18.404/6.840 Michael Sipser
Theory of Computation
Due via Gradescope 2:30pm sharp, Thursday, November 12, 2020 **Problem Set 5**

Read all of Chapter 8.

1. Let $SET\text{-}SPLITTING = \{\langle S, C\rangle \mid S$ is a finite set and $C = \{C_1, \ldots, C_k\}$ is a collection of subsets of S, where the elements of S can be colored *red* or *blue* so every C_i has at least one red element and at least one blue element$\}$. Show that *SET-SPLITTING* is NP-complete.

2. For a cnf-formula ϕ with m variables and c clauses, show that you can construct in polynomial time an NFA with $O(cm)$ states that accepts all nonsatisfying assignments, represented as Boolean strings of length m. Conclude that P $\neq$ NP implies that NFAs cannot be minimized in polynomial time. Here, ***minimizing an* NFA** means finding an NFA with the fewest possible number of states that recognizes the same language as a given NFA.

3. Say that two Boolean formulas are ***equivalent*** if they have the same set of variables and are true on the same set of assignments to those variables (i.e., they describe the same Boolean function). A Boolean formula is ***minimal*** if no shorter Boolean formula is equivalent to it. (For definiteness, say that the length of a Boolean formula is the number of symbols it has.) Let *MIN-FORMULA* be the collection of minimal Boolean formulas.
Show that *MIN-FORMULA* $\in$ PSPACE.

4. (a) Explain why the following argument fails to show that *MIN-FORMULA* $\in$ coNP:
      i. If $\phi \notin$ *MIN-FORMULA*, then ϕ has a smaller equivalent formula.
      ii. An NTM can verify that $\phi \in \overline{MIN\text{-}FORMULA}$ by guessing that formula.
   (b) Show (despite part a) that if P = NP, then *MIN-FORMULA* $\in$ P.

5. For any positive integer x, let $x^{\mathcal{R}}$ be the integer whose binary representation is the reverse of the binary representation of x. (Assume no leading `0`s in the binary representation of x.) Define the function $\mathcal{R}^+ \colon \mathcal{N} \longrightarrow \mathcal{N}$ where $\mathcal{R}^+(x) = x + x^{\mathcal{R}}$.
   (a) Let $A_2 = \{\langle x, y\rangle \mid \mathcal{R}^+(x) = y\}$. Show $A_2 \in$ L.
   (b) Let $A_3 = \{\langle x, y\rangle \mid \mathcal{R}^+(\mathcal{R}^+(x)) = y\}$. Show $A_3 \in$ L.

6. Show that A_{NFA} is NL-complete.

7.* (optional) Let B be the language of properly nested parentheses and brackets. For example, `([()()]()[])` is in B but `([)]` is not. Show that B is in L.