Here are some practice problems you might use in preparing for the final exam. I'm sorry to say that I won't have a chance to post solutions to these problems before the final. I will try to post solutions to the final itself by the evening of May 26, but I can't make that a definite promise.

1. Let $E = (0, 1)$, a subset of $\mathbb{R}$ with the usual metric. Prove directly from the definition (without using the Heine-Borel theorem) that E is *not* compact. (This means you need to find a collection $\{U_\alpha\}$ of open subsets of $\mathbb{R}$ so that E is contained in the union of all the $\{U_\alpha\}$, but E is *not* contained in the union of any finite subcollection of the $\{U_\alpha\}$.)

2. Let X be the metric space of all rational numbers in $[0, 1]$. Find a subset E of X such that $E \neq X$, $E \neq \emptyset$, but E is both open and closed in X.

3. Give an example of a subset of $\mathbb{R}$ having exactly two limit points.

4. Give an example of a real-valued differentiable function f on $\mathbb{R}$ for which f' is not continuous.

5. (20 points) X is a metric space and

$$f\colon X \to X$$

is a function from X to X. A *fixed point* of f is a point $x \in X$ such that $f(x) = x$. Prove that every continuous function from $[0, 1]$ to $[0, 1]$ has a fixed point. (Hint: you want to show that the continuous function $f(x) - x$ is equal to zero somewhere. Use the Intermediate Value Theorem.)

6. Suppose that $f\colon X \to X$ is any continuous function, and that $x_0 \in X$. Define a sequence $x_1, x_2, x_3 \ldots$ of points in X by

$$x_{n+1} = f(x_n) \qquad (n \geq 0).$$

Prove that *if* the sequence $\{x_n\}$ converges to a limit point $x \in X$, then $f(x) = x$.

7. This problem concerns Riemann sums for integrating the function x on the interval $[a, b]$. You may need to use the formula

$$\sum_{j=1}^{n} j = n(n+1)/2.$$

For each positive integer n, consider the partition of $[a, b]$ into n equal parts:

$$P_n = (a = x_0 < x_1 < \cdots < x_n = b), \qquad x_i = a + i(b-a)/n.$$

a) Calculate the upper sum $U(P_n, x)$.
b) Calculate the lower sum $L(P_n, x)$.
c) Deduce from these two calculations (not using the Fundamental Theorem of Calculus) that $\int_a^b x dx = (b^2 - a^2)/2$.

8. Suppose that f is a continuous function on $[0, 2\pi]$. Recall that the mth Fourier coefficient of f is by definition

$$c_m(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-imx}\, dx \qquad (m \in \mathbb{Z}).$$

Suppose that the series of real numbers

$$\sum_{n=1}^{\infty} |c_n| + |c_{-n}|$$

converges.

a) Deduce that the Fourier series $\sum_{m=-\infty}^{\infty} c_m e^{imx}$ converges uniformly to a continuous function $F(x)$.

b) Show that $c_m(F) = c_m(f)$.

9. Suppose $\{s_n\}$ is a sequence of real numbers. Define a new sequence $\{\sigma_n\}$ by

$$\sigma_n = \frac{1}{n}(s_1 + \cdots s_n),$$

the average of the first n elements of the first sequence.

a) Prove that if $\lim_{n \to \infty} s_n = s$, then $\lim_{n \to \infty} \sigma_n = s$.

b) Find an example of a sequence $\{s_n\}$ that has no limit, for which $\{\sigma_n\}$ converges. (Hint: make $\{s_n\}$ bounce back and forth between two values.)

c) Can there be an unbounded seqence $\{s_n\}$ for which $\{\sigma_n\}$ converges?