Model Predictive Control

7. Reference Tracking and Disturbance Rejection

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Reference Tracking based on Target Calculation

- **Discrete-Time Linear Time-Invariant (LTI) System**

$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k)$	state equation	(7.1)
$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k)$	measured output equation	(7.2)
$\boldsymbol{y}_{\boldsymbol{r}}(k) = \boldsymbol{C}_{\boldsymbol{r}}\boldsymbol{x}(k)$	controlled output equation	(7.3)

- **Symbols**

$\boldsymbol{x}(k) \in \mathbb{R}^n$ state vector	$\boldsymbol{u}(k) \in \mathbb{U} \subseteq \mathbb{R}^m$ input vector
$\boldsymbol{y}(k) \in \mathbb{Y} \subseteq \mathbb{R}^p$ measured output vector	$\boldsymbol{y}_{\boldsymbol{r}}(k) \in \mathbb{R}^{p_r}$ controlled output vector
$\boldsymbol{A} \in \mathbb{R}^{n \times n}$ system matrix	$\boldsymbol{B} \in \mathbb{R}^{n \times m}$ input matrix
$\boldsymbol{C} \in \mathbb{R}^{p \times n}$ measured output matrix	$\boldsymbol{C}_{\boldsymbol{r}} \in \mathbb{R}^{p_r \times n}$ controlled output matrix

- **Remarks**
  - The measured output $\boldsymbol{y}(k)$ is used for the observer
  - The controlled output $\boldsymbol{y}_{\boldsymbol{r}}(k)$ is considered for reference tracking

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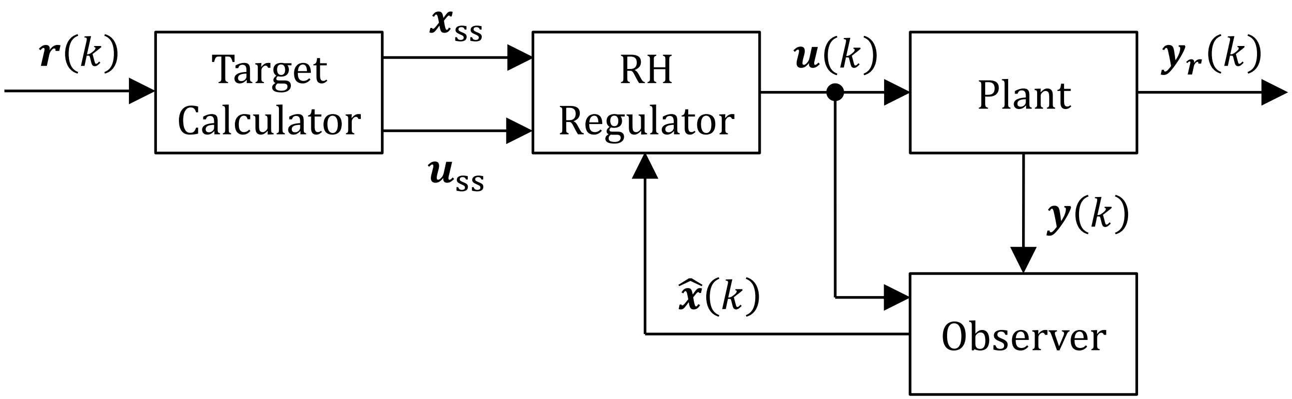
- **Structure**



- **Objective**
  - Control the discrete-time LTI system such that $\boldsymbol{y}_{\boldsymbol{r}}(k) \to \boldsymbol{r}(k)$ if $\boldsymbol{r}(k) \to$ const. as $k \to \infty$
- **Approach**
  - The target calculator computes the target state $\boldsymbol{x}_{\mathrm{ss}}$ and the target input $\boldsymbol{u}_{\mathrm{ss}}$
  - The RH regulator controls the discrete-time LTI system to the target pair $(\boldsymbol{x}_{\mathrm{ss}}, \boldsymbol{u}_{\mathrm{ss}})$

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- **Approach**
  - Consider that the discrete-time LTI system (7.1)/(7.3) is in the steady target state $\boldsymbol{x}_{\mathrm{ss}}$, i.e.

$$\begin{aligned} \boldsymbol{x}_{\mathrm{ss}} = \boldsymbol{A}\boldsymbol{x}_{\mathrm{ss}} + \boldsymbol{B}\boldsymbol{u}_{\mathrm{ss}} \quad &\Leftrightarrow \quad (\boldsymbol{I}_{n\times n} - \boldsymbol{A})\boldsymbol{x}_{\mathrm{ss}} - \boldsymbol{B}\boldsymbol{u}_{\mathrm{ss}} = \boldsymbol{0}_{n\times 1} \\ \boldsymbol{y}_{r\mathrm{ss}} = \boldsymbol{C}_r\boldsymbol{x}_{\mathrm{ss}} = \boldsymbol{r} \quad &\Leftrightarrow \quad \boldsymbol{C}_r\boldsymbol{x}_{\mathrm{ss}} = \boldsymbol{r} \end{aligned}$$

  - Rewrite the equations in matrix form, i.e.

$$\begin{pmatrix} \boldsymbol{I}_{n\times n} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C}_r & \boldsymbol{0}_{p_r\times m} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{\mathrm{ss}} \\ \boldsymbol{u}_{\mathrm{ss}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0}_{n\times 1} \\ \boldsymbol{r} \end{pmatrix} \tag{7.4}$$

  - The steady target pair $(\boldsymbol{x}_{\mathrm{ss}}, \boldsymbol{u}_{\mathrm{ss}})$ can be then calculated from (7.4) provided that a solution exists
  - Introduce the state deviation and input deviation

$$\tilde{\boldsymbol{x}}(k) = \hat{\boldsymbol{x}}(k) - \boldsymbol{x}_{\mathrm{ss}}, \quad \tilde{\boldsymbol{u}}(k) = \boldsymbol{u}(k) - \boldsymbol{u}_{\mathrm{ss}}$$

  - This leads to the discrete-time LTI state equation

$$\tilde{\boldsymbol{x}}(k+1) = \hat{\boldsymbol{x}}(k+1) - \boldsymbol{x}_{\mathrm{ss}} = \boldsymbol{A}\hat{\boldsymbol{x}}(k) + \boldsymbol{B}\boldsymbol{u}(k) - (\boldsymbol{A}\boldsymbol{x}_{\mathrm{ss}} + \boldsymbol{B}\boldsymbol{u}_{\mathrm{ss}}) = \boldsymbol{A}\tilde{\boldsymbol{x}}(k) + \boldsymbol{B}\tilde{\boldsymbol{u}}(k) \tag{7.5}$$

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- **Approach**

  the discrete-time quadratic cost function

$$\tilde{V}_N\left(\tilde{\boldsymbol{x}}(k), \tilde{\boldsymbol{U}}(k)\right) = \tilde{\boldsymbol{x}}^T(k+N)\boldsymbol{P}\tilde{\boldsymbol{x}}(k+N) + \sum_{i=0}^{N-1} \tilde{\boldsymbol{x}}^T(k+i)\boldsymbol{Q}\tilde{\boldsymbol{x}}(k+i) + \tilde{\boldsymbol{u}}^T(k+i)\boldsymbol{R}\tilde{\boldsymbol{u}}(k+i) \quad (7.6)$$

  the state and input constraints

$$\begin{aligned} &\boldsymbol{C}(\tilde{\boldsymbol{x}}(k+i) + \boldsymbol{x}_{ss}) \in \mathbb{Y}, i = 1,2,\dots,N \\ &\tilde{\boldsymbol{u}}(k+i) + \boldsymbol{u}_{ss} \in \mathbb{U}, \quad i = 0,1,\dots,N-1 \end{aligned} \quad (7.7)$$

  – Rewrite Problem 4.1/5.1 w.r.t. the state equation (7.5), cost function (7.6) and constraints (7.7), i.e.

$$\min_{\tilde{\boldsymbol{U}}(k)} \tilde{V}_N(\tilde{\boldsymbol{x}}(k), \tilde{\boldsymbol{U}}(k))$$

$$\text{subject to} \begin{cases} \tilde{\boldsymbol{x}}(k+i+1) = \boldsymbol{A}\tilde{\boldsymbol{x}}(k+i) + \boldsymbol{B}\tilde{\boldsymbol{u}}(k+i), i = 0,1,\dots,N-1 \\ \boldsymbol{C}(\tilde{\boldsymbol{x}}(k+i) + \boldsymbol{x}_{ss}) \in \mathbb{Y}, \quad i = 1,2,\dots,N \\ \tilde{\boldsymbol{u}}(k+i) + \boldsymbol{u}_{ss} \in \mathbb{U}, \quad i = 0,1,\dots,N-1 \end{cases} \quad (7.8)$$

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- **Remark on the Optimization Problem**
  - Problem (7.8) can be formulated as a quadratic program using the methods from Chapter 4 and 5
  - Problem (7.8) relates to a regulation problem, i.e. $\tilde{\boldsymbol{x}}(k) \to \mathbf{0}, \tilde{\boldsymbol{u}}(k) \to \mathbf{0}$ as $k \to \infty$
  - The reference tracking problem is, however, simultaneously addressed since

    $$\tilde{\boldsymbol{x}}(k) \to \mathbf{0}, \tilde{\boldsymbol{u}}(k) \to \mathbf{0} \Rightarrow \hat{\boldsymbol{x}}(k) \to \boldsymbol{x}_{\mathrm{ss}}, \boldsymbol{u}(k) \to \boldsymbol{u}_{\mathrm{ss}} \Rightarrow \boldsymbol{y}_{\boldsymbol{r}}(k) \to \boldsymbol{r}$$

- **Remark on the Target Calculator**
  - Generally it is not possible to control the state $\boldsymbol{x}(k)$ to an arbitrary target state $\boldsymbol{x}_{\mathrm{ss}}$
  - E.g. it is not possible to maintain a constant position and a constant velocity of a car simultaneously
  - A sufficient condition for the existence of a solution of (7.4) for any reference input $\boldsymbol{r}$ is that

    $$\begin{pmatrix} \boldsymbol{I}_{n\times n} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C}_{\boldsymbol{r}} & \mathbf{0}_{p_{\boldsymbol{r}}\times m} \end{pmatrix} \in \mathbb{R}^{(n+p_r)\times(n+m)} \text{ has full rank } n + p_{\boldsymbol{r}}$$

  - This implies that $\boldsymbol{C}_{\boldsymbol{r}}$ must have full rank and number of controlled outputs $p_{\boldsymbol{r}} \leq$ number of inputs m
  - The solution may not be unique

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- **Remark on the Constrained Case**
  - For the constrained case target pair $(\boldsymbol{x}_{\mathrm{ss}}, \boldsymbol{u}_{\mathrm{ss}})$ must fulfill output constraint $\mathbb{Y}$ and input constraint $\mathbb{U}$
  - For this purpose the target calculator based on (7.4) must be modified to

$$\min_{\boldsymbol{x}_{\mathrm{ss}},\boldsymbol{u}_{\mathrm{ss}}} \frac{1}{2}\left((\boldsymbol{C}_r\boldsymbol{x}_{\mathrm{ss}} - \boldsymbol{r})^T\boldsymbol{Q}_{\mathrm{ss}}(\boldsymbol{C}_r\boldsymbol{x}_{\mathrm{ss}} - \boldsymbol{r}) + (\boldsymbol{u}_{\mathrm{ss}} - \boldsymbol{u}_{\mathrm{ss}}^{\mathrm{unc}})^T\boldsymbol{R}_{\mathrm{ss}}(\boldsymbol{u}_{\mathrm{ss}} - \boldsymbol{u}_{\mathrm{ss}}^{\mathrm{unc}})\right)$$

$$\text{subject to} \begin{cases} \begin{pmatrix} \boldsymbol{I}_{n\times n} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C}_r & \boldsymbol{0}_{p_r\times m} \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_{\mathrm{ss}} \\ \boldsymbol{u}_{\mathrm{ss}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0}_{n\times 1} \\ \boldsymbol{r} \end{pmatrix} \\ \boldsymbol{C}\boldsymbol{x}_{\mathrm{ss}} \in \mathbb{Y} \\ \boldsymbol{u}_{\mathrm{ss}} \in \mathbb{U} \end{cases} \tag{7.9}$$

    where $\boldsymbol{u}_{\mathrm{ss}}^{\mathrm{unc}}$ is the target input resulting from (7.4) for the unconstrained case and $\boldsymbol{Q}_{\mathrm{ss}} = \boldsymbol{Q}_{\mathrm{ss}}^T \succcurlyeq \boldsymbol{0}$ and $\boldsymbol{R}_{\mathrm{ss}} = \boldsymbol{R}_{\mathrm{ss}}^T \succ \boldsymbol{0}$ are weighting matrices
  - Problem (7.9) can be formulated as a quadratic program
  - Feasibility of Problem (7.9) is discussed in [RM09, Section 1.5.1]

Reference Tracking based on Target Calculation

- **Remark on Receding Horizon Control**

  1. Estimate the current state $\hat{\boldsymbol{x}}(k)$
  2. Solve Problem (7.9) for the given reference input $\boldsymbol{r}(k)$ to determine the target pair $(\boldsymbol{x}_{\mathrm{ss}}, \boldsymbol{u}_{\mathrm{ss}})$
  3. Solve Problem (7.8) for $\tilde{\boldsymbol{x}}(k) = \hat{\boldsymbol{x}}(k) - \boldsymbol{x}_{\mathrm{ss}}$ to determine the optimal input sequence $\widetilde{\boldsymbol{U}}^*(k)$
  4. Compute first element of the optimal input sequence $\tilde{\boldsymbol{u}}^*(k) = (\boldsymbol{I}_{m\times m} \quad \boldsymbol{0}_{m\times m} \quad \cdots \quad \boldsymbol{0}_{m\times m})\widetilde{\boldsymbol{U}}^*(k)$
  5. Implement the optimal input $\boldsymbol{u}^*(k) = \tilde{\boldsymbol{u}}^*(k) + \boldsymbol{u}_{\mathrm{ss}}$
  6. Increment the time instant $k \coloneqq k + 1$ and go to 1.

- **Further Remarks**

  – The extension for state constraints and controlled output constraints is straightforward

  – The structure on Slide 7-3 is essentially equivalent to the state-command structure on Slide 2-49ff

  – Disturbances and uncertainties in $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}_{\boldsymbol{r}}$ lead to a steady-state error or offset

  – More details and references are given in [RM09, Section 1.5.1], [BBM17, Section 12.7], and [MR93]

Literature

Miscellaneous

[MR93] Kenneth R. Muske and James B. Rawlings. Model predictive control with linear models. *AIChE Journal*, 39(2):262–287, 1993.

[PR03] Gabriele Pannocchia and James B. Rawlings. Disturbance models for offset-free model predictive control. *AIChE Journal*, 49(2):426–437, 2003.