

## EE364b Spring 2020 Homework 4

Due Friday 5/8 at 11:59pm via Gradescope

- 4.1 Consider the Bregman divergence  $D(x, y) = \sum_i x_i \log(x_i/y_i) - (x_i - y_i)$ , which is known as the generalized KL divergence.

Show that the projection with respect to this Bregman divergence on the simplex, *i.e.*,  $\Delta_n = \{x \in \mathbf{R}_+^n \mid \mathbf{1}^T x = 1\}$ , amounts to a simple renormalization  $y \rightarrow y/\|y\|_1$ .

- 4.2 *Coordinate-wise descent.* In the coordinate-wise descent method for minimizing a convex function  $f$ , we first minimize over  $x_1$ , keeping all other variables fixed; then we minimize over  $x_2$ , keeping all other variables fixed, and so on. After minimizing over  $x_n$ , we go back to  $x_1$  and repeat the whole process, repeatedly cycling over all  $n$  variables. (There are many obvious variations on this, such as block coordinate-wise descent and random coordinate-wise descent.)

- (a) Show that coordinate-wise descent fails for the function

$$f(x) = |x_1 - x_2| + 0.1(x_1 + x_2).$$

(In particular, verify that the algorithm terminates after one step at the point  $(x_2^{(0)}, x_2^{(0)})$ , while  $\inf_x f(x) = -\infty$ .) Thus, coordinate-wise descent need not work, for general convex functions.

- (b) Now consider coordinate-wise descent for minimizing the specific function  $\phi(x) = f(x) + \lambda\|x\|_1$ , where  $f$  is smooth and convex, and  $\lambda \geq 0$ . Assuming  $f$  is strongly convex (say) it can be shown that the iterates converge to a fixed point  $\tilde{x}$ . You will now show that the fixed point  $\tilde{x}$  is optimal, *i.e.*, minimizes  $\phi$ . Thus, coordinate-wise descent works for  $\ell_1$ -regularized minimization of a differentiable function.

- (i) Show that  $\tilde{x}$  satisfies

$$0 \in \frac{\partial f}{\partial x_i}(\tilde{x}) + \lambda \mathcal{I}_i, \quad i = 1, \dots, n,$$

where  $\mathcal{I}_i$  is the subdifferential of  $|\cdot|$  at  $\tilde{x}_i$ :  $\mathcal{I}_i = \{-1\}$  if  $\tilde{x}_i < 0$ ,  $\mathcal{I}_i = \{+1\}$  if  $\tilde{x}_i > 0$ , and  $\mathcal{I}_i = [-1, 1]$  if  $\tilde{x}_i = 0$ .

- (ii) Show that the condition in (i) implies that  $\tilde{x}$  minimizes  $\phi$ .

- (c) Work out an explicit form for coordinate-wise descent for  $\ell_1$ -regularized least-squares, *i.e.*, for minimizing the function

$$\|Ax - b\|_2^2 + \lambda\|x\|_1.$$

You might find the deadzone function

$$\psi(u) = \begin{cases} u - 1 & u > 1 \\ 0 & |u| \leq 1 \\ u + 1 & u < -1 \end{cases}$$

useful. Generate some data and try out the coordinate-wise descent method. Check the result against the solution found using CVX, and produce a graph showing convergence of your coordinate-wise method.

- 4.3 *High dimensional problems, mirror descent, and gradient descent.* We consider using mirror descent versus projected subgradient descent to solve the non-smooth minimization problem

$$\text{minimize } f(x) = \max_{i \in \{1, \dots, m\}} \{a_i^T x + b_i\} \quad \text{subject to } x \in \Delta_n = \{z \in \mathbf{R}_+^n \mid z^T \mathbf{1} = 1\}.$$

Implement mirror descent with the choice  $h(x) = \sum_{i=1}^n x_i \log x_i$  and projected subgradient descent for this problem. (You will need to project onto the simplex efficiently for this to be a reasonable method at all.) You will compare the performance of these two methods.

Generate random problem data for the above objective with  $a_i$  drawn as i.i.d.  $N(0, I_{n \times n})$  (multivariate normals) and  $b_i$  drawn i.i.d.  $N(0, 1)$ , where  $n = 500$  and  $m = 50$ . Solve the problem using CVX (or `Convex.jl` or `CVXPY`), then run mirror descent and projected gradient descent on the same data for 100 iterations. Run each method with constant stepsizes  $\alpha \in \{2^{-12}, 2^{-11}, \dots, 2^6, 2^7\}$ . Repeat this 25 times, then plot the average optimality gap  $f(x^k) - f(x^*)$  or  $f_{\text{best}}^k - f(x^*)$  as a function of iteration for the best stepsize (chosen by smallest optimality gaps) for each method. Which method gives the best performance?

- 4.4 *SGD versus dual averaging.* A support vector machine problem has an objective of the form

$$f(x) = \frac{1}{N} \sum_{i=1}^N \max\{1 - a_i^T x, 0\}.$$

We consider using the dual averaging method to minimize the function  $f$  over the probability simplex in  $\mathbf{R}^n$ , i.e.,  $\Delta_n = \{x \in \mathbf{R}_+^n \mid \mathbf{1}^T x = 1\}$ . This constraint set is a heuristic for finding a sparse solution  $x^*$ , i.e., one with many zero entries. Implement projected stochastic gradient descent and stochastic dual averaging. That is, implement a method that at each iteration, draws a single sample of the  $a_i$  vectors, computes an associated stochastic subgradient  $g^k$ , and then performs one of the following two updates:

$$\begin{aligned} \text{SGD} \quad x^{k+1} &= \operatorname{argmin}_{x \in \Delta_n} \left\{ \|x - (x^k - \alpha_k g^k)\|_2^2 \right\} \\ \text{Dual averaging} \quad x^{k+1} &= \operatorname{argmin}_{x \in \Delta_n} \left\{ \langle z^k, x \rangle + \frac{1}{2\alpha_k} \|x\|_2^2 \right\}, \end{aligned}$$

where  $z^k = \sum_{i=1}^k g^i$ . For stochastic gradient descent, use the stepsizes  $\alpha_k = 1/\sqrt{nk}$ , and for stochastic dual averaging use stepsizes  $\alpha_k = 1/\sqrt{k}$ .

Use the data in `dual_averaging_data.py` (for Python) to generate the matrix  $A = [a_1 \cdots a_N]^T \in \mathbf{R}^{N \times n}$ . Run each method for 200 steps, and give two plots: one with the gaps  $f(x^k) - f(x^*)$  as a function of iteration  $k$  for each of the methods, the other with the number of non-zero entries in  $x^k$  as a function of  $k$ . (If your projection does not produce exact zeros, truncate any coordinates with  $|x_j| < 10^{-5}$  to zero.) You should see that one of the two methods results in far fewer non-zero entries than the other.