18.404 Recitation 9

Nov 6, 2020

Today's Topics

- Review: Savitch's Theorem
- Review: PSPACE reduction
- Review: TQBF is PSPACE-Complete
- Prove: GG $\in$ PSPACE
- Prove: NL $\subseteq$ SPACE($\log^2$(n))
- Review: NL $\subseteq$ P
- BIPARTITE $\in$ coNL

Review: Savitch's Theorem

Conclusion: PSPACE = NPSPACE

Proof: Convert NPSPACE NTM to PSPACE TM by only using square more space

Idea:

- Use $\text{LADDER}_{\text{DFA}} \in$ PSPACE construction where "words" in the LADDER are actually computation histories.
- Test for LADDER from start to accept configuration

Review: Savitch's Theorem (cont.)

Proof: Deterministic TM M simulating NTM N which uses f(n) space

M = "On input <c_i,c_j,b>

1. If b = 1, accept if $c_i \rightarrow c_j$ is a valid state transition for NTM N
2. For b > 1, repeat for all configs c_{mid} that use f(n) space
   a. Recursively test $c_i \rightarrow_{b/2} c_{mid}$, $c_{mid} \rightarrow_{b/2} c_j$
   b. Accept if both accept
3. Reject if all fail"

Test if N accepts w by testing <c_{start},c_{end},t> where t = # configs = $|Q| \cdot f(n) \cdot |\Sigma|^{f(n)}$

Review: Savitch's Theorem (Space Analysis)

Recall: M = "On input <c_i,c_j,b>

1. If b = 1, accept if $c_i \rightarrow c_j$ is a valid state transition for NTM N
2. For b > 1, repeat for all configs c_{mid} that use f(n) space
   a. Recursively test $c_i \rightarrow_{b/2} c_{mid}$, $c_{mid} \rightarrow_{b/2} c_j$
   b. Accept if both accept
3. Reject if all fail"

Each recursion level stores 1 config = O(f(n)) space.

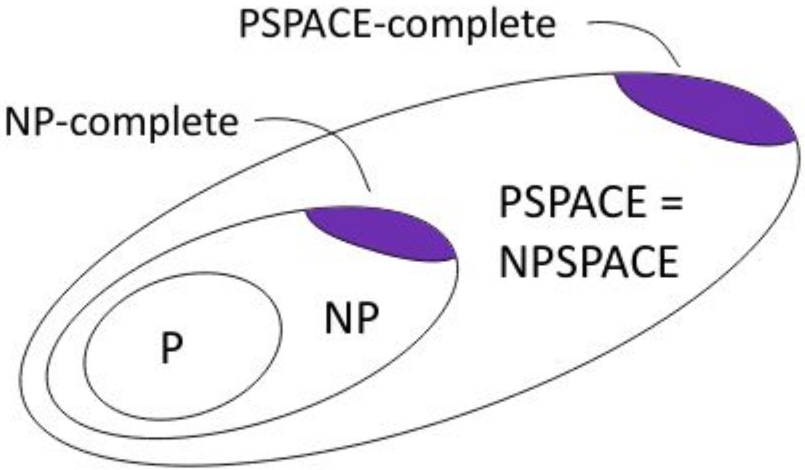
Number of levels = log(t) = O(f(n))

Total: $O(f^2(n))$ space

Review: PSPACE Reduction

Definition: B is PSPACE-Complete if

1. B ∈ PSPACE
2. For all A ∈ PSPACE, $A \leq_p B$



Note: Reduction uses *p* for polynomial **time** reduction machine

Review: TQBF is PSPACE-Complete

Goal: Compute if M on w *accepts* where:

- M is a TM which runs in PSPACE
- Reduction in TQBF uses polynomial time (aka $\leq_p$)

Note: Many of the initial attempts presented in class were not reducible in polynomial time. Usually there was an exponential blow up somewhere

Review: TQBF is PSPACE-Complete (cont.)

Use Cook-Levin transition boolean formula: $\varphi_{ci,cj,1}$ is well-defined

Reduction: (Recursively defined)

$\varphi_{Ci,Cj,b} = \exists c_{mid} [\, \forall (c_g, c_h) \in \{ (c_i, c_{mid}), (c_{mid}, c_j) \} [\, \varphi_{Cg,Ch,b/2} \,]\,]$

The $\varphi_{Cg,Ch,b/2}$ eventually recurses down to $\varphi_{Ci,Cj,1}$ which is well-defined

So: $\varphi_{M,w} = \varphi_{Cstart,Caccept,t}$ where $t = d^{f(n)}$ (# configs before repeat)

Size Analysis: Each recursive level adds $O(f(n))$. #levels = $\log(d^{f(n)}) = O(f(n))$

Size is $O(f^2(n))$

Review: TQBF is PSPACE-Complete (Space Analysis)

Recall:

$$\varphi_{Ci,Cj,b} = \exists c_{mid} \left[\forall (c_g, c_h) \in \{ (c_i, c_{mid}), (c_{mid}, c_j) \} \left[\varphi_{Cg,Ch,b/2} \right] \right]$$

Size Analysis:

- Each recursive level adds constant number of configs to QBF: $O(f(n))$
- #levels = $\log(d^{f(n)}) = O(f(n))$

So: Size is $O(f^2(n))$

Prove: GG ∈ PSPACE

Idea: Develop polynomial-space recursive algorithm determining which player has a winning strategy

Proof: M = “On input <G, n_{start}>

1. If n_{start} has no outgoing edges, *reject* since no available move (signalling loss)
2. Remember list of nodes $[n_1, \ldots, n_i]$ reachable from n_{start} through a single edge
3. Remove n_{start} and all edges connecting it to form graph G’
4. For every $n_j \in [n_1, \ldots, n_i]$, call M(G’, n_j) (signalling the moves of the opponent)
5. If any call return *accept*, means that opponent can always win, so we lose and therefore *reject*. Otherwise *accept* since we have a winning path.

Prove: NL $\subseteq$ SPACE($\log^2(n)$)

Same proof using Savitch's Theorem

Proof: M = "On input <c_i,c_j,b>

1. If b = 1, accept if $c_i \rightarrow c_j$ is a valid state transition for NTM N
2. For b > 1, repeat for all configs c_{mid} that use **log(n)** space
    a. Recursively test $c_i \rightarrow_{b/2} c_{mid}$, $c_{mid} \rightarrow_{b/2} c_j$
    b. Accept if both accept
3. Reject if all fail"

Prove: NL $\subseteq$ SPACE($\log^2(n)$) (Space Analysis.)

Recall: M = “On input <c_i,c_j,b>

1. If b = 1, accept if $c_i \rightarrow c_j$ is a valid state transition for NTM N
2. For b > 1, repeat for all configs c_{mid} that use **log(n)** space
    a. Recursively test $c_i \rightarrow_{b/2} c_{mid}$, $c_{mid} \rightarrow_{b/2} c_j$
    b. Accept if both accept
3. Reject if all fail”

Each recursion level stores 1 config = O(log(n)) space.

Number of levels = log(t) = O(log(n))

Total: $O(\log^2(n))$ space

Review: NL $\subseteq$ P

Define a configuration graph $G_{M,w}$ for M on w which has:

- nodes for all configurations M on w
- edges for all valid transitions $c_i \rightarrow c_j$

Utilize TM T deciding PATH $\in$ P

Run T on $\langle G_{M,w}, c_{start}, c_{accept} \rangle$

Review: NL $\subseteq$ P (Time Analysis)

Recall: Graph $G_{M,w}$ has:

- nodes for all configurations M on w
- edges for all valid transitions $c_i \rightarrow c_j$

Theorem: Constructing $G_{M,w}$ can be done in polynomial time

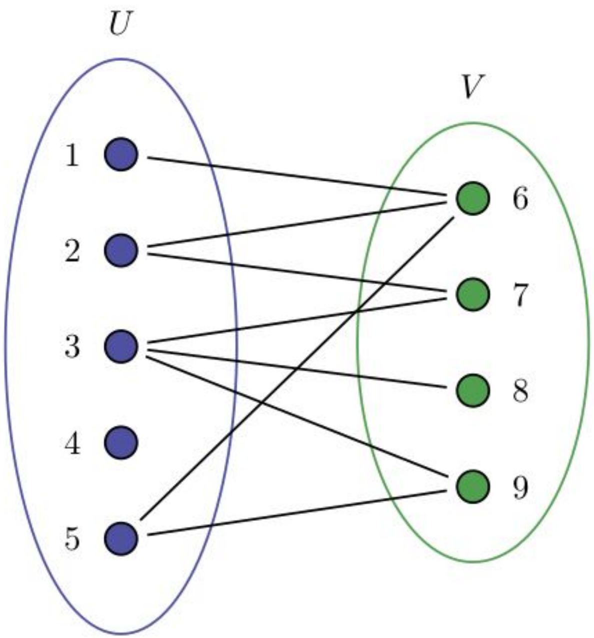
Proof: Configurations space for NL is log(n), therefore at most $\log(d^{f(n)}) = f(n)$ steps for all possible executions. Which is polynomial in time

Also: PATH is also in P

BIPARTITE ∈ coNL

Definition: An undirected graph is bipartite if:

- the nodes of the graph can be split into two groupings
- edges only span groupings



BIPARTITE $\in$ coNL (cont.)

Lemma: A graph is bipartite *iff* G does not contain a cycle of odd number nodes

Backward direction: In a cycle, the nodes of the same parity (even/odd) belong to the same grouping

Forward direction: In order to form a cycle, need to leave grouping and come back to origin grouping. This forces an even parity always.

BIPARTITE $\in$ coNL (cont.)

In coNL means NOT-BIPARTITE in NL. Define NL TM M for NOT-BIPARTITE

M = “On input <G>:

1. Nondet. guess node u, and remember it
2. Remember *prev* = u
3. For i = 1 … #nodes:
   a. Nondet. guess node v
   b. If edge between *prev* and v does not exist, *reject*
   c. If v = u and i is odd, *accept*
   d. If edge exists, set *prev* = v
4. If loop ends without accepting, *reject*”

BIPARTITE ∈ coNL (Space Analysis)

Recall: M = “On input <G>:

1. Nondet. guess node u, and remember it
2. Remember *prev* = u
3. For i = 1 … #nodes:
    a. Nondet. guess node v
    b. If edge between *prev* and v does not exist, *reject*
    c. If v = u and i is odd, *accept*
    d. If edge exists, set *prev* = v
4. If loop ends without accepting, *reject*”

Only remember u,v, and *prev* -- constant space. Rember counter i which is log(n)