18.100B Problem Set 8

Due in class Wednesday, May 1. You may discuss the problems with other students, but you should write solutions entirely on your own.

1. This is a version of problems 9 and 10 in the text (pages 166-167). Let $S = \{x_1, x_2, \dots\}$ be a countable subset of $(0, 1)$. (For example, S might consist of all the rational numbers between 0 and 1.) Define a real-valued function f on $[0, 1]$ by

$$f(x) = \sum_{\{n | x_n \leq x\}} 2^{-n}.$$

The notation means that the sum extends exactly over those positive integers n for which $x_n \leq x$.

a) Show that f is a well-defined increasing function on $[0, 1]$, that $f(0) = 0$, and $f(1) = 1$. (By Theorem 6.9, it follows that f is Riemann-integrable.)

b) Show that f is continuous at x if and only if $x \notin S$.

c) Suppose that $x \notin \overline{S}$. Prove that f is differentiable at x.

d) Suppose S is dense in $[0, 1]$. Does the derivative $f'(x)$ exist for any value of x? (In this case part (c) doesn't provide any places where the derivative exists. This question is quite a bit harder than any of the others; don't worry if you can't make any progress on it.)

2. Text, page 165, number 4.

3. Text, page 168, number 14. (If you're artistically inclined, you might try defining $x_N(t)$ and $y_N(t)$ to be the Nth partial sums of the series defining x and y, and then trying to sketch the parametric curves $\Phi_N(t) = (x_N(t), y_N(t))$ for $N = 1$ and 2.)