Notes, Solutions etc.

for George E. Martin's "Counting - The Art of Enumerative Combinatorics"

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Contents

1. Elementary Enumeration

Combination Formula

The number of ways to choose r objects from n distinct objects is given by:

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Also written as ${}_nC_r$ or $\binom{n}{r}$.

Quickies - I

1. Addition principle: $6 + 8 = 14$ ways.
2. Same principle, although different fruits are indistinguishable in their own class: $1 + 1 = 2$ ways.
3. 3 ways.
4. 2 B's, 2 G's, or 1 B and 1 G: 3 ways.
5. 6 students total (3 boys + 3 girls) and we choose 2: $C(6,2) = 15$ ways.
6. 1 way, since any orange we do not pick is indistinguishable from any other orange that we did not pick in a different scenario.
7. $C(6,5) = 6$ ways.
8. $C(6,1) = 6$ ways.
9. We need to pick exactly 5 fruits. Let's consider picking i oranges and $(5-i)$ apples where $0 \leq i \leq 5$:

- 0 oranges, 5 apples
- 1 orange, 4 apples
- 2 oranges, 3 apples
- 3 oranges, 2 apples
- 4 oranges, 1 apple
- 5 oranges, 0 apples

Total: 6 ways.

10. Counting the different ways to pick each fruit:

- For oranges: 0 to 9 (10 choices)
- For apples: 0 to 6 (7 choices)

Therefore the total choice combinations are $10 \times 7 = 70$ ways. But we have to substract the one case where we pick 0 of both fruits, so we have $70 - 1 = 69$ ways.

Permutation Formula

The number of ways to arrange r objects from n distinct objects (order matters) is given by:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Also written as ${}_nC_r$ or $A(n,r)$.

Relationship between Permutation and Combination:

Since permutations consider order while combinations do not, we have:

$$P(n,r) = r! \times C(n,r)$$

This is because for each combination of r objects, there are $r!$ ways to arrange them.

Quickies - II

1. Multiplication principle: We pick 1 Latin book from 5 and 1 Greek book from 7: $\mathbf{5 \times 7 = 35}$ ways.
2. Each letter can be any of the 26 letters: $\mathbf{26^2}$ ways.
3. Since we can't repeat letters, we have 26 choices for the first letter and 25 for the second: $\mathbf{26 \times 25 = 650}$ ways.
4. $\mathbf{21 \times 5 = 105}$ ways.
5. $\mathbf{3 \times 8 = 24}$ ways.
6. $\boldsymbol{P(5,2) = 20}$ ways. (We permute here since the arrangement matters)
7. $\boldsymbol{C(5,2) = 10}$ ways.
8. $\mathbf{26^4}$ ways.
9. Pick any row (5 choices) and any column (7 choices): $\mathbf{5 \times 7 = 35}$ ways.
10. $\boldsymbol{m \times n}$ ways.
11. Coin has 2 outcomes, die has 6 outcomes: $\mathbf{2 \times 6 = 12}$ ways.
12. $\mathbf{2 \times 6 \times 52 = 624}$ ways.
13. $\mathbf{4!}$ ways (since each ace is distinct).
14. $\mathbf{13!}$ ways.

A Discussion Question

Question: How many ways can a pair of dice fall?

Solution for this depends on how the question means, or how we interpret, "ways":

Distinguishable Dice: If we can tell the dice apart (e.g., one red die and one blue die), then each die can show any of 6 faces independently. Using the multiplication principle: $\mathbf{6 \times 6 = 36}$ ways.

This counts (1,2) and (2,1) as different outcomes since the first number represents the red die and the second represents the blue die.

Indistinguishable Dice (Unordered Pairs): If the dice are identical and we only care about which numbers appear, then we're counting unordered pairs. The possible outcomes are: (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,5), (5,6), (6,6)

This gives us $\boldsymbol{\binom{6+2-1}{2} = \binom{7}{2} = 21}$ ways (stars and bars approach).

Possible Sums: If we only care about the sum of the dice, there are 11 possible sums: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

Pigeonhole Principle

The pigeonhole principle is a fundamental counting principle:

> If n objects are placed into m containers where $n > m$, then at least one container must contain more than one object.
>
> This can be generalized to: if n objects are placed into m containers, then at least one container contains at least $\lceil \frac{n}{m} \rceil$ objects.

Pigeonhole Problems

1. First, draw a white sock. Then, draw a black sock. The next sock drawn will have to match one of these, thus giving us a pair. Therefore, we need **3** socks.

2. Same as last question, we have 4 different suits. We can draw one each from all of them, but the 5th drawn card will have to match one of the suits. Thus, we need **5** cards.

3. Given 365 possible birthdays, we can have two people for each birthdate ($\mathbf{365 \times 2 = 730}$), and one more person, who has to be born on one of these 365 days, resulting in 3 people having the same birthday. Therefore, we need **731** people.

4. We have 4 colors: 12 red, 20 white, 7 blue, 8 green balls. To avoid having 10 balls of the same color, we can take at most 9 from each color that has at least 9 balls. We can take all 7 blue balls, all 8 green balls, 9 red balls, and 9 white balls, giving us $\mathbf{7 + 8 + 9 + 9 = 33}$ balls. The 34th ball must be either red or white, giving us our 10th ball of that color. Therefore, we need **34** balls.

5. We can pick 1 person from each couple, gicing us n people. The next person picked has to be the un-picked half of any couple, giving us $n + 1$ people to ensure atleast one couple is picked.

6. Each person can have between 0 and 19 mutual friends (since there are 19 other people in the room). However, if one person has 0 mutual friends and another has 19 mutual friends, this creates a contradiction: if someone has 19 mutual friends, then everyone else is their friend, so no one can have 0 mutual friends. Therefore, there are only 19 possible values for the number of mutual friends. With 20 people and 19 possible values, by the pigeonhole principle, at least 2 people must have the same number of mutual friends. QED.

7. Consider any 5 lattice points. We can partition all lattice points into 4 classes based on the parity of their coordinates:
   - Class 1: (even, even)
   - Class 2: (even, odd)
   - Class 3: (odd, even)
   - Class 4: (odd, odd)

   Two points are in the same class if and only if their x-coordinates have the same parity and their y-coordinates have the same parity. Since we have 5 points and only 4 classes, by the pigeonhole principle, at least 2 of the 5 points must be in the same class.

   If two points (x_1, y_1) and (x_2, y_2) are in the same class, then their midpoint $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ has integer coordinates, since both $x_1 + x_2$ and $y_1 + y_2$ are even. This midpoint lies on the segment connecting the two points, so we have found a lattice point on one of our 10 segments. QED.

8. Label the 6 people as vertices of a complete graph K_6. Color each edge red if the corresponding people know each other, blue if they are strangers.

   Pick any vertex v. It has 5 edges connecting to other vertices. By the pigeonhole principle, at least $\lceil \frac{5}{2} \rceil = 3$ edges have the same color.

   Without loss of generality, assume at least 3 edges from v are red, connecting v to vertices a, b, and c. Now consider the triangle formed by a, b, and c:
   - If any edge of this triangle is red, then we have a red triangle (3 mutual acquaintances)
   - If all edges of this triangle are blue, then we have a blue triangle (3 mutual strangers)

   In either case, we have found the required set of 3. QED.

Ramsey Theory

Ramsey theory studies the conditions under which order must appear in large enough structures. The fundamental question is: how large must a structure be to guarantee that it contains a particular substructure?

> **Ramsey Number $R(m, n)$:** The smallest number N such that if we color the edges of the complete graph K_N with two colors (red and blue), then either there exists a red clique of size m or a blue clique of size n.

Problem 8 from the previous section demonstrates that $R(3, 3) = 6$. This means that in any group of 6 people, we can always find either 3 mutual acquaintances or 3 mutual strangers, and 6 is the smallest number for which this is guaranteed.

Arrangements with Repetition

When arranging objects where some are identical, we must account for the fact that swapping identical objects doesn't create a new arrangement:

> If we have n total objects consisting of n_1 identical objects of type 1, n_2 identical objects of type 2, ..., n_k identical objects of type k, then the number of distinct arrangements is:
>
> $$\frac{n!}{n_1! \times n_2! \times \ldots \times n_k!}$$
>
> where $n_1 + n_2 + \ldots + n_k = n$.

Reasoning: Start with $n!$ total arrangements, as if all objects were distinct. However, since the n_1 objects of type 1 can be arranged among themselves in $n_1!$ ways without creating new distinct arrangements, we divide by $n_1!$. Apply the same logic for each type of identical object.

This formula naturally reduces to $n!$ when all objects are distinct (each $n_i = 1$) and to 1 when all objects are identical ($n_1 = n$, all other $n_i = 0$).

n Choose r by Way of MISSISSIPPI

1. All 6 letters are distinct, so we have $6! = 720$ ways.
2. All 6 letters are distinct (subscripts make them different), so we have $6! = 720$ ways.
3. 3 distinct A's, 2 identical E's, and 1 F. $\frac{6!}{2!} = 360$ ways.

4. We have 3 identical A's, 2 distinct E's (due to subscripts), and 1 J. The number of arrangements is $\frac{6!}{3!} = \mathbf{120}$ ways.
5. 3 identical A's, 2 identical E's, and 1 F. $\frac{6!}{3!\times 2!} = \mathbf{60}$ ways.
6. 1 B, 3 A's, and 2 N's. $\frac{6!}{1!\times 3!\times 2!} = \mathbf{60}$ ways.
7. 3 A's, 2 B's, 4 C's, and 1 D. $\frac{10!}{3!\times 2!\times 4!\times 1!} = \mathbf{12600}$ ways.
8. $\frac{11!}{2!\times 2!\times 2!}$ ways.
9. $\frac{11!}{1!\times 4!\times 4!\times 2!}$ ways.
10. 4 A's, 3 G's, and 6 distinct letters (total 13 objects). The number of arrangements is $\frac{13!}{4!\times 3!}$.
11. $\frac{13!}{4!\times 3!}$ ways.
12. First arrange the 3 subjects: $\mathbf{3!}$ ways. Then arrange books within each subject: $\mathbf{4! \times 3! \times 6!}$ ways. Total arrangements: $\mathbf{3! \times 4! \times 3! \times 6! = 622080}$.
13. We need to arrange n letters where r are C's and $(n-r)$ are R's. The number of arrangements is $\frac{n!}{r!(n-r)!} = \binom{n}{r}$.
14. Selecting r persons from n persons is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.
15. Selecting r distinguishable objects from n distinguishable objects is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ (This is the same as the last question, even though the objects are distinguishable. This is because the order of selection does not matter).

Circular Arrangements

When arranging objects in a circle, we must account for the fact that rotations of the same arrangement are considered identical:

The number of ways to arrange n distinct objects in a circle is $(n-1)!$.

Reasoning: Consider any linear arrangement of n people. When we place them in a circle, this single arrangement can be rotated n different ways around the circle, but all these rotations represent the same circular seating arrangement. Since there are $n!$ linear arrangements, and each circular arrangement corresponds to n linear arrangements, we have $\frac{n!}{n} = (n-1)!$ distinct circular arrangements.

Equivalently, we can fix one person's position (to eliminate rotational symmetry) and arrange the remaining $(n-1)$ people in the remaining positions, giving us $(n-1)!$ arrangements.

The Round Table

1. Using the circular arrangement formula: $(8-1)! = 7! = \mathbf{5040}$ ways.
2. $(12-1)! = 11!$ ways.
3. Treat each couple as a single unit. We have 8 units to arrange in a row: $\mathbf{8!}$ ways. Within each couple, the 2 persons can be arranged in $\mathbf{2!}$ ways. Total: $2^8 \times 8!$ ways.
4. The couples can be arranged in $(8-1)! = 7!$ ways, and within each couple there are $2!$ arrangements. Total: $2^8 \times 7!$ ways.
5. We have $4 + 7 + 10 = 21$ people total. Using the circular arrangement formula: $(21-1)! = 20!$ ways.

6. First, arrange the 8 R's in a circle: $\mathbf{(8-1)! = 7!}$ ways. This creates 8 gaps between consecutive R's where we can place the C's. To ensure no 2 C's are adjacent, we must choose 4 of these 8 gaps for our C's: $\binom{8}{4}$ ways. Total: $\mathbf{7! \times \binom{8}{4}}$ ways.

Homework

1. $\binom{11}{5}$ ways.
2. $\binom{52}{5}$ ways.
3. $\binom{52}{13}$ ways.
4. A full house requires three-of-a-kind and a pair:
   - Choose rank for three-of-a-kind: $\mathbf{13}$ ways
   - Choose 3 cards from 4 of that rank: $\binom{4}{3}$ ways
   - Choose different rank for pair: $\mathbf{12}$ ways
   - Choose 2 cards from 4 of that rank: $\binom{4}{2}$ ways

   Total: $\mathbf{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}$ ways.
5. $\mathbf{2^{10} - 1}$ ways.
6. $\frac{\mathbf{13!}}{\mathbf{4! \times 4! \times 4! \times 1!}}$ ways.
7. Total ways minus same-subject pairs: $\binom{16}{2} - \binom{5}{2} - \binom{7}{2} - \binom{4}{2}$ ways.
8. Total combinations excluding choosing none: $\mathbf{6 \times 8 - 1 = 47}$ ways.
9. Arrange 21 consonants first: $\mathbf{21!}$ ways. This creates 22 gaps for the 5 vowels. Choose 5 gaps: $\binom{22}{5}$ ways. Arrange vowels in chosen positions: $\mathbf{5!}$ ways.

   Total: $\mathbf{21! \times \binom{22}{5} \times 5!}$ ways.
10. First letter has 26 choices, each subsequent letter has 25 choices (cannot repeat previous): $\mathbf{26 \times 25^9}$ ways.
11. Use complement: total 10-element subsets minus those with no consecutive letters. For no consecutive letters, we choose 10 positions from an effective alphabet of size $\mathbf{26 - 9 = 17}$:

    $\binom{26}{10} - \binom{17}{10}$ ways.
12. $\mathbf{7! \times \binom{8}{5} \times 5!}$ ways.
13. $\mathbf{(7-1)! \times \binom{7}{5} \times 5!}$ ways.