CS 553

CRYPTOGRAPHY

Lecture 10

More on Linear Cryptanalysis

Instructor

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Image Source: Google

- The idea of linear masks
- The notion of approximation
- Expressing key bits in terms of plaintext and ciphertexts
- Approximating the non-linear component
- Extending the approximation to other associate parts of a simple cryptosystem
- Using the linear approximation to mount a KPA
- Recovering a single bit of key material

Sypher00B

Moving on to a more complex but still toy cryptosystem:

- Sypher00B encrypts 4 bits with **three** 4 bit keys

S-box

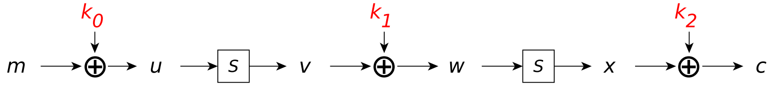
x	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
$S(x)$	f	e	b	c	6	d	7	8	0	3	9	a	4	2	1	5

Encryption

$$m \xrightarrow{\oplus\, k_0} u \xrightarrow{\boxed{S}} v \xrightarrow{\oplus\, k_1} w \xrightarrow{\boxed{S}} x \xrightarrow{\oplus\, k_2} c$$

- Again, same as Sypher002 with a different SBox

- KPA assumption: attacker knows message m and ciphertext c



- WLOG the following holds for any mask α, β, γ. Why?

$$(\alpha \cdot m) = (\alpha \cdot k_0) \oplus (\alpha \cdot u) \tag{1}$$

$$(\beta \cdot v) = (\beta \cdot k_1) \oplus (\beta \cdot w) \tag{2}$$

$$(\gamma \cdot x) = (\gamma \cdot k_2) \oplus (\gamma \cdot c) \tag{3}$$

- We assume: we can find α, β, γ such that

$$\alpha \cdot u = \beta \cdot S[u] = \beta \cdot v \qquad \text{Holds with prob. } p_1 \neq \frac{1}{2}$$

$$\beta \cdot w = \gamma \cdot S[w] = \gamma \cdot x \qquad \text{Holds with prob. } p_2 \neq \frac{1}{2}$$

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$$m \longrightarrow \overset{\color{red}{k_0}\,\downarrow}{\oplus} \longrightarrow u \longrightarrow \boxed{S} \longrightarrow v \longrightarrow \overset{\color{red}{k_1}\,\downarrow}{\oplus} \longrightarrow w \longrightarrow \boxed{S} \longrightarrow x \longrightarrow \overset{\color{red}{k_2}\,\downarrow}{\oplus} \longrightarrow c$$

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$$\begin{tikzcd}
 & k_0 \arrow[d] & & & & k_1 \arrow[d] & & & & k_2 \arrow[d] & \\
m \arrow[r] & \oplus \arrow[r] & u \arrow[r] & \boxed{S} \arrow[r] & v \arrow[r] & \oplus \arrow[r] & w \arrow[r] & \boxed{S} \arrow[r] & x \arrow[r] & \oplus \arrow[r] & c
\end{tikzcd}$$

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- Using Eqn. (1) – (3)

$$(\alpha \cdot m) \oplus (\beta \cdot v) \oplus (\gamma \cdot x) = (\alpha \cdot u) \oplus (\beta \cdot w) \oplus (\gamma \cdot c) \oplus (\alpha \cdot k_0) \oplus (\beta \cdot k_1) \oplus (\gamma \cdot k_2)$$

- Rearranging

$$(\alpha \cdot m) \oplus (\beta \cdot v) \oplus (\gamma \cdot x) \oplus (\alpha \cdot u) \oplus (\beta \cdot w) \oplus (\gamma \cdot c) = (\alpha \cdot k_0) \oplus (\beta \cdot k_1) \oplus (\gamma \cdot k_2)$$

- Note RHS is a constant, for LHS, we know:

$$\left.\begin{array}{lll} (\alpha \cdot u) & = (\beta \cdot v) & \text{with prob. } p_1 \\ (\beta \cdot w) & = (\gamma \cdot x) & \text{with prob. } p_2 \end{array}\right\} \text{taken over all } u \text{ and } w$$

- Possibility to remove intermediate variables:

$$\alpha \cdot u \qquad \beta \cdot v \qquad \beta \cdot w \qquad \gamma \cdot x$$

What is the probability that the approximation holds when all the internal variables have canceled out?

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The Independence Assumption

Events $(\alpha \cdot u) = (\beta \cdot v)$ and $(\beta \cdot w) = (\gamma \cdot x)$ are independent

The Possibilities

Case 1

$$(\alpha \cdot u) = (\beta \cdot v)$$
$$(\beta \cdot w) = (\gamma \cdot x)$$

Case 2

$$(\alpha \cdot u) = (\beta \cdot v) \oplus 1$$
$$(\beta \cdot w) = (\gamma \cdot x) \oplus 1$$

Implication: $(\alpha \cdot m) \oplus (\gamma \cdot c) = (\alpha \cdot k_0) \oplus (\beta \cdot k_1) \oplus (\gamma \cdot k_2)$

Case 3

$$(\alpha \cdot u) = (\beta \cdot v)$$
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- Favorable events:
$$\begin{cases} \text{Case 1} \rightarrow \text{Prob.} = p_1 \times p_2 \\ \text{Case 2} \rightarrow \text{Prob.} = (1-p_1) \times (1-p_2) \end{cases}$$
- Probability of linear approximation: $\boxed{p_1 p_2 + (1-p_1)(1-p_2)}$
- If $p_1 = \frac{1}{2} + \epsilon_1$ and $p_2 = \frac{1}{2} + \epsilon_2$, then

$$\begin{aligned} & p_1 p_2 + (1-p_1)(1-p_2) \\ =& 1 - p_1 - p_2 + 2p_1 p_2 \\ =& 1 - \frac{1}{2} - \epsilon_1 - \frac{1}{2} - \epsilon_2 + 2\left(\frac{1}{4} + \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2} + \epsilon_1\epsilon_2\right) \\ =& \frac{1}{2} + \boxed{2\epsilon_1\epsilon_2} \end{aligned}$$

Intuition

What would the general case look like?

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Intuition

What would the general case look like?

- Extending this to m independent events with probabilities $p_i, i = 1, \cdots, m$, we have:

$$\frac{1}{2} + 2^{m-1} \prod_{i=1}^{m} \left(p_i - \frac{1}{2} \right)$$

- How would you define the event in the general case?
  - What is actually piling-up?
  - What happens when constituent events are true?

The piling-up lemma allows us to compute the **bias** of a set of combined linear approximations provided that the constituent linear approximations are **independent**.

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- How?

The Linear Approximation Table

This table lists the probabilities that the **sum** of certain input bits of a equals the sum of certain output bits of $S[a]$.

- Each entry gives us the linear characteristic for a pair of input-output masks

$$\alpha \xrightarrow{S} \beta$$

- And also the associated **bias**

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	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
1	-2	.	2	.	-2	4	-2	2	4	2	.	-2	.	2	.
2	2	-2	.	-2	.	.	2	2	4	.	2	4	-2	-2	.
3	4	2	2	-2	2	.	.	.	.	2	-2	-2	-2	.	4
4	.	-2	2	2	-2	.	.	-4	.	2	2	2	2	.	4
5	-2	2	.	2	4	.	2	-2	4	.	-2	.	2	-2	.
6	-2	.	2	.	2	4	2	2	-4	2	.	2	.	-2	.
7	.	.	.	4	.	-4	.	.	.	.	4	.	4	.	.
8	.	-2	2	-4	.	2	2	-4	.	-2	-2	.	.	2	-2
9	-2	-6	.	.	2	-2	.	2	.	.	-2	-2	.	.	2
a	-2	.	-6	-2	.	2	.	-2	.	2	.	.	-2	.	2
b	.	.	.	2	-2	2	-2	.	.	-4	-4	2	-2	-2	2
c	.	.	.	-2	-2	-2	-2	.	.	4	-4	2	2	-2	-2
d	-2	.	2	2	.	-2	.	-2	.	2	.	.	$\boxed{-6}$	.	-2
e	2	-2	.	.	2	2	-4	-2	.	.	2	-2	.	-4	-2
f	-4	2	2	-4	.	-2	-2	.	.	-2	2	.	.	-2	2

Highest Bias for $\alpha = \beta = \gamma = \mathtt{d}$

- The chosen characteristic: $\mathtt{d} \xrightarrow{S} \mathtt{d} \xrightarrow{S} \mathtt{d}$
- For Sypher00B this implies:

$$(\mathtt{d} \cdot m) \oplus (\mathtt{d} \cdot c) = (\mathtt{d} \cdot k_0) \oplus (\mathtt{d} \cdot k_1) \oplus (\mathtt{d} \cdot k_2)$$

- Associated prob.

$$\begin{aligned} \Pr\left(\mathtt{d} \xrightarrow{S} \mathtt{d} \xrightarrow{S} \mathtt{d}\right) &= \frac{1}{8} \times \frac{1}{8} + \frac{7}{8} \times \frac{7}{8} \\ &= \frac{25}{32} \\ &= \frac{1}{2} + \frac{9}{32} \end{aligned}$$

- Attacker collects N KPs to calculate $(\mathtt{d} \cdot m) \oplus (\mathtt{d} \cdot c)$
- Based on counter values determine if $(k_0 \oplus k_1 \oplus k_2) \cdot d \stackrel{?}{=} 0/1$

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Is recovering single key-bit material enough?

- Can we do more? How?
- How about more linear approximations?
- Fine, but may not be always available

Better if we can use one but recover more than one

- How to deduce more key bits?
- Sypher00C leads the way

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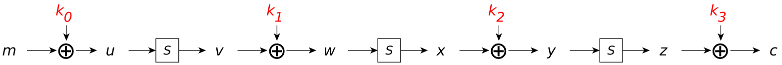
- How about more linear approximations?
- Fine, but may not be always available

Better if we can use one but recover more than one

- How to deduce more key bits?
- Sypher00C leads the way



- Sypher00B $\leftarrow$ Sypher00B with extra round
- Use characteristic from Sypher00B for first two rounds

$$(\mathtt{d} \cdot m) \oplus (\mathtt{d} \cdot y) = (\mathtt{d} \cdot k_0) \oplus (\mathtt{d} \cdot k_1) \oplus (\mathtt{d} \cdot k_2)$$

- Holds with prob. $\frac{1}{2} + \frac{9}{32}$
- How to handle last round?

Intuition

Guess k_3 and invert last round

- Repeat the attack used for Sypher00B for every guess of k_3

Sypher00C

$$m \longrightarrow \overset{k_0}{\oplus} \longrightarrow u \longrightarrow \boxed{S} \longrightarrow v \longrightarrow \overset{k_1}{\oplus} \longrightarrow w \longrightarrow \boxed{S} \longrightarrow x \longrightarrow \overset{k_2}{\oplus} \longrightarrow y \longrightarrow \boxed{S} \longrightarrow z \longrightarrow \overset{k_3}{\oplus} \longrightarrow c$$

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- For a given ciphertext c, attacker computes

$$y' = S^{-1}[c \oplus i]$$

for every guess $k_3 = i$
- Uses the corresponding message m to compute

$$(\mathtt{d} \cdot m) \oplus (\mathtt{d} \cdot y')$$

- For each guess i, he maintains two counters T_0^i and T_1^i

$$T_0^i\texttt{++} \;\; \texttt{if} \;\; (\mathtt{d} \cdot m) \oplus (\mathtt{d} \cdot y') = 0$$

$$T_1^i\texttt{++} \;\; \texttt{if} \;\; (\mathtt{d} \cdot m) \oplus (\mathtt{d} \cdot y') = 1$$

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How to distinguish?

- For the correct guess, $k_3 = \nu$ (say), expected value of

$$\begin{cases} T_0^{\nu} \leftarrow \frac{N}{2} + \frac{9N}{32} \\ T_1^{\nu} \leftarrow \frac{N}{2} - \frac{9N}{32} \end{cases}$$

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- What else?
- Looking at actual values of T_0^i and T_1^i in the highest imbalanced counter value of $(k_0 \oplus k_1 \oplus k_2) \cdot d$ is recovered
- For Sypher00C, largest counter indicates value of $(k_0 \oplus k_1 \oplus k_2) \cdot d$

How many counters?

- Last round inversion works for Sypher00C

Point to Ponder

Can the same be done for the first round?

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Point to Ponder

Can the same be done for the first round?

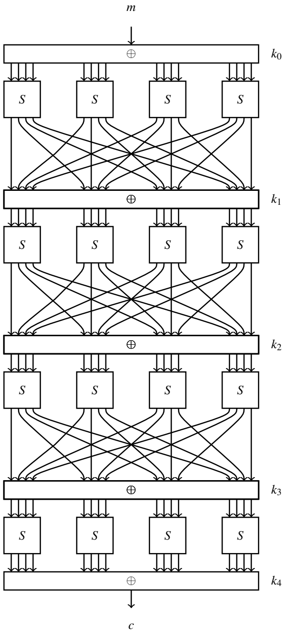
What is estimate for number of KPs required: N?

- For Single-bit recovery a good estimate is

$$N = c\left|p - \frac{1}{2}\right|^{-2} \qquad \text{or} \qquad N = c|\epsilon|^{-2}$$

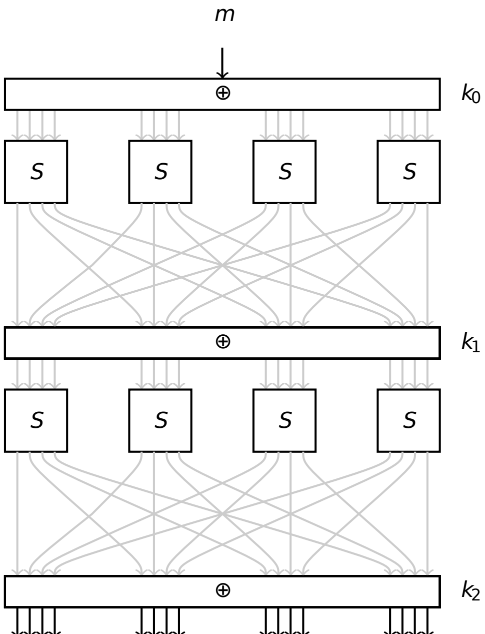
  where $\epsilon \rightarrow$ bias
- Constant $c \geq 2$ varies with block cipher and attack
- c for single-bit recovery will definitely be less than that for multiple-bit recovery
  - Think about #counters to choose from

What about Sypher00D?

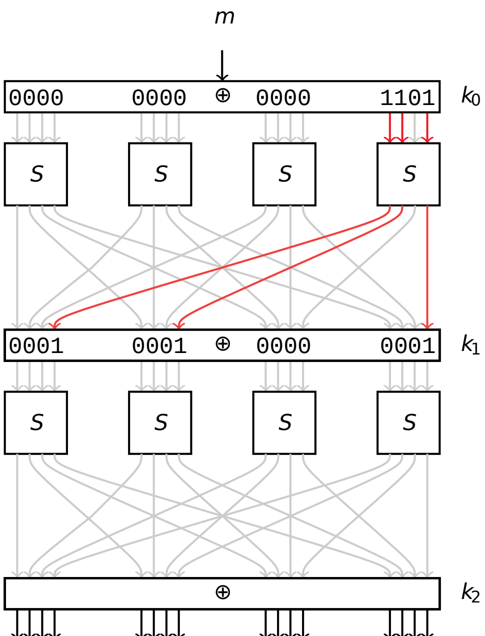


- Sbox same as Sypher00A-C
- Permutation same as Sypher004 in DC lecture
- Number of rounds is 4

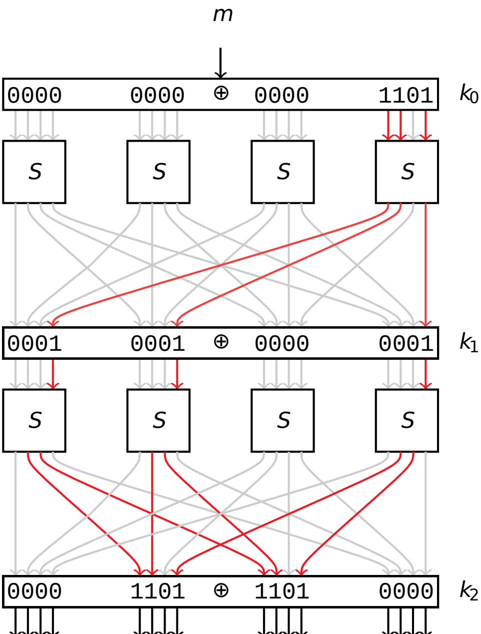
First 2 Rounds



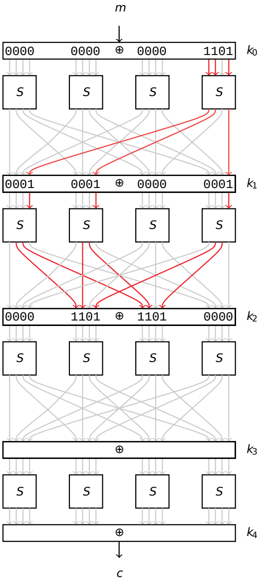
$(0,0,0,d) \xrightarrow{\mathcal{R}} (1,1,0,1)$ $\quad p_1 = \frac{1}{2} - \frac{6}{16}$



$(1,1,0,1) \xrightarrow{\mathcal{R}} (0,d,d,0) \qquad p_2 = \frac{1}{2} + 2^2\left(\frac{4}{16}\right)^3 = \frac{1}{2} + \frac{1}{16}$



2-Round Linear Characteristic



- $(0,0,0,d) \xrightarrow{\mathcal{R}} (1,1,0,1) \xrightarrow{\mathcal{R}} (0,d,d,0)$

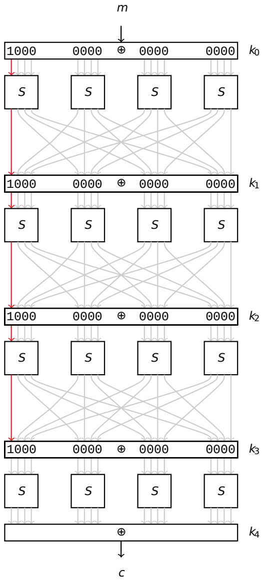
$$p_1 = \frac{1}{2} - \frac{6}{16} = \frac{1}{8}$$

$$p_2 = \frac{1}{2} + 2^2\left(\frac{4}{16}\right)^3 = \frac{1}{2} + \frac{1}{16} = \frac{9}{16}$$

- Prob. of 2-round characteristic:

$$\frac{1}{8} \times \frac{9}{16} + \frac{7}{8} \times \frac{7}{16} = \frac{29}{64} = \frac{1}{2} - \frac{3}{64}$$

- And so on.



- Prob. of one round characteristic:

$$(8,0,0,0) \xrightarrow{\mathcal{R}} (8,0,0,0) : \frac{1}{2} - \frac{4}{16}$$

Iterative Characteristic

Input mask = Output mask

- Using piling-up lemma we have prob. for $(8,0,0,0) \xrightarrow{3\mathcal{R}} (8,0,0,0)$

$$\frac{1}{2} + 2^2\left(\frac{1}{4}\right)^3 = \frac{9}{16} = \frac{1}{2} + \frac{1}{16}$$

- Key recovery as illustrated before