Notes, Solutions etc.

for George E. Martin's "Counting - The Art of Enumerative Combinatorics"

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Contents

1. Elementary Enumeration

Combination Formula

The number of ways to choose $\boldsymbol{r}$ objects from $\boldsymbol{n}$ distinct objects is given by:

$$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Also written as ${}_nC_r$ or $\binom{n}{r}$.

Quickies - I

1. Using the addition principle: $\mathbf{6 + 8 = 14}$ ways.

2. Same logic as problem 1: $\mathbf{6 + 8 = 14}$ ways.

3. We can pick any one of the three types of letters: A, B, or C. Therefore, $\mathbf{3}$ ways.

4. We need to consider all possible combinations:
- 2 B's and 0 G's: $C(3, 2) = 3$ ways
- 1 B and 1 G: $C(3, 1) \times C(3, 1) = 3 \times 3 = 9$ ways
- 0 B's and 2 G's: $C(3, 2) = 3$ ways

Total: $\mathbf{3 + 9 + 3 = 15}$ ways.

5. We have 6 students total (3 boys + 3 girls) and need to pick 2: $C(6, 2) = 15$ ways.

6. $C(6, 5) = 6$ ways.

7. $C(6, 5) = 6$ ways.

8. $C(6, 1) = 6$ ways.

9. We need to pick exactly 5 fruits. Let's consider picking i oranges and $(5 - i)$ apples where $0 \leq i \leq 5$:

- 0 oranges, 5 apples: $C(7, 0) \times C(8, 5) = 1 \times 56 = 56$
- 1 orange, 4 apples: $C(7, 1) \times C(8, 4) = 7 \times 70 = 490$
- 2 oranges, 3 apples: $C(7, 2) \times C(8, 3) = 21 \times 56 = 1176$
- 3 oranges, 2 apples: $C(7, 3) \times C(8, 2) = 35 \times 28 = 980$
- 4 oranges, 1 apple: $C(7, 4) \times C(8, 1) = 35 \times 8 = 280$
- 5 oranges, 0 apples: $C(7, 5) \times C(8, 0) = 21 \times 1 = 21$

Total: $\mathbf{56 + 490 + 1176 + 980 + 280 + 21 = 3003}$ ways.

10. Total ways to pick any subset from the fruits = $2^9 \times 2^6 = 2^{15} = 32768$ ways.

Ways to pick nothing = $\mathbf{1}$ way.

Ways to pick at least 1 piece = $\mathbf{32768 - 1 = 32767}$ ways.

Permutation Formula

The number of ways to arrange $\boldsymbol{r}$ objects from $\boldsymbol{n}$ distinct objects (order matters) is given by:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Also written as ${}_nC_r$ or $A(n,r)$.

Relationship between Permutation and Combination:

Since permutations consider order while combinations do not, we have:

$$P(n,r) = r! \times C(n,r)$$

This is because for each combination of r objects, there are $r!$ ways to arrange them.

Quickies - II

1. Using the multiplication principle: We pick 1 Latin book from 5 and 1 Greek book from 7: $\mathbf{5 \times 7 = 35}$ ways.

2. Each position in the 2-letter word can be any of the 26 letters: $\mathbf{26 \times 26 = 676}$ ways.

3. First letter has 26 choices, second letter has 25 remaining choices (must be different): $\mathbf{26 \times 25 = 650}$ ways.

4. We have 21 consonants and 5 vowels. Consonant first, then vowel: $\mathbf{21 \times 5 = 105}$ ways.

5. Using the multiplication principle (similar to problem 1): $\mathbf{3 \times 8 = 24}$ ways.

6. This is an arrangement problem. We need to arrange 2 people in 5 chairs: $P(5,2) = \frac{5!}{3!} = 5 \times 4 = 20$ ways.

7. This is a combination problem (order doesn't matter for chairs): $C(5,2) = 10$ ways.

8. Similar to problem 2: $\mathbf{26^4 = 456776}$ ways.

9. We can pick any row (5 choices) and any column (7 choices): $\mathbf{5 \times 7 = 35}$ ways.

10. Similar to problem 9: $m \times n$ ways.

11. Coin has 2 outcomes, die has 6 outcomes: $\mathbf{2 \times 6 = 12}$ ways.

12. Similar to problem 11, but with an additional card choice: $\mathbf{2 \times 6 \times 52 = 624}$ ways.

13. We need to arrange 4 aces in a row: $\mathbf{4! = 24}$ ways.

14. Similar to problem 13, but with 13 spades: $\mathbf{13!}$ ways.

A Discussion Question

Question: How many ways can a pair of dice fall?

Solution for this depends on how the question means, or how we interpret, "ways":

Distinguishable Dice: If we can tell the dice apart (e.g., one red die and one blue die), then each die can show any of 6 faces independently. Using the multiplication principle: $\mathbf{6 \times 6 = 36}$ ways.

This counts (1,2) and (2,1) as different outcomes since the first number represents the red die and the second represents the blue die.

Indistinguishable Dice (Unordered Pairs): If the dice are identical and we only care about which numbers appear, then we're counting unordered pairs. The possible outcomes are: (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,5), (5,6), (6,6)

This gives us $\boldsymbol{\binom{6+2-1}{2} = \binom{7}{2} = 21}$ ways (stars and bars approach).

Possible Sums: If we only care about the sum of the dice, there are 11 possible sums: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

The “correct” answer depends entirely on what the question is really asking!