respectively and construct the disjunction of the remaining clauses. The constructed clause is a resolvent of C_1 and C_2.
    - $C_1 = P \vee Q \vee R$
    - $C_2 = \neg P \vee \neg S \vee T$
    - What is a resolvent of C_1 and C_2?

- *Resolution Principle* is another way of showing that an argument is correct.
- Definitions:
  - Literal: A variable or a negation of a variable is called a literal.
  - Sum and Product: A disjunction of literals is called a sum and a conjunction of literals is called a product.
  - Clause: A disjunction of literals is called a clause.
  - Resolvent: For any two clauses C_1 and C_2, if there is a literal L_1 in C_1 that is complementary to literal L_2 in C_2, then delete L_1 and L_2 from C_1 and C_2 respectively and construct the disjunction of the remaining clauses. The constructed clause is a resolvent of C_1 and C_2.
    - $C_1 = P \vee Q \vee R$
    - $C_2 = \neg P \vee \neg S \vee T$
    - What is a resolvent of C_1 and C_2? $Q \vee R \vee \neg S \vee T$

Theorem

Given two clauses C_1 and C_2, a resolvent C of C_1 and C_2 is a logical consequence of C_1 and C_2.

- Example: Modus ponens $(P \wedge (P \rightarrow Q) \rightarrow Q)$
  - C_1: P
  - C_2: $\neg P \vee Q$
  - The resolvent of C_1 and C_2 is Q which is a logical consequence of C_1 and C_2.

Theorem

Given two clauses C_1 and C_2, a resolvent C of C_1 and C_2 is a logical consequence of C_1 and C_2.

Definition (Resolution principle and refutation)

Given a set S of clauses, a (resolution) deduction of C from S is a finite sequence $C_1, ..., C_k$ of clauses such that each C_i either is a clause in S or a resolvent of clauses preceding C and $C_k = C$. A deduction of $\Box$ (empty clause) is called a *refutation*.

Theorem

Given two clauses C_1 and C_2, a resolvent C of C_1 and C_2 is a logical consequence of C_1 and C_2.

Definition (Resolution principle and refutation)

Given a set S of clauses, a (resolution) deduction of C from S is a finite sequence $C_1, ..., C_k$ of clauses such that each C_i either is a clause in S or a resolvent of clauses preceding C and $C_k = C$. A deduction of $\square$ (empty clause) is called a *refutation* of a proof of S.

- If there is an argument where $P_1, ..., P_r$ are the premises and C is the conclusion, to get a proof using resolution principle, put $P_1, ..., P_r$ in clause form and add to it $\neg C$ in clause form. From this sequence, if $\square$ can be derived, the argument is valid.

- If there is an argument where $P_1, ..., P_r$ are the premises and C is the conclusion, to get a proof using resolution principle, put $P_1, ..., P_r$ in clause form and add to it $\neg C$ in clause form. From this sequence, if $\square$ can be derived, the argument is valid.
- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

  - What are the clauses?

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

  - C_1: $\neg T \vee M \vee E$
  - C_2 : $\neg S \vee \neg E$
  - C_3 : T
  - C_4 : S
  - C_5 : $\neg M$

- Example:

$$\begin{array}{l} T \to (M \vee E) \\ S \to \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

- C_1: $\neg T \vee M \vee E$
- C_2 : $\neg S \vee \neg E$
- C_3 : T
- C_4 : S
- C_5 : $\neg M$
- C_6 : $\neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

  - C_1: $\neg T \vee M \vee E$
  - C_2 : $\neg S \vee \neg E$
  - C_3 : T
  - C_4 : S
  - C_5 : $\neg M$
  - C_6 : $\neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)
  - C_7 : $M \vee \neg S$ (resolvent of C_3 and C_6)

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

  - C_1: $\neg T \vee M \vee E$
  - C_2 : $\neg S \vee \neg E$
  - C_3 : T
  - C_4 : S
  - C_5 : $\neg M$
  - C_6 : $\neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)
  - C_7 : $M \vee \neg S$ (resolvent of C_3 and C_6)
  - C_8 : M (resolvent of C_4 and C_7)

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

  - C_1: $\neg T \vee M \vee E$
  - $C_2 : \neg S \vee \neg E$
  - $C_3 : T$
  - $C_4 : S$
  - $C_5 : \neg M$
  - $C_6 : \neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)
  - $C_7 : M \vee \neg S$ (resolvent of C_3 and C_6)
  - $C_8 : M$ (resolvent of C_4 and C_7)
  - $C_9 : \square$ (resolvent of C_5 and C_8)
- Hence, from the resolution principle, the argument is valid.

Rules of Inference for Quantified Statements

Rule of inference	Name
$\dfrac{\forall x\ P(x)}{\therefore ?}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore ?}$	Universal generalization
$\dfrac{\exists x\ P(x)}{\therefore ?}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore ?}$	Existential generalization
$\dfrac{\begin{array}{l}\forall x(P(x) \rightarrow Q(x)) \\ P(a) \text{ where } a \text{ is a particular element in the domain}\end{array}}{\therefore ?}$	Universal modus ponens
$\dfrac{\begin{array}{l}\forall x(P(x) \rightarrow Q(x)) \\ \neg Q(a) \text{ where } a \text{ is a particular element in the domain}\end{array}}{\therefore ?}$	Universal modus tollens

Table: Rules of inference for quantified statements

Rule of inference	Name
$\forall x\ P(x)$ ——— $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c ——— $\therefore \forall x\ P(x)$	Universal generalization
$\exists x\ P(x)$ ——— $\therefore P(x)$ for some element c	Existential instantiation
$P(c)$ for some element c ——— $\therefore \exists x\ P(x)$	Existential generalization
$\forall x(P(x) \rightarrow Q(x))$ $P(a)$ where a is a particular element in the domain ——— $\therefore Q(a)$	Universal modus ponens
$\forall x(P(x) \rightarrow Q(x))$ $\neg Q(a)$ where a is a particular element in the domain ——— $\therefore \neg P(a)$	Universal modus tollens

Table: Rules of inference for quantified statements

- Use rules of inference for quantified statements to show the premises "*A student in this class has not read the book,*" and "*Everyone in this class passed the first exam*" imply the conclusion "*Someone who has passed the first exam has not read the book.*"

End