18.404/6.840 Lecture 5

Last time:
- Context free grammars (CFGs)
- Context free languages (CFLs)
- Pushdown automata (PDA)
- Converting CFGs to PDAs

Today:
- Proving languages not Context Free
- Turing machines
- T-recognizable and T-decidable languages

Posted:
- Solutions to PSet 1
- PSet 2

Equivalence of CFGs and PDAs

Recall Theorem: A is a CFL iff some PDA recognizes A

$\longrightarrow$ Done.

$\longleftarrow$ Need to know the fact, not the proof

Corollaries:

1) Every regular language is a CFL.
2) If A is a CFL and B is regular then $A \cap B$ is a CFL.

Proof sketch of (2):
While reading the input, the finite control of the PDA for A simulates the DFA for B.

Note 1: If A and B are CFLs then $A \cap B$ may not be a CFL (will show today).
Therefore the class of CFLs is not closed under $\cap$.

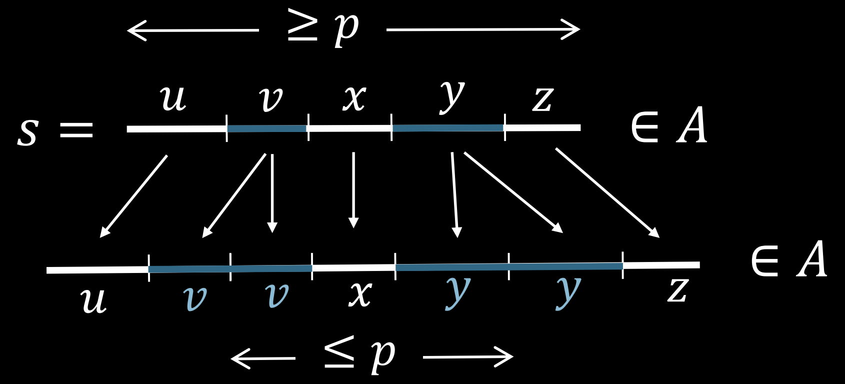
Note 2: The class of CFLs is closed under $\cup, \circ, *$ (see Pset 2).

Proving languages not Context Free

Let $B = \{0^k 1^k 2^k \mid k \geq 0\}$. We will show that B isn't a CFL.

Pumping Lemma for CFLs: For every CFL A, there is a p such that if $s \in A$ and $|s| \geq p$ then $s = uvxyz$ where

1) $uv^i xy^i z \in A$ for all $i \geq 0$
2) $vy \neq \varepsilon$
3) $|vxy| \leq p$



Informally: All long strings in A are pumpable and stay in A.

Pumping Lemma – Proof

Pumping Lemma for CFLs: For every CFL A, there is a p such that if $s \in A$ and $|s| \geq p$ then $s = uvxyz$ where

1) $uv^i xy^i z \in A$ for all $i \geq 0$
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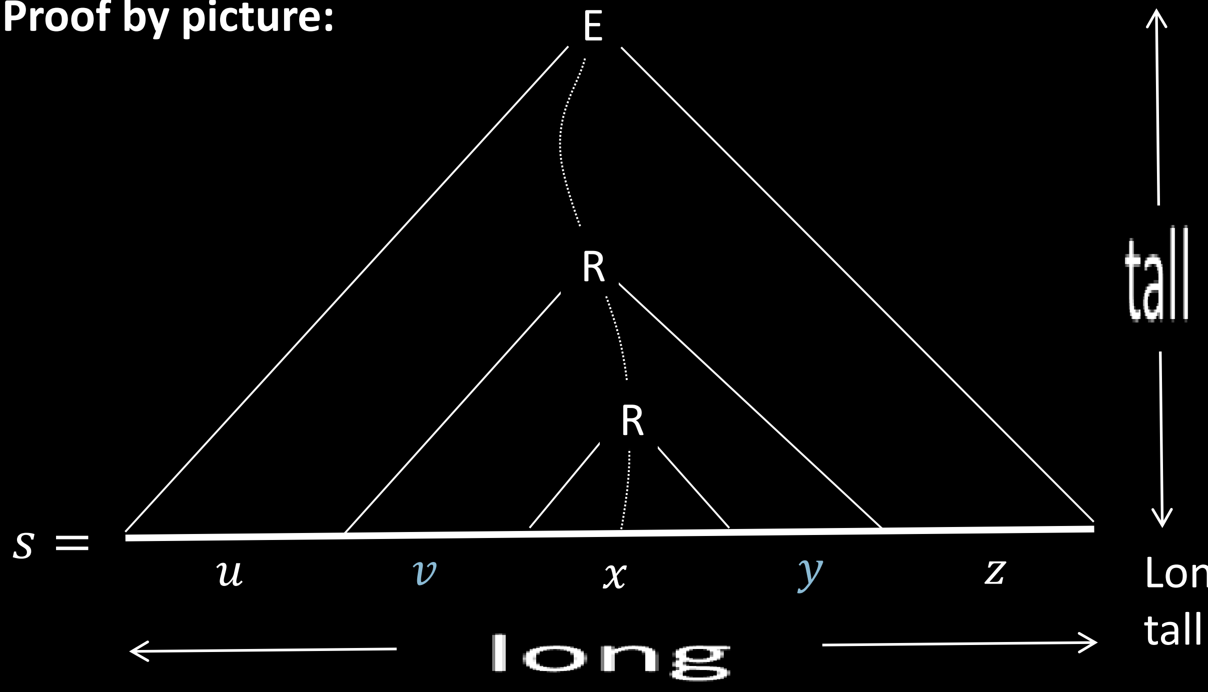
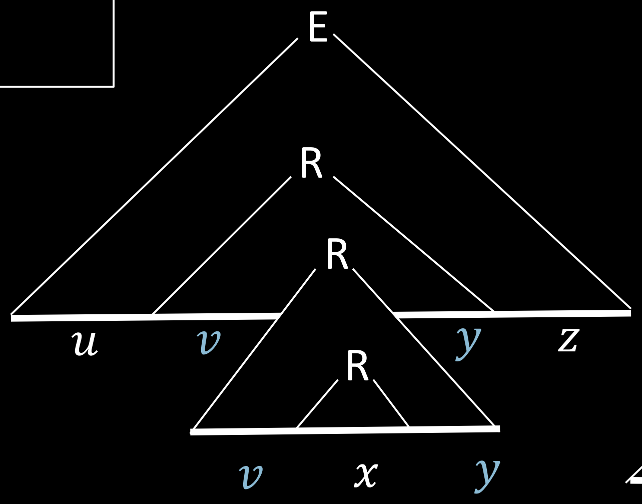
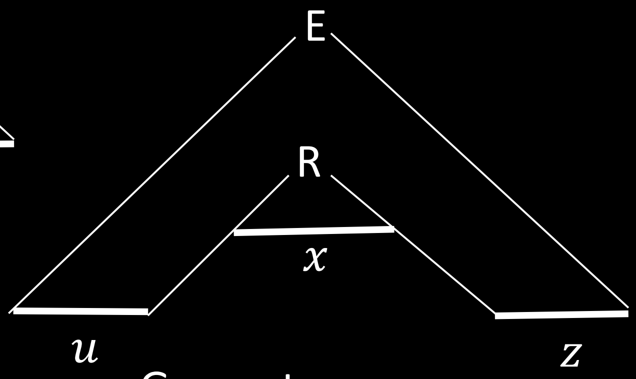
Proof by picture:



Long $s \to$
tall parse tree



Generates $uvvxyyz$
$= uv^2xy^2z$



Generates uxz
$= uv^0xy^0z$

"cutting and pasting" argument

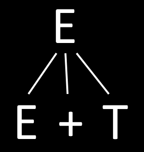
Pumping Lemma – Proof details

For $s \in A$ where $|s| \geq p$, we have $s = uvxyz$ where:

1) $uv^ixy^iz \in A$ for all $i \geq 0$
2) $vy \neq \varepsilon$
3) $|vxy| \leq p$

Let $b =$ the length of the longest right hand side of a rule (E → E+T)

$\quad = $ the max branching of the parse tree



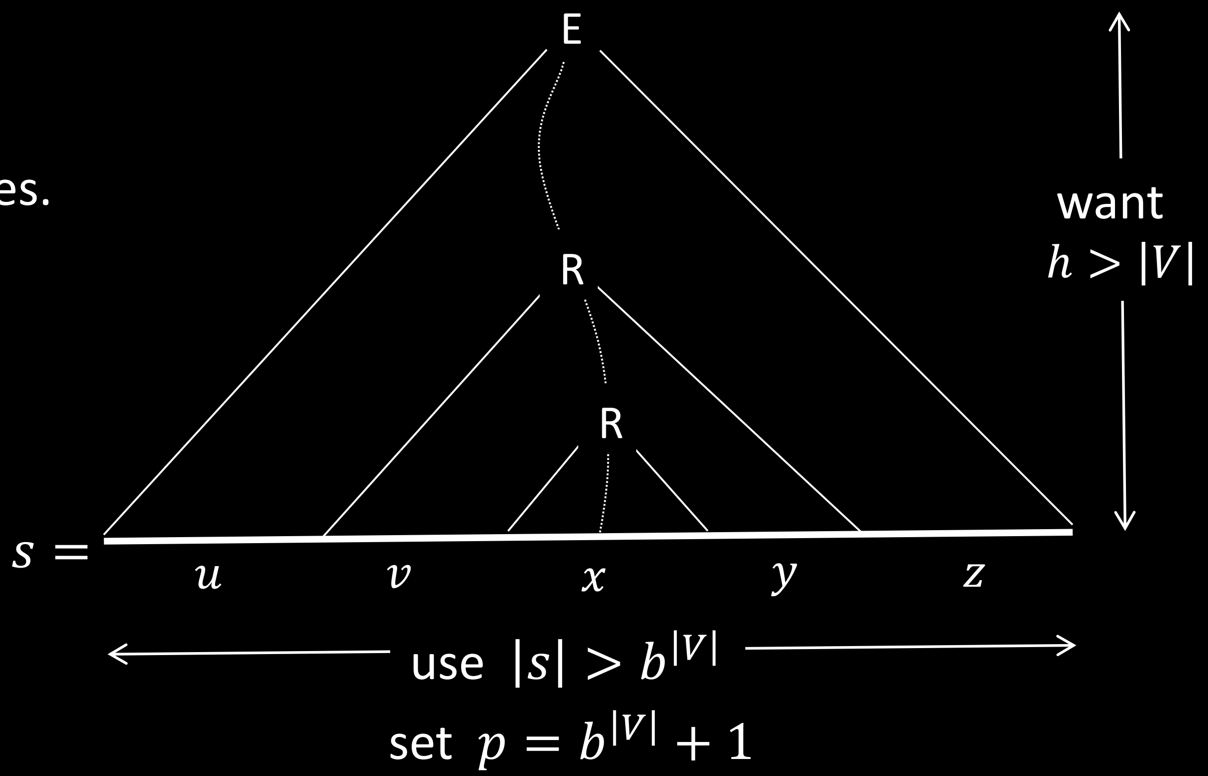
Let $h =$ the height of the parse tree for s.

A tree of height h and max branching b has at most b^h leaves.
So $|s| \leq b^h$.

Let $p = b^{|V|} + 1$ where $|V| = \#$ variables in the grammar.

So if $|s| \geq p > b^{|V|}$ then $|s| > b^{|V|}$ and so $h > |V|$.

Thus at least $|V| + 1$ variables occur in the longest path.
So some variable R must repeat on a path.



Example 1 of Proving Non-CF

Pumping Lemma for CFLs: For every CFL A, there is a p such that if $s \in A$ and $|s| \geq p$ then $s = uvxyz$ where

1) $uv^ixy^iz \in A$ for all $i \geq 0$
2) $vy \neq \varepsilon$
3) $|vxy| \leq p$

Let $B = \{0^k1^k2^k \mid k \geq 0\}$

Show: B is not a CFL

Check-in 5.1

Let $A_1 = \{0^k1^k2^l \mid k, l \geq 0\}$ (equal #s of 0s and 1s)

Let $A_2 = \{0^l1^k2^k \mid k, l \geq 0\}$ (equal #s of 1s and 2s)

Observe that PDAs can recognize A_1 and A_2. What can we now conclude?

a) The class of CFLs is not closed under intersection.

b) The Pumping Lemma shows that $A_1 \cup A_2$ is not a CFL .

c) The class of CFLs is closed under complement.

$s = 00\cdots0011\cdots1122\cdots22$

$u \mid v \mid x \mid y \mid z$

$\leftarrow \leq p \rightarrow$

Example 2 of Proving Non-CF

Pumping Lemma for CFLs: For every CFL A, there is a p such that if $s \in A$ and $|s| \geq p$ then $s = uvxyz$ where

1) $uv^ixy^iz \in A$ for all $i \geq 0$
2) $vy \neq \varepsilon$
3) $|vxy| \leq p$

Let $F = \{ww \mid w \in \Sigma^*\}$. $\Sigma = \{0,1\}$.

Show: F is not a CFL.

Assume (for contradiction) that F is a CFL.
The CFL pumping lemma gives p as above. Need to choose $s \in F$. Which s?

Try $s_1 = 0^p10^p1 \in F$.

$s_1 = 000\cdots001000\cdots001$

$u \quad v \quad x \quad y \quad z$

$\leftarrow \leq p \rightarrow$

Try $s_2 = 0^p1^p0^p1^p \in F$.
Show s_2 cannot be pumped $s_2 = uvxyz$ satisfying the 3 conditions.
Condition 3 implies that vxy does not overlap two runs of 0s or two runs of 1s.
Therefore, in uv^2xy^2z, two runs of 0s or two runs of 1s have unequal length.
So $uv^2xy^2z \notin F$ violating Condition 1. Contradiction! Thus F is not a CFL.

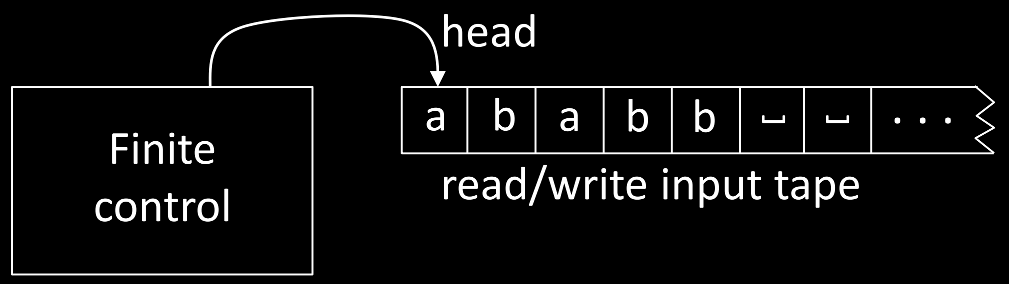
$s_2 = 0\cdots01\cdots10\cdots01\cdots1$

$u \quad v \quad x \quad y \quad z$

$\leftarrow \leq p \rightarrow$


Coffee
Break

Turing Machines (TMs)

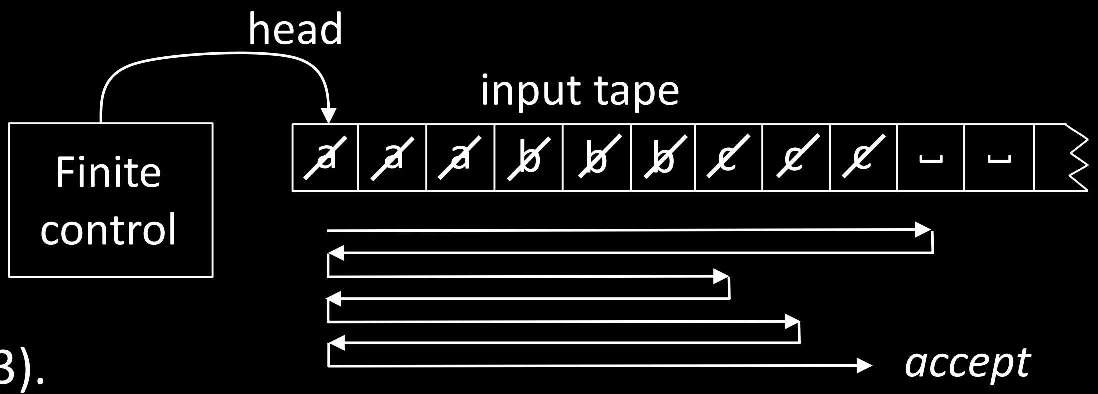


1) Head can read and write
2) Head is two way (can move left or right)
3) Tape is infinite (to the right)
4) Infinitely many blanks “␣“ follow input
5) Can accept or reject any time (not only at end of input)

TM – example

TM recognizing $B = \{\mathrm{a}^k\mathrm{b}^k\mathrm{c}^k \mid k \geq 0\}$

1) Scan right until ␣ while checking if input is in $\mathrm{a}^*\mathrm{b}^*\mathrm{c}^*$, *reject* if not.
2) Return head to left end.
3) Scan right, crossing off single a, b, and c.
4) If the last one of each symbol, *accept*.
5) If the last one of some symbol but not others, *reject*.
6) If all symbols remain, return to left end and repeat from (3).



Check-in 5.2

How do we get the effect of “crossing off” with a Turing machine?

a) We add that feature to the model.

b) We use a tape alphabet $\Gamma = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \cancel{\mathrm{a}}, \cancel{\mathrm{b}}, \cancel{\mathrm{c}}, \sqcup\}$.

c) All Turing machines come with an eraser.

TM – Formal Definition

Defn: A Turing Machine (TM) is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{acc}}, q_{\mathrm{rej}})$

Σ input alphabet

Γ tape alphabet ($\Sigma \subseteq \Gamma$)

δ: $\mathrm{Q} \times \Gamma \to Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ (L = Left, R = Right)

$\delta(q, \mathrm{a}) = (r, \mathrm{b}, \mathrm{R})$

On input w a TM M may halt (enter q_{acc} or q_{rej}) or M may run forever ("loop").

So M has 3 possible outcomes for each input w:

1. *Accept* w (enter q_{acc})
2. *Reject* w by halting (enter q_{rej})
3. *Reject* w by looping (running forever)

Check-in 5.3

This Turing machine model is deterministic.
How would we change it to be nondeterministic?

a) Add a second transition function.

b) Change δ to be δ: $\mathrm{Q} \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\})$

c) Change the tape alphabet Γ to be infinite.

Check-in 5.3

TM Recognizers and Deciders

Let M be a TM. Then $L(M) = \{w |\ M \text{ accepts } w\}$.

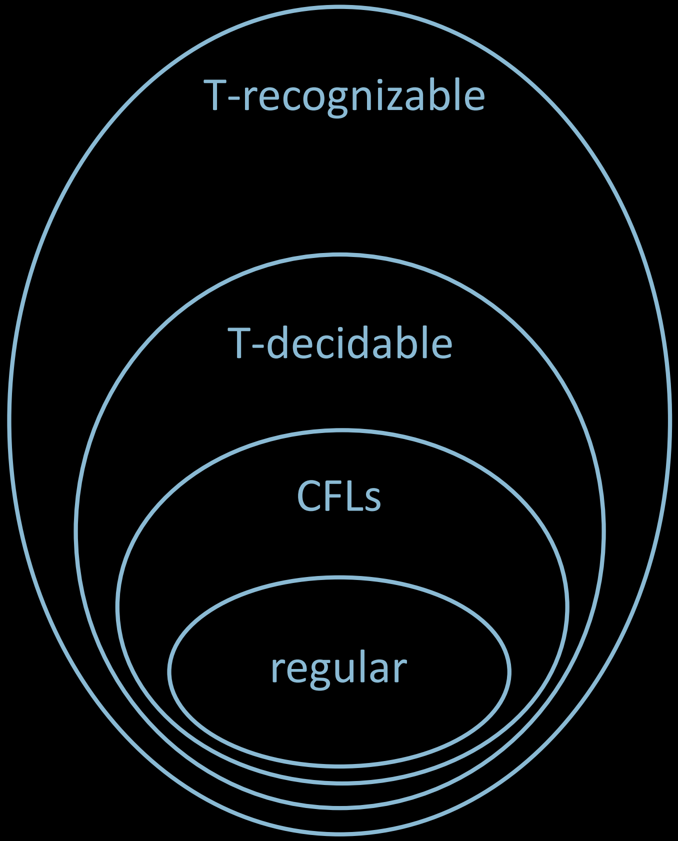
Say that M recognizes A if $A = L(M)$.

Defn: A is Turing-recognizable if $A = L(M)$ for some TM M.

Defn: TM M is a decider if M halts on all inputs.

Say that M decides A if $A = L(M)$ and M is a decider.

Defn: A is Turing-decidable if $A = L(M)$ for some TM decider M.



Quick review of today

1. Proved the CFL Pumping Lemma as a tool for showing that languages are not context free.
2. Defined Turing machines (TMs).
3. Defined TM deciders (halt on all inputs).
4. T-recognizable and T-decidable languages.