RAMAKRISHNA MISSION VIDYAMANDIRA
(AFFILIATED TO UNIVERSITY OF CALCUTTA)
(BELUR MATH , HOWRAH , 711 202)

B.SC. PROJECT DISSERTATION

Some Properties of Bessel Functions and Their Applications

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April 26, 2025

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Acknowledgement

I express my sincere gratitude to Dr. Subhankar Roy for his invaluable guidance, support, and encouragement throughout this project.

I thank my institution, Ramakrishna Mission Vidyamandira, and my faculty members for providing me with the necessary resources and a conducive learning environment.

Lastly, I am deeply grateful to my family and friends for their constant support and motivation.

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April 26, 2025

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Abstract

This project explores the fundamental properties of Bessel functions and their significant applications in solving mathematical and physical problems. Beginning with a review of Bessel's differential equation and the construction of its solutions, the study proceeds to derive key recurrence relations and generating functions. Various expansions involving Bessel functions are presented, along with trigonometric representations. The orthogonality properties of Bessel functions are discussed in detail, leading to Fourier-Bessel expansions, which are crucial in solving boundary value problems. Finally, the applications of Bessel functions to problems involving Laplace's equation in cylindrical coordinates are demonstrated, such as steady-state heat flow.

Contents

Chapter 1

Preliminaries on Bessel Functions

This chapter provides a brief recap of the basic properties of Bessel Functions and their associated differential equations, laying the foundation for the detailed discussions in the subsequent chapters.

- The differential equation of the form $y'' + \frac{1}{x}y' + (1 - \frac{n^2}{x^2})y = 0$ is known as Bessel's Equation of order n.
- The solution of the above differential equation is indicated by $J_n(x)$ and $J_{-n}(x)$, where $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}$ and $J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{2r-n}$
- $J_n(x)$ is called the Bessel functions of the first kind of order n.
- $J_{-n}(x) = (-1)^n J_n(x)$ when n is an integer, and when n is not an integer, $J_n(x)$ and $J_{-n}(x)$ are independent of each other.
- The two independent solutions of Bessel's equation may be taken to be $J_n(x)$ and $Y_n(x) = \frac{cos\, n\pi\; J_n(x) - J_{-n}(x)}{sin\, n\pi}$, for non-integer values of n.
- When n is an integer, this definition becomes problematic because $sin\, n\pi = 0$. For that case, we take $Y_n(x) = \lim_{v \to n} \frac{cos\,(v\pi)\; J_v(x) - J_{-v}(x)}{sin\,(v\pi)}$
- $Y_n(x)$ is known as Bessel function of order n of the second kind. $Y_n(x)$ is also called the Neumann function of order n and is denoted by $N_n(x)$.
- The Laplace equation in cylindrical coordinates assumes the following form:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0$$

- The solution to the above Laplace equation is

$$u(r, \theta, z) = J_n(\lambda r)(c_1 e^{\lambda z} + c_2 e^{-\lambda z})(c_3\, cos\, n\theta + c_4\, sin\, n\theta)$$

- Recurrence Relations (Formulae) of $J_n(x)$.

  I. $\dfrac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$

  II. $\dfrac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$

  III. $J_n'(x) = J_{n-1}(x) - \dfrac{n}{x} J_n(x)$

  IV. $J_n'(x) = \dfrac{n}{x} J_n(x) - J_{n+1}(x)$

  V. $J_n'(x) = \dfrac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$

  VI. $J_{n-1}(x) - J_{n+1}(x) = \dfrac{2n}{x} J_n(x)$

- More Relation linking Recurrence Relations

  (i) $\left(\dfrac{1}{x}\dfrac{d}{dx}\right)^m (x^n J_n) = x^{n-m} J_{n-m}$ where m is positive integer and $m < n$.

  (ii) $\left(\dfrac{1}{x}\dfrac{d}{dx}\right)^m (x^{-n} J_n) = (-1)^m x^{-n-m} J_{n+m}$.

  (iii) $J_n(x) = (-1)^n x^n \left(\dfrac{1}{x}\dfrac{d}{dx}\right)^m J_0(x)$, n being a positive integer.

  (iv) $\dfrac{d}{dx}\{x\, J_n(x)\, J_{n+1}(x)\} = x\{J_n^2(x) - J_{n+1}^2(x)\}$

  (v) $x^2 J_n''(x) = (n^2 - n - x^2) J_n(x) + x J_{n+1}(x)$, where n = 0, 1, 2, ...

  (vi) $\dfrac{J_{n+1}}{J_n} = \dfrac{(x/2)}{(n+1)-}\,\dfrac{(x/2)^2}{(n+2)-}\,\dfrac{(x/2)^3}{(n+3)-}\cdots\cdots$ or

$$\frac{J_{n+1}}{J_n} = \frac{(x/2)}{(n+1) - \left[\dfrac{(x/2)^2}{(n+2) - \left\{\dfrac{(x/2)^3}{(n+3) - \cdots\cdots}\right\}}\right]}$$

Chapter 2

Integral Methods on Recurrence Relations of Bessel Functions

Relation 1.

$$\int_0^x x^{n+1}\, J_n(x)\, dx = x^{n+1}\, J_{n+1}(x)\ , \quad n > -1$$

$\rightarrow$ We know from previously shown Recurrence Relation that

$$\frac{d}{dx}\left\{x^n\, J_n(x)\right\} = x^n\, J_{n-1}(x)$$

Putting $n+1$ in place of n, we get

$$\frac{d}{dx}\left\{x^{n+1}\, J_{n+1}(x)\right\} = x^{n+1}\, J_n(x)$$

Integrating both sides wrt 'x' between 0 and x, we get

$$\int_0^x x^{n+1}\, J_n(x)\, dx = \left[x^{n+1}\, J_n(x)\right]_0^x = x^{n+1}\, J_{n+1}(x)$$

Relation 2.

$$J_{n+1}(x) = x\int_0^1 J_n(xy)\, y^{n+1}\, dy$$

$\rightarrow$ Let $xy = t$ `so that` $xdy = dt$ (Keeping x as constant)
Changing the above values in the given integration, we get

$$\begin{aligned} x\int_0^1 J_n(t)\left(\frac{t}{x}\right)^{n+1}\frac{dt}{x} &= x^{-n-1}\int_0^1 J_n(t)\, t^{n+1}\, dt \\ &= x^{-n-1}.\left[t^{n+1}\, J_{n+1}(t)\right]_0^x \\ &= x^{-n-1}.x^{n+1}\, J_{n+1}(x) \\ &= J_{n+1}(x) \end{aligned}$$

Relation 3.

$$\int_0^x x^{-n}\, J_{n+1}(x)\, dx = \frac{1}{2^n\, \Gamma(n+1)} - x^{-n}\, J_n \ , \ n > 1$$

$\rightarrow$ Previously from the Recurrence Relation, we have $\frac{d}{dx}[x^{-n}\, J_n(x)] = -x^{-n}\, J_{n+1}(x)$
Integrating both sides wrt x between the limits 0 and x, we get

$$\left[-x^{-n}\, J_n(x)\right]_0^x = \int_0^x x^{-n}\, J_{n+1}(x)\, dx \qquad or \ \lim_{x\to 0} \frac{J_n(x)}{x^n} - x^{-n}\, J_n(x) = \int_0^x x^{-n}\, J_{n+1}(x)\, dx \quad .... (1)$$

$$But \qquad \lim_{x\to 0} \frac{J_n(x)}{x^n} = \frac{1}{x^n}\, \frac{x^n}{2^n\, \Gamma(n+1)} \left[1 - \frac{x^n}{2^n\, (n+1)} + ...\right] = \frac{1}{2^n\, \Gamma(n+1)} \ .... (2)$$

Therefore putting the value of (2) in (1), we get

$$\int_0^x x^{-n}\, J_{n+1}(x)\, dx = \frac{1}{2^n\, \Gamma(n+1)} - x^{-n}\, J_n(x)$$

Relation 4.

$$\int_0^\infty x^{-n}\, J_{n+1}(x)\, dx = \frac{1}{2^n\, \Gamma(n+1)} \ , \ n > -\frac{1}{2}$$

$\rightarrow$ Previously from the Recurrence Relation, we have $\frac{d}{dx}[x^{-n}\, J_n(x)] = -x^{-n}\, J_{n+1}(x)$
Integrating both sides wrt x between the limit 0 to ∞, we get

$$\int_0^\infty x^{-n}\, J_{n+1}(x)\, dx = -\left[x^{-n}\, J_n(x)\right]_0^\infty = \frac{1}{2^n\, \Gamma(n+1)} - \lim_{x\to\infty} \frac{J_n(x)}{x^n} \qquad From\ Previous\ Relation$$

Now for the enormous value of x, we know that $J_n(x)$ approximates the value to

$$J_n(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} cos\left\{x - \left(n + \frac{1}{2}\right)\frac{\pi}{2}\right\} \ , \ n > \frac{1}{2}$$

$$\Rightarrow \quad \lim_{x\to\infty} \frac{J_n(x)}{x^n} = 0$$

$$Thus \quad \int_0^\infty x^{-n}\, J_{n+1}(x)\, dx = \frac{1}{2^n\, \Gamma(n+1)}$$

Relation 5.

$$\int_0^b x\, J_0(ax)\, dx = \frac{b}{a}\, J_1(ab)$$

$\rightarrow$ Previously from the Recurrence Relation, we have

$$\frac{d}{dx}\left\{x^n\, J_n(x)\right\} = x^n\, J_{n-1}(x)$$

Putting $n = 1$, we get

$$\frac{d}{dx}\{x\,J_1(x)\} = x\,J_0(x)$$

Integrating both sides wrt x, we get

$$\int_0^x x\,J_0(x)\,dx = x\,J_1(x) \qquad .....(1)$$

Now, let us assume that ax=t such that adx=dt.
Putting the above value in the given problem, we get

$$\begin{aligned}\int_0^b x\,J_0(ax)\,dx &= \int_0^{ab} \frac{t}{a}\,J_0(t)\,\frac{dt}{a} \\ &= \frac{1}{a^2}\int_0^{ab} t\,J_0(t)\,dt \\ &= \frac{1}{a^2}\,[t\,J_1(t)]_0^{ab} \qquad \text{From (1)} \\ &= \frac{1}{a^2}\,[ab\,J_1(ab) - 0] \\ &= \frac{b}{a}\,J_1(ab)\end{aligned}$$

Relation 6.

$$\int_a^b J_0(x)\,J_1(x)\,dx = \frac{1}{2}\left[J_0^2(a) - J_0^2(b)\right]$$

$\rightarrow$ Previously, from the Recurrence Relation, we have

$$x^{-n}\,J_{n+1}(x) = -\frac{d}{dx}\{x^{-n}\,J_n(x)\}$$

Putting $n = 0$, we have

$$J_1(x) = -\frac{d}{dx}\{J_0(x)\}$$

Now,

$$\begin{aligned}\int_a^b J_0(x)\,J_1(x)\,dx &= -\int_a^b J_0(x)\,J_0'(x)\,dx \\ &= -\left[\frac{J_0^2(x)}{2}\right]_a^b \\ &= \frac{1}{2}\left[J_0^2(a) - J_0^2(b)\right]\end{aligned}$$

Relation 7. Expressing $\int J_3$ in terms of J_1 and J_0.

$\rightarrow$ Previously, from the Recurrence Relation, we have

$$x^{-n}\, J_{n+1} = -\frac{d}{dx}\{x^{-n}\, J_n\}$$

Integrating both sides wrt x, we get

$$\int x^{-n} J_{n+1}\, dx = -x^{-n}\, J_n$$

Now,

$$\begin{aligned}\int J_3 &= \int x^2\, \left(x^{-2}\, J_3\right)\, dx \\ &= x^2\, \left(-x^{-2}\, J_2\right) - \int 2x\, \left(-x^{-2}\, J_2\right)\, dx \\ &= -J_2 + 2\int \left(x^{-1}\, J_2\right)\, dx \\ &= -J_2 + 2\left(-x^{-1}\, J_1\right) + c \quad \text{where c is an arbitrary constant} \\ &\qquad\qquad\qquad\qquad \text{.......(1)}\end{aligned}$$

Again

$$\begin{aligned}&\left(\frac{2n}{x}\right)\, J_n = J_{n-1} + J_{n+1} \\ &\Rightarrow 2\, x^{-1}\, J_1 = J_0\, +\, J_2 \qquad \text{Putting n=1} \\ &\Rightarrow J_2 = 2\, x^{-1}\, J_1\, -\, J_0\end{aligned}$$

Putting the value of J_2 in (1), we get

$$\begin{aligned}\int J_3 &= -\left(2x^{-1}\, J_1\, -\, J_0\right) - 2x^{-1}\, J_1 + c \\ &= -4x^{-1}\, J_1 + J_0 + c \quad \text{where c is an arbitrary constant}\end{aligned}$$

Relation 8.

$$\int J_{n+1}\, dx = \int J_{n-1}\, dx - 2\, J_n$$

$\rightarrow$ From the Recurrence Relation, we have

$$2\, J_n' = J_{n-1}\, -\, J_{n+1}$$

or

$$J_{n+1} = J_{n-1}\, -\, 2\, J_n'$$

Integrating, we get

$$\int J_{n+1}\, dx = \int J_{n-1}\, dx - 2\, J_n$$

Chapter 3

Generating Functions and Trigonometric Expansions Involving Bessel Functions

3.1 Generating Function for the Bessel Function $J_n(x)$

Theorem: Show that $J_n(x)$ is the coefficient of z^n in the expansion of $exp\left\{\frac{1}{2}x\left(z-\frac{1}{z}\right)\right\}$, (or, $e^{\frac{1}{2}x\left(z-\frac{1}{z}\right)}$), i.e.,

$$exp\left\{\frac{1}{2}x\left(z-\frac{1}{z}\right)\right\} = \sum_{n=-\infty}^{\infty} z^n\, J_n(x)$$

Proof: We have

$$exp\left\{\frac{1}{2}x\left(z-\frac{1}{z}\right)\right\} = e^{\frac{xz}{2}-\frac{x}{2z}} = e^{\frac{xz}{2}}.e^{-\frac{x}{2z}}$$

$$= \left[1+\left(\frac{x}{2}\right)z+\left(\frac{x}{2}\right)^2\frac{z^2}{2!}+....+\left(\frac{x}{2}\right)^n\frac{z^n}{n!}+\left(\frac{x}{2}\right)^{n+1}\frac{z^{n+1}}{(n+1)!}+...\right]\times$$

$$\left[1-\left(\frac{x}{2}\right)z^{-1}+\left(\frac{x}{2}\right)^2\frac{z^{-2}}{2!}+....+\left(\frac{x}{2}\right)^n\frac{(-1)^n z^{-n}}{n!}+\left(\frac{x}{2}\right)^{n+1}\frac{(-1)^{n+1}z^{-(n+1)}}{(n+1)!}+...\right] \qquad .....(1)$$

The coefficient of z^n in the product (1) is obtained by multiplying the coefficients of z^n, z^{n+1}, z^{n+2}, ... in the first bracket with the coefficient of z^0, z^{-1}, z^{-2}, ... in the second bracket, respectively.

So, coefficient of z^n in product (1) $= \left(\frac{x}{2}\right)^n\frac{1}{n!}-\left(\frac{x}{2}\right)^{n+2}\frac{1}{(n+1)!}+\left(\frac{x}{2}\right)^{n+4}\frac{1}{(n+2)!\,2!}-....$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$= J_n(x)$$

Similarly, the coefficient of z^{-n} in the product (1) is obtained by multiplying the coefficients of z^{-n}, z^{-n-1}, z^{-n-2}, ... in the first bracket with the coefficient of z^0, z^1, z^2, ... in the second bracket, respectively.

So, coefficient of z^{-n} in (1) is $= \left(\frac{x}{2}\right)^n \frac{(-1)^n}{n!} + \left(\frac{x}{2}\right)^{n+1} \frac{(-1)^{n+1}}{(n+1)!}\frac{x}{2} + \left(\frac{x}{2}\right)^{n+2} \frac{(-1)^{n+2}}{(n+2)!\,2!}\left(\frac{x}{2}\right)^2 + ...$

$$= (-1)^n \left[\left(\frac{x}{2}\right)^n \frac{1}{n!} - \left(\frac{x}{2}\right)^{n+2} \frac{1}{(n+1)!} + \left(\frac{x}{2}\right)^{n+4} \frac{1}{(n+2)!\,2!} - ....\right]$$
$$= (-1)^n J_n(x)$$
$$= J_{-n}(x)$$

Finally, the coefficient of z^0 in the product (1) is obtained by multiplying the coefficients of z^0, z^1, z^2, ... in the first bracket with the coefficient of z^0, z^{-1}, z^{-2}, ... in the second bracket, respectively and thus

$$= 1 - \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^4 \left(\frac{1}{2!}\right)^2 - \left(\frac{x}{2}\right)^6 \left(\frac{1}{3!}\right)^2$$
$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - ...$$
$$= J_0(x)$$

We observe that the coefficients of z^0, z^1, z^{-1}, z^2, z^{-2}, ..., z^n, z^{-n},.... are $J_0(x)$, $J_1(x)$, $J_{-1}(x)$, $J_2(x)$, $J_{-2}(x)$, ..., $J_n(x)$, $J_{-n}(x)$, ... respectively.
Thus, (1) gives us

$$exp\left\{\frac{1}{2}x\left(z - \frac{1}{z}\right)\right\} = \sum_{n=-\infty}^{\infty} z^n\, J_n(x)$$

3.2 Trigonometric Function Involving Bessel Function

Representation: Expand the following Trigonometric Function as shown below

$$\begin{array}{rl}
(i) & cos(x\, sin\,\phi) = J_0 + 2\,J_2\, cos\,2\phi + 2\,J_4\, cos\,4\phi + ... \\
(ii) & sin(x\, sin\,\phi) = 2\,J_1\, sin\,\phi + 2\,J_3\, sin\,3\phi + 2\,J_5\, sin\,5\phi + ... \\
(iii) & cos(x\, cos\,\phi) = J_0 - 2\,J_2\, cos\,2\phi + 2\,J_4\, cos\,4\phi + ... \\
(iv) & sin(x\, cos\,\phi) = 2\,J_1\, cos\,\phi - 2\,J_3\, cos\,3\phi + 2\,J_5\, cos\,5\phi - ...... \\
(v) & cos\ x = J_0 - 2J_2 + 2J_4....... \\
(vi) & sin\ x = 2J_1 - 2J_3 + 2J_5.......
\end{array}$$

Proof: We know that

$$e^{\frac{x}{2}\cdot\left(z-\frac{1}{z}\right)} = J_0 + \left(z - \frac{1}{z}\right)J_1 + \left(z^2 + \frac{1}{z^2}\right)J_2 + \left(z^3 - \frac{1}{z^3}\right)J_3 + ... \qquad ....(1)$$

Let $z = e^{i\phi}$ so that $z^n = e^{in\phi}$ and $z^{-n} = e^{-in\phi}$. Then (1) gives

$$e^{\frac{x}{2}\cdot\left(e^{i\phi}-e^{-i\phi}\right)} = J_0+\left(e^{i\phi}-e^{-i\phi}\right)J_1+\left(e^{2i\phi}+e^{-2i\phi}\right)J_2+\left(e^{3i\phi}-e^{-3i\phi}\right)J_3+.... \qquad ....(2)$$

Since $cos\, n\phi = (e^{ni\phi}+e^{-ni\phi})/2$ and $sin\, n\phi = (e^{ni\phi}-e^{-ni\phi})/2i$

$$e^{xi\, sin\,\phi} = J_0 + (2i\,\cdot\, sin\,\phi)\,J_1 + (2\,\cdot\, cos\,2\phi)\,J_2 + (2i\,\cdot\, sin\,3\phi)\,J_3 + ...$$

$$or,\; cos(x\, sin\,\phi)+isin(x\, sin\,\phi) = (J_0+2\,J_2\, cos\,2\phi+......)+i(2\,J_1\, sin\,\phi+2\,J_3\, sin\,3\phi+......) \qquad ....(3)$$

Part (i). Equating real parts in (3), we get

$$cos(x\, sin\,\phi) = J_0 + 2\,J_2\, cos\,2\phi + 2\,J_4\, cos\,4\phi + ... \qquad .....(4)$$

Part (ii). Equating imaginary parts in (3), we get

$$sin(x\, sin\,\phi) = 2\,J_1\, sin\,\phi + 2\,J_3\, sin\,3\phi + 2\,J_5\, sin\,5\phi + ... \qquad .....(5)$$

Part (iii). Replacing ϕ by $\pi/2-\phi$ in (4) and simplifying, we get

$$cos(x\, cos\,\phi) = J_0 - 2\,J_2\, cos\,2\phi + 2\,J_4\, cos\,4\phi + ... \qquad .....(6)$$

Part (iv). Replacing ϕ by $\pi/2-\phi$ in (5) and simplifying, we get

$$sin(x\, cos\,\phi) = 2\,J_1\, cos\,\phi - 2\,J_3\, cos\,3\phi + 2\,J_5\, cos\,5\phi - ...... \qquad .....(7)$$

Part (v). Replacing ϕ by 0 in (6), we get

$$cos\; x = J_0 - 2J_2 + 2J_4.......$$

Part (vi). Replacing ϕ by 0 in (7), we get

$$sin\; x = 2J_1 - 2J_3 + 2J_5.......$$

3.3 Examples involving Sections 3.1 and 3.2

Ex 1: Derive $x\, sin\, x$ using Bessel's Representation.
Proof:

We know that $$cos(x\; sin\,\phi) = J_0 + 2\,J_2\, cos\,2\phi + 2\,J_4\, cos\,4\phi + ... \qquad ....(1)$$

Differentiating (1) w.r.t. 'ϕ' , $$-sin(x\; sin\,\phi)\cdot x\; cos\,\phi = 0 - 2\cdot 2\,J_2\, sin\,2\phi - 2\cdot 4\,J_4\, sin\,4\phi - ...... \qquad ....(2)$$

Differentiating (2) w.r.t. 'ϕ' , $$-cos(x\; sin\,\phi)\cdot(x\; cos\,\phi)^2 + sin(x\; sin\,\phi)\cdot x\; sin\,\phi$$
$$= -2\cdot 2^2\,J_2\, cos\,2\phi - 2\cdot 4^2\,J_4\, cos\,4\phi - 2\cdot 6^2\,J_6\, cos\,6\phi - ...... \qquad ....(3)$$

Replacing ϕ by $\pi/2$ in (3), we get $$x\, sin\, x = 2(2^2J_2 - 4^2J_4 + 6^2J_6 - ....)$$

Ex 2. Prove the following Bessel's Integrals

$$(i)\ J_n(x) = \frac{1}{\pi}\int_0^{\pi} cos(n\phi - x\, sin\phi)\, d\phi, \text{ where n is a positive integer}$$
$$(ii)\ J_n(x) = \frac{1}{\pi}\int_0^{\pi} cos(n\phi - x\, sin\phi)\, d\phi, \text{ where n is any integer}$$
$$(iii)\ J_0(x) = \frac{1}{\pi}\int_0^{\pi} cos(x\, sin\phi)\, d\phi = \frac{1}{\pi}\int_0^{\pi} cos(x\, cos\phi)\, d\phi$$
$$(iv) \text{ Deduce that } J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^2}{2^2 \cdot 4^2} - .... = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2^r \cdot r!)^2}$$

Proof: Part (i): We shall use the following results

$$\int_0^{\pi} cos\, m\phi\, cos\, n\phi\, d\phi = \int_0^{\pi} sin\, m\phi\, sin\, n\phi\, d\phi = \begin{cases} \pi/2 & \text{when m = n} \\ 0 & \text{when m} \neq \text{n} \end{cases} \qquad ....(1)$$

We also know

$$cos(x\, sin\, \phi) = J_0 + 2\, J_2\, cos\, 2\phi + 2\, J_4\, cos\, 4\phi + ... \qquad ....(2)$$
$$sin(x\, sin\, \phi) = 2\, J_1\, sin\, \phi + 2\, J_3\, sin\, 3\phi + +2\, J_5\, sin\, 5\phi + ... \qquad ....(3)$$

Multiplying both sides of (2) by $cos\ n\phi$ and then integrating w.r.t. 'ϕ' between limits 0 to π and using (1), we have

$$\int_0^{\pi} cos(x\, sin\, \phi)\, cos\, n\phi\, d\phi \quad \begin{aligned} &= 0, \text{ if n is odd} \quad ......(4) \\ &= \pi J_n, \text{ if n is even} \, ...(5) \end{aligned}$$

Multiplying both sides of (3) by $sin\ n\phi$ and then integrating w.r.t. 'ϕ' between limits 0 to π and using (1), we have

$$\int_0^{\pi} sin(x\, sin\, \phi)\, sin\, n\phi\, d\phi \quad \begin{aligned} &= \pi J_n, \text{ if n is odd} \quad ......(6) \\ &= 0, \text{ if n is even} \quad ......(7) \end{aligned}$$

Taking n as odd or even, Adding (4) and (6), or adding (5) and (7) respectively, we get the following

$$\int_0^{\pi} [cos(x\, sin\, \phi)\, cos\, n\phi \ + \ sin(x\, sin\, \phi)\, sin\, n\phi]\, d\phi = \pi J_n$$

or $$\int_0^{\pi} cos(n\phi - x\, sin\phi)\, d\phi = \pi J_n \qquad or \qquad J_n(x) = \frac{1}{\pi}\int_0^{\pi} cos(n\phi - x\, sin\phi)\, d\phi \quad .....(8)$$

Part (ii): Let n be any integer. Then, as in part (i), if n is a positive integer, we have

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} cos(n\phi - x\, sin\phi)\, d\phi$$

Next, let n be a negative integer so that n = - m, where m is a positive integer. To prove the required result for a negative integer, we prove that

$$J_{-m}(x) = \frac{1}{\pi}\int_0^{\pi} cos(-m\phi - x\, sin\phi)\, d\phi \qquad .....(9)$$

Let $\phi = \pi - \theta$ so that $d\phi = -d\theta$. Then, we have
R.H.S. of (9)

$$= \frac{1}{\pi}\int_0^{\pi} cos\{-m(\pi-\theta) - x\, sin(\pi-\theta)\}\,(-d\theta) = \frac{1}{\pi}\int_0^{\pi} cos[(m\theta - x\, sin\,\theta) - m\pi]\, d\theta$$
$$= \frac{1}{\pi}\int_0^{\pi} [cos(m\theta - x\, sin\,\theta)\, cos\, m\pi \;+\; sin(m\theta - x\, sin\,\theta)\, sin\, m\pi]\, d\theta$$
$$= \frac{1}{\pi}\int_0^{\pi} (-1)^m\, cos(m\theta - x\, sin\,\theta)\, d\theta \qquad \text{[Since sin m}\phi = 0 \text{ and cos m}\phi = (-1)^m \text{]}$$
$$= \frac{1}{\pi}(-1)^m \int_0^{\pi} cos(m\theta - x\, sin\,\theta)\, d\theta = (-1)^m J_m(x)$$
$$= J_{-m}(x) = \text{ L.H.S. of (9)}$$

Thus (9) shows that the results hold for a negative integer too.
Therefore, the result is true for all integers.
Part (iii): Integrating (2) w.r.t. 'ϕ' between the limits 0 to π and using the result

$$\int_0^{\pi} cos\, p\phi\, d\phi = 0, \text{ if p is an even integer, we have}$$

$$\int_0^{\pi} cos(x\, sin\,\phi)\, d\phi = J_0 \cdot \int_0^{\pi} d\phi + 0 + 0 + ... = \pi \cdot J_0$$

or, $$J_0 = \frac{1}{\pi}\int_0^{\pi} cos(x\, sin\,\phi)\, d\phi \qquad ......(10)$$

Replacing ϕ by π/2 - ϕ in (2) and running the above process, we get

$$J_0 = \frac{1}{\pi}\int_0^{\pi} cos(x\, cos\phi)\, d\phi \qquad ......(11)$$

Combining (10) and (11), we get

$$J_0 = \frac{1}{\pi}\int_0^{\pi} cos(x\, sin\,\phi)\, d\phi = \frac{1}{\pi}\int_0^{\pi} cos(x\, cos\phi)\, d\phi$$

Part (iv): From (11), we get

$$J_0 = \frac{1}{\pi}\int_0^{\pi} cos(x\, cos\phi)\, d\phi$$
$$= \frac{1}{\pi}\int_0^{\pi} \left(1 - \frac{x^2 cos^2\phi}{2!} + \frac{x^4 cos^4\phi}{4!} - ....\right)\, d\phi$$

Also, we know that
$$\int_0^{\pi} cos^{2n}\phi\, d\phi = \frac{1\cdot 3\cdot 5....(2n-1)}{2\cdot 4\cdot 6....2n}\,\pi$$
This gives
$$\begin{aligned} J_0 &= \frac{1}{\pi}\left[\pi - \frac{x^2}{2!}\cdot\frac{1}{2}\,\pi + \frac{x^4}{4!}\cdot\frac{1\cdot 3}{2\cdot 4}\,\pi - \frac{x^6}{6!}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\pi + ...\right] \\ &= 1 - \frac{x^2}{2^2} + \frac{x^2}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + ... \\ &= \sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2^r\cdot r!)^2} \end{aligned}$$

Ex 3: Use the Generating Function to show that $J_n(-x) = (-1)^n J_n(x)$

Proof: We have
$$\sum_{n=-\infty}^{\infty} J_n(x)\, z^n \;=\; exp\left\{\frac{x}{2}\left(z-\frac{1}{z}\right)\right\} \qquad .....(1)$$

Replacing x by $-$x in (1), we get
$$\sum_{n=-\infty}^{\infty} J_n(-x)\, z^n \;=\; exp\left\{-\frac{x}{2}\left(z-\frac{1}{z}\right)\right\} = exp\left\{\frac{x}{2}\left(-z-\frac{1}{-z}\right)\right\} = \sum_{n=-\infty}^{\infty} J_n(x)\,(-z)^n,\ by\ (1)$$
$$or \qquad \sum_{n=-\infty}^{\infty} J_n(-x)\, z^n \;=\; \sum_{n=-\infty}^{\infty} J_n(x)\,(-1)^n z^n$$
Equating the coefficient of z^n from both sides in the above equation, we have $J_n(-x) = (-1)^n J_n(x)$

Ex 4: Use the Generating function to show that $J_n(x+y) = \displaystyle\sum_{n=-\infty}^{\infty} J_r(x)\cdot J_{n-r}(y)$

Proof: By the generating function $exp\left\{\frac{1}{2}(x+y)\left(z-\frac{1}{z}\right)\right\} = \displaystyle\sum_{n=-\infty}^{\infty} J_n(x+y)\, z^n$ ....(1)

So, we see that $J_n(x+y)$ is the coefficient of z^n in the R.H.S. of (1). Now, we need to obtain the coefficient of z^n in the L.H.S. of (1) in order to get the required result.

$$\begin{aligned} \text{So, L.H.S. of (1) is} &= exp\left\{\frac{1}{2}(x+y)\left(z-\frac{1}{z}\right)\right\} \\ &= exp\left\{\frac{1}{2}x\left(z-\frac{1}{z}\right)\right\}\cdot exp\left\{\frac{1}{2}y\left(z-\frac{1}{z}\right)\right\} \\ &= \sum_{r=-\infty}^{\infty} J_r(x) z^r \cdot \sum_{s=-\infty}^{\infty} J_s(y) z^s \end{aligned}$$

$$= \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} J_r(x) J_s(y) z^{r+s}$$

For a fixed value of r, we get z^n by taking r + s = n, i.e., s = n – r. Thus, keeping r fixed, the coefficient of z^n in (1) is $J_r(x)J_{n-r}(y)$. So the total coefficient of z^n will be given by summing all such terms from n = – ∞ to n = ∞ and is given by

$$\sum_{n=-\infty}^{\infty} J_r(x) \cdot J_{n-r}(y)$$

Hence, comparing both sides of (1), we get $\quad J_n(x+y) = \sum_{n=-\infty}^{\infty} J_r(x) \cdot J_{n-r}(y)$

Ex 5: If $a > 0$, prove that

$$\int_0^{\infty} e^{-ax} J_0(bx)\, dx = \frac{1}{\sqrt{(a^2+b^2)}}$$

Proof: From the examples shown above, we know that

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} cos\,(x\, sin\, \phi)\, d\phi$$

So, we can now say that $\quad J_0(bx) = \frac{1}{\pi} \int_0^{\pi} cos\,(bx\, sin\, \phi)\, d\phi$

Now,

$$\begin{aligned}
\int_0^{\infty} e^{-ax} J_0(bx)\, dx &= \int_0^{\infty} e^{-ax} \left\{ \frac{1}{\pi} \int_0^{\pi} cos\,(bx\, sin\, \phi)\, d\phi \right\} dx \\
&= \frac{1}{\pi} \int_0^{\infty} \left\{ \int_0^{\pi} e^{-ax}\, cos\,(bx\, sin\, \phi)\, d\phi \right\} dx \\
&= \frac{1}{\pi} \int_0^{\pi} \left\{ \int_0^{\infty} e^{-ax}\, cos\,(bx\, sin\, \phi)\, dx \right\} d\phi \\
&= \frac{1}{\pi} \int_0^{\pi} \left\{ \int_0^{\infty} e^{-ax}\, \frac{e^{i\,bx\,sin\,\phi} + e^{-i\,bx\,sin\,\phi}}{2}\, dx \right\} d\phi \\
&= \frac{1}{2\pi} \int_0^{\pi} \left\{ \int_0^{\infty} [e^{-(a-i\,b\,sin\,\phi)x} + e^{-(a+i\,b\,sin\,\phi)x}]\, dx \right\} d\phi \\
&= \frac{1}{2\pi} \int_0^{\pi} \left[\frac{e^{-(a-i\,b\,sin\,\phi)x}}{-(a-i\,b\,sin\,\phi)} + \frac{e^{-(a+i\,b\,sin\,\phi)x}}{-(a+i\,b\,sin\,\phi)} \right]_0^{\infty} d\phi \\
&= \frac{1}{2\pi} \int_0^{\pi} \left[\frac{1}{a-i\,b\,sin\,\phi} + \frac{1}{a+i\,b\,sin\,\phi} \right] d\phi \\
&= \frac{a}{\pi} \int_0^{\pi} \frac{d\phi}{a^2 + b^2 sin^2\phi} = \frac{a}{\pi} \int_0^{\pi} \frac{cosec^2\phi\, d\phi}{a^2 cosec^2\phi + b^2}
\end{aligned}$$

$$= \frac{a}{\pi}\int_0^{\pi} \frac{cosec^2\phi\, d\phi}{a^2(1+cot^2\phi)+b^2} = \frac{2a}{\pi}\int_0^{\pi/2} \frac{cosec^2\phi\, d\phi}{(a^2+b^2)+a^2cot^2\phi}$$

$$= \frac{2a}{\pi}\int_{\infty}^{0} \frac{(-dt)}{(a^2+b^2)+a^2t^2} \qquad \text{Putting cot } \phi = \text{t so that } - cosec^2\phi\, d\phi = \text{dt}$$

$$= \frac{2a}{\pi a^2}\int_0^{\infty} \frac{dt}{t^2+[\sqrt{a^2+b^2}/a]^2} = \frac{2}{\pi a}\cdot\frac{1}{\sqrt{a^2+b^2}/a}\left[tan^{-1}\frac{t}{\sqrt{a^2+b^2}/a}\right]_0^{\infty}$$

$$= \frac{2}{\pi\sqrt{a^2+b^2}}\left(\frac{\pi}{2}-0\right) = \frac{1}{\sqrt{a^2+b^2}}$$

Ex 6: Using the generating function, prove that $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

Proof: Generating function is given by $\sum_{n=-\infty}^{\infty} J_n(x)z^n = e^{(x/2)\times(z-1/z)}$ .....(1)

Differentiating both sides of (1) w.r.t x,

$$\sum_{n=-\infty}^{\infty} J_n'(x)z^n = e^{(x/2)\times(z-1/z)} \times (1/2)\times(z-1/z)$$

$$\text{or,} \quad 2\sum_{n=-\infty}^{\infty} J_n'(x)z^n = (z-z^{-1})\sum_{n=-\infty}^{\infty} J_n(x)z^n$$

$$= \sum_{n=-\infty}^{\infty} J_n(x)(z^{n+1}-z^{n-1}) \quad \text{(using (1))}$$

$$\text{or,} \quad 2\sum_{n=-\infty}^{\infty} J_n'(x)z^n = \sum_{n=-\infty}^{\infty} J_n(x)z^{n+1} - \sum_{n=-\infty}^{\infty} J_n(x)z^{n-1} \qquad .....(2)$$

Equating the coefficients of z^n on both sides of (2) yields

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Ex 7: Show that $\int_0^y \frac{x\, sin\, ax}{(y^2-x^2)^{1/2}}\, dx = \frac{\pi y}{2} J_1(ay)$

Proof: We know that

$$sin(x\, sin\,\phi) = 2\, J_1\, sin\,\phi + 2\, J_3\, sin\, 3\phi + 2\, J_5\, sin\, 5\phi + ... \qquad .....(1)$$

Multiplying both sides of (1) by $\sin\phi$ and then integrating between the limits 0 and π, we have

$$\int_0^{\pi} \sin(x\sin\phi)\sin\phi\, d\phi = J_1\int_0^{\pi}(2\sin^2\phi)\, d\phi + J_2\int_0^{\pi}(2\sin\phi\sin 3\phi)\, d\phi + \cdots$$

$$
\begin{aligned}
&= J_1 \int_0^{\pi} (1 - \cos 2\phi)\, d\phi + J_2 \int_0^{\pi} (\cos 2\phi - \cos 4\phi)\, d\phi + \cdots \\
&= J_1 \left[\phi - \frac{\sin 2\phi}{2} \right]_0^{\pi} + J_2 \left[\frac{\sin 2\phi}{2} - \frac{\sin 4\phi}{4} \right]_0^{\pi} + \cdots \\
&= J_1 \pi, \quad \text{since the remaining terms vanish.}
\end{aligned}
$$

Thus,

$$\pi J_1(x) = \int_0^{\pi} \sin(x \sin \phi) \sin \phi \, d\phi \qquad .....(2)$$

Let $F(\phi) = \sin(x \sin \phi) \sin \phi$. Then, clearly $F(\pi - \phi) = F(\phi)$.
Hence, using a property of definite integrals, (2) yields

$$\pi J_1(x) = 2 \int_0^{\pi/2} \sin(x \sin \phi) \sin \phi \, d\phi$$

or

$$\pi J_1(ay) = 2 \int_0^{\pi/2} \sin(ay \sin \phi) \sin \phi \, d\phi \qquad .....(3)$$

Put $y \sin \phi = x$, so that $y \cos \phi \, d\phi = dx$ and hence

$$
\begin{aligned}
d\phi &= \frac{dx}{y \cos \phi} \\
&= \frac{dx}{y(1 - \sin^2 \phi)^{1/2}} \\
&= \frac{dx}{y(1 - x^2/y^2)^{1/2}} \\
&= \frac{dx}{(y^2 - x^2)^{1/2}}.
\end{aligned}
$$

Hence (3) yields

$$
\begin{aligned}
\pi J_1(ay) &= 2 \int_0^{y} \left(\sin ax \times \frac{x}{y} \times \frac{1}{(y^2 - x^2)^{1/2}} \right) dx \\
\Rightarrow \quad \frac{\pi y}{2} J_1(ay) &= \int_0^{y} \frac{x \sin ax \, dx}{(y^2 - x^2)^{1/2}}.
\end{aligned}
$$

Ex 8: Verify directly from the representation

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} cos(x\, sin\phi)\, d\phi$$

that $J_0(x)$ satisfies the Bessel's equation in which n=0.
Proof: Let

$$y = J_0(x) = \frac{1}{\pi} \int_0^{\pi} cos(x\, sin\phi)\, d\phi \qquad .....(1)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\pi}\int_0^{\pi} sin(x\, sin\phi)\, sin\phi\, d\phi \qquad .....(2)$$

$$and \qquad \frac{d^2y}{dx^2} = -\frac{1}{\pi}\int_0^{\pi} cos(x\, sin\phi)\, sin^2\phi\, d\phi \qquad .....(3)$$

Evaluating the R.H.S. of (2) using integration by parts, we have the following

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{1}{\pi}\left[\{-sin(x\, sin\phi)\, cos\phi\}_0^{\pi} + \int_0^{\pi} cos(x\, sin\phi)\, x\ \cdot cos^2\phi\, d\phi\right] \\
&= \frac{1}{\pi}\,(sin(x\ \cdot sin\,\pi)cos\,\pi - sin(x\ \cdot sin\,0)cos\,0) - \frac{x}{\pi}\int_0^{\pi} cos(x\, sin\phi)\, x\ \cdot cos^2\phi\, d\phi \\
&= -\frac{x}{\pi}\int_0^{\pi} cos(x\, sin\phi)\, x\ \cdot (1 - sin^2\,\phi)\, d\phi \\
&= -\frac{x}{\pi}\int_0^{\pi} cos(x\, sin\phi)\, d\phi + \frac{x}{\pi}\int_0^{\pi} cos(x\, sin\phi)\, sin^2\phi\, d\phi \\
&= -xy - x\frac{d^2y}{dx^2} \qquad \text{[from (1) and (3)]}
\end{aligned}$$

$$or, \quad \frac{d^2y}{dx^2} + \frac{1}{x}\,\frac{dy}{dx} + y = 0 \qquad \text{which is Bessel's equation for n = 0}$$

Hence $J_0(x) = \frac{1}{\pi}\int_0^{\pi} cos(x\, sin\phi)\, d\phi$ satisfies the Bessel's equation of the zeroth order.

Chapter 4

Orthogonality of Bessel Function and its Application

4.1 Orthogonality of Bessel Function

Theorem: If λ_i and λ_j are the roots of $J_n(\lambda a) = 0$, then

$$\int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx = \begin{cases} 0, & \text{if i} \neq \text{j} \\ \frac{a^2}{2}\, J_{n+1}^2(\lambda_i a), & \text{if i = j} \end{cases}$$

Proof: **Case 1.** Let i $\neq$ j, i.e. let λ_i and λ_j are the unequal roots of $J_n(\lambda a) = 0$

$$\begin{array}{rcl} J_n(\lambda_i a) = 0 & and & J_n(\lambda_j a) = 0 \\ Let,\, u(x) = J_n(\lambda_i x) & and & v(x) = J_n(\lambda_j x) \qquad .....(1) \end{array}$$

Then u and v are the Bessel functions satisfying the Bessel Function

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0$$

Therefore, putting u and v in the above equation, we can say that

$$\begin{array}{rl} & x^2u'' + xu' + (\lambda_i^2x^2 - n^2)u = 0 \qquad .....(2) \\ and & x^2v'' + xv' + (\lambda_j^2x^2 - n^2)v = 0 \qquad .....(3) \end{array}$$

Multiplying (2) by v and (3) by u and then subtracting both, we get

$$\begin{array}{rl} & x^2(vu'' - uv'') + x(vu' - uv') + x^2(\lambda_i^2 - \lambda_j^2)uv = 0 \\ or, & x(vu'' - uv'') + (vu' - uv') = x(\lambda_j^2 - \lambda_i^2)uv \\ or, & x\dfrac{d}{dx}(vu' - uv') + (vu' - uv') = x(\lambda_j^2 - \lambda_i^2)uv \\ or, & \dfrac{d}{dx}\left\{x(vu' - uv')\right\} = x(\lambda_j^2 - \lambda_i^2)uv \qquad .....(4) \end{array}$$

Integrating (4) w.r.t. x within the limit 0 to a, we get

$$(\lambda_j^2 - \lambda_i^2) \int_0^a xuv\, dx = [x(vu' - uv')]_0^a \qquad .....(5)$$

Using (1), (5) reduces to

$$\begin{aligned}(\lambda_j^2 - \lambda_i^2) \int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx &= [x\{J_n(\lambda_j x)\, J_n'(\lambda_i x) \,-\, J_n(\lambda_i x)\, J_n'(\lambda_j x)\,\}]_0^a \\ &= a\{J_n(\lambda_j a)\, J_n'(\lambda_i a) \,-\, J_n(\lambda_i a)\, J_n'(\lambda_j a)\,\} \\ &= 0 \qquad \text{[Since } \lambda_i \text{ and } \lambda_j \text{ are the roots of } J_n(\lambda a) = 0]\end{aligned}$$

Therefore, for $\lambda_i \neq \lambda_j$, the above equation gives

$$\int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx = 0, \qquad \text{when i} \neq \text{j}$$

Case 2. Let i = j (equal roots). Multiplying (2) by $2u'$, we have

$$\begin{aligned} & 2x^2u''u' + 2xu'^2 + 2(\lambda_i^2 x^2 - n^2)uu' = 0 \\ or, \quad & \frac{d}{dx}\{x^2u'^2 - n^2u^2 + \lambda_i^2 x^2 u^2\} - 2\lambda_i^2 xu^2 = 0 \\ or, \quad & \frac{d}{dx}\{x^2u'^2 - n^2u^2 + \lambda_i^2 x^2 u^2\} = 2\lambda_i^2 xu^2 \qquad .....(6)\end{aligned}$$

Integrating (6) w.r.t. 'x' from 0 to a,

$$2\lambda_i^2 \int_0^a xu^2\, dx = \left[x^2u'^2 - n^2u^2 + \lambda_i^2 x^2 u^2\right]_0^a \qquad .....(7)$$

Using (1), (7) reduces to

$$\begin{aligned}2\lambda_i^2 \int_0^a xJ_n^2(\lambda_i x)\, dx &= \left[x^2\{J_n'(\lambda_i x)\}^2 - n^2 J_n^2(\lambda_i x) + \lambda_i^2 x^2 J_n^2(\lambda_i x)\right]_0^a \\ &= a^2 \left[\{J_n'(\lambda_i x)\}^2\right]_{x=a} \qquad \text{[Since } \lambda_i \text{ is the root of } J_n(\lambda a) = 0 \text{ and } J_n(0) = 0] \\ & \qquad .....(8)\end{aligned}$$

From Recurrence Relations, we know

$$\frac{d}{dx} J_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

Replacing x by $\lambda_i x$, we get

$$\begin{aligned} & \frac{d}{d(\lambda_i x)} J_n(\lambda_i x) = \frac{n}{(\lambda_i x)} J_n(\lambda_i x) - J_{n+1}(\lambda_i x) \\ or, \quad & \frac{1}{\lambda_i}\frac{d}{dx} J_n(\lambda_i x) = \frac{n}{\lambda_i x} J_n(\lambda_i x) - J_{n+1}(\lambda_i x) \\ or, \quad & J_n'(\lambda_i x) = \frac{n}{x} J_n(\lambda_i x) - \lambda_i J_{n+1}(\lambda_i x)\end{aligned}$$

$$or, \qquad \left[\{J_n'(\lambda_i x)\}^2\right]_{x=a} = \left[\left\{\frac{n}{x}J_n(\lambda_i x) - \lambda_i J_{n+1}(\lambda_i x)\right\}^2\right]_{x=a} = \{0 - \lambda_i J_{n+1}(\lambda_i a)\}^2$$
$$= \lambda_i^2 J_{n+1}^2(\lambda_i a)$$

Using this value in (8) and dividing both the resulting equations by $2\lambda_i^2$, we get

$$\int_0^a xJ_n^2(\lambda_i x)\,dx = \frac{a^2}{2}J_{n+1}^2(\lambda_i a)$$

Combining both cases, we get

$$\int_0^a x\,J_n(\lambda_i x)\,J_n(\lambda_j x)\,dx = \begin{cases} 0, & \text{if } i \neq j \\ \frac{a^2}{2}\,J_{n+1}^2(\lambda_i a), & \text{if } i = j \end{cases}$$

4.2 Fourier-Bessel Expansion for $f(x)$

Statement: If f(x) is defined in the region $0 \le x \le a$ and has an expansion of the form

$$f(x) = \sum_{i=1}^{n} c_i\,J_n(\lambda_i x)$$

where the λ_i are the roots of the equation $J_n(\lambda a) = 0$, then

$$c_i = \frac{2\int_0^a x\,f(x)\,J_n(\lambda_i x)dx}{a^2 J_{n+1}^2(\lambda_i a)}$$

Proof: We have,

$$f(x) = \sum_{i=1}^{n} c_i\,J_n(\lambda_i x) \qquad .....(1)$$

Multiplying both sides of (1) by $xJ_n(\lambda_j x)$, we get

$$xf(x)J_n(\lambda_j x) = \sum_{i=1}^{n} c_i\,x\,J_n(\lambda_i x)\,J_n(\lambda_j x) \qquad .....(2)$$

Integrating both sides of (2) w.r.t. 'x' from 0 to a, we get

$$\int_0^a xf(x)J_n(\lambda_j x) = \sum_{i=1}^{n} c_i \int_0^a x\,J_n(\lambda_i x)\,J_n(\lambda_j x) \qquad .....(3)$$

From the orthogonality property of Bessel functions, we have

$$\int_0^a x\,J_n(\lambda_i x)\,J_n(\lambda_j x)\,dx = \begin{cases} 0, & \text{if } i \neq j \\ \frac{a^2}{2}\,J_{n+1}^2(\lambda_j a), & \text{if } i = j \end{cases} \qquad .....(4)$$

Using (4), (3) reduces to

$$\int_0^a xf(x)J_n(\lambda_j x) = c_j\frac{a^2}{2}\,J_{n+1}^2(\lambda_j a) \qquad .....(5)$$

Replacing j by i in (5), we get

$$\int_0^a xf(x)J_n(\lambda_i x) = c_i\frac{a^2}{2}\,J_{n+1}^2(\lambda_i a) \qquad or, \qquad c_i = \frac{2\int_0^a x\,f(x)\,J_n(\lambda_i x)dx}{a^2 J_{n+1}^2(\lambda_i a)}$$

4.3 Problems Related to Fourier-Bessel Expansion

Ex 1. Expand the function f(x) = 1, $0 \le x \le a$ in a series of the form $\sum_{i=1}^{\infty} c_i\, J_0(\lambda_i x)$, where λ_i are the roots of the equation $J_0(\lambda a) = 0$

Sol. Given f(x) = 1 = $\sum_{i=1}^{\infty} c_i\, J_0(\lambda_i x)$ .....(1)

where $J_0(\lambda a) = 0$.

Then we know that

$$c_i = \frac{2\int_0^a x\, f(x)\, J_0(\lambda_i x) dx}{a^2 J_1^2(\lambda_i a)} = \frac{2\int_0^a x\, J_0(\lambda_i x) dx}{a^2 J_1^2(\lambda_i a)}, \text{ as f(x) = 1} \qquad ....(2)$$

Let $\lambda_i x = t$ so that $dx = dt/\lambda_i$. Then, we have

$$\begin{aligned}\int_0^a x\, J_0(\lambda_i x)\, dx &= \frac{1}{\lambda_i^2}\int_0^{a\lambda_i} t\, J_0(t)\, dt \\ &= \frac{1}{\lambda_i^2}\left[t\, J_1(t)\right]_0^{a\lambda_i} = \frac{1}{\lambda_i^2}\left[a\lambda_i\, J_1(a\lambda_i) - 0\right], \text{ as } J_1(0) = 0 \\ &= \frac{a}{\lambda_i} J_1(a\lambda_i) \qquad ......(3)\end{aligned}$$

Using (3), (2) becomes $$c_i = \frac{2 \times (a/\lambda_i) \times J_1(a\lambda_i)}{a^2 J_1^2(a\lambda_i)} = \frac{2}{a\lambda_i J_1(a\lambda_i)} \qquad .....(4)$$

Using (4), (1) becomes $$1 = \frac{2}{a}\sum_{i=1}^{\infty} \frac{J_0(\lambda_i x)}{\lambda_i J_1(\lambda_i a)},\ 0 \le x \le a$$

Ex 2. Expand x in a series of the form $\sum_{i=1}^{\infty} c_i\, J_1(\lambda_i x)$ valid in the region $0 \le x \le 1$, where λ_i are the roots of the equation $J_1(\lambda) = 0$

Sol. Given f(x) = x = $\sum_{i=1}^{\infty} c_i\, J_1(\lambda_i x)$ .....(1)

where $J_1(\lambda) = 0$.

Then we know that (with a = 1)

$$c_i = \frac{2\int_0^1 x\, f(x)\, J_1(\lambda_i x) dx}{J_2^2(\lambda_i)} = \frac{2\int_0^1 x^2\, J_1(\lambda_i x) dx}{J_2^2(\lambda_i)}, \text{ as f(x) = x} \qquad ....(2)$$

Let $\lambda_i x = t$ so that $dx = dt/\lambda_i$. Then, we have

$$\begin{aligned}\int_0^1 x^2\, J_1(\lambda_i x)\, dx &= \frac{1}{\lambda_i^3}\int_0^{\lambda_i} t^2\, J_1(t)\, dt \\ &= \frac{1}{\lambda_i^3}\left[t^2\, J_2(t)\right]_0^{\lambda_i} = \frac{1}{\lambda_i^3}\left[\lambda_i^2\, J_2(\lambda_i) - 0\right], \text{ as } J_2(0) = 0 \\ &= \frac{1}{\lambda_i} J_2(\lambda_i) \qquad ......(3)\end{aligned}$$

Using (3), (2) becomes $$c_i = \frac{2}{\lambda_i J_2(\lambda_i)} \qquad .....(4)$$

Using (4), (1) becomes $$x = 2\sum_{i=1}^{\infty} \frac{J_1(\lambda_i x)}{\lambda_i J_2(\lambda_i)},\ 0 \leq x \leq 1$$

Chapter 5

Applications of Bessel Functions to Laplace's Equation in Cylindrical Coordinates

Problem 1. A homogeneous thermally conducting cylinder occupies the region $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq h$, where r, θ, z are cylindrical coordinates. The top z = h and the lateral surface r = a are held at $0°$, while the base z = 0 is held at $100°$. Assuming there are no sources of heat generation within the cylinder, determine the steady-state temperature distribution within the cylinder.

Solution. The temperature u must be a single-valued continuous function. The steady state temperature satisfies the Laplace equation inside the cylinder. To compute the temperature distribution inside the cylinder, we have to solve the following Boundary Value Problem:

$$\begin{aligned} PDE: \quad & \nabla^2 u = 0 \\ BCs: \quad & u = 0 \quad on \;\; z = h \\ & u = 0 \quad on \;\; r = a \\ & u = 100 \quad on \;\; z = 0 \end{aligned}$$

The general solution of the Laplace equation in cylindrical coordinates is

$$u(r, \theta, z) = J_n(\lambda r)(c_1 e^{\lambda z} + c_2 e^{-\lambda z})(c_3 \, cos \, n\theta + c_4 \, sin \, n\theta)$$

Since the face z = 0 is maintained at $100°$ and since the other face and lateral surface of the cylinder are maintained at $0°$, the temperature at any point inside the cylinder is independent of θ. This is possible only when n = 0 in the general solution. Thus,

$$u(r, z) = J_0(\lambda r)(Ae^{\lambda z} + Be^{-\lambda z})$$

Using the BC: $u = 0 \quad on \;\; z = h$, we get

$$0 = J_0(\lambda r)(Ae^{\lambda h} + Be^{-\lambda h})$$

implying thereby $Ae^{\lambda h} + Be^{-\lambda h} = 0$, from which $B = \frac{Ae^{\lambda h}}{e^{-\lambda h}}$.
Therefore, the solution is

$$u(r, z) = \frac{J_0(\lambda r)\, A}{e^{-\lambda h}}(e^{\lambda(z-h)} + e^{-\lambda(z-h)})$$

or

$$u(r, z) = J_0(\lambda r) A_1 sinh\, \lambda(z-h)$$

where $A_1 = 2A/e^{-\lambda h}$. Now using the BC: $u = 0 \quad on \ \ r = a$, we have

$$0 = A_1 J_0(\lambda a) sinh\, \lambda(z-h)$$

implying $J_0(\lambda a) = 0$, which has infinitely many positive roots. Denoting them by ξ_n, we have $\xi_n = \lambda a$, and therefore, $\lambda = \frac{\xi_n}{a}$.
Thus, the solution is

$$u(r, z) = A_1 J_0\left(\frac{\xi_n r}{a}\right) sinh\left[\frac{\xi_n}{a}(z-h)\right], \qquad\qquad n = 1, 2, ...$$

Using the principle of superposition, we have

$$u(r, z) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\xi_n r}{a}\right) sinh\left[\frac{\xi_n}{a}(z-h)\right]$$

The BC: $u = 100 \quad on \ \ z = 0$ gives

$$100 = \sum_{n=1}^{\infty} A_n sinh\left(-\frac{\xi_n h}{a}\right) J_0\left(\frac{\xi_n r}{a}\right)$$

which is a Fourier-Bessel series. Multiplying both sides with $rJ_0(\xi_m r/a)$ and integrating, we get

$$100\int_0^a rJ_0\left(\frac{\xi_m r}{a}\right) dr = \sum_{n=1}^{\infty} A_n sinh\left(-\frac{\xi_n h}{a}\right)\int_0^a rJ_0\left(\frac{\xi_n r}{a}\right) J_0\left(\frac{\xi_m r}{a}\right) dr$$

Using the orthogonality property of Bessel function, we have

$$100\int_0^a rJ_0\left(\frac{\xi_m r}{a}\right) dr = A_n sinh\left(-\frac{\xi_n h}{a}\right)\frac{a^2}{2}J_1^2(\xi_n)$$

Therefore,

$$\begin{aligned} A_n &= \frac{200}{a^2 sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)}\int_0^a rJ_0\left(\frac{\xi_m r}{a}\right) dr \\ &= \frac{200}{{\xi_n}^2 sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)}\int_0^{\xi_n} xJ_0(x)\, dx \quad \left[\text{Taking } \frac{\xi_m r}{a} = x, \quad dr = \frac{a}{\xi_n}dx\right] \end{aligned}$$

$$= \frac{200}{{\xi_n}^2 sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} [xJ_1(x)]_0^{\xi_n} \qquad \text{[Using the Recurrence Relation]}$$

$$= \frac{200}{\xi_n sinh\left(-\frac{\xi_n h}{a}\right) J_1(\xi_n)}$$

Hence, the required temperature distribution inside the cylinder is

$$u(r,z) = 200 \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\xi_n r}{a}\right) sinh\left[\frac{\xi_n}{a}(z-h)\right]}{\xi_n sinh\left(-\frac{\xi_n h}{a}\right) J_1(\xi_n)}$$

where ξ_n are the positive zeros of $J_0(\xi)$.

Problem 2. Find the potential u inside the cylinder $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq h$. If the potential on the top, z = h, and the lateral surface, r = a, are held at 0, while the base, z = 0, the potential is given by $u(r,\theta,z) = V_0(1 - r^2/a^2)$, where r, θ, z are cylindrical coordinates.

Solution. The potential u must be a single-valued continuous function and satisfy the Laplace equation inside the cylinder. To compute the potential inside the cylinder, we have to solve the following Boundary Value Problem:

$$\begin{aligned} PDE: &\quad \nabla^2 u = 0 \\ BCs: &\quad u = 0 \quad on \ \ z = h \\ &\quad u = 0 \quad on \ \ r = a \\ &\quad u = V_0\left(1 - \frac{r^2}{a^2}\right) \quad on \ \ z = 0 \end{aligned}$$

The general solution of the Laplace equation in cylindrical coordinates is

$$u(r,\theta,z) = J_n(\lambda r)(c_1 e^{\lambda z} + c_2 e^{-\lambda z})(c_3 \cos n\theta + c_4 \sin n\theta)$$

Since the face z = 0 has potential $V_0(1 - r^2/a^2)$, which is purely a fraction of r and is independent of θ and since the other face and lateral surface of the cylinder are at zero potential, the potential at any point inside the cylinder is independent of θ. This is possible only when n = 0 in the general solution. Thus,

$$u(r,z) = J_0(\lambda r)(Ae^{\lambda z} + Be^{-\lambda z})$$

Using the BC: $u = 0 \quad on \ \ z = h$, we get

$$0 = J_0(\lambda r)(Ae^{\lambda h} + Be^{-\lambda h})$$

implying thereby $Ae^{\lambda h} + Be^{-\lambda h} = 0$, from which $B = \frac{Ae^{\lambda h}}{e^{-\lambda h}}$.
Therefore, the solution is

$$u(r,z) = \frac{J_0(\lambda r)\, A}{e^{-\lambda h}}(e^{\lambda(z-h)} + e^{-\lambda(z-h)})$$

or

$$u(r,z) = A_1 J_0(\lambda r) sinh\, \lambda(z-h)$$

where $A_1 = 2A/e^{-\lambda h}$. Now using the BC: $u = 0 \quad on \;\; r = a$, we have

$$0 = A_1 J_0(\lambda a) sinh\, \lambda(z-h)$$

implying $J_0(\lambda a) = 0$, which has infinitely many positive roots. Denoting them by ξ_n, we have $\xi_n = \lambda a$, and therefore, $\lambda = \frac{\xi_n}{a}$.
Thus, the solution is

$$u(r,z) = A_1 J_0\left(\frac{\xi_n r}{a}\right) sinh\left[\frac{\xi_n}{a}(z-h)\right], \qquad\qquad n = 1, 2, ...$$

Using the principle of superposition, we have

$$u(r,z) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\xi_n r}{a}\right) sinh\left[\frac{\xi_n}{a}(z-h)\right]$$

The last BC: $u = V_0\left(1 - \frac{r^2}{a^2}\right) \quad on \;\; z = 0$ gives

$$V_0\left(1 - \frac{r^2}{a^2}\right) = \sum_{n=1}^{\infty} A_n sinh\left(-\frac{\xi_n h}{a}\right) J_0\left(\frac{\xi_n r}{a}\right)$$

which is a Fourier-Bessel series. Multiplying both sides with $rJ_0(\xi_m r/a)$ and integrating, we get

$$V_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right) rJ_0\left(\frac{\xi_m r}{a}\right) dr = \sum_{n=1}^{\infty} A_n sinh\left(-\frac{\xi_n h}{a}\right) \int_0^a rJ_0\left(\frac{\xi_n r}{a}\right) J_0\left(\frac{\xi_m r}{a}\right) dr$$

Using the orthogonality property of Bessel function, we have

$$V_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right) rJ_0\left(\frac{\xi_m r}{a}\right) dr = A_n sinh\left(-\frac{\xi_n h}{a}\right) \frac{a^2}{2} J_1^2(\xi_n)$$

Therefore,

$$A_n = \frac{2V_0}{a^2 sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^a \left(1 - \frac{r^2}{a^2}\right) rJ_0\left(\frac{\xi_m r}{a}\right) dr$$

$$= \frac{2V_0}{{\xi_n}^4 sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^{\xi_n} (\xi_n^2 - x^2) x J_0(x)\, dx \quad \left[\text{Taking } \frac{\xi_m r}{a} = x, \quad dr = \frac{a}{\xi_n} dx\right]$$

$$= \frac{2V_0}{{\xi_n}^4 sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^{\xi_n} (\xi_n^2 - x^2)\, d[x J_1(x)] \quad [\text{Using the Recurrence Relation}]$$

$$= \frac{4V_0}{{\xi_n}^4 sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} \left[x^2 J_2(x)\right]_0^{\xi_n} \quad [\text{Using Integration by Parts}]$$

$$= \frac{4V_0 J_2(\xi_n)}{{\xi_n}^2 sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)}$$

Now, using the recurrence relation $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$ for n = 1 and x = ξ_n, we get

$$J_0(\xi_n) + J_2(\xi_n) = \frac{2}{\xi_n} J_1(\xi_n)$$

$$or, \qquad J_2(\xi_n) = \frac{2}{\xi_n} J_1(\xi_n) \qquad [\text{Since } J_0(\xi_n) = 0]$$

Therefore

$$A_n = \frac{8V_0 J_1(\xi_n)}{{\xi_n}^3 sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} = \frac{8V_0}{{\xi_n}^3 sinh\left(-\frac{\xi_n h}{a}\right) J_1(\xi_n)}$$

Hence, the required potential inside the cylinder is

$$u(r,z) = \sum_{n=1}^{\infty} \frac{8V_0 J_0\left(\frac{\xi_n r}{a}\right) sinh\left[\frac{\xi_n}{a}(z-h)\right]}{{\xi_n}^3 sinh\left(-\frac{\xi_n h}{a}\right) J_1(\xi_n)}$$

where ξ_n are the positive zeros of $J_0(\xi)$.

Conclusions and Future Directions

Conclusion

In this project, we comprehensively studied the properties and applications of Bessel functions. Starting with their recurrence relations and generating functions, we examined their essential role in expanding trigonometric functions and solving integrals. We also demonstrated the orthogonality property of Bessel functions, which enables Fourier-Bessel expansions for representing functions over bounded domains. These expansions were then applied to solve boundary value problems involving the Laplace equation in cylindrical coordinates, emphasizing the applicability of Bessel functions in real-world physics problems like steady-state heat conduction and potential distribution. Future work can involve exploring the behaviour of Bessel functions of the second kind, modified Bessel functions, and their roles in more complex partial differential equations such as wave equations or diffusion processes in non-cylindrical geometries.

Further Directions

The study of Bessel functions provides a strong foundation for solving boundary value problems, particularly in cylindrical coordinate systems. Building upon this work, several further directions can be pursued:

1. **Study of Bessel Functions of the Second Kind:**
   While this project focused on Bessel functions of the first kind, exploring Bessel functions of the second kind (Neumann functions) could provide a more complete understanding, especially for problems where solutions exhibit singular behaviour at the origin.

2. **Modified Bessel Functions and Their Applications:**
   Modified Bessel functions arise naturally in problems involving heat conduction in cylindrical coordinates with time dependence, such as in transient heat flow and diffusion processes. Extending the study to include $I_n(x)$ and $K_n(x)$ functions would broaden the application range.

3. **Numerical Computation of Bessel Functions:**
   Analytical solutions are often difficult or impossible for complex geometries or boundary conditions. Learning numerical methods to compute Bessel functions and their zeros can support simulations in engineering and physics.

4. **Applications to Wave and Quantum Mechanics Problems:**
   Bessel functions play a crucial role in wave propagation problems (such as electromagnetic waves in circular waveguides) and quantum mechanical systems with radial symmetry (spherical Bessel functions). Exploring these topics would highlight the deeper significance of Bessel functions.

5. **Fourier-Bessel Series and Signal Processing:**
   Fourier-Bessel series expansions have applications in signal processing and image reconstruction. Investigating these modern applications would connect classical mathematics to cutting-edge technology.

6. **Higher-Dimensional Problems and Spherical Coordinates:**
   Moving beyond cylindrical coordinates to spherical coordinates involves spherical Bessel functions. Studying problems in three-dimensional settings, such as acoustic waves in spherical enclosures, would be a natural and interesting extension.

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