Partial derivative Contd ..

Higher order

$z = f(x,y)$

$$f_x \Big|_{(a,b)} \lim_{n \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y \Big|_{(a,b)} = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}\Big|_{(a,b)} = \lim_{h \to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h}$$

$$f_{xx} = \frac{\partial f_x}{\partial x}\Big|_{(a,b)}$$

$$\cancel{\frac{\partial^2 f}{\partial y}} \; \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}\Big|_{(a,b)} \quad 2$$

$$\lim_{h \to 0} \frac{f_y(a,b+h) - f_y(a,b)}{h}$$

2nd order partial derivative

mixed order

$$\frac{\partial^2 f}{\partial x \partial y}\Bigg|_{(a,b)} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\Bigg|_{(a,b)}$$

$$= \frac{\partial f_y}{\partial x} = \lim_{h \to 0} \frac{f_y(a+h, b) - f(a,b)}{h}$$

$$(f_{xy})_{(a,b)} = \lim_{h \to 0} \frac{f_x(a, b+h)\ f_x(a,b)}{h}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Questions

2nd Order Mixed partial Derivatives

$f(2x - 3y)^3$

$f_x = 3(2x-3y)^2 \frac{\partial}{\partial x}(2x-3y)$

$= 6(2x-3y)^2$

$f_y = \cancel{8(2x-3y)^2} \Rightarrow -9(2x-3y)^2$

$f_{xx} = 12(2x-3y)(2) = 24(2x-3y)$

$f_{yy} = \frac{\partial f_y}{\partial y} = -18(2x-3y)(-3$

$\Rightarrow 54(2x-3y)$

$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(f_x)$

$\Rightarrow f_x = 6(2x-3y)^2$

$12(2x-3y)\frac{\partial}{\partial y}(2x-3y)$

$\Rightarrow -36(2x-3y)$

$\rightarrow f_{yx} = (f_y)_x = \frac{\partial}{\partial x}(f_y)$

$\Rightarrow -18(2x-3y)(2)$

$\Rightarrow -36(2x-3y)$

② $f = \frac{x}{x^2+y^2}$ $(x,y) \neq (0,0)$

$f_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$

$= \frac{y^2 - x^2}{(x^2+y^2)^2}$

$f_y = \cancel{(x^2+y^2)x}$ $x\left(\frac{-1}{(x^2+y^2)}\right)\frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right)$

$\Rightarrow \frac{-2xy}{(x^2+y^2)^2}$

$$f_{xx} = \frac{(x^2+y^2)(-2x) - (y^2-x^2)2}{}$$

$$\Rightarrow \frac{-2x(x^2+y^2) - 4x(y^2-x^2)}{(x^2+y^2)^3}$$

$$\Rightarrow \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

$$f_{xy} = \frac{\partial}{\partial y} f_x$$

$$\Rightarrow \frac{(x^2+y^2)(2y) - \cancel{(x^2+y^2)}\, 2y(y^2-x^2)(2)(x^2+y^2)}{(x^2+y^2)^4}$$

$$\Rightarrow \frac{-2y^3 + 6yx^2}{(x^2+y^2)^3}$$

Q 3variable

$f_{xyz} = x^x y^y z^z \quad x.y z \Rightarrow 0$

$f_x \quad \log f = x\log x + y\log y + z\log z$

$\frac{1}{f} \cdot f_x = x\frac{1}{x} + \log x . x + 1 = 0$

$f_x = f\left[1 + \log x\right]$

$f_y = f\left[1 + \log y\right]$

$f_{xx} = \frac{\partial}{\partial x} f_x$

$f_x = f\left[1 + \log x\right)$

$f\left(\frac{1}{x}\right] + (1+\log x) f_x$

$\frac{f}{x} + f(1+\log x)^2$

$f_{yy} = \frac{f}{y} + f(1+\log y)^2$

Q $f_{xz} = (f_x)_z = \frac{\partial}{\partial z}(f_x)$

$f_x = f(1+\log x)$

$\frac{\partial}{\partial z}(f_x) = (1+\log x) f_z$

$= (1+\log x) f (1+\log z)$

$f((1+\log x)(1+\log z))$

f = function of x, y, z

Q. $f = e^{3x+4y} \cos 5z$

$f_x = (e^{3x+4y} \cos 5z)_{\frac{\partial}{\partial x}}(3x+4y \cos 5z)$

$f_{x} = (e^{3x+4y} \cos 5z)\, 3 \cdot e^{3x+4y} \cos 5z$

$f_{xx} = 9\, e^{3x+4y} \cos 5z$

$f_y = 4 \cdot e^{3x+4y} \cos 5z$

$f_{yy} = 16\, e^{3x+4y} \cos 5z$

$f_z = \left(e^{3x+4y}\right)(-\sin 5z)\,5.$

$f_{zz} = -25\left(e^{3x+4y}\right)(\cos 5z).$

Add $f_{xx} + f_{yy} + f_{zz} = 0$

$\Rightarrow$ Laplace equation

$$f(x,y) = \begin{cases} \dfrac{xy\,(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Proove mixed partial derivative for 2nd order is NOT equal

$f_{xy}(0,0) \neq f_{yx}(0,0)$

Q what condition for $f_{xy} \neq f_{yx}$

Eulers theoram :

if $f(x,y)$ and its partial deriv . .

f_x f_y f_{xy} f_{yx} are defined through

open region containg point (a,b)

and all continuous at (a,b) then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} \Rightarrow \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(f_{yx}\right) \Rightarrow (f_{yx})_x$$

$$= f_{xyx}$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

similarly

$$\frac{\partial^3 f}{\partial x^3} = f_{yyy}$$

Differentiability

$x : x_0 \longrightarrow x_0 + \Delta x$ when

if x is differentiable

$y = f(x) \longrightarrow f'(x_0)\Delta x + \varepsilon \Delta x$

$\varepsilon \rightarrow 0$

$\Delta x \rightarrow 0$

$\Delta y = f(x_0 + \Delta x) - f(x_0)$

2 variable function

$z = f(x, y)$

$x : x_0 \longrightarrow x_0 + \Delta x$

$y : y_0 \longrightarrow y_0 + \Delta y$

$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 y_0)$

if this condition holds then

$$\boxed{\begin{aligned} \Delta z &= f_x(x_0 y_0)\Delta x + f_y(x_0 y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\ &\varepsilon,\ \varepsilon_1(\Delta x, \Delta y) \rightarrow 0 \\ &\varepsilon_2(\Delta x, \Delta y) \rightarrow 0 \end{aligned}}$$

function is differentiable at $x_0 y_0$

Total differential of z

$$\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Necessary condition
Existence of partial derivative at Point P.

Sufficient

f_x and f_y of $f(x,y)$ are continuous at every point in area (open region) R.

Q $Z = \tan^{-1}\left(\frac{y}{x}\right)$

find total differential ie $\underline{dz}$ at x_0, y_0

$dz = f_x(x_0\, y_0)\,\Delta x + f_y(x_0\, y_0)\,\Delta y$

$f_x = \frac{\partial}{\partial x} f(x)$ since here it is Z we can say Z_x

$Z_x = \frac{1}{1+(y/x)^2}\frac{\partial}{\partial x}(y/x)$

$\frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right)$ Similarly $Z_y = \frac{x}{x^2+y^2}$

$= \frac{-y}{x^2+y^2}$.

$Z_y(a,b) = \frac{a}{a^2+b^2}$

$Z_x(a,b) = \frac{-b}{a^2+b^2}$

$dz = \left(\frac{-b}{a^2+b^2}\right)\Delta x + \left(\frac{a}{a^2+b^2}\right)\Delta y$

Q2. $U = (x^2+y^2+z^2)^{-3/2}$, $(x,y,z) \neq (0,0,0)$

find total differential of U at (a,b,c)

$U_x = \frac{-3}{2}(x^2+y^2+z^2)^{-5/2}(2x)$

$= -3x(x^2+y^2+z^2)^{-5/2}$

$\Rightarrow -3b \,/\, (a^2+b^2+c^2)^{5/2}$

Similar for other variables

$$dy = \frac{-3a\,\Delta x}{(a^2+b^2+c^2)^{5/2}} + \frac{3b\,\Delta y}{(a^2+b^2+c^2)^{5/2}} + \frac{-3c\,\Delta z}{(a^2+b^2+c^2)^{5/2}}$$

Proving differentiability at any point (a,b)

① Existence of partial derivative

② Eq 1 must hold (total differential)

Q $f(x,y) = x^2 + y^2$

$f_x = 2x \quad f_x\big|_{a,b} = 2a$

$f_y = 2y \quad f_y\big|_{a,b} = 2b$.

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \quad \text{①}$$

for point (a,b)

$$= 2x\Delta x + 2y\Delta y \text{ on}$$

$$2a\Delta x + 2b\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \quad \text{②}$$

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

$$(a+\Delta x)^2 + (b+\Delta y)^2 - a^2 - b^2$$

$$\Delta x^2 + 2a\Delta x + \Delta y^2 + 2b\Delta y \quad \text{③}$$

from ② and ③

$$\Delta x^2 + 2a\Delta x + \Delta y^2 + 2b\Delta y = 2a\Delta x + 2b\Delta y + \varepsilon_1 \Delta x + \varepsilon_1 \Delta y$$

$\varepsilon_1 = \Delta x$ definitely
$\varepsilon_2 = \Delta y$

$\varepsilon_i \to 0$ if Δx or $\Delta y \to 0$

$\therefore$ Equation is differentiable.

$\lim_{h \to 0} \left(\frac{0 - 0}{h}\right) = 0$

$f_y \quad \lim_{h \to 0} \frac{(0, 0+h) - f(0,0)}{h}$

$\Rightarrow \frac{0 - 0}{0} = 0$

Revise
(1) first order derivative
(2) $\varepsilon \to 0 \ \forall \ \Delta x$ and Δy

$\therefore f_x$ and f_y exist

Q $f(x,y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$f_x|_{(0,0)} ; \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h}$

$\Delta Z = f_x(x,y)\Delta x + f_y(0,0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$= \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$\Delta Z = f(0 + \Delta x, 0 + \Delta y) - f(0,0)$

$$\frac{\Delta x\,\Delta y\,(\Delta x^2 - \Delta y^2)}{\Delta x^2 + \Delta y^2} - f(0,0) \text{ ie } 0 \text{ (zero)}$$

$f(0,0) = 0$.

this is equal to

$\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$$e_1 = \frac{\Delta y\,\Delta x^2}{\Delta x^2 + \Delta y^2} \qquad \varepsilon_2 = \frac{-\Delta x\,\Delta y^2}{\Delta x^2 + \Delta y^2}$$

ε_1 and ε_2 may not be unique.

As $\lim\limits_{\Delta x, \Delta y \to (0,0)}$ $\lim$

Now we need to prove that limit exist ie when $\Delta x, \Delta y \to 0$, $\varepsilon_1 \varepsilon_2 \to 0$

Easist polar coordinate

$\Delta x = r\cos\theta \qquad \Delta y = r\sin\theta$.

$$\lim_{r \to 0} \frac{r^2\cos^2\theta\, r\sin\theta}{r^2} = 0$$

$\lim$

$\Delta\ \delta\ \epsilon$ method

$$|\cos^2\theta \sin\theta - 0| = |r\cos^2\theta \sin\theta|$$

$$\leq |r| = \delta$$

$\delta = \epsilon$; then $|\cos^2\theta\, r\sin\theta| < \epsilon$ when $|r| < \delta$

Another method to prove differentiability.

$$\Delta z = f_x(x_0 y_0)\Delta x + f_y(x_0 y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Let $\Delta \rho = \sqrt{\Delta x^2 + \Delta y^2}$

as $\Delta x, \Delta y \to 0$, $\Delta \rho \to 0$

$$dz = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\Delta z - dz = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\frac{\Delta z - dz}{\Delta \rho} = \frac{\varepsilon_1 \Delta x}{\Delta \rho} + \frac{\varepsilon_2 \Delta y}{\Delta \rho}$$

as $\Delta \rho \to 0$

$\varepsilon_1, \varepsilon_2 \to 0$

$\therefore$ finite Quantity ; $\Delta \rho$

$\rho < N$

$\therefore$

$$\frac{\Delta z - dz}{\Delta \rho} = 0$$

Illustration

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & x,y \neq (0,0) \\ 0 & x,y = (0,0) \end{cases}$$

objective ① Proove 1st order partial derivative exist

② thee is not differentiable. (Non differential)

① take $f(x,y)\Big|_{(0,0)}$ $\lim_{h \to 0} \dfrac{f(0+h, 0) - f(0,0)}{h}$

$\lim_{h \to 0} \dfrac{h^3 - 0}{h} - \dfrac{0}{h}$ $\Rightarrow \dfrac{h^3}{h^3} = 1$

$\lim_{h \to 0} \dfrac{f(0, 0+h) - f(0,0)}{h}$

$\lim_{h \to 0} \dfrac{-h^3}{h^3} = -1$ $\therefore$ limit Exists.

② To proove differentiable we need.

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho} = 0$$

$$\Delta z = f(0+\Delta x, 0+\Delta y) - f(x,y)$$

$$= \frac{\Delta x^3 - \Delta y^3}{\Delta x^2 + \Delta y^3} \quad \text{①}$$

$$dz = f_x(0,0)\Delta x + f_y(0,0)\Delta y$$

$$\Delta x - \Delta y$$

We just saw $f_x = 1$, $f_y = -1$

$$\lim_{\Delta\rho \to 0} \frac{\frac{\Delta x^3 - \Delta y^3}{\Delta x^2 + \Delta y^2} - \Delta x + \Delta y}{\Delta \rho}$$

we changed limit here b co'z of can b rep in my

$$\lim_{\Delta x \Delta y \to 0} \frac{\Delta x^3 - \Delta y^3 - \Delta x^3 + \Delta y \Delta x^2 - \Delta x \Delta y^2 + \Delta y^3}{(\Delta x^2 + \Delta y^2)^{3/2}}$$

$$\lim_{\Delta x \Delta y \to 0} \frac{\Delta x^2 \Delta y - \Delta x \Delta y^2}{(\Delta x^2 + \Delta y^2)^{3/2}}$$; function will

if this lim exist $\longrightarrow$ then differentiable at that point

So we try / check that it dont exist.

Most easy $\rightarrow$ path dependent shoo kar do

path
$\Delta y = m \Delta x$

$$\lim_{\Delta x \to 0} \frac{\Delta x^2 m \Delta x - \Delta x \, m^2 \Delta x^2}{(\Delta x^2 + m^2 \Delta x^2)^{3/2}}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{\Delta x^3}(m - m^2)}{\boxed{\cancel{\Delta x^3}}\left[(1 + m^2)\right]^{3/2}}$$

$$\lim_{\Delta x \to 0} \frac{m - m^2}{(1 + m^2)^{3/2}}$$

so lim depends on m

lim does not exist.

Differentiability Properties

$df = f_x(x_0, y_0)\,dx + f_y(x_0 y_0)\,dy$

(total differential of f)

(1) ~~Near (1, 2)~~

$f(x,y) = x^2 - xy + y^2 - 3$

$f_y(1,4) = + + 4 = 8.$

$df = \cancel{\Delta x \Delta y}$

$0.dx + 3.dy$ at a point $(1,2)$

$= 3dy$

Q. $X = \sqrt{298^2 + (401)^2}$

$f(x,y) = \sqrt{x^2 + y^2}$

$X_0 = 300 \quad y_0 = 400$

$dx = -2 \quad dy = +1$

$\sqrt{(x_0 + dx) + (y_0 + dy)}$

$df = f_x(x_0 y_0)\,dx + f_y(x_0 y_0)\,dy$

$f_x \Big|_{(300, 400)} \frac{300}{500} = 3/5$

$$f_y\Big|_{(300,400)} = \frac{400}{500} = 4/5$$

$$df = 3/5\,(-2) + 4/5\,(-1)$$

$$= \frac{-6+4}{5} = \underline{\underline{-0.4}}$$

$$df = \underline{f(298,401) - f(300,400)}$$

$$f\,(x_0+\Delta x)\,(y_0+\Delta x) + (x_0, y_0)$$

$$\Rightarrow -0.4 + 500$$

$$x = 499.96.$$

Q. $f(2,3) = 5$

$f_x(2,3) = 3$

$f_y(2,3) = 7$

$$f(1.98, 3.01) = ?$$

$dx = -(.02)$ | $df = f_x(2,3)\,dx + f_y(2,3)\,dy$

$dy = 0.01$

$$\Rightarrow 3(-.02) + 7(.01)$$

$$f(1.98, 3.01) - f(2,3)$$

$$df + f(2,3)$$

$$(0.9+5)$$

$$= \cancel{0.04 +}$$

$= df + f$

$= df + f(2,3)$

$= 0.01 + 5$

$\Rightarrow 5.01$

Q Absolute, relative

% change

① Absolute change

true estimate

	True	Estimate
Absolute	Δf	df
Relative change	$\dfrac{\Delta f}{f(x_0 y_0)}$	$\dfrac{df}{f(x_0 y_0)}$
Percentage change	$\dfrac{\Delta f}{f(x_0 y_0)} \times 100$	$\dfrac{df}{f(x_0 y_0)} \times 100$

Question Var. r and h change from initial value $(r_0 h_0) = (1, 5)$ by the amount $dr = 0.03$ and $dh = -.01$

Estimate the change in function $V = \pi r^2 h$

~~$dV = f(x_0, y_0)\Delta x + f(x_0 y_0)\Delta y$~~

V is a function of r and h.

$$dV = V_r(r_0 h_0)\,dr + V_h(r_0 h_0)\,dh$$

$V_r = 2\pi r h \quad V_r(1,5) = 10\pi$

$V_h = \pi r^2 \;;\; V_h(1,5) = \pi$

$dV = 10\pi(.03) + \pi(-.01)$
$= .3\pi - .01\pi$
$= 0.29\pi \longrightarrow 0.2\pi$

Relative $= \dfrac{dV}{V} = \dfrac{0.2\pi}{\pi(r^2 h)}$

$r_0 h_0 = (1, 5) \Rightarrow \dfrac{0.2}{1^2 5} = \dfrac{0.2}{5}$

$= 1/25$ or
$= 0.04$

% change = Relative $\times 100 \Rightarrow .04 \times 100$
$= \underline{4\%}$

Question :

find % Error in computed area of ellipse when error of 2% is made in increasing the major, minor axes

Given area $= \pi ab$

$dA = A_a da + A_b db$

$dA = \pi b da + \pi a db$

change $\dfrac{dA}{A} = \dfrac{\pi b da + \pi a db}{\pi ab}$

For a Relative change

$\Rightarrow \dfrac{\Delta f}{f(x_0, y_0)}$ this is if we are finding $f(x_0, y_0)$ at a point

$$\frac{dA}{A} = \frac{da}{a} + \frac{db}{b}$$

$$\frac{dA}{A} \times 100 = \frac{da}{a} \times 100 + \frac{db}{b} \times 100$$

$\frac{da}{a} \times 100 \to 2\%$, $\frac{db}{b} \times 100 \to 2\%$

4% ✓

otherwise relative change $= \dfrac{\Delta f}{f}$

$\Delta f = f_a dA + f_b dB$

if $f = f(A, B)$

Q. power consumed given by $P = E^2/R$ (watts)

if $E = 80$ volt, $R = 5$ ohms how much power will change

if E inc by 3, R inc by 0.1

$E_0 = 80$

$R_0 = \cancel{80}$ 5 ohm

$dE = +3V$

$dR = .1\,\Omega$

$dP = P_E\,dE + P_R\,dR$

$\dfrac{2E}{R} dE + \left(\dfrac{-E^2}{R^2}\right) dR$

$dP = P_E\, dE + P_R\, dR$

$= \frac{2E}{R} dE + \frac{-E^2}{R^2} dR$

$= \frac{2E}{R}(+3) + \frac{E^2}{R^2}(0.1)$

$\frac{2(80)}{5}(3) + \frac{(80)^2}{25}(.01)$

$= \frac{6(80)}{5} + \frac{64}{25}$

$\frac{30(80)+64}{25}$

$\Rightarrow$

$\Rightarrow \frac{2464}{25} =$

$dI = I_V dV + I_r dR$

$I_V = \frac{1}{R}, \quad I_R = \frac{-V}{R^2}$

$dI = \frac{dV}{R} + \left(\frac{-V}{R^2}\right) dR$

Question

Let $I = V/R$

Voltage drops from 24 to 23

Resistance drops from 100 to 80

Will I increase or decrease? Express in percentage.

$1/24 = 4.01\%$

$\frac{20}{100} = 20\%$

$\frac{dI}{I} = \frac{dV}{R(V)}(\cancel{R}) + \frac{-\cancel{V}}{R^2} \frac{dR \times R}{\cancel{V}}$

$= \frac{dV}{V} - \frac{dR}{R}$

$= 4\% - 20\%$

$\Rightarrow -16\%$

I decreases by 16%

Q $y = uv$ u measured with 2% Error
and v measured with 3% Error
what is % Error in y.

$dy = y_u\,du + y_v\,dv$

$\frac{dy}{y} = \frac{du}{u} + \frac{dv}{v}$

% change $\frac{dy}{y}(100)$ $\frac{du}{u}(100) + \frac{dv}{v}(100)$

$\Rightarrow$ 2% + 3% [Remember only % are given]

$\Rightarrow$ 5%

Question : Radius and height of cone are measured with 1% error.
find the % change in Surface area (curved)
if measured $r = 3ft$ $h = 4ft$
Also max % Error in area of cone?

Solution : curved $S = \pi r l$

$r_0 = 3$ $h_0 = 4$ $\frac{dh}{h}\times 100 = 1$

$\frac{dr}{r}\times 100 = 1$

$dS = S_r\,dr + S_l\,dl$

$S_r = \pi l$ $S_l = \pi r$

Chain Rule

say $z = f(x)$

$\omega = f(x, y)$

$$\frac{dz}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt}$$

(chain rule)

$x = f(t)$
$y = f(t)$

Assumption :

$\left.\begin{matrix} \omega \\ x \\ y \end{matrix}\right\}$ all these are defferentiable function.

$$\frac{d\omega}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$\frac{dx}{dt} \Rightarrow$ Complete derivative bcoz $x =$ fuction of single variable ie t

Any proof?

$t : t_0 \to t_0 + \Delta t$

$\left.\begin{matrix} x \to \Delta x \\ y \to \Delta y \\ \omega \to \Delta \omega \end{matrix}\right\}$ corresponding changes

4 point p $(x(t_0), y(t_0))$

$$\Delta \omega = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \varepsilon_2 \to 0$ as $\Delta x \, \Delta y \to 0$

div by Δt

$$\frac{\Delta \omega}{\Delta t} = \left(\frac{\partial f}{\partial x}\right)_p \frac{\Delta x}{\Delta t} + \left(\frac{\partial f}{\partial y}\right)_p \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \frac{\varepsilon_2 \Delta y}{\Delta t}$$

take $\Delta t \longrightarrow 0$

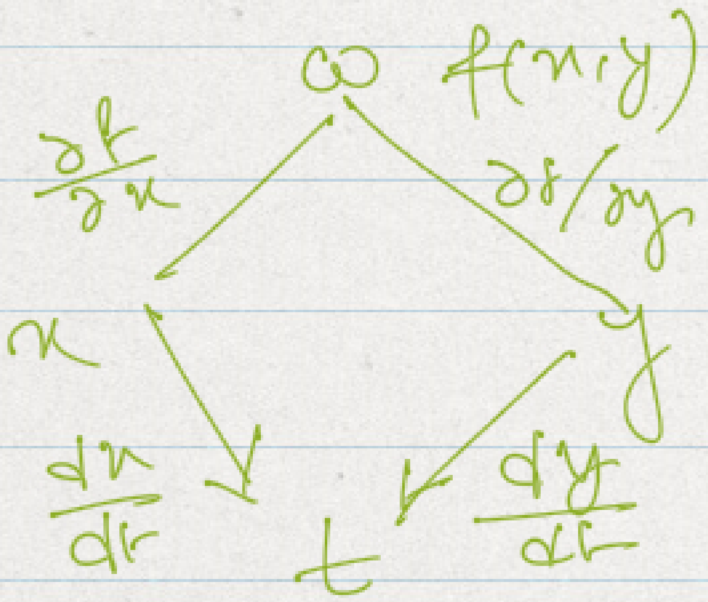
$$\cancel{\Delta w}\left(\frac{dw}{dt}\right)_{t_0} = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t}$$

$$= \left(\frac{\partial f}{\partial x}\right)_P \left(\frac{dx}{dt}\right)_{t_0} + \left(\frac{\partial f}{\partial y}\right)_P \left(\frac{dy}{dt}\right)_{t_0} + 0 + 0$$

$\because \Delta t \to 0$ so $\varepsilon_1 \varepsilon_2 \to 0$

$$\frac{dw}{dt} = \left(\frac{\partial f}{\partial x}\right)_P \left(\frac{dx}{dt}\right)_{t_0} + \left(\frac{\partial f}{\partial y}\right)_P \left(\frac{dy}{dt}\right)_{t_0}$$

Tree Diagram to Remember



Tree diagram

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Q $\omega = x\cos y + e^{-x}\sin y$

$x = t^2+1,$
$y = 2t$

$\frac{d\omega}{dt} = ??$

$= \frac{\partial \omega}{\partial x}\frac{dx}{dt} + \frac{\partial \omega}{\partial y}\frac{dy}{dt}$

$\Rightarrow \cos y - e^{-x}\sin y\,(2t) + 2\left[(-x\sin y) + e^{-x}\cos y\right]$

$\left(\frac{d\omega}{dt}\right)_{t=0} \Rightarrow \cos 0 - e\sin 0 + 2\left[-\sin 0 + e\cos 0\right]$

when $t=0$ $x=1$
$y=0$

$\Rightarrow \frac{2}{e}$ $\sin 0 = 0$
$\cos 0 = 1$

Q $\omega = x^2 + 2xy^2 + y^3$; $x = e^t$
$y = \cos t$

$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial x}\frac{dx}{dt} + \frac{\partial \omega}{\partial y}\frac{dy}{dt}$ at $t=0$
$x=1,\ y=1$

$\Rightarrow (2x+2x)e^t + (4xy+3y^2)(-\sin t)$

whole term $=0$ since $t=0$
$\sin 0 = 0$

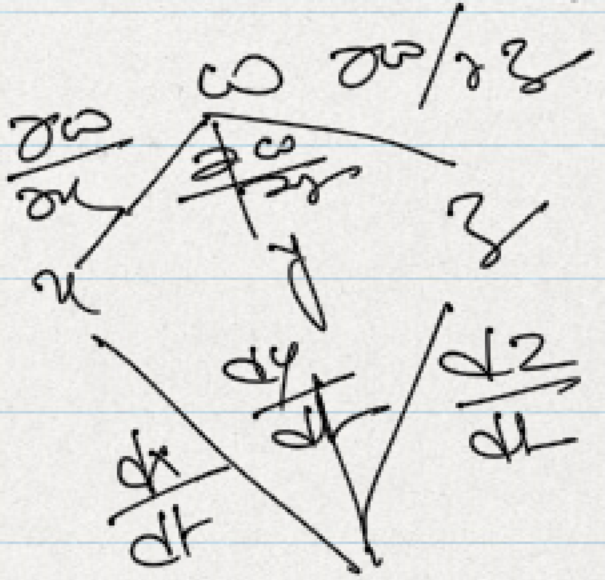
$\Rightarrow \left(\frac{d\omega}{dt}\right)_{t=0} = 4e$

PTO

Q

$\omega = f(x, y, z)$ $\quad x = f(t)$, $y = f(t)$, $z = f(t)$

$$\frac{d\omega}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$



$\omega = f(x_1, x_2, x_3 \dots x_n)$ $\quad x_i = x_i(t) \ \forall i$

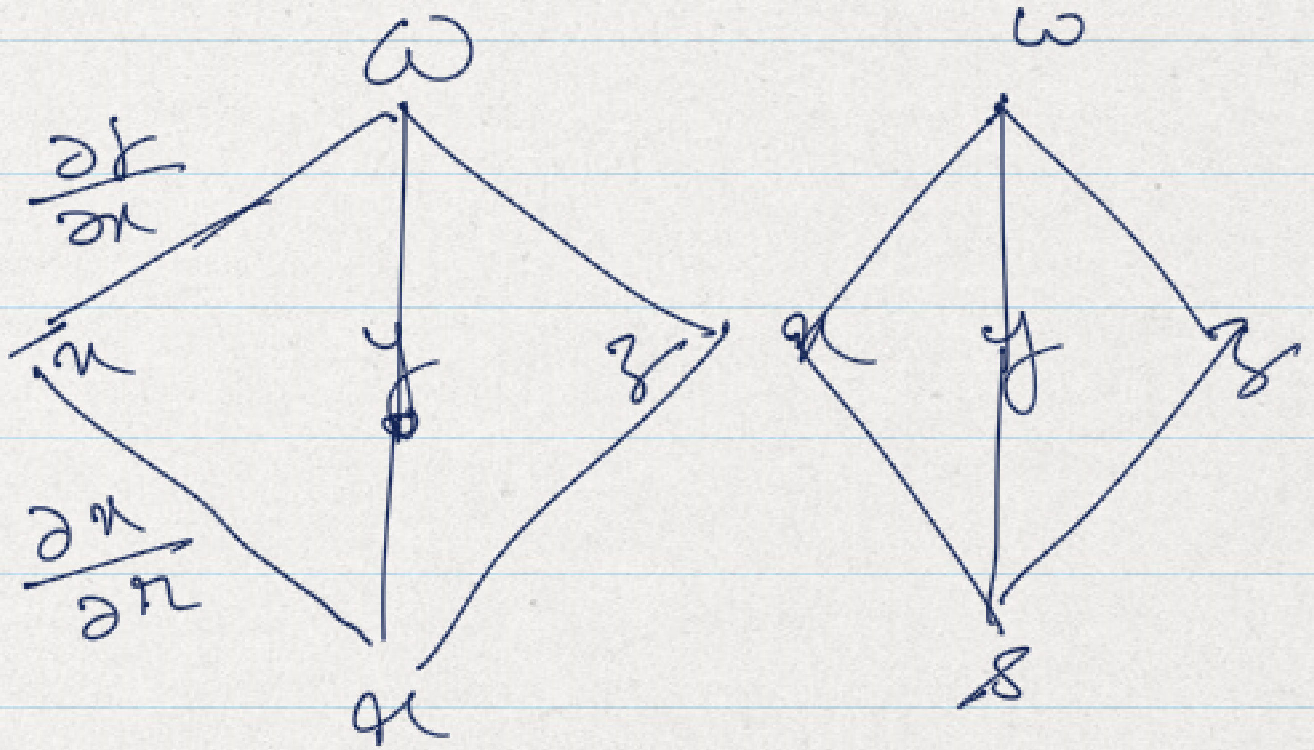
$$\frac{d\omega}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \dots \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{dt}$$

$\omega = f(x, y, z)$ $\quad x = \varphi(r, s)$, $y = \varphi(r, s)$, $z = \varphi(r, s)$

NOW x, y, z = function of two variable (independent)

$\frac{d\omega}{ds}, \frac{d\omega}{dr} \rightarrow$ it will become $\frac{\partial \omega}{\partial r}$ and $\partial\omega/\partial s$



$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

Question

$$w = 2ye^{x} - \ln z,$$

$$x = \ln(t^2+1)$$

$$y = \tan^{-1} t$$

$$z = e^{t}$$

Compute

$$\left.\frac{dw}{dt}\right|_{t=1}$$

at $t = 1$

$x = \ln 2$

$y = \tan^{-1} 1 = \pi/4$

$z = e$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \cdots\cdots$$

$$2ye^{x}\frac{1\times 2t}{(t^2+1)} +$$

$$2e^{x}\frac{1}{(1+t^2)} + \left(\frac{-1}{z}\right)e^{t}$$

$$\left.\frac{dw}{dt}\right|_{t=1} = \left(\frac{2\pi}{4} \times 2\right) 2\left(\frac{1}{2}\right) + 2(2)\frac{1}{2} + -\frac{1}{e}e$$

$$= \pi + 2 - 1 \Rightarrow \underline{\underline{\pi + 1}}$$

Question

$$w = \frac{x+y}{z} \qquad x = \cos^2 t$$

$$y = \sin^2 t$$

$$z = 1/t$$

At $t=3$ $x = \cos^2 3$

$y = \sin^2 3$

$z = 1/3$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \cdots\cdots$$

$$\left(\frac{1}{z}\right)\left(2\cos t\,(-\sin t)\right) + \frac{1}{z}(2\sin t\cos t) + \frac{(x+y)}{-z^2}\left(\frac{-1}{t^2}\right)$$

$$\Rightarrow \frac{dw}{dt} = \frac{x+y}{z^2 t^2}$$

$$\left(\frac{dw}{dt}\right)_{t=3} \frac{\cos^2 3 + \sin^2 3\,(\cancel{9})}{\cancel{9}}$$

$$= \cos^2 3 + \sin^2 3 = \underline{\underline{1}}$$

$z = 4e^{x}\ln y.$ $\quad x = \ln(r\cos\theta)$

$y = r\sin\theta$

$(r,\theta) = (2, \pi/4)$

Evaluate

$\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$? (chain Rule)

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}.$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$(4e^{x}\log y)\left(\frac{1}{r}\right) + \frac{4e^{x}}{y}(\sin\theta)$$

$$4e^{x}\ln y\left(\frac{1}{\cos\theta}\right)(-\sin\theta) + \frac{4e^{x}}{y}(r\cos\theta)$$

$\Rightarrow$ at $(r,\theta) = (2, \pi/4)$

$x = \ln\left(2\frac{1}{\sqrt{2}}\right) = \ln\sqrt{2} = \frac{1}{2}\log 2$

$y = r\sin\theta,\ 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$

Put x, y

get Answer.

Question

$\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v} = ?$

when $u = 0, v = 1$ if $w = \sin(xy) + x\sin y$

; $x = u^2 + v^2$
$y = uv$

Question

$\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ if $w = xy + yz + zx$

; $x = u + v$
$y = u - v$
$z = uv$ at
$(u, v) = \left(\frac{1}{2}, 1\right)$

Question

$\frac{\partial z}{\partial p}$ $\frac{\partial z}{\partial q}$ if $z = x^3 + y^3 - 3x^2y + 6xy^2$

$x = u^2 + v^2$
$y = u^2 - v^2$

$u = p - q$
$v = p^2 + pq$
$(p, q) = (1, 0)$

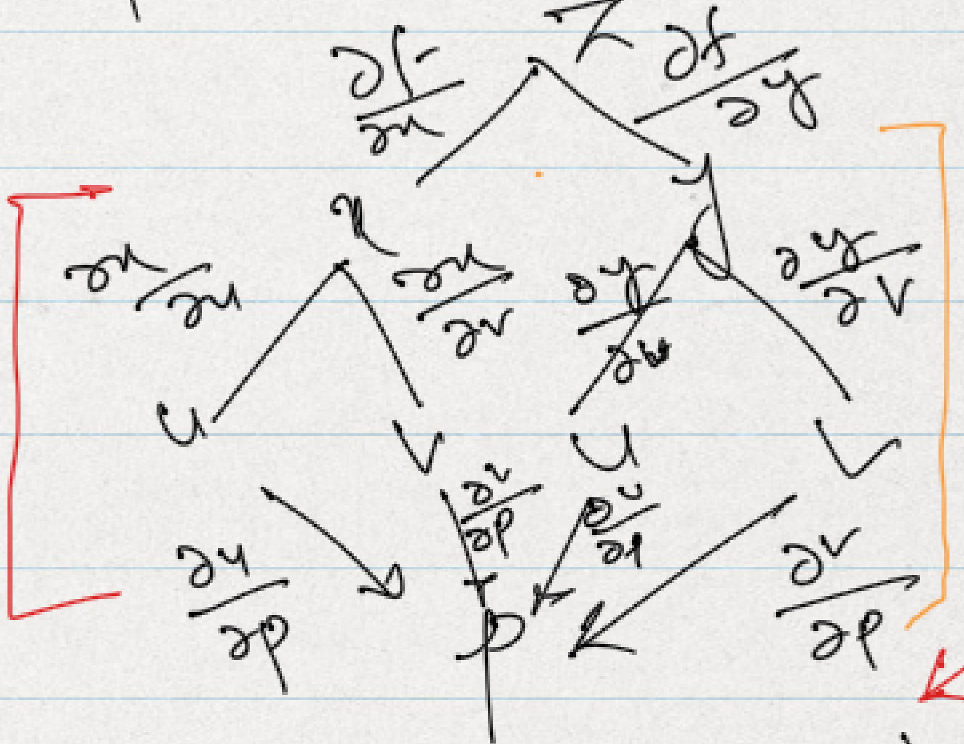
Solve

$z \rightarrow f(x, y)$; $x \rightarrow f(u, v)$
$y \rightarrow (u, v)$;
$u \rightarrow f(p, q)$
$v \rightarrow f(p, q)$

$\therefore z \longrightarrow f(p, q)$

Compute $\frac{\partial z}{\partial p}, \frac{\partial z}{\partial q}$??

at $p=1, q=0$.



Second Branch

first Branch

first Branch

$$\frac{\partial z}{\partial p} = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial p}\right) + \frac{\partial f}{\partial y}\left[\frac{\partial y}{\partial v}\frac{\partial v}{\partial p} + \frac{\partial y}{\partial u}\frac{\partial u}{\partial p}\right]$$

$$\frac{\partial z}{\partial q} = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial q}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial q}\right)$$

$$\left\{\begin{array}{l} \text{At } p=1\ q=0 \\ u=1,\ v=1;\ x=2 \\ \qquad\qquad\quad\ y=0 \end{array}\right\}$$

$\frac{\partial z}{\partial p}\ \frac{\partial z}{\partial q}$ of $z = x^3+y^3-3x^2y+6xy^2$

$x = u^2+v^2$

$y = u^2-v^2$

$u = p-q$

$v = p^2+pq$

$(p,q) = (1,0)$

$\frac{\partial z}{\partial p}\ \frac{\partial z}{\partial q}$ if $z = x^3 + y^3 - 3x^2y + 6xy^2$

$x = u^2 + v^2$ $\quad u = p - q$

$y = u^2 - v^2$ $\quad v = p^2 + pq$

$(p, q) = (1, 0)$

$$\frac{\partial z}{\partial p} \Rightarrow (3x^2 - 6xy + 6y^2)\left[(2u) + 2v(2p+q)\right]$$

$$+ (3y^2 - 3x^2 + 12xy)\left[(2u) + (-2v)(2p+q)\right]$$

$$\Rightarrow (3+\ )\left[2 + 2(2)\right] + \left[-3\right]\left[2 - 2(2)\right]$$

$$3(6) + 3(2)$$

$$\Rightarrow \underline{\underline{24}} \qquad \frac{\partial z}{\partial p} = \underline{\underline{24}}$$

Implicit differentiation

(Properties of chain rule)

Let function (x, y) be differentiable at equation $f(x, y) = 0$ defines y implicitly as function of x. then

$$w = f(x, y) = 0 \Rightarrow \frac{dw}{dx} = 0$$

$$\frac{dw}{dx} \Rightarrow \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

Q. $x^y + y^x = \alpha$ $\quad \alpha =$ constant

compute $\frac{dy}{dx}$?

let $f(x,y)$ as $= x^y + y^x - \alpha = 0$

$$\frac{dy}{dx} = -\left[\frac{yx^{y-1} + y^x \ln x}{x^y \ln x + xy^{x-1}}\right] \qquad \boxed{\frac{dy}{dx} = \frac{-f_x}{f_y}}$$

Q $xe^y + \sin(xy) + y - \ln 2 = 0$

let $f(x,y) = xe^y + \sin(xy) + y - \ln 2 = 0$

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$-\left[\frac{xe^y + \cos(xy)\cdot y}{xe^y + \cos(xy)\, x + 1}\right]$$

\# 3 variable $\quad f = f(x,y,z)$

$$\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

we know $\frac{\partial y}{\partial x} = 0$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\times\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-f_x}{f_z}$$

Similarly $\frac{\partial z}{\partial y} = \frac{-f_y}{f}$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{f_y}{f_z}}$$

Q. $F = z^3 + xy + yz + y^3 - 2 = 0$

compute $\dfrac{\partial z}{\partial x} = -\dfrac{f_x}{f_z}$

$$= -\left(\frac{y}{3z^2+y}\right) = \frac{\partial z}{\partial x}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = -\left(\frac{1}{4}\right)$$

directly differentiate

$$3z^2 z_x + y + yz_x = 0$$

$$\frac{\partial z}{\partial x} = 3z^2 z_x + y + yz_x \qquad z_x = -\frac{y}{3z^2+y}$$

$W = f(x-y,\ y-z,\ z-x)$

Prove $f_x + f_y + f_z = 0$

$W = f(u, v, w)$

$u = x-y$

$v = y-z$

$w = z-x$

$$\frac{\partial W}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$f_u(1) + f_v(0) + f_w(-1)$$

$$\frac{\partial W}{\partial x} \Rightarrow f_u - f_w$$

$$\frac{\partial W}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$

$$f_u(-1) + f_v(1) + f_w(0)$$

$$\Rightarrow f_v - f_u$$

$$\frac{\partial W}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}$$

$$\frac{\partial W}{\partial z} \Rightarrow f_u(0) + f_v(-1) + f_w(1)$$

$$= f_w - f_v$$

$$f_x + f_y + f_z = \begin{matrix} f_u + f_v + f_w \\ - f_u - f_v - f_w \end{matrix}$$

$$= 0$$

Q $z = f(x,y)$ $\quad x = u\cos\alpha - v\sin\alpha$

$y = u\sin\alpha + v\cos\alpha$

$\because \alpha =$ constant

$x = f(u,v)$

$y = f(u,v)$

Prove: $\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = f_x(\cos\alpha) + f_y(\sin\alpha)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = f_x(-\sin\alpha) + f_y(\cos\alpha)$$

$$f_u^2 + f_v^2 = f_x^2\cos^2\alpha + f_y^2\sin^2\alpha + 2f_xf_y\cos\alpha\sin\alpha$$
$$+ f_x^2\sin^2\alpha + f_x\cos^2\alpha - 2f_xf_y\cos\alpha\sin\alpha$$

$$= f_x^2(\sin^2 + \cos^2) + f_y$$

$$\boxed{f_u^2 + f_v^2 \Rightarrow f_x^2 + f_y^2}$$

Question Laplace Equation

$w = f(u,v)$ given f satisfy laplace

ie $f_{uu} + f_{vv} = 0$

$u = \frac{x^2-y^2}{2}$, $v = xy$

Prove that: $w_{xx} + w_{yy} = 0$

ie $\frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial x}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial w}{\partial y}\right) = 0$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$f_u\left(\frac{2x}{2}\right) + f_v(y) \Rightarrow f_u(x) + f_v(y)$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}(w_x)$$

$$= \frac{\partial}{\partial x}\left(x f_u + y f_v\right)$$

$u = f(x,y)$
$v = f(x,y)$
$\therefore$ product rule

$\Rightarrow \left(x\frac{\partial}{\partial x}f_u + f_u\right) + y\frac{\partial}{\partial x}f_v$

$\Rightarrow$ chain rule $+ y\left(\frac{\partial}{\partial x}f_u\right.$

$\Rightarrow x\left[\frac{\partial}{\partial u}f_u \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v}f_u\frac{\partial v}{\partial x}\right] + f_u + y\left[\frac{\partial}{\partial u}f_u\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}f_v \cdot \frac{\partial v}{\partial x}\right]$

$\Rightarrow x\left[x f_{uu} + f_{uv}\, y\right] + f_u + y\left[x f_{vu} + f_{vv}\, y\right]$

$x^2 f_{uu} + y^2 f_{vv} + 2xy f_{uv} + f_u$

Similarly:

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + f_v u$$

$$= f_u(-y) + f_v(x)$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left[f_u(-y) + x f_v\right]$$

$$= f_u(-1) + x\frac{\partial}{\partial y} f_v$$

$$-y\frac{\partial}{\partial y} f_u + f_u(-1) +$$

chain Rule $\Rightarrow$ $-y\left[\frac{\partial f_u}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_u}{\partial v}\frac{\partial v}{\partial y}\right] - f_u$

$$+ x\left[\frac{\partial f_v}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_v}{\partial v}\frac{\partial v}{\partial y}\right]$$

$$\Rightarrow -y\left[f_{uu}(-y) + x f_{uv}\right] - f_u$$

$$+ x\left[-y(f_{uv}) + f_{vv} \cdot x\right]$$

$$\Rightarrow x^2 f_{vv} + y^2 f_{uu} - 2xy f_{uu} - f_u = w_{yy}$$

$$w_{xx} + w_{yy} = x^2 + y^2 (f_{uu} + f_{vv})$$

$$= 0.$$

VIMP to tell which variables are dependent; which are independent.

Example: $w = x^2 + y^2 + z^2$, $z = x^2 + y^2$

find $\frac{\partial w}{\partial x}$?

Basic Assumption
w = dependent ; x = independent.

Others, z, y = don't know !!

Dependent	independent	
w, z	x, y	1
w, y	x, z	2

for case 1

$$\frac{\partial w}{\partial x} = 2x + 0 + 2z z_x$$
$$= 2x + 2z(2x)$$
$$2x + 4xz$$
$$2x + 4x(x^2 + y^2) = 4x^3 + 4xy^2 + 2x$$

for case 2

$$\frac{\partial w}{\partial x} = 2x + 2y y_x + 0$$

$$y^2 = z - x^2$$
$$2y \cdot y_x = -2x$$

$$\frac{\partial w}{\partial x} = 2x + -2x$$
$$= 0$$

Notation for independent

Case 1 $\left(\frac{\partial w}{\partial x}\right)_y$

Case 2 $\left(\frac{\partial w}{\partial x}\right)_z$

Question:

$\omega = x^2 + y - z + \sin t$;

$x + y = t$

$\left(\frac{\partial \omega}{\partial y}\right)_{(x,z)} = 1 + \cos t \quad t y$

$t = x + y$

$\frac{\partial t}{\partial y} = 1$

$\left(\frac{\partial \omega}{\partial y}\right)_{(x,z)} = 1 + \cos t \cdot \times 1$

$= 1 + \cos t$

$= y_t - 0 + \cos t$

$y = t - x$

$\frac{\partial y}{\partial t} = 1$

$\left(\frac{\partial \omega}{\partial t}\right)_{(x,z)} =$

$= 1 + \cos t$

~~Ques~~ find $\left(\frac{\partial \omega}{\partial y}\right)_x$, $\left(\frac{\partial \omega}{\partial y}\right)_z$

at $(\omega, x, y, z) = (4, 2, 1, -1)$

if $\omega = x^2 y^2 - yz - z^3$ and

$x^2 + y^2 + z^2 = 6$