18.100B Final Exam Solutions

1. (20 points) Let

$$E = \{0, 1, 1/2, 1/3, 1/4, \dots\}$$

be the set of real numbers consisting of zero and the inverses of the positive integers. Prove that E is compact directly from the definition (without using the Heine-Borel theorem).

Let $\{U_\alpha | \alpha \in A\}$ be an open cover of E. One of these sets, say U_{α_0}, must contain 0. Because U_{α_0} is open, it contains the ball of some positive radius r around 0. Choose n so large that $1/n \leq r$; then all of the points $1/(n+1), 1/(n+2), 1/(n+3), \dots$ are contained in U_{α_0}. For each of the remaining n points $1/j$ $(j = 1, 2, \dots, n)$, choose an open set U_{α_j} containing $1/j$. Then E is covered by the $n+1$ sets $U_{\alpha_0}, U_{\alpha_1}, \dots, U_{\alpha_n}$.

2. (20 points) Let $X = \mathbb{R}$. Define a real-valued function on $\mathbb{R} \times \mathbb{R}$ by

$$d(x, y) = (x - y)^2.$$

Is d a metric on $\mathbb{R}$? (Prove your answer).

This is not a metric. It does have the first two properties (distances are non-negative, and distances between distinct points are strictly positive), but it fails to satisfy the triangle inequality. To see this, take $x = -1$, $y = 0$, $z = 1$; then $d(x, y) = d(y, z) = 1$, but $d(x, z) = 4$, which is not less than or equal to $d(x, y) + d(y, z)$.

3. (20 points) Recall that the length of a vector $v \in \mathbb{R}^n$ can be defined in terms of the inner product by

$$\| v \| = (v \cdot v)^{1/2}.$$

The Cauchy-Schwartz inequality says that if x and y are vectors in $\mathbb{R}^n$, then $|x \cdot y| \leq \| x \| \| y \|$. Using this fact, prove the triangle inequality

$$\| u + v \| \leq \| u \| + \| v \| .$$

Both sides of the inequality we want to prove are non-negative, so it is enough to prove the inequality for their squares:

$$\| u + v \|^2 \leq \| u \|^2 + \| v \|^2 + 2 \| u \| \| v \| .$$

The left side of this proposed inequality is

$$(U + v) \cdot (u + v) = u \cdot u + v \cdot v + 2u \cdot v.$$

The right side is

$$u \cdot u + v \cdot v + 2 \| u \| \| v \| .$$

So the inequality we want is equivalent to

$$u \cdot v \leq \| u \| \| v \|,$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX which is the Cauchy-Schwartz inequality.

4. (20 points) Give an example of a metric space X and an open subset of X having exactly three points.

Take X to be the subspace of $\mathbb{R}$ consisting of the three points 0, 1, and 2. Then X is open in X, and has exactly three points.

5. (20 points) Suppose X is a metric space and

$$f: X \to X$$

is a function from X to X. We say that f is a *weak contraction* if

$$d(f(x_1), f(x_2)) \leq d(x_1, x_2) \qquad (x_1, x_2 \in X).$$

Prove that any weak contraction is continuous.

Given $x \in X$ and $\epsilon > 0$, we must find $\delta > 0$ so that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$. We can choose $\delta = \epsilon$.

6. (20 points) Suppose that $f: X \to X$ is any continuous function, and that $x_0 \in X$. Define a sequence $x_1, x_2, x_3 \ldots$ of points in X by

$$x_{n+1} = f(x_n) \qquad (n \geq 0).$$

Prove that *if* the sequence $\{x_n\}$ converges to a limit point $x \in X$, then $f(x) = x$.

If F is continuous and z_n converges to z, then $F(z_n)$ converges to $F(z)$. Applying this general fact in this case, we deduce that the sequence $f(x_n)$ converges to $f(x)$. But $f(x_n) = x_{n+1}$, so x_{n+1} converges to $f(x)$. But the sequence x_{n+1} obviously has the same limit as x_n, namely x; so $x = f(x)$.

7. (40 points) We say that $f: X \to X$ is a *strong contraction* if there is a number $r < 1$ such that

$$d(f(x_1), f(x_2)) \leq r d(x_1, x_2) \qquad (x_1, x_2 \in X).$$

Suppose f is a strong contraction, and $x_0 \in X$. Define a sequence $x_1, x_2, x_3 \ldots$ of points in X by

$$x_{n+1} = f(x_n) \qquad (n \geq 0).$$

Finally, define $A = d(x_0, f(x_0))$.

a) Prove that $d(x_0, x_n) \leq A(1 - r^n)/(1 - r) \leq A/(1 - r)$.

Since f reduces distances by at least a factor of r, we see that

$$d(f(x_0), f(f(x_0))) \leq rA, \quad d(f(f(x_0)), f(f(f(x_0)))) \leq r^2 A,$$

and in general

$$d(x_n, x_{n+1}) \leq r^n A.$$

By the triangle inequality,

$$d(x_0, x_n) \leq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \leq A(1 + r + r^2 \cdots + r^{n-1}).$$

Summing the geometric series gives the formula we want.

b) Prove that $d(x_m, x_{m+n}) \leq Ar^m/(1-r)$.

Apply f m times to the pair of points (x_0, x_n); this gives (x_m, x_{m+n}). Each application shrinks the distance by a factor of at least r.

c) Prove that $\{x_n\}$ is a Cauchy sequence.

Given $\epsilon > 0$, choose N so large that $Ar^N/(1-r) < \epsilon$; this is possible since r^N tends to 0. If $p, q \geq N$, then (if say $p \leq q$)

$$d(x_p, x_q) \leq Ar^p/(1-r) \leq Ar^N/(1-r) < \epsilon,$$

as we wished to show.

d) Suppose that X is a complete metric space. Prove that there is a point $x \in X$ such that $f(x) = x$. (Such an x is called a *fixed point*.)

Take any x_0, and define a sequence x_n as above. It's Cauchy by (c), and therefore converges by the definition of complete. By problem 6, the limit point x satisfies $f(x) = x$.

8. (20 points) Prove that if x and y are any real numbers, then

$$|\sin x - \sin y| \leq |x - y|.$$

(You may use standard facts about trigonometric functions and their derivatives.)

The derivative of sin is cos. By the mean value theorem, there is a point z between x and y so that

$$\sin x - \sin y = (x - y)\cos z.$$

Take absolute values, and use $|\cos| \leq 1$.

9. (20 points) If f is a Riemann-integrable function on $[0, 2\pi]$, recall that the mth Fourier coefficient of f is by definition

$$c_m(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-imx}dx \qquad (m \in \mathbb{Z}).$$

Suppose that $f(x) = \exp(\exp(ix))$; here as usual $\exp z$ means e^z. Prove that there is a constant A so that

$$|c_m(f)| \leq A/m^2 \qquad (m \neq 0).$$

(Hint: it's possible to compute $c_m(f)$ explicitly, and to solve the problem in that way. But it's easier to use an abstract argument given in class.)

The argument given in class was to integrate by parts in the definition of c_m, using $u = f$ and $dv = e^{-imx}$ (so that $v = (-1/im)e^{-imx}$. (What's needed to make this work is that f is periodic of period 2π, so that the terms at the ends of the interval cancel, and that the derivative is Riemann integrable.) Thus

$$c_m(f) = \frac{-1}{2\pi}\int_0^{2\pi} f'(x)(-1/im)e^{-imx}\,dx.$$

Repeating this gives

$$c_m(f) = \frac{1}{2\pi}\int_0^{2\pi} f''(x)(-1/m^2)e^{-imx}\,dx.$$

Taking absolute values everywhere gives

$$|c_m(f)| \leq (1/m^2)\frac{1}{2\pi}\int_0^{2\pi} |f''(x)|\,dx.$$

This is what we want, with A equal to the average value of $|f''|$.

In fact the negative Fourier coefficients are all zero, and for the non-negative ones

$$c_m = 1/m!.$$

(Can you see this without calculating anything?)