1 Convex Sets

1.1 Every Convex Combinition is in the Convex Set

Proof. This is done by proving by induction.
The definition of a convex set says, for any 2 points in set C, their convex combinition is in the set. Therefore, for the case of $k = 2$, there is:

$$\theta_1 x_1 + \theta_2 x_2 \in C$$

For any k, suppose $\sum_i^k \theta_i x_i = x^* \in C$, the $k+1$ case is :

$$\theta_1 x_1 + ... + \theta_k x_k + \theta_{k+1} x_{k+1} = x^* + \theta_{k+1} x_{k+1} \in C$$

□

1.2 Convexity from Line Intersection

Proof. The lemma is intuitively right. Taking the intersection of a set is to "slice" it with a line or hyperplane. All of the slices must be convex in order for the set to be convex. The intersection says that the condition for a line and affine/convex set must hold at the same moment.
Condition for a line:

$$w^T x = b \tag{1}$$

Condition for a affine set:

$$\theta_i x_i + \theta_j x_j \in C, \theta_i + \theta_j = 1 \tag{2}$$

Plug in the condition of the line:

$$w^T \theta_i x_i + w^T \theta_j x_j = w^T (\theta_i x_i + \theta_j x_j) = (\theta_i + \theta_j) b = b \tag{3}$$

The above shows indication from both sides. For any affine set, the intersection with any line is affine, while for any line, if the intersections are all affine, the set is affine.

□

1.3 Midpoint Convexity Implies Convexity

Proof. For any $x, y \in C$, there is:

$$\frac{1}{2} x + \frac{1}{2} y \in C$$

Apply this recursively to every point in the set. Every point can be represented by a sum times a half. Neverthless, we try to express every point on a line segment with divisions by 2.

$$\theta x + (1 - \theta) y = x + \theta (y - x)$$

The key is to proof that θ can take continuous value along the interval $[0, 1]$. By taking infinite times of division, a point on the line segment can be represented by the power series of $1/2$:

$$\theta = \sum_i c_i 2^{-i}$$

$\square$

1.4 Convex Hull is the Intersection of All

Proof. This is to proof that any convex set containing S also contains conf S, and no larger. $\square$

1.5 Distance between Hyperplanes

The distance between the hyperplanes is the length of the projection of the difference between 2 vectors on the 2 planes on the direction of the normal vector. Suppose there are 2 vectors on the 2 planes $a^T x_1 = b_1, a^T x_2 = b_2$. The length of the projection is given by the inner product between the vectors:

$$l = \frac{a^T(x_1 - x_2)}{||a||_2} = \frac{b_1 - b_2}{||a||_2}$$

1.6 Halfspaces Contain each Other

This is to express the condition that a vector is contained by a set as well as another.

$$a^T x \leq b \Rightarrow \tilde{a}^T x \leq \tilde{b}$$

$a, \tilde{a}$ must be parrelle to each other, or the condition won't hold. Therefore:

$$\lambda a = \tilde{a}, a^T x \leq b \Rightarrow a^T x \leq \frac{1}{\lambda}\tilde{b}$$

In order for this to hold:

$$b \leq \frac{1}{\lambda}\tilde{b}$$

1.7 Mid Half Space

The boundary is $||x - a||_2 = ||x - b||_2$, which is a line by a theorem of geometry. Neverthless, we perform the computation here.

$$||x - a||_2^2 = \sum_j (x_j - a_j)^2, ||x - a||_2^2 - ||x - b||_2^2 = \sum_j (b_j - a_j)(2x_j - (a_j + b_j))$$

The sum is represented as an inner product:

$$\sum_j (b_j - a_j)(2x_j - (a_j + b_j)) = (b-a)^T(2x - (a+b))$$

Express in the form of $c^T x = d$:

$$(b-a)^T 2x = (b-a)^T(a+b) = b^2 - a^2$$

1.8 Express Polyhedra

(a)

$$x = y_1 a_1 + y_2 a_2 = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Ay$$

Write the inverse form:

$$y = A^{-1}x$$

Plug in the range in y:

$$A^{-1}x \succeq (-1,-1)^T, A^{-1}x \preceq (1,1)^T$$

(b) Compose the 3 expressions together:

$$\begin{pmatrix} a_1 \\ a_2 \\ 1 \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \\ 1 \end{pmatrix}$$

(c) Expand the norm:

$$\sum_i^n y_i^2 = 1$$

1.9 Voronoi Sets

(a) Expand norm:

$$(x - x_0)^2 \leq (x_i - x_0)^2 \Rightarrow 2(x_i - x_0)^T x = 2x^T(x_i - x_0) \leq x_i^T x_i - x_0^T x_0$$

To the right form:

$$A_{K\times n} = \begin{pmatrix} \vdots \\ x_i^T - x_0^T \\ \vdots \end{pmatrix}, b_{K\times 1} = \begin{pmatrix} \vdots \\ x_i^2 - x_0^2 \\ \vdots \end{pmatrix}$$

(b)

1.10 Quadratic

(a) Plug in the definition of convex sets:

$$x$$

(b)