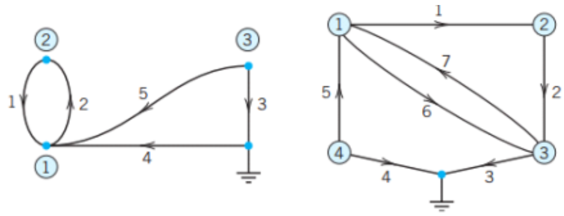
Linear Algebra Matrices

04 May 2021 20:15



Gauss Elimination

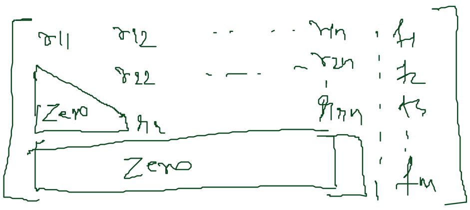
② 2 steps in gauss elimination

① Forward elimination → Reduce $Ax = b$ to upper triangular matrix (or REF)

i.e. $Tx = b'$ } Reduced

② Backward Substitution → solve $Tx = b'$ for x. Same x is the solution for A. {given that of is unique}

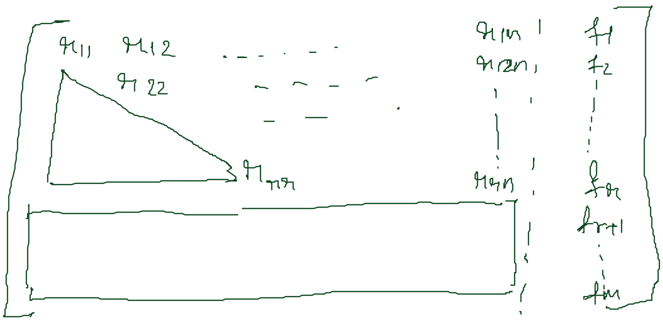
Before the back substitution the augmented matrix (Reduced) will look like this :



m rows
n columns

- Reduced matrix = upper triangular
- first r - non zero rows
- RHS may have last m rows = zero

upto r we have non zero rows
∴ r = rank of matrix



★ If any of the last m rows are Non zero : then System of Equations is inconsistent

★ If m = no of rows then Complexity = $O(m^3)$

① Inconsistency Eg: Reduced matrix →
$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$f_3 \neq 0$

∴ $r < m$
r = no of Equations.
m = no of rows.

$r = 2$
$m = 3$

∴ $Rx = f$ is inconsistent
∴ $Ax = B$ is inconsistent and ⇒ No Solution

② Consistency with $f_r = 0$

(2) Consistency with $f_r = 0$

Consider an example of Reduced matrix

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow f_r = 0$$

Here also $r < m$; $r = 2$, $m = 3$

but $f_2 = 0$ ∴ (underdetermined) System is Consistent and have infinite many solution.

In order to solve this:

Assume $x_3 = C$

Here Backward Substitution :

$$x_1 + x_2 + x_3 = 3 \Rightarrow x_1 + x_2 = 3 - C$$
$$-2x_2 + x_3 = -1 \qquad -2x_2 = -1 - C$$
$$x_2 = \frac{1+C}{2}$$

$$\Rightarrow x_1 + \left(\frac{1+C}{2}\right) = 3 - C$$

$$x_1 = 3 - C - \left(\frac{1+C}{2}\right) \Rightarrow = \frac{6 - 2C - 1 - C}{2}$$

$$x_1 \Rightarrow \frac{5 - 3C}{2}$$

$$\therefore X = \begin{bmatrix} C \\ (1+C)/2 \\ (5-3C)/2 \end{bmatrix}$$

∴ Consistent = atleast 1 solution

Inconsistent = No solution

If $f_r = 0$ infinite solution

rank = rank (reduced) → unique solution

Overdetermined if $m > n$ ←

determined if $m = n$ ← unique solution

underdetermined if $m < n$ ← infinite solution

pitfalls of Gauss elimination method

(1) Division by zero - when pivot element is 0. even if we exchange rows there may be case where we divide by zero.

$$\begin{bmatrix} 0 & 2 & 8 \\ 4 & 6 & 7 \\ 2 & 1 & 6 \end{bmatrix}$$

[Be Careful] solution : Pivoting

(2) Rounding off error -

Computers carry limited significant figure so it might round off few decimal points. Resulting in inaccurate answer.

★ Pivoting : Replacing the 0 pivot element row with the row with large number

★ Scaling - used to reduce the round off error accuracy

★ Partial pivoting : Search largest element then replace it with pivot element

A Scaling - Use the round off error and improve accuracy

A Partial pivoting: Search largest element then replace it with pivot element

A Use 5 significant figure to reduce rounding off error

Counting no of operations: To judge the Quality of numerical method:

- Space complexity
- time complexity

Eg, Consider 4x4 matrix $\begin{bmatrix} a_{11} & & \\ & & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_4 \end{bmatrix}$ Agumented $[A \mid b]$

$\Rightarrow \begin{bmatrix} a_{11} & & b_1 \\ & \ddots & \vdots \\ & a_{44} & b_4 \end{bmatrix}$

Consider worst case
No zero value initially

$\boxed{R_2 \leftarrow R_2 - \frac{a_{21}}{a_{11}} R_1}$ $\longrightarrow$ first element of $R_2 \to 0$

1 div

5 multiply
5 add

Now a_{21} is 0
$\therefore$ 4 $\times$
4 +

A Doing for all rows:

$R_3 \leftarrow R_3 - \frac{a_{31}}{a_{11}} R_1$

$R_4 \leftarrow R_4 - \frac{a_{41}}{a_{11}} R_1$

total 3 division
4x3 multiply
4x3 add

$\therefore$ $n-2$: divisions
$(n-1)(n-2)$ = multiplications
$(n-1)(n-2)$ additions

$\therefore$ total no of ops required is

$$2 \sum_{k=1}^{n-1} (n-k)(n-k+1) = O(n^3)$$

Total no of divisions $= O(n^2)$

$$= \sum_{k=1}^{n-1} (n-k) \tfrac{k}{2} = O(n^2)$$

In back Substitution:

no of operations: $\left[2 \sum_{k=1}^{n} (n-k)\right] + n = O(n^2)$

Consider: $n = 1000$, $n = 100000$