6.840 Recitation 10

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1 Strongly Connected Graph

STRONGLY-CONNECTED $\in$ NL-Complete

We define the language STRONGLY-CONNECTED = $\{\langle G \rangle | G$ is a directed graph and there exists a path from every node in G to every other node in $G\}$

We first show that strongly connected is in NL.

TM M_1 = "On input $\langle G \rangle$:
- Check that $\langle G \rangle$ is a valid encoding of a directed graph, else reject.
- Loop through all pairs of nodes (u, v) that are in G and run the nondeterministic procedure used in the PATH language NTM decider to determine if there exists a path starting from u and ending at v. On the branches that have not found a path and reject, M_1 rejects. If some branch finds a path, instead of accepting it will continue with the next pair of nodes.
- If we have looped through all the pairs of nodes and have not rejected (meaning we have found paths between all of them), then ACCEPT."

Now, we show that PATH $\leq_L$ STRONGLY-CONNECTED. Reminder: PATH = $\{\langle G, s, t \rangle | G$ is a directed graph and there exists a path from s to $t\}$.

Given the graph G from the PATH language input, we create a new graph G' that is identical to G with some additional edges:
- Add an edge from node t to every node in the graph.
- Add an edge from every node in the graph to node s.

Now, if there exists a path from s to t in G, G' is strongly connected, as we can get from any node u in G' to any other node v in G' by taking the edge from u to s, then taking the path from s to t and finally taking the edge from t to v. Conversely, if G' is strongly connected, then there exists a path from s

to t in G, because the edges we added to construct G' do not help us on our journey from s to t, they only help us in getting from t or anywhere else back to s. Hence, the reduction works. It uses log-space because when outputting the new graph G', we only have to make sure to add the extra edges in G''s description which is very straightforward.

2 $\overline{2 - SAT}$

We show that PATH $\leq_L \overline{2 - SAT}$.

For every node u in the graph G from PATH, we assign a variable x_u in the boolean formula ϕ we are constructing. For every edge between u and v in G, we add the clause $(x_u \rightarrow x_v) \equiv (\overline{x_u} \bigvee x_v)$. We also make sure that x_s is true by adding the clause $(x_s \bigvee x_s)$ and we add the clause $(x_t \rightarrow \overline{x_s}) \equiv (\overline{x_t} \bigvee \overline{x_s})$.

Now, if we have a path from s to t, x_s is true and because every node connected to s is true, x_t ends up being true. But we also have the clause $(x_t \rightarrow \overline{x_s})$, implying that because x_t is true, then x_s must be false. This cannot happen, so ϕ is unsatisfiable. Conversely, if ϕ is unsatisfiable, this means that there must be a path from s to t in G. If there was no such path, then we could set x_s along with all the variables corresponding to nodes reachable from s as true and all other nodes including t false and would have a satisfying assignment.