P231: Mathematical Methods in Graduate Physics

Flip Tanedo
flip.tanedo@ucr.edu
Department of Physics & Astronomy, University of California, Riverside, CA 92521

Abstract

This is a crash course on mathematical methods necessary to succeed in the first-year physics graduate curriculum at UC Riverside. The focus is how to solve differential equations using Green's functions.

Contents

1 Introduction: Why mathematical methods?

Physics 231: Methods of Theoretical Physics is a course for first-year physics and astronomy graduate students. It is a 'crash course' in mathematical methods necessary for graduate lec 01

courses in electrodynamics, quantum mechanics, and statistical mechanics. It is a *boot camp* rather than a rigorous theorem–proof mathematics class. Where possible, the emphasis is on physical intuition rather than mathematical precision.

1.1 Green's functions

Our primary goal is to solve linear differential equations:

$$\mathcal{O}f(x) = s(x) \ . \tag{1.1}$$

In this equation, $\mathcal{O}$ is a *differential operator* that encodes some kind of physical dynamics[1], $s(x)$ is the *source* of those dynamics, and $f(x)$ is the system's physical *response* that we would like to determine. The solution to this equation is:

$$f(x) = \mathcal{O}^{-1}s(x) \ . \tag{1.2}$$

Simply writing that is deeply unsatisfying! In this course, we think carefully about what $\mathcal{O}^{-1}$ actually *means* and how we can calculate it. As you may have guessed, $\mathcal{O}^{-1}$ is the **Green's function** for the differential operator $\mathcal{O}$.

We approach problem by analogy to linear algebra, where a linear transformation[2] A acts on a vector to give equations like

$$A\mathbf{v} = \mathbf{w} \ , \tag{1.3}$$

whose solution is

$$\mathbf{v} = A^{-1}\mathbf{w} \ . \tag{1.4}$$

We connect the notion of a linear differential operator to a matrix in an infinite dimensional space to give a working definition of $\mathcal{O}^{-1}$. We then pull out a bag of tricks from complex analysis to actually solve $\mathcal{O}^{-1}s(x)$ given $\mathcal{O}$ and $s(x)$.

1.2 This is not what I expected from a math methods course

This is a course in mathematical methods for *physicists*. We will not solve *every* class of differential equation that is likely to pop up in your research careers[3]—that's neither feasible nor particularly enjoyable. This is also not a course in formal proofs—there are plenty of excellent textbooks for you to learn those formal proofs to your heart's content[4]. The goal of this course is to weave together ideas that are not often connected explicitly in undergraduate physics courses in the United States: linear algebra, differential equations, complex analysis.

[1] A **differential operator** is just something built out of derivatives that can act on a function. The differential operator may contain coefficients that depend on the variable that we are differentiating with respect to; for example, $\mathcal{O} = (d/dx)^2 + 3x\,(d/dx)$. Pop quiz: is this operator *linear*? The first term is squared...

[2] Recall that as physicists, 'linear transformation' is a fancy way of saying 'matrix.'

[3] That would be a course on mathematical methods for *engineers*.

[4] ... and as a graduate student, you should feel well equipped and encouraged to learn all of the necessary material *you* need for *your* research and interests—whether or not they show up in your coursework.

These ideas are not necessarily new—in fact, I *expect* you have seen many them often—but rather we will take a big view of how the interconnection of these ideas come up over and over again in our description of nature.

Do not be surprised if we only mention Bessel functions in passing. Do not think less of our efforts if we do not determine Wronksians or go beyond a single Riemann sheet. As graduate students, it is *your* responsibility to be able to grab your favorite textbook to apply mathematics as needed to your research. *This course* is about the larger narrative that is not often shared explicitly in those books. It is the 'knack for math' that physicists are, as a culture, rather proud of. It is what tends to make us employable in Silicon Valley while simultaneously terrible at splitting the bill at a restaurant.

1.3 The totally not-mathematical idea of mathematical niceness

I find it useful to appeal to the notion of a **nice** mathematical situation. This is not a formal idea, and it is one many things mathematicians find ridiculous about me. But as a physicist, the concept of mathematical *niceness* is helpful.

The physical systems that we spend the most time thinking about are all *nice*. While our mathematical cousins may spend years proving every exceptional case to a theorem, we tend to be happy to push onward as long as mathematical results are true for the *nice* cases. Nice mathematical models make tidy predictions. Then we can Taylor expand about these nice predictions to make better predictions. When doing this, we sometimes say *perturbation theory* multiple times in case someone watching us does not think are being rigorous enough.

This is not to say that nature cares at all about our physical models. Every once in a while, we *do* have to worry about the exceptional cases because our models fail to accommodate what is *actually* happening in nature. Those scenarios are the most interesting of all. That's when our mathematical formalism grabs us by the collar and says, *listen to me—something important is happening and it probably has to do with nature!* This often happens when a calculation tells us that a physical result is infinite.

Exercise 1.1 *Consider the potential that an electron feels in the hydrogen atom:*

$$V(r) = -\frac{\alpha}{r} \ . \tag{1.5}$$

As the electron–proton separation goes to zero, $r \to 0$, the potential goes to infinity. Classical electrodynamics is telling us that something curious is happening. What actually happens? (And why didn't you ask this question when you were in high school?)

In this course we focus on *nice* functions and *nice* operators and *nice* boundary conditions, etc. For the most part, this is what we need to make progress on our physical models and it's worth spending our time learning to work with *nice* limits. Leave the degenerate cases to the mathematicians for now. Eventually, though, you may find yourself in a situation where physics demands *not nice* mathematics. In that case—and only when the physics demands it—you will be ready to poke and prod at the mathematical curiosity until the underlying *physics* reason for the not-niceness is apparent.

1.4 Physics versus Mathematics

Let's make one point clear: lec 02

$$\text{Physics} \neq \text{Mathematics} \ . \tag{1.6}$$

This is a truth in many different respects[5]:

- Physicists are rooted in experimental results[6].
- Physicists Taylor expand to their hearts' content—sometimes even when the expansion is not formally justified[7].
- Physicists use explicit coordinates, mathematicians abhor this. Even worse, we pick a basis and decorate every tensor with indices[8].
- Physicists seek to uncover a truth about *this* universe.

1.5 The most important binary relation

When we write equations, the symbol that separates the left-hand side from the right-hand side is a binary relation. We use binary relations like $=$ or $\neq$. Sometimes to make a point we'll write $\cong$ or $\equiv$ or $\doteq$ to mean something like 'definition' or 'tautologically equivalent to' or some other variant of *even more equal than equal*.

As physicists the most important binary relation is none of those things[9]. Usually what we really care about is in $\sim$.[10] This tells how how something *scales*. If I double a quantity on the right-hand side, how does the quantity on the left-hand side scale? Does it depend linearly? Quadratically? Non-linearly? The answer encodes something important about the underlying physics of the system. It's the reason why *imagine the cow is a sphere* is a popular punchline in a joke about physicists.

By the way, implicit in this is the idea that in this class, we will not care about stray factors of 2. As my adviser used to say, if you're worried about a factor of 2, then your additional homework is to figure out that factor of 2.[11]

1.6 Units

There is another way in which physics is different from mathematics. It is far more prosaic. *Quantities in physics have units.* We don't just deal with numbers, we deal with kilograms, electron volts, meters. It turns out that dimensional analysis is a big part of what we do as physicists.

[5] The astronomer Fritz Zwicky would perhaps call this a *spherical truth*; no matter how you look at it, the statement is still true.

[6] Even theorists? *especially* theorists.

[7] https://johncarlosbaez.wordpress.com/2016/09/21/struggles-with-the-continuum-part-6/

[8] Those who are not trained may be intimidated by physics because of all the indices we use. Ironically, physicists are often intimidated by mathematics because of the conspicuous absence of any indices.

[9] https://xkcd.com/2343/

[10] I use this the same way as $\propto$, which is completely different from 'approximately,' $\approx$.

[11] That being said, you're reading these notes and find an error, do let me know about it.

2 Dimensional Analysis

You may be be surprised how far you can go in physics by thinking deeply about dimensional analysis. Here we'll only get you started. To go one step further, you may read more about the Buckingham Pi theorem[12] or dive into neat applications[13]. lec 03

2.1 Converting Units

Imagine that you have three apples. This is a number (three) an a unit (apple). The meaning of the unit depends on what you're using it to measure. For example, if apples are \$1 each, then you could use an apple as a unit of currency. The way to do this is to simply *multiply by one*:

$$(3 \text{ apples}) \times \left(\frac{\$\ 1}{\text{apple}} \right) = \$3 \ . \tag{2.1}$$

We have used the fact that the exchange rate is simply the statement that

$$1 \text{ apple} = \$1 \qquad \Rightarrow \qquad 1 = \frac{\$1}{1 \text{ apple}} \ . \tag{2.2}$$

You can do a similar thing for [kilo-]calories or any other conversion rate.

All that matters is that the conversion is constant. Indeed, the constants of nature make very good 'exchange rates.' For example, in high-energy physics we like to use **natural units**. This is the curious statement that

$$\hbar = c = 1 \ . \tag{2.3}$$

At face value, this doesn't make sense. $\hbar$ has units of action, c is a speed, and 1 is dimensionless. However, because nature gives us a *fundamental* unit of action and a *fundamental* unit of speed, we may use them as conversion factors (exchange rates),

$$c = 3 \times 10^{10} \text{ cm/s} \ . \tag{2.4}$$

If $c = 1$, then this means

$$1 \text{ s} = 3 \times 10^{10} \text{ cm} \ . \tag{2.5}$$

This, in turn, connects a unit of time to a unit of distance. By measuring time, the constant c automatically gives us an associated distance. The physical relevance of the distance is tied to the nature of the fundamental constant: one second (or 'light-second') is the distance that a photon travels in one second. Observe that this only works because c is a constant.

[12] https://aapt.scitation.org/doi/10.1119/1.1987069

[13] https://aapt.scitation.org/doi/full/10.1119/1.3535586, http://inspirehep.net/record/153032?ln=en

2.2 Quantifying units

We use the notation that a physical quantity Q has **dimension** $[Q]$ that can be expressed in terms of units of length, mass, and time:

$$[Q] = L^a M^b T^c \ . \tag{2.6}$$

The dimension is the statement of the powers a, b, and c. You may want to also include units of, say, electric charge. Sticklers may pontificate about whether electric charge formally carries a new unit or not.

Example 2.1 *What are the units of force? We remember that* $\mathbf{F} = m\mathbf{a}$*, so*

$$[\mathbf{F}] = [m][\mathbf{a}] = M \times LT^{-2} = L^1 M^1 T^{-2} \ . \tag{2.7}$$

Life is even easier in **natural units**, where $c = 1$ means that units of length and time are 'the same' and $\hbar = 1$ means that units of time and energy (mass) are inversely related. In natural units, one typically write $[Q]$ to mean the mass-dimension of a quantity. To revert back to conventional units, one simply multiplies by appropriate factors of $1 = c$ and $1 = \hbar$.

Example 2.2 *What are the units of force in natural units? From* (2.7) *we multiply by one to convert length and time into mass dimensions:*

$$[\mathbf{F}] = [c^{-3}\hbar\mathbf{F}] = M^2 \ . \tag{2.8}$$

In natural units we say $[\mathbf{F}] = 2$*. Recall that energy and mass have the same dimension, which you may recall from the Einstein relation* $E^2 = m^2c^4 + p^2c^2$*.*

2.3 Usage: Sanity Check

The simplest use of dimensional analysis is to check your work. The following expression is obviously wrong:

$$1 + (3 \text{ cm}) \ . \tag{2.9}$$

This does not make sense. You cannot sum terms with different dimensions. Similarly, $\sin(3 \text{ cm})$ does not make sense. What about $e^{5 \text{ cm}}$? This doesn't make sense because

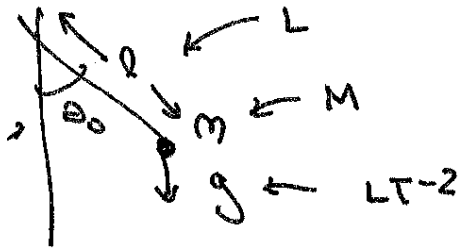
$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots \tag{2.10}$$

Since each term comes with a different power of x, the argument of the exponential must be dimensionless.

Exercise 2.1 *Consider the energy spectrum of light emitted from some constant source—a distant star, the ongoing annihilation of dark matter in the galactic center, a laser in the Hemmerling lab. The spectrum encodes how many photons are emitted per unit time. We can plot this spectrum as a curve on a graph. We can even normalize the curve so that it integrates to one photon. This means we only care about the distribution of energy, not the absolute amount. The horizontal axis of such a plot is the photon energy. What are the units of the vertical axis?*

2.4 Usage: Solving problems

Here's a common problem in introductory physics. Assume you have a pendulum with some [sufficiently small] initial displacement θ_0. What's the period, τ of the pendulum? We draw a picture like this:



From dimensional analysis, we know that the period has dimensions of time, $[\tau] = T$. The problem gives us a length $[\ell] = L$ and the gravitational acceleration, $[g] = LT^{-2}$. Note that $[\theta_0] = 1$ is dimensionless. This means that the only way to form a quantity with dimensions of time is to use $g^{-1/2}$. This leaves us with a leftover $L^{-1/2}$, which we can fix by inserting a square root of ℓ:

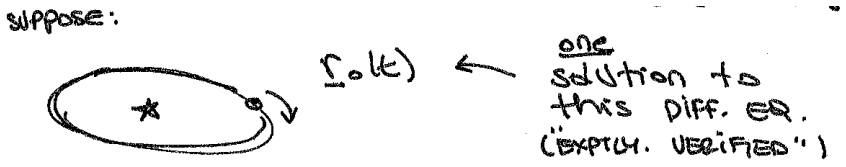
$$\tau \sim g^{-1/2}\ell^{1/2} \ . \tag{2.11}$$

If we wanted to be fancy, we can make this an equal sign by writing a function of the other dimensionless quantities in the problem:

$$\tau = f(\theta_0)\sqrt{\frac{\ell}{g}} \ . \tag{2.12}$$

2.5 Scaling

A large part of physics has to do with scaling relations. Here's a somewhat contrived example of how this works[14]. Suppose you have some static, central potential $U(\mathbf{r})$. Maybe it's some planet orbiting a star.



The force law gives:

$$m\ddot{\mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}} \ . \tag{2.13}$$

[14]This is adapted from section 11 of V. I. Arnold's *Mathematical Methods of Classical Mechanics*, one of my favorite differential geometry textbooks because it's disguised as a book on mechanics.

Suppose we are given a solution, $\mathbf{r}_0(t)$. Perhaps this is a trajectory that is experimentally verified. Dimensional analysis gives a way to scale this solution into other solutions. For example, let us scale time by defining a new variable t':

$$t \equiv \alpha t' \ . \tag{2.14}$$

If the potential is static, then only the left-hand side of the force law changes. Even though the right-hand side formally has dimensions of time $\sim T^{-2}$, it does not transform because those units are carried in a constant, perhaps G_N, not a $(d/dt)^2$ like the left-hand side. The left-hand side of the force law gives:

$$m\left(\frac{d}{dt}\right)^2 \mathbf{r}_0(t) = m\alpha^{-2}\left(\frac{d}{dt'}\right)^2 \mathbf{r}_0(\alpha t') \ . \tag{2.15}$$

This begs us to define a new mass $m' = m\alpha^{-2}$. We thus have

$$m'\left(\frac{d}{dt'}\right)^2 \mathbf{r}_0(\alpha t') = -\frac{\partial U}{\partial \mathbf{r}_0} \ . \tag{2.16}$$

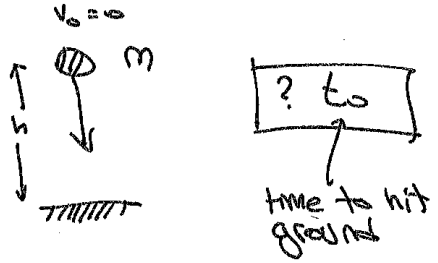
What this tells us is that $\mathbf{r}_1(t') \equiv \mathbf{r}_0(\alpha t')$ is a solution in the same potential that traces the same trajectory but at α times the speed and with mass m'. Changing labels $t' \to t$ for a direct comparison:

$$m'\left(\frac{d}{dt}\right)^2 \mathbf{r}_1(t) = -\frac{\partial U}{\partial \mathbf{r}_1} \ , \tag{2.17}$$

which is indeed[15] (2.13) with a new mass m' and a trajectory $\mathbf{r}_1(t') \equiv \mathbf{r}_0(\alpha t')$. For example, if $\alpha = 2$, then $\mathbf{r}_1(t)$ traces the same trajectory at double the velocity with one fourth of the mass.

2.6 Error Estimates

This section is based on a lovely *American Journal of Physics* article by Craig Bohren.[16] Let's go back to another high school physics problem.



[15] We were able to swap $\mathbf{r}_0$ with $\mathbf{r}_1$ simply because U only depends on the position.
[16] https://doi.org/10.1119/1.1574042

Suppose you drop a mass m from height h that is initially at rest. How long before this hits the ground? You can integrate the force equation to get

$$t_0 = \sqrt{\frac{2h}{g}} \ . \tag{2.18}$$

This is the *exact* answer *within our model* of the system. The model made several assumptions. The mass is a point mass, the gravitational acceleration is constant at all positions, there is no air resistance, etc. In fact, we *know* that if we do an experiment, our result will almost certainly *not* be t_0. All we know is that t_0 is probably a good approximation of what the actual answer is.

How good of an approximation is it?

One way to do this is to do the next-to-leading order ('NLO') calculation, taking into account a more realistic (and hence more complicated) model and then compare to t_0. But this is stupid. Why do we need to do a *hard* calculation to justify doing an *easy* one? If we're going to do the hard calculation anyway, what's the point of ever doing the easy one?

What we really want is an error estimate. The error is

$$\epsilon = \frac{t_1 - t_0}{t_0} \ . \tag{2.19}$$

This is a dimensionless quantity that determines how far off t_0 is from a more realistic calculation, t_1. Ideally we don't actually have to do work to get t_1.

Let's assume that we're not completely nuts and that we're in a regime where the error is small[17]. Then the error is a function of some dimensionless parameters, ξ, in the system. We define these ξ so that as $\xi \to 0$, $\epsilon(\xi) \to 0$. In other words, the approximation gets better as the ξ are made smaller. By Taylor expansion:

$$\epsilon(\xi) = \epsilon(0) + \epsilon'(0)\xi + \mathcal{O}(\xi^2) \ . \tag{2.20}$$

By assumption $\epsilon(0) = 0$ and $\mathcal{O}(\xi^2)$ is small. We can then make a reasonable *assumption* that the dimensionless value $\epsilon'(0)$ is $\mathcal{O}(1)$. This tells us that the error goes like $\epsilon(\xi) \sim \xi$.

By the way $\mathcal{O}(1)$ is read "order one" and is fancy notation for the order of magnitude. Numbers like 0.6, 2, and π are all $\mathcal{O}(1)$. A number like $4\pi^2$, on the other hand, is $\mathcal{O}(10)$. The assumption that a dimensionless number is $\mathcal{O}(1)$ is reasonable. When nature gives you a dimensionless parameter that is both (a) important and (b) very different from $\mathcal{O}(1)$, then there's a good chance that it's trying to tell you something about your model. Good examples of this are the cosmological constant, the strong CP phase, and the electroweak hierarchy problem[18].

[17] Note the error has to be dimensionless in order for us to be able to call it 'small,' otherwise it begs the question of 'small with respect to what¿

[18] There are also 'bad' examples. The ratio of the angular size of the moon to the angular size of the sun is unity to very good approximation. This is quite certainly a coincidence. Our universe appears to be in an epoch where the density of matter, radiation, and dark energy all happen to be in the same ballpark. Our cosmological models imply that this is purely a coincidence. It would be very curious if this were not the case. As an exercise, you can explore (and critique) the appearance of the anthropic principle in physics.

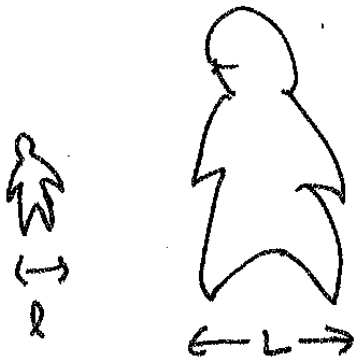
Here's how it works in practice. One effect that we miss in our toy calculation of t_0 is that the earth is round with radius R. This means that assuming a constant g is an approximation. We have two choices for a dimensionless parameter ξ:

$$\xi = \frac{h}{R} \qquad \text{or} \qquad \xi = \frac{R}{h} \ . \tag{2.21}$$

There is an obvious choice: $\xi = h/R$, because we know that as h is made smaller (drop the ball closer to the ground) or R becomes bigger (larger radius of Earth) then the constant g approximation gets better. We thus expect that the corrections from the position-dependence of g go like $\mathcal{O}(h/R)$.

2.7 Bonus: Allometry

There's a fun topic called **allometry**. This is basically dimensional analysis applied to biology. A typical example is to consider two people who have roughly the same shape but different characteristic lengths, ℓ and L:



Exercise 2.2 *If both people exercised at the same rate, which one loses more absolute weight? By how much? Let's assume that weight loss is primarily from the conversion of organic molecules into carbon dioxide.*

Exercise 2.3 *David Hu won his first IgNobel prize for determining that mammals take about 21 seconds to urinate, largely independently of their size*[19]*. Can you use dimensional analysis to argue why this would be the case? It may be helpful to refer to the paper*[20]*; as you read this, figure out which terms are negligible (and in what limits), identify the assumptions of the mathematical model (scaling of the bladder and urethra), and prove the approximate scaling relation. Make a note to yourself of which steps were non-trivial and where one may have naively mis-modeled the system. By the way, David Hu won a second IgNobel prize for understanding how wombats poop.*

[19] I learned about this in his excellent popular science book, *How To Walk on Water and Climb Up Walls.*
[20] https://doi.org/10.1073/pnas.1402289111

3 Linear Algebra Review

As physicists, linear algebra is part of our DNA, from the vector calculus in our first electrodynamics course to quantum mechanics. So why should we patronize ourselves with yet another review of linear algebra? We want to understand Green's functions the inverse of a matrix. The 'matrix' in question is the differential operator $\mathcal{O}$ in (1.1). This is important:

$$\text{differential operator} = \infty\text{-dimensional matrix} \,. \tag{3.1}$$

If differential operators are matrices, what vector space do these matrices act on? These matrices act on a space of functions, which turns out to be a vector space:

$$\text{function space} = \infty\text{-dimensional vector space} \,. \tag{3.2}$$

Don't be intimidated by terminology like *function space*; this is just an abstract place where functions live. Just recall back to your intuition from 3D Euclidean vector space, $\mathbb{R}^3$: any 3-vector $\mathbf{v}$ lives in the vector space $\mathbb{R}^3$. If we transform $\mathbf{v}$ by a linear transformation A, you get a new vector $\mathbf{w} = A\mathbf{v} \in \mathbb{R}^3$ that is also in the vector space.

Weird things can happen when we extend our intuition from finite things to infinite things[21], but for this course we'll try to draw as much intuition as we can from finite dimensional linear algebra to apply it to infinite dimensional function spaces.

3.1 The basics

A **linear transformation** A acts on a vector $\mathbf{v}$ as $A\mathbf{v}$. This transformation satisfies

$$A(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha A\mathbf{v} + \beta A\mathbf{w} \,. \tag{3.3}$$

Here α and β are numbers. This is conventionally matrix multiplication. The result is also a vector. One way that we like to think about vectors is as columns of elements:

$$\begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^N \end{pmatrix} \,, \tag{3.4}$$

where N is the **dimension** of the vector space. Our notation is that v^i refers to the i^{th} component of $\mathbf{v}$. Sometimes—as physicists—we refer to v^i as the vector itself, which is a slight abuse of notation that occasionally causes confusion.

In this course we always assume a nice orthonormal basis. In this case, $(\mathbf{v}+\mathbf{w})^i = v^i + w^i$.

Exercise 3.1 *Convince yourself that adding vectors becomes more complicated in polar coordinates. Namely,* $(\mathbf{v} + \mathbf{w})^i \neq v^i + w^i$.

[21] For example, the Hilbert Hotel puzzle.

Because the linear transformation of a vector is another vector, we know that the sequential application of linear transformations is itself a linear transformation. This is a bombastic way of saying that you can multiply matrices to produce a matrix. Here's how it works in two dimensions. A transformation that takes vectors into vectors takes the following form:

$$A = \begin{pmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{pmatrix} \ . \tag{3.5}$$

We have introduced upper and lower indices; for now treat this as a definition. This sometimes causes confusion. So here are some guidelines:

- Treat the upper and lower indices as a definition. The components of the linear transformation A are *defined* by $A^i{}_j$ where i is the row number and j is the column number.
- We have not yet explained the significance of the heights, but for now we mandate that the first index is always upper and the lower index is always lower. The following objects do not (yet) make sense: $A^1{}_2$ and *not* A_{12}, A^{12}, or $A_1{}^2$.
- We will soon define *additional machinery* to raise and lower indices shortly. This is takes us from a vector space to a metric space.
- The heights of the indices are a convenient shorthand notation that we will elucidate shortly; it is related to the choice of upper indices in (3.4).
- All of this may be familiar from special relativity. Extra credit if you realize that this should also be familiar from quantum mechanics.

If you're squeamish about the indices, don't worry: the elements of A have two indices, the first one is written a little higher than the second one. This notation is neither mathematics nor physics, it's a convention that we use for future convenience.

The action of a linear transformation A on a vector $\mathbf{v}$ is:

$$A\mathbf{v} = \begin{pmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} A^1{}_1 v^1 + A^1{}_2 v^2 \\ A^2{}_1 v^1 + A^2{}_2 v^2 \end{pmatrix} \ . \tag{3.6}$$

Look at this carefully. The components of the new vector $(A\mathbf{v})^i$ are sums. In each term, the second/lower index of an A element multiplies the component of $\mathbf{v}$ with the same index. The first/upper index of A tells you whether that term should is in $(A\mathbf{v})^1$ or $(A\mathbf{v})^2$.

A generic component of $(A\mathbf{v})$ is

$$(A\mathbf{v})^i = \sum_j A^i{}_j v^j = A^i{}_j v^j \quad \text{(Einstein convention)} \ . \tag{3.7}$$

On the right-hand side we use Einstein notation: *we implicitly sum over repeated upper/lower indices*. We will use this notation from now on. If you are at all in doubt about this, please work out the 2×2 case carefully and compare to the succinct notation above.

Exercise 3.2 *Consider three-dimensional Euclidean space, $\mathbb{R}^3$. A linear transformation A on this space is a 3×3 matrix with elements of the form $A^i{}_j$. Explicitly write out the second component of the vector $A\mathbf{v}$. This is a sum of three terms.*

If A and B are linear transformations, then $A + B$ is a linear transformation. The components of $A + B$ are simply the piecewise sum of the corresponding components of A and B:

$$(A + B)^i{}_j = A^i{}_j + B^i{}_j \ . \tag{3.8}$$

3.2 Linear Transformations and Vector Spaces

Let's be a little more pedantic. We need to move past the idea that a vector $\mathbf{v}$ is some *column of numbers*. A vector space is abstract and we need to to start thinking of vector spaces more generally. The layer of abstraction is encoded in the basis vectors, which we write as $\mathbf{e}_{(i)}$. For a space of dimension N, there are N such vectors indexed by the subscript. Let us more formally write the vector $\mathbf{v}$ as

$$\mathbf{v} = v^1\mathbf{e}_{(1)} + v^2\mathbf{e}_{(2)} + \cdots = v^i\mathbf{e}_{(i)} \ . \tag{3.9}$$

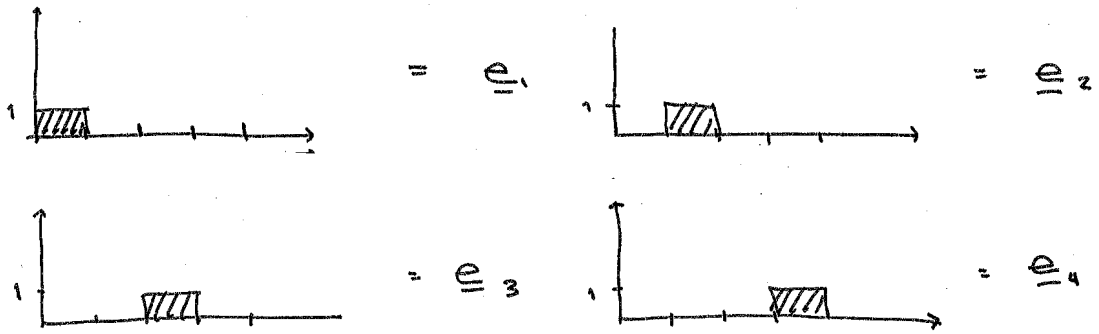
These basis vectors may be unit vectors in space. In the 'column of numbers' representation, they can be unit column vectors, e.g.

$$\mathbf{e}_{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{e}_{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \cdots \ . \tag{3.10}$$

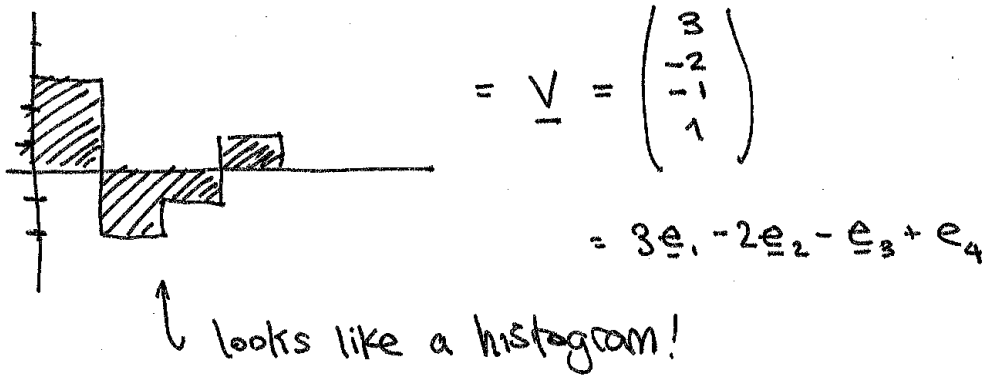
With this basis, (3.9) gives (3.4) But these may be more general objects. For example, you can specify a color of light by specifying the red/green/blue content. We could have $\mathbf{e}_{(1)}$ be a unit amount of red light, $\mathbf{e}_{(2)}$ be a unit amount of green light, and $\mathbf{e}_{(3)}$ be a unit amount of blue light. Then a 3-vector $\mathbf{v}$ would correspond to light of a particular color. This color space is a vector space.

3.3 A funny vector space: histogram space

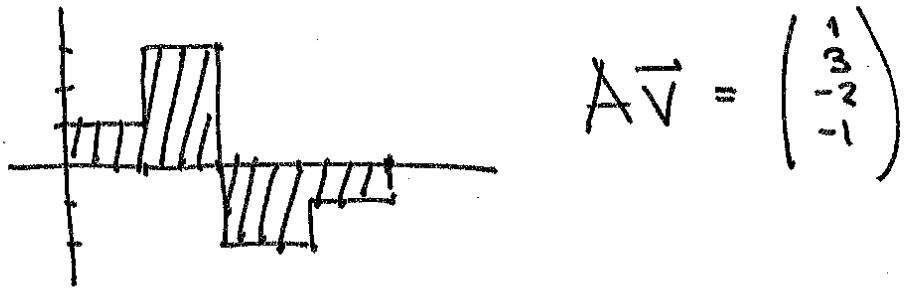
Here's a funny vector space that we're going to use as a pedagogical crutch. Imagine histogram-space. The basis vectors are:



This is a basis for a histogram over unit bins from $x = 0$ to $x = 4$. A vector in this space is, for example:



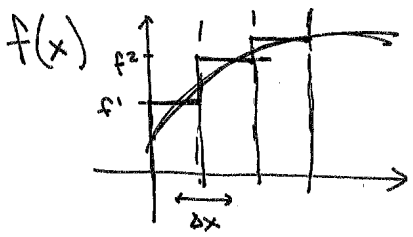
We can perform a linear transformation A on $\mathbf{v}$ which outputs another vector. Let's say it's this:



Exercise 3.3 *From the image above, can you derive what A is?*

The answer to the above exercise is *no*. Please make sure you convince yourself why: there are many different transformations that convert to old histogram into the new histogram. If you're not convinced: the matrix A is 4×4 and thus has 16 entries that we need to define. The matrix equation $A\mathbf{v} = \mathbf{w}$ for known vectors $\mathbf{v}$ and $\mathbf{w}$ encodes only four equations.

The power of this admittedly strange formalism is that we can think of these histograms as approximations of continuous functions:



Thus a vector in this approximate (discretized) *function* space is

$$\mathbf{f} = \begin{pmatrix} f^1 \\ f^2 \\ \vdots \\ f^N \end{pmatrix} . \tag{3.11}$$

3.4 Derivative Operators

Our discretized function space allows us to define a [forward] derivative[22]:

$$\mathbf{f}' = \frac{1}{\Delta x}\begin{pmatrix} f^2 - f^1 \\ f^3 - f^2 \\ \vdots \\ f^{i+1} - f^i \\ \vdots \end{pmatrix} . \tag{3.12}$$

This is familiar if you've ever had to manually program a derivative into a computer program. Note that the right-hand side looks like a linear transformation of $\mathbf{f}$. In other words, we expect to be able to write a matrix D so that

$$\mathbf{f}' = D\mathbf{f} . \tag{3.13}$$

One problem is apparent: what happens at the 'bottom' of the vector? What is the last component of the derivative, ${\mathbf{f}'}^N$? Formally, this is

$$(f')^N = \frac{1}{\Delta x}(f^{N+1} - f^N) \tag{3.14}$$

but now we have no idea what f^{N+1} is. That was never a component in our vector space. There is no $\mathbf{e}_{(N+1)}$ basis vector. This demonstrates and important lesson that we'll need when we move more formally to function spaces:

Boundary conditions are part of the definition of the function space.

That was so important that I put the whole damn sentence in boldface and set it in the middle of the line. The significance of boundary conditions may be a bit surprising—but think of this as part of the definition of which functions we allow into our function space.

Example 3.1 *When one first learns about Fourier series with Dirichlet boundary conditions, one finds that the Fourier expansion* only *contains sines. The solution to the wave equation in such a system is some function that is zero at each endpoint. So the function space relevant to the system is composed only of functions that are zero at each endpoint.*

For now let's assume **Dirichlet boundary conditions**. A convenient way to impose this is to define what happens to all functions outside the domain of the function space:

$$f^{i>N} = f^{i<1} = 0 . \tag{3.15}$$

[22] One could have also defined a backward derivative where $(f')^i \sim f^i - f^{i-1}$. Note that you *cannot* try to make this symmetric by defining a 'centered' derivative like $(f')^i \sim f^{i+1/2} - f^{i-1/2}$ because there's no such thing as a fractional index. If you tried to write $(f')^i \sim f^{i+1} - f^{i-1}$ you're making a worse approximation. If you're like me, the fact that there's some asymmetry in how we define the first derivative is deeply unsettling. There's something to this intuition!

This solves the problem of the derivative on the last component:

$$(f')^N = \frac{1}{\Delta x}(f^{N+1} - f^N) = \frac{-f^N}{\Delta x} \ . \tag{3.16}$$

Alternatively, we could have also imposed **periodic boundary conditions**:

$$f^i = f^{i+kN} \qquad\qquad k \in \mathbb{Z} \ . \tag{3.17}$$

This would then give

$$(f')^N = \frac{1}{\Delta x}(f^{N+1} - f^N) = \frac{1}{\Delta x}(f^1 - f^N) \ . \tag{3.18}$$

Periodic boundary conditions amount to wrapping the x-axis into a circle. Older folks sometimes call this *Asteroids* boundary conditions. I'd also accept *Star Control* boundary conditions. Periodic boundary conditions show up *all* the time in physics. Sometimes they show up in obvious places, like the Brillouin zone of a crystal lattice. Other times they show up in not-so-obvious places like the boundary conditions of the known universe. In addition to being crucial for a well-defined function space, the boundary conditions of a system establish its topology[23].

Exercise 3.4 *We don't know anything about the universe outside the Hubble radius. Why do you think it would be reasonable in a physical model to* assume *that it has periodic boundary conditions? Hint: what would happen to the x-momentum of an asteroid in the classic arcade game* Asteroids *if the game did not have periodic boundary conditions?*

The second derivative may be defined symmetrically:

$$(f'')^i = \frac{(f^{i+1} - f^i) - (f^i - f^{i-1})}{\Delta x^2} \ . \tag{3.19}$$

You may pontificate about the reason why the first derivative does have a symmetric discretization while the second derivative does.

3.5 Derivatives in other function space bases

There are other ways to write a discrete basis of functions. Here's a natural one for functions that are up to second-order polynomials:

$$\mathbf{e}_{(0)} = 1 \qquad\qquad \mathbf{e}_{(1)} = x \qquad\qquad \mathbf{e}_{(2)} = x^2 \ . \tag{3.20}$$

[23] I cannot over-emphasize the importance of topology in contemporary physics. Most of the physics you will learn in your first year graduate courses are intrinsically *local* because the laws of physics are causal. Topological quantities are *global*, they are integrals over an entire space. Because winding numbers (and their higher-dimensional cousins) are quantized, they are robust against perturbations. The number of holes in a donut is one, whether or not it's been slightly squished in the box. By the way, the best donuts in Southern California are from *Sidecar Doughnuts* in Costa Mesa. Get the Basil Eggs Benedict donut before 11am; you can thank me later.

Let's sidestep questions about orthonormality for the moment. Clearly linear combinations of these basis functions can produce any quadratic function:

$$f(x) = ax^2 + bx + c \qquad \Rightarrow \qquad \mathbf{f} = \begin{pmatrix} c \\ b \\ a \end{pmatrix} \ . \tag{3.21}$$

The derivative operator has an easy representation in this space:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{3.22}$$

We can see that

$$D\mathbf{f} = \begin{pmatrix} b \\ 2a \\ 0 \end{pmatrix} \qquad D^2\mathbf{f} = \begin{pmatrix} 2a \\ 0 \\ 0 \end{pmatrix} \qquad D^3\mathbf{f} = 0 \ . \tag{3.23}$$

The last line is, of course, the realization that the third-derivative of a quadratic function vanishes. Feel free to attach mathy words to this like *kernel*.

There are other bases that we may use for function space. A particularly nice one that we will use over and over is the Fourier basis, which we usually refer to as *momentum space*. The basis vectors are things like sines, cosines, or oscillating exponentials. These do not vanish for any power of D.

3.6 Locality

Notice that in the histogram basis, the derivative matrix D is sparse: it is zero everywhere away from the diagonal. The only non-zero elements on the i^{th} row are around the $(i \pm 1)^{\text{th}}$ column. Higher powers of D sample further away, but the non-zero elements are always clustered near the diagonal.

This is simply a notion of **locality**. Remember the Taylor expansion:

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \cdots \ . \tag{3.24}$$

If we think about the histogram as a discretization of a continuous function, then it is clear what the higher derivatives are doing. Given a function $f(x) = \mathbf{f}$, one might like to know about the function around some point x_0 corresponding to some index i. That is: $f^i = f(x_0)$. If you'd like to learn more about the function around that point, one can express the derivative at x_0. Thus $D\mathbf{f}$ says something about the slow, $D^2\mathbf{f}$ says something about the curvature, and so on. Because each successive power of D samples terms further away from f^i, you can tell that these higher order terms are learning about the function further and further away from x_0.

Now think about the types of differential equations that you've encountered in physics. They often include one or two derivatives. You hardly ever see three, four, or more derivatives[24]. There's a reason for this: at the scales that we can access experimentally, nature appears to be local. Our mathematical models of nature typically have locality built in[25].

[24] With some thought, it may also be clear why spatial derivatives typically appear squared.

[25] A recent counterexample: https://www.quantamagazine.org/physicists-discover-geometry-underlying-particle-

Physics at one spacetime point should not depend on spacetime points that are far away.

This may be familiar from the idea of causality—the idea that A *causes* B therefore A must have happened *before* B. One of the key results in special relativity is that causality can be tricky if two events do not occur at the same spacetime point. More carefully, A can only cause B if there is a timelike separation of the appropriate sign. If we want to build causal theories of nature, then the dynamics at x_0 should not rely on what is happening at x_1, a finite distance away.[26]

3.7 Row Vectors and all that

In high school we did not distinguish between row vectors and column vectors. They both seemed to convey the same information—they were simply one-dimensional arrays of numbers. Row vectors are just 'tipped over.' Such a tipping-over is convenient since you could apply the elementary schools rules of 'matrix multiplication' have the row vector act on a column vector:

$$\begin{pmatrix} w_1 & w_2 & \cdots \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \end{pmatrix} = w_1 v^1 + w_2 v^2 + \cdots \ . \tag{3.25}$$

In fact, this is like $\mathbf{w}^T$ is a function that acts *linearly* on its argument, $\mathbf{v}$:

$$\mathbf{w}^T(\mathbf{v}) = w_1 v^1 + w_2 v^2 + \cdots \ . \tag{3.26}$$

Perhaps you see why we wrote the row vector components with lower indices, w_i so that we may use Einstein summation notation: $\mathbf{w}^T\mathbf{v} = w_i v^i$.

Indeed, let is be a bit more formal about this. This layer of formalism is uncharacteristic of our approach in this course, but this underpins so much of the mathematical structure of our physical theories that it is worth getting right from the beginning. Let V be a vector space. It contains vectors, $\mathbf{v}$. Sometimes these are called contravariant vectors or kets. They have basis vectors $\mathbf{e}_{(i)}$.

Now introduce a related but *completely distinct* vector space called V^*. This is the space of **dual vectors** to V. A **dual vector** is what you may know as a **row vector**, a **co-vector** a **ket**, a **covariant vector**, or a [differential] **one-form**. These are all words for the *same idea*. A dual vector, say $(\mathbf{w}^T)$ is a *linear function that takes vectors and spits out numbers*:

$$\mathbf{w}^T \in V^* \Rightarrow \mathbf{w}^T : V \to \mathbb{R} \ . \tag{3.27}$$

Don't think about $\mathbf{w}^T$ as some kind of operation on a vector $\mathbf{w} \in V$; at least not yet. For now the '^{T}' is just part of the name of $\mathbf{w}^T$. The two spaces V and V^* are totally different. We haven't said anything about how to turn elements of V into elements of V^* or vice versa. It should be clear that there is a sense of 'duality' here: the vectors V are also linear functions that take a dual vector and spit out a number.

[26] This is different from saying that information cannot propagate from x_0 to x_1; such propagation could come from some causal excitation of the electromagnetic field traveling every infinitesimal distance between the two positions. This is reminiscent of the classical Zeno's paradox.

Let us call the basis of dual vectors $\widetilde{\mathbf{e}}^{(i)}$. This notation is cumbersome, so we'll change to something different soon. The upper index is deliberate. The defining property of $\widetilde{\mathbf{e}}^{(i)}$ is:

$$\widetilde{\mathbf{e}}^{(i)}\left(\mathbf{e}_{(j)}\right) \equiv \mathbf{e}_{(j)}\left(\widetilde{\mathbf{e}}^{(i)}\right) \equiv \delta^i_j \ . \tag{3.28}$$

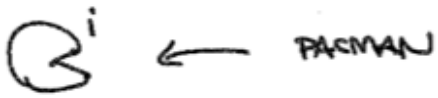
One may check that this gives

$$\left(w_i\widetilde{\mathbf{e}}^{(i)}\right)\left(v^j\mathbf{e}_{(j)}\right) = w_i v^j \delta^i_j = w_i v^i = w_1 v^1 + w_2 v^2 + \cdots \ . \tag{3.29}$$

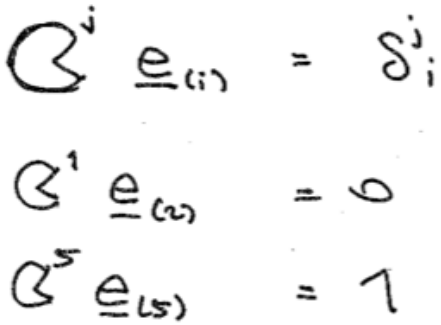
All that we've done here is defined basis vectors that carry the intrinsic *vector-ness* or *dual-vector-ness* through their relations (3.28). We have 'derived' the contraction of a lower-index object with an upper-index object, and hence our summation convention, in terms of these basis vectors.

3.8 Dual vectors as vector-eaters

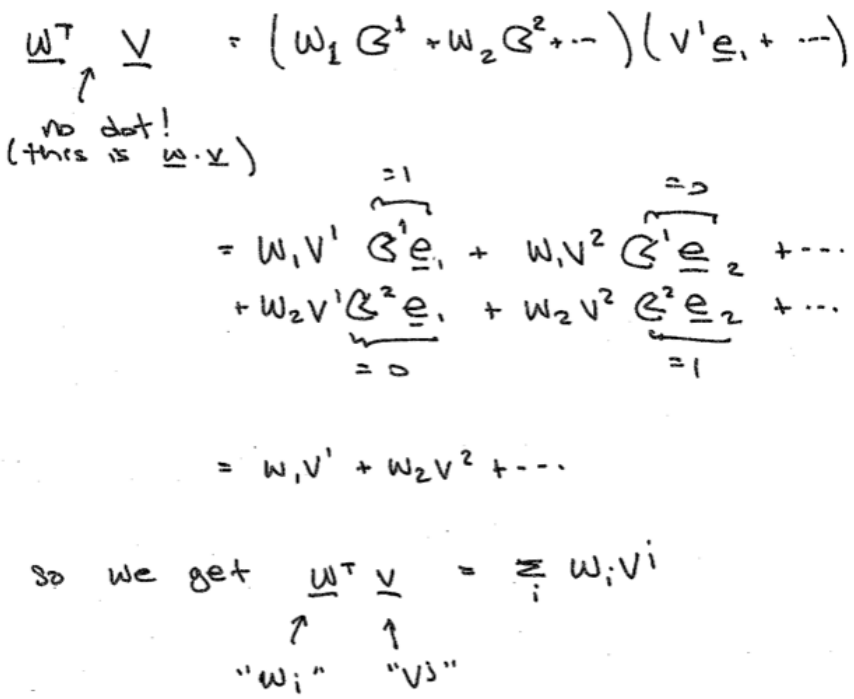
It is perhaps useful to use a slightly different notation based on Pac-Man. Rather than writing $\widetilde{\mathbf{e}}^{(i)}$, lets write the basis dual vectors as



In this notation, the action of a basis dual vector on a basis vector is simply Pac-Man eating the basis vectors:



So we can write (3.26) and (3.29) as



3.9 Orthonormal Bases

At this point we should take a deep breath and state explicitly that we've been assuming an orthonormal basis. In this course we will continue to use an orthonormal basis. You may object to this and say that you used to believe in orthonormal bases until you were forced to write down the gradient (or worse, the Laplacian) in spherical coordinates. In other words, in principle one could imagine a basis where (3.28) does not hold. There are many things to be said about this, none of them are particularly edifying without a full discussion. With no apologies, I'll make the following [perhaps perplexing] remarks:

1. There is no such thing as a 'position vector.' Positions refer to some base space, whereas vectors (like differential operators) act on the tangent space at a point of that base space.
2. A given tangent space is 'nice' and has a nice orthonormal basis.
3. That basis may not be the same for neighboring tangent spaces (perhaps due to coordinates, perhaps due to intrinsic curvature).

In this course these nuances will not come up. In the rest of your life you'll still have to deal with curvilinear coordinates. But suffice it to say that our study of function space will be nice an orthonormal. We haven't yet given an adequate definition of 'orthonormality,' so let's take (3.28) as a working definition.

3.10 Bra-Ket Notation

There is neither any physics nor mathematics contained in a choice of notation. However, a convenient notation does simplify our lives. Let us introduce bra-ket notation. In this

notation, we denote vectors by kets:

$$|v\rangle = v^i|i\rangle \ , \tag{3.30}$$

where $|i\rangle = \mathbf{e}_{(i)}$ is the basis of vectors that span the vector space V. There is nothing new or different about this object, $\mathbf{v} = |v\rangle$.

We denote dual vectors (row-vectors, one-forms) as bras:

$$\langle w| = w_i\langle i| \ , \tag{3.31}$$

where $\langle i| = \widetilde{\mathbf{e}}^{(i)}$. The orthonormality of this basis is encoded in

$$\langle i|j\rangle = \delta^i_j \ . \tag{3.32}$$

In bra-ket notation a linear transformation A has a basis

$$A = A^i{}_j|i\rangle\langle j| \ . \tag{3.33}$$

The notation $|i\rangle\langle j|$ is shorthand for the **tensor product** $|i\rangle \otimes \langle j|$. If the $\otimes$ doesn't mean anything to you, that's fine. It doesn't mean much to me either. Maybe you can replace it with the word '*and*' so that $|i\rangle \otimes \langle j|$ means you have a basis ket $|i\rangle$ *and* an basis bra $\langle j|$ that are somehow stuck together but aren't acting on each other. Matrix multiplication proceeds as before:

$$A\mathbf{v} = A|v\rangle = A^i{}_j|i\rangle\langle j|v^k|k\rangle = A^i{}_jv^k|i\rangle\langle j|k\rangle = A^i{}_jv^k|i\rangle\delta^j_k = A^i{}_jv^j|i\rangle \ . \tag{3.34}$$

Observe that the power of the notation is clear: the object with the index v^i is just a number. It commutes with everything. All of the vector-ness is carried in the basis objects: the bras, kets, and ket-bras. Those do not commute. But they have a well defined way in which kets act on bras (or vice versa).[27]

3.11 Eigenvectors are nice

Give a sufficiently *nice* linear transformation, A, there is a particularly convenient basis: the eigenvectors of A. These are kets $|\lambda\rangle$ such that

$$A|\lambda\rangle = \lambda|\lambda\rangle \ . \tag{3.35}$$

In other words, A acts on the eigenvector by rescaling. The rescaling coefficient is the eigenvalues. For *nice* transformations (see Section 1.3), there is a complete set of such vectors to span the vector space.

If you write a general vector $|v\rangle$ in terms of this eigenbasis,

$$|v\rangle = v^i|\lambda_{(i)}\rangle \ , \tag{3.36}$$

[27] This is where the $\oplus$ notation is handy. It keeps track of which kets/bras might hit which other bras/kets. This falls under the name of multi-linear algebra.

Then the action of A on this vector is easy:

$$A|v\rangle = \sum_i \lambda_{(i)} v^i |\lambda_{(i)}\rangle \ . \tag{3.37}$$

In fact, assuming that all of the eigenvalues are non-zero, even the matrix inverse is easy:

$$A^{-1}|v\rangle = \sum_i \lambda_{(i)}^{-1} v^i |\lambda_{(i)}\rangle \ . \tag{3.38}$$

The first time you see this should have brought a deep joy to your life: if you can decompose a matrix (linear transformation) into its eigenvectors and eigenvalues, then taking the inverse transformation is simple.

3.12 Linearity of Inverse Operators

Given a linear operator A, the inverse operator A^{-1} is defined by

$$A^{-1}A = \mathbb{1}_{N\times N} \ . \tag{3.39}$$

For an N-dimensional vector space, this represents N^2 different equations: one for each element. The inverse operator, by the way, is also linear. Let's remind ourselves of what this means. A linear transformation, when written as a matrix, is simply stating what that linear trasformation does to your basis vectors. If a matrix B has elments

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = B^i{}_j |i\rangle\langle j| \ , \tag{3.40}$$

then this simply means that acting on basis vectors $\mathbf{e}_{(1)} = |1\rangle$ and $\mathbf{e}_{(2)} = |2\rangle$ gives

$$B|1\rangle = a|1\rangle + c|2\rangle \tag{3.41}$$
$$B|2\rangle = b|1\rangle + d|2\rangle \ . \tag{3.42}$$

In column vector notation:

$$B\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} \qquad\qquad B\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix} \ . \tag{3.43}$$

So knowing the action on basis vectors is the same as knowing the transformation itself. Suppose I told you the action of the inverse transformation A^{-1} on your basis vectors, vis-a-vis (3.42):

$$A^{-1}|1\rangle = x|1\rangle + y|2\rangle \tag{3.44}$$
$$A^{-1}2\rangle = z|1\rangle + w|2\rangle \ . \tag{3.45}$$

Then you know exactly how A^{-1} acts on a general vector $|s\rangle = s^1|1\rangle + s^2|2\rangle$:

$$A^{-1}|s\rangle = A^{-1}\left(s^1|1\rangle + s^2|2\rangle\right) \tag{3.46}$$
$$= s^1 A^{-1}|1\rangle + s^2 A^{-1}|2\rangle \tag{3.47}$$
$$= (s^1 x + s^2 z)|1\rangle + (s^1 y + s^2 z)|2\rangle \ . \tag{3.48}$$

You can now keep this in mind when we say we want to solve $A|\psi\rangle = |s\rangle$. If we knew the action of A^{-1} on some basis of the space, then the problem is simple:

$$|\psi\rangle = \psi^i |i\rangle = \left(A^{-1}\right)^i{}_j \, |i\rangle\langle j| \, s^k |k\rangle \tag{3.49}$$

$$= \left(A^{-1}\right)^i{}_j \, s^k \, |i\rangle\langle j||k\rangle \tag{3.50}$$

$$= \left(A^{-1}\right)^i{}_j \, s^j \, |i\rangle \ . \tag{3.51}$$

We can write this as an equation for each component:

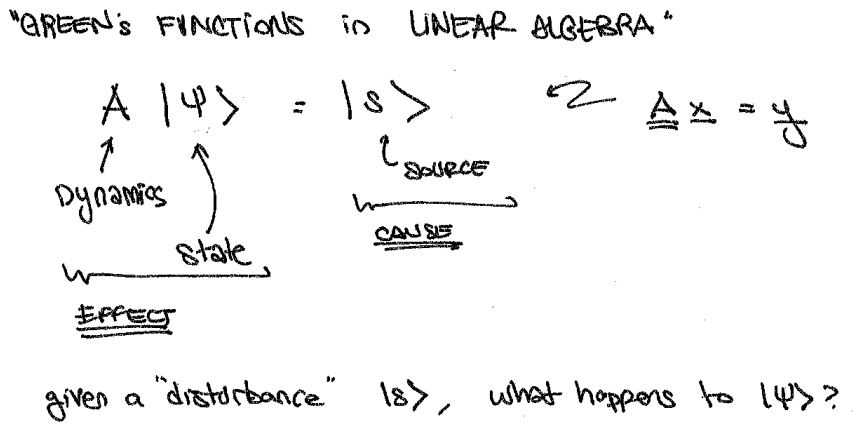
$$\psi^i = \sum_j \left(A^{-1}\right)^i{}_j \, s^j \ . \tag{3.52}$$

We've restored the explicit sum over j as a convenient reminder. The quantity $\left(A^{-1}\right)^i{}_j$ is what we would like to identify with a Green's function.

3.13 The Green's Function Problem

Going back to the big picture: recall that we want to solve differential equations of the form $\mathcal{O}f(x) = s(x)$. If we had a sense of the *eigenfunctions* of $\mathcal{O}$, then we could expand $s(x)$ in a basis of those eigenfunctions and then apply $\mathcal{O}^{-1}$ to both sides.

The analog is this:



The operator A encodes the *physics* of the system, the underlying dynamics. This is presumably local: it is a near-diagonal matrix coming from one or two powers of derivatives. The ket $|s\rangle$ is the source. This is the thing that *causes* the dynamics. The ket $|\psi\rangle$ is some state that we would like to determine. (3.38) is telling us that to invert a differential operator $\mathcal{O}$, it may be useful to decompose it into **eigenfunctions**.

By the way, *this* is where all of the 'special functions' in your physics education show up. The reason why you would ever care about Bessel functions (of various kinds) or Legendre polynomials is simply that they are the eigenfunctions of differential operators that we care about[28]. Sometimes we confuse mathematical physics with 'properties of special functions.'

[28] In fact, they're mostly the eigenfunctions of the same differential operator in different coordinate systems. Do you know which differential operator? I'll give you a guess. It's starts with a 'har-' and ends with a 'monic oscillator.'

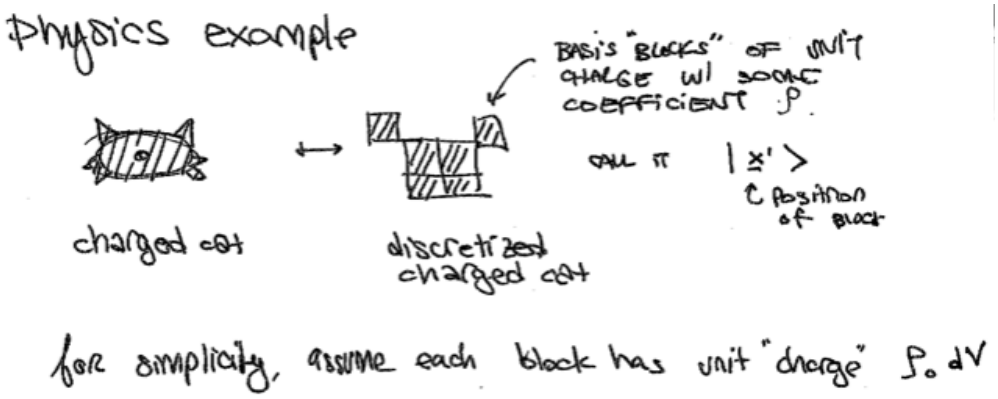
I do not care about special functions; perhaps with the notable exception of the Γ-function. Seriously, to screw special functions. The real intuition for what we're doing is evident in the few-dimensional harmonic oscillator. All the Bessel-schmessel function-ology that will pain you in your Jackson E&M course are just *technical details*.

Example 3.2 *Consider a differential operator $A \to \mathcal{O} = (d/dx)^2$. There's a basis of nice eigenvectors $\xi_{(k)}$:*

$$\xi_{(k)} = \sin(kx) \qquad\qquad \mathcal{O}\xi_{(k)} = -k^2\xi_{(k)}. \tag{3.53}$$

I've deliberately avoided normalizing for now. From this you can see that writing a function as a ***Fourier Series*** *is simply a change of basis to eigenfunctions of $(d/dx)^2$. Can you see how you would 'invert' this operator acting on a general function $f(x)$ with a set of Fourier coefficients $f(x) = \sum_k c_k \sin(kx)$?*

Example 3.3 *A common example in electrostatics is the triboelectric effect. Go ahead, take a moment to look it up on Wikipedia. The image is very relevant. If you pet your cat with hard rubber*[29]*, the cat builds up some electrostatic charge. For simplicity, let's model this system as a bunch of Lego blocks arranged in a cat-shape where each block has a constant electric charge. Let's say that each block has volume dV and constant charge density ρ_0 so that each block has charge $\rho_0 dV$. The entire cat is described by a charge density $\rho(x)$ where $\rho(x) = 0$ if you're outside the cat and $\rho(x) = \rho_0$ if you're inside the cat.*



You know how this problem works. You want to solve for the electrostatic potential, $\Phi(x)$, given the charge density $\rho(x)$. The relevant equation is

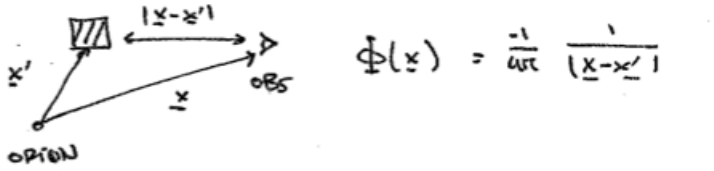
$$\nabla^2\Phi(\mathbf{x}) = -\rho(\mathbf{x}) \ , \tag{3.54}$$

where $\epsilon_0 = 1$ in convenient units. This looks like a tricky differential equation to solve, but the fist week of our undergraduate electrodynamics course taught us that the potential from a unit point charge *is*

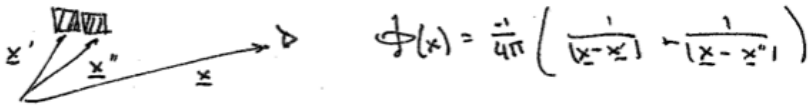
$$\Phi(\mathbf{x}) = \frac{-1}{4\pi}\frac{1}{|\mathbf{x} - \mathbf{x}'|} \ . \tag{3.55}$$

The relevant diagram is

[29] I do not recommend doing this.



If we know the potential from a single point charge, then we can invoke linearity—specifically, the linearity of ∇^2)—to write down the potential for two unit point charges:



At this point, you should start to see how this looks just like (3.48). The result is

$$\Phi(\mathbf{x}) = \sum_{\mathbf{x}'} \frac{-1}{4\pi} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \rho(\mathbf{x}') dV \;\rightarrow\; \int d^3\mathbf{x}' \frac{-1}{4\pi} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \rho(\mathbf{x}') \; . \tag{3.56}$$

This last example is really useful. You've seen this calculation before, but please review it carefully from the perspective of the action of $(\nabla^2)^{-1}$. Just as we are able to build a 'Lego cat' out of unit blocks, each of those unit blocks comes with an electrostatic potential. The solution to $\nabla^2\Phi(x) = -\rho(x)$ is to simply sum together those point-source solutions in the same way that one sums together the point sources (assembles the blocks) to model the finite source.

Exercise 3.5 *In the example above, what plays the role of the basis vectors?*

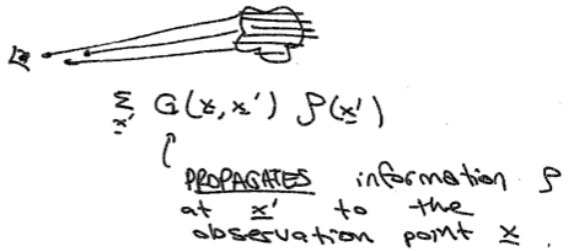
Compare (3.56) to (3.52). The sum over positions $\mathbf{x}'$ that becomes an integral over $d^3\mathbf{x}'$ is completely analogous to the sum over the index j. $\rho(\mathbf{x}')$ plays the role of a component of the source, s^j. On the left-hand side, $\Phi(\mathbf{x})$ plays the role of ψ^i. We note that the index i is replaced by the position $\mathbf{x}$. Evidently, the inverse of ∇^2 is simply

$$(\nabla^2)^{-1} \equiv G(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \; . \tag{3.57}$$

Here we've written out $G(\mathbf{x},\mathbf{x}')$, the Green's function for ∇^2. Observe that the Green's function has two arguments in the same way that the inverse matrix $(A^{-1})^i{}_j$ has two indices. The $\mathbf{x}'$ 'index' is integrated/summed over—it scans each 'building block' of the source in the same way that we sum over j in the index contraction $(A^{-1})^i{}_j\, s^j$. In other words, we may heuristically write (3.57) as

$$\Phi^i \sim \sum_j G^i{}_j \rho^j \; . \tag{3.58}$$

At this point, it's useful to point out that the Green's function is often called a **propagator**. As a function, $G(x,x')$ propagates the information of the source at x' to the observer at x assuming some dynamics—the operator for which G is a Green's function.



Please, please, please make sure you understand this example. This will be the key starting point from which we will generalize our study of Green's functions in physics.

3.14 Remark: Implicit assumption of linearity

In everything we're building we're assuming that the dynamics that relates the source $|s\rangle$ and the state $|\psi\rangle$ is *linear*. That's why we can think of the differential operator $\mathcal{O}$ as a matrix. If $|\psi_0$ is the effect of source $|s_0\rangle$, then you know by linearity that $2|\psi_0\rangle$ is the effect of a source that is twice as strong, $2|s_0\rangle$.

Linearity is *not* a truth of nature, yet we're spending all of this course developing techniques for dealing with linear dynamics! The fact of the matter is that a good chunk of physics *is* linear—and that's good because those are the parts that we can solve using our standard toolkit. Most of the frontier of physics has to do with how to deal with the *non*-linearities. There are a few options here: numerical solution, perturbation expansions about a linear solution (Feynman diagrams), and looking for topological invariants.

3.15 Metrics on Finite Dimensional Vector Spaces

Thus far we have introduced vector spaces. The dual vector space is a set of linear functions that act on elements of a vector space; these are bras/row-vectors/one-forms. Let us now introduce a new piece of machinery: a **metric**. This is also known as an **inner product** or a **dot product**. A space with a metric is called a metric space. We only state this fact to emphasize that we are *adding this structure by hand*. Vector spaces don't come with metrics—someone makes up a metric and slaps it onto the vector space.

The **metric** is a function that takes two vectors and spits out a number. It is linear in each argument. In other words, a metric g is:

$$g: \; V \times V \to \mathbb{R} \; . \tag{3.59}$$

Occasionally one may want a metric defined such that the output is a complex number. We thus have:

$$g(\alpha\mathbf{v} + \beta\mathbf{w}, \delta\mathbf{x} + \gamma\mathbf{y}) = \alpha\delta g(\mathbf{v},\mathbf{x}) + \alpha\gamma g(\mathbf{v},\mathbf{y}) + \beta\delta g(\mathbf{w},\mathbf{x}) + \beta\gamma g(\mathbf{w},\mathbf{y}) \; . \tag{3.60}$$

One more special assumption about the metric is that it is **symmetric**[30]:

$$g(\mathbf{v},\mathbf{w}) = g(\mathbf{w},\mathbf{v}) \; . \tag{3.61}$$

[30] Actually it is conjugate symmetric, $g(\mathbf{v},\mathbf{w}) = g(\mathbf{w},\mathbf{v})^*$. This distinction will be important for function spaces.

In indices one may write

$$g = g_{ij}\langle i| \otimes \langle j| \tag{3.62}$$

so that

$$g(\mathbf{v}, \mathbf{w}) = g_{ij} v^i w^j \ . \tag{3.63}$$

Here we see the usefulness of the $\otimes$ notation. It tells us that the bras and kets resolve as follows:

$$g_{ij}\langle i| \otimes \langle j| \left(v^k|k\rangle\right) \left(w^\ell|\ell\rangle\right) = g_i j v^k w^\ell \langle i|k\rangle\langle j|\ell\rangle = g_i j v^k w^\ell \delta^i_k \delta^j_\ell = g_{ij} v^i w^j \ . \tag{3.64}$$

If this is your first time seeing it, please re-read (3.64) carefully to see exactly how the bras and kets resolve themselves. For ordinary Euclidean space in flat coordinates, the metric is simply the unit matrix: $g_{ij} = \mathrm{diag}(1, \cdots, 1)$. In Minkowksi space there's a relative minus sign between space and time. In curvilinear coordinates things get ugly.

Here's the neat thing about metrics. We can take a metric g and pre-load it with a vector $\mathbf{v}$:

$$g(\mathbf{v}, \qquad) \ . \tag{3.65}$$

In fact, we may then define a function with respect to this pre-loaded metric:

$$f(\mathbf{w}) = g(\mathbf{v}, \mathbf{w}) \tag{3.66}$$

Observe that $f(\mathbf{w})$ is a linear function that takes elements of V and returns a number. In other words, this is a *dual vector* (row-vector, one-form, element of V^*). The metric has allowed us to *convert vectors into dual vectors*:

$$g(\mathbf{v}, \qquad) = g_{ij} v^i \langle j| \ . \tag{3.67}$$

Similarly, one may define an inverse metric g^{-1} such that $g^{-1}g = \mathbb{1}$. In a slight abuse of notation, the inverse metric is written with two upper indices: g^{ij}. Note that we do not write the '$^{-1}$.' The inverse metric will *raise* the index on a lower-index object, while the metric *lowers* the index of an upper-index object.[31]

3.16 The dot/inner product, orthonormality

You are already familiar with the 'obvious' way to take two vectors and spit out a number. This is the dot product of two vectors, also known as the inner product. They are equivalent to each other and equivalent to the action of the metric on a pair of vectors;

$$\mathbf{v} \cdot \mathbf{w} = \langle \mathbf{v}, \mathbf{w} \rangle = g(\mathbf{v}, \mathbf{w}) \ . \tag{3.68}$$

[31] Of course: what's really happening is that the metric has a basis $\langle i| \otimes \langle j|$ while the inverse metric has a basis $|i\rangle \otimes |j\rangle$.

The norm of a vector $\mathbf{v}$ is simply the square root of $g(\mathbf{v}, \mathbf{v})$. By the linearity of the metric, we can decompose the dot product into the action of the metric on basis vectors:

$$g(\mathbf{v}, \mathbf{w}) = g(v^1 \mathbf{e}_{(1)} + v^2 \mathbf{e}_{(2)} + \cdots , w^1 \mathbf{e}_{(1)} + w^2 \mathbf{e}_{(2)} + \cdots) \tag{3.69}$$

$$= v^1 w^1 g(\mathbf{e}_{(1)}, \mathbf{e}_{(1)}) + v^1 w^2 g(\mathbf{e}_{(1)}, \mathbf{e}_{(2)}) + \cdots , \tag{3.70}$$

where our assumption of an *orthonormal basis*[32] is the statement that

$$g(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}) = \pm \delta^i_j \ . \tag{3.71}$$

The $\pm$ depends on the signature of the metric. As physicists, this is the distinction between 'space' and 'time.' For Euclidean space the sign is always plus. For Minkowski spacetime, there's some choice of sign for the timelike direction and the opposite sign for the spacelike direction. The choice is a convention, there's no 'right' choice[33]. When (3.71) is not true the basis is not orthonormal. Sometimes this happens when your space is curvy (general relativity); other times you just have curvilinear coordinates for a flat space (e.g. polar coordinates). I'm glossing over several subtleties here, among them is the idea that position vectors do not exist[34].

Exercise 3.6 *Speaking of the notion of curved space: Consider a sheet of paper as an idealized two-dimensional surface. If you tape opposite edges of the paper together to make a cylinder, is the two-dimensional space curved or flat? For example, if two-dimensional beings lived on the paper like in Edwin Abbot–Abbot's novella* Flatland*, would they say that their space is curved?*

Exercise 3.7 *The cylinder from the previous exercise is an example of periodic boundary conditions. For those familiar with special relativity, this entire section has probably been a big review. Here's a fun puzzle to think about while you skim. What happens to the twin paradox if the universe were periodic in some spatial direction? The usual resolution to the twin paradox is that the twin that 'turns around' must change inertial frame. However, if there were a periodic direction, neither twin has to 'turn around' for them to meet once again and compare clocks*[35]*.*

Example 3.4 *Given two vectors* $\mathbf{v}$ *and* $\mathbf{w}$*, there are two equivalent ways of thinking about how they may be combined into a number. First, one can lower the index of one of the vectors with the metric* $v_i \equiv g_{ij} v^i$ *and then contract the 'row vector'* v_i *with the 'column vector'* w^j*:* $v_i w^i$*. I am of course abusing notation by conflating the components* v_i *and* w^j *with the row/column vectors. The second way of doing this is taking the two vectors and contracting them directly with the metric:* $v^i w^j g_{ij}$*. Please make sure you are comfortable that these 'two ways' are exactly the same thing.*

[32] Not an 'oprhan normal' basis like Zoom's auto-transcription claims I said.

[33] The right choice is $+$ for timelike and $-$ for spacelike.

[34] Formally, vectors live in the tangent space of a manifold. They are 'intrinsically' tangent vectors (velocities) of some trajectory on the manifold with some 'time' parameter. The position on the manifold is a coordinate, but it is not a *vector*. For more on what I mean by this, I recommend `hep-th/0611201`, Arnold's *Mathematical Methods of Classical Mechanics*, or a graduate general relativity textbook.

[35] For a nice solution, see: https://doi.org/10.1080/00029890.2001.11919789

3.17 Comment: Why physicists like indices

Einstein's summation convention tells us that we should see repeated upper/lower indices as contracted pairs—we should treat them differently. In contrast, the un-paired indices tell us something about the physical quantity to which those indices are attached. This is why—despite what our mathematician colleagues tell us—we love indices[36] Vectors are objects that transform in a well defined way with respect to rotations:

$$v^i \to v'^i = R^i{}_j v^j \ ,$$

where R is a rotation matrix. Similarly, one-forms/column vectors transform 'oppositely':

$$w_k \to w'_k = \left(R^T\right)^\ell{}_k \, w_\ell \ . \tag{3.72}$$

Observe that the rotation matrix is *transposed* when acting on an *lower* index object. This is consistent with the idea that the dot product is invariant under rotations:

$$v^i w_i \to v'^i w'_i = R^i{}_j v^j \left(R^T\right)^\ell{}_i \, w_\ell = \left[\left(R^T\right)^\ell{}_i R^i{}_j\right] v^j w_\ell = v^j w_j \ , \tag{3.73}$$

where we've used the fact that the matrix in the square brackets is simply δ^ℓ_j which is evident from $R^T R = \mathbb{1}$.

Example 3.5 *We exploited something neat about the index notation in (3.73). Note that there aren't any 'vectorial' objects in (3.73)—all there is are a bunch of numbers (components of vectorial objects). We used the fact that we could re-arrange these products to make it clear how they contract. That's why we were able to move the $(R^T)^\ell{}_i$ term all the way to the left.*

A neat lesson here is that

> Indices tell us how an object transforms under 'rotations'.

Here 'rotations' mean whatever the appropriate symmetry is. In quantum mechanics, for example, the rotations are unitary transformations. If an object has indices, then it transforms. If an object has no indices (or is only composed of contracted indices) then it is a scalar with respect to those transformations. This 'indexology' is the backbone of how lazy physicists apply representations of symmetry groups[37]. It's especially useful in particle physics[38].

The generalization to tensors is hopefully clear. An object with some number of upper indices and some number of lower indices transforms with several factors of the rotation matrix and its transpose:

$$T^{i_1 i_2 \cdots i_n}{}_{j_1 j_2 \cdots j_m} \to R^{i_1}{}_{k_1} R^{i_2}{}_{k_2} \cdots R^{i_n}{}_{k_n} (R^T)^{\ell_1}{}_{j_1} (R^T)^{\ell_2}{}_{j_2} \cdots (R^T)^{\ell_m}{}_{j_m} T^{k_1 k_2 \cdots k_n}{}_{\ell_1 \ell_2 \cdots \ell_m} \ . \tag{3.74}$$

Yes, we are so wealthy with indices that our indices have indices.

Exercise 3.8 *How does the moment of inertia tensor transform under a rotation R?*

[36] Recent Nobel laureate Roger Penrose has a clever notation that replaces indices with lines; https://en.wikipedia.org/wiki/Penrose_graphical_notation.

[37] https://github.com/Tanedo/Physics262-2019

[38] https://sites.google.com/ucr.edu/p165/

3.18 Adjoints and Hermitian Conjugates

Because the metric lowers the index of a vector and produces a dual vector, you may want to think about this as 'tipping over' a column vector—almost like a transpose, right? We should be careful with this notion. Suppose we have a vector $\mathbf{w} = A\mathbf{v}$. How do we express the components of the *dual vector* w_i with respect to the elements of the dual vector v_i?

$$w_i = g_{ij}w^j = g_{ij}A^j{}_k v^k = A_{jk}v^k = A_j{}^k v_k \equiv v_k (A^T)^k{}_j \ . \tag{3.75}$$

In the second-to-last step we have used $g^{ij}g_{jk} = \delta^i_k$. In the last step we have *defined* the transpose of a real matrix in the usual way: swap the first and second indices. These happen to have heights that come along for the ride. For convenience we moved the v_k to the left-side of the expression—it should be clear that this does not affect the expression at all (just write out the sum explicitly). We've gone through these gymnastics simply to motivate the notion that

$$\mathbf{w}^T = \mathbf{v}^T A^T \ . \tag{3.76}$$

Notice that the T *acts* on the matrix A. This is in contrast to our notation where the T in $\mathbf{w}^T$ is simply a label to remind us that the object is a dual (row) vector.

For complex vector spaces—for example, the space of wavefunctions in quantum mechanics—we need to be a little bit more careful. We won't say much about complex vector spaces in general, but us simply state that the 'transpose' is generalized to the Hermitian conjugate. You may be most familiar with this in bra-ket notation from quantum mechanics:

$$\langle x| = |x\rangle^\dagger = \langle x, \quad \rangle \ , \tag{3.77}$$

where $\langle x, \quad \rangle$ is simply the inner product on the complex space with one component pre-loaded[39]. You should now see why the Hermitian conjugate isn't just a 'transpose' but also a complex conjugation. The **norm** of a vector is $||\mathbf{v}|| = \langle \mathbf{v}, \mathbf{v}\rangle$. We would like the **norm** to be positive definite[40], even when $\mathbf{v}$ is complex. If $\mathbf{v}$ has a complex phase, then there needs to be a 'complex conjugation' built into the notion of an inner product.

Example 3.6 *While this may sound like hand-waving, you already know this from quantum mechanics where the wavefunction $\psi(x)$ is complex. You know that a nice wavefunction is normalized, $\langle \psi, \psi\rangle = 1$, which we interpret as:*

$$\int dx\, \psi^*(x)\psi(x) = 1 \ . \tag{3.79}$$

The $\psi^*(x)$ in the integrand is related to the complex conjugation implicit when going from a ket $|\psi\rangle$ to a bra $\langle\psi|$.

[39] By the way, this motivates the bra-ket notation for vectors and dual-vectors:

$$\langle x|y\rangle = \langle x, y\rangle \ . \tag{3.78}$$

In the above equation, the left-hand side is a bra (dual vector) acting on a ket (vector), while the right-hand side is the inner product acting on two kets. This is simply repeating the statement in Example 3.4.

[40] The notable exception is when you have a space with nontrivial signature like Minkowski space where the relative sign between space and time is relevant.

We will work this out more carefully when we re-introduce function spaces systematically—but all of this should feel familiar.

Let us write **adjoint** to mean transpose for real spaces and Hermitian conjugate for complex spaces. The importance of the adjoint is not converting vectors into dual vectors, but rather the action on operators. Armed with a metric/inner product, the adjoint $A^\dagger$ of an operator A satisfies

$$\langle A^\dagger w, v\rangle = \langle w, Av\rangle \ . \tag{3.80}$$

In a given inner product, the adjoint shifts the action from one vector to the other.

Example 3.7 *From this definition, it should be clear that*

$$\langle w|Av\rangle = \langle A^\dagger w|v\rangle \ . \tag{3.81}$$

This equation does not invoke the metric in contrast to (3.80)*. Of course, the dual vector* $\langle w|$ *is related to the vector* $|w\rangle$ *by the metric.*

Exercise 3.9 *If you are mathematically inclined, prove the existence, uniqueness, and linearity of the adjoint* $A^\dagger$ *of a linear operator* A*.* Hint: see Byron & Fuller chapter 4.4.

The adjoint/Hermitian conjugate is important because of the result that **self-adjoint operators**, $A^\dagger = A$, have real eigenvalues and a complete set of orthogonal eigenvectors. Then we can use the strategy of Section 3.11 to use eigenvectors of A to simplify the solution of $A\mathbf{v} = \mathbf{w}$.

4 Function Space

Function spaces are vector spaces where the vectors are functions. We introduced 'histogram space' in Section 3.3 as a crude model of a finite dimensional function spaces. An alternative finite dimensional model is the space of polynomials up to some degree, which we introduced in Section 3.5. Our models of nature are typically continuous[41]. This requires *infinite* dimensional function spaces, or Hilbert spaces. lec 04

4.1 The Green's Function Problem in Function Space

The Green's function can be defined by analogy to the finite-dimensional inverse transformation. The finite-dimensional linear system $A\mathbf{v} = \mathbf{w}$ can be solved by applying the inverse transformation A^{-1} in the same way that the continuum (infinite-dimensional) system $\mathcal{O}\psi(x) = s(x)$ can be solved with the Green's function $G(x, y)$ of the operator $\mathcal{O}$:

$$v^i = \sum_i \left(A^{-1}\right)^i{}_j w^j \qquad \Rightarrow \qquad \psi(x) = \int dy\, G(x, y)s(y) \ . \tag{4.1}$$

We've explicitly written out the sum over the dummy index i to emphasize the analogy to the integration over the dummy variable y. The arguments of the functions play the role of 'continuum indices.'

[41] This does not mean that nature is fundamentally continuous. There is a deep sense in which our models are valid whether or not nature is continuous so long as any granularity is smaller than our experimental probes.

4.2 Differential Operators

Linear transformations on function space are differential operators. In principle you can imagine linear transformations that are not differential operators, for example a finite translation. However, because our models of nature are typically *local* and *causal*, the linear transformations that we obtain from physical models are differential operators[42].

Let's write a general differential operator as:

$$\mathcal{O} = p_0(x) + p_1(x)\frac{d}{dx} + p_2(x)\left(\frac{d}{dx}\right)^2 + \cdots \tag{4.2}$$

where the $p_i(x)$ are polynomials. Sometimes we will write this as $\mathcal{O}_x$ to make it clear that the argument of the polynomials is x and the variable with which we are differentiating is x.

Exercise 4.1 *Explain why (4.2) is a linear operator acting on function spaces.*

Exercise 4.2 *A confused colleague argues to you that (4.2) cannot possibly be 'linear.' Just look at it, your colleague says: the functions $p_i(x)$ are polynomials—those aren't* linear*! There are also powers of derivatives—how is that possibly linear? Explain to your colleague why the $p_i(x)$ does not have to be linear nor is one restricted to finite powers of derivatives for the operator $\mathcal{O}$ to be a linear operator acting on function space.*

Technically (4.2) is called a **formal operator** because we haven't specified the boundary conditions of the function space. Recall in our discretized 'histogram space' in Section 3.3 that we had to be careful about how to define the derivative acting on the boundaries of the space. A differential operator along with boundary conditions is called a **concrete operator**.

4.3 Inner Product

There's a convenient inner product that you may be familiar with from quantum mechanics. For two functions $f(x)$ and $g(x)$ in your function space, define the inner product to be

$$\langle f, g\rangle = \int dx\, f^*(x)g(x)\ . \tag{4.3}$$

Example 4.1 *Wave functions in 1D quantum mechanics obey this norm. For an infinite domain, we typically restrict to square-integrable functions meaning that $|f|^2$ goes to zero fast enough at $\pm\infty$ so that the integral $\langle f, f\rangle$ is finite.*

[42] This is not to say that finite transformations are somehow not permitted. The dynamics that govern our models of nature, however, only dictate how information is transmitted infinitesimally in space and time. Propagation forward in time by some finite interval is described by the exponentiation of infinitesimal forward time translations. This is, of course, why the time-translation operator in quantum mechanics is $e^{i\hat{H}t}$, where the Hamiltonian H is described as a local function with perhaps one or two derivative operators.

Sometimes the inner product is defined with respect to a **weight** function $w(x)$:

$$\langle f, g\rangle_w = \int dx\, w(x)\, f^*(x) g(x)\ . \tag{4.4}$$

There's nothing mysterious about inner products with weights. They typically boil down to the fact that one is not using Cartesian coordinates.

Example 4.2 *Have you met the Bessel functions? If not, you're in for a treat in your electrodynamics course. The Bessel functions satisfy a funny orthogonality relation with weight $w(x) \sim x$ because they show up as the radial part of a solution when using polar coordinates. When you separate variables, $d^2x = r dr\, d\theta$, we see that the measure over the radial coordinate r carries a* weight r.

We will assume unit weight until we go to higher spatial dimensions[43].

4.4 Dual Vectors

What are the 'dual functions' (dual vectors, bras) in function space? These are linear functions on act on functions and spit out numbers. Taking inspiration from (3.67), these are integrals that are pre-loaded with some factors. Assuming unit weight:

$$\langle f| = \langle f, \qquad \rangle = \int dx\, f^*(x)\, [\text{ insert ket here }]\ . \tag{4.5}$$

4.5 Adjoint operators

What is the adjoint of a differential operator? The definition of the adjoint (3.80) and the function space inner product (4.3) give us a hint. We define $\mathcal{O}^\dagger$ by the property

$$\int dx\ [\mathcal{O} f(x)]^*\, g(x) = \int dx\, f^*(x)\, \left[\mathcal{O}^\dagger g(x)\right]\ . \tag{4.6}$$

The strategy is: given an inner product (integral) over f^* and g where there is some stuff ($\mathcal{O}$) acting on g, can we re-write this as an integral with no stuff acting on g and some *other* stuff acting on f^*? If so, then the 'other stuff' is the adjoint $\mathcal{O}^\dagger$.

Example 4.3 *What is the adjoint of the derivative operator, $\mathcal{O} = d/dx$? Assume an interval $x \in [a, b]$ and Dirichlet boundary conditions, $f(a) = f(b) = 0$. There's a simple way to do this: integrate by parts.*

$$\int dx\ \left[\frac{d}{dx} f(x)\right]^*\, g(x) = -\int dx\, f^*(x) \left[\frac{d}{dx} g(x)\right] + [f^*(x) g(x)]_a^b\ = -\int dx\, f^*(x) \left[\frac{d}{dx} g(x)\right]\ . \tag{4.7}$$

From this we deduce that

$$\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}\ . \tag{4.8}$$

[43] My dissertation focused on theories of extra dimensions. I also noticed that my weight increased in my final year of graduate school as I spent most of my time writing about extra dimensions and eating cafe pastries.

We will be especially interested in **self-adjoint** (Hermitian) operators for which

$$\mathcal{O}^\dagger = \mathcal{O} \ . \tag{4.9}$$

This is, as we mentioned for the finite-dimensional case, because self-adjoint operators are *nice*: they have real eigenvalues and orthogonal eigenvectors. Since most physical values are real eigenvalues of some operator, one may expect that the differential operators that show up in physics are typically self-adjoint.

Exercise 4.3 *We saw above that the derivative operator is not self-adjoint. What is an appropriate self-adjoint version of the derivative operator?* Hint: what is the momentum operator in quantum mechanics?[44]

Example 4.4 *Consider $\mathcal{O} = -\partial_x^2$ defined on the domain $x \in [0,1]$ with the boundary conditions $f(0) = f(1) = 0$. Is this operator self-adjoint? We want to check of $\langle f, \mathcal{O}g\rangle = \langle Of, g\rangle$. We have one trick: integration by parts. Let's see how this works.*

$$\langle f, \mathcal{O}g\rangle = -\int dx\, f^*(x)\partial^2 g(x) \ . \tag{4.10}$$

This is compared to

$$\langle \mathcal{O}f, g\rangle = -\int_0^1 dx\ \left[\partial^2 f(x)\right] * g(x) \tag{4.11}$$

$$= -\left(\partial f(x)\right)^* g(x)\big|_0^1 + \int_0^1 dx\ \left[\partial f(x)\right]^* \partial g(x) \tag{4.12}$$

$$= f^*(x)\partial^2 g(x)\big|_0^1 - \int_0^1 dx\, f^*(x)\partial^2 g(x) \tag{4.13}$$

$$= -\int_0^1 dx\, f^*(x)\partial^2 g(x) \ . \tag{4.14}$$

And so we see that indeed $(-\partial^2)^\dagger = -\partial^2$.

Exercise 4.4 *In the previous example, what is the significance of the overall sign of the operator?* Hint: the sign doesn't matter, it's because we typically think of $-\partial^2$ and its higher-dimensional derivatives as the square of the momentum operator.

Example 4.5 *The **eigenfunctions** f_n of $-\partial^2$ defined on $x \in [0,1]$ with Dirichlet boundary conditions are simply*

$$f_n(x) = A_n \sin(n\pi x) \qquad\qquad \lambda_n = -n^2\pi^2 \ , \tag{4.15}$$

where λ_n is the associated eigenvalue and A_n is some normalization that. These eigenfunctions are orthonormal in the following sense:

$$\langle f_n, f_m\rangle = \int_0^1 dx\ \sin(n\pi x)\sin(m\pi x) = \frac{A_n A_m}{2}\delta_{nm} \ , \tag{4.16}$$

from which we deduce that the normalization is $A_n = \sqrt{2}$. That's basically all there is to know about Fourier series.

[44] https://aapt.scitation.org/doi/abs/10.1119/1.9932

Exercise 4.5 *A function $g(x)$ defined on an interval $x \in [0,1]$ with Dirichlet boundary conditions can be written with respect to the Fourier basis (4.15). In ket notation, the n^{th} component of g with respect to this basis is*

$$g^n = \langle f_n | g \rangle \ . \tag{4.17}$$

Confirm that this is precisely what you know from Fourier series. In other words, we can decompose $g(x)$ as

$$g(x) = \sum_n \langle f_n | g \rangle f_n(x) \ . \tag{4.18}$$

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4.6 Completeness in Function Space

We rarely have much to say about the unit matrix in linear algebra. However, much like when we discussed units, we can squeeze a lot out of inserting the identity in our mathematical machinations. In order to help with translate this to function space, let's review how it works in finite dimensional vector spaces. The unit matrix is $\mathbb{1}$ and may be written:

$$\mathbb{1} = \sum_i |i\rangle\langle i| \ , \tag{4.19}$$

where $|i\rangle$ and $\langle j|$ are basis (dual-)vectors.

Exercise 4.6 *Take a moment and convince yourself that (4.19) is true and obvious. It may be helpful to explicitly write out $|i\rangle\langle j|$ as a matrix.*

Exercise 4.7 *Suppose you have a two-dimensional Euclidean vector space. Show that (4.19) is true for the basis*

$$|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4.20}$$

$$\langle 1| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \qquad \langle 2| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \ . \tag{4.21}$$

In fact, (4.19) defines what it means that a set of basis vectors is **complete**. You can write any vector $|v\rangle$ with respect to the basis $|i\rangle$—the components are simply

$$v^i = \langle i | v \rangle \tag{4.22}$$

so that

$$|v\rangle = \sum_i |i\rangle\langle i|v\rangle \ , \tag{4.23}$$

which we recognize as nothing more than 'multiplying by the identity.'

What does completeness look like in function space?

Let $e_{(n)}(x)$ be a set of basis functions. The basis is **complete** if

$$\sum_n \left[e_{(n)}(x)\right]^* e_{(n)}(y) = \delta(x-y) \ . \tag{4.24}$$

Compare this *very carefully* with the completeness relation (4.19). The sum over i in the finite-dimensional case has been relabeled into a sum over n in the function space—this is just my preference[45]. The $\mathbb{1}$ has been replaced by a Dirac δ-function, $\delta(x-y)$. Let's confirm that this makes sense. The *multiply by one* completeness relation (4.23) in function space is

$$|g\rangle = \sum_n |e_{(n)}\rangle\langle e_{(n)}|g\rangle \qquad\qquad \langle e_{(n)}|g\rangle = \int dy\, [e_{(n)}(y)]^* g(y) \ . \tag{4.25}$$

We have deliberately changed the name of the integration variable to y to avoid confusion; since this variable is integrated over it's simply a *dummy variable* and it doesn't matter what we name it—the quantity $\langle e_{(n)}|g\rangle$ is independent of y because y is integrated over[46]. Writing this out explicitly as functions:

$$g(x) = \sum_n \left[\int dy\, e^*_{(n)}(y) g(y)\right] e_{(n)}(x) \ . \tag{4.26}$$

The factor in the square brackets is simply $\langle e_{(n)}|g\rangle$, which is just a *number*—it has no functional dependence on x. If this seems unusual, please refer back to Example 4.5 and Exercise 4.5.

By the way, you'll often hear people (perhaps even me) say that the Dirac δ function is not strictly an *function* but rather a **distribution**—this means that it only makes sense when it is integrated over. As physicists we'll sometimes be sloppy and talk about physical quantities that could be Dirac δ-functions. There is *never* an appropriate, measurable physical quantity that is described by a $\delta(x)$. Anything with a $\delta(x)$ is an object that was meant to be integrated over. When you imagine that a point charge density is a δ-function, this is only because you will eventually integrate over it to determine the total charge. This is precisely what we saw in the charged cat in Example 3.3. If you ever calculate a *measurable* quantity to be $\delta(x)$ check your work. If you ever find $\delta(x)^2$, then go home, it's past your bed time.

Example 4.6 *One can vaguely motivate the δ-function as the unit matrix by appealing to the 'histogram basis' of discretized function space. In an ordinary finite-dimensional vector space, unit matrix can be written as*

$$\mathbb{1} = |1\rangle\langle 1| + |2\rangle\langle 2| + \cdots = \sum_{i,j} \delta^j_i |i\rangle\langle j| \ . \tag{4.27}$$

[45] I think this is because we will deal with complex functions and I want to avoid using i as an index. But if we're being honest, it's just become a habit.

[46] By the way, this should ring a bell from our summation convention. When an upper and lower tensor index are contracted, the resulting object behaves as if it didn't have those indices: $A^i{}_j v^j$ behaves as a vector with one upper index.

The Dirac δ-function in histogram space is analogous to

$$\delta(x - x') \to \delta_x^{x'} |x\rangle\langle x'| \ , \tag{4.28}$$

where x and x' are discrete bins on the right-hand side. Thus for a discretized function $f = f(x_1)|x_1\rangle + f(x_2)|x_2\rangle + \cdots$, one has

$$\int dy\, \delta(x - y) f(y) = f(x) \longrightarrow \sum_j \delta_{x_j}^{x_i} |x_i\rangle\langle x_j|f\rangle = f(x_i)|x_i\rangle \ . \tag{4.29}$$

4.7 Orthonormality in Function Space

One should contrast the notion of completeness of a a basis this with that of **orthonormality** of the basis. Orthonormality is the statement that

$$\langle i|j\rangle = \delta_i^j \ . \tag{4.30}$$

Completeness has to do with the 'outer product' $|i\rangle\langle i|$ while orthonormality has to do with the 'inner product' $\langle i|i\rangle = \langle i, i\rangle$. The function space generalization of orthonormality is[47]

$$\langle e_{(n)}|e_{(m)}\rangle = \int dx\, e^*_{(n)}(x) e_{(m)}(x) = \delta_{nm} \ . \tag{4.31}$$

Exercise 4.8 *Why does (4.31) have a Kronecker δ with discrete indices when (4.24) has a Dirac δ? Please make sure you can answer this; it establishes the conceptual foundation of the analogy between finite- and infinite-dimensional vector spaces.*

For the completeness relation, we sum over the same eigenfunction label n for a function and its conjugate evaluated at different continuous positions. For the orthonormality relation, we integrate over the positions of two different eigenfunction indices, n and m.

Do not confuse the eigenfunction label with the index of a vector. If this is confusing, please refer back to Exercise 4.7. You may be stuck thinking about basis vectors in the Cartesian basis—this is the analog of thinking about basis functions in the 'histogram basis' of Section 3.3. What we want to do is generalize to more convenient bases, like the eigenfunctions of differential operators (e.g. the Fourier basis for $-\partial^2$).

4.8 Completeness and Green's Functions

The utility of the completeness relation should be clear. If you happen to have a nice (self-adjoint) linear differential operator $\mathcal{O}$ with a nice (complete, orthogonal) eigenfunctions $e_{(n)}$

[47] If you're a purist, you'll note that δ_{nm} should really be written as δ_m^n because the dual basis vector has an upper index. While this may be true, I'm making the present notational choice because the object that we would call $\widetilde{\mathbf{e}}^{(n)}$ really does contain $e^*_{(n)}(x)$, the complex conjugate of $e_{(n)}(x)$.

and eigenvalues λ_n, then we can expand any function $\psi(x)$ with respect to these eigenfunctions. Then it is easy to invert the differential equation $\mathcal{O}\psi(x) = s(x)$ to determine the response $\psi(x)$ to a source $s(x)$:

$$\psi(x) = \mathcal{O}^{-1} \sum_n \langle e_{(n)}|s\rangle e_{(n)}(x) = \sum_n \frac{\langle e_{(n)}|s\rangle}{\lambda_n} e_{(n)}(x) \ , \tag{4.32}$$

where we've simply used (3.38). The inner product $\langle e_{(n)}|s\rangle$ is an overlap integral between known functions:

$$\psi(x) = \int dy \sum_n \frac{e^*_{(n)}(y) e_{(n)}(x)}{\lambda_n} s(y) \ , \tag{4.33}$$

where we have rearranged terms rather suggestively. This is now in the same form as our prototype Green's function example (3.56).

Referring back to (4.1), we see that our completeness relation—that is, our trick of inserting unity—in (4.33) tells us an explicit form for the Green's function of a differential operator $\mathcal{O}$ if you know the eigenfunctions and eigenvalues of that operator:

$$G(x,y) = \sum_n \frac{e^*_{(n)}(y) e_{(n)}(x)}{\lambda_n} \ . \tag{4.34}$$

This is formally an infinite sum and so is only practically useful if each term is successively smaller.

4.9 Green's Function by Completeness: why is this helpful?

If you look at (4.34) and think about our goals for the class, you may say *hooray! We're done.* After all, given the Green's function $G(x,x')$ for a given differential operator[48] $\mathcal{O}_x$, then we know how to *invert* $\mathcal{O}_x$. So for any source $s(x)$ and differential equation $\mathcal{O}_x\psi(x) = s(x)$, we can find $\psi(x)$ by

$$\psi(x) = \int dx'\, G(x,x') s(x') \ . \tag{4.35}$$

The integral is over the domain on which we've defined the function space and subject to functions satisfying the boundary conditions. We interpreted the integral over x' as an integral over the source configuration. Armed with an expression for $G(x,x')$, we can simply perform the overlap integral with $s(x')$—numerically if needed—and that gives us $\psi(x)$. Easy! What are we missing?

First, all of this *assumed* that you know the eigenfunctions of $\mathcal{O}$. This is actually a fairly safe assumption. There are only so many differential operators that matter in physics, especially since the physically motivated operators are typically self-adjoint and respect many symmetries. In fact, there are so few of these that their eigenfunctions are all famous—so when you're slogging through electrodynamics dealing with spherical harmonics, Bessel

[48] We've explicitly written the x in $\mathcal{O}_x$ to indicate that derivatives are with respect to that variable, *not* x'.

functions, and Legendre polynomials—you know that these special functions are 'special' because they're eigenfunctions of variations of the Laplacian that show up in physics over and over again. They are so important that ancient graduate students had to use them *before* one could just plug them into *Mathematica*[49]. All that is to say for any differential equation that you will probably ever care about in physics, the eigenfunctions are probably known and their properties are well documented[50].

Okay, so if the identification of eigenfunctions is not a problem, why isn't (4.34) the end of this course? One reason is that it is an *infinite* series. The differential operator is a 'matrix' in infinite-dimensional space, so there are an infinite number of eigenfunctions that space the space. If you're like me, you really only want to deal with one or two terms—very rarely is it worth it to have to go to many more terms[51]. This means that the infinite sum is only practical is each successive term is a small correction to the previous terms. While this is not always the case, this may start to sound familiar to you. Let's see it in action with an example.

Example 4.7 *The eigenfunctions for the angular part of the Laplacian, ∇^2, in spherical coordinates are the* ***spherical harmonics****, $Y_{\ell m}(\theta, \varphi)$. When you tack on the radial piece, the Green's function for the Laplacian in spherical coordinates is*

$$G(\mathbf{r}, \mathbf{r}') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} Y_{\ell m}(\theta, \varphi) Y^*_{\ell m}(\theta', \varphi') \frac{r_<^\ell}{r_>^{\ell+1}} \ , \tag{4.36}$$

where $r_> = \max(\mathbf{r}, \mathbf{r}')$ and $r_< = \min(\mathbf{r}, \mathbf{r}')$. To be concrete you can assume that $r > r'$ so that $r_> = r$ and $r_< = r'$. This corresponds to an observer further away from the origin than the source. Remind yourself of where expressions just like this *show up in electrodynamics—for example, a charged cat curled up into a small lump near the origin of your coordinate system.*

Note that (4.36) *has* two *sums over eigenfunction 'labels' m and ℓ. That's okay—this simply generalizes the case of a single sum. Clearly this expression has the form of a completeness relation with the radial piece tacked on.*

The upshot of having his expression is that you can take any *source $\rho(\mathbf{r})$, such as that lump of charged cat, and write a closed form expression for the state (e.g. the electrostatic*

[49] Have you ever heard of Gradshteyn and Ryzhik? When I was a student there was a story that most of it was written while the authors were bored in Siberia. In Cornell the theoretical physics journal club used to be called the Gradsteyn seminar because it would "integrate the knowledge of the graduate student participants." (Source: Michael Peskin, private communication.) Anyway, if you've made it this far in the footnote: you should consider running a journal club with your lab/classmates. It may be the best preparation you can give yourself for being a young academic.

[50] By the way, if you were expecting this class to be about the properties of Bessel functions and all that, then forget it! I find nothing fun about that. We're going to stick to good old sines and cosines because *all* of the essential intuition is already there. If you deeply understand the orthogonality, completeness, projections onto trigonometric functions, then you can 'read' the special functions as generalizations of the trigonometric functions for their respective differential operators. By the way, beware of any young person who seems to know the Bessel function properties *a little too well*... that person has probably been through some shit.

[51] One notable local exception is Prof. Hai-Bo Yu's work on self-interacting dark matter calculations. In the resonant regime, some of these numerical results require sums over hundreds of partial waves.

potential):

$$\Phi(\mathbf{r}) = \int d^3\mathbf{r}' \frac{r_<^\ell}{r_>^{\ell+1}} \left[\sum_{\ell,m} \frac{1}{2\ell+1} Y_{\ell m}(\theta,\varphi) Y^*_{\ell m}(\theta',\varphi') \right] \rho(\mathbf{r}') \ , \tag{4.37}$$

where the expression in the bracket has a special name:

$$P_\ell(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}') = \sum_{\ell,m} \frac{1}{2\ell+1} Y_{\ell m}(\theta,\varphi) Y^*_{\ell m}(\theta',\varphi') \ . \tag{4.38}$$

The P_ℓ's are called Legendre polynomials[52] *In the limit where $r \gg r'$, the expression takes the following form:*

$$\Phi(\mathbf{r}) = \sum_{\ell,m} \frac{1}{2\ell+1} \frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}} \left[\int d^3\mathbf{r}' \, r'^\ell Y^*_{\ell m}(\theta',\varphi') \rho(\mathbf{r}') \right] \ , \tag{4.39}$$

where now the term in the brackets is purely a property of the source. Do you recognize what it is? This is simply the ***multipole expansion*** *of the charged, lumpy cat. Observe that each successive term in the sum is suppressed by an additional power of r'/r. As long as $r \gg r'$—that is, as long as we are far away from the charged, lumpy cat—we can approximate its electrostatic potential as the sum of a monopole term, dipole term, etc.*

What we see from the above example is that in the limit where there is a small parameter, the Green's function series expression (4.34) coming from the completeness of eigenfunctions can be seen as a Taylor expansion.

4.10 Patching a Green's function together

There is another clever[53] way of solving for Green's functions. We'll leave most of this work to your homework, but let's sketch the procedure.

Recall that Green's functions are the analogs to inverse matrices in a finite dimensional vector space. In other words,

$$A(A^{-1}) = A^i{}_j (A^{-1})^j{}_k = \mathbb{1}^i{}_k = \delta^i_k \ . \tag{4.40}$$

The infinite dimensional version of this is

$$\mathcal{O}_x G(x,x') = \delta(x-x') \ . \tag{4.41}$$

[52] Once when I was teaching a class of undergraduates in electromagnetism I asked them if they knew what these special functions, P_ℓ were called. One of them enthusiastically shouted, *ooh! Is that a Pessel function*? That's when I learned to appreciate the joy of serendipity in teaching.

[53] 'Clever' is not always a positive word. A mathematical technique that is *clever* may have an aesthetic quality that we can appreciate, but it's not practically useful if you have to be *clever* to know to use it. We would rather prefer something that is general and systematic. By the way, this is the reason that high-energy experimentalists all know how to use version control software for their thousand-person publications while theorists have a hard time working simultaneously on a draft between three people.

Example 4.8 *Let's do a quick 'sanity' check for why $\delta(x-x')$ could plausibly play the role of an identity element, δ^i_k. When δ^i_k acts on a vector, it acts on each component as (writing the sum explicitly):*

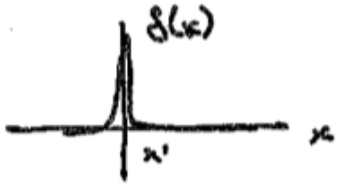
$$\sum_k \delta^i_k v^k = v^i \ . \tag{4.42}$$

Recalling that finite-dimensional indices are arguments in function space, the analog for the $\delta(x-x')$ acting on a function f is

$$\int dx' \, \delta(x-x') f(x') = f(x) \ . \tag{4.43}$$

If we compare (4.41) to the class of equations that we wanted to solve, $\mathcal{O}\psi(x) = s(x)$, we realize that the Green's function $G(x,x')$ is simply the *state* ψ at position x coming from an idealized δ-function source at position x'. Of course, there's no such thing as Dirac δ-function sources in nature, so we emphasize that this interpretation should not be taken literally[54]. Heuristically, the Green's function equation looks like:

$$\mathcal{O}_x G(x,x') = \quad . \tag{4.44}$$



Note that the source is *zero* for everywhere. This means that everywhere to the left of $x = x'$ is described by a homogeneous equation,

$$\mathcal{O}_x G_<(x,x') = 0 \ . \tag{4.45}$$

Further, everything to the right of $x = x'$ is described by *another* homogeneous equation,

$$\mathcal{O}_x G_>(x,x') = 0 \ . \tag{4.46}$$

These are different equations for *different functions*: $G_<$ and $G_>$ are two different functions that obey homogeneous equations *in their respective domains*. $G_<(x,x')$ is *not* defined for $x > x'$. Usually solving homogeneous equations is easier[55].

The strategy then is to solve for $G_<(x,x')$ and $G_>(x,x')$ as functions of x and *patch them together*:

$$G(x,x') = \begin{cases} G_<(x,x') & \text{if } x < x' \\ G_>(x,x') & \text{if } x > x' \end{cases} \ . \tag{4.47}$$

[54] In undergraduate electrodynamics we say that the Coulomb potential is the result of a δ-function point source... but you don't actually believe that electrons are δ functions in charge do you? If you do, take some time to think about this. By the way, this is related to Exercise 1.1.

[55] I wouldn't really know, but it seems to take less time on *Mathematica* so there you go.

Here x' is just a spectator variable—we're keeping it fixed. For simplicity, you may even want to shift your coordinates so that $x' = 0$. When we do this, we usually have a second order differential equation—some variant of the Laplacian because that's 99% of what we do—so we need to have enough boundary conditions to fix our cofficients. Since we have two functions in a second order differential equation, we need *four* boundary conditions. When we defined the Green's function problem, presumably we are considering functions over some interval $x, x' \in [a, b]$. This gives boundary conditions at a and b, which may even be at $a = -\infty$ and $b = \infty$. The two additional boundaries are obtained at $x = x'$. These come from requiring the continuity of the solutions

$$G_<(x', x') = G_>(x', x') \tag{4.48}$$

and a 'jump condition' between the first derivatives of the soltuiosn:

$$\lim_{\epsilon \to 0} \int_{x'-\epsilon}^{x'+\epsilon} dx \mathcal{O}_x G_(x, x') = 1 \ , \tag{4.49}$$

where this comes from simply integrating the defining equation $\mathcal{O}_x G(x, x') = \delta(x - x')$ over a sliver around $x = x'$. Since $\mathcal{O}_x$ is assumed to be second order, the jump condition reduces to saying that the first derivatives of $G_<$ and $G_>$ are discontinuous at $x = x'$ by a certain amount. Applying these boundary conditions then gives a piece-wise solution for the Green's function.

4.11 Where we're going

Our primary goal in this course is to find the Green's function $G(x, x')$ given a differential operator $\mathcal{O}$. There are three primary ways to do this:

1. **Eigenfunctions and completeness**, Section 4.9. Assuming one knows the eigenfunctions of the differential operator, this gives a series solution for the Green's function. It is practically useful only when the series is convergent.
2. **Patching**, Section 4.10. This method assume that one can solve the *homogeneous* differential equation $\mathcal{O}_x G(x, x') = 0$ and then produces a piece-wise solution to the *inhomogeneous* differential equation that defines the Green's function, $\mathcal{O}_x G(x, x') = \delta(x - x')$. This is practically useful in one dimension where the boundary conditions where the pieces are connected are easy to define.
3. **Fourier transform and its cousins**. This will be the topic of the rest of our course. We convert the differential equation into an *algebraic equation* in momentum space. Aspects of the causal structure of the system that are manifested in complex momentum space. Furthermore, one can use contour integrals to do the 'hard work.'

Recently I filled a hole in my undergraduate education and used a fourth method called the *method of variations* to solve inhomogeneous differential equations. The method is sketched out in Appendix A. We won't have anything further to say about that here, except that it turns out you can get a faculty job in theoretical particle physics without knowing how to use it.

Example 4.9 *This problem is from Matthews & Walker Section 9-4. Consider a unit string with frequency $k = \omega/c$ and Dirichlet boundary conditions at $x = 0, 1$; where we note that we are using units of 'length of the string.' The differential operator describing standing waves is*

$$\mathcal{O}_x = \frac{d^2}{dx^2} + k^2 \ . \tag{4.50}$$

Let's solve this using eigenfunctions. We know the normalized eigenfunctions from Example 4.5:

$$f_n(x) = \sqrt{2}\sin(n\pi x) \qquad \mathcal{O}_x f_n(x) = \left(-n^2\pi^2 + k^2\right) f_n(x) \equiv \lambda_n f_n(x) \ . \tag{4.51}$$

By the way, it should have been obvious *that these are the eigenfunctions and eigenvalues, even though $\mathcal{O}_x$ is* not *the same as d^2/dx^2. Using our completeness relation, the Green's function is*

$$G(x, x') = \sum_n \frac{f^*(x) f(x')}{\lambda_n} = 2\sum_n \frac{\sin(n\pi x)\sin(n\pi x')}{k^2 - n^2\pi^2} \ . \tag{4.52}$$

Thus the solution to the system with some inhomogeneous source $s(x)$

$$\left[\frac{d^2}{dx^2} + k^2\right] f(x) = s(x) \tag{4.53}$$

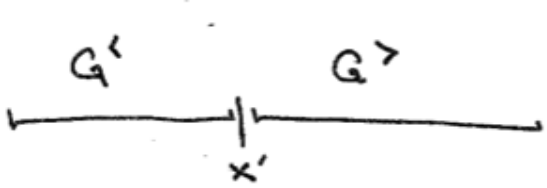
is simply

$$f(x) = \int_0^1 dx'\, G(x, x') s(x') \ . \tag{4.54}$$

Example 4.10 *Let's do the same example with the patching method. In this case we start with the equation (the analog of $A(A^{-1}) = \mathbb{1}$):*

$$\left[\frac{d^2}{dx^2} + k^2\right] G(x, x') = \delta(x - x') \tag{4.55}$$

We now separate the domain into $x < x'$ and $x > x'$, with a priori *independent solutions $G_<(x, x')$ and $G_>(x, x')$:*



Applying the Dirichlet boundary conditions at $x = 0, 1$ gives

$$G(x, x') = \begin{cases} G_<(x, x') = a\sin(kx) & \text{for } x < x' \\ G_>(x, x') = b\sin\left(k(x-1)\right) & \text{for } x > x' \end{cases} \ . \tag{4.56}$$

Make sure you understand why $G_>(x,x')$ has a factor of $(x-1)$ and not x; this is simply the boundary condition at $x=1$ without setting $b=0$. The two coefficients a and b must be fixed by the matching at $x=x'$. To do this, integrate the second order differential equation over a sliver around x':

$$\int_{x'-\varepsilon}^{x'+\varepsilon} dx \ , \left[\frac{d^2}{dx^2}+k^2\right] G(x,x') = \int_{x'-\varepsilon}^{x'+\varepsilon} dx\, \delta(x-x') \ . \tag{4.57}$$

Note that the k^2G term in the integrand vanishes since it scales like ε. The integral of the second derivative is simple since it is simply the integral of a derivative, $\int dx\, d/dx(G') = \int d(G') = G'$ so that

$$\frac{d}{dx}G\Big|_{x'-\varepsilon}^{x'+\varepsilon} = G'_>(x',x') - G'_<(x',x') = 1 \ . \tag{4.58}$$

This is the jump condition of the first derivative. If we integrate the jump condition once more over a sliver between $x \pm \varepsilon$ gives the continuity condition:

$$G_<(x',x') = G_>(x',x') \ . \tag{4.59}$$

The jump and continuity conditions give

$$ka\cos(kx') + 1 = kb\cos\left(k(x'-1)\right) \tag{4.60}$$

$$a\sin(kx') = b\sin\left(k(x'-1)\right). \tag{4.61}$$

One can solve for the coefficients:

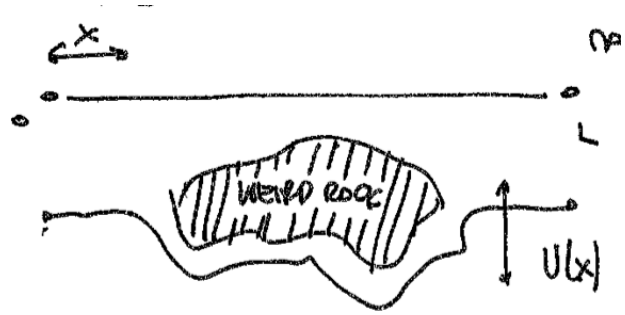
$$a = \frac{\sin\left(k(x'-1)\right)}{k\sin k} \qquad\qquad b = \frac{\sin kx'}{k\sin k} \ . \tag{4.62}$$

Plugging this all in gives

$$G(x,x') = \frac{1}{k\sin k}\begin{cases}\sin kx\, \sin\left(k(x'-1)\right) & \text{if } x<x' \\ \sin kx'\, \sin\left(k(x-1)\right) & \text{if } x>x' \ .\end{cases} \tag{4.63}$$

Observe that you find two *different* expressions for G by these methods. Please confirm—perhaps numerically—that these indeed represent the same function $G(x,x')$.

Exercise 4.9 *This example is from Butkov, chapter 12.1. Consider a taut string of length L under the load of a weirdly-shaped rock:*



The force equation for the vertical displacement of the string, $u(x)$, is

$$Tu''(x) = F(x), \tag{4.64}$$

where T is the tension and $F(x)$ is the force-per-unit-length. One may write this more simply as $u''(x) = f(x) = F(x)/T$. Show that the Green's function may be written as a piecewise function

$$G(x,x') = \begin{cases} \left(\frac{x'-L}{L}\right) x & \text{if } x < x' \\ \left(\frac{x-L}{L}\right) x' & \text{if } x > x' \end{cases} . \tag{4.65}$$

Suppose you replace the weird rock by a square paper weight of size $x < L$ in the middle of the string. Sketch what the paper weight on the string looks like.

5 Complex Analysis Review

5.1 Teaser: why are we doing this?

lec 12

We now switch gears to complex analysis. One of the results that I hope you'll come to appreciate in your study of physics is the rich role of complex numbers (and their generalizations) in describing nature. Our study of complex analysis will focus on motivating the residue theorem to calculate integrals in complex space. This, in turn, is useful when we represent functions in *momentum space* by Fourier transforming. For example, a Green's function $G(x,x')$ can be written with respect to the Fourier momentum variable k by

$$G(x,x') = \int đ k\, e^{-ikx} \widetilde{G}(k,x') \ , \tag{5.1}$$

where I've used a convenient notation where $đ = d/2\pi$. This, in turn is useful because our defining relation for the Green's function is (4.41):

$$\mathcal{O}_x G(x,x') = \delta(x-x') \ , \tag{5.2}$$

which we may then write (Fourier transforming both sides) as

$$\mathcal{O}_x \int đ k\, e^{-ikx} \widetilde{G}(k,x') = \int đ k\, e^{-ikx} e^{ikx'} \ . \tag{5.3}$$

On the right-hand side we've written the $\delta(x-x')$ in its Fourier representation. We have assumed that $\mathcal{O}_x$ is a polynomial in derivatives with respect to x:

$$\mathcal{O}_x = \sum_{n=0}^{\infty} p_n(x) \left(\frac{d}{dx}\right)^n \equiv P\left(x, \frac{d}{dx}\right) \ . \tag{5.4}$$

What's powerful about this is that that when we apply a differential operator in x onto the Fourier representation of a function, say (5.1), then the derivatives only 'see' the exponential factor e^{-ikx}. This means that derivatives are particularly easy:

$$\left(\frac{d}{dx}\right)^n e^{-ikx} = (-ik)^n \ . \tag{5.5}$$

This means that we may write the differential operator $\mathcal{O}_x$ in (5.4) when it acts on the Fourier representation as

$$\mathcal{O}_x = \sum_{n=0}^{\infty} p_n(x)\,(-ik)^n \equiv P\,(x, -ik)\ , \tag{5.6}$$

so that the defining relation for G, (5.3), may in turn be simplified:

$$\int đ\,k\, e^{-ikx} P(x, -ik)\widetilde{G}(k, x') = \int đ\,k\, e^{-ik(x-x')}\ , \tag{5.7}$$

which suggests a simple expression for the Fourier coefficients:

$$\widetilde{G}(k, x') = e^{ikx'}\,[P(x, -ik)]^{-1}\ , \tag{5.8}$$

where one may find $G(x, x')$ from simply taking the inverse Fourier transform. There will turn out to be a few neat features of this approach, but before getting there, let's start from the very beginning.

lec 13

5.2 Complex Numbers

A complex number z may be decomposed into real and imaginary parts

$$z = x + iy\ , \tag{5.9}$$

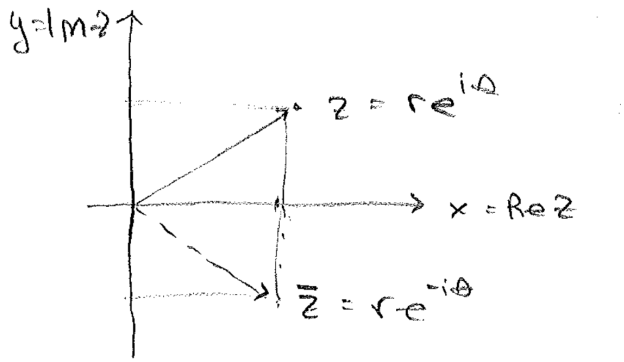
where x and y are both real numbers. We may also define the complex conjugate as

$$\bar{z} = z^* = x - iy\ . \tag{5.10}$$

One may also write these variables in a polar notation,

$$z = re^{i\theta} \qquad\qquad \bar{z} = re^{-i\theta}\ , \tag{5.11}$$

where $r \in [0, \infty]$ and $\theta \in [0, 2\pi]$:



Complex numbers live in the space of complex numbers $\mathbb{C}$, a *two-dimensional* real space with coordinates (x, y) in $\mathbb{R}^2$ augmented by an additional rule for multiplication (take two complex numbers and return a complex number) that is called the *complex structure*. This is basically the definition $i^2 = -1$, though one may generalize to higher complex dimensions.

5.3 Complex functions

Complex functions take complex numbers and return complex numbers. Inspired by the two-dimensional-ness of $\mathbb{C}$, let us write a complex function as $f(z) = f(x, y)$ where $z = x + iy$. Because $f(x, y)$ is, itself, a complex number for a given $z = x + iy$, we can decompose f into two real-valued functions

$$f(z) = f(x, y) = u(x, y) + iv(x, y) \ . \tag{5.12}$$

This is clearly a map from $\mathbb{C} \to \mathbb{C}$ and so we can simply "do multivariable calculus" on this space.

If we forget the complex structure, then this *seems* to simply be the calculus of maps between $\mathbb{R}^2 \to \mathbb{R}^2$ where

$$f(x, y) = u(x, y)\hat{\mathbf{x}} + v(x, y)\hat{\mathbf{y}} \ . \tag{5.13}$$

In that case, we need to collect the partial derivatives. We know that we can write the variation of f as

$$\delta f(x, y) = \delta u(x, y)\hat{\mathbf{x}} + \delta v(x, y)\hat{\mathbf{y}} = \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right)\hat{\mathbf{x}} + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right)\hat{\mathbf{y}} \ . \tag{5.14}$$

Remembering our complex structure, we can write this succinctly as

$$\delta f = \frac{\partial f}{\partial z}\delta z + \frac{\partial f}{\partial \bar{z}}\delta \bar{z} \ , \tag{5.15}$$

where we're using

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial (iy)}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial (-iy)}\right) \tag{5.16}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \ . \tag{5.17}$$

The fact that partial derivatives combine like this may be familiar[56].

So far we have emphasized that $f(x, y)$ behaves like a function in two real dimensions. Critically, in some ways we want to think about functions $f(z)$ that are functions of 'one' [complex] dimension rather than two real dimensions. In two real dimensions, calculus came with a notion of directional derivative. When we wanted the rate of change for a function, we had to ask *in what direction* are we checking the rate of change. At a given function may change by a positive amount in the x-direction, but a negative amount in the y-direction, for example. This is in contrast to one-dimensional calculus where each point had an unambiguous derivative that represented how much the function changed in the positive x direction with respect to a point x_0:

$$\delta f(x) = \frac{df(x_0)}{dx}(x - x_0) + \cdots \ . \tag{5.18}$$

[56] This comes from the underlying mathematical structure. The partial derivatives are basis vectors for the *tangent space* at some point (x, y). These partial derivatives ask: how much does the function I'm acting on change if I move by one unit in this direction?

This is in contrast to the change in a general complex function (5.15). In fact, if we explicitly wrote some expansion with respect to a point z_0, it looks a little funny:

$$\delta f(z) = \frac{\partial f}{\partial z}(z - z_0) + \frac{\partial f}{\partial \bar{z}}(\bar{z} - \bar{z}_0) + \cdots \ . \tag{5.19}$$

What is going on here? Why should $f(z)$ be a function of one complex variable z but the expansion seems to depend not only on $(z - z_0)$ but also $(z - z_0)^*$?

5.4 Analytic complex functions are nice

This leads us to a definition of "nice" complex functions. The following terms are all more-or-less equivalent for this sense of nice-ness: **[complex] differentiable**, **analytic**, **holomorphic**, **regular**. This is simply the restriction that the $\partial f/\partial \bar{z}$ term should vanish so that the variation $f(z)$ only depends on δz and not $\delta \bar{z}$. In other words, an analytic function admits usual single-dimensional definition of the derivative,

$$\frac{df(z_0)}{dz} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \ . \tag{5.20}$$

The key aspect of this is that there is only *one* value of df/dz at z_0 no matter how you approach z_0. In other words, it doesn't matter if $\delta z = z - z_0$ is coming from the positive/negative real/imaginary direction. The value of df/dz is the same.

For most of these lectures we will use the term **analytic** to refer this property. Let us see what happens when we impose $df/d\bar{z} = 0$ onto the component functions $u(x, y)$ and $v(x, y)$:

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)[u(x, y) + iv(v, y)] = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \ . \tag{5.21}$$

This implies the **Cauchy–Riemann equations**, which are equivalent to the function f being analytic:

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ . \tag{5.22}$$

Thus far we've talked about functions being analytic or not. It turns out that analytic functions are *so* nice that there really aren't that many physically relevant functions that are analytic *everywhere*[57]. Instead, we will often talk about functions that are analytic in most places but are not analytic (differentiable) in other places. Indeed, the *non*-analyticity of Green's functions is a key result in this class.

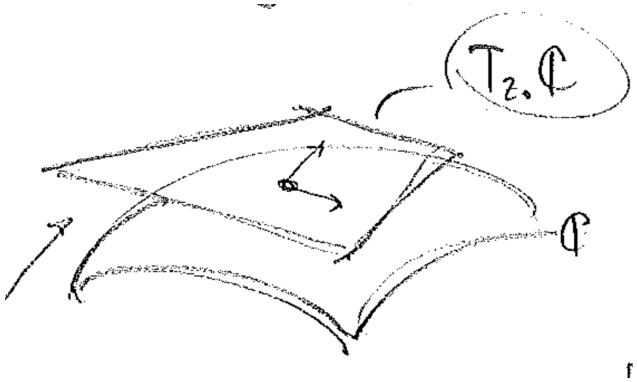
5.5 A geometric point of view (optional)

For culture, let us comment on a geometric perspective of analyticity/complex-differentiability. The differential of a function, df, is a **differential one-form**. If you recall some of our

[57] The analog here is people who are nice. Most people are nice most of the time. But very few people are nice *all* of the time. I suspect Fred Rogers may be one of the few who could plausibly be ~~analytic~~ *nice* everywhere.

nomenclature for dual vectors, a one-form is a kind of 'row vector': it is a linear function that takes in a vector and spits out a number. If df is such a dual vector, what is the vector space upon which it acts?

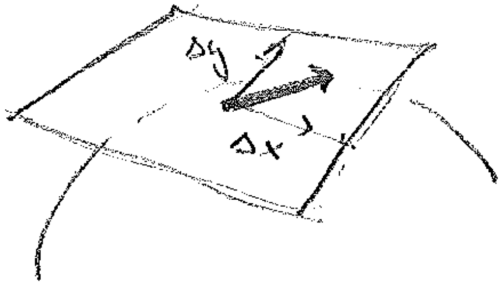
Let's specifically consider the differential of f at a specific point z_0 in the complex plane: $df(z_0)$. This is not a number. In order to get a number from this, you have to feed it a vector: I claim this vector is an element of the tangent plane[58] of the complex plane at z_0: $T_{z_0}\mathbb{C}$. We can sketch this if you allow me to artificially 'curve' the complex plane to help make it easy to distinguish:



Using the jargon of differential geometry, we call the complex plane the *base manifold*. The set of all tangent spaces is called a *tangent bundle*[59]. Each tangent space is a vector space 'attached' to a single point of the base space.

The object $df(z_0)$ is a dual vector on the vector space[60] $T_{z_0}(\mathbb{C})$. The vectors of this space are 'velocities' of trajectories that pass through z_0. For now let's think of them as quantities

$$\mathbf{v} = \Delta x + i\Delta y \ . \tag{5.23}$$



[58] The tangent plane is exactly what it sounds like: if you have a curved surface like a globe, the tangent plane at z_0 and glued a rigid piece of cardboard to one point, z_0. Now imagine that the piece of paper has graph paper on it.

[59] More general types of vector spaces can be attached to each point of the base manifold. These are called fiber bundles. They are the underlying mathematical structure of mechanics and reflect why Hamilton's equations are so damn special. They are also the mathematical structure that defines gauge theories and so address the question "what is a force?"

[60] The standard notation is to write a dual vector of a space V as a vector in the dual space V^*, so one could write $df(z_0) \in T^*_{z_0}(\mathbb{C})$.

Then we may write the linear action of $df(z_0)$ on $\mathbf{v} \in T_{z_0}(\mathbb{C})$ as

$$df(z_0)\,[\mathbf{v}] = f'(z_0)\Delta x + f'(z_0) i \Delta y \ , \tag{5.24}$$

where we assume that f is analytic so that $f'(z_0)$ is unambiguously defined (independent of the direction of δz) as in (5.20).

However, if we think about this complex space as a two-dimensional real space, the following is also true:

$$df(z_0)\,[\mathbf{v}] = \left.\frac{\partial f}{\partial x}\right|_{z_0} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{z_0} \Delta y \tag{5.25}$$

Comparing (5.24) and (5.24) gives

$$\frac{\partial f}{\partial x} = f'(z_0) = -i\frac{\partial f}{\partial y} \ . \tag{5.26}$$

Recalling that $f = u + iv$ then gives

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \ . \tag{5.27}$$

Setting the real and imaginary parts equal to one another give, as expected, the Cauchy–Riemann equations.

5.6 Analytic and harmonic

One more comment is in order. If a function $f(z) = u(x, y) + iv(x, y)$ is analytic, then its real and imaginary parts are independently 2D harmonic with respect to $\mathbb{R}^2$:

$$\partial_x^2 u(x, y) + \partial_y^2 u(x, y) = 0 \qquad\qquad \partial_x^2 v(x, y) + \partial_y^2 v(x, y) = 0 \ . \tag{5.28}$$

Harmonic functions show up all over the place—though admittedly you probably care mostly about three-dimensional harmonic functions. The above realization may be helpful, however, for 3D systems for which the third dimension is trivial. For example, you may want to consider fluid flow across an airplane wing. One can take the limit where the wing is infinitely long so that the fluid dynamics reduces to the two-dimensional motion around a cross section of the wing. Alternatively, maybe you care about an electrostatic system which are similarly 'effectively' 2D. In this case you can 'promote' your real harmonic variable (a fluid dynamics potential or electrostatic potential) to an analytic function and then bear the full power of complex analysis to attack the problem. One of the most fascinating—if outdated—methods is called **conformal mapping** by which one can solve for a harmonic function with weird boundary conditions by mapping to much simpler boundary conditions. This topic is beautiful and gives a first, intuitive example of *conformal transformations* which are a pillar of theoretical physics (in any discipline). The technique is how engineers in the pre-personal computer era designed airplane wings. For experimentalists, it is pretty neat that UCR's own Nathan Gabor has created an analogous micromagnetic 'solver' of this very same problem[61].

[61] https://arxiv.org/abs/2002.07902

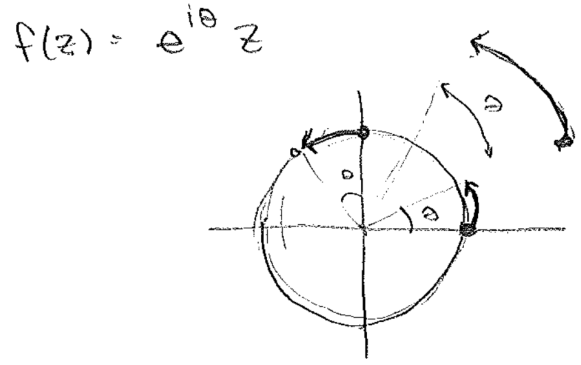
5.7 Complex functions as maps

A complex function $f(z)$ takes in a complex number and returns a complex number. These are maps of the complex plane to itself. It is useful to build an intuition for this rather trivial-sounding statement: complex functions 'deform' the complex plane.

Example 5.1 *Consider the function that multiplies a complex number by a constant phase,*

$$f(z) = e^{i\theta} z \ . \tag{5.29}$$

This takes each point and rotates it in the counterclockwise direction by θ.

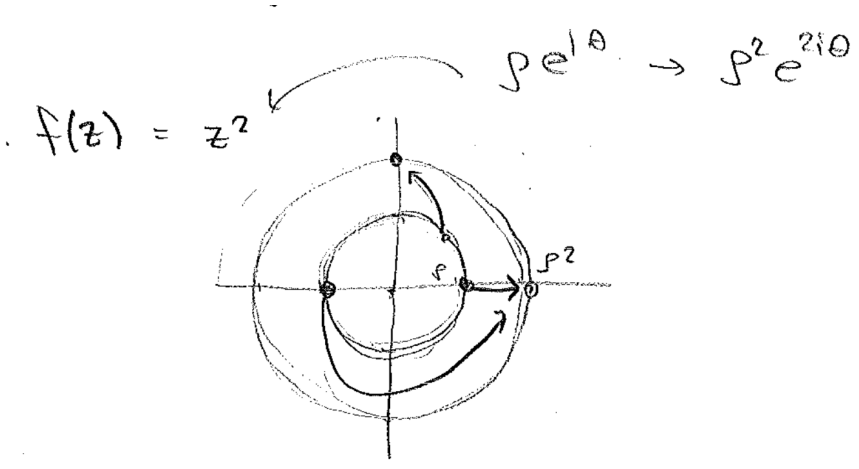


Exercise 5.1 *What happens to points under $f(z) = z + z_0$?*

Example 5.2 *Now consider the simplest non-trivial polynomial, the function that squares its argument:*

$$f(z) = z^2 \ . \tag{5.30}$$

For a point $z = \rho e^{i\theta}$, $f(z) = \rho^2 e^{2i\theta}$. This squares the modulus (length) of each point and rotates according to how much the point is already rotated relative to the positive real axis.



Points that are inside the unit circle get pulled closer to the origin while points outside the unit circle are pushed away from the origin. Points in the first quadrant (positive real and imaginary parts) are sent to points in the upper half plane (positive imaginary part).

Exercise 5.2 *Show that under the map $f(z) = z^2$, a vertical line gets mapped onto a parabola:*

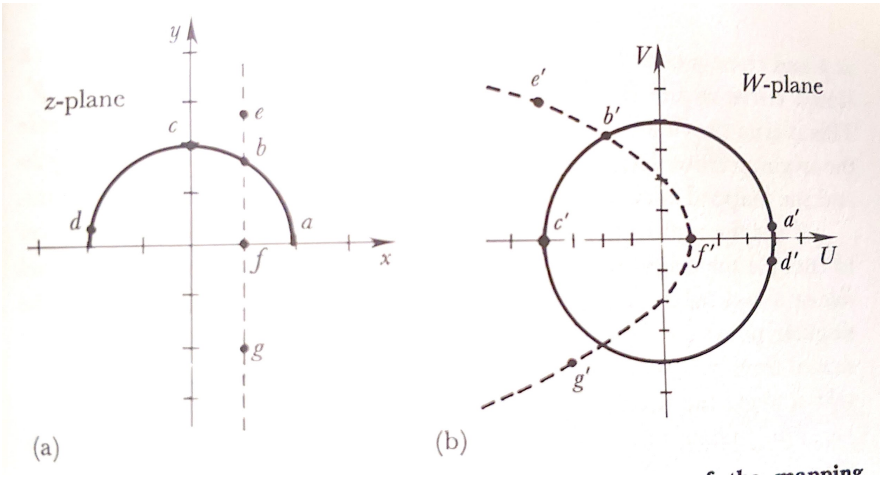
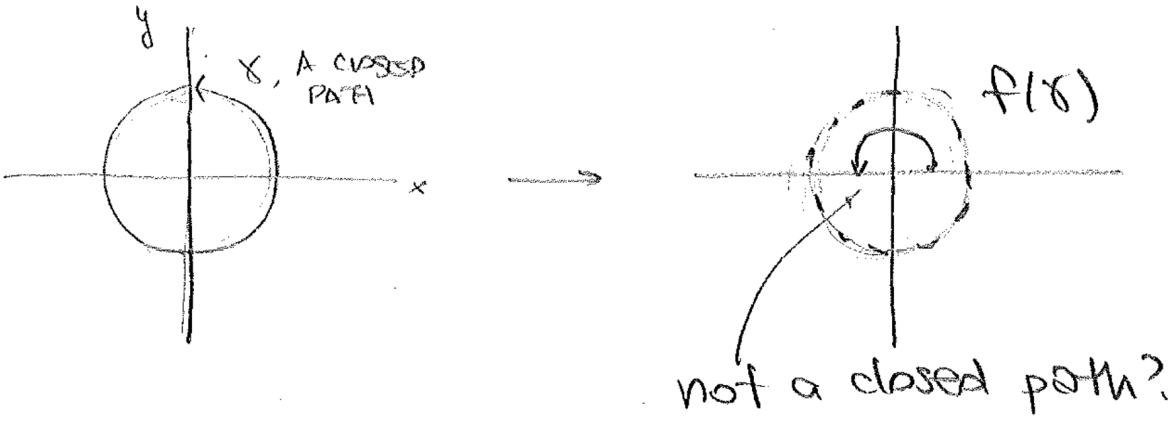


Figure from Matthews and Walker, points a, b, etc. are mapped onto a′, b′ etc. The W-plane corresponds to the image of the complex plan under z^2.

Now let's consider a curious case. What about the square root function?

$$f(z) = \sqrt{z} \ . \tag{5.31}$$

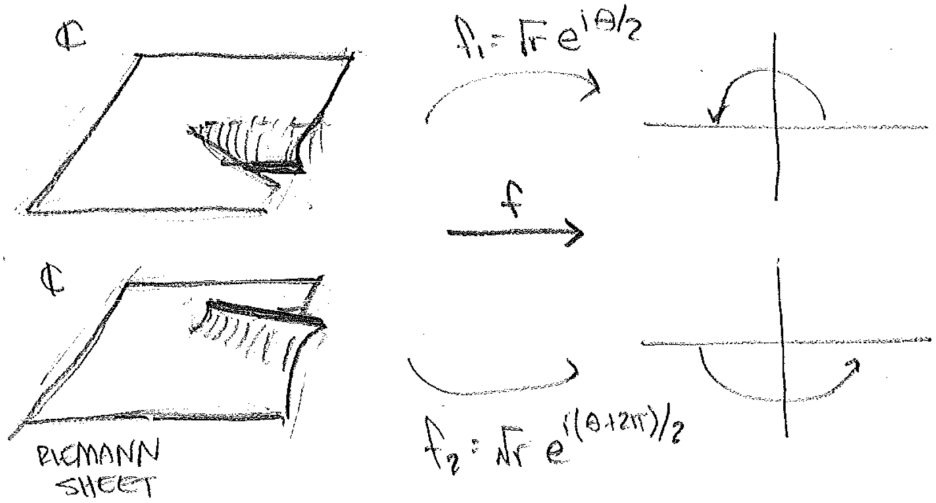
Something very weird happens here. Consider a closed path $\gamma(t) = e^{it}$ with path parameter $t \in [0, 2\pi]$. This just means consider a bunch of points corresponding to $\gamma(t)$ with a bunch of values of t in the specified range. Clearly this corresponds to a circle. However, when we put those points through the square root function, these only map onto *half* of the circle.



The technical fact that this happens is not surprising at all, you can see that $f(\gamma(t)) = e^{it/2}$ which only spans the upper half circle. But a deeper question bubbles to the surface: if we're thinking about complex functions as maps of the complex plane to itself, what does it mean that $f(z) = \sqrt{z}$ appears to have 'lost' the entire lower half of $\mathbb{C}$? Or more strangely, how is it that a *closed path* has been mapped to an *open path* with endpoints?

It's clear that if we wanted to reach the *lower* half plane in the iamage, we'd need $\theta \in [0, 4\pi]$. Algebraically that means that $f(\gamma(t)) = e^{it/2}$ covers the entire complex plane *only* if we allow angles from 0 to 4π [62]. In some sense, this is totally ridiculous, since $\rho e^{i\theta} \equiv \rho e^{i(\theta+2\pi)}$. I put that third bar on the equal sign because they're supposed to be *that* equal.

If, say, $z_1 = e^{i\pi/3}$ and $z_2 e^{7i\pi/3}$ are supposed to be the *same* point, $z_1 \equiv z_2$, how is it that $f(z_1) = e^{i\pi/6}$ and $f(z_2) = e^{7i\pi/6}$ so that $f(z_1) \neq f(z_2)$? Stated differently, functions are supposed to be single valued. It appears that $f(z) = \sqrt{z}$ is *multivalued* since the same point $z_1 = z_2$ is mapped onto two different points. One way to do this is to *define* additional structure and extend the domain (pre-image) of f. In order to do this, we take *two* copies of the complex plane and stitch them together:



Each copy of the complex plane is called a **Riemann sheet**. We 'glue' these sheets together by defining that $\rho e^{i(\theta+2\pi)}$ of one sheet maps onto $\rho e^{i\theta}$ of the other sheet. Implicitly this chooses the positive real axis as a place where we *cut* each sheet and glue them together. We call this a **branch cut**. One could, of course, have chosen the domain of each sheet to be any interval, say $\theta \in [-\pi, \pi]$ and picked a different branch cut ($\theta_{\text{branch}} = \pi$).

Exercise 5.3 *Is $f(z) = \sqrt{z}$ analytic on the extended complex plane (two Riemann sheets glued together at, say, $\theta = 2\pi$)?*

Take a moment to ponder the above exercise. As you may suspect, the relevant question isn't whether a function $f(z)$ 'is analytic,' but rather *where* is that function analytic? Is it analytic (differentiable) anywhere? Sure—when stay within one Riemann sheet, say $\theta \in (0, 2\pi)$ and staying away from the boundary, $f(z) = \sqrt{z}$ is perfectly differentiable. Further, there's

[62] You may be familiar with another case where 'rotating by 2π' only takes you halfway—spinors in relativistic quantum mechanics. You should know that the 'double cover' nature of the spinor is (to the best of my view) independent of the present discussion of Riemann sheets. That double cover has to do with the universal cover of the Poincaré group. I refer to the literature on the Wigner theorem, see e.g. volume 1 of Weinberg's *Quantum Theory of Fields*. For an unrelated cute spinor paper, see https://www.jstor.org/stable/2318771.

nothing special about the branch cut—we could have placed it anywhere. Thus it doesn't seem like there's *any* place at which $f(z)$ is *not* analytic.

This should make sense: analyticity is about whether a function is differentiable at a given point. This is inherently a *local* notion. The misbehavior of $f(z) = \sqrt{z}$—the necessity of Riemann sheets and a branch cut—are *global* issues when we try to explore the *whole* space. The interplay between local properties (like derivatives and Taylor expansions) and global properties (like integrals over a closed surface or topology) is a critical theme in mathematical physics. The function $f(z) = \sqrt{z}$ is analytic everywhere, but that doesn't mean we don't have to be careful. One of the lessons from this section is that the existence of a branch cut can be critically important if you happen to take trajectories that are *loops* in the complex plane. For those who know where this is going: whenever there are branch cuts, you have to be careful with your contour integrals. This typically happens whenever you have a function that is not some series of integer powers of z.

As a final example, let's consider the complex logarithm. We'll use the notation of Byron and Fuller and write log to denote the complex natural logarithm and ln to denote the 'usual' natural logarithm acting on real numbers. For $z = re^{i\theta}$, the complex logarithm satisfies

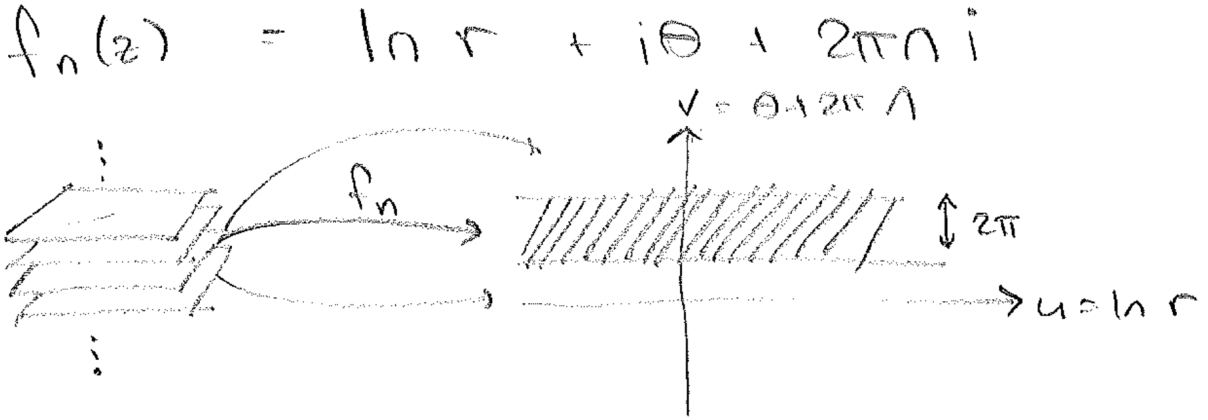
$$\log z = \ln r + i\theta \tag{5.32}$$

$$\log(z_1 z_2) = \ln(r_1 r_2) + i(\theta_1 + \theta_2) = \log z_1 + \log z_2 \ . \tag{5.33}$$

Example 5.3 *What does the domain of the complex logarithm look like? As a function, $f(z) = u(z) + iv(z)$, we see that $v(z) = \theta$ and $v(z_1 z_2) = \theta_1 + \theta_2$. This means that if we take a trajectory that keeps going around the origin, $v(z)$ just keeps increasing*[63]*. Each time you go around you must be on a new Riemann sheet.*

Rather than having an infinite number of Riemann sheets, an alternative way of thinking about this is to imagine an infinite number of equally valid 'complex logarithm' functions labeled by n:

$$f_n(z) = \ln r + i\theta + 2\pi n i \ . \tag{5.34}$$



[63] This is where you can start humming 'stairway to heaven.'

The image of the $f_n(z)$ corresponds to a horizontal strip with $v \in (2\pi n, 2\pi(n_1))$.

Example 5.4 *Is the complex logarithm analytic? Everywhere but at $z = 0$.*

Let us end by repeating the main point of introducing branch cuts:

Do not integrate across a branch cut!

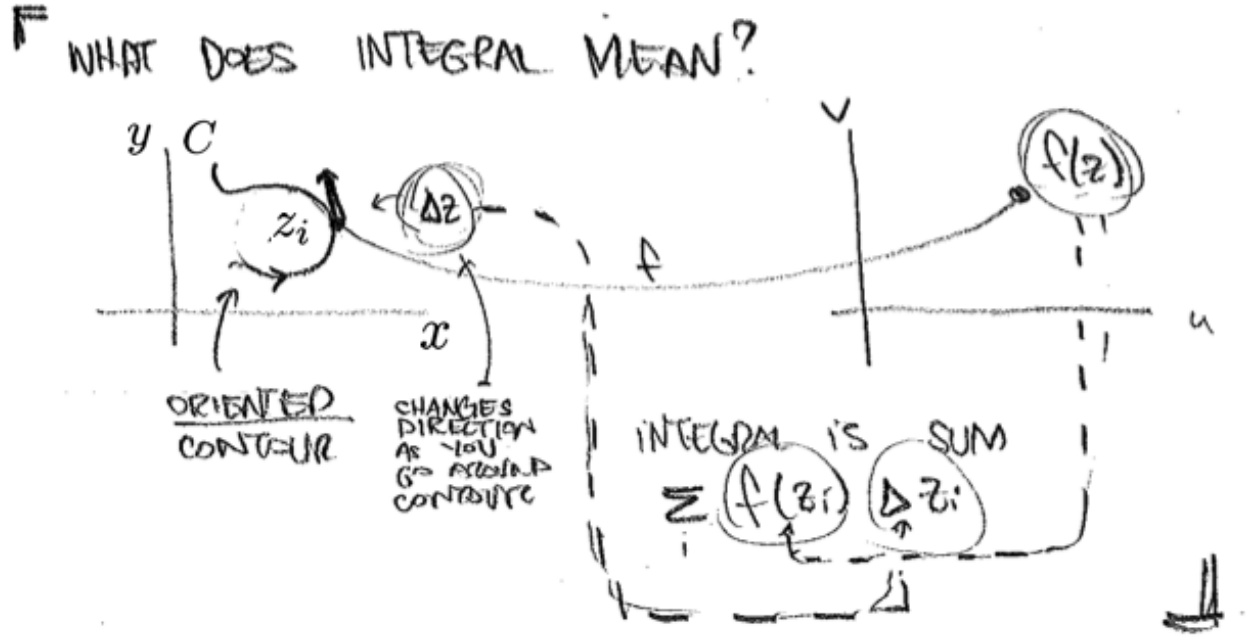
5.8 Integration on the Complex Plane

Let's now focus on integrating analytic functions. Oops, I just mis-spoke a bit there. What I meant to write was that we want to integrate functions in a region of $\mathbb{C}$ where those functions are analytic. Rarely are functions simply 'analytic' or 'not-analytic.' The functions we care about will be analytic in most places, but non-analytic in others.

When we integrate on the complex plane, $\mathbb{C}$, we have to define a **contour**—this is just the curve, C, that we are integrating over. This is similar to doing a "line integral" in $\mathbb{R}^2$: you are doing a 'one-dimensional integral' over a two[ish]-dimensional space. An integral over the contour C is:

$$\int_C dz\, f(z) = \int_C (dx + idy)\ , [u(x,y) + iv(x,y)] = \int_C [u(x,y)dx - v(x,y)dy] + i \int_C [u(x,y)dy + v(x,y)dx] \ . \tag{5.35}$$

Written this way, we are simply doing calculus on $\mathbb{R}^2$ and keeping track of funny factors of i. It may be helpful to remind ourselves what this means:



The integral along a curve can be thought of as two-dimensional version of a Riemann sum. Recall that the one-dimensional Riemann sum approximated the integral as a sum of terms $f(x_i)\Delta x$. In more than one dimension, the quantity Δx is promoted to a 'vector'[64]. In other words, we may write

$$\int_C dz\, f(z) = \sum_i \Delta z_i\, f(z_i)\ , \tag{5.36}$$

where Δz has a real and imaginary part: it *points* to a direction in the complex plane. In the picture above, the left-hand side shows the curve C on which we are integrating. Consider some point z_i on the curve. That point is mapped by the function f to a point $f(z_i) = u(z_i) + iv(z_i)$. At that point, the curve also has a differential element, Δz_i which is tangent to the curve C at point z_0 and has some fixed infinitesimal length for all i. The *direction* of Δz matters. It contributes to the overall phase of the term $\Delta z_i\, f(z_i)$.

Example 5.5 *Consider the function $f(z) = z^2 + i$. Consider a specific point, $z_0 = 1 + i$. If we consider different curves C_i that pass through z_0, the integral $\int_{C_i} dz\, f(z)$ will have different contributions at z_0 depending on the* direction of the curve *as it passes through z_0. To see this, consider the specific term in the sum (5.36) corresponding to $z_i = z_0$. If the curve C happens to be moving in the positive vertical (pure imaginary) direction, then the contribution to the sum from this point is*

$$\int_C dz\, f(z) = \cdots + i\Delta y\left[(1+i)^2 + i\right] + \cdots\ . \tag{5.37}$$

However, if the curve C happens to be moving in the horizontal (pure real) direction, the contribution to the sum from this point is

$$\int_C dz\, f(z) = \cdots + \Delta x\left[(1+i)^2 + i\right] + \cdots\ . \tag{5.38}$$

Exercise 5.4 *Suppose f is itself a derivative of an analytic function, $f(z) = dF(z)/dz$. Convince yourself that if C is a smooth connected path between two points z_0 and z_N, then the integral of $f(z)$ over C is*

$$\int_C dz\, f(z) = F(z_N) - F(z_0)\ , \tag{5.39}$$

as you would expect from real-number calculus.

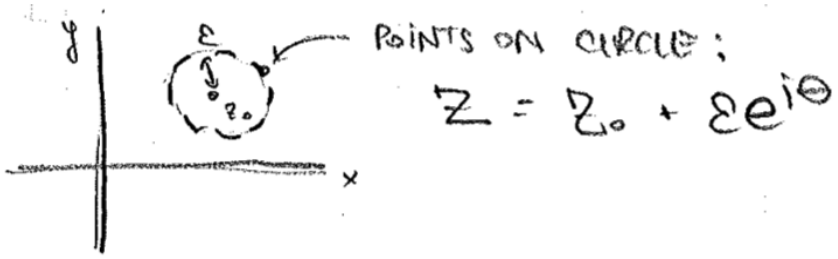
5.9 Cauchy Integral Theorem

As we may have referred to earlier—function that are analytic everywhere are too nice. Have you ever read a novel where everyone just got along nicely? Not very interesting. Complex functions are the same—it turns out that integrals of functions around closed curves in domains where they are analytic end up being zero.

[64] As I write this I feel sick to my stomach. In fact, Δx is *not* promoted to a 'vector,' though it is promoted to something that points in a direction. More properly, it is a [differential] one-form which is more like an *dual vector* in the sense referenced in Section 5.5.

5.9.1 Little tiny circles

Let's show this starting from the simplest possible case. Consider a function f that is analytic in some region $R \in \mathbb{C}$. The boundary of this region is a curve $C = \partial R$. First consider the integral of f around a small circle of radius ε around some point z_0:



In other words, C can be parameterized by the angle θ. We can write points z on the curve as

$$z(\theta) = z_0 + \varepsilon e^{i\theta} \qquad\qquad dz = i\varepsilon e^{i\theta} d\theta \ . \tag{5.40}$$

Then the integral around the little circle around z_0 is

$$\oint_C dz\, f(z) = \int_0^{2\pi} i\varepsilon e^{i\theta} d\theta\, f\left(z_0 + \varepsilon e^{i\theta}\right) \ . \tag{5.41}$$

Now we observe that since f is analytic, we can differentiate it—which means we can write it as a Taylor expansion:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \mathcal{O}(\varepsilon^2) \ , \tag{5.42}$$

where we recognize that $z - z_0 = \varepsilon e^{i\theta}$. Plugging this in gives

$$\oint_C dz\, f(z) = \int_0^{2\pi} i\varepsilon e^{i\theta} d\theta\, f(z_0) + \int_0^{2\pi} i\varepsilon^2 e^{2i\theta} d\theta\, f'(z_0) + \cdots \ . \tag{5.43}$$

Observe that the only θ-dependence in the integrand shows up in factors of $e^{ni\theta}$ for positive integers n. However, we note that

$$\int_0^{2\pi} d\theta e^{in\theta} = \frac{1}{in}\left(e^{2\pi in} - e^0\right) = 0 \ . \tag{5.44}$$

What we find is that

$$\oint_C dz\, f(z) = 0 \tag{5.45}$$

for the contour C being a small circle around some arbitrary point z_0 inside the region R in which f is analytic.

Exercise 5.5 *Convince yourself that it didn't matter that C is a small circle. It could have been any small shape.*

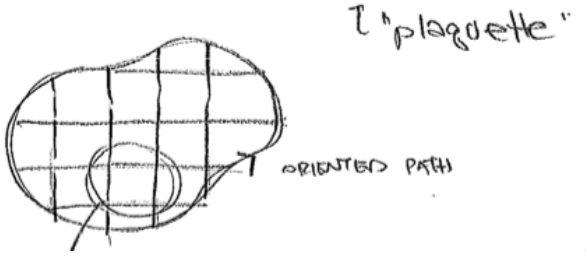
Notice that we did not rely on $\varepsilon \to 0$ in order to motivate this argument. The circle didn't actually have to be small. We used $\varepsilon \to 0$ to motivate the idea of truncating the Taylor series. But armed with (5.44), we realize that *every* term in the Taylor series will vanish, no matter how high the power since that simply corresponds to some larger positive integer n. At this point you should start to wonder whether there may be any loopholes in this argument.

5.9.2 Finite regions

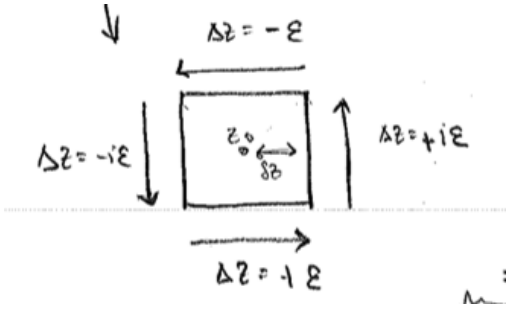
Let's now prove a more general version of this known as the **Cauchy Integral Theorem**. If f is analytic in a connected region $R \in \mathbb{C}$ with some boundary $C = \partial R$, then

$$\oint_{C=\partial R} dz\, f(z) = 0 \ , \tag{5.46}$$

even if R is some *finite* region, not just some infinitesimally small circle. From the discussion of the case where C is a little circle, you may have already guessed this. Let's show this more carefully. For any such region R, let's break it up into boxes:



The boundary of a region ∂R has some orientation. By convention a positive orientation corresponds to counterclockwise. Now that we've carved up the region R into a grid like Manhattan[65], we integrate around the boundary of one of these *plaquettes*:



We've assumed that each plaquette has characteristic size ε. Note that for each side of the plaquette the orientation of Δz is different. Let z_0 correspond to the center of this plaquette. Since the function is analytic, we can write the function as a Taylor function around the plaquette:

$$f(z) = f(z_0) + f'(z_0)(z - z_0)^2 + \cdots \tag{5.47}$$

This means that we can approximate the integral around the square as

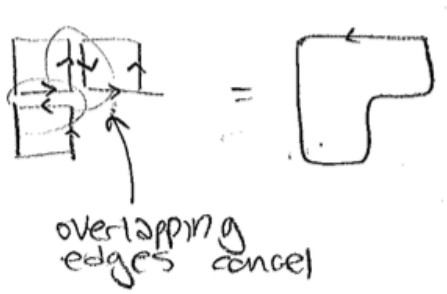
$$\oint_{\text{plaquette}} dz\, f(z) = f(z_0)\left(\Delta z_1 + \Delta z_2 + \Delta z_3 + \Delta z_4\right) + \cdots \tag{5.48}$$

where the Δz_i are given in the figure above. It should be clear that the sum of the Δz_i is zero since opposite sides give equal and opposite contributions. We have shown that to leading order the integral of an analytic function f around a little box around it is zero—not too surprising given our previous result for integrating around a little circle.

[65] Search for 'Manhattanhenge.' It has nothing to do with this course, but this is a photogenic consequence of the Manhattan grid.

Exercise 5.6 *What about the next-to-leading order contribution*[66]*? Show that the next-to-leading order contribution, which goes like* $\sum_i f'(z_0)\delta z_i \Delta z_i$ *vanishes. Here the* δz_i *is the separation from* z_0 *to the* i^{th} *side, as shown in the figure above.*

The above statement is true about each plaquette. However, note that when we piece the plaquettes together, the sum of the integrals around each little boundary corresponds to the integral around the boundary of the combined region:



One can see that neighboring boundaries have opposite orientations so that the integrals along those regions cancel. This means that the integral around a the boundary C of finite region R is equivalent to the sum of the integrals around the little plaquettes that tile that region:

$$\oint_C dz\, f(z) = \sum_i \oint_{\text{plaquette}_i} dz\, f(z) = 0\ . \tag{5.49}$$

Since the integral around the boundary of each plaquette is zero, the integral along C is zero. This proves the Cauchy integral theorem, which may now be stated as follows:

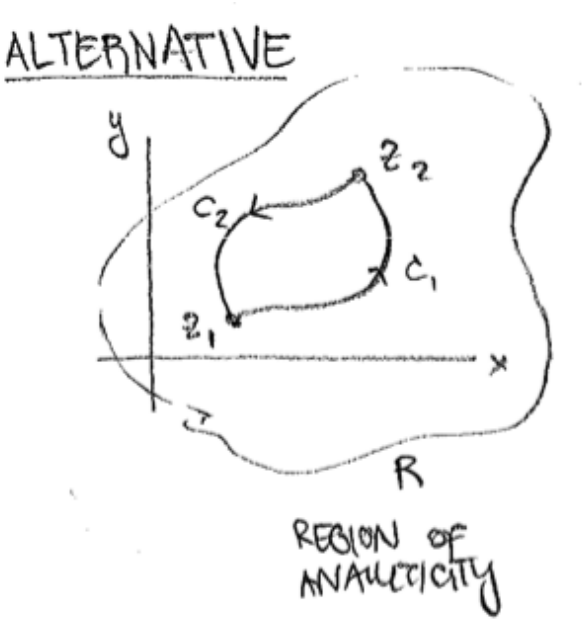
> Analytic functions are *so* nice that they're boring.

In other words, if a function f is analytic in a connected region R and you try to integrate over a closed path C that is inside R, then the result is zero.

5.10 An alternative argument (optional)

For those who are mathematically inclined, here's a geometrically-motivated argument for Cauchy's integral theorem. Consider two points z_1 and z_2 in a connected region R where a function f is analytic. Now consider any path C_1 that connects z_1 to z_2 and any other path C_2 that connects z_2 to z_1. The orientation matters.

[66] Life advice: "What about the next-to-leading order contribution" is a good question to ask whenever you have shown that the leading order contribution is zero.



In the region of analyticity, a function $f(z)$ has an anti-derivative $F(z)$ such that $f(z) = dF(z)/dz$, recall Exercise 5.4. Then we know that

$$\int_{C_1} dz\, f(z) = F(z_2) - F(z_1) \qquad \int_{C_2} dz\, f(z) = F(z_1) - F(z_2)\ . \tag{5.50}$$

We deduce that the integral of the combined path $C = C_1 + C_2$ is zero. The above argument is true without having to specify what $F(z)$ is and for *any* paths C_1 and C_2 that share common endpoints. We thus prove Cauchy's Integral Theorem.

Commentary This argument is highlights a general idea in differential geometry, which is the generalized Stokes' theorem. In words, one can state this as:

> The integral of the derivative of some function f over some domain D is simply the function evaluated on the boundary of the domain, ∂D. The ∂D is notation for 'boundary of D.'

Technically it is relevant for the domain D to be sufficiently *nice*; this includes the domain being connected and having a reasonable boundary. Note that the boundary ∂D is always oriented—when we integrate over an interval $[a, b] \in \mathbb{R}$, the overall sign depends on which boundary is 'on top' of the integral and which is 'on the bottom.' In other words, it matters if we integrate over $[a, b]$ or $[b, a]$.

We write formally as follows:

$$\int_D df = \int_{\partial D} f\ . \tag{5.51}$$

Here the domain D is an n-dimensional space and df is called a *differential n-form*. From the name you should guess that it is related to the idea of a *one-form* as a dual vector; see the

optional discussion in Section 5.5. This is a type of tensor with n indices that is contracted with the 'volume form' of the n-dimensional space. Formally it looks like this:

$$df = (\partial_{\mu_1} f_{\mu_2 \cdots \mu_n})\, dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n} \ . \tag{5.52}$$

The quantity $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$ is the volume form and is an *oriented* n-dimensional differential volume element. The funny wedge symbols ($\wedge$) represent an antisymmetric tensor product. This should not be so surprising: recall that the vector triple product $\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$ is a totally antisymmetric combination of vectors that produces the volume of the parallelogram with edges corresponding to the three vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$. This is precisely $\mathbf{v} \wedge \mathbf{w} \wedge \mathbf{u}$. The n-dimensional wedge product generalizes this notion to n-dimensional volumes.

On the right-hand side of (5.52) is an integral over the *boundary* of D. This is an oriented $(n-1)$-dimensional space. The integrand is the differential $(n-1)$-form. We thus see that the integral of df over D is related to a lower-dimensional integral of its primitive, f, on the boundary of D.

Stokes' theorem is the underpinning of everything you've seen in vector calculus. In one dimension:

$$\int_a^b dx\, \frac{df(x)}{dx} = \int_{f(a)}^{f(b)} df = f(b) - f(a) \ . \tag{5.53}$$

In two dimensions, when you have a vector field $\mathbf{f}(x)$, the appropriate derivative (and the one that pops out of differential form notation) is the curl, $\nabla \times \mathbf{f}(x)$. This becomes an integral with respect to the differential surface element $d\hat{S} = \hat{\mathbf{n}} dS$, which may be chosen to be oriented in the z-direction perpendicular to the plane[67]. Then we have the usual Green's theorem:

$$\int_S dS\, \hat{\mathbf{n}} \cdot (\nabla \times \mathbf{f}(x)) = \oint_{C=\partial S} \mathbf{f}(x) \cdot d\mathbf{x} \ , \tag{5.54}$$

where $d\mathbf{x}$ is a differential line element along the oriented curve C that is the boundary of the integration region S. Finally, the divergence theorem in 3D relates the divergence of a vector field to its value on the surface:

$$\int_V dV\, \nabla \cdot \mathbf{f}(x) = \int_{S=\partial V} dS\, \hat{\mathbf{n}} \cdot \mathbf{f}(x) \ , \tag{5.55}$$

where $\hat{\mathbf{n}}$ is the unit normal vector at each point on the surface S bounding the volume V. These three versions of the 'fundamental theorem of calculus' all look tantalizingly similar—and here we see that in fact, they're all manifestations of the general Stokes' theorem.

Observe that the 'boundary of a boundary' is nothing. If you have a disc, the boundary is a circle. The circle itself has no endpoints—in contrast to an interval. We can start to see some of the neat features of differential geometry when we look at Stokes' theorem, (5.52), from this perspective. Indeed, one can read Stokes' theorem as a relation between the differential operator d acting on an integrand and the 'boundary operator' ∂ acting on the space. The 'boundary of a boundary = 0' mantra can be loosely translated into 'derivative

[67] Alternatively, $\hat{\mathbf{n}}$ is the unit normal of the 2D curved surface in a 3D space.

of a derivative' is zero; where we are not being technically rigorous at all. You've already seen variants of this in vector calculus:

$$\nabla \times \nabla f(x) = 0 \qquad\qquad \nabla \cdot \nabla \times \mathbf{f}(x) = 0 \ , \tag{5.56}$$

and so forth. I used to think vector calculus was very challenging because there seemed to be so many different rules for how to differentiate and integrate. It turns out that differential geometry unifies this nicely into a general mathematical structure that, when applied to specific dimensions of space, produce all of the funny things you learn in undergraduate electrodynamics. It should not surprise you that this mathematical structure has a lot to say about the structure of theories like electrodynamics and gravity[68].

Exercise 5.7 *To see how this structure shows up in a simple physical system, look up Montgomery's treatment of the Falling Cat Problem. Similarly, Shapere and Wilczek's treatment of swimming animals at low Reynolds number. Wilczek famously won the Nobel prize for his contributions to the gauge theory of the strong nuclear force; a theory based on precisely this type of geometric structure we've discussed here. A final reference is to look up the Parallel Parking Problem, which was a topic of discussion in the first version of this P231 class that I taught in 2016. In that problem, one asks how a car can move transversely given that its only degrees of freedom are to move forward/backward and to turn the steering wheel. For the next two years students raised concerns that this class was hopelessly mathematical—so now here we are at the tail end of an exercise with no specific directions in an optional section of the lecture notes.*

5.11 Cauchy's Integral Formula

Cauchy's Theorem tells us that integrating functions over domains where they are analytic is boring. Cauchy's Integral Formula[69] is the first step to something that is decidedly *not* boring. The formula applies to a function f that is analytic in some connected domain and a path C entirely contained in that domain:

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \frac{f(z)}{(z - z_0)} \ . \tag{5.57}$$

This is the statement that the value of some function at some point z_0 is related to an integral of the function around the point. In fact, maybe this shouldn't be so surprising: the right-hand side looks like some kind of average of the function in the neighborhood of the function. Given the relation between analytic functions and harmonic functions, this sounds plausible. On the right-hand side there's a factor of $(2\pi(z - z_0))^{-1}$ which indeed looks like one is averaging over the circumference around the point z_0. The factor of i is curious. Those who are familiar with all of this will notice the 'famous' combination $(2\pi i)$.

Let is highlight that the integrand on the right-hand side is

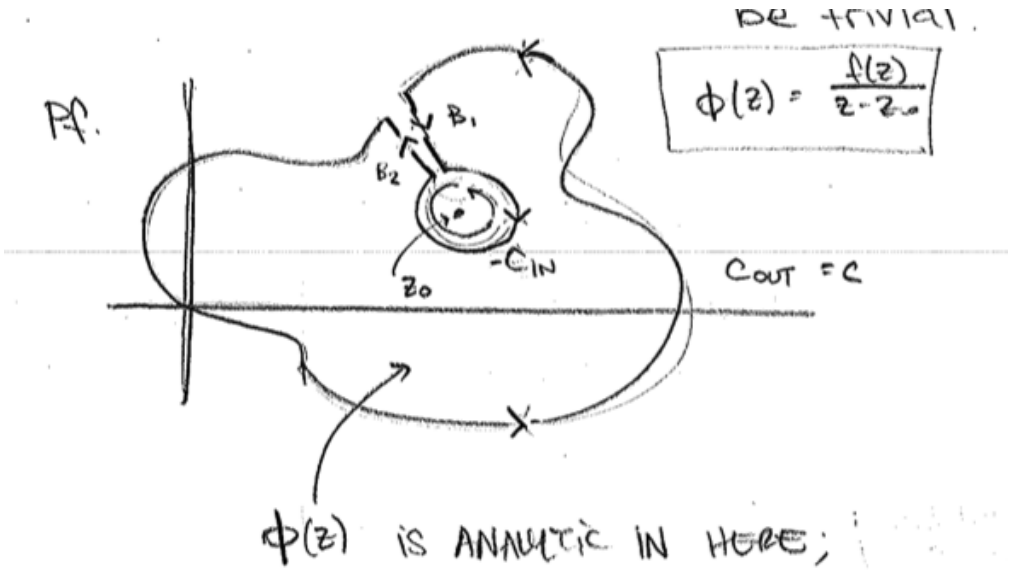
$$g(z) = \frac{f(z)}{z - z_0} \ . \tag{5.58}$$

[68] One of my favorite introductions: https://arxiv.org/abs/hep-th/0611201v1.

[69] At this point you wonder: how is it that Cauchy got his name on everything in this business?

Unlike $f(z)$, $g(z)$ is absolutely *not* analytic 'everywhere' in the region that we're looking at. Most notably, it is not analytic at $z = z_0$. What's the derivative of $g(z)$ at the singularity? Heck if I know[70]. This is important because it is our first example of a function that is analytic in a region up to a single point; we say that the function $g(z)$ has a **pole** at $z = z_0$.

Since the integrand $g(z)$ has a pole, we probably shouldn't integrate over it. No problem, our integration contour C away from z_0. However, z_0 still punctures our domain over which the integrand is analytic[71]. To see what this does, let's consider the integral on the right-hand side.



Let us call $C_{\text{out}} = C$, the original contour over which we're integrating. It is oriented counter clockwise by assumption. Because we're curious about the singularity at $z = z_0$, let's deform the contour by building a little bridge B_1 that heads towards z_0, then a little circle $-C_{\text{in}}$ that goes around the pole[72], and then a little bridge B_2 that returns to C_{out} where we originally left it. Clearly the integral over B_1 and B_2 cancel because

$$B_1 = -B_2 \tag{5.59}$$

as paths. Note, however, that the region enclosed by the total curve $C_{\text{out}} + B_1 - C_{\text{in}} + B_2$ is a region in which $g(x)$ is totally analytic. The $-C_{\text{in}}$ boundary separates the pole from the region enclosed[73]. This means that Cauchy's Theorem holds. Because the bridge integrals

[70] Is it infinity? I suppose, but the notion of infinity in complex space can be a little tricky. At any rate, usually 'infinity' is not the kind of answer that inspires much confidence.

[71] We've given our domain a topology. This, by the way, perhaps the one of the lamest things you can give someone. Once when I was a child I was sad when my parents got me a pair of socks for my birthday. I'd have been even more disappointed with topology. At least the socks kept my feet warm.

[72] Note the minus sign! We define C_{in} to have positive/counter-clockwise orientation. In the picture above, we see that we traverse this little circle in the clockwise direction, so we put a minus sign on C_{in}.

[73] An engineer, a physicist, and a mathematician are tasked to optimize the amount of space surrounded by a finite length of fencing. The engineer builds a square pen since that makes it simple to construct. The physicist mumbles something about variational principles and builds a circle, stating that it optimizes the area enclosed for fixed perimeter. The mathematician takes the fence, throws away most of it, and then makes a tiny enclosure. The mathematician then carefully steps inside and says, "I declare myself to be on the outside."

over B_1 and B_2 cancel, the theorem tells us that

$$\oint_{C_{\text{out}}} dz\, g(z) - \oint_{C_{\text{in}}} dz\, g(z) = 0 \ , \tag{5.60}$$

where the minus sign came from $\oint_{-C_{\text{in}}} = -\oint_{C_{\text{in}}}$. The first term is precisely the right-hand side of (5.57). Apparently the second term is supposed to be $f(z_0)$. We can evaluate the second term along the contour by parameterizing C_{in} as

$$C_{\text{in}}: \quad z(\theta) = z_0 + \varepsilon e^{i\theta} \ , \tag{5.61}$$

which goes around C_{in} for $\theta \in [0, 2\pi]$. Now watch carefully. The relevant quantities in our integrand are:

$$dz = i\varepsilon e^{i\theta} d\theta \qquad\qquad z - z_0 = \varepsilon e^{i\theta} \ . \tag{5.62}$$

Now watch carefully:

$$\oint_{C_{\text{in}}} dz\, g(z) = \oint_{C_{\text{in}}} dz\, \frac{f(z)}{z - z_0} = i \oint_{C_{\text{in}}} d\theta f(z) = 2\pi i f(z_0). \tag{5.63}$$

Note that once we wrote the integral with respect to $d\theta$, this is just an ordinary 'real' integral where the integrand happens to have complex numbers in it. What is critical is that the powers of $e^{i\theta}$ canceled. Compare this to what happened in (5.44), which was the analogous critical step for showing that the integral of analytic functions vanishes. In (5.63) we ended up with *no* factors of $e^{i\theta}$ so that the $d\theta$ integral ended up being non-zero.

Putting this all together gives the desired result,

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz\, \frac{f(z)}{z - z_0} \ . \tag{5.64}$$

The star of this discussion is not the analytic function $f(z)$; rather it is the analytic-up-to-a-pole function $g(z)$. Unlike our *boring* scenario of functions that are analytic in a given domain, interesting stuff happens when our functions have singularities. We get to live dangerously and dance around theses singularities. In general, we will refer to singularities that go like $(z - z_0)^{-n}$ to be poles at $z = z_0$. The positive integer n is called the order of the pole. The case $n = 1$ is called a **simple pole**.

5.12 From Taylor to Laurent

Functions with poles are clearly not analytic *everywhere*. However, they're pretty close to being analytic. They're analytic except for isolated poles. A function that is analytic up to poles is called **meromorphic**. If you're like me, you should classify these as *nice, but not too nice* functions. They're just not-nice enough to be interesting. If you're keeping up with the 'big picture,' you'll recall that the Fourier transform of a differential operator's Green's function (5.8) appears to be in this class.

In a region where f is analytic, differentiability meant that one could write a Taylor expansion: a series of terms that go like $(z - z_0)^n$ for positive integers n. When f is merely *meromorphic*, the Taylor expansion is generalized to a **Laurent expansion**:

$$f(z) = \sum_{n=-N}^{\infty} a_n(z_0)(z - z_0)^n \ , \tag{5.65}$$

where N is the order of the pole at z_0 (if there is one).

5.13 The Residue Theorem: a first look

Now we arrive at our mail tool. Suppose $f(z)$ is meromorphic in some region of the complex plane. In fact, suppose $f(z)$ has a simple pole at z_0; the function $g(z)$ in (5.58) is a function precisely of this type. Consider the integral of $f(z)$ around a closed contour that goes around the pole once. For example, the pole is in some connected region R and the contour is the boundary of this region, $C = \partial R$. Applying the Laurent expansion about z_0 for this meromorphic function f gives

$$\oint_C dz\, f(z) = \oint_C dz \left[\sum_{n<0} a_n(z - z_0)^n + \sum_{n\geq 0} a_n(z - z_0)^n \right] \ . \tag{5.66}$$

All we have done is separated the positive-power terms of the Laurent expansion from the negative power terms. We know that the positive power terms integrate to zero because those terms are analytic in R. Thanks, Cauchy's Theorem.

What about the term with negative powers? Since we assumed that z_0 is a simple pole, we know that only the a_{-1} term is non-zero in this Laurent expansion about z_0. That means that we can write the integral as

$$\oint_C dz\, f(z) = \oint_C dz\, \frac{a_{-1}}{z - z_0} \ . \tag{5.67}$$

The coefficient a_{-1} of the Laurent expansion about a pole is called the **residue** of the function f at the pole z_0. We will use the notation $\mathrm{Res}_f(z_0)$.

Now recall Cauchy's Integral Theorem. For the sake of clarity (I have used f too many times), let us write the integral theorem with respect to an analytic-in-this-neighborhood function h:

$$\oint_C \frac{h(z)}{z - z_0} = 2\pi i\, h(z_0) \ , \tag{5.68}$$

we've gone ahead and re-arranged some terms. Comparing this to (5.67), we may take $h(z) = a_{-1} = \mathrm{Res}_f(z_0)$. This means that $h(z_0) = \mathrm{Res}_f(z_0)$ so that we ultimately have

$$\oint_C dz\, f(z) = 2\pi i\, \mathrm{Res}_f(z_0) \ . \tag{5.69}$$

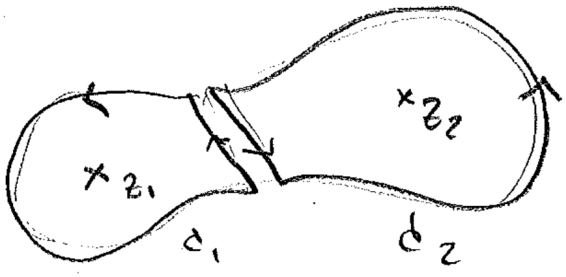
We will generalize this shortly, but the main idea is in this simple example. If you can identify the a_{-1} coefficient of a meromorphic function at its pole, then you can easily integrate the function around the pole. As long as there aren't any other poles in the neighborhood, it doesn't matter what your contour is as long as you are going around counter-clockwise and you go around exactly once.

5.14 The Residue Theorem: more carefully

The residue theorem is our primary tool for calculating integrals on the complex plane. Let's see how it generalizes. Consider a meromorphic function $g(z)$ with two simple poles:

$$g(z) = \frac{h(z)}{(z-z_1)(z-z_2)} \qquad\qquad h(z) \text{ analytic over } \mathbb{C} \ . \tag{5.70}$$

The simple poles are located at $z = z_1$ and one at $z = z_2$. Consider a contour C that encloses both poles. The integral of $g(z)$ along C is simply $2\pi i$ times the sum of the residues of the two simple poles. This is easy to see from dividing C into the sum of two paths:



Here C_1 is a closed contour that encloses z_1 but not z_2. Similarly, C_2 is a closed contour that encloses z_2 but not z_1. Observe that C_1 and C_2 overlap along a line that separates the two poles. Because both C_1 and C_2 are positively oriented, the integrals along this line cancel. From this it is clear that

$$\oint_C dz\, g(z) = \oint_{C_1} dz\, g(z) + \oint_{C_2} dz\, g(z) = 2\pi i \left(\text{Res}_f(z_1) + \text{Res}_f(z_2)\right) \ . \tag{5.71}$$

This gives a more general form of the residue theorem. The integral of a meromorphic function $f(z)$ around a closed contour C that is the boundary of some region of the complex plane[74] is

$$\oint_C dz\, f(z) = 2\pi i \sum_{i \in \text{poles}} \text{Res}_f(z_i) \ , \tag{5.72}$$

where the sum is over the poles of f at z_i enclosed inside C.

Exercise 5.8 *How would the residue theorem change if the contour C were oriented in the opposite direction? What if the contour circled the poles multiple times? What if the contour circled some poles some number of times, and other poles a different number of times?*

[74] Note that this means that C is positively oriented and only circles around the region once.

5.15 Non-simple poles (a case study in being careful)

So far we've focused only on meromorphic functions that have *simple* poles. What about higher order singularities? Here's a tangible example:

$$\oint_C dz\, f(z) = ? \qquad\qquad f(z) = \frac{1}{(z-2i)^2}\ , \tag{5.73}$$

where C is a positively oriented contour that circles the second-order pole $z_0 = 2i$. Because $f(z)$ has no simple pole at z_0, it looks like there's no contribution to the integral. To check this, we recall Cauchy's integral formula, (5.63). The reason why simple poles contributed to the contour integral is the observation that for an integer, n, the integral of $e^{in\theta}d\theta$ behaves as follows:

$$\int_0^{2\pi} d\theta\, e^{in\theta} = \begin{cases} 0 & \text{if } n \neq 0 \\ 2\pi & \text{if } n = 0 \end{cases}\ . \tag{5.74}$$

Following the same in the previous sub-sections, the integral in (5.73) about a contour C that circles $z_0 = 2i$ once is equivalent to the integral over smaller contour C' that is a small circle that surrounds z_0. Parameterize C' by the angular variable:

$$z(\theta) = z_0 + \varepsilon e^{i\theta} \qquad\qquad dz = i\varepsilon e^{i\theta}\, d\theta\ . \tag{5.75}$$

Then the integral is:

$$\oint_C dz\, f(z) = \oint_{C'} dz\, f(z) = \int_0^{2\pi} i\varepsilon e^{i\theta} d\theta\, \frac{1}{\varepsilon^2 e^{2i\theta}} = \frac{i}{\varepsilon}\int_0^{2\pi} d\theta\, e^{-i\theta} = 0\ . \tag{5.76}$$

So this is all consistent with the mantra of *find the a_{-1} coefficient of the Laurent expansion, that's the residue.*

We should be careful, though. If we're too slick we can convince ourselves of wrong things. For example, suppose we wanted to generalize (5.73) by changing the numerator of $f(x)$:

$$\oint_C dz\, f(z) = ? \qquad\qquad f(z) = \frac{h(z)}{(z-2i)^2}\ , \tag{5.77}$$

where $h(z)$ is an analytic function. For the sake of argument, let's assume that $h(z)/(z-2i)^2$ has been simplified so that there are no common factors between the numerator and denominator[75]. One might think that when we shrink the contour from C to C', we can approximate $h(z) = h(z_0)$ so that

$$\oint_{C'} dz\, f(z) = \oint_{C'} i\varepsilon e^{i\theta} d\theta\, \frac{h(z_0)}{(z-2i)^2} = \frac{i}{\varepsilon} h(z_0)\int_0^{2\pi} d\theta\, e^{-i\theta} = 0\,? \tag{5.78}$$

In general, this argument is *wrong*. It is wrong even though the (5.76) happens to be true.

[75] For example, if $h(z) = (z-2i)$ then clearly $f(z)$ has a simple pole at $z_0 = 2i$ so the integral picks up a non-zero residue.

Exercise 5.9 *Can you spot the incorrect assumption?*

The incorrect assumption is that we could approximate $h(z) = h(z_0)$ because the little circle C' is always close to z_0. The intuition is fine, but we weren't careful enough: even though $h(z)$ is analytic everywhere, we should remember write a Taylor expansion:

$$h(z) = h(z_0) + h'(z_0)(z - z_0) + \cdots \tag{5.79}$$

Even if our intent is to only keep the first term, it's a good habit to remember that the Taylor expansion contains many terms. Let's see what happens:

$$\oint_{C'} dz\, f(z) = \oint_{C'} i\varepsilon e^{i\theta} d\theta\, \frac{h(z_0) + h'(z_0)(z - z_0) + \cdots}{(z - 2i)^2} \tag{5.80}$$

$$= \frac{ih(z_0)}{\varepsilon} \int_0^{2\pi} d\theta\, e^{-i\theta} + ih'(z_0) \int_0^{2\pi} d\theta + \cdots \tag{5.81}$$

While the first term vanishes, the second term is clearly non-zero since it's a contour integral. Indeed, one ends up with

$$\oint_{C'} dz\, \frac{h(z)}{(z - 2i)^2} = 2\pi i h'(z_0)\ , \tag{5.82}$$

where evidently $\mathrm{Res}_f(2i) = h'(2i)$ is the residue of $f(z) = h(z)/(z - 2i)^2$ at $z_0 = 2i$.

Exercise 5.10 *Why don't higher order terms in* (5.81) *do not contribute to the contour integral.*

The purpose of this example was to show that it can be a little tricky to identify the residue of a function by simply 'looking at the denominator.' You will derive an explicit formula in your homework. Fortunately, we will rarely consider non-simple poles in this course.

Let's go through a few examples. These are from the third edition of Boas, chapter 14.6.

Example 5.6 *Let $f(z) = \cot z$. Find the residue of f at $z = 0$.*

To solve this, we can write out the numerator and denominator of the cotangent:

$$\cot z = \frac{\cos z}{\sin z} = \frac{1 - z^2/2 + \cdots}{z - z^3/3!}\ . \tag{5.83}$$

From this we see that $z \to 0$ is indeed a simple pole, and one may write the residue as

$$\mathrm{Res}_f(0) = \lim_{z \to 0} z \frac{1}{z} = 1\ . \tag{5.84}$$

Example 5.7 *Let $f(z) = \cot^2 z$. Find the residue of f at $z = 0$.*

Writing out the numerator and denominator again:

$$\cot^2 z == \frac{1 - z^2 + \cdots}{z^2 - 2z^3/3! + \cdots} \sim \frac{1}{z^2} + \mathcal{O}(1)\ . \tag{5.85}$$

From this we deduce that $\mathrm{Res}_f(0) = 0$.

Exercise 5.11 *Why were we able to be slick in Example (5.7) after making a big deal about being careful in (5.81)?*

Exercise 5.12 *Let $f(z) = z\cot^2 z$. Find the residue of f at $z = 0$.*

Exercise 5.13 *Let $f(z)$ be*

$$f(z) = \left(\sum_{n=0}^{\infty} c_n z^n\right) \cot^2 z \tag{5.86}$$

Show that the residue of f at $z = 0$ is c_1.

5.16 The Killer App: Real Integrals

The *killer app*[76] for complex contour integrals is to solve integrals along the *real line*. It is often the case that some of the real functions that we would like to integrate happen to have poles in the complex plane. In this case, the integral along the real line can be completed into a closed contour C in the complex plane. If we can separate the contribution of the real line from the rest of the contour, then we can use the residue theorem to do the integral without any of the hard work of, uh, integrating.

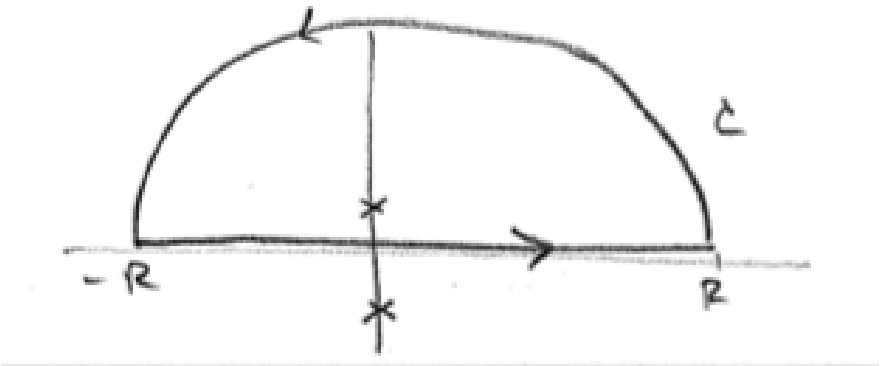
Example 5.8 *Consider the real function*

$$f(x) = \frac{1}{x^2+1} \ . \tag{5.87}$$

Typically we care about real functions with real arguments. However, let's analytically continue *$f(x)$ into a meromorphic function $f(z)$*

$$f(z) = \frac{1}{z^2+1} \ . \tag{5.88}$$

This is the (unique[77]) complex function that is analytic and agrees with the original function on the real line. We immediately notice that $z^2 + 1 = (z + i)(z - i)$ so that $f(z)$ has simple poles at $z = \pm i$. Consider the following contour which includes the real interval $x \in [-R, R]$ as part of it:



[76] This is an old phrase from the first dot-com bubble.
[77] The proof is left to you to derive or look up.

We will care about the limit $R \to \infty$ where the integral includes the entire real line. Observe that the contour only encloses the pole at $z = i$. Let us integrate the function around C. From the residue theorem, we have

$$\oint_C dz \, \frac{1}{(z+i)(z-i)} = 2\pi i \mathrm{Res}_f(i) = \pi \ , \tag{5.89}$$

where we've used $\mathrm{Res}_f(i) = 1/2i$. However, we can also write the integral as a sum of the real integral plus a counter-clockwise arc of radius R:

$$\oint_C dz \, f(z) = \int_{-R}^{R} dx \, f(x) + \int_0^{\pi} iRe^{i\theta} d\theta \, f\left(Re^{i\theta}\right) \ . \tag{5.90}$$

where in the second term we've used the parameterization $z(\theta) = Re^{i\theta}$. Note that the first term on the right-hand side is the 'ordinary' purely real integral. The second term behaves as follows:

$$\int_0^{\pi} iRe^{i\theta} d\theta \, f\left(Re^{i\theta}\right) = \frac{Re^{i\theta}}{R^2 e^{2i\theta} + 1} d\theta \ . \tag{5.91}$$

As we take the limit $R \to \infty$, the integral goes like $\sim 1/R \to 0$. Thus as long as we actually care about the integral along the entire real line, $R \to \infty$, we have

$$\oint_C dz \, \frac{1}{(z+i)(z-i)} = \pi = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} \ . \tag{5.92}$$

And so we find

$$\int_{-\infty}^{\infty} dx \, \frac{dx}{x^2+1} = \pi \ . \tag{5.93}$$

This is an absolutely correct result about a real *integral that we sneakily derived without actually doing the integral. I assume one may check this by the appropriate trigonometric substitution*[78] *Observe that the critical step was that the curved part of the contour (the one that actually had imaginary parts) went to zero in a well defined limit.*

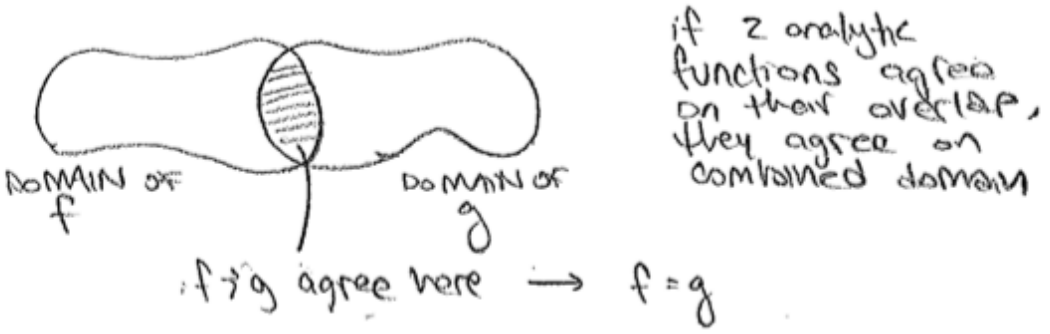
Exercise 5.14 *What result do you get when you solve for the same integral of function $f(x) = (x^2+1)^{-1}$ over the real line, but this time you use the contour C' that encircles the lower half of the complex plane?*

Let's address the question of **analytic continuation**[79]. We started with a purely real function $f(x)$ and 'generalized it' to a complex function $f(z)$ that happened to have poles in the complex plane. Because we used the residue theorem, the details of these poles (residue and whether they're enclosed by C) are critical for performing the integral. So a good question is: is the *complex function* $f(z)$ even uniquely defined given the *real* function $f(x)$?[80]

[78] Just kidding, you can just check it on *Mathematica*. You can even tell people that you did the trig substitution if it makes you feel better.

[79] See Appel section 5.3 for a more careful discussion.

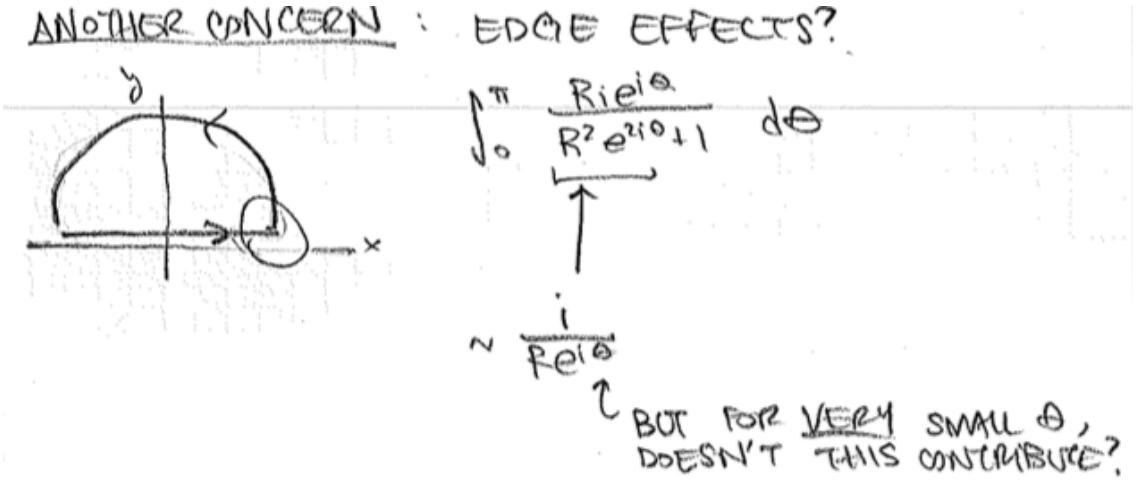
[80] My notation is a little glib and assumes the answer. If you want, you can write $f_{\mathbb{C}}$ and $f_{\mathbb{R}}$ to differentiate the two functions that are different in principle.



There is a handy theorem that states that when two analytic functions agree on their domain of overlap, then they agree on their common domain. For *meromorphic* functions, we can just imagine the domain where those singularities are not included. In our example above, $f(x)$ is analytic along the real line and $f(z)$ is a function that trivially (by construction) is analytic and matches $f(x)$ on the real line. Then the theorem says that $f(z)$ and $f(x)$ 'agree' in the entire domain of analyticity; so one can *analytically continue* $f(x)$ to the complex plane as long as one avoids any singularities.

The gist of the proof is that if two functions f and g agree in some overlapping domain of analyticity, then their difference $h(z) \equiv f(z) - g(z)$ is also an analytic function in this domain. Analyticity means one has finite radius of convergence because it has a well defined Taylor expansion. One can then use this to argue that $h(z)$ can be defined beyond the overlapping domain. Continuity requires that $h(z)$ remains zero even outside the overlapping domain, which ensures that $f(z) = g(z)$. In other words, there is a unique analytic continuation of a function to a maximal domain of analyticity.

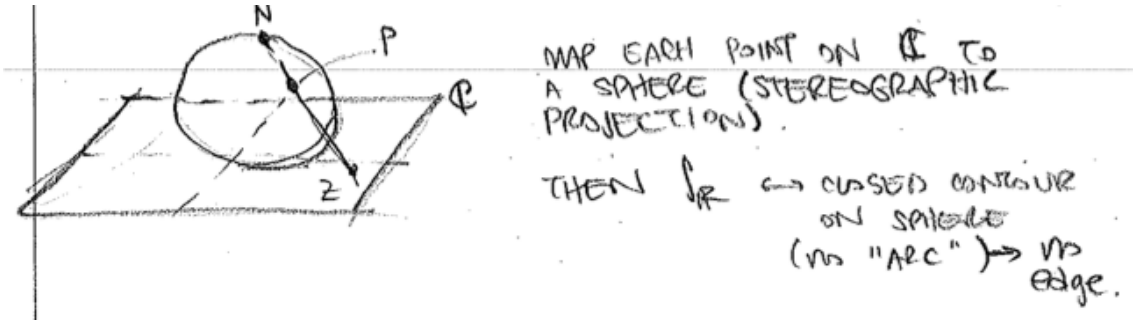
There's another effect you may wonder about. Are there some 'edge' effects from the corner of the contour?



A hand-waving answer is to say that we take the $R \to \infty$ limit 'first' and so that corner is pushed off into infinity. This should be totally unsatisfying since mathematical physics *rarely* depends on the order in which limits are taken.

A better answer is that the complex plane *with infinity* is a slightly different object from the complex plane without infinity. One way to see this is to note that the complex plane

can be mapped one-to-one onto a sphere. This is called *stereographic projection*. The picture is this:



The idea is to place a unit sphere at the origin of $\mathbb{C}$. The south pole of the sphere is touching $z = 0$. Then for any point $z \in \mathbb{C}$, one can draw a line from z to the north pole of the sphere. There is a single unique point on the sphere that intersects that line. This maps the entire complex plane onto the sphere—except for the north pole of the sphere. We identify the north pole with complex infinity. That's right: all infinities are the same: ∞, $i\infty$, $\infty e^{2\pi/3}$, $-\infty$, etc. From this point of view, the real line maps onto the prime meridian of the sphere and there there are no 'corners' in the contour. In fact, there's no arc component, either! The integral along the real line is a closed contour with respect to the stereographic projection. Note that the orientation of the contour depends on which hemisphere you are figuratively standing in.

5.17 The Main Example: follow this carefully

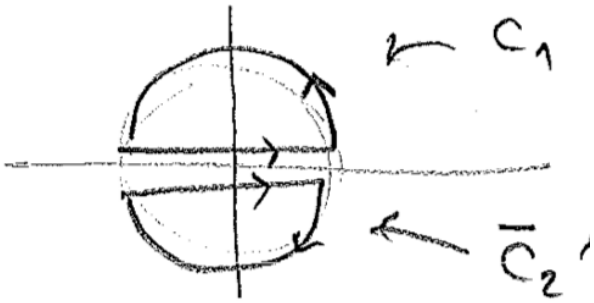
Let's try another instructive example. Here's an real integral that we would like to solve using contour integral techniques:

$$\int dx\, f(x) = \int_{-\infty}^{\infty} dx\, \frac{2\cos x}{x^2+1}\ . \tag{5.94}$$

We can analytically continue this into a complex function by simply replacing the real variable with a complex variable, $x \to z$. This complex function $f(z)$ is analytic on the real line. The presence of simple poles at $z = \pm i$ mean that it is meromorphic with respect to the whole complex plane. It is convenient—for reasons that will be clear momentarily—to write the cosine as a sum of exponentials:

$$f(z) = \frac{e^{iz}+e^{-iz}}{(z+i)(z-i)} = \frac{e^{iz}}{(z+i)(z-i)} + \frac{e^{-iz}}{(z+i)(z-i)} \equiv f_+(z) + f_-(z)\ . \tag{5.95}$$

With some foresight, we have separated $f(z) = f_+(z) + f_-(z)$ into two different functions. There are now two obvious choices for convenient integration contours that include the real line with the appropriate orientation:



We label the two contours are C_1 and $\bar{C}_2$. The bar over $\bar{C}_2$ is to remind us of the clockwise orientation; alternatively one could[81] have written $\bar{C}_2 = -C_2$. Does it matter which contour we pick?

Our criteria for a convenient contour are that (1) the contour includes the real line with the appropriate orientation and (2) the rest of the contour integrates to zero. That way we can use the residue theorem:

$$\int_C f(z)dz = \int_{-\infty}^{\infty} dx\, f(x) + \int_{\text{arc}} dz\, f(z) = \int_{-\infty}^{\infty} dx\, f(x) = \sum_{i=\text{poles}} \text{Res}_f(z_i)\ , \tag{5.96}$$

where we recall that i is runs over the poles enclosed by the contour C. The condition that the integral over the arc vanishes determines which contour we take. We may conveniently parameterize the arcs as follows:

$$z(\theta) = R\cos\theta + iR\sin\theta \tag{5.97}$$

where for C_1 the arc is given by $\theta \in [0, \pi]$ while for $\bar{C}_2$ the arc is $\theta \in [0, -\pi]$. We assume the limit $R \to \infty$. The arc integrals are then:

$$\int_{\text{arc}} dz\, f_\pm(z) = \int_{\text{arc}} d\theta\, \frac{e^{\pm iz}}{z^2+1}\ . \tag{5.98}$$

When we plug in the parameterization (5.97), we see that the denominator of $f_\pm$ scales like R^2. There's an additional factor of R in the numerator coming from dz, and so it may appear that *either* C_1 or $\bar{C}_2$ could be used. However, recall that $f_\pm(z) \sim e^{\pm iz}$ and exponentials beat polynomials. Plugging in $z(\theta)$ into the exponentials gives

$$e^{\pm iz} = e^{\pm i(R\cos\theta + iR\sin\theta)}\ , \tag{5.99}$$

where the difference between C_1 and $\bar{C}_2$ is the range (and direction) of θ.

- C_1 lives in the upper half plane so the imaginary part of z is positive: $R\sin\theta > 0$.
- Similarly, C_2 lives in the lower half plane so the imaginary part of z is negative: $R\sin\theta < 0$.

[81] This all boils down to notation. You are free to be creative with notation, but the underlying quest is *clarity*. Notation doesn't change the underlying mathematics or physics, but a good notation make your ideas clearer—perhaps even to yourself.

This makes it easy to read off the convergence properties of the exponential

$$e^{\pm iz} = (\text{oscillating})e^{\mp R\sin\theta} \ . \tag{5.100}$$

The oscillating part is not relevant: since $|e^{i\varphi}| = 1$, this factor doesn't actually affect the magnitude of the contribution of the integrand, just the phase. The $\exp(\mp R\sin\theta)$ factor, on the other hand, is everything.

- $e^{-R\sin\theta}$ is exponentially suppressed when $\sin\theta > 0$, this corresponds to the upper half plane and so we use the C_1 arc to complete the integration contour of the $f_+(z) \sim \exp(-R\sin\theta)$ integrand.
- $e^{+R\sin\theta}$ is exponentially suppressed when $\sin\theta < 0$, this corresponds to the lower half plane and so we use the $\bar{C}_2$ arc to complete the integration contour of the $f_-(z) \sim \exp(+R\sin\theta)$ integrand.

Because the e^{-R} exponential suppression defeats the polynomial R^{-1} scaling, the vanishing of the integral along the arc is determined only by this exponential factor: $\lim_{R\to\infty} e^{-R}/R \to 0$. We thus have the following approach to the original real integral:

$$\int f(x) = \int_{-\infty}^{\infty} dx\, \frac{2\cos x}{x^2+1} = \int_{-\infty}^{\infty} dx\, \frac{e^{+ix}}{x^2+1} + \int_{-\infty}^{\infty} dx\, \frac{e^{-ix}}{x^2+1} \tag{5.101}$$

$$= \int_{C_1} dz\, \frac{e^{+iz}}{z^2+1} + \int_{\bar{C}_2} dz\, \frac{e^{-iz}}{z^2+1} \tag{5.102}$$

$$= 2\pi i \text{Res}_{f_+}(i) - 2\pi i \text{Res}_{f_-}(-i) \ . \tag{5.103}$$

In the second step we have used the fact that the integral along the real line can be extended by large arcs along C_1 or $\bar{C}_2$ as long as the integral along those arcs vanishes. The residues are straightforward:

$$\text{Res}_{f_\pm}(\pm i) = \pm\frac{1}{2ie} \tag{5.104}$$

so that the final result is

$$\int_{-\infty}^{\infty} dx\, f(x) = \frac{2\pi}{e} \ . \tag{5.105}$$

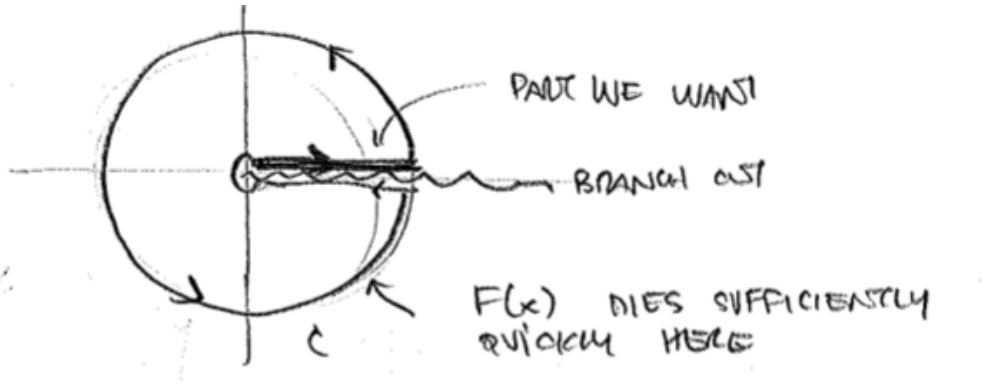
5.18 A fancy example

Here's an example of the residue theorem used to calculate an integral along just the positive real line,

$$\int_0^{\infty} dx\, x^{1/3} F(x) \ . \tag{5.106}$$

We can trivially analytically continue the integrand by replacing $x \to z$. Let us assume that $F(x)$ is a meromorphic function with poles away from the positive real line. Let us further assume further that $F(z)$ is exponentially suppressed for $|z| = R \gg 1$, say $F(z) \sim e^{-R}$.

Note that the $x^{1/3} \to z^{1/3}$ factor is problematic. It tells us that there's an *branch cut* in our integrand: if we start at some point and went a full circle around the origin, say $z(\theta) = e^{i\theta}$ for $\theta \in [0, 2\pi]$, then $z^{1/3}$ doesn't return to the origin. This means you probably need to extend the complex plane into Riemann sheets. Practically what this means is that we have to pick a branch cut that prevents us from taking contours that are 'full circles' around the origin. We have the freedom to pick the orientation of this branch cut, but in this case it is convenient to pick it along the positive real line[82]. Let's take a peculiar contour that includes the integration region we want, $x \in [0, \infty)$, then circles the entire complex plane, and returns back to zero along $x \in (\infty, 0]$.



In the past, we'd have said that the integral along $x \in [0, \infty)$ should cancel that from $x \in (\infty, 0]$. However, we now notice that there's a branch cut separating those two regions. Indeed, you may already see that the *integrand* is not identical along those two lines.

Construct the following contour integral:

$$\oint_C dz\, z^{1/3} F(z) = \int_0^\infty dx x^{1/3} F(x) + \int_{\text{arc}} dz\, z^1/3 F(z) + \int_\infty^0 dx \left(x e^{2\pi i}\right)^{1/3} F\left(e^{2\pi i}\right) \ . \quad (5.107)$$

The left-hand side is simple to evaluate with the residue theorem and will depend on the poles of the unspecified function $F(z)$. On the right-hand side, the integral along the arc vanishes by the assumptions we've made about $F(z)$ becoming exponentially small for large arguments, $|z| = R \gg 1$. Finally, we notice that the third term on the right-hand side we've written $z = x e^{2\pi i}$; this is critical since it includes the phase that z picks up when traversing the large circular path. Since $F(z)$ is meromorphic, we may simply replace $F(x \exp(2\pi i)) \to F(x)$. However, the $z^{1/3}$ needs some care. We find:

$$2\pi i \sum_{j=\text{poles}} z_j^{1/3} \text{Res}_F(z_j) = \left(1 - e^{2\pi i/3}\right) \int_0^\infty dx\, x^{1/3} F(x) \ . \quad (5.108)$$

The $-e^{2\pi i/3}$ came from the 1/3-power of the phase. The minus sign comes from $\int_\infty^0 dx = -\int_0^\infty dx$. Converting the difference of exponentials into a sine gives

$$\left(1 - e^{2\pi i/3}\right) = 2e^{i\pi/3} \cdot \frac{1}{2}\left(e^{-i\pi/3} - e^{i\pi/3}\right) = -2e^{i\pi/3} \sin(\pi/3) \ . \quad (5.109)$$

[82] If you want to be careful to make sure that you're not interfering with the original integral, you can put the branch cut just below the real line by some vanishingly small amount and take that amount to zero at the end of the calculation.

We thus find an expression for the original integral:

$$\int_0^\infty dx\, x^{1/3} F(x) = \sum_{j=\text{poles}} \frac{\pi i e^{i\pi/3} z_j^{1/3} \text{Res}_F(z_j)}{\sin(\pi/3)} \ . \tag{5.110}$$

We won't worry too much about branch cuts in this course. However, they do occasionally show up in physical manifestations, such as in dispersion relations[83].

Acknowledgements

A Method of Variations

The method of variations is a way to solve inhomogeneous differential equations starting from a basis of solutions to the homogeneous differential equation. We specialize to the case of second-order differential operators, though the technique is valid for higher-order operators.

A.1 First order equation for coefficients

Consider the second-order inhomogeneous linear differential equation

$$\mathcal{O}f(x) = a_2(x) f''(x) + a_1(x) f'(x) + a_0(x) = g(x) \ . \tag{A.1}$$

One can determine the solution to this equation from varying with respect to the solutions to the homogeneous differential equation, $u(x)$ and $v(x)$. Write the solution to the inhomogeneous equation as

$$f(x) = c(x)u(x) + d(x)v(x) \tag{A.2}$$

Make the ansatz that

$$c'(x)u(x) + d'(x)v(x) = 0 \ . \tag{A.3}$$

Then the inhomogeneous differential equation reduces to

$$\mathcal{O}f = a_2(x)\left(c'(x)u'(x) + d'(x)v'(x)\right) = g(x) \ . \tag{A.4}$$

This gives expressions for c' and d':

$$c'(x) = \frac{-v(x)g(x)/a_2(x)}{u(x)v'(x) - u'(x)v(x)} \qquad d'(x) = \frac{u(x)g(x)/a_2(x)}{u(x)v'(x) - u'(x)u(x)} \ , \tag{A.5}$$

where one may recognize the denominators to be the Wronksian, $W = uv' - u'v$. For simplicity, write $\widetilde{g}(x) \equiv g(x)/a_2(x)$.

[83] See, e.g., https://arxiv.org/abs/1610.06090 for a peek at how it can show up in quantum field theory.

A.2 Integrating and boundary conditions

Suppose that the inhomogeneous differential equation has boundary conditions at x_1 and x_2 given by differential operators $\mathcal{B}_{1,2}$:

$$\mathcal{B}_1 f(x) = \alpha_1 f'(x_1) + \beta_1 f(x_1) = 0 \qquad \mathcal{B}_2 f(x) = \alpha_2 f'(x_2) + \beta_2 f(x_2) = 0 \ . \tag{A.6}$$

These boundary conditions are, in general, first order for a second order inhomogeneous differential equation. For example, these may come from integrating a second-order equation of motion by parts, leaving a first-order operator on either boundary. The ansatz (A.3) simplifies these boundary conditions:

$$\mathcal{B}_i f(x) = c(x_i)\mathcal{B}_i u(x) + d(x_i)\mathcal{B}_i v(x) = 0 \ . \tag{A.7}$$

For simplicity, let us write $(\mathcal{B}_i u) = \alpha_i u'(x_i) + \beta_i f(x_i)$ as a constant that depends on u and u' evaluated at the boundary x_i. We define $(\mathcal{B}_i v)$ similarly.

Integrate the first-order differential equations for the coefficients (A.5) using these boundary conditions. The simplest approach is to integrate a convenient linear combinations:

$$\mathcal{I}_1 = \int_{x_1}^{x} dy\, c'(y)(\mathcal{B}_1 u) + d'(y)(\mathcal{B}_1 v) = c(x)(\mathcal{B}_1 u) + d(x)(\mathcal{B}_1 v) \ , \tag{A.8}$$

where the lower limit vanishes because $(\mathcal{B}_i f) \equiv 0$. Similarly, one finds

$$\mathcal{I}_2 = \int_{x}^{x_2} dy\, c'(y)(\mathcal{B}_2 u) + d'(y)(\mathcal{B}_2 v) = -c(x)(\mathcal{B}_2 u) - d(x)(\mathcal{B}_2 v) \ . \tag{A.9}$$

The integrals in (A.8) and (A.9) are equivalently expressed using the expressions for $c'(x)$ and $d'(x)$ in (A.5):

$$\mathcal{I}_1 = \int_{x_1}^{x} dy\, \left[-(\mathcal{B}_1 u)v(y) + (\mathcal{B}_1 v)u(y)\right] \frac{\widetilde{g}(y)}{W(y)} \tag{A.10}$$

$$\mathcal{I}_2 = \int_{x}^{x_2} dy\, \left[-(\mathcal{B}_2 u)v(y) + (\mathcal{B}_2 v)u(y)\right] \frac{\widetilde{g}(y)}{W(y)} \ . \tag{A.11}$$

Combining this with the right-hand sides of (A.8) and (A.9) then gives the desired integral solutions for the coefficient functions:

$$\left[(\mathcal{B}_1 u)\,(\mathcal{B}_2 v) - (\mathcal{B}_2 u)\,(\mathcal{B}_1 v)\right] c(x) = (\mathcal{B}_2 v)\,\mathcal{I}_1 + (\mathcal{B}_1 v)\,\mathcal{I}_2 \tag{A.12}$$

$$\left[(\mathcal{B}_2 u)\,(\mathcal{B}_1 v) - (\mathcal{B}_1 u)\,(\mathcal{B}_2 v)\right] d(x) = (\mathcal{B}_2 u)\,\mathcal{I}_1 + (\mathcal{B}_1 u)\,\mathcal{I}_2 \ . \tag{A.13}$$

Define the left-hand side prefactors:

$$(\text{den.}) \equiv (\mathcal{B}_1 u)\,(\mathcal{B}_2 v) - (\mathcal{B}_2 u)\,(\mathcal{B}_1 v) \ , \tag{A.14}$$

so that we have

$$f(x) = \frac{(\mathcal{B}_2 v)\,\mathcal{I}_1 + (\mathcal{B}_1 v)\,\mathcal{I}_2}{(\text{den.})} u(x) - \frac{(\mathcal{B}_2 u)\,\mathcal{I}_1 + (\mathcal{B}_1 u)\,\mathcal{I}_2}{(\text{den.})} v(x) \tag{A.15}$$

Collecting the terms by the integration range gives:

$$(\text{den.})f(x) = \int_{x_1}^{x} dy \frac{\widetilde{g}(y)}{W(y)} \left[-(\mathcal{B}_1 u)(\mathcal{B}_2 u)v(x)v(y) + (\mathcal{B}_1 v)(\mathcal{B}_2 v)u(x)u(y)\right. \tag{A.16}$$

$$\left. -(\mathcal{B}_1 u)(\mathcal{B}_2 v)u(x)v(y) + (\mathcal{B}_1 v)(\mathcal{B}_2 u)v(x)u(y)\right] + \cdots \tag{A.17}$$

$$= \int_{x_1}^{x} dy \frac{\widetilde{g}(y)}{W(y)} \left[(\mathcal{B}_1 u)v(y) - (\mathcal{B}_1 v)u(y)\right]\left[(\mathcal{B}_2 u)v(x) - (\mathcal{B}_2 v)u(x)\right] + \cdots \ . \tag{A.18}$$

The $\cdots$ represent the $\mathcal{I}_2$ terms. Performing the same grouping for the $\mathcal{I}_2$ terms then gives:

$$(\text{den.})f(x) = \int_{x_1}^{x} dy \frac{\widetilde{g}(y)}{W(y)} \left[(\mathcal{B}_1 u)v(y) - (\mathcal{B}_1 v)u(y)\right]\left[(\mathcal{B}_2 u)v(x) - (\mathcal{B}_2 v)u(x)\right] + \tag{A.19}$$

$$\int_{x}^{x_2} dy \frac{\widetilde{g}(y)}{W(y)} \left[(\mathcal{B}_1 v)u(x) - (\mathcal{B}_1 u)v(x)\right]\left[(\mathcal{B}_2 v)u(y) - (\mathcal{B}_2 u)v(y)\right] \ . \tag{A.20}$$

These two terms can be combined rather nicely:

$$f(x) = \frac{1}{(\text{den.})} \int_{x_1}^{x_2} dy \, \frac{\widetilde{g}(y)}{W(y)} \left[(\mathcal{B}_1 u)v(x_<) - (\mathcal{B}_1 v)u(x_<)\right]\left[(\mathcal{B}_2 u)v(x_>) - (\mathcal{B}_2 v)u(x_>)\right] \tag{A.21}$$

where we introduce the convenient notation that

$$x_< = \min(x, y) \qquad\qquad x_> = \max(x, y) \ . \tag{A.22}$$

B Fourier Conventions

This section is inspired by an excellent post on `physics.stackexchange`[84]. There are many different conventions for Fourier transforms. The danger is that you accidentally use one convention to do the Fourier transform and a different convention for the inverse transform.

There are two choices one can make when defining a Fourier transform convention; we parameterize these choices by real numbers a and b. The Fourier transform $\widetilde{f}(\omega)$ of a function $f(t)$ is

$$\widetilde{f}(\omega) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} dt \, e^{ib\omega t} f(t) \ . \tag{B.1}$$

We see that a tells us about the (2π) factors and b tells us about the argument of the basis function $e^{ib\omega t}$. With this basis, the inverse Fourier transform is

$$f(t) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{\infty} d\omega \, e^{-ib\omega t} f(\omega) \ . \tag{B.2}$$

[84] https://physics.stackexchange.com/a/308248

One may check that the inverse Fourier transform of a Fourier transform gives the original function:

$$\widetilde{\widetilde{f}} = \frac{|b|}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-ib\omega t}\int_{-\infty}^{\infty} ds\, e^{ib\omega s} f(s) \tag{B.3}$$

$$= \frac{|b|}{2\pi}\int ds\, f(z)\int d\omega\, e^{ib\omega(s-t)} \tag{B.4}$$

$$= \int ds\, \delta(s-t) f(s)\ , \tag{B.5}$$

where we have used $\int d\xi \exp(2\pi i x\xi) = \delta(x)$.

The convention that we will choose for the *time–frequency* [inverse] Fourier transform is

$$f(t) = \int_{-\infty}^{\infty} đ\omega e^{-i\omega t}\widetilde{f}(\omega) \qquad\qquad đ\omega \equiv \frac{d\omega}{2\pi}\ . \tag{B.6}$$

This corresponds to $a = -1$ and $b = 1$. The corresponding transform for the frequency-domain function is

$$\widetilde{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t} f(t)\ . \tag{B.7}$$

All of this generalizes to higher dimensions: you simply Fourier transform each dimension. In fact, one is free to use a different Fourier transform convention for each direction. We can use this freedom to pick a convention that 'automatically' fits our conventions for spacetime. In particular, given a four-vector $x = (t, \mathbf{x})$ and its conjugate four-momentum $p = (\omega, \mathbf{k})$, one may choose to Fourier transform as follows:

$$f(x) = \int đ\omega đ^3\mathbf{k} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}\widetilde{f}(p)\ . \tag{B.8}$$

With this convention, the basis function is simply

$$e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} = e^{-ip\cdot x}\ , \tag{B.9}$$

where $p \cdot x$ is the usual Minkowski dot product, $p_\mu x^\mu$. This makes it clear that the basis function is Lorentz invariant. The Fourier transform would still respect the spacetime symmetries even if we had not chosen a convenient notation—it just wouldn't be as simple to see.

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