18.404/6.840 Lecture 8

Last time:

- Decision procedures for automata and grammars
  , , , , , are decidable
  is T-recognizable

Today:

- is undecidable
- The diagonalization method
- is T-unrecognizable
- The reducibility method
- Other undecidable languages

Pset 3 will be posted soon
Continue to have chat-breaks; will try to keep them short.

Recall: Acceptance Problem for TMs

Let is a TM and accepts

Today's Theorem: is not decidable

Proof uses <u>the diagonalization method</u>,
so we will introduce that first.

The Size of Infinity

How to compare the relative sizes of infinite sets?

Cantor (~1890s) had the following idea.

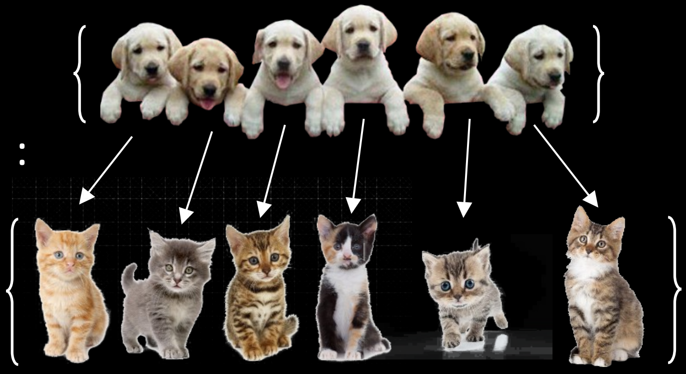
Defn: Say that set and <u>have the same size</u> if there is
a $\underbrace{\text{one-to-one}}_{\text{“injective”}}$ and $\underbrace{\text{onto}}_{\text{Range}\ \text{“surjective”}}$ function

We call such an a <u>1-1 correspondence</u>

Informally, two sets have the same size if we can pair up their members.

This definition works for finite sets.

Apply it to infinite sets too.



Countable Sets

Let and let

Show and   have the same size

Let

Show and have the same size

	1	2	3	4	…
1	1/1	1/2	1/3	1/4	
2	2/1	2/2	2/3	2/4	…
3	3/1	3/2	3/3	3/4	
4	4/1	4/2	4/3	4/4	

Defn: A set is <u>countable</u> if it is finite or it has the same size as .

Both and are countable.

is Uncountable – Diagonalization

Let all real numbers (expressible by infinite decimal expansion)

Theorem: is uncountable

Proof by contradiction via diagonalization: Assume is countable

So there is a 1-1 correspondence

1	
2	
3	
4	
5	
6	
7	

Diagonalization

Demonstrate a number that is missing from the list.

0.851

differs from the th number in the th digit
so cannot be the th number for any .

Hence is not paired with any It is missing from the list.

Therefore is not a 1-1 correspondence.

is Uncountable – Corollaries

Let all languages

Corollary 1: is uncountable

Proof: There's a 1-1 correspondence from to so they are the same size.

Observation: is countable.

Let all Turing machines

Observation: is countable.

Because is a TM.

Corollary 2: Some language is not decidable.

Because there are more languages than TMs.

We will show some specific language is not decidable.

Check-in 8.1

Hilbert's 1st question asked if there is a set of intermediate size between and . Gödel and Cohen showed that we cannot answer this question by using the standard axioms of mathematics.

How can we interpret their conclusion?

(a) We need better axioms to describe reality.

(b) Infinite sets have no mathematical reality so Hilbert's 1st question has no answer.

Check-in 8.1

is undecidable

Recall is a TM and accepts

Theorem: is not decidable

Proof by contradiction: Assume some TM decides .

So on

Use to construct TM

"On input

1. Simulate on input
2. *Accept* if rejects. *Reject* if accepts."

accepts iff doesn't accept .

accepts iff doesn't accept .

Contradiction.

Why is this proof a diagonalization?

All TMs	All TM descriptions:				. . .
	acc				
		rej			
			acc		
				acc	


Coffee
Break

Check-in 8.2

Recall the Queue Automaton (QA) defined in Pset 2.
It is similar to a PDA except that it is deterministic
and it has a queue instead of a stack.

Let is a QA and accepts

Is decidable?

(a) Yes, because QA are similar to PDA and is decidable.

(b) No, because “yes” would contradict results we now know.

(c) We don’t have enough information to answer this question.

is T-unrecognizable

Theorem: If and are T-recognizable then is decidable

Proof: Let TM and recognize and .

Construct TM deciding .

"On input

1. Run and on in parallel until one accepts.
2. If accepts then *accept*.
   If accepts then *reject*."

Corollary: is T-unrecognizable

Proof: is T-recognizable but also undecidable

Check-in 8.3

From what we've learned, which closure properties can we prove for the class of T-recognizable languages?

Choose all that apply.

(a) Closed under union.

(b) Closed under intersection.

(c) Closed under complement.

(d) Closed under concatenation.

(e) Closed under star.

The Reducibility Method

Use our knowledge that is undecidable to show other problems are undecidable.

Defn: halts on input

Theorem: is undecidable

Proof by contradiction, showing that is reducible to :

Assume that is decidable and show that is decidable (false!).

Let TM decide .

Construct TM deciding .

"On input

1. Use to test if on halts. If not, reject.
2. Simulate on until it halts (as guaranteed by).
3. If has accepted then *accept*.
   If has rejected then *reject*.

TM decides , a contradiction. Therefore is undecidable.

Quick review of today

1. Showed that and are not the same size
   to introduce the Diagonalization Method.
2. is undecidable.
3. If and are T-recognizable then is decidable.
4. is T-unrecognizable.
5. Introduced the Reducibility Method to show that
   is undecidable.