# 18.404/6.840 Lecture 18

**Last time:**

- Space complexity
- SPACE$(f(n))$, NSPACE$(f(n))$, PSPACE, NPSPACE
- Relationship with TIME classes

**Today:**

- Review $LADDER_{\mathrm{DFA}} \in$ PSPACE
- Savitch's Theorem: NSPACE$(f(n)) \subseteq$ SPACE$(f^2(n))$
- PSPACE-completeness
- $TQBF$ is PSPACE-complete

shrink me →

**Posted:** Pset 4 solutions, Pset 5

# Review: SPACE Complexity

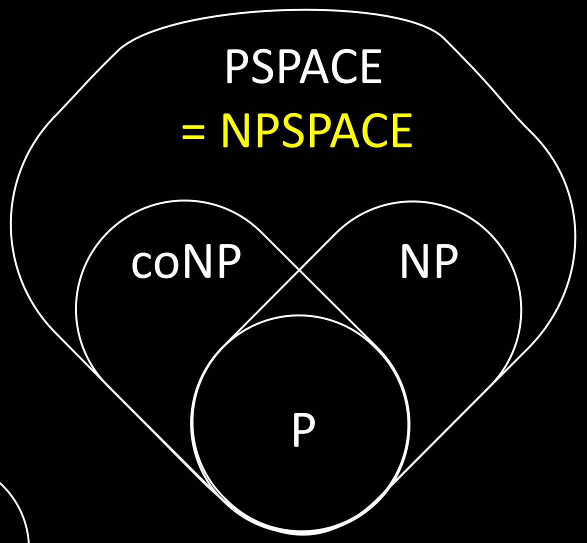
**Defn:** Let $f\colon \mathbb{N} \rightarrow \mathbb{N}$ where $f(n) \geq n$. Say TM $M$ runs in space $f(n)$ if $M$ always halts and uses at most $f(n)$ tape cells on all inputs of length $n$.

An NTM $M$ runs in space $f(n)$ if all branches halt and each branch uses at most $f(n)$ tape cells on all inputs of length $n$.

$\text{SPACE}\big(f(n)\big) = \{B \mid$ some 1-tape TM decides $B$ in space $O\big(f(n)\big)\}$
$\text{NSPACE}\big(f(n)\big) = \{B \mid$ some 1-tape NTM decides $B$ in space $O\big(f(n)\big)\}$

$\text{PSPACE} = \bigcup_k \text{SPACE}(n^k)$ "polynomial space"
$\text{NPSPACE} = \bigcup_k \text{NSPACE}(n^k)$ "nondeterministic polynomial space"

Today: PSPACE = NPSPACE




Or possibly: P = NP = coNP = PSPACE

# Review: $LADDER_{\text{DFA}} \in$ PSPACE

Theorem: $LADDER_{\text{DFA}} \in$ SPACE$(n^2)$

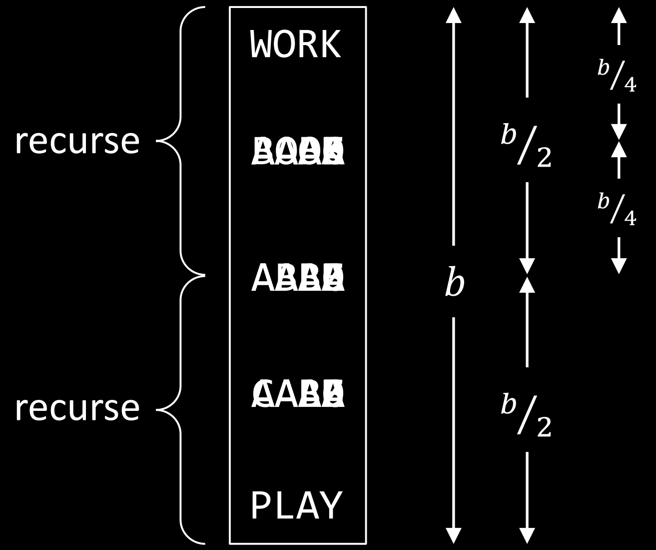
Proof: Write $u \xrightarrow{b} v$ if there's a ladder from $u$ to $v$ of length $\leq b$.

Here's a recursive procedure to solve the bounded DFA ladder problem:

$BOUNDED\text{-}LADDER_{\text{DFA}} = \{\langle B, u, v, b\rangle \mid B$ a DFA and $u \xrightarrow{b} v$ by a ladder in $L(B)\}$

$B\text{-}L$ = "On input $\langle B, u, v, b\rangle$ Let $m = |u| = |v|$.

1. For $b = 1$, *accept* if $u, v \in L(B)$ and differ in $\leq 1$ place, else *reject*.
2. For $b > 1$, repeat for each $w \in L(B)$ of length $|u|$
3. Recursively test $u \xrightarrow{b/2} w$ and $w \xrightarrow{b/2} v$ [division rounds up]
4. *Accept* both accept.
5. *Reject* [if all fail]."




Test $\langle B, u, v\rangle \in LADDER_{\text{DFA}}$ with $B\text{-}L$ procedure on input $\langle B, u, v, t\rangle$ for $t = |\Sigma|^m$

Space analysis:

Each recursive level uses space $O(n)$ (to record $w$).

Recursion depth is $\log t = O(m) = O(n)$.

Total space used is $O(n^2)$.

# PSPACE = NPSPACE

**Savitch's Theorem:** For $f(n) \geq n$, $\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n))$

Proof: Convert NTM $N$ to equivalent TM $M$, only squaring the space used.

For configurations $c_i$ and $c_j$ of $N$, write $c_i \xrightarrow{b} c_j$ if can get from $c_i$ to $c_j$ in $\leq b$ steps.

Give recursive algorithm to test $c_i \xrightarrow{b} c_j$:

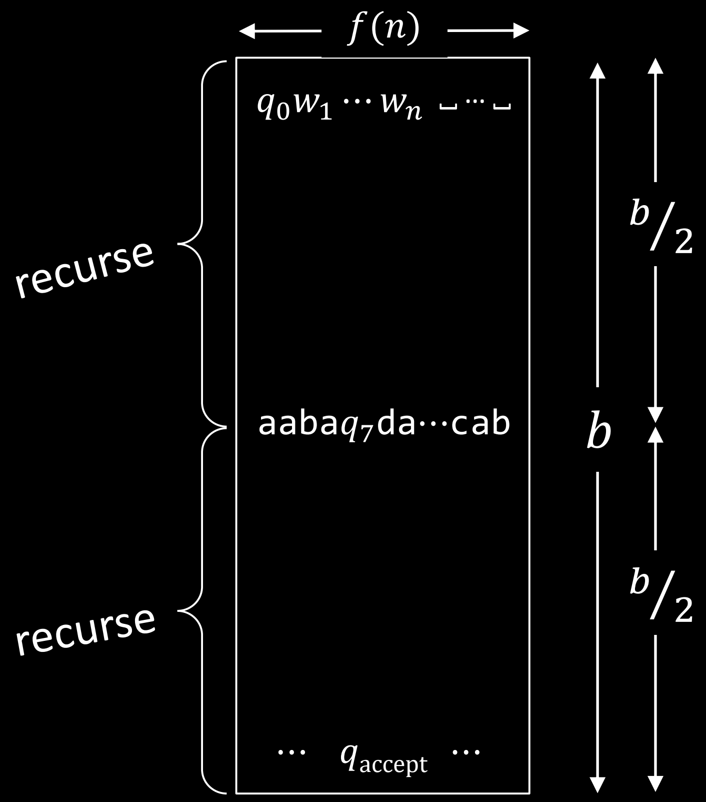
$M$ = "On input $c_i, c_j, b$ [goal is to check $c_i \xrightarrow{b} c_j$]

1. If $b = 1$, check directly by using $N$'s program and answer accordingly.
2. If $b > 1$, repeat for all configurations $c_{\text{mid}}$ that use $f(n)$ space.
3. Recursively test $c_i \xrightarrow{b/2} c_{\text{mid}}$ and $c_{\text{mid}} \xrightarrow{b/2} c_j$
4. If both are true, *accept*. If not, continue.
5. *Reject* if haven't yet accepted."

Test if $N$ accepts $w$ by testing $c_{\text{start}} \xrightarrow{t} c_{\text{accept}}$ where $t$ = number of configurations = $|Q| \times f(n) \times d^{f(n)}$

Each recursion level stores 1 config = $O(f(n))$ space.

Number of levels = $\log t = O(f(n))$. Total $O(f^2(n))$ space.



# PSPACE-completeness

**Defn:** $B$ is <u>PSPACE-complete</u> if

1) $B \in$ PSPACE
2) For all $A \in$ PSPACE, $A \leq_{\mathrm{P}} B$

If $B$ is PSPACE-complete and $B \in$ P then P $=$ PSPACE.

**Check-in 18.1**

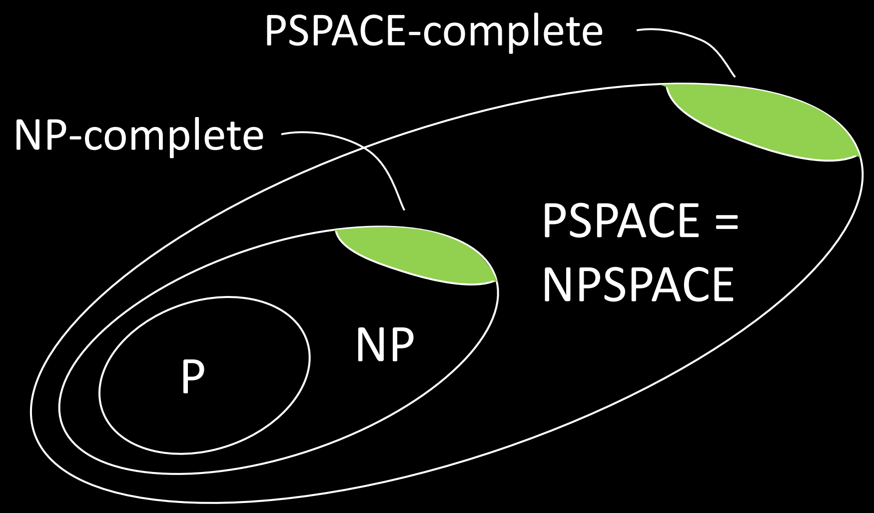
Knowing that $TQBF$ is PSPACE-complete, what can we conclude if $TQBF \in$ NP? Check all that apply.

(a) P = PSPACE

(b) NP = PSPACE

(c) P = NP

(d) NP = coNP




Think of complete problems as the "hardest" in their associated class.

Check-in 18.1


Coffee
Break

# $TQBF$ is PSPACE-complete

Recall: $TQBF = \{\langle\phi\rangle |\ \phi$ is a QBF that is TRUE$\}$

**Examples:** $\phi_1 = \forall x\, \exists y\, [(x \vee y) \wedge (\overline{x} \vee \overline{y})] \in TQBF$ [TRUE]
$\phi_2 = \exists y\, \forall x\, [(x \vee y) \wedge (\overline{x} \vee \overline{y})] \notin TQBF$ [FALSE]

**Theorem:** $TQBF$ is PSPACE-complete

Proof: 1) $TQBF \in$ PSPACE ✓

2) For all $A \in$ PSPACE, $A \leq_{\mathrm{P}} TQBF$

Let $A \in$ PSPACE be decided by TM $M$ in space $n^k$.

Give a polynomial-time reduction $f$ mapping $A$ to $TQBF$.

$f : \Sigma^* \rightarrow$ QBFs

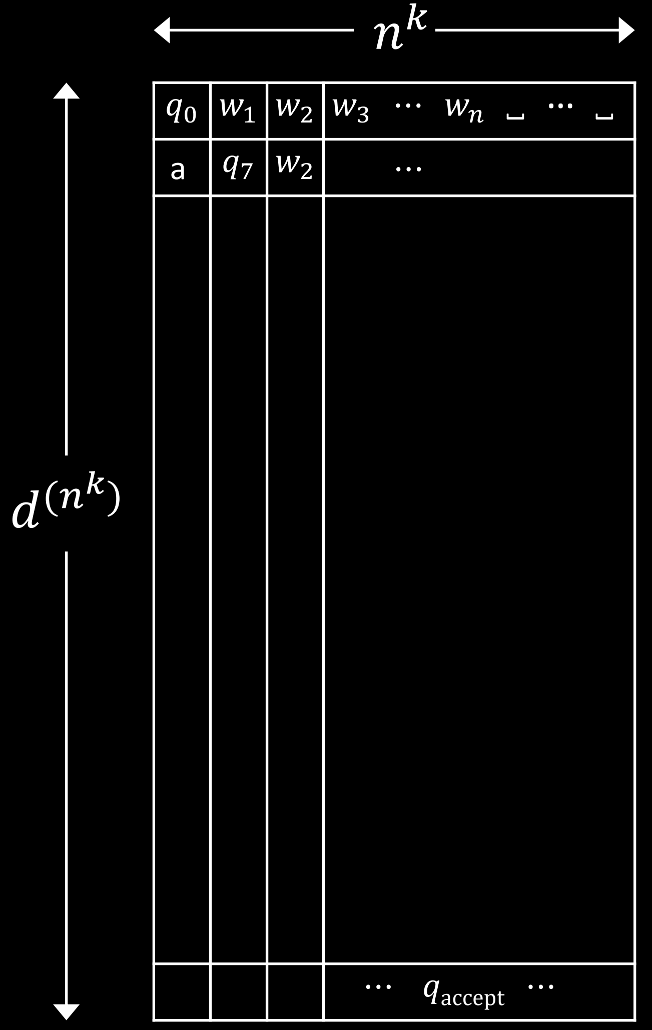
$f(w) = \langle\phi_{M,w}\rangle$

$w \in A$ iff $\phi_{M,w}$ is TRUE

Plan: Design $\phi_{M,w}$ to "say" $M$ accepts $w$. $\phi_{M,w}$ simulates $M$ on $w$.

# Constructing $\phi_{M,w}$: 1st try

Tableau for $M$ on $w$




Recall: A tableau for $M$ on $w$ represents a computation history for $M$ on $w$ when $M$ accepts $w$.
Rows of that tableau are configurations.

$M$ runs in space $n^k$, its tableau has:

- $n^k$ columns (max size of a configuration)
- $d^{(n^k)}$ rows (max number of steps)

Constructing $\phi_{M,w}$. Try Cook-Levin method.
Then $\phi_{M,w}$ will be as big as tableau.
But that is exponential: $n^k \times d^{(n^k)}$.
Too big! ☹

# Constructing $\phi_{M,w}$: 2nd try

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For configs $c_i$ and $c_j$ construct $\phi_{c_i, c_j, b}$ which "says" $c_i \xrightarrow{b} c_j$ recursively.

$$\phi_{c_i, c_j, b} = \underbrace{\exists c_{\text{mid}}}_{} \left[ \phi_{c_i, c_{\text{mid}}, b/2} \ \wedge \ \phi_{c_{\text{mid}}, c_j, b/2} \right]$$

$\exists x_1, x_2, \cdots, c_l$
as in Cook-Levin

$\exists c_{\text{mid}}[\phi_{\ ,\ , b/4} \ \wedge \ \phi_{\ ,\ , b/4}]$ $\quad$ $\exists c_{\text{mid}}[\phi_{\ ,\ , b/4} \ \wedge \ \phi_{\ ,\ , b/4}]$

⋮ $\quad$ ⋮ $\quad$ $\exists c_{\text{mid}}[\phi_{\ ,\ , b/8} \cdots]$

$\phi_{\ ,\ , 1}$ defined as in Cook-Levin

$$\phi_{M,w} = \phi_{c_{\text{start}}, c_{\text{accept}}, t}$$
$$t = d^{(n^k)}$$

**Size analysis:**
Each recursive level doubles number of QBFs.
Number of levels is $\log d^{(n^k)} = O(n^k)$.
→ Size is exponential. ☹

## Check-in 18.2

Why shouldn't we be surprised that this construction fails?

(a) We can't define a QBF by using recursion.

(b) It doesn't use ∀ anywhere.

(c) We know that $TQBF \notin$ P.

Check-in 18.2

# Constructing $\phi_{M,w}$: 3rd try

$$\phi_{c_i,\, c_j,\, b} = \exists c_{\text{mid}} \underbrace{\left[\, \phi_{c_i,\, c_{\text{mid}},\, b/2} \;\wedge\; \phi_{c_{\text{mid}},\, c_j,\, b/2} \,\right]}_{\forall (c_g, c_h) \in \{ (c_i, c_{\text{mid}}), (c_{\text{mid}}, c_j) \} [\phi_{c_g, c_h, b/2}]}$$

$$\forall\left(c_g, c_h\right) \in \left\{ \left(c_i, c_{\text{mid}}\right), \left(c_{\text{mid}}, c_j\right) \right\} \left[\phi_{c_g,\, c_h,\, b/2}\right]$$

$\forall (x \in S)\ [\,\psi\,]$
is equivalent to
$\forall x\ [(x \in S) \longrightarrow \psi]$

$\vdots$

$\phi_{\,,\,,1}$ defined as in Cook-Levin

$\phi_{M,w} = \phi_{c_{\text{start}},\, c_{\text{accept}},\, t}$

$t = d^{(n^k)}$

**Size analysis:**

Each recursive level <u>adds</u> $O(n^k)$ to the QBF.

Number of levels is $\log d^{(n^k)} = O(n^k)$.

→ Size is $O\left(n^k \times n^k\right) = O\left(n^{2k}\right)$ ☺

## Check-in 18.3

Would this construction still work if $M$ were nondeterministic?

(a) Yes.

(b) No.

# Quick review of today

1. $LADDER_{\mathrm{DFA}} \in$ PSPACE
2. Savitch's Theorem: $\mathrm{NSPACE}\big(f(n)\big) \subseteq \mathrm{SPACE}\big(f^2(n)\big)$
3. $TQBF$ is PSPACE-complete