

CS553 Cryptography

BitBees

Question 8

Given a 3-D tic-tac-toe board, where players take turns placing X 's and O 's, the game typically ends when one player completes a line or diagonal; that is, when they manage to place their symbols in three cells that form a line or diagonal in the grid. The only twist: the game continues until every cell contains a symbol, and the goal is to arrange the symbols to minimize the number of completed lines or diagonals.

$\text{isX}_{ijk} \in [0, 1]$: Does cell (i, j, k) contain an X ($\text{isX} = 1$) or an O ($\text{isX} = 0$)?

$\text{isLine}_l \in [0, 1]$: Does line/diagonal l contain 3 of the same symbol?

Objective Function

Lines: Minimize the number of completed lines or diagonals

$$\text{Minimize } Z = \sum_{l \in \text{Lines}} \text{isLine}_l \quad (1)$$

Constraints

Take turns: The board must contain 14 X 's and 13 O 's (X goes first).

$$\sum_{ijk} \text{isX}_{ijk} = 14 \quad (2)$$

Lines: For a line to not be complete, one cell must have a different value. The simple observation here is that the sum of the corresponding 3 binary variables would be 3 if they are all X and 0 if they were all O . We need to forbid those outcomes whenever $\text{isLine}_l == 0$. Note that l_0 is the first cell in line l , l_1 is the second, and l_2 is the third.

$$\text{isLine}_l == 0 \implies \text{isX}[l_0] + \text{isX}[l_1] + \text{isX}[l_2] \geq 1 \quad \forall l \in \text{Lines} \quad (3)$$

$$\text{isLine}_l == 0 \implies \text{isX}[l_0] + \text{isX}[l_1] + \text{isX}[l_2] \leq 2 \quad \forall l \in \text{Lines} \quad (4)$$